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PARAMETER-DEPENDENT ESTIMATES FOR MIXED-ORDER BOUNDARY VALUE PROBLEMS

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Abstract. In this paper we prove parameter-dependent a priori estimates for mixed-order boundary value problems of rather general structure. In particular, the diagonal operators are not assumed to be of the same order. Our assumptions on the structure of the boundary value problem covers the case of Dirichlet type boundary conditions.

1. Introduction

Let us consider the model problem for a general mixed-order system of the form

$$A(D)u(x) - \lambda u(x) = f(x) \text{ in } \mathbb{R}^n_+, \tag{1.1}$$

$$B_j(D)u(x) = g_j(x) \text{ on } \mathbb{R}^{n-1} \text{ for } j = 1, \dots, \widetilde{N}, \tag{1.2}$$

Here $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x_n > 0\}$ denotes the half-space with boundary $\partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}$, $u(x) = (u_1(x), \dots, u_N(x))^T$ and $f(x) = (f_1(x), \dots, f_N(x))^T$ are N-dimensional vector functions defined in $\mathbb{R}^n_+$, $g_j(x)$ are scalar functions defined on $\mathbb{R}^{n-1}$. The matrix $A(D) = \big(A_{jk}(D)\big)_{j,k=1,\dots,N}$ is a mixed-order system of differential operators, and $B_j(D)$ is a $1 \times N$-matrix operator for $1 \le j \le \widetilde{N}$. The Douglis-Nirenberg structure of A is given by integers $\{s_j\}_{j=1,\dots,N}$ and $\{t_j\}_{j=1,\dots,N}$ satisfying $s_1 \ge \dots \ge s_n \ge 0$ and $t_1 \ge \dots \ge t_N \ge 0$. Setting $m_j = s_j + t_j$, we assume, for simplicity, that $m_1 > \dots > m_N > 0$.

In the sequel we will impose conditions which will ensure that m_j is even for $j = 1, \dots, N$ and set $N_j := \frac{1}{2}(m_1 + \dots + m_j)$ for $j = 1, \dots, N$ and $\widetilde{N} := N_N$. Then the mixed-order structure of the boundary conditions is given by a sequence $\{\sigma_j\}_{j=1,\dots,\widetilde{N}}$ of integers with $\sigma_j < 0$ for $j = 1, \dots, \widetilde{N}$. We then assume $\operatorname{ord} A_{jk} \le s_j + t_k$, $j, k = 1, \dots, N$, and $\operatorname{ord} B_{jk} \le \sigma_j + t_k$, $j = 1, \dots, \widetilde{N}$, $k = 1, \dots, N$. As we will study the model problem only, we further assume that $A_{jk} = 0$ if $\operatorname{ord} A_{jk} < s_j + t_k$ and $B_{jk} = 0$ if $\operatorname{ord} B_{jk} < \sigma_j + t_k$. Further the operators A and B_j are assumed to have constant coefficients,

$$A_{jk}(D) = \sum_{|\alpha| = s_j + t_k} a_\alpha^{jk} D^\alpha, \quad j, k = 1, \dots, N,$$

$$B_{jk}(D) = \sum_{|\alpha| = \sigma_j + t_k} b_\alpha^{jk} D^\alpha, \quad j = 1, \dots, \widetilde{N},\ k = 1, \dots, N.$$

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Here we used the standard multi-index notation $D^\alpha = (-i)^{|\alpha|}\partial_{x_1}^{\alpha_1}\dots\partial_{x_n}^{\alpha_n}$, $\alpha = (\alpha_1,\dots,\alpha_n)$. We will show that under suitable parameter-ellipticity conditions, unique solvability and a priori estimates for the solution hold. First results in this direction were obtained by Kozhevnikov in [K1]. In this paper the author deals with a system of pseudodifferential operators on a closed manifold; by introducing the so-called Kozhevnikov conditions, the author is able to establish a priori estimates and spectral results. In subsequent works [K2], [K3], the author deals with genuine boundary value problems, imposing, however, conditions on the dimension of the system ($N = 2$ in [K2]) or assuming triangular form of the boundary matrix. In the paper [DMV] by Denk, Mennicken, and Volevich, and [DV] by Denk and Volevich, the Newton polygon method was used to establish a priori-estimates for rather general systems. However, also in the paper [DV] the authors impose severe restrictions on the order of the boundary operators. Both papers [K3] and [DV] are not able to deal with the important problem where $s_j = t_j = t'_j$ for $j = 1,\dots,N$ (here $\{t'_j\}_{j=1}^N$ denotes a monotonic decreasing sequence of positive integers) and the boundary conditions are of Dirichlet type (see [ADN, Section 2], [G, p.448]). In the present paper, we establish a priori estimates for solutions of (1.1), (1.2) under boundary conditions which include Dirichlet type conditions. In this way, we also generalize results from Agranovich, Denk, and Faierman [ADF] concerning scalar problems.

Let us mention that we restrict ourselves to the model problem (1.1), (1.2). The general case of boundary value problems in bounded domains with coefficients with limited smoothness, as well as less restrictive assumptions on the order structure, are treated in our paper [DF]; the present note should be seen as a simplified and shortened version of [DF].

The structure of the paper is as follows. In Section 2 we introduce some terminology and definitions concerning the boundary value problem (1.1), (1.2) and present the main result, Theorem 2.6 below. The proof of Theorem 2.6 can be found in Sections 3 and 4.

2. Assumptions and main results

Let us first introduce some notation. For $1 < p < \infty$ and $s \in \mathbb{N}\cup\{0\}$, let $W_p^s(\mathbb{R}^n_+)$ stand for the standard L_p-Sobolev space with norm

$$\|u\|_{s,p,\mathbb{R}^n_+} = \Big(\sum_{|\alpha|\le s}\int_{\mathbb{R}^n_+}|D^\alpha u(x)|^p\,dx\Big)^{1/p}.$$

For $1 \le j \le N$ and $\lambda \in \mathbb{C}\setminus\{0\}$, we define the parameter-dependent norm

$$|\!|\!|u|\!|\!|^{(j)}_{s,p,\mathbb{R}^n_+} = \|u\|_{s,p,\mathbb{R}^n_+} + |\lambda|^{s/m_j}\|u\|_{0,p,\mathbb{R}^n_+} \quad \text{for } u \in W_p^s(\mathbb{R}^n_+).$$

We will also deal with the Bessel potential spaces $H_p^s(\mathbb{R}^n_+)$ for $s \in \mathbb{Z}$, $s < 0$, and the related parameter-dependent norm $|\!|\!|u|\!|\!|^{(j)}_{s,p,\mathbb{R}^n} = \|F^{-1}\langle\xi,\lambda\rangle_j^s Fu\|_{0,p,\mathbb{R}^n}$ and $|\!|\!|u|\!|\!|^{(j)}_{s,p,\mathbb{R}^n_+} = \inf |\!|\!|v|\!|\!|^{(j)}_{s,p,\mathbb{R}^n}$, where the infimum is taken over all $v \in H_p^s(\mathbb{R}^n)$ for which $u = v\big|\mathbb{R}^n_+$, F denotes the Fourier transformation in $\mathbb{R}^n(x \to \xi)$ and $\langle\xi,\lambda\rangle_j = (|\xi|^2 + |\lambda|^{2/m_j})^{1/2}$ (see [GK, Section 1], [T, p. 177]).

On the boundary $\mathbb{R}^{n-1}$, the trace spaces $W_p^{s-1/p}(\mathbb{R}^{n-1})$, $s \in \mathbb{N}$, are defined in a standard way (see, e.g., [ADF, Section 2] and [Gr, p.20]). For $\lambda \in \mathbb{C}\backslash\{\,0\,\}$ and

$1 \le j \le N$, we set

$$|\!|\!| u |\!|\!|^{(j)}_{s-1/p,p,\mathbb{R}^{n-1}} = \|u\|_{s-1/p,p,\mathbb{R}^{n-1}} + |\lambda|^{(s-1/p)/m_j} \|u\|_{0,p,\mathbb{R}^{n-1}} \quad \text{for } u \in W_p^{s-1/p}(\mathbb{R}^{n-1}).$$

To formulate the parameter-ellipticity conditions on (A, B), we set for $\xi \in \mathbb{R}^n$ and $1 \le r \le N$

$$\mathcal{A}_{11}^{(r)}(\xi) = (A_{jk}(\xi))_{j,k=1,\dots,r}\,, \qquad \mathcal{A}_{12}^{(r)}(\xi) = (A_{jk}(\xi))_{\substack{j=1,\dots,r\\ k=r+1,\dots,N}}\,,$$
$$\mathcal{A}_{21}^{(r)}(\xi) = (A_{jk}(\xi))_{\substack{j=r+1,\dots,N\\ k=1,\dots,r}}\,, \text{ and} \qquad \mathcal{A}_{22}^{(r)}(\xi) = (A_{jk}(\xi))_{j,k=r+1,\dots,N}\,.$$

Also for $\xi \in \mathbb{R}^n$ and $1 \le r \le \ell_1, \ell \le N$, let

$$\mathcal{B}^{(r,\ell)}(\xi) = (B_{jk}(\xi))_{\substack{j=N_{\ell-1}(1-\delta_{r,\ell})+1,\dots,N_\ell\\ k=1,\dots,N}}\,,$$
$$\mathcal{B}_{\ell_1,1}^{(r,\ell)}(\xi) = (B_{jk}(\xi))_{\substack{j=N_{\ell-1}(1-\delta_{r,\ell})+1,\dots,N_\ell\\ k=1,\dots,\ell_1}}\,,$$
$$\mathcal{B}_{\ell_1,2}^{(r,\ell)}(\xi) = (B_{jk}(\xi))_{\substack{j=N_{\ell-1}(1-\delta_{r,\ell})+1,\dots,N_\ell\\ k=\ell_1+1,\dots,N}}\,,$$

where $\delta_{r,\ell}$ is the Kronecker delta. In addition we let I_r denote the r-dimensional unit matrix and $I_{r,0} = \operatorname{diag}(0,\dots,0,1) \in \mathbb{R}^{r\times r}$ for $r = 1,\dots,N$.

Definition 2.1 ([K1], [DMV])**.** Let $\mathcal{L}$ be a closed sector in the complex plane with vertex at the origin. Then the operator $A(D) - \lambda I_N$ will be called parameter-elliptic in $\mathcal{L}$ if $\det\big(\mathcal{A}_{11}^{(r)}(\xi) - \lambda I_{r,0}\big) \neq 0$ for $\xi \in \mathbb{R}^n \setminus \{0\}$ and $\lambda \in \mathcal{L}$, $r = 1,\dots,N$.

In the sequel we let $\mathbb{C}_\pm = \{z \in \mathbb{C}, \operatorname{Im} z \gtrless 0\}$.

Definition 2.2. Suppose that the operator $A(D) - \lambda I_N$ is parameter-elliptic in the sector $\mathcal{L}$. Then the operator $A(D) - \lambda I_N$ will be called properly parameter-elliptic in $\mathcal{L}$ if the following conditions are satisfied.

1. The polynomial $\det\big(\mathcal{A}_{11}^{(r)}(\xi', z) - \lambda I_{r,0}\big)$ has precisely N_r zeros lying in $\mathbb{C}_+$ for $\xi' \in \mathbb{R}^{n-1} \setminus \{0\}$ and $\lambda \in \mathcal{L}$, $r = 1,\dots,N$.
2. The polynomial $\det\big(\mathcal{A}_{11}^{(r)}(0, z) - \lambda I_{r,0}\big)$ has precisely $N_r - N_{r-1}$ zeros lying in $\mathbb{C}_+$ for $\lambda \in \mathcal{L} \setminus \{0\}$, $r = 2,\dots,N$.

Remark 2.3. Referring to Condition (1) of Definition 2.2, we know from [AV, Section 2] that $\det(\mathcal{A}_{11}^{(r)}(\xi', z) - \lambda I_{r,0})$ has precisely N_r zeros in $\mathbb{C}_+$ if $r = 1$ or if $r > 1$ and $n > 2$. Turning next to Condition (2) of the definition, it is clear that the number of zeros of the determinant in $\mathbb{C}_+$ (resp. $\mathbb{C}_-$) does not depend upon λ. Hence it follows from an expansion of the determinant in powers of z and λ that Condition (2) always holds if m_r is even and there is a $\lambda \in \mathcal{L}\setminus\{0\}$ such that $-\lambda \in \mathcal{L}$. Lastly we mention at this point that it is also clear from what was said above that Condition (2) is always satisfied if the operator $A(D)$ is essentially upper triangular (see Definition 2.5 below).

Definition 2.4. We say that the boundary problem (1.1), (1.2) is parameter-elliptic in $\mathcal{L}$ if $A(x, D) - \lambda I_N$ is properly parameter-elliptic in $\mathcal{L}$ and the following conditions are satisfied. According to the notation introduced above, let

$$\mathcal{B}_{r,1}^{(r,r)}(\xi', D_n) = (B_{jk}(\xi', D_n))_{\substack{j=1,\dots,N_r \\ k=1,\dots,r}}, \quad 1 \le r \le N,$$

$$\mathcal{B}_{\ell,1}^{(r,\ell)}(\xi', D_n) = (B_{jk}(\xi', D_n))_{\substack{j=N_{\ell-1}+1,\dots,N_\ell \\ k=1,\dots,\ell}}, \quad 1 \le r < \ell \le N.$$

Then

(1) the boundary problem on the half-line

$$\mathcal{A}_{11}^{(r)}(\xi', D_n)v(x_n) - \lambda I_{r,0}v(x_n) = 0 \quad \text{for } x_n > 0,$$
$$\mathcal{B}_{r,1}^{(r,r)}(\xi', D_n)v(x_n) = 0 \quad \text{at } x_n = 0,$$
$$|v(x_n)| \to 0 \quad \text{as } x_n \to \infty,$$

has only the trivial solution for $\xi' \in \mathbb{R}^{n-1} \setminus \{0\}$, $\lambda \in \mathcal{L}$ and $1 \le r \le N$;

(2) the boundary problem on the half-line

$$\mathcal{A}_{11}^{(\ell)}(0, D_n)v(x_n) - \lambda\, I_{\ell,0}v(x_n) = 0 \quad \text{for } x_n > 0,$$
$$\mathcal{B}_{\ell,1}^{(r,\ell)}(0, D_n)v(x_n) = 0 \quad \text{at } x_n = 0,$$
$$|v(x_n)| \to 0 \quad \text{as } x_n \to \infty,$$

has only the trivial solution for $\lambda \in \mathcal{L} \setminus \{0\}$, $1 \le r < \ell \le N$.

For our purposes we need to introduce some further terminology. For $1 \le j \le \widetilde{N}$, let $\pi(j) = r$ if $N_{r-1} < j \le N_r$, where $N_0 = 0$. In addition we let $\langle \xi \rangle = \big(1 + |\xi|^2\big)^{1/2}$, $\langle \xi' \rangle = \big(1 + |\xi'|^2\big)^{1/2}$, and $\langle \xi', \lambda \rangle_j = \big(|\xi'|^2 + |\lambda|^{2/m_j}\big)^{1/2}$ for $1 \le j \le N$. We also require the following definition.

Definition 2.5. We say that the operator $A(D)$ is essentially upper triangular if $a_\alpha^{jk} = 0$ for $|\alpha| = s_j + t_k$, $1 \le k \le j-1, \ell = 2, \dots, N$. Likewise we say that the operator $B(D) = (B_{jk}(D))_{\substack{j=1,\dots,\widetilde{N} \\ k=1\dots,N}}$ is essentially upper triangular if $b_\alpha^{jk} = 0$ for $|\alpha| = \sigma_j + t_k, N_{\ell-1} < j \le N_\ell, 1 \le k \le \ell - 1, \ell = 2, \dots, N$.

We are now in a position to state the main result of this paper, namely Theorem 2.6 below, which will be proved in Sections 3 and 4. In this theorem we will require the further assumption, which will be made precise in Definition 4.4 below, that the operators $A(D)$ and $B(D)$ are compatible. Hence for the moment let us state that this condition will always be satisfied if $B(D)$ is of Dirichlet type or if the operators $A(D)$ and $B(D)$ are essentially upper triangular.

Theorem 2.6. *Suppose that the boundary problem* (1.1), (1.2) *is parameter-elliptic in $\mathcal{L}$. Suppose also that the operators $A(D)$ and $B(D)$ are compatible. In addition, suppose that $B(D)$ is essentially upper triangular. Then there exists a constant $\lambda^0 = \lambda^0(p) > 1$ such that for $\lambda \in \mathcal{L}$ with $|\lambda| \ge \lambda^0$, the boundary problem* (1.1), (1.2) *has a unique solution $u \in \prod_{j=1}^N W_p^{t_j}(\mathbb{R}_+^n)$ for every $f \in \prod_{j=1}^N H_p^{-s_j}(\mathbb{R}_+^n)$ and $g = \big(g_1, \dots, g_{\widetilde{N}}\big)^T \in \prod_{j=1}^{\widetilde{N}} W_p^{-\sigma_j - 1/p}(\mathbb{R}^{n-1})$, and the a priori estimate*

$$\sum_{j=1}^N |||u_j|||_{t_j,p,\mathbb{R}_+^n}^{(j)} \le C \left(\sum_{j=1}^N |||f_j|||_{-s_j,p,\mathbb{R}_+^n}^{(j)} + \sum_{j=1}^{\widetilde{N}} |||g_j|||_{-\sigma_j - 1/p,p,\mathbb{R}^{n-1}}^{(\pi(j))} \right) \tag{2.1}$$

holds, where the constant C does not depend upon the f_j, g_j, and λ.

Remark 2.7. It can be shown by standard arguments (cf. [AV]) and the results of Section 3 that in fact the estimate (2.1) is 2-sided, i.e., an estimate reverse to (2.1) holds.

3. Proof of the main theorem

Following a standard approach in elliptic theory, we will prove Theorem 2.6 by studying first the whole space equation and then the equation in the half-space. The main technical issue, the proof of Proposition 3.4 below, can be found in Section 4. In the following, $C, C_1, C_2, \dots, \widetilde{C}_1, \widetilde{C}_2, \dots$ stand for unspecific constants which may vary at each time of their appearance.

Let us first consider the whole-space equation

$$A(D)u(x) - \lambda\, u(x) = f(x) \text{ for } x \in \mathbb{R}^n \text{ and } \lambda \in \mathcal{L} \setminus \{0\}. \tag{3.1}$$

We then have the following two results.

Proposition 3.1. *Suppose that* $u \in \prod_{j=1}^N W_p^{t_j}(\mathbb{R}^n)$ *and that* (3.1) *holds. Then* $f \in \prod_{j=1}^N H_p^{-s_j}(\mathbb{R}^n)$ *and* $\sum_{j=1}^N |\!|\!| f_j |\!|\!|^{(j)}_{-s_j,p,\mathbb{R}^n} \le C \sum_{j=1}^N |\!|\!| u_j |\!|\!|^{(j)}_{t_j,p,\mathbb{R}^n}$.

Proposition 3.2. *Suppose that the operator* $A(D) - \lambda\, I_N$ *is parameter-elliptic in* $\mathcal{L}$ *and that* $f \in \prod_{j=1}^N H_p^{-s_j}(\mathbb{R}^n)$. *Then there exists the constant* $\lambda^0 > 0$ *such that for* $\lambda \in \mathcal{L}$ *with* $|\lambda| \ge \lambda^0$, *the differential equation* (3.1) *has a unique solution* $u \in \prod_{j=1}^N W_p^{t_j}(\mathbb{R}^n)$ *and* $\sum_{j=1}^N |\!|\!| u_j |\!|\!|^{(j)}_{t_j,p,\mathbb{R}^n} \le C \sum_{j=1}^N |\!|\!| f_j |\!|\!|^{(j)}_{-s_j,p,\mathbb{R}^n}$.

We will only prove Proposition 3.2 as the proof of Proposition 3.1 follows directly from the definition and the Mikhlin-Lizorkin multiplier theorem.

Proof of Proposition 3.2. Under our assumptions we know from [DMV] and [K1] that for $\xi \in \mathbb{R}^n$ and $\lambda \in \mathcal{L}$ with $|\lambda| \ge \lambda^0$, $A(\xi) - \lambda\, I_N$ is invertible and

$$\Big| \det\left(A(\xi) - \lambda\, I_N\right) \Big| \ge C \prod_{j=1}^N \langle \xi, \lambda \rangle_j^{m_j}.$$

Furthermore, if we put $\left(A(\xi) - \lambda\, I_N\right)^{-1} = \left(\widetilde{a}_{j,k}(\xi,\lambda)\right)_{j,k=1}^N$, then the $\widetilde{a}_{j,k}(\xi,\lambda)$ are rational functions of their arguments, while it also follows from the references just cited that for any multi-index α whose entries are either 0 or 1,

$$|\xi^\alpha D_\xi^\alpha \widetilde{a}_{j,k}(\xi,\lambda)| \le C \langle \xi \rangle^{s_j + t_k} \langle \xi, \lambda \rangle_j^{-m_j} \langle \xi, \lambda \rangle_k^{-m_k}$$

for all $\xi \in \mathbb{R}^n$ whose components are all non-zero.

Now observe that under Fourier transformation (3.1) becomes

$$A(\xi) Fu(\xi) - \lambda\, Fu(\xi) = Ff(\xi).$$

Furthermore, in light of what was said above, we conclude that this equation has a unique solution in the space of tempered distributions on $\mathbb{R}^n$ given by $Fu(\xi) = \left(A(\xi) - \lambda\, I_N\right)^{-1} Ff(\xi)$. Hence all of the assertions of the proposition follow immediately from this last result, the definitions of the terms involved, and the Mikhlin-Lizorkin multiplier theorem. □

Let us next fix our attention upon the boundary problem

$$A(D)u(x) - \lambda\, u(x) = 0 \text{ for } x \in \mathbb{R}^n_+, \tag{3.2}$$

$$B_j(D)u(x) = g_j(x') \text{ at } x_n = 0,\ j = 1, \dots, \widetilde{N}, \tag{3.3}$$

with $\lambda \in \mathbb{C} \setminus \{0\}$.

Proposition 3.3. *Suppose that* $u \in \prod_{j=1}^N W_p^{t_j}(\mathbb{R}_+^n)$, *that* (3.3) *holds, and that* $B(x,D)$ *is essentially upper triangular. Then*

$$g = (g_1, \dots, g_{\widetilde{N}})^T \in \prod_{j=1}^{\widetilde{N}} W_p^{-\sigma_j - 1/p}(\mathbb{R}^{n-1})$$

and

$$\sum_{j=1}^{\widetilde{N}} |\!|\!| g_j |\!|\!|^{(\pi(j))}_{-\sigma_j - \frac{1}{p}, p, \mathbb{R}^{n-1}} \le C \sum_{j=1}^{N} |\!|\!| u_j |\!|\!|^{(j)}_{t_j, p, \mathbb{R}_+^n}. \tag{3.4}$$

Proof. Let $1 \le j \le \widetilde{N}$. Then it follows from [ADF, Proposition 2.2] that

$$\begin{aligned} |\!|\!| g_j |\!|\!|^{(\pi(j))}_{-\sigma_j - \frac{1}{p}, p, \mathbb{R}^{n-1}} &\le C_1 \Big|\!\Big|\!\Big| \sum_{k=1}^{N} \sum_{|\alpha| = \sigma_j + t_k} b_\alpha^{jk} D^\alpha u_k \Big|\!\Big|\!\Big|^{(\pi(j))}_{-\sigma_j, p, \mathbb{R}_+^n} \\ &\le C_2 \sum_{k=\pi(j)}^{N} \sum_{|\alpha| = \sigma_j + t_k} \Big(\| D^\alpha u_k \|_{-\sigma_j, p, \mathbb{R}_+^n} + |\lambda|^{\frac{-\sigma_j}{m_{\pi(j)}}} \| D^\alpha u_k \|_{0, p, \mathbb{R}_+^n} \Big) \\ &\le C_3 \sum_{k=\pi(j)}^{N} \Big(\| u_k \|_{t_k, p, \mathbb{R}_+^n} + |\!|\!| u_k |\!|\!|^{(\pi(j))}_{t_k, p, \mathbb{R}_+^n} \Big) \\ &\le 2\, C_3 \sum_{k=\pi(j)}^{N} |\!|\!| u_k |\!|\!|^{(\pi(j))}_{t_k, p, \mathbb{R}_+^n}. \end{aligned}$$

Hence all the assertions of the proposition follow from this last result. □

We now come to the main result of this section.

Proposition 3.4. *Suppose that the boundary problem* (1.1), (1.2) *is parameter-elliptic in* $\mathcal{L}$. *Suppose also that the operators* $A(D)$ *and* $B(D)$ *are compatible. Then there exists a constant* $\lambda^0 = \lambda^0(p) > 1$ *such that for* $\lambda \in \mathcal{L}$ *with* $|\lambda| \ge \lambda^0$, *the boundary problem* (3.2), (3.3) *has a unique solution* $u \in \prod_{j=1}^N W_p^{t_j}(\mathbb{R}_+^n)$ *for every* $g = (g_1, \dots, g_{\widetilde{N}})^T \in \prod_{j=1}^{\widetilde{N}} W_p^{-\sigma_j - 1/p}(\mathbb{R}^{n-1})$, *and the a priori estimate*

$$\sum_{j=1}^{N} |\!|\!| u_j |\!|\!|^{(j)}_{t_j, p, \mathbb{R}_+^n} \le C \sum_{j=1}^{\widetilde{N}} |\!|\!| g_j |\!|\!|^{(\pi(j))}_{-\sigma_j - 1/p, p, \mathbb{R}^{n-1}}$$

holds.

The proof of this proposition will be given in Section 4. The proof of the main result Theorem 2.6 is now a consequence of Propositions 3.2, 3.3, and 3.4.

Proof of Theorem 2.6. Let us first fix our attention upon the problem in the half-space

$$A(D)u(x) - \lambda\, u(x) = f(x) \text{ for } x \in \mathbb{R}_+^n \text{ and } \lambda \in \mathcal{L} \setminus \{0\}. \tag{3.5}$$

Then from a consideration of the pairing between $H_p^{-s_j}(\mathbb{R}_+^n)$, equipped with the norm $|\!|\!| \cdot |\!|\!|^{(j)}_{-s_j, p, \mathbb{R}_+^n}$, and its dual $\mathring{W}_{p'}^{s_j}(\mathbb{R}_+^n)$, equipped with the norm $|\!|\!| \cdot |\!|\!|^{(j)}_{s_j, p', \mathbb{R}_+^n}$, $1 \le$

$j \le N$, $p' = p/(p-1)$ (see [GK, Theorem 1.1]), we see that there exists a $\lambda^0 > 0$ such that for $\lambda \in \mathcal{L}$ with $|\lambda| \ge \lambda^0$, the differential equation (3.5) has a solution $u \in \prod_{j=1}^N W_p^{t_j}(\mathbb{R}^n_+)$ and $\sum_{j=1}^N \|\!|u_j|\!\|^{(j)}_{t_j,p,\mathbb{R}^n_+} \le C \sum_{j=1}^N \|\!|f_j|\!\|^{(j)}_{-s_j,p,\mathbb{R}^n_+}$.

Here we used the existence of an extension operator from the half-space to the whole space. In fact, it follows from [T, Lemma 2.9.3, p.218] and [GK, Eqn.(1.27)] that there is a $\widetilde{f} \in \prod_{j=1}^N H_p^{-s_j}(\mathbb{R}^n)$ such that $\widetilde{f}\big|\mathbb{R}^n_+ = f$ and

$$\sum_{j=1}^N \|\!|\widetilde{f}_j|\!\|^{(j)}_{-s_j,p,\mathbb{R}^n} \le C \sum_{j=1}^N \|\!|f_j|\!\|^{(j)}_{-s_j,p,\mathbb{R}^n_+}.$$

Hence if $\widetilde{u}$ denotes the solution of (3.1) when f there is replaced by $\widetilde{f}$ and $u = \widetilde{u}\big|\mathbb{R}^n_+$, then the assertion follows directly from Proposition 3.2.

In this way we reduced problem (1.1), (1.2) to problem (3.2), (3.3), modifying the right-hand sides g_j. Now the proof of Theorem 2.6 follows by standard arguments from Propositions 3.3 and 3.4. □

4. Proof of Proposition 3.4

For the proof of Proposition 3.4, we use a partition of unity in the space $\mathbb{R}^{n-1}$ of the covariable ξ'. More precisely, let us fix $\lambda \in \mathcal{L}$ with $|\lambda| > 1$ sufficiently large and let $\{\epsilon_j\}_1^N$ denote a sequence of numbers satisfying $0 < \epsilon_j < 1$, $j = 1, \dots, N$ (the magnitudes of λ and the ϵ_j will be specified below). We will establish estimates of the solution of (3.2), (3.3) for ξ' belonging to one of the regions

$$\mathcal{U}_0 := \{\xi' \in \mathbb{R}^{n-1} : |\xi'| \le \tfrac{7}{8}\epsilon_1 |\lambda|^{1/m_1}\},$$
$$\mathcal{U}_r := \{\xi' \in \mathbb{R}^{n-1} : \tfrac{1}{8}\epsilon_r |\lambda|^{1/m_r} \le |\xi'| \le \tfrac{7}{8}\epsilon_{r+1} |\lambda|^{1/m_{r+1}}\},\ r = 1, \dots, N-1,$$
$$\mathcal{U}_N := \{\xi' \in \mathbb{R}^{n-1} : \tfrac{1}{8}\epsilon_N |\lambda|^{1/m_N} \le |\xi'|\}.$$

As the proof of the a priori estimates in the regions $\mathcal{U}_0$ and $\mathcal{U}_N$ are similar to the corresponding proof in $\mathcal{U}_r$, $1 \le r \le N-1$, we will only consider the latter case. Therefore, throughout this section we will assume that the conditions of Proposition 3.4 hold, that $\lambda \in \mathcal{L}$ with $|\lambda|$ sufficiently large, and that $1 \le r \le N-1$ is fixed.

Our first result concerns the zeros of $\det(A(\xi', z) - \lambda I_N)$ as a polynomial in z.

Proposition 4.1. *For every fixed ϵ_r we can choose numbers ϵ^0 sufficiently small and λ^0 sufficiently large so that for $\epsilon_{r+1} \le \epsilon^0$ and $|\lambda| \ge \lambda^0$, $\det(A(\xi', z) - \lambda I_N)$ has precisely $\widetilde{N}$ zeros, say $\{z_j^{(r)}(\xi', \lambda)\}_{j=1}^{\widetilde{N}}$, lying in $\mathbb{C}_+$ and satisfying*

$$\operatorname{Im} z_j^{(r)}(\xi',\lambda) \ge C_1 \langle \xi', \lambda\rangle_r, \qquad |z_j^{(r)}(\xi',\lambda)| \le C_2 \langle \xi', \lambda\rangle_r,\ j = 1, \dots, N_r,$$
$$\operatorname{Im} z_j^{(r)}(\xi',\lambda) \ge C_1 \langle \xi', \lambda\rangle_\ell, \qquad |z_j^{(r)}(\xi',\lambda)| \le C_2 \langle \xi', \lambda\rangle_\ell,\ j = N_{\ell-1}, \dots, N_\ell$$

for $\ell = r+1, \dots, N$, where $C_2\langle \xi', \lambda\rangle_\ell < C_1 \langle \xi', \lambda\rangle_{\ell+1}$ for $\ell = r, \dots, N-1$.

Proof. To begin with let us observe that

$$A(\xi', z) - \lambda I_N = \begin{pmatrix} \mathcal{A}_{11}^{(r)}(\xi', z) - \lambda I_r & \mathcal{A}_{12}^{(r)}(\xi', z) \\ \mathcal{A}_{21}^{(r)}(\xi', z) & \mathcal{A}_{22}^{(r)}(\xi', z) - \lambda I_{N-r} \end{pmatrix}$$

and that

$$\mathcal{A}_{11}^{(r)}(\xi', z) - \lambda I_r = \mathcal{A}_{11}^{(r)}(\xi', z) - \lambda I_{r,0} - \lambda(I_r - I_{r,0}).$$

Then as a consequence of our hypotheses we know that $\det(\mathcal{A}_{11}^{(r)}(\xi',z)-\lambda I_{r,0})$ has precisely N_r zeros lying in $\mathbb{C}_+$ and that there is a closed contour $\gamma_r^+(\xi',\lambda)\subset\mathbb{C}_+$ containing all these zeros in its interior such that for $z\in\gamma_r^+(\xi',\lambda)$ we have $\operatorname{Im} z\geq C_1\langle\xi',\lambda\rangle_r, |z|\leq C_2\langle\xi',\lambda\rangle_r$, and

$$C_3\langle\xi',\lambda\rangle_r^{2N_r}\leq\Big|\det\Big(\mathcal{A}_{11}^{(r)}(\xi',z)-\lambda I_{r,0}\Big)\Big|\leq C_4\langle\xi',\lambda\rangle_r^{2N_r}.$$

By expanding the determinant, it is easy to show that for $z\in\gamma_r^+(\xi',\lambda)$,

$$\Big|\det\big(\mathcal{A}_{11}^{(r)}(\xi',z)-\lambda\,I_{k_r}\big)-\det\big(\mathcal{A}_{11}^{(r)}(\xi',z)-\lambda I_{r,0}\big)\Big|\leq$$
$$C\sum_{\ell=1}^{k_{r-1}}\sum_{1\leq i(1)<\ldots<i(\ell)\leq k_{r-1}}\prod_{k=1}^{\ell}\left(\frac{|\lambda|^{1/m_{i(k)}}}{\langle\xi',\lambda\rangle_r}\right)^{m_{i(k)}}\langle\xi',\lambda\rangle_r^{2N_r}.$$

Similarly, one can show by the Laplace method of expanding a determinant, that

$$\big|\det\big(A(\xi',z)-\lambda\,I_N\big)-\det\big(\mathcal{A}_{11}^{(r)}(\xi',z)-\lambda\,I_r\big)\lambda^{N-r}\big|$$
$$\leq C(\epsilon_{r+1},\lambda)\,\big|\det(\mathcal{A}_{11}^{(r)}(\xi',z)-\lambda\,I_r)\big|\cdot|\lambda|^{N-r}$$

with a factor $C(\epsilon_{r+1},\lambda)$ which can be made arbitrarily small if ϵ_{r+1} is chosen small enough and $|\lambda|$ is chosen large enough. Hence it follows from Rouché's theorem that in this case, $\det\big(A(\xi',z)-\lambda\,I_N\big)$ has precisely N_r zeros contained in $\gamma_r^+(\xi',\lambda)$.

For $\ell=r+1,\ldots,N$, we write

$$A(\xi',z)-\lambda\,I_N=\begin{pmatrix}\mathcal{A}_{11}^{(\ell)}(\xi',z)-\lambda\,I_\ell & \mathcal{A}_{12}^{(\ell)}(\xi',z)\\ \mathcal{A}_{21}^{(\ell)}(\xi',z) & \mathcal{A}_{22}^{(\ell)}(\xi',z)-\lambda\,I_{N-\ell}\end{pmatrix}\tag{4.1}$$

and

$$\mathcal{A}_{11}^{(\ell)}(\xi',z)-\lambda\,I_\ell=\mathcal{A}_{11}^{(\ell)}(0,z)-\lambda\,I_{\ell,0}+\mathcal{A}_{11}^{(\ell)}(\xi',z)-\mathcal{A}_{11}(0,0,z)-\lambda(I_\ell-I_{\ell,0}).$$

Then as a consequence of our hypotheses we know that $\det(\mathcal{A}_{11}^{(\ell)}(0,z)-\lambda I_{\ell,0})$ has precisely $N_\ell-N_{\ell-1}$ zeros lying in $\mathbb{C}_+$ and that there is a closed contour $\gamma_{r,\ell}^+(\lambda)\subset\mathbb{C}_+$ containing all these zeros in its interior such that for $z\in\gamma_{r,\ell}^+(\lambda)$ we have $\operatorname{Im} z\geq C_1'|\lambda|^{\frac{1}{m_\ell}}, |z|\leq C_2'|\lambda|^{\frac{1}{m_\ell}}$ and

$$C_3'|\lambda|^{\frac{2N_\ell}{m_\ell}}\leq\Big|\det\Big(\mathcal{A}_{11}^{(\ell)}(0,z)-\lambda\,I_{\ell,0}\Big)\Big|\leq C_4'|\lambda|^{\frac{2N_\ell}{m_\ell}}.$$

Furthermore, we can show that for $z\in\gamma_{r,\ell}^+(\lambda)$,

$$|\det(\mathcal{A}_{11}^{(\ell)}(\xi',z)-\lambda\,I_\ell)-\det(\mathcal{A}_{11}^{(\ell)}(0,z)-\lambda I_{\ell,0})|$$
$$\leq C\left(|\lambda|^{\frac{1}{m_{\ell-1}}-\frac{1}{m_\ell}}+\delta_{\ell,r+1}\epsilon_{r+1}\right)|\lambda|^{\frac{2N_\ell}{m_\ell}}.$$

In a similar way as before, we obtain

$$\big|\det(A(\xi',z)-\lambda\,I_N)-\det(\mathcal{A}_{11}^{(\ell)}(\xi',z)-\lambda\,I_\ell)\lambda^{N-\ell}\big|$$
$$\leq C'(\epsilon_{r+1},\lambda)\,\big|\det(\mathcal{A}_{11}^{(\ell)}(\xi',z)-\lambda\,I_\ell)\big|\cdot|\lambda|^{N-\ell}$$

with a factor $C'(\epsilon_{r+1},\lambda)$ which can be made small for small ϵ_{r+1} and large $|\lambda|$. It follows again from Rouché's theorem that then $\det(A(0,\xi',z)-\lambda\,I_N)$ has precisely $N_\ell-N_{\ell-1}$ zeros contained in $\gamma_{r,\ell}^+(\lambda)$. Hence since ℓ was chosen arbitrary, this completes the proof of the proposition. □

Still assuming that $\xi' \in \mathcal{U}_r$, $1 \le r \le N-1$, and ϵ^0 has been chosen sufficiently small and λ^0 sufficiently large, it follows from [V] that the set of solutions of the differential equation

$$A(\xi', D_n)u(x_n) - \lambda\, u(x_n) = 0 \text{ for } x_n > 0 \tag{4.2}$$

which decay exponentially at ∞ form a vector space of dimension $\widetilde{N}$. Furthermore, this vector space is precisely the direct sum of the vector space (of dimension N_r) spanned by the columns of the matrix

$$\int_{\gamma_r^+(\xi',\lambda)} e^{ix_n z}\,(A(\xi',z) - \lambda\, I_N)^{-1}\,\big(I_N, zI_N, \ldots, z^{m_1-1}I_N\big)\,dz, \tag{4.3}$$

and the $N-r$ vector spaces (of dimension $N_\ell - N_{\ell-1}$, $\ell = r+1, \ldots, N$) spanned by the columns of each of the matrices

$$\int_{\gamma_{r,\ell}^+(\lambda)} e^{ix_n z}\,(A(\xi',z) - \lambda\, I_N)^{-1}\,\big(I_N, zI_N, \ldots, z^{m_1-1}I_N\big)\,dz$$

for $\ell = r+1, \ldots, N$.

Proposition 4.2. *We can choose the constants ϵ^0 and λ^0 of Proposition 4.1 sufficiently small and sufficiently large, respectively, so that for $\xi' \in \mathcal{U}_r$ we have*

$$\operatorname{rank} \int_{\gamma_r^+(\xi',\lambda)} \mathcal{B}^{(r,r)}(\xi',z)\,(A(\xi',z) - \lambda\, I_N)^{-1}\,\big(I_N, zI_N, \ldots, z^{m_1-1}I_N\big)\,dz = N_r,$$

$$\operatorname{rank} \int_{\gamma_{r,\ell}^+(\lambda)} \mathcal{B}^{(r,\ell)}(\xi',z)\,(A(\xi',z) - \lambda\, I_N)^{-1}\,\big(I_N, zI_N, \ldots, z^{m_1-1}I_N\big)\,dz$$

$$= N_\ell - N_{\ell-1} \quad \textit{for } \ell = r+1, \ldots, N. \tag{4.4}$$

The proof of this proposition is based on a thorough study of the integrals (4.4) in comparison with the analog integrals corresponding to the boundary value problems appearing in Definition 2.4. For the details, we refer the reader to [DF, Section 3].

In light of Proposition 4.2 and from what was said in the text preceding that proposition, we are now in a position to present some results pertaining to the solutions of (4.2).

Proposition 4.3. *Suppose that ϵ^0 and λ^0 have been chosen small enough and large enough, respectively, so that the conclusions of Proposition 4.2 hold. Then for $\xi' \in \mathcal{U}_r$, the differential equation (4.2) has $\widetilde{N}$ linearly independent solutions, $\{w^{(r,\nu)}(\xi', x_n, \lambda)\}_{\nu=1}^{\widetilde{N}}$, which decay exponentially at ∞ and satisfy*

$$\mathcal{B}^{(r,\ell)}(\xi', D_n)W^{(r,\ell)}(\xi', 0, \lambda) = I_{N_\ell - N_{r,\ell-1}} \quad \textit{for } \ell = r, \ldots, N, \tag{4.5}$$

where $N_{r,\ell-1} = (1-\delta_{r,\ell})N_{\ell-1}$, $W^{(r,\ell)}(\xi', x_n, \lambda)$ denotes the $N \times (N_\ell - N_{r,\ell-1})$ matrix function whose columns are precisely the $w^{(r,\nu)}(\xi', x_n, \lambda)$ for $\nu = N_{r,\ell-1}+1, \ldots, N_\ell$. Furthermore, we have the representations

$$W^{(r,r)}(\xi', x_n, \lambda) = \int_{\gamma_r^+(\xi',\lambda)} e^{ix_n z}\big(A(\xi',z) - \lambda I_N\big)^{-1} G^{(r,r)}(\xi', z, \lambda)dz,$$

$$W^{(r,\ell)}(\xi', x_n, \lambda) = \int_{\gamma_{r,\ell}^+(\lambda)} e^{ix_n z}\big(A(\xi',z) - \lambda I_N\big)^{-1} G^{(r,\ell)}(\xi', z, \lambda)dz,$$

$$\textit{for } \ell = r+1, \ldots, N, \tag{4.6}$$

where for $r \le \ell \le N$

$$G^{(r,\ell)}(\xi', z, \lambda) = \begin{pmatrix} \widetilde{G}^{(r,\ell)}(\eta'_{r,\ell}, \zeta_{r,\ell}, \mu_{r,\ell}, \rho_{r,\ell}) \\ 0 \cdot I_{N-\ell} \end{pmatrix},$$

$$\widetilde{G}^{(r,\ell)}(\eta'_{r,\ell}, \zeta_{r,\ell}, \mu_{r,\ell}, \rho_{r,\ell}) = \rho_{r,\ell}^{-1} \operatorname{diag}(\rho_{r,\ell}^{s_1}, \dots, \rho_{r,\ell}^{s_\ell}) K^{(r,\ell)}(\eta'_{r,\ell}, \zeta_{r,\ell}, \mu_{r,\ell}, \rho_{r,\ell}) \times \operatorname{diag}(\rho_{r,\ell}^{-\sigma_{N_{r,\ell-1}+1}}, \dots, \rho_{r,\ell}^{-\sigma_{N_\ell}}),$$

$\rho_{r,r} = \langle \xi', \lambda \rangle_r$, $\rho_{r,\ell} = |\lambda|^{1/m_\ell}$ *for* $\ell > r$, $\eta'_{r,\ell} = \rho_{r,\ell}^{-1}\xi'$, $\zeta_{r,\ell} = \rho_{r,\ell}^{-1} z$, $\mu_{r,\ell} = \rho_{r,\ell}^{-m_\ell}\lambda$, *and*

$$K^{(r,\ell)}(\eta'_{r,\ell}, \zeta_{r,\ell}, \mu_{r,\ell}, \rho_{r,\ell}) = \left(K_{jk}^{(r,\ell)}(\eta'_{r,\ell}, \zeta_{r,\ell}, \mu_{r,\ell}, \rho_{r,\ell}) \right)_{\substack{j=1,\dots,k_\ell \\ k=N_{r,\ell-1}+1,\dots,N_\ell}},$$

such that for each pair j, k, $K_{jk}^{(r,\ell)}(\eta'_{r,\ell}, \zeta_{r,\ell}, \mu_{r,\ell}, \rho_{r,\ell})$ *is a finite sum of a product of a power of* $\zeta_{r,\ell}$ *and an expression of rational type in* $\eta'_{r,\ell}, \zeta_{r,\ell}, \mu_{r,\ell}$ *and* $\rho_{r,\ell}$ *(i.e., a rational function of terms which are integrals of rational functions of the components of* $\eta'_{r,\ell}, \zeta_{r,\ell}, \mu_{r,\ell}$ *and* $\rho_{r,\ell}$*) and is bounded in modulus by a constant depending only upon* $\eta'_{r,r}$ *and* $\mu_{r,r}$ *if* $\ell = r$ *and only upon* $\rho_{r,\ell}$ *if* $\ell > r$*.*

Proof. We will only consider the case $\ell = r$, the assertion for the case $\ell > r$ will follow in a similar way. From Proposition 4.2 we know that there exist matrices J and $Z(z)$ such that the $N_r \times N_r$ matrix

$$\Lambda(\xi', \lambda) := \int_{\gamma_r^+(\xi',\lambda)} \mathcal{B}^{(r,r)}(\xi', z) \left(A(\xi', z) - \lambda I_N\right)^{-1} \begin{pmatrix} J \\ 0 \end{pmatrix} Z(z) dz$$

is invertible where J denotes an $r \times N_r$ matrix with the property that each of its columns has precisely one non-zero component, namely 1, and $Z(z) = \operatorname{diag}(z^{q(1)}, \dots z^{q(N_r)})$, where the $q(j)$ denote non-negative integers not exceeding $m_1 - 1$. Moreover, using the homogeneities of A and B, by scaling arguments we obtain that $\Lambda(\xi', \lambda)$ can be written in the form

$$\Lambda(\xi', \lambda) = \rho \operatorname{diag}\left(\rho^{\sigma_1}, \dots, \rho^{\sigma_{N_r}}\right)\left(K_1(\eta', \mu) + K_2(\eta', \mu, \rho)\right) \times \operatorname{diag}\left(\rho^{q(1)-s_1}, \dots, \rho^{q(N_r)-s_{N_r}}\right), \tag{4.7}$$

where $\rho = \langle \xi', \lambda \rangle_r$, $\eta' = \rho^{-1}\xi'$, $\mu = \rho^{-m_r}\lambda$. Additionally we have

$$|\det K_1(\eta', \mu)| \ge c_1 > 0,$$

and the elements of the matrix $K_2(\eta', \mu, \rho)$ are expressions of rational type of the arguments and are bounded by

$$\epsilon_{r+1}^{m_{r+1}} + |\lambda|^{\frac{m_{r+1}}{m_r} - 1} + |\lambda|^{1 - \frac{m_{r-1}}{m_r}}. \tag{4.8}$$

Therefore, for small ϵ_{r+1} and large $|\lambda|$, the matrix $\Lambda(\xi', \lambda)$ is invertible and

$$\Lambda(\xi', \lambda)^{-1} = \rho^{-1} \operatorname{diag}\left(\rho^{s_1 - q(1)}, \dots, \rho^{s_{N_r} - q_{N(r)}}\right) \times (I_{N_r} + K_3(\eta', \mu, \rho)) K_1(\eta', \mu)^{-1} \operatorname{diag}\left(\rho^{-\sigma_1}, \dots, \rho^{-\sigma_{N_r}}\right),$$

where $K_3(\eta', \mu, \rho)$ is an $N_r \times N_r$ matrix function defined by

$$I_{N_r} + K_3(\eta', \mu, \rho) = \left(I_{N_r} + K_1(\eta', \mu)^{-1} K_2(\eta', \mu, \rho)\right)^{-1}$$

and where each entry of $K_3(\eta',\mu,\rho)$ is a function of its arguments of rational type and is bounded in modulus by the expression (4.8). Hence it follows that

$$\begin{aligned}\int_{\gamma_r^+(\xi',\lambda)} &\mathcal{B}^{(r,r)}(\xi',z)\big(A(\xi',z)-\lambda I_N\big)^{-1}\begin{pmatrix}\widetilde{G}^{(1)}(\eta',\zeta,\mu,\rho)+\widetilde{G}^{(2)}(\eta',\zeta,\mu,\rho)\\ 0\cdot I_{N-k_r}\end{pmatrix}dz\\ &= I_{N_r},\end{aligned}$$

where

$$\begin{aligned}\widetilde{G}^{(1)}(\eta',\zeta,\mu,\rho)&=\rho^{-1}\operatorname{diag}(\rho^{s_1},\dots,\rho^{s_r})JZ(\zeta)K_1(\eta',\mu)^{-1}\operatorname{diag}(\rho^{-\sigma_1},\dots,\rho^{-\sigma_{N_r}}),\\ \widetilde{G}^{(2)}(\eta',\zeta,\mu,\rho)&=\rho^{-1}\operatorname{diag}(\rho^{s_1},\dots,\rho^{s_r})JZ(\zeta)K_3(\eta',\mu,\rho)K_1(\eta',\mu)^{-1}\\ &\qquad\times\operatorname{diag}(\rho^{-\sigma_1},\dots,\rho^{-\sigma_{N_r}}).\end{aligned}\tag{4.9}$$

□

Let us denote by $W^{(r)}(\xi',x_n,\lambda)$ the $N\times\widetilde{N}$ matrix function whose columns are precisely the $w^{(r,\nu)}(\xi',x_n,\lambda)$. For $r\le j\le N$, let

$$\begin{aligned}\mathcal{I}_{r,j}(\xi',\lambda)&=\Big(\mathcal{B}^{(r,\ell)}(\xi',D_n)W^{(r,\ell_1)}(\xi',0,\lambda)\Big)_{\ell,\ell_1=j}^{N},\\ \widetilde{\mathcal{I}}_{r,j}(\xi',\lambda)&=\Big(\widetilde{\mathcal{B}}^{(r,\ell)}(\xi',D_n)\widetilde{W}^{(r,\ell_1)}(\xi',0,\lambda)\Big)_{\ell,\ell_1=j}^{N},\end{aligned}\tag{4.10}$$

where

$$\begin{aligned}\widetilde{\mathcal{B}}^{(r,\ell)}(\xi',D_n)&=\operatorname{diag}\big(\rho_{r,\ell}^{-\sigma_{N_{r,\ell-1}+1}},\dots,\rho_{r,\ell}^{-\sigma_{N_\ell}}\big)\mathcal{B}^{(r,\ell)}(\xi',D_n),\\ \widetilde{W}^{(r,\ell_1)}(\xi',0,\lambda)&=W^{(r,\ell_1)}(\xi',0,\lambda)\operatorname{diag}\big(\rho_{r,\ell_1}^{\sigma_{N_{r,\ell_1-1}+1}},\dots,\rho_{r,\ell_1}^{\sigma_{N_{\ell_1}}}\big).\end{aligned}$$

Then we can write

$$\begin{aligned}B(\xi',D_n)W^{(r)}(\xi',0,\lambda)&=\mathcal{I}_{r,r}(\xi',\lambda)\\ &=\operatorname{diag}\big(\widetilde{\rho}_1^{\sigma_1},\dots,\widetilde{\rho}_{\widetilde{N}}^{\sigma_{\widetilde{N}}}\big)\widetilde{\mathcal{I}}_{r,r}(\xi',\lambda)\operatorname{diag}\big(\widetilde{\rho}_1^{-\sigma_1},\dots,\widetilde{\rho}_{\widetilde{N}}^{-\sigma_{\widetilde{N}}}\big),\end{aligned}\tag{4.11}$$

where

$$\widetilde{\rho}_\nu=\begin{cases}\rho_{r,r} & \text{for } 1\le\nu\le N_r,\\ \rho_{r,\ell} & \text{for } N_{\ell-1}<\nu\le N_\ell,\ \ell=r+1,\dots,N.\end{cases}$$

We remark at this point that as a consequence of Proposition 4.5 below, it will be seen that for $\ell\ne\ell_1$, $\widetilde{\mathcal{B}}^{(r,\ell)}(\xi',D_n)\widetilde{W}^{(r,\ell_1)}(\xi',0,\lambda)$ is an $(N_\ell-N_{r,\ell-1})\times(N_{\ell_1}-N_{r,\ell_1-1})$ matrix function whose entries are products of powers of $\rho_{r,\ell}/\rho_{r,\ell_1}$ and an expression of rational type in the components of η'_{r,ℓ_1}, μ_{r,ℓ_1}, and ρ_{r,ℓ_1}.

Definition 4.4. Suppose that the boundary problem (1.1), (1.2) is parameter-elliptic in $\mathcal{L}$ and that $\lambda\in\mathcal{L}$. In addition, suppose that with respect to this boundary problem all hypotheses of Proposition 4.3 hold. Then bearing in mind the definitions of the various terms introduced above, we say that the operators $A(D)$ and $B(D)=(B_1(D),\dots,B_{\widetilde{N}}(D))^T$ are compatible if for each r satisfying $0\le r<N$ and for each pair of integers ℓ,ℓ_1 satisfying $r\le\ell_1<N$, $\ell_1+1\le\ell\le N$, each entry of the matrix $\widetilde{\mathcal{B}}^{(r,\ell)}(\xi',D_n)\widetilde{W}^{(r,\ell_1)}(\xi',0,\lambda)$ is bounded in modulus by a constant depending only upon $(\eta'_{r,r},\mu_{r,r})$ and $\mu_{r,\ell}$ for $\ell>r$.

Proposition 4.5. *Suppose that the hypotheses of Proposition 4.3 hold and that the operators $A(D)$ and $B(D)$ are compatible. Suppose also that $0 \le r < N$. Then for ϵ^0 sufficiently small and λ^0 sufficiently large the matrix function $\widetilde{\mathcal{I}}_{r,r}(\xi',\lambda)$ of* (4.11)*, and hence the matrix function $\mathcal{I}_{r,r}(\xi',\lambda)$ are invertible and we have*

$$\mathcal{I}_{r,r}(\xi',\lambda)^{-1} = \operatorname{diag}\big(\widetilde{\rho}_1^{\sigma_1},\dots,\widetilde{\rho}_{\widetilde{N}}^{\sigma_{\widetilde{N}}}\big)\widetilde{\mathcal{I}}_{r,r}(\xi',\lambda)^{-1}\operatorname{diag}\big(\widetilde{\rho}_1^{-\sigma_1},\dots,\widetilde{\rho}_{\widetilde{N}}^{-\sigma_{\widetilde{N}}}\big),$$

where $|\det\widetilde{\mathcal{I}}_{r,r}(\xi',\lambda)| > \frac{1}{2}$, while the entries of $\widetilde{\mathcal{I}}_{r,r}(\xi',\lambda)^{-1}$ are rational functions of the $\rho_{r,\ell}$ and expressions of rational type in $\eta'_{r,\ell}, \mu_{r,\ell}$, and $\rho_{r,\ell}$ for $\ell \ge r$ and are bounded in modulus by a constant depending only upon $(\eta'_{r,r},\mu_{r,r})$ and $\mu_{r,\ell}$ for $\ell > r$. Lastly, the operators $A(D)$ and $B(D)$ are always compatible if
(i) the boundary conditions (1.2) *are of Dirichlet type or if*
(ii) the operators $A(D)$ and $B(D)$ are both essentially upper triangular.

Proof. We will only prove the proposition for the case $1 \le r < N$; the case $r = 0$ can be similarly treated. Accordingly, let us fix our attention upon (4.10) and suppose that $\ell \ne \ell_1$.

Suppose that $\ell > \ell_1$. Then by hypothesis the entries of the matrix $\widetilde{\mathcal{B}}^{(r,\ell)}(\xi',D_n)$ $\widetilde{W}^{(r,\ell_1)}(\xi',0,\lambda)$ are bounded in modulus by a constant depending only on $(\eta'_{r,r},\mu_{r,\ell})$ and $\mu_{r,\ell}$ for $\ell > r$. Let us now show that this boundedness condition is always satisfied if the boundary conditions (1.2) are of Dirichlet type. Indeed, if this latter condition holds, then every entry of $\mathcal{B}^{(r,\ell)}_{\ell_1,1}(\eta'_{r,\ell_1},\zeta_{r,\ell_1})$ is 0, while the only non-zero entries of $\mathcal{B}^{(r,\ell)}_{\ell_1,2}(\eta'_{r,\ell_1},\zeta_{r,\ell_1})$ are those lying in rows $N_\ell-\nu$, $\nu = 0,\dots,N_\ell-N_{\ell-1}-1$, and in column ℓ. Then recalling from Section 1 that we now have $s_j = t_j = t'_j$ for $j = 1,\dots,N$, it follows from the foregoing results that the entry in the $(N_\ell-\nu)$-th row and the ℓ-th column of $\widetilde{\mathcal{B}}^{(r,\ell)}(\xi',D_n)\widetilde{W}^{(r,\ell_1)}(\xi',0,\lambda)$ is bounded in modulus by

$$C|\lambda|^{-\frac{1}{2}-\frac{t'_{k_\ell}}{t'_{k_{\ell_1}}}-\frac{\nu^\dagger}{2}\left(\frac{1}{t'_{k_\ell}}-\frac{1}{t'_{k_{\ell_1}}}\right)}\Big(|\lambda|^{\frac{t'_{k_\ell}}{t'_{k_{\ell_1}}}}+\delta_{r,\ell_1}\epsilon_{r+1}\Big),$$

where $0 \le \nu^\dagger \le t'_{k_\ell}-1$ and the constant C depends only upon $(\eta'_{r,r},\mu_{r,r})$ and $\mu_{r,\ell}$. Our claim concerning Dirichlet boundary conditions are an immediate consequence of this last result.

Note also that when $A(D)$ and $B(D)$ are essentially upper triangular, then every entry of $\mathcal{B}^{(r,\ell)}_{\ell_1,1}(\eta'_{r,\ell_1},\zeta_{r,\ell_1})$ is 0, and hence the entries of $\widetilde{\mathcal{B}}^{(r,\ell)}(\xi',D_n)\widetilde{W}^{(r,\ell_1)}(\xi',0,\lambda)$ are all 0. Thus the boundedness condition also holds under the cited conditions concerning $A(D)$ and $B(D)$.

Let us now consider the case $\ell_1 > \ell$. Then it is a simple matter to deduce that for this case each entry of

$$\widetilde{\mathcal{B}}^{(r,\ell)}(\xi',D_n)\widetilde{W}^{(r,\ell)}(\xi',0,\lambda)$$

is bounded in modulus by

$$C\Big(|\lambda|^{\frac{1}{m_\ell}-\frac{1}{m_{\ell_1}}}+\delta_{r+1,\ell_1}\epsilon_{r+1}+(1-\delta_{\ell_1,N})|\lambda|^{\frac{m_{\ell_1+1}}{m_{\ell_1}}-1}\Big), \tag{4.12}$$

where the constant C depends only upon $\mu_{r,\ell}$.

Returning again to (4.10), let us next introduce the block row and column matrices

$$\widetilde{\mathcal{I}}^{(1,2)}_{r,j}(\xi',\lambda) = \Big(\widetilde{\mathcal{B}}^{(r,j)}(\xi',D_n)\widetilde{W}^{(r,\ell_1)}(\xi',0,\lambda)\Big)^{N}_{\ell_1=j+1},$$
$$\widetilde{\mathcal{I}}^{(2,1)}_{r,j}(\xi',\lambda) = \Big(\widetilde{\mathcal{B}}^{(r,\ell)}(\xi',D_n)\widetilde{W}^{(r,j)}(\xi',0,\lambda)\Big)^{N}_{\ell=j+1},$$

respectively. Then a factorization in the sense of Schur gives

$$\begin{aligned}\widetilde{\mathcal{I}}_{r,j}(\xi',\lambda) &= \begin{pmatrix} I_{N_j-N_{r,j-1}} & 0 \\ \widetilde{\mathcal{I}}^{(2,1)}_{r,j}(\xi',\lambda) & I_{\widetilde{N}-N_j}\end{pmatrix}\\ &\times \begin{pmatrix} I_{N_j-N_{r,j-1}} & 0 \\ 0 & \widetilde{\mathcal{I}}_{r,j+1}(\xi',\lambda)-\widetilde{\mathcal{I}}^{(2,1)}_{r,j}(\xi',\lambda)\widetilde{\mathcal{I}}^{(1,2)}_{r,j}(\xi',\lambda)\end{pmatrix}\\ &\times \begin{pmatrix} I_{N_j-N_{r,j-1}} & \widetilde{\mathcal{I}}^{(1,2)}_{r,j}(\xi',\lambda) \\ 0 & I_{\widetilde{N}-N_j}\end{pmatrix}.\end{aligned}$$

Since we already know that each entry of $\widetilde{\mathcal{I}}^{(2,1)}_{r,j}(\xi',\lambda)$ is bounded in modulus by a constant depending only upon $(\eta'_{r,r},\mu_{r,r})$, that each entry of $\widetilde{\mathcal{I}}^{(1,2)}_{r,j}(\xi',\lambda)$ is bounded in modulus by the quantity (4.12), and that $\widetilde{\mathcal{I}}_{r,N}(\xi',\lambda) = I_{\widetilde{N}-N_{r,N-1}}$, an inductive argument involving Schur factorizations shows that we may choose ϵ^0 sufficiently small and λ^0 sufficiently large so that all the assertions of the proposition concerning $\widetilde{\mathcal{I}}_{r,r}(\xi',\lambda)$ hold. This completes the proof of the proposition for the case under consideration. □

As a consequence of Proposition 4.5, we are now in a position to prove Proposition 3.4. We let $\mathcal{S}(\mathbb{R}^{n-1})$ denote the Schwartz space of rapidly decreasing functions on $\mathbb{R}^{n-1}$

Proof of Proposition 3.4. Under Fourier transform F' with respect to x', the boundary value problem (3.2), (3.3) is transformed into the boundary value problem

$$A(\xi',D_n)F'u(\xi',x_n) - \lambda F'u(\xi',x_n) = 0 \quad \text{for } x_n > 0, \tag{4.13}$$
$$B(0,\xi',D_n)F'u(\xi',x_n) = F'g^{(r)}(\xi') \quad \text{at } x_n = 0. \tag{4.14}$$

We assume that ϵ^0 and λ^0 have been chosen so that the conclusions of Proposition 4.3 and 4.5 hold.

(i) We first assume that $g \in \mathcal{S}(\mathbb{R}^{n-1})^{\widetilde{N}}$ and that $F'g$ has support in the region $\mathcal{U}_r$ defined above for some fixed $r = 0,\dots,N$. From Proposition 4.3 and 4.5 it follows that for a fixed $\lambda \in \mathcal{L}$ with $|\lambda| \geq \lambda_0$, the boundary value problem (4.13), (4.14) has a solution in the vector space decaying exponentially at ∞, say $\widetilde{u}(\xi',x_n,\lambda)$, which is unique for $\xi' \in \mathcal{U}_r$ and is given by

$$\begin{aligned}\widetilde{u}(\xi',x_n,\lambda) &= W^{(r)}(\xi',x_n,\lambda)\mathcal{I}_{r,r}(\xi',\lambda)^{-1}F'g(\xi')\\ &= \sum_{\ell=r}^{N} W^{(r,\ell)}(\xi',x_n,\lambda)\mathcal{I}^{(\ell)}_{r,r}(\xi',\lambda)^{-1}F'g(\xi').\end{aligned}$$

Here $W^{(r,\ell)}$ is defined in (4.6) and $\mathcal{I}^{(\ell)}_{r,r}(\xi',\lambda)^{-1}$ denotes the matrix whose rows are precisely the rows $N_{r,\ell-1}+1,\dots,N_\ell$ of the matrix $\mathcal{I}_{r,r}(\xi',\lambda)^{-1}$. For fixed

$\ell = r, \dots, N$, we therefore have to consider the term

$$\widetilde{u}^{(\ell)}(\xi', x_n, \lambda) = W^{(r,\ell)}(\xi', x_n, \lambda)\mathcal{I}^{(\ell)}_{r,r}(\xi', \lambda)^{-1}F'g(\xi').$$

Following a standard approach in elliptic theory (Volevich trick), we write

$$\widetilde{u}^{(\ell)}(\xi', x_n, \lambda) = -\int_0^\infty \partial_{y_n}\big[W^{(r,\ell)}(\xi', x_n + y_n, \lambda)\mathcal{I}^{(\ell)}_{r,r}(\xi', \lambda)^{-1}F'h(\xi', y_n)\big]\,dy_n$$

where we defined the extension $h \in \prod_{j=1}^{\widetilde{N}} W_p^{-\sigma_j}(\mathbb{R}^n_+)$ of g by

$$h(x, \lambda) = \begin{cases} \big(F'^{-1}\exp(-\langle\xi'\rangle x_n)F'g\big)(x) & \text{if } \ell = r, \\ \big(F'^{-1}\exp(-\langle\xi', \lambda\rangle_\ell x_n)F'g\big)(x) & \text{if } \ell > r \end{cases}$$

(see [ADF, Prop. 3.2]). Arguing as in [V, Section 5], using the representation (4.6) from Proposition 4.3 and the Mikhlin-Lizorkin multiplier theorem, we obtain for $u^{(\ell)}(x, \lambda) := F'^{-1}\widetilde{u}^{(\ell)}(\xi', x_n, \lambda)$ the a priori estimate

$$\sum_{j=1}^{N} \|\!| u_j^{(\ell)} \|\!|^{(j)}_{t_j,p,\mathbb{R}^n_+} \le C_\ell \sum_{j=1}^{\widetilde{N}} \|\!| g_j \|\!|^{(\pi(j))}_{-\sigma_j - 1/p, p, \mathbb{R}^{n-1}}.$$

Summing up over ℓ, we obtain the desired estimate for the case under consideration.

(ii) To prove the a priori estimate for general $g \in \prod_{j=1}^{\widetilde{N}} W_p^{-\sigma_j-1/p}(\mathbb{R}^{n-1})$, we first approximate g by a Schwartz function in a standard way and then use a partition of unity with respect to ξ' subordinated to the regions $\mathcal{U}_0, \dots, \mathcal{U}_N$. For details, we refer the reader to [DF]. Note that unique solvability for fixed λ, $|\lambda| \ge \lambda_0$, with λ^0 large enough, follows from [DV, Thm. 3.5] and the above considerations. □

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