A system of equations for magnetoelastic plates

Dissertation

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Zusammenfassung

Diese Arbeit beschäftigt sich mit dem reduzierten magnetoelastischem Plattensystem

$$\left|\begin{array}{l}\omega_{tt} - \gamma\Delta\omega_{tt} + d\Delta^2\omega - \alpha\,(\operatorname{rot}\operatorname{rot} h)\cdot h = 0 \quad \text{in} \;\; Q \\ h_t + \operatorname{rot}\operatorname{rot} h + \beta \operatorname{rot}\operatorname{rot}(hw_t) = 0 \quad \text{in} \;\; Q \\ \operatorname{div} h = 0 \;\; \text{in} \;\; Q \\ h\cdot\eta(t,x_1,x_2) = \eta\times\operatorname{rot} h(t,x_1,x_2) = 0 \;\; \text{on} \;\; \Sigma \\ w = \frac{\partial w}{\partial \eta} = 0 \;\; \text{on} \;\; \Sigma \\ w(x,0) = w_0, \;\; w_t(x,0) = w_1, \;\; h(x,0) = h_0\end{array}\right.$$

wobei $Q = \Omega \times (0,T)$ und $\Sigma = \partial\Omega \times (0,T)$. Der Skalar w bezeichnet die Verschiebung, $h = (h_1, h_2)$ das magnetische Feld und Ω ein beschränktes Gebiet im $\mathbb{R}^2$ der Klasse $\mathcal{C}^2$. Diese Gleichungen wurden aus [9], [10] and [11] hergeleitet. Aus mathematischer Sicht kann das betrachtete System gesehen werden als die Verkopplung des Plattensystems (vom Schrödinger-Typ), das Schwingungen elastischer Wellen in einer dünnen Platte modelliert, mit einer parabolischen Gleichung für das magnetische Feld.

Im einleitenden ersten Abschnitt führen wir Notationen, Definitionen, Lemmata und Sätze ein, die wir brauchen werden. Im zweiten Abschnitt folgen wir [9], um das lineare und das nichtlineare System magnetoelastischer Gleichungen für eine dünne Platte von gleichmäßiger Dicke $2h$ zu finden.

Man nimmt an, dass die Platte aus unserer Überlegung eine mittlere Fläche zwischen ihrer Ober- und Unterseite hat, welche im Gleichgewicht das Gebiet Ω in der Ebene $x_3 = 0$ besitzt. Wir nehmen an, die Dicke der Platte sei so klein, dass wir nicht mit den 3-dimensionalen magnetoelastischen Gleichungen für Festkörper mit Lösungen, die von (t, x_1, x_2, x_3) abhängen, zu tun haben werden. Sondern wir möchten ein System von zwei reduzierten verkoppelten Gleichungen mit Lösungen, die nur von (t, x_1, x_2) abhängen, finden mit $(x_1, x_2) \in \Omega$.

Nach dem wir das Gleichungssystem gefunden haben, konzentrieren wir uns zuerst auf das lineare System

$$
\left|
\begin{array}{c}
\omega_{tt} - \gamma \Delta \omega_{tt} + d\Delta^2 \omega - \alpha \left(\operatorname{rot} \operatorname{rot} h\right) \cdot \vec{H} = 0 \quad \text{in} \;\; Q \\
h_t + \operatorname{rot} \operatorname{rot} h + \beta \operatorname{rot} \operatorname{rot} \left(\vec{H} \cdot w_t\right) = 0 \quad \text{in} \;\; Q \\
\operatorname{div} h = 0 \;\; \text{in} \;\; Q \\
h \cdot \eta(t, x_1, x_2) = \eta \times \operatorname{rot} h(t, x_1, x_2) = 0 \;\; \text{on} \;\; \Sigma \\
w = \frac{\partial w}{\partial \eta} = 0 \;\; \text{on} \;\; \Sigma \\
w(x,0) = w_0, \quad w_t(x,0) = w_1, \quad h(x,0) = h_0
\end{array}
\right.
$$

wobei $\vec{H} = (H_1, H_2)$ ein konstantes magnetisches Feld ist. So betrachten wir in Sektion 3 die Existenz und Eindeutigkeit von Lösungen des linearen Systems, was wir durch Unterscheidung der Fälle $\gamma = 0$ und $\gamma > 0$ bewerkstelligen. Wir benutzen hier Halbgruppentheorie und die Galerkinmethode. Für den Fall $\gamma = 0$ haben wir mit Halbgruppentheorie nach [19] gearbeitet und für den Fall $\gamma > 0$ entsprechend nach [16].

Im Sektion 4 betrachten wir die Existenz von rein imaginären Eigenwerten des linearen Systems, um Information über das asymptotische Verhalten der Lösungen zu bekommen. Wir reduzieren das Problem auf die Frage nach der Existenz von nichttrivialen Eigenfunktionen für die Bilaplacesche mit Randbedingungen und eine Zusatzbedingung in Ω. Die Existenz von solchen Eigenfunktionen bleibt offen. Einerseits wäre das schwingende Verhalten bewiesen, wenn nichttriviale Eigenfunktionen existieren würden. Wenn andererseits nur die triviale Lösung existieren würde, könnte man mit dem Lasalleschen Invarianzprinzip wie in [19] ableiten dass, jede Lösung endlicher Energie des linearen Systems im Energieraum gegen Null konvergiert wenn $t \longrightarrow \infty$.

Im Abschnitt 5 arbeiten wir mit dem nichtlinearen System magnetoelastischer Platten. Wir folgten [4], um die Existenz von Lösungen mit dem Galerkinverfahren zu beweisen. Wir brauchten Regularität der Anfangswerte, um unser Existenzergebnis zu beweisen. Eine Fallunterscheidung nach γ war nicht nötig. Es wurde Existenz von Lösungen für kleine Daten bewiesen. Die Eindeutigkeit bleibt offen.

Introduction

In this work we are going to study the reduced magnetoelastic plate system

$$
\left|
\begin{array}{c}
\omega_{tt} - \gamma\Delta\omega_{tt} + d\Delta^2\omega - \alpha\,(\operatorname{rot}\operatorname{rot} h)\cdot h = 0 \qquad \text{in} \quad Q \\
\\
h_t + \operatorname{rot}\operatorname{rot} h + \beta \operatorname{rot}\operatorname{rot}(hw_t) = 0 \qquad \text{in} \quad Q \\
\\
\\
\operatorname{div} h = 0 \quad \text{in} \quad Q \\
\\
h \cdot \eta(t, x_1, x_2) = \eta \times \operatorname{rot} h(t, x_1, x_2) = 0 \quad \text{on} \quad \Sigma \\
\\
w = \frac{\partial w}{\partial \eta} = 0 \quad \text{on} \quad \Sigma \\
w(x, 0) = w_0, \quad w_t(x, 0) = w_1, \quad h(x, 0) = h_0
\end{array}
\right.
$$

where $Q = \Omega \times (0, T)$ and $\Sigma = \partial\Omega \times (0, T)$. The scalar w denotes the displacement, while $h = (h_1, h_2)$ denotes the magnetic field and Ω is a bounded domain of $\mathbb{R}^2$ of class $\mathcal{C}^2$. These equations have been derived from [9], [10] and [11]. From a mathematical point of view, the system under consideration may be viewed as the coupling between the plate system (which is of Schrödinger type) modelling vibrations of elastic waves in a thin plate and a parabolic equation for the magnetical field.

The first section is devoted to preliminaries, where we introduce notations, definitions, Lemmas and Theorems which we will need. In the second section we follow [9] to find the linear and nonlinear system of magnetoelastic equations for a thin plate of uniform thickness $2h$.

It is assumed that the plate being considered has a middle surface midway between its faces which, in equilibrium, occupies the region Ω in the plane $x_3 = 0$. We consider a very small thickness and so we do not work with 3-dimensional magnetoelastic equations for a solid bodies with solutions which depend on (t, x_1, x_2, x_3). But we will find a system of two reduced coupled equations with solutions which depend on (t, x_1, x_2), where $(x_1, x_2) \in \Omega$.

After we have found the system of equations, we restrict first our attention to the linear system

$$
\left|
\begin{array}{c}
\omega_{tt} - \gamma \Delta \omega_{tt} + d\Delta^2 \omega - \alpha \left(\operatorname{rot} \operatorname{rot} h \right) \cdot \vec{H} = 0 \quad \text{ in } \; Q \\
h_t + \operatorname{rot} \operatorname{rot} h + \beta \operatorname{rot} \operatorname{rot} \left(\vec{H} \cdot w_t \right) = 0 \quad \text{ in } \; Q \\
\operatorname{div} h = 0 \; \text{ in } \; Q \\
h \cdot \eta(t, x_1, x_2) = \eta \times \operatorname{rot} h(t, x_1, x_2) = 0 \; \text{ on } \; \Sigma \\
w = \frac{\partial w}{\partial \eta} = 0 \; \text{ on } \; \Sigma \\
w(x,0) = w_0, \;\; w_t(x,0) = w_1, \;\; h(x,0) = h_0
\end{array}
\right.
$$

where $\vec{H} = (H_1, H_2)$ is a constant magnetic field, so in section 3 it is studied the existence and uniqueness of solutions of the linear system, that is done in two cases, $\gamma = 0$ and $\gamma > 0$. Here we use semigroup theory as well as the Galerkin method. We have followed [19] for the case $\gamma = 0$ with semigroup theory and [16] for the case $\gamma > 0$, respectively.

In section 4 we study the existence of purely imaginary eigenvalues for the linear system in order to obtain some information about the asymptotical behavior of the solutions. We find that the problem is reduced to the possible existence of non trivial eigenfunctions for the bilaplacian with boundary conditions and an extra condition in Ω. The existence of such eigenfunctions remains open . In one hand if there are non trivial eigenfunctions then oscillatory behavior is proved. On the other hand if there exists only the trivial solution, with the Lasalle invariance principle as in [19] we could deduce that every solution of finite energy of the linear system converges to zero in the energy space as $t \longrightarrow \infty$.

In section 5, we work with the nonlinear system of magnetoelastic plates. Following [4] we have proved the existence of solutions with the Galerkin method. We have needed regularity for the initial values in order to prove our existence result. The two cases study for γ was not needed. Existence of solutions was proved for small data. Uniqueness remain open.

1 Preliminaries

The rotation of a vector field $u : \mathbb{R}^2 \longmapsto \mathbb{R}^2$ is defined as the scalar

$$\operatorname{rot}\begin{pmatrix} u^1 \\ u^2 \end{pmatrix} := \partial_1 u^2 - \partial_2 u^1$$

Where $\partial_j = \frac{\partial}{\partial x_j}$ for $j = 1, 2$.
If the rotation of a scalar field f is defined in $\mathbb{R}^2$ as

$$\operatorname{rot} f := \begin{pmatrix} \partial_2 f \\ -\partial_1 f \end{pmatrix},$$

then the formula

$$\Delta = \triangledown \operatorname{div} - \operatorname{rot}\operatorname{rot}$$

holds in two and three space dimensions.
In following we denote with $L^2(\Omega), H_0^i(\Omega), H^s(\Omega),\ i = 1, 2,\ s \in \mathbb{R}$, the same standard spaces as in Adams [1]. We denote by $(\cdot, \cdot)$ and $|\cdot|$ the scalar product and norm in $[L^2(\Omega)]^2$ and $L^2(\Omega)$, and by $((\cdot, \cdot))_s$ and $\|\cdot\|_s$ the scalar product and norm in $[H^s(\Omega)]^2$ and $H^s(\Omega)$.
We introduce the Λ operator using [14] and then give some applications.

The Λ operator
Let X and Y be two separable Hilbert spaces, with $X \subset Y$, X dense in Y with continuous injection. Let $(,)_X$ and $(,)_Y$ be the scalar products in X and Y, respectively.
The space X may be defined as the domain of an unbounded self-adjoint, positive operator Λ in Y (in fact Λ is not unique!), X having a norm equivalent to the graph norm

$$\left(\|u\|_Y^2 + \|\Lambda u\|_Y^2\right)^{\frac{1}{2}}, \quad u \in \mathcal{D}(\Lambda) = X. \tag{1.1}$$

We define $\mathcal{D}(S)$ as the set of u's such that the antilinear form

$$v \longmapsto (u, v)_X, \quad v \in X \tag{1.2}$$

is continuous in the topology induced by Y. Then

$$(u, v)_X = (Su, v)_Y, \tag{1.3}$$

which defines S as an unbounded operator in Y with domain $\mathcal{D}(S)$.
It can be verified that $\mathcal{D}(S)$ is dense in Y and S is self-adjoint and strictly positive.

In fact,

$$(Sv,v)_Y = \|v\|_X^2 \geq c\|v\|_Y^2. \tag{1.4}$$

Using the spectral decomposition of self-adjoint operators, the powers S^θ of $S,\ \ \theta \in \mathbb{R}$, may be defined.

In particular, we shall use

$$\Lambda = S^{\frac{1}{2}} \tag{1.5}$$

The operator Λ is self-adjoint and positive in Y, with domain X.
From (1.3) and (1.5) we deduce that

$$(u,v)_X = (\Lambda u, \Lambda v)_Y, \quad \forall u,v \in X. \tag{1.6}$$

Remark: The operator S depends on the choice of the scalar products in X and Y (without changing the topologies of X and Y) and therefore Λ also depends of these scalar products.

Applications:
I:
Let Ω be simply connected. Let

$$Y := \Big\{ g \in \left[L^2(\Omega)\right]^2; \ \ \operatorname{div} h = 0 \quad \text{in} \quad \Omega, \quad g\cdot\eta = 0 \quad \text{on} \quad \partial\Omega \Big\}$$

the space with the norm $\|\cdot\|_Y := |\cdot|_{L^2(\Omega)^2}$ and

$$V = \big\{ g \in Y; \quad \operatorname{rot} g \in L^2(\Omega) \big\}$$

with the norm

$$\|\cdot\|_V := \Big(|\operatorname{rot} g|^2_{L^2(\Omega)} + |g|^2_{[L^2(\Omega)]^2} \Big)^{\frac{1}{2}},$$

where the condition $g\cdot\eta = 0$ is well defined as $(g\cdot\eta, f)_{\partial\Omega} = 0$ for all $f \in H^1(\Omega)$.
We have that $V \subset Y$, with dense, continuous and compact injection. Then the bilinear form $b(f,g) = (\operatorname{rot} f, \operatorname{rot} g)$, which is coercive in V when Ω is simply connected (see [7], [13]), generates the operator $B := \operatorname{rot}\operatorname{rot}$ with domain

$$\mathcal{D}(B) = \{ g \in V / \ \ Bg \in Y \}$$

and there exists the spectral family of eigenfunctions for the problem

$$\begin{cases} B\psi_n = \lambda_n \psi_n \quad \text{in} \ \ \Omega \\ \\ \qquad \psi_n \in \mathcal{D}(B) \end{cases}$$

II:
We know that $H_0^2(\Omega) \subset L^2(\Omega)$ with dense, continuous and compact injection, $H_0^2(\Omega)$ endowed with the norm $\|\cdot\|_2$ and $L^2(\Omega)$ endowed with the usual norm $|\cdot| := |\cdot|_{L^2(\Omega)}$.
We have the bilinear form in $H_0^2(\Omega) \times H_0^2(\Omega)$

$$a(u, v) = (\Delta u, \Delta v)$$

which is coercive in $H_0^2(\Omega)$ and generate the operator $A := \Delta^2$ with domain

$$\mathcal{D}(A) = \left\{ v \in H_0^2(\Omega) / \ \ Av \in L^2(\Omega) \right\}$$

and there exists the spectral family of eigenfunctions for the problem

$$\begin{cases} A\varphi_n = \rho_n \varphi_n \quad \text{in} \ \ \Omega \\ \\ \qquad \varphi_n \in \mathcal{D}(A) \end{cases}$$

We will need the following lemma later.

Lemma 1.1 *(Bihari's lemma or generalization of the Bellman lemma, see [2]). Let $Y(x), F(x)$ be positive continuous functions in $a \leq x \leq b$ and $k \geq 0, M \geq 0$, further let $\omega \to \omega(u)$ be a non-negative increasing continuous function for $u \geq 0$. Then the inequality*

$$Y(x) \leq k + M \int_a^x F(t)\omega(Y(t))dt \qquad (a \leq x \leq b)$$

implies the inequality

$$Y(x) \leq R^{-1}\left(R(k) + M \int_a^x F(t)dt \right) \qquad (a \leq x \leq b)$$

where

$$R(u) = \int_{u_0}^u \frac{dt}{\omega(t)} \qquad (u_0 > 0, \ u \geq 0)$$

and R^{-1} denotes the inverse function to R.

Theorem 1.1 *(Aubin-Lions theorem, see [15]) Let* B_0, B, B_1 *Banach spaces, with* B_0, B_1 *reflexive. If* $B_0 \subset B \subset B_1$ *and the immersion of* B_0 *in* B *is compact, then for all* $1 < p_0, p_1 < \infty$ *the space*

$$W = \{u \in L^{p_0}(0,T;B_0), \quad u' \in L^{p_1}(0,T;B_1)\}$$

endowed with the norm

$$\|u\|_W = \|u\|_{L^{p_0}(0,T;B_0)} + \|u\|_{L^{p_1}(0,T;B_1)}.$$

is a Banach space and

$$W \subset L^{p_0}(0,T;B) \quad \textit{with compact immersion.}$$

2 The system of magnetoelastic plate equations

In the section we find the system of magnetoelastic equations for a thin plate of uniform thickness $2h$. In the first part we obtain the linear system and in the second part we obtain the nonlinear one.

2.1 The linear system

The classical plate theory must be modified in order to take into account the effect of the applied magnetic field.
We consider a thin plate P, of thickness $2h$, with a median surface $x_3 = 0$. We use the classical plate theory of Love and Kirchoff. To obtain this model (in small displacement theory) we assume (a) a linear strain displacement relation (strain tensor)

$$\epsilon_{kl} = \frac{1}{2}\left(u_{k,l} + u_{l,k}\right)$$

where $u = u(x_1, x_2, x_3, t) = (u_1, u_2, u_3)$, and (b) the linear filaments of the plate initially perpendicular to the middle surface remain straight and perpendicular to the deformed middle surface and undergo neither contraction nor extension. (Thus transverse shear effects are neglected.) Hypothesis (b) imposes a nonlinear relationship between the displacements $\{u_i\}$ and $\{v_i, w\}$. If this relationship is linearized we obtain the approximate relations (that is, correct up to terms of order h^2)

$$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}; \qquad \begin{array}{l} u_i = v_i(x_1, x_2, t) - x_3 \frac{\partial w}{\partial x_i}(x_1, x_2, t); \quad i = 1, 2 \\ \\ u_3 = w(x_1, x_2, t) \end{array} \tag{2.1}$$

For the total magnetic field we have (see [9])

$$H = \underline{H} + b, \qquad \text{where} \ \ \underline{H} = (H_1, H_2, H_3)$$

Here $\underline{H}$ is a static bias magnetic field called the primary magnetic field, while b is the induced magnetic field. Likewise, the displacement fields u_1, u_2, and u_3 are considered to be small. The deformation is measured from the initial configuration which carries the $\underline{H}$-field but no initial stress. The constitutive equation for the bias field $\underline{H}$, is $B_0 = \underline{H} + M^0 = \mu_0 \underline{H}$, where B is the magnetic induction, M the magnetization field and μ_0 is constant.

The bias field satisfies the Maxwell's equation

$$\triangledown \times \underline{H} = 0.$$

We will assume that $\underline{H}$ is constant.
The differential equation for the magnetic body force and stress-strain relations is (see [9])

$$\sigma_{kl,k} + \tilde{F}_l - \rho_0 \ddot{u}_l = 0 \tag{2.2}$$

where in (2.2) we assume that

$$\tilde{F}_l = (\mu_0 - 1)(b_{k,l} - b_{l,k})H_k \tag{2.3}$$

(see [9] (8.14.10)) is the magnetic body force,

$$\sigma_{kl} = \lambda_e \epsilon_{rr} \delta_{kl} + 2\mu_e \epsilon_{kl}; \tag{2.4}$$

are the stress-strain relations in isotropic media λ_e, μ_e the elastic moduli and ρ_0 is the (constant) mass density per unit of volume.

Equations $(2.2 - 2.3)$ are valid for three-dimensional isotropic magnetoelastic solids. In order to obtain equations for thin plates, we employ (2.1) and perform the following two sets of operations on $(2.2 - 2.3)$:

(i) We integrate these equations across the thickness of the plate.
(ii) We multiply the first two components $(l = 1, 2)$ of these equations by x_3 and then integrate them across the thickness of the plate.

It is customary in thin plate theory to assume that the transverse normal stress σ_{33} is negligible compared to the other stresses. Thus we set $\sigma_{33} = 0$ (More precisely is $\sigma_{33} = O(h^2)$), which gives $\epsilon_{33} = -\lambda\epsilon_{\gamma\gamma}/(\lambda_e + 2\mu_e)$. Using this, the stress-strain relation (2.4) becomes

$$\sigma_{\alpha\beta} = \frac{E}{1-\nu^2}\left[\nu\epsilon_{\gamma\gamma}\delta_{\alpha\beta} + (1-\nu)\epsilon_{\alpha\beta}\right]$$

$$\sigma_{\alpha 3} = \sigma_{33} = 0, \quad \alpha, \beta, \gamma = 1, 2, \tag{2.5}$$

where

$$\epsilon_{\alpha\beta} = \frac{1}{2}(v_{\alpha,\beta} + v_{\beta,\alpha}) - x_3\omega_{,\alpha\beta} \ , \tag{2.6}$$

E is the Young's Modulus and ν Poisson's ratio $0 < \nu < \frac{1}{2}$.
Operations (i) and (ii) introduce the planar forces $N_{\alpha\beta}$, vertical shears Q_α, and couples $M_{\alpha\beta}$, as defined by

$$N_{\alpha\beta} = \int_{-h}^{h} \sigma_{\alpha\beta} dx_3, \qquad Q_\alpha = \int_{-h}^{h} \sigma_{\alpha 3} dx_3, \qquad M_{\alpha\beta} = \int_{-h}^{h} \sigma_{\alpha\beta} x_3 dx_3.$$

Using (2.5) and (2.6) we obtain

$$N_{\alpha\beta} = \frac{Eh}{1-\nu^2}[2\nu v_{\gamma,\gamma}\delta_{\alpha\beta} + (1-\nu)(v_{\alpha,\beta} + v_{\beta,\alpha})],$$

$$M_{\alpha\beta} = -D[\nu w_{,\gamma\gamma}\delta_{\alpha\beta} + (1-\nu)w_{,\alpha\beta}]$$

where D is the flexural rigidity defined by

$$D := \frac{2Eh^3}{3(1-\nu^2)}.$$

We note that because of (2.5), Q_α vanishes. However, generally, by considering the effect of the shear deformations this bring additional terms of order x_3^2 to the expressions (2.1). These effects become important for thick plates. But it's not our case.
The operations (i) and (ii) on (2.2) and elimination give (see [9])

$$\frac{Eh}{1-\nu}\left(v_{i,ij} + \frac{1-\nu}{1+\nu}v_{j,ii}\right) + \tilde{f}_j - 2\rho_0 h \ddot{v}_j = 0, \qquad i,j = 1,2 \tag{2.7}$$

$$D\Delta^2 w - f + 2\rho_0 h\ddot{w} - \frac{2}{3}\rho_0 h^3 \Delta\ddot{w} = 0 \tag{2.8}$$

where,

$$\tilde{f}_k = \int_{-h}^{h} \tilde{F}_k dx_3, \qquad f = \int_{-h}^{h} (\tilde{F}_3 + x_3\tilde{F}_{\beta,\beta})dx_3. \tag{2.9}$$

Equation (2.7) is the differential equation for the planar motions of the plate, and (2.8) is that for the flexural motions. We note that these equations are coupled with the magnetic fields through $\tilde{f}_k$, f and the boundary conditions.

Let η represent the unit normal of the undeformed surface under consideration, and $\tilde{\eta}$ represent its increment due to deformation. For $\tilde{\eta}$ we have (see [9]),

$$\tilde{\eta} = (u_{i,j}\eta_i\eta_j\delta_{kl} - u_{l,k})\eta_l$$

for example, on the upper and lower surfaces $x_3 = \pm h$ of the plate

$$\eta = \pm\vec{i}_3, \quad \tilde{\eta}_i = \mp w_{,i}, \quad \tilde{\eta}_3 = 0, \quad i = 1, 2.$$

On the contour surface $\partial\Omega$ of the plate, we have

$$\eta = (\eta_i, 0),$$

$$\tilde{\eta}_i = (\tilde{\eta}_{ij} - x_3\tilde{m}_{ij})\eta_j, \qquad \tilde{\eta}_3 = \frac{\partial w}{\partial x_j}\eta_j,$$

where

$$\tilde{\eta}_{ij} = v_{\gamma,r}\eta_\gamma\eta_r\delta_{ij} - v_{j,i}, \qquad \tilde{m}_{ij} = w_{\gamma,r}\eta_\gamma\eta_r\delta_{ij} - w_{,ji}, \qquad \gamma, r, i, j = 1, 2.$$

Now, from (2.2) we compute

$$\begin{aligned}
\tilde{F}_1 &= (\mu_0 - 1)\left[\left(\tfrac{\partial b_2}{\partial x_1} - \tfrac{\partial b_1}{\partial x_2}\right)H_2 + \left(\tfrac{\partial b_3}{\partial x_1} - \tfrac{\partial b_1}{\partial x_3}\right)H_3\right] \\
\tilde{F}_2 &= (\mu_0 - 1)\left[\left(\tfrac{\partial b_1}{\partial x_2} - \tfrac{\partial b_2}{\partial x_1}\right)H_1 + \left(\tfrac{\partial b_3}{\partial x_2} - \tfrac{\partial b_2}{\partial x_3}\right)H_3\right] \\
\tilde{F}_3 &= (\mu_0 - 1)\left[\left(\tfrac{\partial b_1}{\partial x_3} - \tfrac{\partial b_3}{\partial x_1}\right)H_1 + \left(\tfrac{\partial b_2}{\partial x_3} - \tfrac{\partial b_3}{\partial x_2}\right)H_2\right]
\end{aligned}$$

In order to obtain our reduced system, we assume

$$b = \begin{pmatrix} h_{01} \\ h_{02} \\ h_{03} \end{pmatrix} + x_3\begin{pmatrix} h_{11} \\ h_{12} \\ h_{13} \end{pmatrix}, \quad h_{ij}(t, x_1, x_2), \quad i = 0, 1; \; j = 1, 2, 3;$$

so, we must compute the force for this assumed b

$$\tilde{F}_1 = (\mu_0 - 1)\left\{\left[\left(\tfrac{\partial h_{02}}{\partial x_1} - \tfrac{\partial h_{01}}{\partial x_2}\right)H_2 + \tfrac{\partial h_{03}}{\partial x_1}H_3\right] + x_3\left[\left(\tfrac{\partial h_{12}}{\partial x_1} - \tfrac{\partial h_{11}}{\partial x_2}\right)H_2 + \tfrac{\partial h_{13}}{\partial x_1}H_3\right]\right\}$$

$$\tilde{F}_2 = (\mu_0 - 1)\left\{\left[\left(\tfrac{\partial h_{01}}{\partial x_2} - \tfrac{\partial h_{02}}{\partial x_1}\right)H_1 + \tfrac{\partial h_{03}}{\partial x_1}H_3\right] + x_3\left[\left(\tfrac{\partial h_{11}}{\partial x_2} - \tfrac{\partial h_{12}}{\partial x_1}\right)H_1 + \tfrac{\partial h_{13}}{\partial x_2}H_3\right]\right\}$$

$$\tilde{F}_3 = (\mu_0 - 1)\left\{\left[-\tfrac{\partial h_{03}}{\partial x_1}H_1 - \tfrac{\partial h_{03}}{\partial x_2}H_2\right] + x_3\left[-\tfrac{\partial h_{13}}{\partial x_1}H_1 - \tfrac{\partial h_{03}}{\partial x_2}H_2\right]\right\}$$

and now we use these force in (2.9) to compute f

$$\int_{-h}^{h} x_3 \frac{\partial}{\partial x_1} \tilde{F}_1 dx_3 = \frac{2h^3}{3}(\mu_0 - 1)\left[\left(\frac{\partial^2 h_{12}}{\partial x_1^2} - \frac{\partial^2 h_{11}}{\partial x_2 \partial x_1}\right) H_2 + \frac{\partial^2 h_{13}}{\partial x_1^2} H_3\right]$$

$$\int_{-h}^{h} x_3 \frac{\partial}{\partial x_2} \tilde{F}_2 dx_3 = \frac{2h^3}{3}(\mu_0 - 1)\left[\left(\frac{\partial^2 h_{11}}{\partial x_2^2} - \frac{\partial^2 h_{12}}{\partial x_1 \partial x_2}\right) H_1 + \frac{\partial^2 h_{13}}{\partial x_2^2} H_3\right]$$

$$\int_{-h}^{h} \tilde{F}_3 dx_3 = 2h(\mu_0 - 1)\left[-\frac{\partial h_{03}}{\partial x_1} H_1 - \frac{\partial h_{03}}{\partial x_2} H_2\right] \tag{2.10}$$

then by (2.8) – (2.10) and $\frac{\partial h_{03}}{\partial x_3} = 0$ we have

$$D\Delta^2 w + \alpha \left[\left(\frac{\partial^2 h_{12}}{\partial x_1^2} - \frac{\partial^2 h_{11}}{\partial x_1 \partial x_2}\right) H_2 + \left(\frac{\partial^2 h_{11}}{\partial x_2^2} - \frac{\partial^2 h_{12}}{\partial x_1 \partial x_2}\right) H_1 + \Delta h_{13} H_3\right] + \\ \lambda \triangledown h_{03} \cdot (H_1, H_2, H_3) + + a\ddot{w} - b\Delta\ddot{w} = 0 \tag{2.11}$$

where $\alpha = \frac{2h^3}{3}(\mu_0 - 1)$ and $\lambda = -2h(\mu_0 - 1)$, $\quad a = 2\rho_0 h$, $\quad \gamma = \frac{2}{3}\rho_0 h^3$, $\quad d := D$

and we rewrite (2.11) to have

$$d\Delta^2 w - \alpha \operatorname{rot}\operatorname{rot} h_1 \cdot (H_1, H_2, H_3) + \lambda \triangledown h_{03} \cdot (H_1, H_2, H_3) + \\ + a\ddot{w} - \gamma\Delta\ddot{w} = 0 \tag{2.12}$$

now, we divide with a and we use the same names for all constants, and so we had obtained our first reduced equation

$$\ddot{w} - \gamma\Delta\ddot{w} + d\Delta^2 w - \alpha \operatorname{rot}\operatorname{rot} h_1 \cdot \underline{H} + \lambda \triangledown h_{03} \cdot \underline{H} = 0 \tag{2.13}$$

We will reduce now the magnetical equation

$$\begin{aligned} & b_t - \Delta b - \beta \operatorname{rot}[u_t \times \underline{H}] = 0 \ \text{ in } \Omega \times (0, T), \\ & \operatorname{div} b = 0 \ \text{ in } \Omega \times (0, T), \\ & b \cdot \eta(x_1, x_2, x_3) = 0 \text{ on } \partial\Omega \times (0, T); \\ & \operatorname{rot} b \times \eta(x_1, x_2, x_3) = 0, \text{ on } \partial\Omega \times (0, T) \end{aligned} \tag{2.14}$$

to a equation where the unknown functions depend only on (x_1,x_2). In order to do that, we recall

$$b=h_0+x_3h_1,\quad \text{for } h_i:=(h_{ij}(t,x_1,x_2))\;\;,\quad i=0,1;\quad j=1,2,3, \tag{2.15}$$

now, we use (2.1) to compute

$$\mathrm{rot}[u_t\times \underline{H}]=\begin{pmatrix} \frac{\partial}{\partial x_2}\left[(v_t^1,v_t^2)\cdot(H_1,H_2)^{\perp}\right]-\frac{\partial}{\partial x_1}\omega_t H_3 \\ \\ -\frac{\partial}{\partial x_1}\left[(v_t^1,v_t^2)\cdot(H_1,H_2)^{\perp}\right]-\frac{\partial}{\partial x_2}w_t H_3 \\ \\ -\operatorname{div}(v_t^1,v_t^2)\,H_3+\triangledown\omega_t\cdot(H_1,H_2)\end{pmatrix}+x_3\begin{pmatrix}-\frac{\partial}{\partial x_2}\left[\triangledown\omega_t\cdot(H_1,H_2)^{\perp}\right] \\ \\ \frac{\partial}{\partial x_1}\left[\triangledown\omega_t\cdot(H_1,H_2)^{\perp}\right] \\ \\ \Delta\omega_t H_3\end{pmatrix}$$

where $\triangledown\omega_t:=(\frac{\partial w_t}{\partial x_1},\frac{\partial w_t}{\partial x_2})$, and so we can write

$$\mathrm{rot}[u_t\times\underline{H}]=\mathrm{rot}\left[\begin{pmatrix}\dot{v_1}\\ \dot{v_2}\\ \dot{w}\end{pmatrix}\times\underline{H}\right]-H_3\begin{pmatrix}\frac{\partial}{\partial x_1}w_t\\ \frac{\partial}{\partial x_2}w_t\\ 0\end{pmatrix}-x_3\,\mathrm{rot}\,\mathrm{rot}\,(\underline{H}\cdot w_t) \tag{2.16}$$

Let $\vec{H}=(H_1,H_2)$, $\vec{H}^{\perp}=(H_2,-H_1)$, $v=(v_1,v_2)$. We obtain from (2.14), (2.15) and (2.16) two vectorial equations by
$i)$ integration of (2.14) in x_3 from $-h$ to h, and
$ii)$ multiplication of (2.14) with x_3 and then integration in x_3 from $-h$ to h, respectively

$$
\left|\begin{array}{l}
h_{0t} - \Delta h_0 - \beta \begin{pmatrix} \frac{\partial}{\partial x_2}\left[v_t \cdot \vec{H}^{\perp}\right] - \frac{\partial}{\partial x_1}\omega_t H_3 \\ \\ -\frac{\partial}{\partial x_1}\left[v_t \cdot \vec{H}^{\perp}\right] - \frac{\partial}{\partial x_2} w_t H_3 \\ \\ -\operatorname{div} v_t H_3 + \nabla\omega_t \cdot \vec{H} \end{pmatrix} = 0 \\ \\
h_{1t} - \Delta h_1 - \beta \underbrace{\begin{pmatrix} -\frac{\partial}{\partial x_2}\left[\nabla\omega_t \cdot \vec{H}^{\perp}\right] \\ \\ \frac{\partial}{\partial x_1}\left[\nabla\omega_t \cdot \vec{H}^{\perp}\right] \\ \\ \Delta\omega_t H_3 \end{pmatrix}}_{-\operatorname{rot}\operatorname{rot}(\underline{H} w_t)} = 0
\end{array}\right. \tag{2.17}
$$

First case :

If we assumed $\underline{H} = (0, 0, H_3)$ i.e.. $\vec{H} = (0, 0)$ we would obtain

$$
\left|\begin{array}{l}
h_{0t} - \Delta h_0 - \beta \begin{pmatrix} -\frac{\partial}{\partial x_1}\omega_t H_3 \\ \\ -\frac{\partial}{\partial x_2} w_t H_3 \\ \\ -\operatorname{div} v_t H_3 \end{pmatrix} = 0 \\ \\
h_{1t} - \Delta h_1 - \beta \begin{pmatrix} 0 \\ \\ 0 \\ \\ \Delta\omega_t H_3 \end{pmatrix} = 0
\end{array}\right. \tag{2.18}
$$

we would have interest only in the second equation of the system (2.18), because only w_t appear. And in the same equation, the first and second component, we can see the heat equation, while the third component together with (2.13) gives us a system similar to the thermoelastic plate, and this is not our interest.

Second case :

On the other hand we have, by (2.13) and the second equation in (2.17) that our interesting system is

$$\left| \begin{array}{l} \omega_{tt} - \gamma\Delta\omega_{tt} + d\Delta^2\omega - \alpha\,(\mathrm{rot}\,\mathrm{rot}\,h_1)\cdot \underline{H} + \lambda\triangledown h_{03}\cdot \underline{H} = 0 \\ \\ h_{1t} - \Delta h_1 + \beta\,\mathrm{rot}\,\mathrm{rot}\,(\underline{H}w_t) = 0 \end{array} \right.$$

We assume $h_{03} = 0$ to obtain

$$\left| \begin{array}{l} \omega_{tt} - \gamma\Delta\omega_{tt} + d\Delta^2\omega - \alpha\,(\mathrm{rot}\,\mathrm{rot}\,h_1)\cdot \underline{H} = 0 \\ \\ h_{1t} - \Delta h_1 + \beta\,\mathrm{rot}\,\mathrm{rot}\,(\underline{H}w_t) = 0 \end{array} \right. \tag{2.19}$$

But

$$\Delta h_1 = \triangledown\,\mathrm{div}\,h_1 - \mathrm{rot}\,\mathrm{rot}\,h_1$$

and

$$0 = \mathrm{div}\,b = \mathrm{div}\,\underbrace{[h_0 + x_3 h_1]}_{b} = \mathrm{div}\,h_0 + \mathrm{div}(x_3 h_1) = \mathrm{div}\,h_0 + h_{13} + x_3\,\mathrm{div}\begin{pmatrix} h_{11} \\ h_{12} \\ 0 \end{pmatrix}$$

so by multiplication with x_3 and integration from $-h$ to h in x_3

$$\mathrm{div}\underbrace{\begin{pmatrix} h_{11} \\ h_{12} \\ h_{13} \end{pmatrix}}_{h_1} = 0$$

because h_{13} is independent of x_3.
The condition $b \cdot \eta = 0$ becomes

$$(h_0 \cdot \eta + x_3 h_1 \cdot \eta) = 0$$

We multiply by x_3 and integrate from $-h$ to h in x_3 to obtain the new condition $h_1 \cdot \eta = 0$.
Now, we work with the second boundary condition , so $\mathrm{rot}\,b \times \eta = 0$ becomes

$$\mathrm{rot}\,h_0 \times \eta + \mathrm{rot}[x_3 h_1] \times \eta = 0 \tag{2.20}$$

and

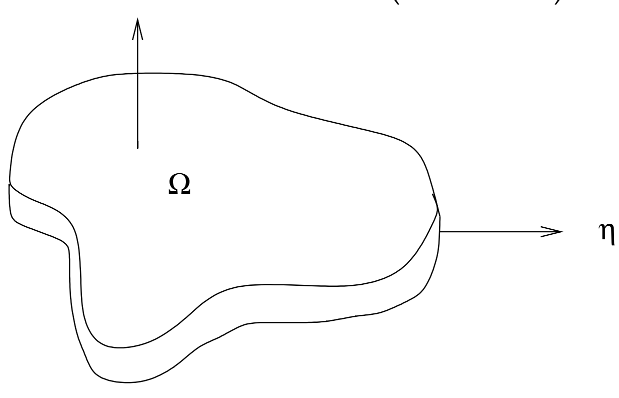
$$\text{rot}[x_3 h_1] = x_3(\partial_2 h_{13}, -\partial_1 h_{13}, \text{rot}\begin{pmatrix} h_{11} \\ h_{12} \end{pmatrix}) + (-h_{12}, h_{11}, 0)$$

$$= x_3 \,\text{rot}\begin{pmatrix} 0 \\ 0 \\ h_{13} \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ \text{rot}\begin{pmatrix} h_{11} \\ h_{12} \end{pmatrix} \end{pmatrix} + (-h_{12}, h_{11}, 0)$$

If we multiply (2.20) by x_3 and integrate once from $-h$ to h in x_3, we obtain

$$\text{rot}\begin{pmatrix} 0 \\ 0 \\ h_{13} \end{pmatrix} \times \eta + \begin{pmatrix} 0 \\ 0 \\ \text{rot}\begin{pmatrix} h_{11} \\ h_{12} \end{pmatrix} \end{pmatrix} \times \eta = 0$$

so, we must also assume that $h_{13} = 0$ in order to obtain the boundary condition, and we consider $\eta = \begin{pmatrix} \eta_1(x_1, x_2) \\ \eta_2(x_1, x_2) \\ 0 \end{pmatrix}$. This assumption brings $H_3 = 0$ too.



so, with this assumption, we can say

$$\begin{pmatrix} 0 \\ 0 \\ \text{rot}\begin{pmatrix} h_{11} \\ h_{12} \end{pmatrix} \end{pmatrix} \times \eta = 0$$

but

$$\begin{pmatrix} 0 \\ 0 \\ \text{rot}\begin{pmatrix} h_{11} \\ h_{12} \end{pmatrix} \end{pmatrix} \times \eta = 0 \Rightarrow -\text{rot}\, h_1 \eta_2 = 0 \quad \text{and} \quad \text{rot}\, h_1 \eta_1 = 0$$

and because we can not have $\eta_1 = 0 = \eta_2$, $\operatorname{rot} h_1$ must be equal 0. Therefore, we have found our desireded reduced magnetoelastic plate system

$$\left| \begin{array}{c} \omega_{tt} - \gamma\Delta\omega_{tt} + d\Delta^2\omega - \alpha\,(\operatorname{rot}\operatorname{rot} h)\cdot\vec{H} = 0 \\ \\ h_t + \operatorname{rot}\operatorname{rot} h + \beta\operatorname{rot}\operatorname{rot}\left(\vec{H}\cdot w_t\right) = 0 \end{array} \right.$$

where

$$h := h_1 = (h_{11}, h_{12}, 0), \quad \vec{H} = (H_1, H_2, 0), \quad \eta = (\eta_1, \eta_2, 0)$$

or better to say,

$$\left| \begin{array}{c} \omega_{tt} - \gamma\Delta\omega_{tt} + d\Delta^2\omega - \alpha\,(\operatorname{rot}\operatorname{rot} h)\cdot\vec{H} = 0 \quad \text{in} \;\; Q \\ \\ h_t + \operatorname{rot}\operatorname{rot} h + \beta\operatorname{rot}\operatorname{rot}\left(\vec{H}\cdot w_t\right) = 0 \quad \text{in} \;\; Q \\ \\ \operatorname{div} h = 0 \;\; \text{in} \;\; Q \\ \\ h\cdot\eta(t,x_1,x_2) = \eta\times\operatorname{rot} h(t,x_1,x_2) = 0 \;\; \text{on} \;\; \Sigma \\ \\ w = \frac{\partial w}{\partial\eta} = 0 \;\; \text{on} \;\; \Sigma \\ w(x,0) = w_0, \quad w_t(x,0) = w_1, \quad h(x,0) = h_0 \end{array} \right. \tag{2.21}$$

where $Q = \Omega\times(0,T)$ and $\Sigma = \partial\Omega\times(0,T)$.

2.2 The nonlinear system

For the nonlinear system we shall replace the constant magnetic field $\vec{H}$ by the magnetic field h. So the nonlinear system under consideration will be

$$
\left|
\begin{array}{l}
\omega_{tt} - \gamma \Delta \omega_{tt} + d\Delta^2 \omega - \alpha \, (\mathrm{rot}\,\mathrm{rot}\, h) \cdot h = 0 \quad \text{in} \quad Q \\
h_t + \mathrm{rot}\,\mathrm{rot}\, h + \beta \,\mathrm{rot}\,\mathrm{rot}\,(h w_t) = 0 \quad \text{in} \quad Q \\
\mathrm{div}\, h = 0 \quad \text{in} \quad Q \\
h \cdot \eta(t, x_1, x_2) = \eta \times \mathrm{rot}\, h(t, x_1, x_2) = 0 \quad \text{on} \quad \Sigma \\
w = \frac{\partial w}{\partial \eta} = 0 \quad \text{on} \quad \Sigma \\
w(x, 0) = w_0, \quad w_t(x, 0) = w_1, \quad h(x, 0) = h_0
\end{array}
\right. \tag{2.22}
$$

where $Q = \Omega \times (0, T)$ and $\Sigma = \partial\Omega \times (0, T)$. We will work with this system in the last section.

3 Existence and uniqueness of solutions for the linear system

In this section we are going to analyse the existence and uniqueness for the linear system in two cases $\gamma > 0$ and $\gamma = 0$. For the case $\gamma > 0$ we use first the semigroup theory and then Galerkin method. The case $\gamma = 0$ is also treated by semigroup theory.

3.1 Weak formulation of the problem

Let Ω be a bounded simply connected domain of $\mathbb{R}^2$ of class $\mathcal{C}^2$. Let us assume that $\{w, h\}$ is a classical solution of problem (2.21), say $w \in \mathcal{C}^4(\bar{Q})$, $h \in \mathcal{C}^2(\bar{Q})$. Then it is easily verified that the following equalities are satisfied:

$$\left| \begin{array}{l} \frac{d}{dt}\left(\omega_t, \varphi\right) + \gamma \frac{d}{dt}\left(\triangledown w_t, \triangledown\varphi\right) + d\left(\Delta w, \Delta\varphi\right) - \alpha\left(\operatorname{rot} h, \operatorname{rot}\left(\varphi\vec{H}\right)\right) = 0 \qquad \forall\varphi \text{ in } H_0^2(\Omega) \\ \\ \qquad\qquad \frac{\alpha}{\beta}\frac{d}{dt}\left(h, \chi\right) + \frac{\alpha}{\beta}\left(\operatorname{rot} h, \operatorname{rot}\chi\right) + \alpha\left(\operatorname{rot}\left(\vec{H} w_t\right), \operatorname{rot}\chi\right) = 0 \qquad \forall\chi \text{ in } V, \end{array} \right. \tag{3.1}$$

In order to prove existence and uniqueness of weak solutions of (2.21) we need some definitions.
We use the bilaplacian operator

$$A : H_0^2(\Omega) \longmapsto H^{-2}(\Omega),$$

with domain $H_0^2(\Omega) \cap H^4(\Omega)$ and which is generated by the scalar product $a(u, v) = (\Delta u, \Delta v)$ in $H_0^2(\Omega)$, $Au = \Delta^2 u$.
We define the space (see [19], [3])

$$Y = \{g \in [L^2(\Omega)]^2; \ \ \operatorname{div} g = 0 \ \text{ in } \ \Omega; \ \ g \cdot \eta = 0 \ \text{ on } \ \partial\Omega\}$$

and (see [13])

$$Y = \mathcal{D}_o^o = \mathcal{D}^o \cap \mathcal{D}_o,$$

where

$$\mathcal{D}^o = \left\{g \in [L^2(\Omega)]^2; \ \forall f \in H^1(\Omega); \quad (g, \triangledown f) = -\left(\operatorname{div} g, f\right)\right\}$$

and

$$\mathcal{D}_o = \left\{g \in [L^2(\Omega)]^2; \operatorname{div} g = 0\right\}$$

We can also define (see [20]) the space Y as

$$Y = \text{ the closure of } \mathcal{V} \text{ in } [L^2(\Omega)]^2$$

where

$$\mathcal{V} = \left\{ f \in [\mathcal{D}(\Omega)]^2; \ \ \operatorname{div} f = 0 \right\}$$

Let

$$\mathcal{W} = \left\{ \varphi; \ \varphi \in \left(\mathcal{C}^\infty(\bar{\Omega}) \right)^2, \ \ \operatorname{div} \varphi = 0, \ \eta \cdot \varphi = 0 \text{ on } \partial\Omega \right\}$$

We let $(s \geq 0)$ $V_s =$ closure of $\mathcal{W}$ in $(H^s(\Omega))^2$. V_s is a Hilbert space with scalar product $((\cdot,\cdot))_s$ induced from $(H^s(\Omega))^2$.
Now, we define the space

$$V = \{ f \in Y; \ \ \operatorname{rot} f \in L^2(\Omega) \}$$

and V can be defined also as (see [3])

$$V = \{ f \in [H^1(\Omega)]^2; \ \ \operatorname{div} f = 0 \ \text{ in } \ \Omega; \ \ f \cdot \eta = 0 \quad \text{on} \ \ \partial\Omega \}$$

We note that $V_1 = V$ and $V_0 = Y$.
The space Y is endowed with the norm in $[L^2(\Omega)]^2$; $\|f\|_Y = |f|_{[L^2(\Omega)]^2}$, and the space V with the norm $\|f\|_V = (\|f\|_Y^2 + |\operatorname{rot} f|^2)^{\frac{1}{2}}$.
With these norms we have $V \hookrightarrow Y$ with continuous, dense and compact immersion.
This, together with the scalar product in $V \times V$, $b(u,v) = (\operatorname{rot} u, \operatorname{rot} v)_{L^2}$ and from the assumption that Ω is simply connected (see [13] p. 157, [7] p. 360), we have that b is coercive; therefore it generates the operator B

$$Bg := -\Delta g = \operatorname{rot} \operatorname{rot} g,$$

with

$$\mathcal{D}(B) = \{ g \in V; \ \ \exists f \in Y \ \text{ such that } \ b(g,v) = (f,v)_Y, \quad \forall v \in V \}$$

or in other words

$$\mathcal{D}(B) \ = \ \{ g \in V; \ \ Bg \in Y \} = \left\{ g \in [H^2(\Omega)]^2 \cap Y; \ \ Bg \in Y \right\}$$

We note

$$[L^2(\Omega)]^2 = Y \oplus Y^{\perp} = Y \oplus \{g \in [L^2(\Omega)]^2; \quad g = \triangledown p, \quad p \in H^1(\Omega)\}.$$

We will prove that

Lemma 3.1 $\mathcal{D}(B) = \{g \in [H^2(\Omega)]^2 \bigcap Y; \quad \eta \times \mathrm{rot}\, g = 0 \quad on \quad \partial\Omega\}$

Proof: $(\supset)$ Let $g \in [H^2(\Omega)]^2 \bigcap Y$ so that $\eta \times \mathrm{rot}\, g$ on $\partial\Omega$. We will prove that $Bg \in Y$.
For all $f \in H^1(\Omega)$ we have:

$$(\mathrm{rot}\,\mathrm{rot}\, g, \triangledown f) = -(\mathrm{div}\,\mathrm{rot}\,\mathrm{rot}\, g, f) + (\mathrm{rot}\,\mathrm{rot}\, g \cdot \eta, f)_{\partial\Omega}$$

and

$$(\mathrm{rot}\,\mathrm{rot}\, g, \triangledown f) = (\mathrm{rot}\, g, \mathrm{rot}\, \triangledown f) + (\eta \times \mathrm{rot}\, g, \triangledown f)_{\partial\Omega} = 0.$$

Therefore $(\mathrm{rot}\,\mathrm{rot}\, g \cdot \eta, f)_{\partial\Omega} = 0 \ \ \forall f \in H^1(\Omega)$.

$(\subset)$ Let $g \in \mathcal{D}(B)$. For all $f \in [H^1(\Omega)]^2$ we have $f = f_1 + f_2$, where $f_2 = \triangledown p, \quad p \in H^1(\Omega), \quad$ and so $\quad \mathrm{rot}\, f_2 = 0$.
Let $f \in [H^1(\Omega)]^2$ then $\mathrm{rot}\, f \in L^2(\Omega)$, we have therefore $\mathrm{rot}\, f_1 \in L^2(\Omega)$ and so $f_1 \in V$. Since $g \in \mathcal{D}(B)$ we have

$$(\mathrm{rot}\, g, \mathrm{rot}\, f_1) = (\mathrm{rot}\,\mathrm{rot}\, g, f_1)\,, \quad \forall f_1 \in V$$

but we have also

$$\begin{aligned}(\mathrm{rot}\, g, \mathrm{rot}\, f) &= (\mathrm{rot}\,\mathrm{rot}\, g, f) + (\eta \times \mathrm{rot}\, g, f)_{\partial\Omega} \\ &= (\mathrm{rot}\,\mathrm{rot}\, g, f_1) + (\mathrm{rot}\,\mathrm{rot}\, g, f_2) + (\eta \times \mathrm{rot}\, g, f)_{\partial\Omega}\end{aligned}$$

and therefore

$$0 = (\mathrm{rot}\,\mathrm{rot}\, g, f_2) + (\eta \times \mathrm{rot}\, g, f)_{\partial\Omega}$$

but, since

$$(\mathrm{rot}\,\mathrm{rot}\, g, f_2) = -\,(\mathrm{div}\,\mathrm{rot}\,\mathrm{rot}\, g, p) + (\mathrm{rot}\,\mathrm{rot}\, g \cdot \eta, p)_{\partial\Omega} = 0$$

because $Bg \in Y$, we get

$$(\eta \times \operatorname{rot} g, f)_{\partial\Omega} = 0, \quad \forall f \in [H^1(\Omega)]^2 \ \ \square$$

We now define the operator

$$Cg := \operatorname{rot}\operatorname{rot} g$$

with

$$\mathcal{D}(C) = \{g \in U,\ Cg \in Y\}$$

where

$$U := \{g \in [L^2(\Omega)]^2; \quad \operatorname{rot} g \in L^2(\Omega),\ \operatorname{div} g \in L^2(\Omega), \quad g \cdot \eta = 0 \ \text{ on } \ \partial\Omega\}$$

In other words

$$\mathcal{D}(C) = \left\{g \in [H^1(\Omega)]^2; \ \ g \cdot \eta = 0,\ Cg \in Y\right\}$$

We observe that $\left[H_0^2(\Omega)\right]^2 \not\subset \mathcal{D}(B)$, because $\operatorname{div}\left[H_0^2(\Omega)\right]^2 \neq 0$ in Ω generally. But we can prove that $\left[H_0^2(\Omega)\right]^2 \hookrightarrow \mathcal{D}(C)$.

Remark: This is a problem if we want use Theorem 2.2 in [12] for set #1, in order to prove that this semigroup should be analytic because since $\left[H_0^2(\Omega)\right]^2 \not\subset \mathcal{D}(B)$, condition H.2 not holds.

Lemma 3.2 $[H_0^2(\Omega)]^2 \subset \mathcal{D}(C)$

Proof: Let $w \in H_0^2(\Omega)$, $\implies$ $w\vec{H} \in [H_0^2(\Omega)]^2 \implies w\vec{H} \in U$.

Let $v = \operatorname{rot}(w\vec{H})$, $\implies$ $v \in H_0^1(\Omega) \Longrightarrow \operatorname{rot} v \in Y$

In fact, from

$$\forall f \in H^1(\Omega) \quad (\operatorname{rot} v, \triangledown f) = -\left(\operatorname{div}\operatorname{rot} v, f\right) + (\operatorname{rot} v \cdot \eta, f)_{\partial\Omega} = (\operatorname{rot} v \cdot \eta, f)_{\partial\Omega}$$

and

$$(\operatorname{rot} v, \triangledown f) = (v, \operatorname{rot} \triangledown f) - (\eta \times v, \triangledown f)_{\partial\Omega} = 0$$

we have that $\operatorname{rot} v \cdot \eta = 0$ on $\partial\Omega$ because $\operatorname{rot} v \cdot \eta$ is well defined in $H^{-\frac{1}{2}}(\partial\Omega)$, i.e. $w\vec{H} \in \mathcal{D}(C)$. $\square$

We use all these definitions in order to write the system (3.1) in operator form

$$\left| \begin{array}{l} \omega_{tt} - \gamma A^{\frac{1}{2}} \omega_{tt} + dA\omega - \alpha\,(Bh) \cdot \vec{H} = 0 \\ \\ \quad \frac{\alpha}{\beta} h_t + \frac{\alpha}{\beta} Bh + \alpha C \left(\vec{H} \cdot w_t \right) = 0 \end{array} \right. \tag{3.2}$$

where $h := (h_{11}, h_{12}, 0)$, $\vec{H} = (H_1, H_2, 0)$, $\eta = (\eta_1, \eta_2, 0)$.

Now, we distinguish two cases. In the first case we will have $\gamma > 0$, and we find that there exists a unique solution of problem (3.2) in the space of energy $\mathcal{H} = H^2_{0d} \times H^1_0(\Omega) \times Y_{\frac{\alpha}{\beta}}$; for second case we have $\gamma = 0$ and we find that the problem (3.2) has a unique solution in the space of energy $X = H^2_{0d} \times L^2(\Omega) \times Y_{\frac{\alpha}{\beta}}$.

Lemma 3.3

$$\int_\Omega [\mathrm{rot}\,\mathrm{rot}\, f] \cdot \vec{H} v dx = \int_\Omega f \cdot \mathrm{rot}\,\mathrm{rot} \left(\vec{H} v \right) dx, \quad \forall f \in \left[H^2(\Omega) \right]^2 \cap Y, \;\; \forall v \in H^2_0(\Omega)$$

Proof: Partial integration and $v \in H^2_0(\Omega)$.

3.2 The case $\gamma > 0$, semigroup theory

When $\gamma > 0$ in the system of equations (2.21) the term w_{tt} is not alone and we have $w_{tt} - \gamma\Delta w_{tt}$ in the system. In order to overcome this inconvenient situation we will follow [16] which use a special frame in semigroup theory.
We set the space X_1 on page 7 to be H^2_{0d} that is $H^2_0(\Omega)$ endowed with the scalar product

$$a_1(u,v) = (u,v)_{H^2_{0d}} := d(\Delta u, \Delta v)_{L^2}, \quad \forall u,v \in H^2_0(\Omega)$$

We set also Y_1 to be $H^1_0(\Omega)$ with

$$c_1(u,v) = \gamma(\nabla u, \nabla v)_{L^2} + (u,v)_{L^2}, \quad \forall u,v \in H^1_0(\Omega)$$

We define $X_2 = V$, with

$$b(f,g) = (\operatorname{rot} f, \operatorname{rot} g)_{L^2}, \quad \forall f,g \in V$$

and $Y_2 = Y_{\frac{\alpha}{\beta}}$ with

$$c_2(h,g) = (h,g)_{Y_{\frac{\alpha}{\beta}}} := \frac{\alpha}{\beta}(h,g)_Y$$

And so there exists the operator

$$B := \frac{\beta}{\alpha} \operatorname{rot} \operatorname{rot}$$

so that

$$b(f,g) = c_2(B^{\frac{1}{2}} f, B^{\frac{1}{2}} g) \quad \forall f,g \in X_2.$$

On the other hand, there also exists the operator

$$A_1 := (I - \gamma\Delta)^{-1}\Delta^2 : H^4(\Omega) \cap H^2_0(\Omega) \longrightarrow H^2(\Omega) \cap H^1_0(\Omega)$$

so that

$$a_1(u,v) = c_1(A_1^{\frac{1}{2}} u, A_1^{\frac{1}{2}} v), \qquad \forall u,v \in X_1$$

where

$$A_1^{\frac{1}{2}} : H^2_0(\Omega) \to H^1_0(\Omega).$$

We define the operator

$$\left[A_1^{\frac{1}{2}}\right]^2 : [H^2_0(\Omega)]^2 \to \left[H^1_0(\Omega)\right]^2$$

where

$$\left[A_1^{\frac{1}{2}}\right]^2 := \begin{bmatrix} A_1^{\frac{1}{2}} & 0 \\ 0 & A_1^{\frac{1}{2}} \end{bmatrix}$$

We also have $A_1^{\frac{-1}{2}} : H_0^1(\Omega) \to H_0^2(\Omega)$ and $C : [H_0^2(\Omega)]^2 \to Y_{\frac{\alpha}{\beta}}$ hence we have

$$C\left[A_1^{\frac{-1}{2}}\right]^2 : \left[H_0^1(\Omega)\right]^2 \to Y_{\frac{\alpha}{\beta}}$$

With $C = \frac{\beta}{\alpha}\,\mathrm{rot}\,\mathrm{rot}$ formally.
The variational system is

$$\left|\begin{array}{l} c_1(\omega_{tt}, \hat{w}) - \alpha c_2(h, C(\hat{w}\vec{H})) + a_1(w, \hat{w}) = 0 \ \ \forall \hat{w} \in H_0^2(\Omega) \\ \\ c_2(h_t, \hat{h}) + \frac{\alpha}{\beta} c_2(Bh, \hat{h}) + \alpha c_2(C(\vec{H}\cdot w_t), \hat{h}) = 0 \ \ \forall \hat{h} \in V \end{array}\right.$$

We suppose that $\{w, h\}$ is a regular solution of our problem (2.21), i.e. h is at least $\left[\mathcal{C}^2(\bar{Q})\right]^2$. The pair $\{w, h\}$ is then a solution of the variational equation

$$\left|\begin{array}{l} c_1(\omega_{tt}, \hat{w}) - \alpha c_2(h, C(\hat{w}\vec{H})) + c_1(A_1^{\frac{1}{2}} w, A_1^{\frac{1}{2}} \hat{w}) = 0 \ \ \forall \hat{w} \in H_0^2(\Omega) \\ \\ \qquad c_2(h_t + \frac{\alpha}{\beta} Bh + \alpha C(\vec{H}\cdot w_t), \hat{h}) = 0 \ \ \forall \hat{h} \in V \end{array}\right. \tag{3.3}$$

We would like to write the first equation in (3.3) all in the form $c_1(\ldots, \hat{w})$. In order to do that, let $\hat{w} \in H_0^2(\Omega)$ then

$$\tilde{w} := \left[A_1^{\frac{1}{2}}\right]^2 (\hat{w}\vec{H}) = [A_1^{\frac{1}{2}}\hat{w}]\vec{H} \in \left[H_0^1(\Omega)\right]^2$$

therefore we can write

$$\begin{aligned} c_2(h, C(\hat{w}\vec{H})) &= \tfrac{\alpha}{\beta}(h, C(\hat{w}\vec{H}))_Y \\ &= \tfrac{\alpha}{\beta}\left(h, C\left[A_1^{\frac{-1}{2}}\right]^2 \tilde{w}\right)_Y \end{aligned}$$

and by the definition of adjoint operator

$$= \sum_{i=1}^{2} c_1\left(\left[\left[C\left[A_1^{\frac{-1}{2}}\right]^2\right]^* h\right]_i, [A_1^{\frac{1}{2}}\hat{w}]H_i\right)$$

$$= c_1\left(\left[\left[C\left[A_1^{\frac{-1}{2}}\right]^2\right]^* h\right]\cdot\vec{H}, A_1^{\frac{1}{2}}\hat{w}\right) \tag{3.4}$$

Now well, we can add $c_1(A_1^{\frac{1}{2}}w, A_1^{\frac{1}{2}}\hat{w})$ to (3.4), so

$$\alpha c_2(h, C(\hat{w}\vec{H})) - c_1(A_1^{\frac{1}{2}}w, A_1^{\frac{1}{2}}\hat{w}) = c_1\left(-A_1^{\frac{1}{2}}w + \alpha\left[C\left[A_1^{\frac{-1}{2}}\right]^2\right]^* h\cdot\vec{H}, A_1^{\frac{1}{2}}\hat{w}\right)$$

but $A_1^{\frac{1}{2}}$ is a self-adjoint operator and in the case where

$$-A_1^{\frac{1}{2}}w + \alpha\left[C\left[A_1^{\frac{-1}{2}}\right]^2\right]^* h\cdot\vec{H}$$

belongs to $\mathcal{D}(A_1^{\frac{1}{2}})$, we would have

$$c_1\left(-A_1^{\frac{1}{2}}w + \alpha\left[C\left[A_1^{\frac{-1}{2}}\right]^2\right]^* h\cdot\vec{H}, A_1^{\frac{1}{2}}\hat{w}\right)$$
$$= c_1(A_1^{\frac{1}{2}}\left[-A_1^{\frac{1}{2}}w + \alpha\left[C\left[A_1^{\frac{-1}{2}}\right]^2\right]^* h\cdot\vec{H}\right], \hat{w})$$

where from (3.3) and density of $H_0^2(\Omega)$ in $H_0^1(\Omega)$ we obtain the system

$$\begin{cases} v_t = w_{tt} = A_1^{\frac{1}{2}}\left[-A_1^{\frac{1}{2}}w + \alpha\left[C\left[A_1^{\frac{-1}{2}}\right]^2\right]^* h\cdot\vec{H}\right] \text{ in } H_0^1(\Omega),\ \forall t>0 \\ h_t = -\frac{\alpha}{\beta}Bh - \alpha C\left[\dot{w}\vec{H}\right] \text{ in } Y_{\frac{\alpha}{\beta}},\ \forall t>0 \end{cases}$$

Let $(w, v, h) \in \mathcal{H} = H_{0d}^2 \times H_0^1(\Omega) \times Y_{\frac{\alpha}{\beta}}$, with $v = \omega_t$.

Then, we can write the system as

$$\begin{pmatrix} w \\ v \\ h \end{pmatrix}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & A_1^{\frac{1}{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ -A_1^{\frac{1}{2}}\bullet & 0 & \alpha\left[C\left[A_1^{\frac{-1}{2}}\right]^2\right]^*\bullet\cdot\vec{H} \\ 0 & -\alpha C\left[\cdot\vec{H}\right] & -\frac{\alpha}{\beta}B \end{pmatrix}\begin{pmatrix} w \\ v \\ h \end{pmatrix}$$

Let $\mathcal{A} := \begin{pmatrix} 0 & 1 & 0 \\ A_1^{\frac{1}{2}}\left[-A_1^{\frac{1}{2}}\bullet + \alpha\left[C\left[A_1^{\frac{-1}{2}}\right]^2\right]^*\bullet\cdot\vec{H}\right] & & \\ 0 & -\alpha C\left[\cdot\vec{H}\right] & -\frac{\alpha}{\beta}B \end{pmatrix}$, then

$$\mathcal{A}z = \begin{pmatrix} v \\ \\ A_1^{\frac{1}{2}}\left[-A_1^{\frac{1}{2}}w + \alpha\left[C\left[A_1^{\frac{-1}{2}}\right]^2\right]^* h \cdot \vec{H}\right] \\ \\ -\alpha C\left[v\vec{H}\right] - \frac{\alpha}{\beta}Bh \end{pmatrix}_X$$

We define $\mathcal{D}\left(\mathcal{A}\right)$ equal to be

$$\left\{\begin{pmatrix} w \\ v \\ h \end{pmatrix} \in \mathcal{H}/\ v \in H_0^2(\Omega),\quad h \in \mathcal{D}(B),\quad -A_1^{\frac{1}{2}}w + \alpha\left[C\left[A_1^{\frac{-1}{2}}\right]^2\right]^* h \cdot \vec{H} \in H_0^2(\Omega)\right\}$$

Lemma 3.4 *i)* $\mathcal{D}\left(\mathcal{A}\right)$ *is dense in* $\mathcal{H}$ *and* $\mathcal{A}$ *is dissipative.*
ii) $0 \in \rho(\mathcal{A})$ *with* $\rho(\mathcal{A})$ *being the resolvent set of* $\mathcal{A}$.

Proof: We have that $\mathcal{A}$ is dissipative

$$\begin{aligned} Re\left(\mathcal{A}z, z\right)_{\mathcal{H}} &= Re\left[a_1\left(v, w\right) - a_1\left(w, v\right) + \alpha c_1\left(\left[C\left[A_1^{\frac{-1}{2}}\right]^2\right]^* h \cdot \vec{H}, A_1^{\frac{1}{2}}v\right)\right. \\ &\qquad \left. -\alpha c_2\left(C\left[v\vec{H}\right], h\right) - \tfrac{\alpha}{\beta}c_2\left(Bh, h\right)\right] \\ &= -\tfrac{\alpha}{\beta}b\left(h, h\right) + \alpha Re\left[c_2\left(h, C\left[v\vec{H}\right]\right) - c_2\left(C\left[v\vec{H}\right], h\right)\right] \end{aligned}$$

i.e.

$$Re\left(\mathcal{A}z, z\right)_{\mathcal{H}} = -\tfrac{\alpha}{\beta}|\operatorname{rot} h|^2 \leq 0.$$

Now, we prove that $0 \in \rho(\mathcal{A})$.

For any $F = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \in \mathcal{H}$, the equation $\mathcal{A}z = F$ has a unique solution $z = \begin{pmatrix} w \\ v \\ h \end{pmatrix}$:

$$\begin{cases} v = f_1 \\ \\ A_1^{\frac{1}{2}}\left[-A_1^{\frac{1}{2}}w + \alpha\left[C\left[A_1^{\frac{-1}{2}}\right]^2\right]^* h \cdot \vec{H}\right] = f_2 \\ \\ -\alpha C\left[v\vec{H}\right] - \frac{\alpha}{\beta}Bh = f_3 \end{cases}$$

$\Rightarrow \quad v := f_1,$

$$Bh = -f_3 + C\left[v\vec{H}\right] \in Y \quad \Rightarrow h \in \mathcal{D}\left(\mathcal{B}\right), \quad h := B^{-1}\left[-f_3 + C\left[v\vec{H}\right]\right]$$

and

$$-A_1^{\frac{1}{2}}w + \alpha\left[C\left[A_1^{\frac{-1}{2}}\right]^2\right]^* h \cdot \vec{H} = A_1^{\frac{-1}{2}} f_2$$

$$-A_1^{\frac{1}{2}}w = \left[A_1^{\frac{-1}{2}} f_2 - \alpha\left[C\left[A_1^{\frac{-1}{2}}\right]^2\right]^* h \cdot \vec{H}\right]$$

$$\Rightarrow \quad \omega := -A_1^{\frac{-1}{2}}\left[A_1^{\frac{-1}{2}} f_2 - \alpha\left[C\left[A_1^{\frac{-1}{2}}\right]^2\right]^* h \cdot \vec{H}\right] \in H_0^2(\Omega)$$

$$\Rightarrow \quad \|z\|_{\mathcal{H}} \leq K\|(f_1, f_2, f_3)\|_{\mathcal{H}}$$

with some $K > 0$ independent of F. Therefore, $\mathcal{A}$ is closed and $0 \in \rho(\mathcal{A})\Box$

As a corollary (see Theorem 1.2.4 in [16]) we obtain that $\mathcal{A}$ generates a C_0-semigroup of contracctions $\left\{e^{\mathcal{A}t}\right\}_{t\geq 0}$.

Theorem 3.1 *Let* $(\ w(t), w_t(t), h(t)\ ) = e^{\mathcal{A}t}z_0$. *If* $z_0 = (w_0, w_1, h_0) \in \mathcal{H}$, *then problem (2.21) is globally well-posed in energy space* $\mathcal{H}$ *and the (unique) weak solution* $z = (w, w_t, h)$ *of problem* $\frac{dz}{dt} = \mathcal{A}z$ *belongs to* $\mathcal{C}([0,\infty);\mathcal{H})$.
Moreover, if $z_0 \in \mathcal{D}\left(\mathcal{A}\right)$, *then the solution* (w, w_t, h) *belongs to*

$$\mathcal{C}([0,\infty);\mathcal{D}\left(\mathcal{A}\right))\bigcap\mathcal{C}^1([0,\infty);\mathcal{H})$$

3.3 The case $\gamma > 0$, Galerkin approximation

In the following section we are going to treat the case $\gamma > 0$ using the Galerkin method. As a basis in the space $H_0^2(\Omega)$ we choose the eigenfunctions $\varphi_i \in H_0^2(\Omega)$ of the spectral problem for the operator $A := \Delta^2$ relative to the bilinear form $a(u,v) = (\Delta u, \Delta v)$ as in application II on page 9.

For the space V we choose the eigenfunctions $\psi_i \in V$ of the spectral problem for the operator $B := \operatorname{rot}\operatorname{rot}$ relative to the bilinear form $b(f,g) = (\operatorname{rot} f, \operatorname{rot} g)$ as in application I in page 9.

Let W_n and V_n be the generated spaces by the n first eigenvectors $\varphi_1, \varphi_2, ..., \varphi_n$ and $\psi_1, \psi_2, ..., \psi_n$ respectively. The approximate problem is to determine $w_n(t) \in W_n$ and $h_n(t) \in V_n$ with

$$w_n(t) = \sum_{j=1}^{n} f_{jn}(t)\varphi_j, \quad h_n(t) = \sum_{j=1}^{n} g_{jn}(t)\psi_j$$

so that they are solutions of the distributional system

$$\begin{cases} \frac{d}{dt}(\dot{w}_n, v) + \gamma \frac{d}{dt}(\nabla \dot{w}_n, \nabla v) + d(\Delta w_n, \Delta v) - \alpha\left(\operatorname{rot} h_n, \left[v\vec{H}\right]\right) = 0 \\ \frac{d}{dt}(h_n, g) + (\operatorname{rot} h_n, \operatorname{rot} g) + \beta\left(\operatorname{rot}\left[\dot{w}_n\vec{H}\right], \operatorname{rot} g\right) = 0 \\ \quad \text{in} \quad \mathcal{D}'(0,T), \quad \forall v \in W_n, \quad \forall g \in V_n \\ w_n(0) = w_0^n = \sum_{j=1}^{n} a_j\varphi_j \longrightarrow w_0 \ \text{ in } H_0^2(\Omega) \\ \dot{w}_n(0) = w_1^n = \sum_{j=1}^{n} b_j\varphi_j \longrightarrow w_1 \ \text{ in } H_0^1(\Omega) \\ h_n(0) = h_0^n = \sum_{j=1}^{n} c_j\psi_j \longrightarrow h_0 \ \text{ in } Y \end{cases} \tag{3.5}$$

If we take $v = \varphi_i$ and $g = \psi_i$ we have

$$\begin{cases} \frac{d}{dt}(\dot{w}_n, \varphi_i) + \gamma \frac{d}{dt}(\nabla \dot{w}_n, \nabla \varphi_i) + d(\Delta w_n, \Delta \varphi_i) - \alpha\left(\operatorname{rot} h_n, \operatorname{rot}\left[\varphi_i\vec{H}\right]\right) = 0 \\ \\ \frac{d}{dt}(h_n, \psi_i) + (\operatorname{rot} h_n, \operatorname{rot} \psi_i) + \beta\left(\operatorname{rot}\left[\dot{w}_n\vec{H}\right], \operatorname{rot} \psi_i\right) = 0 \\ \\ \quad \text{in} \quad \mathcal{D}'(0,T), \quad 1 \le i \le n \end{cases} \tag{3.6}$$

first equation in (3.6) times $\dot{f}_{in}$, second equation times $\frac{\alpha}{\beta}g_{in}$ and adding for $1 \le i \le n$ we obtain

$$\frac{1}{2}\frac{d}{dt}\left\{|\dot{w}_n|^2 + \gamma|\nabla \dot{w}_n|^2 + d|\Delta w_n|^2 + \frac{\alpha}{\beta}|h_n|^2\right\} + \frac{\alpha}{\beta}|\operatorname{rot} h_n|^2 = 0$$

i.e.

$$\frac{d}{dt}E_m(t) + \frac{\alpha}{\beta}|\operatorname{rot} h_n|^2 = 0$$

and therefore

$$E_m(t) + \frac{\alpha}{\beta}\int_0^t |\operatorname{rot} h_n|^2 ds = E_m(0) \leq E(0),$$

for t independent of m, where

$$E(t) = \frac{1}{2}\left\{|\dot{w}|^2 + \gamma|\nabla\dot{w}|^2 + d|\Delta w|^2 + \frac{\alpha}{\beta}|h|^2\right\}$$

so, we have obtained

(w_n) bounded in $\mathcal{C}(0,T;H_0^2(\Omega))$;
$(\dot{w}_n)$ bounded in $\mathcal{C}(0,T;H_0^1(\Omega))$;
(h_n) bounded in $\mathcal{C}(0,T;Y)$;
(h_n) bounded in $L^2(0,T;V)$.

And from

$$\begin{cases} (I-\gamma\Delta)\ddot{w}_n = -\Delta^2 w_n + \alpha\left[\operatorname{rot}\operatorname{rot} h_n\right]\cdot\vec{H} \\ \\ \dot{h}_n = -\operatorname{rot}\operatorname{rot} h_n - \beta\operatorname{rot}\operatorname{rot}\left[\dot{w}_n\vec{H}\right] \end{cases}$$

we have

$$\|\dot{h}_n\|_{V'} \leq \|\operatorname{rot}\operatorname{rot} h_n\|_{V'} + \beta\|\operatorname{rot}\operatorname{rot}\left[\dot{w}_n\vec{H}\right]\|_{V'} \leq c\|h_n\|_V + k\|\dot{w}_n\|_1$$

$$|\ddot{w}_n| \leq c\|(I-\gamma\Delta)\ddot{w}_n\|_{-2} \leq d\|w_n\|_2 + \alpha|h_n|_{[L^2(\Omega)]^2} \leq kE(0).$$

and hence

$(\ddot{w}_n)$ is bounded in $L^\infty(0,T;L^2(\Omega))$;
$(\dot{h}_n)$ is bounded in $L^2(0,T;V')$.

Which give us, thanks to the theorem 1.1, that $(w_n),(\dot{w}_n)$ converge strongly in $L^2(0,T;L^2(\Omega))$ and (h_n) converges strongly in $L^2(0,T;Y)$
Now without problem we can pass to the limit in (3.5). It follows that there exists a solution $\{w,h\}$ of the linear problem (2.21).

Our next step is to prove that the sequences $(w_n), (\dot{w}_n), (h_n)$ are Cauchy sequences in $\mathcal{C}(0,T;H_0^2(\Omega))$, $\mathcal{C}(0,T;H_0^1(\Omega))$ and $\mathcal{C}(0,T;Y)$ respectively. To do this let us take $(w_n), (w_m), (h_n), (h_m)$ for $m > n$
with

$$w_m = \sum_{i=1}^{m} f_{im}\varphi_i \quad h_m = \sum_{i=1}^{m} g_{im}\psi_i$$

and

$$w_n = \sum_{i=1}^{n} f_{in}\varphi_i = \sum_{i=1}^{m} \bar{f}_{im}\varphi_i, \quad h_n = \sum_{i=1}^{n} g_{in}\psi_i = \sum_{i=1}^{m} \bar{g}_{im}$$

where

$$\bar{f}_{im} = \begin{cases} f_{in}, & i \le n \\ 0, & n < i \le m \end{cases} \qquad \text{and} \qquad \bar{g}_{im} = \begin{cases} g_{in}, & i \le n \\ 0, & n < i \le m \end{cases}$$

and we still have $w_{0n} \longrightarrow w_0$ and $w_{0m} \longrightarrow w_0$ in $H_0^2(\Omega)$, so as $h_{0n} \longrightarrow h_0$ and $h_{0m} \longrightarrow h_0$ in Y.

Let us denote $W_n = w_m - w_n$ and $H_n = h_m - h_n$. Then from de linearity of the problem, we can use the same reasoning as above to get

$$E\begin{pmatrix} W_n \\ H_n \end{pmatrix} \le E\begin{pmatrix} w_{0m} - w_{0n} \\ h_{0m} - h_{0n} \end{pmatrix} \longrightarrow 0, \quad n, m \to \infty$$

In fact, from the system

$$\begin{cases} (\ddot{w}_m - \ddot{w}_n, \varphi_i) + \gamma\left(\triangledown\ddot{w}_m - \triangledown\ddot{w}_n, \triangledown\varphi_i\right) + d\left(\Delta w_m - \Delta w_n, \Delta\varphi_i\right) + \\ \qquad -\alpha\left(\operatorname{rot}\left[h_m - h_n\right], \operatorname{rot}\left[\varphi_i\vec{H}\right]\right) = 0 \\ \frac{\alpha}{\beta}\left(\dot{h}_m - \dot{h}_n, \psi_i\right) + \frac{\alpha}{\beta}\left(\operatorname{rot}\left[h_m - h_n\right], \operatorname{rot}\psi_i\right) + \\ \qquad +\alpha\left(\operatorname{rot}\left[\dot{w}_m - \dot{w}_n\vec{H}\right], \operatorname{rot}\psi_i\right) = 0 \end{cases} \tag{3.7}$$

in which we make first equation times $(f_{im} - \bar{f}_{im})'$, second equation times $g_{im} - \bar{g}_{im}$ and addition in $1 \le i \le m$, we obtain

$$\frac{1}{2}\frac{d}{dt}\left[|\dot{W}_n|^2 + \gamma|\triangledown\dot{W}_n|^2 + |\Delta W_n|^2 + \frac{\alpha}{\beta}|H_n|^2\right] + \frac{\alpha}{\beta}|\operatorname{rot} H_n|^2$$

so that

$$E_n\begin{pmatrix} W_n \\ H_n \end{pmatrix} + \int_0^t \frac{\alpha}{\beta}|\operatorname{rot} H_n|^2 ds = E_n(0) \longrightarrow 0; \quad n, m \to \infty$$

Since

$$W_{0n} = w_{0m} - w_{0n}, \quad W_{1n} = w_{1m} - w_{1n}, \quad H_{0n} = h_{0m} - h_{0n}$$

converge strongly to zero, then $(w_n), (\dot{w}_n), (h_n)$ are Cauchy sequences. So, there are functions

$$w \in \mathcal{C}^1(0,T; H_0^1(\Omega)) \bigcap \mathcal{C}(0,T; H_0^2(\Omega)), \quad h \in \mathcal{C}(0,T; Y),$$

for which we have

$$w_n \longrightarrow w \text{ strongly in } \mathcal{C}^1(0,T; H_0^1(\Omega)) \bigcap \mathcal{C}(0,T; H_0^2(\Omega)),$$

$$h_n \longrightarrow h \text{ strongly in } \mathcal{C}(0,T; Y).$$

Uniqueness:

Let $u_1 = (w_1, \dot{w}_1, h_1)$ and $u_2 = (w_2, \dot{w}_2, h_2)$ be solutions of our system (2.21). Let $W = w_1 - w_2, \quad H = h_1 - h_2$. Then W, H are solutions of

$$\left| \begin{array}{l} \ddot{W} - \gamma\Delta\ddot{W} + d\Delta^2 W - \alpha\left[\operatorname{rot}\operatorname{rot} H\right] \cdot \vec{H} = 0 \\ \frac{\alpha}{\beta}\dot{H} + \frac{\alpha}{\beta}\operatorname{rot}\operatorname{rot} H + \alpha \operatorname{rot}\operatorname{rot}\left[\dot{W}\vec{H}\right] = 0 \\ \operatorname{div} H = 0 \quad \text{in } \Omega \\ H \cdot \eta = 0 \quad \text{on } \partial\Omega \\ \eta \times \operatorname{rot} H = 0 \quad \text{on } \partial\Omega \\ W = \frac{\partial W}{\partial \eta} = 0 \quad \text{on } \partial\Omega \times \mathbb{R}^+ \\ \text{and } \; W(x,0) = W_t(x,0) = 0, \quad h(x,0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{in } \Omega \end{array} \right. \tag{3.8}$$

Multiplying first equation (3.8) by $\dot{W}$, second equation by H, integrate in Ω and adding, we obtain

$$\frac{1}{2}\frac{d}{dt}\left[|\dot{W}|^2 + \gamma|\triangledown\dot{W}|^2 + d|\Delta W|^2 + |H|^2\right] + |\operatorname{rot} H|^2 = 0$$

Now, integration on $[0,T]$ we have:

$$|\dot{W}|^2 + \gamma|\triangledown|^2 + d|\Delta W|^2 + |H|^2 + \int_0^T |\operatorname{rot} H|^2 = 0$$

Therefore $W = 0$ and $H = 0$.

3.4 The case $\gamma = 0$, semigroup theory

We denote by H^2_{0d}, the space $H^2_0(\Omega)$ endowed with the inner product

$$((u,v))_{H^2_0(\Omega)} = d(\Delta u, \Delta v)_{L^2(\Omega)}, \quad \forall u, v \in H^2_0(\Omega)$$

and by $Y_{\frac{\alpha}{\beta}}$, the space Y endowed with the inner product

$$(f,g)_{Y_{\frac{\alpha}{\beta}}} = \frac{\alpha}{\beta}(f,g)_{[L^2(\Omega)]^2}, \quad \forall f, g \in Y.$$

Let $X = H^2_{0d} \times L^2(\Omega) \times Y_{\frac{\alpha}{\beta}}$. From (2.21) for $\gamma = 0$ we have the system

$$\dot{F} = \begin{pmatrix} 0 & 1 & 0 \\ -d\Delta^2 & 0 & \alpha(B\bullet)\cdot\vec{H} \\ 0 & -\beta\,\mathrm{rot}\,\mathrm{rot}\left[\cdot\vec{H}\right] & -\,\mathrm{rot}\,\mathrm{rot} \end{pmatrix} \begin{pmatrix} w \\ v \\ h \end{pmatrix} =: SF$$

with $\quad \mathcal{D}(S) = H^2_0(\Omega) \bigcap H^4(\Omega) \times H^2_0(\Omega) \times \mathcal{D}(B)$

Lemma 3.5 *S is dissipative and range $(I - S) = X$.*

Proof : First of all we check the dissipativity of S :

$$(SF,F)_X = \left(\begin{pmatrix} v \\ -d\Delta^2 w + \alpha(\mathrm{rot}\,\mathrm{rot}\,h)\cdot\vec{H} \\ -\beta\,\mathrm{rot}\,\mathrm{rot}(v\vec{H}) - \mathrm{rot}\,\mathrm{rot}\,h \end{pmatrix}, \begin{pmatrix} w \\ v \\ h \end{pmatrix} \right)_X$$

$$= ((v,w))_{H^2_0(\Omega)} - d\,(\Delta^2 w, v)_{L^2} + \alpha\left((\mathrm{rot}\,\mathrm{rot}\,h)\cdot\vec{H}, v\right)_{L^2} + \\ -\beta\left(\mathrm{rot}\,\mathrm{rot}(v\vec{H}), h\right)_{Y_{\frac{\alpha}{\beta}}} - (\mathrm{rot}\,\mathrm{rot}\,h, h)_{Y_{\frac{\alpha}{\beta}}}$$

Then $Re\,(SF,F)_X = -\dfrac{\alpha}{\beta}\displaystyle\int_\Omega |\,\mathrm{rot}\,h|^2 \leq 0$ for all $F = (w,v,h) \in \mathcal{D}(S)$. In the proof we have used the Lemma 3.3.
Now, let $f = (f_1, f_2, f_3) \in X$ and let us show the existence of (u,v,g) belonging to $\mathcal{D}(S)$ such that

$$(I - S)\begin{pmatrix} u \\ v \\ g \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}.$$

i.e.

$$\begin{cases} u - v = f_1 \in H_0^2(\Omega) \\ v + d\Delta^2 u - \alpha \operatorname{rot}\operatorname{rot} g \cdot \vec{H} = f_2 \in L^2(\Omega) \\ g + \operatorname{rot}\operatorname{rot} g + \beta \operatorname{rot}\operatorname{rot} v \cdot \vec{H} = f_3 \in Y \end{cases}$$

We do $v = u - f_1$, so

$$\begin{cases} u + d\Delta^2 u - \alpha \operatorname{rot}\operatorname{rot} g \cdot \vec{H} = f_2 + f_1 = r_1 \in L^2(\Omega) \\ g + \operatorname{rot}\operatorname{rot} g + \beta \operatorname{rot}\operatorname{rot} u \cdot \vec{H} = f_3 + \beta \operatorname{rot}\operatorname{rot} f_1 \cdot \vec{H} = r_2 \in Y \end{cases} \tag{3.9}$$

taking into account that the right-hand side (r_1, r_2) belongs to $L^2(\Omega) \times Y$. We can use Lax-Milgram's lemma to obtain a weak solution of (3.9). Afterwards, by elliptic regularity we will conclude the assertion. We introduce the following bilinear form in $H_0^2(\Omega) \times V$, where $V = \{g \in Y; \operatorname{rot} g \in L^2(\Omega)\}$

$$a\left((u,g),(\tilde{u},\tilde{g})\right) = \int_\Omega \left[u \cdot \tilde{u} + d\Delta u \Delta \tilde{u} + \tfrac{\alpha}{\beta} g \cdot \tilde{g} + \tfrac{\alpha}{\beta} \operatorname{rot} g \cdot \operatorname{rot} \tilde{g}\right] dx +$$
$$-\alpha \left(\operatorname{rot} g, \operatorname{rot}(\tilde{u}\vec{H})\right)_{L^2} + \alpha \left(\operatorname{rot}(u\vec{H}), \operatorname{rot} \tilde{g}\right)_{L^2(\Omega)}$$

This bilinear form is continuous in $H_0^2(\Omega) \times V$. On the other hand, it is coercive since

$$a\left((u,g),(u,g)\right) = \int_\Omega \left[|u|^2 + d|\Delta u|^2 + \tfrac{\alpha}{\beta}|g|^2 + \tfrac{\alpha}{\beta}|\operatorname{rot} g|^2\right] dx.$$

Therefore, by Lax-Milgram's lemma there exist a unique $(u, g) \in H_0^2(\Omega) \times V$ so that

$$a((u,g),(\hat{u},\hat{w})) = ((r_1,r_2),(\hat{u},\hat{w}))_{H_0^2(\Omega)\times V} \quad , \quad \forall (\hat{u},\hat{w}) \in H_0^2(\Omega) \times V. \tag{3.10}$$

It is easy to see that the solution of (3.10) is a weak solution of (3.9) . On the other hand, $[H_0^2(\Omega)]^2 \subset \mathcal{D}(C)$ therefore

$$u \in H_0^2(\Omega), \Rightarrow \operatorname{rot}\operatorname{rot}(u\vec{H}) \in Y$$

we come back to (3.9) to obtain $\operatorname{rot}\operatorname{rot} g \in Y$ and $\Delta^2 u \in L^2(\Omega)$ so that by regularity arguments $(u, g) \in \mathcal{D}(A) \times \mathcal{D}(B)$. $\square$

As a direct consequence of Lemma 3.5, applying the Lumer-Phillips' Theorem, S is the infinitesimal generator of a strongly continuous semigroup of contractions on X. More precisely:

Theorem 3.2 *Let us consider the problem (2.21) for* $\gamma = 0$. *Let* (w_0, w_1, h_0) *be in the space* $X = H^2_{0d} \times L^2(\Omega) \times Y_{\frac{\alpha}{\beta}}$. *Then, problem (2.21) for* $\gamma = 0$ *is globally well-posed and the (unique) weak solution* (w, w_t, h) *belongs to* $\mathcal{C}([0, \infty), X)$. *Moreover, when* $(w_0, w_1, h_0) \in \mathcal{D}(S)$, *the solution* (w, w_t, h) *belongs to* $\mathcal{C}([0, \infty), \mathcal{D}(S)) \bigcap \mathcal{C}^1([0, \infty), X)$.

4 What about decay and large time behavior for the linear system?

Purely oscillatory behavior is observed if there exist purely imaginary eigenvalues of S because if

$$SW = i\xi W, \quad W \in \mathcal{D}(S)\backslash 0, \quad \xi \in \mathbb{R}\backslash 0 \tag{4.1}$$

then

$$\forall t \geq 0: \quad \|e^{-tS}W\|_X = \|e^{-i\xi t}W\|_X = \|W\|_X. \tag{4.2}$$

We remark that the spaces and the norms have been chosen such that $\frac{1}{2}\|V(t)\|_X^2$ equals the energy of the system, where $V = \begin{pmatrix} w \\ w_t \\ h \end{pmatrix} \equiv \begin{pmatrix} w \\ v \\ h \end{pmatrix}$. Hence (4.2) expresses conservation of energy for data W satisfying (4.1), in particular no decay rate at all, in this case.

Let W satisfy (4.1). Then we conclude

$$-|\operatorname{rot} h|^2 = Re\,\langle SW, W\rangle_X = Re\left(i\xi\|W\|_X^2\right) = 0$$

hence $\operatorname{rot} h = 0$ in $\Omega \times (0,T)$.
Since $\operatorname{rot} h = 0$ and Ω is simply connected then $h = \triangledown p$ in $\Omega \times (0,T)$ for some p. We also know that $\operatorname{div} h = 0$ in $\Omega \times (0,T)$. Therefore $\operatorname{div} \triangledown p = \Delta p = 0$ in $\Omega \times (0,T)$. Since $h \in Y$ then we know that $h \cdot \eta = 0$ on $\partial\Omega \times (0,T)$ consequently

$$\triangledown p \cdot \eta = \frac{\partial p}{\partial \eta} = 0 \quad \Rightarrow \quad \begin{cases} \Delta p = 0 \\ \\ \frac{\partial p}{\partial \eta} = 0 \end{cases}$$

which implies that $p =$ constant and, therefore, $h = \triangledown p \equiv 0$ in $\Omega \times (0,T)$.
This implies

$$\begin{cases} v = i\xi\omega \\ \\ -d\Delta^2\omega = i\xi v \\ \\ C(v\vec{H}) = 0 \end{cases}$$

where $C(v\vec{H}) = 0 \quad \Rightarrow \quad \mathrm{rot}(v\vec{H}) = c_1$, c_1 constant, if we think about

$$\Delta = \triangledown\, \mathrm{div} - \mathrm{rot}\, \mathrm{rot}$$

from which we obtain the following eigenvalue problem

$$\begin{cases} d\Delta^2 v = |\xi|^2 v \\ \\ v = \frac{\partial v}{\partial \eta} = 0 \text{ on } \partial\Omega \end{cases} \tag{4.3}$$

That is, there are purely imaginary eigenvalues if and only if the eigenvalue problem (4.3) has a solution satisfying side condition $\mathrm{rot}(v\vec{H}) = c_1$.
If there does not exist a non trivial solution for (4.3) we can use Lasalle's invariance principle as same as [19] to find that every solution of finite energy of (2.21) converges to zero in the energy space as $t \to +\infty$.
On the other hand, if there exists a non trivial solution for (4.3) then we can tell about oscillatory behavior of solutions of (2.21), and obviously no decay exists. But the existence of solutions of eigenvalue problem (4.3) remains open.

5 Existence of solutions for the nonlinear system

We now look at the nonlinear system of magnetoelastic plates, we define the same spaces Y, V and norms as in the linear case. We consider too the same operators $A := \Delta^2$ and $B := \mathrm{rot}\,\mathrm{rot}$.
Let Ω be a open, bounded, simply connected domain in $\mathbb{R}^2$, with smooth boundary $\partial\Omega$.

$$\begin{cases} \ddot{w} - \gamma\Delta\ddot{w} + d\Delta^2 w - \alpha[Bh]\cdot h = 0 \text{ in } Q \\ \dot{h} + Bh + \beta C[\dot{w}h] = 0 \text{ in } Q \\ \mathrm{div}\, h = 0 \text{ in } Q \\ h\cdot\eta(t,x_1,x_2) = 0 \text{ on } \Sigma \\ \eta \times \mathrm{rot}\, h(t,x_1,x_2) = 0 \text{ on } \Sigma \\ w = \frac{\partial w}{\partial \eta} = 0 \text{ on } \Sigma \\ w(x,0) = w_0, \quad w_t(x,0) = w_1, \quad h(x,0) = h_0 \text{ in } \Omega \end{cases} \tag{5.1}$$

where $Q = \Omega \times (0,T)$ and $\Sigma = \partial\Omega \times (0,T)$.
Let us assume that $\{w, h\}$ is a classical solution of problem (5.1), say $w \in \mathcal{C}^4(\bar{Q})$, $h \in \mathcal{C}^2(\bar{Q})$. Then it is easily verified that the following equalities are satisfied:

$$\left| \begin{array}{ll} \frac{d}{dt}(\dot{w},\varphi) + \gamma\frac{d}{dt}(\nabla\dot{w},\nabla\varphi) + d(\Delta w,\Delta\varphi) - \alpha(\mathrm{rot}\, h, \mathrm{rot}\,(h\varphi)) = 0 & \forall\varphi \text{ in } H_0^2(\Omega) \\ \frac{\alpha}{\beta}\frac{d}{dt}(h,\chi) + \frac{\alpha}{\beta}(\mathrm{rot}\, h, \mathrm{rot}\,\chi) + \alpha(\mathrm{rot}\,(\dot{w}h), \mathrm{rot}\,\chi) = 0 & \forall\chi \text{ in } V, \end{array} \right. \tag{5.2}$$

Lemma 5.1 *(see [17]) Let $\Omega \subset \mathbb{R}^n$ open, bounded and smooth and let $u \in H_0^1(\Omega)\bigcap H^2(\Omega)$. Then:*

$$\frac{\partial u}{\partial x_i} = \eta_i \frac{\partial u}{\partial \eta} \quad \text{on} \quad \partial\Omega$$

$$\text{and} \quad |\nabla u|^2 = \left(\frac{\partial u}{\partial \eta}\right)^2$$

Lemma 5.2 *Let $\dot{w} \in \mathcal{C}(0,T;H_0^2(\Omega))$, $\varphi \in \mathcal{C}(0,T;H_0^1(\Omega))$, $h, g \in V$, then*

i) $(\mathrm{rot}\,\mathrm{rot}(\dot{w}h), g) = (\mathrm{rot}(\dot{w}h), \mathrm{rot}\, g)$

ii) $(\mathrm{rot}\,\mathrm{rot}\, g, h\varphi) = (\mathrm{rot}\, g, \mathrm{rot}(\varphi h))$

Proof: *i*) Partial integration gives us

$$\int_\Omega \operatorname{rot}\operatorname{rot}(\dot{w}h)g dx = \int_\Omega \operatorname{rot}(\dot{w}h)\operatorname{rot} g dx + \int_{\partial\Omega}[\eta\times\operatorname{rot}(\dot{w}h)]g d\tau$$

$$= \int_\Omega \operatorname{rot}(\dot{w}h)\operatorname{rot} g dx + \int_{\partial\Omega}[\eta\times\triangledown\dot{w}\times h]g d\tau + \int_{\partial\Omega}[\eta\times\dot{w}\operatorname{rot} h]g d\tau$$

since

$$\operatorname{rot}(\dot{w}h) = \triangledown\dot{w}\times h + \dot{w}\operatorname{rot} h.$$

The result follows from $\dot{w} = 0$ on $\partial\Omega$, and from Lemma 5.1 $\triangledown\dot{w} = 0$ on $\partial\Omega$.

ii) $\int_\Omega [\operatorname{rot}\operatorname{rot} g]\cdot(\varphi h)dx = \int_\Omega \operatorname{rot} g \operatorname{rot}(\varphi h)dx + \int_{\partial\Omega}[\eta\times\operatorname{rot} g]\varphi h d\tau$

The result follows because $\varphi\in H_0^1(\Omega)$ $\square$

We multiply the first equation of (5.1) times $\dot{w}$, second equation times $\frac{\alpha}{\beta}h$ integration in Ω and addition, we get

$$\int_\Omega \left[d\Delta^2 w\dot{w} - \alpha[Bh]\cdot(h\dot{w}) - \gamma\Delta\ddot{w}\dot{w} + \ddot{w}\dot{w} + \frac{\alpha}{\beta}h_t\cdot h - \frac{\alpha}{\beta}Bh\cdot h + \alpha C(h\dot{w})\cdot h\right]$$

$$= d\frac{1}{2}\frac{d}{dt}|\Delta w|^2 - \alpha\left(\operatorname{rot} h, \operatorname{rot}(h\dot{w})\right) + \gamma\frac{1}{2}\frac{d}{dt}|\triangledown\dot{w}|^2 + \frac{1}{2}\frac{d}{dt}|\dot{w}|^2 + \frac{\alpha}{\beta}\frac{1}{2}\frac{d}{dt}|h|^2$$

$$+\frac{\alpha}{\beta}|\operatorname{rot} h|^2 + \alpha\left(\operatorname{rot}(h\dot{w}), \operatorname{rot} h\right)$$

and so, thanks to Lemma 5.2, we obtain

$$\frac{d}{dt}E(t) = -\frac{\alpha}{\beta}\int_\Omega |roth|^2 dx \le 0\ .$$

where

$$E(t) = \frac{1}{2}\left\{\int_\Omega [d|\Delta w|^2 + \gamma|\triangledown\dot{w}|^2 + |\dot{w}|^2 + \frac{\alpha}{\beta}|h|^2]dx\right\}$$

Therefore, the energy $E(t)$ from solutions of (5.1) decreases along trajectories.

Lemma 5.3 $w\in H^2(\Omega); g,h\in[H^2(\Omega)]^2\bigcap V$:

i) $(C[wh],g)\le\|h\|_2\|w\|_2|g|$

ii) $(C[wh],g)\le c\|w\|_2\|h\|_1\|g\|_1$

iii) $(C[wh], Bg) \leq c\|w\|_2\|h\|_2\|g\|_2$

iv) $(Bg, wh) \leq c\|g\|_2\|w\|_1\|h\|_1$

v) $(Bg, wh) \leq \|g\|_2|w|\|h\|_{L^\infty}$

Proof: *i)*

$$(C[wh], g) = (\operatorname{rot}[\nabla w \times h + w \operatorname{rot} h], g) = \left(\begin{pmatrix} \partial_2(\nabla w \times h) \\ -\partial_1(\nabla w \times h) \end{pmatrix}, g\right)$$

$$= \left(\begin{pmatrix} \partial_2 \nabla \times h \\ -\partial_1 \nabla \times h \end{pmatrix}, g\right) + \left(\begin{pmatrix} \nabla w \times \partial_2 h \\ -\nabla w \times \partial_1 h \end{pmatrix}, g\right)$$

$$\leq \sum_i \|w\|_2 |h|_\infty |g_i| + \sum_i \|\nabla w\|_{L^4} \|\partial_i h\|_{L^4} |g_i|$$

$$\leq \left[|h|_\infty + \|h\|_{\frac{3}{2}}\right] \|w\|_2 |g|$$

Since $H_2(\Omega) \subset H_{\frac{3}{2}} \bigcap L^\infty$

$$\leq \|h\|_2\|w\|_2|g|, \quad \forall h \in \mathcal{D}(B).$$

ii)

$$(C[wh], g) = (\operatorname{rot}[wh], \operatorname{rot} g)$$

$$= (\nabla w \times h, \operatorname{rot} g) + (w \operatorname{rot} h, \operatorname{rot} g)$$

$$\leq \|w\|_{\frac{3}{2}}\|h\|_1\|g\|_1 + |w|_{L^\infty}\|h\|_1\|g\|_1$$

$$\leq c\|w\|_2\|h\|_1\|g\|_1$$

iii) $C[wh] = (\partial_2 \operatorname{rot}(wh), -\partial_1 \operatorname{rot}(wh))$

$$\partial_i \operatorname{rot}(wh) = \partial_i \nabla w \times h + \nabla w \times \partial_i h + \partial_i w \operatorname{rot} h + w \partial_i \operatorname{rot} h$$

$$(\partial_i \operatorname{rot}(wh), [Bg]_i) = (\partial_i \nabla w \times h, [Bh]_i) + (\nabla w \times \partial_i h, [Bh]_i) + (\partial_i w \operatorname{rot} h, [Bh]_i) +$$
$$+ (w \partial_i \operatorname{rot} h, [Bh]_i)$$

$$(C[wh], Bg) \leq \|w\|_2 |h|_\infty \|h\|_2 + \|w\|_2\|h\|_{\frac{3}{2}}\|h\|_2 + \|w\|_{\frac{3}{2}}\|h\|_2^2 + |w|_\infty\|h\|_2^2$$

$$\leq c\|w\|_2\|h\|_2^2.$$

$iv)\,(Bg, wh) \leq \|g\|_2\|w\|_{L^4}\|h\|_{L^4} \quad \square$

We define the continuous trilinear form d_1 as

$$d_1\,(\varphi,\psi,\chi) = \int_\Omega \operatorname{rot}\,[\varphi\psi]\operatorname{rot}\chi dx, \qquad (\varphi,\psi,\chi) \in H^1(\Omega)\times V\times V^{\frac{3}{2}}$$

and the continuous trilinear form d_2 as

$$d_2\,(\varphi,\psi,\chi) = \int_\Omega \operatorname{rot}\varphi\operatorname{rot}\,[\chi\varphi]\,dx, \qquad (\varphi,\psi,\chi) \in V^{\frac{3}{2}}\times V\times H^1(\Omega)$$

We define also the bilinear operators

$$D_1 : H^1(\Omega)\times V \longrightarrow V^{-\frac{3}{2}} \quad \text{ and } \quad D_2 : V^{\frac{3}{2}}\times V \longrightarrow H^{-1}(\Omega)$$

where

$$\langle D_1(\varphi,\psi),\chi\rangle = d_1(\varphi,\psi,\chi) \qquad \langle D_2(\varphi,\psi),\chi\rangle = d_2(\varphi,\psi,\chi)$$

$$D_1(\varphi,\psi) := \operatorname{rot}\operatorname{rot}\,[\varphi\psi] =: C\,[\varphi\psi]$$

$$D_2(\varphi,\psi) := [\operatorname{rot}\operatorname{rot}\varphi]\cdot\psi =: [B\varphi]\cdot\psi$$

Let W_n and V_n be the generated spaces by the n first eigenvectors $\varphi_1,\varphi_2,...,\varphi_n$ and $\psi_1,\psi_2,...,\psi_n$ of spectral problem in application II and I respectively. The approximate problem is to determine $w_n(t)\in W_n$ and $h_n(t)\in V_n$, so that

$$w_n(t) = \sum_{j=1}^n f_{jn}(t)\varphi_j, \quad h_n(t) = \sum_{j=1}^n g_{jn}(t)\psi_j \tag{5.3}$$

For $v\in W_n$, $g\in V_n$ we have the approximation system

$$\begin{cases} (\ddot{w}_n, v) + \gamma\,(\nabla\ddot{w}_n, \nabla v) + (\Delta w_n, \Delta v) - (\operatorname{rot} h_n, \operatorname{rot}\,[vh_n]) = 0 \\ \left(\dot{h}_n, g\right) + (\operatorname{rot} h_n, \operatorname{rot} g) + \beta\,(\operatorname{rot}\,[\dot{w}_n h_n]\,, \operatorname{rot} g) = 0 \\ w_n(0) = w_0^n = \sum_{j=1}^n a_j\varphi_j \longrightarrow w_0 \text{ in } \mathcal{D}(A) \\ \dot{w}_n(0) = w_1^n = \sum_{j=1}^n b_j\varphi_j \longrightarrow w_1 \text{ in } H_0^2(\Omega) \\ h_n(0) = h_0^n = \sum_{j=1}^n c_j\psi_j \longrightarrow h_0 \text{ in } \mathcal{D}(B) \end{cases} \tag{5.4}$$

Proposition 5.1 *If* $h \in L^2(0,T;\mathcal{D}(B))\bigcap L^\infty(0,T;V)$ *then we have* $[Bh]\cdot h \in L^2(0,T;H^{-1}(\Omega))$ *and* $[Bh]\cdot h \in L^1(0,T;L^2(\Omega))$.

Proof: In fact, we can also see that

$$D_2 : \mathcal{D}(B)\times V \longrightarrow H^{-1}(\Omega)$$

and $([Bh]\cdot h,\chi) = ([Bh]\,,\chi h) \leq |Bh|_{[L^2]^2}\|\chi\|_{L^4}\|h\|_{[L^4]^2}$

$$\leq |Bh|_{[L^2]^2}\|\chi\|_1\|h\|_V \text{ for } \chi \in H_0^1(\Omega).$$

Therefore $[Bh]\cdot h \in L^2(0,T;H^{-1}(\Omega))$.

For $h \in \mathcal{D}(B)$ we have too, $([Bh]\cdot h,\chi) \leq \|h\|_{L^\infty}|Bh||\chi|$ for $\chi \in L^2(\Omega)$

and therefore $[Bh]\cdot h \in L^1(0,T;L^2(\Omega))$. □

Lemma 5.4 *:*

$|roth_n|^2 + \int_0^t |Bh_n|^2 ds \leq c|roth_0|^2 + \beta\int_0^t \|\dot{w}_n\|_2\|h_n\|_2^2 ds$

Proof: In the second equation of (5.4) we set $g = Bh_n$ and we use Lemma 5.3 *iii*.

$\left(\dot{h}_n, Bh_n\right) + |Bh_n|^2 + \beta\left(C\left[\dot{w}_n h_n\right], Bh_n\right) = 0$

$\frac{1}{2}\frac{d}{dt}|roth_n|^2 + |Bh_n|^2 + \beta\left(C\left[\dot{w}_n h_n\right], Bh_n\right) = 0$

$|roth_n|^2 + \int_0^t |Bh_n|^2 ds \leq c|roth_0|^2 + \beta\int_0^t \left(C\left[\dot{w}_n h_n\right], Bh_n\right) ds$

$|roth_n|^2 + \int_0^t |Bh_n|^2 ds \leq c|roth_0|^2 + \beta\int_0^t \|\dot{w}_n\|_2\|h_n\|_2^2 ds$ □

Lemma 5.5 *:*

$|\ddot{w}_n(0)|^2 + \gamma|\triangledown\ddot{w}_n(0)|^2 \leq c\left(|Aw_n(0)| + \|h_n(0)\|_2^2\right)^2$

$|\dot{h}_n(0)|^2 \leq c\left(\|h_n(0)\|_2 + \beta\|h_n(0)\|_2\|\dot{w}_n(0)\|_2\right)^2$

Proof: We set in (5.4) $v = \ddot{w}_n(0)$ and $g = \dot{h}_n(0)$ when $t = 0$:

$$|\ddot{w}_n(0)|^2 + \gamma|\triangledown\ddot{w}_n(0)|^2 + d(\Delta w_n(0), \Delta\ddot{w}_n(0)) - \alpha(h_n(0), C[h_n(0)\ddot{w}_n(0)]) = 0$$

$$|\dot{h}_n(0)|^2 + (\operatorname{rot} h_n(0), \operatorname{rot}\dot{h}_n(0)) + \beta(\operatorname{rot}[\dot{w}_n(0)h_n(0)], \operatorname{rot}\dot{h}_n(0)) = 0$$

and so we get

$$\begin{aligned}|\ddot{w}_n(0)|^2 &= -d(\Delta^2 w_n(0), \ddot{w}_n(0)) + \alpha(Bh_n(0)\cdot h_n(0), \ddot{w}_n(0))\\ &\le |Aw_n(0)||\ddot{w}_n(0)| + \|h_n(0)\|_2|h_n(0)|_\infty|\ddot{w}_n(0)|\end{aligned}$$

$$|\ddot{w}_n(0)| \le |Aw_n(0)| + \|h_n(0)\|_2^2$$

$$|\triangledown\ddot{w}_n(0)|^2 \le |Aw_n(0)|\|\ddot{w}_n(0)\|_1 + \|h_n(0)\|_2\|h_n(0)\|_{\frac{3}{2}}\|\ddot{w}_n(0)\|_1$$

$$|\triangledown\ddot{w}_n(0)| \le |Aw_n(0)| + \|h_n(0)\|_2^2$$

therefore

$$|\ddot{w}_n(0)|^2 + \gamma|\triangledown\ddot{w}_n(0)|^2 \le (1+\gamma)\left(|Aw_n(0)| + \|h_n(0)\|_2^2\right)^2$$

and

$$|\dot{h}_n(0)|^2 \le |Bh_n(0)||\dot{h}_n(0)| + \beta\|h_n(0)\|_2\|\dot{w}_n(0)\|_2|\dot{h}_n(0)|$$

$$|\dot{h}_n(0)| \le |Bh_n(0)| + \beta\|h_n(0)\|_2\|\dot{w}_n(0)\|_2. \quad \square$$

For $v = \varphi_i$ and $g = \psi_i$ we consider the problem

$$\begin{cases} (\ddot{w}_n, \varphi_i) + \gamma(\triangledown\ddot{w}_n, \triangledown\varphi_i) + (\Delta w_n, \Delta\varphi_i) - (\operatorname{rot} h_n, \operatorname{rot}[\varphi_i h_n]) = 0 \\ \left(\dot{h}_n, \psi_i\right) + (\operatorname{rot} h_n, \operatorname{rot}\psi_i) + \beta(\operatorname{rot}[\dot{w}_n h_n], \operatorname{rot}\psi_i) = 0 \\ \text{in} \quad \mathcal{D}'(0,T), \quad \text{for } 1 \le i \le m \end{cases} \tag{5.5}$$

Now in system (5.4), we multiply the first equation times $\dot{f}_i(t)$, the second equation times $g_i(t)$ and we add from $i = 1$ to n to obtain the first a priori estimate

$$E_n(t) = \frac{1}{2}\left\{\int_\Omega \left[d|\Delta w_n|^2 + \gamma|\triangledown\dot{w}_n|^2 + |\dot{w}_n|^2 + \frac{\alpha}{\beta}|h_n|^2 dx\right]\right\}$$

$$\frac{d}{dt}E_n(t) = -\frac{\alpha}{\beta}\int_\Omega |\operatorname{rot} h_n|^2 dx$$

$$\Rightarrow \quad E_n(t) + \frac{\alpha}{\beta}\int_0^t |\operatorname{rot} h_n|^2 dx = E_n(0) \le E(0)$$

Therefore

(h_n) bounded in $L^2(0,T;V)\bigcap L^\infty(0,T;Y)$

(w_n) bounded in $L^\infty(0,T;H_0^2(\Omega))$

$(\dot{w}_n)$ bounded in $L^\infty(0,T;H_0^1(\Omega)),\quad \gamma>0$

$(\dot{w}_n)$ bounded in $L^\infty(0,T;L^2(\Omega)),\quad \gamma=0$

But, such a estimate is not enough in regularity for h_n.

So, we must work to find more regularity. In order to do that we have the following theorem

Theorem 5.1 *There exists a constant $c>0$ such that for all (h_0,w_0,w_1) with*

$$\gamma(t)=F<c \tag{5.6}$$

where
$F=|Aw_n(0)|+\|h_n(0)\|_2^2+\|h_n(0)\|_2+\beta\|h_n(0)\|_2\|\dot{w}_n(0)\|_2+\|\dot{w}_n(0)\|_2+\tilde{c}|roth_0|$, $h_0\in\mathcal{D}(B),\quad w_0\in\mathcal{D}(A),\quad w_1\in H_0^2(\Omega)$, *there exists a solution $\{w,h\}$ of the problem (5.2) and*

$h\in L^2(0,T;\mathcal{D}(B))\bigcap L^\infty(0,T;V),\quad \dot{h}\in L^2(0,T;V)\bigcap L^\infty(0,T;L^2(\Omega)),$
$w\in L^2(0,T;H^3(\Omega)),\ \ w\in L^1(0,T;H^4(\Omega))$ *if* $\gamma=0,$
$\dot{w}\in L^\infty(0,T;H_0^2(\Omega)),\quad \ddot{w}\in L^\infty(0,T;H_0^1(\Omega)).$

Proof: We differentiate in t the system (5.4) to obtain

$$\begin{cases}(\ddot{w}_n,\varphi_i)+\gamma\,(\nabla\ddot{w}_n,\nabla\varphi_i)+(\Delta\dot{w}_n,\Delta\varphi_i)-\Big(\operatorname{rot}\dot{h}_n,\operatorname{rot}[h_n\varphi_i]\Big)-\Big(\operatorname{rot}h_n,\operatorname{rot}\Big[\dot{h}_n\varphi_i\Big]\Big)=0\\[2ex] \Big(\ddot{h}_n,\psi_i\Big)+\Big(\operatorname{rot}\dot{h}_n,\operatorname{rot}\psi_i\Big)+\beta\,(\operatorname{rot}[\ddot{w}_nh_n],\operatorname{rot}\psi_i)+\beta\Big(\operatorname{rot}\Big[\dot{w}_n\dot{h}_n\Big],\operatorname{rot}\psi_i\Big)=0\end{cases} \tag{5.7}$$

we make now first equation times $\dddot{f}_{in}$, second equation times $\dot{g}_{in}$ and summation at i to get
$\frac{1}{2}\frac{d}{dt}\Big\{|\ddot{w}_n|^2+\gamma\|\ddot{w}_n\|_1^2+\|\dot{w}_n\|_2^2+|\dot{h}_n|^2\Big\}+|\operatorname{rot}\dot{h}_n|^2+\beta\int_\Omega C\,[\ddot{w}_nh_n]\,\dot{h}_ndx+$

$$-\int_\Omega \left[B\dot{h}_n\right]\ddot{w}_n h_n dx - \int_\Omega \left[Bh_n\right]\dot{h}_n\ddot{w}_n dx + \beta\int_\Omega C\left[\dot{w}_n\dot{h}_n\right]\dot{h}_n = 0$$

then integrate on $[0,T]$

$$|\ddot{w}_n|^2+\gamma\|\ddot{w}_n\|_1^2+\|\dot{w}_n\|_2^2+|\dot{h}_n|^2+\int_0^T|\operatorname{rot}\dot{h}_n|^2ds = |\ddot{w}_n(0)|^2+\gamma\|\ddot{w}_n(0)\|_1^2+\|\dot{w}_n(0)\|_2^2+$$

$$+|\dot{h}_n(0)|^2+\int_0^T\left(\left[Bh_n\right],\dot{h}_n\ddot{w}_n\right)-\beta\int_0^T\left(C\left[\dot{w}_n\dot{h}_n\right],\dot{h}_n\right)$$

by Lemma 5.5 we have

$$\leq \left(|Aw_n(0)|+\|h_n(0)\|_2^2\right)^2+\left(\|h_n(0)\|_2+\beta\|h_n(0)\|_2\|\dot{w}_n(0)\|_2\right)^2+\|\dot{w}_n(0)\|_2^2+$$
$$+\int_0^T\left(\left[Bh_n\right],\dot{h}_n\ddot{w}_n\right)-\beta\int_0^T\left(C\left[\dot{w}_n\dot{h}_n\right],\dot{h}_n\right)$$

$$\leq \left(|Aw_n(0)|+\|h_n(0)\|_2^2\right)^2+\left(\|h_n(0)\|_2+\beta\|h_n(0)\|_2\|\dot{w}_n(0)\|_2\right)^2+\|\dot{w}_n(0)\|_2^2+$$
$$+\int_0^T\|h_n\|_2\|\ddot{w}_n\|_1\|\dot{h}_n\|_1ds+\beta\int_0^T\|\dot{w}_n\|_2\|\dot{h}_n\|_1\|\dot{h}_n\|_1$$

We add now the quantity in Lemma 5.4 to obtain

$$|\ddot{w}_n|^2+\gamma\|\ddot{w}_n\|_1^2+\|\dot{w}_n\|_2^2+|\dot{h}_n|^2+|roth_n|^2+\int_0^T|\operatorname{rot}\dot{h}_n|^2ds+$$

$$+\int_0^t|Bh_n|^2ds\leq F^2+\int_0^T\|h_n\|_2\|\ddot{w}_n\|_1\|\dot{h}_n\|_1ds+\beta\int_0^T\|\dot{w}_n\|_2\|\dot{h}_n\|_1\|\dot{h}_n\|_1+$$
$$+\int_0^t\|\dot{w}_n\|_2\|h_n\|_2^2ds$$

We call

$$g_m(t)=|\ddot{w}_n|^2+\gamma\|\ddot{w}_n\|_1^2+\|\dot{w}_n\|_2^2+|\dot{h}_n|^2+|roth_n|^2+\int_0^T|\operatorname{rot}\dot{h}_n|^2ds+\int_0^t|Bh_n|^2ds$$

and

$$p_m(t)=\left(\int_0^t|Bh_n|^2+\int_0^t\|\dot{h}_n\|_1^2\right)^{\frac{1}{2}}$$

and so

$$g_m(t)\leq F^2+\tfrac{1}{2}\int_0^T\left[\|h_n\|_2^2+\|\dot{h}_n\|_1\right]\|\ddot{w}_n\|_1^2ds+\beta\int_0^T\left[\|\dot{h}_n\|_1^2+\|h_n\|_2^2\right]\|\dot{w}_n\|_2ds$$

$$\leq F^2+a\int_0^T\left[|Bh_n|^2+\|\dot{h}_n\|_1^2\right]\sqrt{g_m(s)}ds$$

Now by Bihari's Lemma 1.1

$$g_m(t)\leq\left[F+a\int_0^T\left(|Bh_n|^2+\|\dot{h}_n\|_1^2\right)ds\right]^2$$

but $p_m(t) \leq (g_m(t))^{\frac{1}{2}}$, then we can write this inequality as

$$p_m(t) \leq F + ap_m^2(t)$$

and then we apply the following lemma (see [4]).

Lemma 5.6 *Assume that certain functions* $p(t), \gamma(t) \in \mathcal{C}([0,T]),\ p(t) \geq 0,\ \gamma(t) \geq 0$, *satisfy the inequality* $p(t) \leq \gamma(t) + ap^2(t)$ *for all* $t \in [0,T]$. *Suppose the condition* $1 - 4a\gamma(t) > 0$ *holds for all* $t \in [0,T]$ *and* $p_1(t)$ *and* $p_2(t)$ *are the roots of the equation* $ap^2 - p + \gamma(t) = 0,\ p_1(t) < p_2(t)$. *Then if* $p(0) < p_1(0)$, *it follows that* $p(t) \leq p_1(t)$ *on* $[0,T]$.

Assume that the condition $1 - 4a\gamma(t) > 0$ holds, i.e. condition (5.6) holds. Then, we get the estimate

$$p_m(t) \leq Cte$$

and hence the estimate

$$g_m(t) \leq Cte$$

and therefore we have

$$\begin{array}{l} (\ddot{w}_n) \text{ bounded in } L^\infty(0,T;H_0^1(\Omega)) \\ (\dot{w}_n) \text{ bounded in } L^\infty(0,T;H_0^2(\Omega)) \\ (\dot{h}_n) \text{ bounded in } L^\infty(0,T;Y) \\ (\dot{h}_n) \text{ bounded in } L^2(0,T;V) \\ (h_n) \text{ bounded in } L^\infty(0,T;V) \\ (h_n) \text{ bounded in } L^2(0,T;\mathcal{D}(\mathcal{B})) \end{array} \tag{5.8}$$

We will now pass to the limit in (5.4).
Let m_0 fix, $v \in W_{m_0}$ and $\theta \in \mathcal{D}'(0,T)$. From Lemma 5.2 we get

$$\int_0^T (\operatorname{rot} h_n, \operatorname{rot}[vh_n])\,\theta(t)dt = \int_0^T (\operatorname{rot}\operatorname{rot} h_n, [vh_n])\,\theta(t)dt$$

$$= \int_0^T (\operatorname{rot}\operatorname{rot} h_n, \theta(t)vh_n)\,dt$$

From (5.8) and Lions Aubin Theorem h_n converges strongly to h in $L^2(0,T;L^2(\Omega))$ and since $\theta \in \mathcal{D}'(0,T)$ and $v \in W_{m_0}$, $\theta v h_n$ converges strongly to $\theta v h$ in $L^2(0,T;Y)$. On the other hand, $\operatorname{rot}\operatorname{rot} h_n$ converges weak to $\operatorname{rot}\operatorname{rot} h$ in $L^2(0,T;Y)$

therefore

$$\int_0^T (\mathrm{rot}\,\mathrm{rot}\, h_n, \theta(t) v h_n)\, dt \longrightarrow \int_0^T (\mathrm{rot}\,\mathrm{rot}\, h, \theta(t) v h)\, dt = \int_0^T (\mathrm{rot}\, h, \mathrm{rot}\,[vh])\, \theta(t) dt$$

For the second equation by Lemma 5.2 we have

$$\begin{aligned} \int_0^T (\mathrm{rot}\,[\dot{w}_n h_n]\,, \mathrm{rot}\, g)\, \theta(t) dt &= \int_0^T (\dot{w}_n h_n, \mathrm{rot}\,\mathrm{rot}\, g)\, \theta(t) dt \\ &= \int_0^T (\dot{w}_n, h_n \cdot \mathrm{rot}\,\mathrm{rot}\, g)\, \theta(t) dt \\ &= \int_0^T (\dot{w}_n, \theta(t) h_n \cdot \mathrm{rot}\,\mathrm{rot}\, g)\, dt \end{aligned}$$

Once h_n converges strongly in $L^2(0,T;Y)$ and for $g \in W_{m_0}$ we have $\mathrm{rot}\,\mathrm{rot}\, g$ smooth enough. This is because W_m is generated by the eigenfunctions of the operator B. So $\theta h_n \cdot \mathrm{rot}\,\mathrm{rot}\, g$ converges strongly in $L^2(0,T;Y)$.
On the other hand $\dot{w}_n$ converges weak in $L^2(0,T;L^2(\Omega))$, therefore

$$\int_0^T (\dot{w}_n, \theta(t) h_n \cdot \mathrm{rot}\,\mathrm{rot}\, g)\, dt \longrightarrow \int_0^T (\dot{w}, \theta(t) h \cdot \mathrm{rot}\,\mathrm{rot}\, g)\, dt = \int_0^T (\mathrm{rot}\,[\dot{w} h]\,, \mathrm{rot}\, g)\, \theta(t) dt$$

And so we have proved that there exists a solution $\{w, h\}$ for the nonlinear problem.

Note that the system (5.4) may be written in operator form

$$\left\{ \begin{array}{c} \ddot{w} - \gamma A^{\frac{1}{2}} \ddot{w} + dAw - \alpha Bh \cdot h = 0 \\ \dot{h} + Bh + \beta C[\dot{w} h] = 0 \end{array} \right. \tag{5.9}$$

Therefore we can easily deduce from Proposition 5.1 that for $\ddot{w} \in L^\infty(0,T;H_0^1(\Omega))$ we have $w \in L^2(0,T;H^3(\Omega))$ and $w \in L^1(0,T;H^4(\Omega))$ if $\gamma = 0$.

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