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Sigma-additivity of quasi-measure extensions of a measure

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Abstract: The situation in which every quasi-measure extension of a given measure is σ-additive is characterized in the case of an atomic or two-valued measure.

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1 Introduction

In general the existence of measure extensions or extremal measure extensions is not guaranteed and can be proved only in special cases (cf. [2], [3]). Quite different is the situation if we look for quasi-measure extensions of a given measure. It is well-known that extensions and even extreme extensions always exist (see [1], section 3.4, [13], p. 194); the set of all quasi-measure extensions is compact and an integral representation holds, see below. This paper deals with the question in which cases *every* quasi-measure extension of a given measure is σ-additive. In section 2 we give a characterization of that situation in the case where the smaller σ-algebra is atomic with respect to the measure. A special case of this is a σ-algebra which is generated by a countable partition. In section 3 these results are applied to two-valued measures. Most results in sections 2 and 3 are part of the diploma thesis of the first author [4] which was finished under guidance of Prof. D. Bierlein at the University of Regensburg.
We fix the following notations (cf. [1],[11]): Let $\mathcal{A}$ and $\mathcal{A}_1$ be algebras [σ-algebras] of subsets of a nonempty set Ω with $\mathcal{A} \subset \mathcal{A}_1$. For $\mathcal{E} \subset 2^\Omega$ we denote by $\mathcal{E}_\beta$ the σ-algebra generated by $\mathcal{E}$. $ba(\mathcal{A})$ [$ca(\mathcal{A})$] is defined as the Banach lattice of all real-valued bounded additive [σ-additive] set functions on $\mathcal{A}$, and we call the elements of the positive cone $ba_+(\mathcal{A})$ quasi-measures and the elements of $ca_+(\mathcal{A})$ measures.
For $\nu \in ba_+(\mathcal{A})$ we set $R(\nu) := \{\nu(A) : A \in \mathcal{A}\}$ and

$$E(\nu, \mathcal{A}_1) := \{\mu \in ba_+(\mathcal{A}_1) : \mu|\mathcal{A} = \nu\}.$$

For a measure $\nu \in ca_+(\mathcal{A})$ we define $E_\sigma(\nu, \mathcal{A}_1) := E(\nu, \mathcal{A}_1) \cap ca(\mathcal{A}_1)$. $ba(\mathcal{A})$ is isometrically isomorphic to the topological dual space of $B(\mathcal{A})$ where $B(\mathcal{A})$ is defined

as the closure of the $\mathcal{A}$-measurable simple functions in the Banach space of all bounded real-valued functions on Ω with the supremum-norm (see [5], section IV.5). This allows us to define the weak*-topology on $ba(\mathcal{A})$. In the following $ba(\mathcal{A})$ will always be endowed with this topology.
In many cases the σ-additivity of all elements of some subset of $ba(\mathcal{A})$ can be reduced to the σ-additivity of the extremal points of this subset. This general principle follows from the Choquet theory and is stated in the following lemma where $\operatorname{ex} M$ stands for the set of all extremal points of M:

Lemma 1 *Let $\mathcal{A}$ be an algebra and M a compact convex subset of $ba(\mathcal{A})$. Then we have $M \subset ca(\mathcal{A})$ if and only if $\operatorname{ex} M \subset ca(\mathcal{A})$.*

Proof. Only the if-part has to be proved. Let $\mu \in M$ and let $\mathcal{B}_0(M)$ be the Baire-σ-algebra of M. Then the theorem of Choquet–Bishop–de Leeuw (see [12], p. 30) tells us that there exists a measure $m : \mathcal{B}_0(M) \cap \operatorname{ex} M \to [0,1]$ with $m(\operatorname{ex} M) = 1$, for which we have $f(\mu) = \int_{\operatorname{ex} M} f(\nu) dm(\nu)$ for every continous linear functional f on $ba(\mathcal{A})$. In particular, we get for every $A \in \mathcal{A}$

$$\mu(A) = \int_{\operatorname{ex} M} \nu(A)\, dm(\nu) \ . \tag{1}$$

For a sequence $A_1 \subset A_2 \subset \ldots$ with $A := \bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$ we obtain from $\operatorname{ex} M \subset ca(\mathcal{A})$ that $\nu(A_n) \to \nu(A)$ for all $\nu \in \operatorname{ex} M$. Because of the compactness of M there exists a constant $c > 0$ with $|\nu(A_n)| \le |\nu|(\Omega) \le c$ for all $n \in \mathbb{N}$ and $\nu \in \operatorname{ex} M$. Using Lebesgue's convergence theorem, we see $\mu(A_n) = \int_{\operatorname{ex} M} \nu(A_n)\, dm(\nu) \to \int_{\operatorname{ex} M} \nu(A)\, dm(\nu) = \mu(A)$ and therefore $\mu \in ca(\mathcal{A})$. □

Examples for the set M in Lemma 1 are $E(\nu, \mathcal{A}_1)$ for some given $\nu \in ba_+(\mathcal{A})$, see the next sections, and the core $C(v)$ of a game $(\Omega, \mathcal{A}, v)$ as defined in [6].

2 Extensions of atomic measures

For an algebra $\mathcal{A}$ and a system $\mathcal{E} \subset \mathcal{A}$ we call an element $A \in \mathcal{A}\backslash\mathcal{E}$ an $\mathcal{E}$-atom if for every $B \in \mathcal{A}$ with $B \subset A$ we have $B \in \mathcal{E}$ or $A\backslash B \in \mathcal{E}$ (cf. [9]). $\mathcal{A}$ is called $\mathcal{E}$-atomic if for every $A \in \mathcal{A}\backslash\mathcal{E}$ there exists an $\mathcal{E}$-atom $B \subset A$. For $\mathcal{E} = \{\emptyset\}$ we obtain the classical definitions of an atom and an atomic algebra. For $\nu \in ba_+(\mathcal{A})$ we define $\mathcal{N}(\nu) := \{A \subset \Omega : \nu^*(A) = 0\}$ (cf. [1], section 4.2). An algebra $\mathcal{A}$ is called ν-atomic if it is $(\mathcal{N}(\nu) \cap \mathcal{A})$-atomic. We set $at(\mathcal{A}) := \{\nu \in ba_+(\mathcal{A}) : \mathcal{A} \text{ is } \nu\text{-atomic}\}$ and call the elements of $at(\mathcal{A})$ atomic quasi-measures. In the following lemma, countable means finite or countably infinite.

Lemma 2 *Let $\mathcal{A}$ be a σ-algebra and $\nu \in at(\mathcal{A})$. Then there exists a countable family of disjoint ν-atoms whose union is Ω.*

Proof. By $A \sim B :\Leftrightarrow \nu(A \triangle B) = 0$ an equivalence relation is defined on the ν-atoms of $\mathcal{A}$. Let $\mathcal{K}'$ be a system of representatives of this equivalence relation. It is easily seen that for every $n \in \mathbb{N}$ there is only a finite number of $K' \in \mathcal{K}'$ with $\nu(K') > \frac{1}{n}$. Thus $\mathcal{K}'$ is countable. Let $N \subset \mathbb{N}$ with $\mathcal{K}' = \{K'_n : n \in N\}$ and $n_0 \in N$. Then by $K_n := K'_n \setminus \bigcup\{K'_j : j < n, j \in N\}$ and $\mathcal{K} := \{K_n : n \in N \setminus \{n_0\}\} \cup \{K_{n_0} \cup (\Omega \setminus \bigcup_{n \in N} K_n)\}$ a partition of Ω with the desired properties is defined. □

Now we consider two σ-algebras $\mathcal{A} \subset \mathcal{A}_1$ and some $\nu \in ca_+(\mathcal{A})$. If ν is an atomic measure, the following theorem gives some conditions under which each quasi-measure extension of ν is σ-additive. Condition *(iii)* and some simplifications in the proof of this theorem are due to Z. Lipecki [8].

Theorem 1 *Let $\mathcal{A} \subset \mathcal{A}_1$ be σ-algebras and $\nu \in ca_+(\mathcal{A}) \cap at(\mathcal{A})$. Then the following conditions are equivalent:*

(i) $E(\nu, \mathcal{A}_1) = E_\sigma(\nu, \mathcal{A}_1)$.

(ii) $\operatorname{ex} E(\nu, \mathcal{A}_1) \subset E_\sigma(\nu, \mathcal{A}_1)$.

(iii) For each increasing sequence $A_1 \subset A_2 \subset \ldots$ in $\mathcal{A}_1$ with $\Omega = \bigcup_{n \in \mathbb{N}} A_n$ we have $\nu_*(A_n) \to \nu(\Omega)$ $(n \to \infty)$.

(iv) $E(\nu, \mathcal{A}_1) \cap at(\mathcal{A}_1) \subset E_\sigma(\nu, \mathcal{A}_1)$.

(v) $E(\nu, \mathcal{A}_1) \cap \{\mu \in at(\mathcal{A}_1) : R(\mu) = R(\nu)\} \subset E_\sigma(\nu, \mathcal{A}_1)$.

(vi) For each ν-atom A and each disjoint sequence $(A_n)_{n \in \mathbb{N}}$ in $\mathcal{A}_1$ with $A = \bigcup_{n \in \mathbb{N}} A_n$ there exists a finite $N \subset \mathbb{N}$ with $\nu_*(\bigcup_{n \in N} A_n) = \nu(A)$.

If one (and therefore each) of the above conditions is fulfilled, every $\mu \in E(\nu, \mathcal{A}_1)$ is atomic.

Proof. For general $\nu \in ca_+(\mathcal{A})$ the equivalence of *(i)* and *(ii)* follows from Lemma 1 and the equivalence of *(ii)* and *(iii)* is shown in [6], Example 9. Now let $\nu \in ca_+(\mathcal{A}) \cap at(\mathcal{A})$. We show that *(v)* implies *(ii)* by proving

$$\operatorname{ex} E(\nu, \mathcal{A}_1) \subset \{\mu \in at(\mathcal{A}_1) : R(\mu) = R(\nu)\} . \tag{2}$$

Indeed, let $\mu \in \operatorname{ex} E(\nu, \mathcal{A}_1)$ and $A \in \mathcal{A}_1 \setminus \mathcal{N}(\mu)$. Due to a well-known theorem of Plachky ([13], Theorem 1) there exists a sequence $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ with $\mu(A \triangle A_n) < \frac{1}{n}$. Using the completeness of $\mathcal{A}$ with respect to the metric induced by $\mu|\mathcal{A}$, we obtain the existence of $\tilde{A} \in \mathcal{A}$ with $\mu(A \triangle \tilde{A}) = 0$. In particular, we have $\mu(A) = \nu(\tilde{A})$ and thus $R(\mu) = R(\nu)$. Because of $\nu \in at(\mathcal{A})$ there exists a ν-atom $K \subset \tilde{A}$ and we see from [14], p. 244, that $K \cap A$ is a μ-atom. *(i)* $\Rightarrow$ *(iv)* and *(iv)* $\Rightarrow$ *(v)* are trivial and thus conditions *(i)* to *(v)* are equivalent.
(iii) $\Rightarrow$ *(vi)*: Let A be a ν-atom and $A = \bigcup_{n \in \mathbb{N}} A_n$ with disjoint $A_n \in \mathcal{A}_1$. For $A'_n := \bigcup_{j \le n} A_j \cup (\Omega \setminus A)$ condition *(iii)* implies $\nu_*(A'_n) \to \nu(\Omega)$ and therefore $\nu_*(\bigcup_{j \le n} A_j) \to \nu(A)$. Because A is a ν-atom there exists some $n \in \mathbb{N}$ with $\nu_*(\bigcup_{j \le n} A_j) = \nu(A)$.

$(vi) \Rightarrow (iii)$: Let $A_1 \subset A_2 \subset \ldots$ be an increasing sequence in $\mathcal{A}_1$ with $\Omega = \bigcup_{n\in\mathbb{N}} A_n$ and $\mathcal{K} \subset \mathcal{A}$ a countable partition of Ω consisting of ν-atoms, see Lemma 2. According to (vi) for every ν-atom $K \in \mathcal{K}$ there exists an $n \in \mathbb{N}$ with $\nu(K) = \nu_*(K \cap A_n)$. Therefore for every finite $\mathcal{K}' \subset \mathcal{K}$ we have some $n \in \mathbb{N}$ with $\nu_*(A_n) \geq \nu_*(\bigcup_{K\in\mathcal{K}'}(K \cap A_n)) = \nu(\bigcup_{K\in\mathcal{K}'} K)$. Using the σ-additivity of ν, we conclude that $\nu_*(A_n) \to \nu(\Omega)$.
Now let one (and therefore each) of the conditions (i) to (vi) be fulfilled and $\mu \in E(\nu, \mathcal{A}_1) = E_\sigma(\nu, \mathcal{A}_1)$. For $A \in \mathcal{A}_1 \backslash \mathcal{N}(\mu)$ there exists a $K \in \mathcal{K}$ (where $\mathcal{K}$ is defined as above) with $\mu(A \cap K) > 0$. Condition (vi) tells us that $\mu|\mathcal{A}_1 \cap K \in at(\mathcal{A}_1 \cap K)$. Therefore there exists a μ-atom $\tilde{A} \subset A \cap K$ and we obtain $\mu \in at(\mathcal{A}_1)$. □

Remark 1 The following example shows that the inclusions in

$$\text{ex}\, E(\nu, \mathcal{A}_1) \subset E(\nu, \mathcal{A}_1) \cap \{\mu \in at(\mathcal{A}_1) : R(\mu) = R(\nu)\} \subset E(\nu, \mathcal{A}_1) \cap at(\mathcal{A}_1)$$

(cf. (2) and condition (v)) can be proper: Let $\mathcal{A} = \{\emptyset, \{1,2\}, \{3\}, \{1,2,3\}\}$, $\nu(\{1,2\}) = \frac{2}{3}$, $\nu(\{3\}) = \frac{1}{3}$ and $\mathcal{A}_1 = 2^{\{1,2,3\}}$. Then for μ_1 defined by $\mu_1(\{n\}) := \frac{1}{3}$ $(n \in \{1,2,3\})$ we have $\mu_1 \in at(\mathcal{A}_1) \cap (E(\nu, \mathcal{A}_1) \backslash \text{ex}\, E(\nu, \mathcal{A}_1))$ and $R(\mu_1) = R(\nu)$. Similarly μ_2 with $\mu_2(\{1\}) = \frac{1}{2}$ and $\mu_2(\{2\}) = \frac{1}{6}$ is in $E(\nu, \mathcal{A}_1) \cap at(\mathcal{A}_1)$ but $R(\mu_2) \neq R(\nu)$.

Corollary. *Let $\mathcal{A} \subset \mathcal{A}_1$ be σ-algebras where $\mathcal{A}$ is generated by a countable partition of Ω and $\nu \in ca_+(\mathcal{A})$. Then $E(\nu, \mathcal{A}_1) = E_\sigma(\nu, \mathcal{A}_1)$ holds if and only if for every atom A of $\mathcal{A}$ with $\nu(A) > 0$ there exists an $n \in \mathbb{N}$ and disjoint atoms $A_1 \ldots, A_n$ of $\mathcal{A}_1$ with $A = \bigcup_{j=1}^n A_j$.*

Proof. Let $E(\nu, \mathcal{A}_1) = E_\sigma(\nu, \mathcal{A}_1)$. We assume that there exists an atom A of $\mathcal{A}$ with $\nu(A) > 0$ and disjoint $A_n \in \mathcal{A}_1 \backslash \{\emptyset\}$ with $A = \bigcup_{n\in\mathbb{N}} A_n$. We apply Theorem 1 (vi) and obtain some finite $N \subset \mathbb{N}$ with $\nu_*(\bigcup_{n\in N} A_n) = \nu(A)$. Therefore there exists some $\tilde{A} \in \mathcal{A}_1$ with $\tilde{A} \subset \bigcup_{n\in N} A_n$ and $\nu(\tilde{A}) > 0$. This gives $\emptyset \neq \tilde{A} \underset{\neq}{\subset} A$ which is a contradiction.
To prove the converse, we show that condition (vi) of Theorem 1 is fulfilled. Let $A \in \mathcal{A}$ be a ν-atom and $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}_1$ be disjoint with $A = \bigcup_{n\in\mathbb{N}} A_n$. We choose an atom $\tilde{A}$ of $\mathcal{A}$ with $\tilde{A} \subset A$ and $\nu(\tilde{A}) = \nu(A)$. As $\tilde{A}$ can be written as a finite union of atoms of $\mathcal{A}_1$, in $\tilde{A} = \bigcup_{n\in\mathbb{N}}(\tilde{A} \cap A_n)$ only finitely many sets on the right side are nonempty. Thus with $N := \{n \in \mathbb{N} : \tilde{A} \cap A_n \neq \emptyset\}$ we obtain $\nu_*(\bigcup_{n\in N} A_n) \geq \nu(\tilde{A}) = \nu(A)$ which was to be shown. □

Remark 2 The following example shows that the corollary is not true for arbitrary $\mathcal{A}$, even if $\mathcal{A}$ is atomic: Let $\mathcal{A}$ be the σ-algebra of all countable subsets of $[0,1]$ and their complements, ν the restriction of the Lebesgue-measure to $\mathcal{A}$ and $\mathcal{A}_1$ the Borel-σ-algebra. Then $\nu(A) = 0$ holds for every atom A of $\mathcal{A}$, but $\text{ex}\, E_\sigma(\nu, \mathcal{A}_1) = \emptyset$ (see [14], Example 1). With $\text{ex}\, E_\sigma(\nu, \mathcal{A}_1) = \text{ex}\, E(\nu, \mathcal{A}_1) \cap E_\sigma(\nu, \mathcal{A}_1)$ and $\text{ex}\, E(\nu, \mathcal{A}_1) \neq \emptyset$ we get $E(\nu, \mathcal{A}_1) \neq E_\sigma(\nu, \mathcal{A}_1)$. For an example with $E_\sigma(\nu, \mathcal{A}_1) = \emptyset$ see [10], p. 444 - 445.

3 Extensions of two-valued measures

The preceeding theorem can be applied to two-valued measures, as in that case every set in $\mathcal{A}\backslash\mathcal{N}(\nu)$ is a ν-atom.

Theorem 2 *Let $\mathcal{A} \subset \mathcal{A}_1$ be σ-algebras and $\nu \in ca_+(\mathcal{A})$ with $R(\nu) = \{0,1\}$. Then the following conditions are equivalent:*

(i) $E(\nu, \mathcal{A}_1) = E_\sigma(\nu, \mathcal{A}_1)$.

(ii) $\{\mu \in E(\nu, \mathcal{A}_1) : R(\mu) = \{0,1\}\} \subset E_\sigma(\nu, \mathcal{A}_1)$.

(iii) For each decreasing sequence $A_1 \supset A_2 \supset \dots$ in $\mathcal{A}_1$ with $\bigcap_{n\in\mathbb{N}} A_n \in \mathcal{N}(\nu)$ there exists an $n \in \mathbb{N}$ with $A_n \in \mathcal{N}(\nu)$.

(iv) There exists a finite partition $\mathcal{K}$ of Ω with $\mathcal{A}_1 = ((\mathcal{A}_1 \cap \mathcal{N}(\nu)) \cup \mathcal{K})_\beta$.

(v) $\operatorname{ex} E(\nu, \mathcal{A}_1)$ *is a finite set.*

(vi) There exists some $n \in \mathbb{N}$ such that for every $\mu \in E(\nu, \mathcal{A}_1)$ the set $R(\mu)$ contains at most n elements.

(vii) For every $\mu \in E(\nu, \mathcal{A}_1) \cap at(\mathcal{A}_1)$ the set $R(\mu)$ is finite.

Proof. The conditions (i) - (iii) are reformulations of the corresponding conditions in Theorem 1, for (ii) we use Plachky's extremality criterion ([13], Theorem 1).
$(iii) \Rightarrow (iv)$: Similar to the proof of Lemma 2, we can define an equivalence relation on the $(\mathcal{N}(\nu) \cap \mathcal{A}_1)$-atoms by $A \sim B :\Leftrightarrow A \bigtriangleup B \in \mathcal{N}(\nu)$. Let $\mathcal{K}'$ be a system of representatives. If $\mathcal{K}'$ contained an infinite sequence $(A_n)_{n\in\mathbb{N}}$, then $B_n := \bigcup_{j=n}^{\infty} A_j$ would be a sequence with $\bigcap_{n\in\mathbb{N}} B_n \in \mathcal{N}(\nu)$, and (iii) implies $B_n \in \mathcal{N}(\nu)$ and therefore $A_n \in \mathcal{N}(\nu)$ for some $n \in \mathbb{N}$, which is a contradiction. This shows that $\mathcal{K}'$ is finite. We define the partition $\mathcal{K}$ of Ω as in the proof of Lemma 2. Now let $A \in \mathcal{A}_1$ be arbitrary and $K \in \mathcal{K}$. If $A \cap K \notin \mathcal{N}(\nu)$ we have $K\backslash A \in \mathcal{N}(\nu)$ and therefore $A \cap K = K\backslash(K\backslash A) \in ((\mathcal{N}(\nu) \cap \mathcal{A}_1) \cup \mathcal{K})_\beta$. Now $A = \bigcup_{K\in\mathcal{K}}(A \cap K)$ shows $\mathcal{A}_1 = ((\mathcal{N}(\nu) \cap \mathcal{A}_1) \cup \mathcal{K})_\beta$.
$(iv) \Rightarrow (v)$: If $\mathcal{K}$ is a partition like in (iv) with n elements, $\operatorname{ex} E(\nu, \mathcal{A}_1)$ has at most n elements.
$(v) \Rightarrow (vi)$: Let $exE(\nu, \mathcal{A}_1) = \{\mu_1, \dots, \mu_n\}$. Using the integral representation (1), we see that every $\mu \in E(\nu, \mathcal{A}_1)$ can be written in the form $\mu = \sum_{j=1}^{n} \alpha_j \mu_j$ with $\alpha_j \geq 0$, $\sum \alpha_j = 1$. Therefore $R(\mu)$ has at most 2^n elements (cf. [1], section 11.1).
$(vi) \Rightarrow (vii)$ is trivial.
$(vii) \Rightarrow (iii)$: Suppose (iii) is false. Then there is a decreasing sequence $(A_n)_{n\in\mathbb{N}} \in \mathcal{A}_1\backslash\mathcal{N}(\nu)$ with $\bigcap_{n\in\mathbb{N}} A_n \in \mathcal{N}(\nu)$. We define $\mathcal{K}' := \{A_n\backslash A_{n+1} : A_n\backslash A_{n+1} \notin \mathcal{N}(\nu)\}$. $\mathcal{K}'$ is infinite. (If this set were finite, we would have some $m \in \mathbb{N}$ with $A_n\backslash A_{n+1} \in \mathcal{N}(\nu)$ for all $n \geq m$ and $A_m = \bigcup_{n\geq m}(A_n\backslash A_{n+1}) \cup (\bigcap_{n\in\mathbb{N}} A_n) \in \mathcal{N}(\nu)$.) Fix $\tilde{K} \in \mathcal{K}'$ and define

$$\mathcal{K} := \{(\Omega\backslash\bigcup\mathcal{K}') \cup \tilde{K}\} \cup \{K \in \mathcal{K}' : K \neq \tilde{K}\}\,.$$

As $\mathcal{K}'$ is a subset of $\mathcal{A}_1 \backslash \mathcal{N}(\nu)$, the same is true for $\mathcal{K}$. Moreover, $\mathcal{K} = \{K_1, K_2, \ldots\}$ is a countably infinite partition of Ω. By $\mu_0(\bigcup_{n\in\mathbb{N}}(A_n \cap K_n)) := \sum_{n\in\mathbb{N}}(\frac{1}{2})^n \nu(A_n)$ we define an atomic measure extension of ν to $(\mathcal{A} \cup \mathcal{K})_\beta = \{\bigcup_{n\in\mathbb{N}}(A_n \cap K_n) : A_n \in \mathcal{A}\}$, cf. [2], Satz 2A. Choose $\mu \in \operatorname{ex} E(\mu_0, \mathcal{A}_1)$. Using (2) with μ_0 instead of ν we see $\mu \in E(\nu, \mathcal{A}_1) \cap at(\mathcal{A}_1)$. But $R(\mu) = R(\mu_0)$ is infinite which contradicts (vii). □

Remark 3 For arbitrary $\nu \in ca_+(\mathcal{A})$, condition (iv) of the preceeding theorem implies condition (i). This follows from [7], Remark 2, since the completion of ν is σ-additive.

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