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Chanyu Shang

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Chanyu Shang
Department of Mathematics and Statistics, University of Konstanz
78457 Konstanz, Germany
on leave from Institute of Mathematics, Fudan University
200433 Shanghai, P.R. China
Email: chanyufd@gmail.com

Abstract

We consider the nonlinear beam equation $u_{tt} + a(x)u_t - f(u_x)_x + u_{xxxx} = 0$ in a bounded interval $(0, 1) \subset R$. The equation has an indefinite damping term, i.e., with a damping function $a = a(x)$ possibly changing sign. For this non-dissipative situation we prove the exponential stability of the corresponding linearized system provided $\bar{a} = \int_0^1 a(x)dx > 0$ and $\| a - \bar{a} \|_{L^2} \leq \tau$, for τ small enough. We shall also demonstrate that the system has the spectrum determined growth (SDG) property for the constant case $a \equiv \bar{a}$. Moreover, we show the global existence of the solution to the corresponding nonlinear system. To our knowledge, this paper is the first to deal with a fourth-order nonlinear evolution equations with indefinite damping.

Keyword: exponential stability; indefinite damping; non-dissipative system

MSC 2000: 35G25; 35B40

1 Introduction

We consider the following nonlinear beam equation

$$u_{tt} + a(x)u_t - f(u_x)_x + u_{xxxx} = 0, \tag{1.1}$$

for a function $u = u(t, x)$, $t \geq 0$, $x \in (0, 1)$, subject to the following initial and boundary conditions

$$u \mid_{t=0} = u_0, \quad u_t \mid_{t=0} = u_1 \tag{1.2}$$

$$u\ |_{x=0,1}= u_{xx}\ |_{x=0,1}= 0 \tag{1.3}$$

We assume that $a \in L^\infty(0,1)$ satisfies

$$\bar{a} := \int_0^1 a(x)dx > 0, \tag{1.4}$$

in particular a may change sign in [0,1] or be zero in open subsets. The nonlinear function f is assumed to be a polynomial function given by

$$f(u_x) = \alpha_1 u_x^5 - \alpha_2 u_x^3 - \alpha_3 u_x \tag{1.5}$$

with α_i being given positive constants. We refer to [13], [12] for a more details explanation of the physical background of $f(u_x)$.

If $a(x) \equiv \bar{a} > 0$, i.e., the constant case, Shang [13] proved the existence of a global attractor in a closed subspace H_β, so the exponential stability of the solution easily follows. Now we focus our attention on the indefinite damping, i.e., a(x) is allowed to change its sign.

Already for the wave equation

$$u_{tt} - u_{xx} + a(x)u_t = 0 \tag{1.6}$$

with Dirichlet boundary conditions, it is a subtle issue to see whether an indefinite damping with the function a just satisfying (1.4) still leads to exponential stability. The non-dissipative case with indefinite a seems to have been posed first by Chen, Fulling, Narcowich and Sun [4] where it was conjectured that if there exists $\gamma > 0$ such that

$$\forall n = 1, 2, ... : \int_0^L a(x)\sin^2(\frac{n\pi x}{L})dx \geq \gamma \tag{1.7}$$

satisfied, then the energy $E(t) = \int_0^L (u_t^2 + u_x^2)dx$ decays exponentially in time. But Freitas [5] found that condition (1.7) on the moments is not sufficient to guarantee exponential stability when $\parallel a \parallel_{L^\infty}$ is large. Replacing a by εa, Freitas and Zuazua [6] proved that when a is of bounded variation and the condition on the moments (1.7) holds, then there is $\varepsilon^* = \varepsilon^*(a)$ such that for all $\varepsilon \in (0, \varepsilon^*)$ the energy decays indeed exponentially. This results was extended to a differential equation of the type

$$u_{tt} - u_{xx} + \varepsilon a(x)u_t + b(x)u = 0 \tag{1.8}$$

by Benaddi and Rao [7]. J.E.Muñoz and R.Racke [10] proved the exponential stability for the system in either

Case (i): $\| a(\cdot) - \bar{a} \|_{L^2}$ is sufficiently small, but $\| a \|_{L^\infty}$ may be large, or

Case (ii): $\| a \|_{L^\infty}$ is small, (a, L) satisfy certain relations, but the moments $\int_0^L a(x)\sin^2(\frac{\pi x}{L})dx$ may be negative.

An extension to the type of equation (1.8) was given by Menz [11].

For the Timoshenko system with indefinite damping, J.E.Muñoz and R.Racke [9] proved that the system was exponentially stable under the same condition as in the positive constant damping case provided $\bar{a} := \int_0^1 a(x)dx > 0$ and $\| a - \bar{a} \|_{L^2}$ small enough. Moveover, [9] also gave a precise description of the decay rate and demonstrated that the system had the spectrum determined growth (SDG) property for the constant case $a \equiv \bar{a}$.

For the fourth-order linear evolution equation with indefinite damping, K.Liu, Z.Liu, R.Rao [8] proved the exponential stability of an abstract nondissipative linear system, which could be applied to several elastic systems including the Euler-Bernoulli beam equation of the type

$$u_{tt} + u_{xxxx} = f(x,t), \quad for \ \ 0 < x < 1, \quad t > 0, \tag{1.9}$$

where $f(\cdot,t) = -\varepsilon K u_t(\cdot,t)$, and

$$[Ku_t](x) = \begin{cases} 2c(x)u_t(x), & 0 < x < \frac{1}{2} \\ 2d(x)u_t(x), & \frac{1}{2} < x < 1 \end{cases} \tag{1.10}$$

[8] proved that if

$$\inf_{n\geq 1} \int_0^{\frac{1}{2}} [d(1-x) + (-1)^{n+1}c(x)](1 - \cos 2n\pi x)dx > 0 \tag{1.11}$$

satisfied, then there exist $\varepsilon^* = \varepsilon^*(c,d)$, such that for $\varepsilon \in (0,\varepsilon^*)$, the exponential stability holds for (1.9).

In this paper, we first consider the linearized equation

$$u_{tt} + a(x)u_t - ku_{xx} + u_{xxxx} = 0, \quad in \ \ (0,\infty) \times (0,1). \tag{1.12}$$

We should see that for our results, we just need a condition on the smallness of $\| a - \bar{a} \|_{L^2}$. In a similar manner to [9], we shall demonstrate that for the constant coefficient case $a \equiv \bar{a}$, the equation has the so-called spectrum determined growth property; that is, after reformulated the system as a first-order system $U_t = AU$, we shall prove that the growth abscissa $\omega_0(A)$ equals the spectral bound $\omega_\sigma(A)$, i.e.,

$$\omega_0(A) = \omega_\sigma(A)$$

with $\omega_0(A)$, $\omega_\sigma(A)$ defined as

$$\omega_0(A) = \inf\{\omega \in R \mid \exists M = M(\omega), \forall \lambda, Re\lambda \geq \omega : \| (\lambda - A)^{-1} \| \leq M\},$$

which was given by Prüss [2] or Huang [3], and

$$\omega_\sigma(A) = \sup\{Re\lambda \mid \lambda \in \sigma(A)\}.$$

For the case of possibly indefinite damping, we shall prove the exponential stability of the corresponding equation provided $\| a - \bar{a} \|_{L^2}$ small enough. In this proof, we shall decompose the fourth-order elliptic equation

$$u_{xxxx} - \kappa u_{xx} + \alpha u = g$$

into two separate two-order equations and then apply the techniques in [9] to estimate the corresponding operators defined by the contraction mapping, where a fixed point argument will be used. Moreover, we show the global existence and uniqueness of the corresponding nonlinear system.

This paper is organized as follows. In section 2, we shall formulate the semigroup setting. In section 3, we shall prove that in the positive constant damping case, the equation has the SDG property. In section 4, we shall prove that in the indefinite damping case, the equation is exponentially stable provided $\| a - \bar{a} \|_{L^2} \leq \varepsilon$ small enough. In section 5, the global existence of the solution to the nonlinear system is investigated.

2 The semigroup setting

We first consider the linearized equation

$$u_{tt} + a(x)u_t - \kappa u_{xx} + u_{xxxx} = 0 \quad in \quad (0, \infty) \times (0, 1), \tag{2.1}$$

$$u \mid_{t=0} = u_0, \quad u_t \mid_{t=0} = u_1, \tag{2.2}$$

$$u \mid_{x=0,1} = u_{xx} \mid_{x=0,1} = 0 \tag{2.3}$$

with $\kappa = -f'(0)$ being a positive constant.

We rewrite problem (2.1)–(2.3) as a first-order system for $U := (u, v)^T$, here we introduce $v = u_t$. Then U satisfies

$$U_t = \mathcal{A}U,$$

$$U(t = 0) = U_0$$

where $U_0 := (u_0, u_1)$, and

$$\mathcal{A}U = \begin{pmatrix} v \\ -a(x)v + \kappa u_{xx} - u_{xxxx} \end{pmatrix}. \tag{2.4}$$

Let

$$\mathcal{H} := \big\{(u,v) \in (H^2 \cap H_0^1) \times L^2 \mid u \mid_{x=0,1} = 0\big\}$$

be the Hilbert space with norm given by

$$\| (u,v)^T \|_{\mathcal{H}} := (\int_0^1 \mid u_{xx} \mid^2 + \mid v \mid^2 dx)^{\frac{1}{2}}$$

Then $\mathcal{A}$ given in (2.4) with domain

$$D(\mathcal{A}) := \big\{(u,v) \in (H^4 \times H_0^1) \times (H^2 \cap H_0^1) \mid u \mid_{x=0,1} = u_{xx} \mid_{x=0,1} = 0, v \mid_{x=0,1} = 0\big\}$$

generates a semigroup $\{e^{\mathcal{A}t}\}_{t\geq 0}$ and $D(\mathcal{A})$ is dense in $\mathcal{H}$.

Lemma 2.1. *For $\kappa \neq j^2\pi^2$, $j \in N$, $\mathcal{A}^{-1}$ is compact.*

Proof. To prove $\mathcal{A}U = F$ is solvable for $F = (f,g)^T \in \mathcal{H}$.

By the definition of $\mathcal{A}$, we have

$$\begin{cases} v = f(x), \\ u_{xxxx} - \kappa u_{xx} = -g(x) + a(x)f(x) := \tilde{g} \end{cases} \tag{2.5}$$

If we let $\omega = u_{xx}$, (2.5) can be rewritten as

$$\omega_{xx} - \kappa\omega = \tilde{g}, \tag{2.6}$$

$$\omega \mid_{x=0,1} = 0. \tag{2.7}$$

Let $\mathcal{N}_\kappa(\tilde{g})$ denote the solution to the Dirichlet problem (2.6)–(2.7), which is well defined since $\kappa \neq j^2\pi^2$, for all $j \in N$. Then we have

$$u_{xx} = \mathcal{N}_\kappa(\tilde{g}),$$

$$u \mid_{x=0,1} = u_{xx} \mid_{x=0,1} = 0$$

From the boundary condition $u \mid_{x=0,1} = 0$, we can see that there exists a point $x_0 \in [0,1]$, such that $u_x(x_0) = 0$. Then we deduce that

$$u(x) = \int_0^x \int_{x_0}^y \mathcal{N}_\kappa(\tilde{g(\tau)}) d\tau dy$$

Using the estimates for $\mathcal{N}_\kappa$ to be proved below, we conclude,

$$\| u_{xx} \|_{L^2} \leq C \| \mathcal{N}_\kappa(\tilde{g(\tau)}) \|_{L^2} \leq C \mid \mathcal{N}_\kappa(\tilde{g(\tau)}) \mid \leq C \| \tilde{g} \|_{L^1}, \tag{2.8}$$

$$\| v \|_{L^2} \leq C \| f \|_{L^2}, \tag{2.9}$$

with C being a positive constant independent of f, g.

Since $a \in L^\infty$, we can also deduce that there exists a positive constant M independent of f, g, such that,

$$\| (u,v)^T \|_{\mathcal{H}} \leq M \| (f,g)^T \|_{\mathcal{H}}$$

Therefore, $\mathcal{A}^{-1} \in \mathcal{L}(\mathcal{H})$ and $0 \in \rho(\mathcal{A})$. Moreover, from the estimates (2.8), (2.9), we have

$$\| v \|_{H^2} \leq C \| (f,g)^T \|_{\mathcal{H}}, \quad \| u \|_{H^4} \leq C \| (f,g)^T \|_{\mathcal{H}}$$

with positive constant C independent of f, g. By the compactness of imbedding operator $H^2(0,1) \hookrightarrow L^2(0,1)$, $H^4(0,1) \hookrightarrow H^2(0,1)$, we can finally conclude that $\mathcal{A}^{-1}$ is also a compact operator. The proof is complete.

□

Remark 2.1. *For the Dirichlet problem appearing in Lemma 2.1, i.e.,*

$$v_{xx} - \kappa v = g$$

$$v \mid_{x=0,1} = 0$$

as well defined if $\sqrt{\kappa}\sinh(\sqrt{\kappa}) \neq 0$, i.e., $\kappa \neq j^2\pi^2$ for $j \in N$.

The Green function for the above equation is

$$G(x,s) = -\frac{1}{\sqrt{\kappa}\sinh(\sqrt{\kappa})}\begin{cases} \sinh(\sqrt{\kappa}x)\sinh(\sqrt{\kappa}(s-1)), & 0 \leq x \leq s \leq 1 \\ \sinh(\sqrt{\kappa}s)\sinh(\sqrt{\kappa}(x-1)), & 0 \leq s \leq x \leq 1 \end{cases}$$

Thus we have the representation of v

$$\begin{aligned} v(x) &= \int_0^1 G(x,s)g(s)ds \\ &= -\frac{1}{\sqrt{\kappa}\sinh(\sqrt{\kappa})}[\sinh(\sqrt{\kappa}(x-1))\int_0^x \sinh(\sqrt{\kappa}s)g(s)ds \\ &+ \sinh(\sqrt{\kappa}x)\int_x^1 \sinh(\sqrt{\kappa}(s-1))g(s)ds] \end{aligned}$$

From the representation of v, in a similar manner to the estimate of $\mathcal{N}_\alpha$ in J.E.Muñoz and R.Racke [9], we have

$$\mid v(x) \mid = \mid \mathcal{N}_\kappa(g) \mid \leq C \| g \|_{L^1} .$$

In the next section we consider the constant coefficient case, i.e., the function $a \equiv \bar{a}$. We write A for the arising constant coefficient operator instead of $\mathcal{A}$ and we shall prove that the SDG property holds for A.

3 The SDG property for constant damping

Now we determine the eigenvalues of A in a way that allows us to determine $\omega_\sigma(A)$ and to estimate the resolvent operators uniformly. Let

$$(A-\lambda)U = 0 \tag{3.1}$$

with $\lambda \in C \setminus \{0\}$ and $U \in D(A)$, then u, v satisfy

$$v = \lambda u, \tag{3.2}$$

$$\lambda v + \bar{a}v - \kappa u_{xx} + u_{xxxx} = 0. \tag{3.3}$$

We reduce the system for (u, v) to a single one for u

$$u_{xxxx} - \kappa u_{xx} + (\lambda^2 + \lambda\bar{a})u = 0, \tag{3.4}$$

$$u\mid_{x=0,1} = u_{xx}\mid_{x=0,1} = 0. \tag{3.5}$$

For (3.4), (3.5), we have a complete orthonormal system of eigenfunctions:
$u_j(x) = \sqrt{2}\sin(\theta_j x)$ with $\theta_j = j\pi$, $j \in N$. Then $\lambda = \lambda_j$ satisfies

$$\lambda^2 + \bar{a}\lambda + \theta_j^4 + \kappa\theta_j^2 = 0,$$

which implies

$$\lambda_{1,2} = \frac{-\bar{a} \pm \sqrt{\bar{a}^2 - 4(\theta_j^4 + \kappa\theta_j^2)}}{2}, \quad \theta_j = j\pi, \quad j \in N \tag{3.6}$$

Then

$$\sigma(A) = \left\{ \frac{-\bar{a} \pm \sqrt{\bar{a}^2 - 4(\theta_j^4 + \kappa\theta_j^2)}}{2}, \quad \theta_j = j\pi, \quad j \in N \right\} \tag{3.7}$$

and

$$\omega_\sigma(A) = \sup\{Re\lambda \mid \lambda \in \sigma(A)\} = \max_{j\in N} Re\left(\frac{-\bar{a} \pm \sqrt{\bar{a}^2 - 4(\theta_j^4 + \kappa\theta_j^2)}}{2}\right) < 0.$$

Next, we shall determine $\omega_0(A)$ and demonstrate the SDG property.

We investigate $\| (\lambda - A)^{-1} \|$ for $Re\lambda > \omega_\sigma$. Let $\lambda \in C$, $Re\lambda > \omega_\sigma + \varepsilon$ for some $\varepsilon > 0$. Consider

$$(\lambda - A)U = F,$$

i.e.,

$$\lambda u - v = f_1,$$

$$\lambda u + \bar{a}v - \kappa u_{xx} + u_{xxxx} = f_2,$$

which can be written as

$$u_{xxxx} - \kappa u_{xx} + (\lambda^2 + \bar{a}\lambda)u = f_2 + \lambda f_1 + \bar{a} f_1 := \tilde{f}, \tag{3.8}$$

$$u \mid_{x=0,1} = u_{xx} \mid_{x=0,1} = 0. \tag{3.9}$$

The boundary condition admits the expansions

$$u(x) = \sum_{j=1}^{\infty} g_j u_j(x), \tag{3.10}$$

with

$$u_j(x) = \sqrt{2}\sin(\theta_j x), \quad \theta_j = j\pi. \tag{3.11}$$

We obtain from (3.8) that

$$g_j \theta_j^4 + \kappa g_j \theta_j^2 + (\lambda^2 + \bar{a}\lambda) g_j = \tilde{f}_j, \tag{3.12}$$

here $\tilde{f}_j$ denote the Fourier coefficients of $\tilde{f}$.

We compute

$$g_j = \frac{\tilde{f}_j}{\theta_j^4 + \kappa \theta_j^2 + (\lambda^2 + \bar{a}\lambda)}. \tag{3.13}$$

Then we need to estimate $\int_0^1 \mid u_{xx} \mid^2 dx$, $\int_0^1 \mid \lambda u \mid^2 dx$ in term of $\| F \|_{\mathcal{H}}^2$.

Hence, we need to prove a bound for

$$I := \frac{\mid \theta_j^2 \mid^2}{\mid \theta_j^4 + \kappa \theta_j^2 + (\lambda^2 + \bar{a}\lambda) \mid^2},$$

and

$$II := \frac{\mid \lambda \mid^2}{\mid \theta_j^4 + \kappa \theta_j^2 + (\lambda^2 + \bar{a}\lambda) \mid^2}.$$

Obviously, the above two terms are bounded for any $\lambda \in C$, $Re\lambda > \omega_\sigma + \varepsilon$ and $\theta_j = j\pi.$, Therefore, we have

$$\int_0^1 \mid u_{xx} \mid^2 dx + \int_0^1 \mid \lambda u \mid^2 dx \le C \| F \|_{\mathcal{H}}^2,$$

with C being a positive constant, which implies

Theorem 3.1. *The SDG property holds for A, i.e.,* $\omega_0(A) = \omega_\sigma(A)$.

4 Exponential stability for indefinite damping

In this section we consider the original equation (2.1)–(2.2), with an indefinite damping $a = a(x)$. It will be shown that the system is exponential stable if $\| a - \bar{a} \|_{L^2}$ is small enough.

Keeping the basic assumption (1.4), i.e.,

$$\bar{a} := \int_0^1 a(x)dx > 0, \tag{4.1}$$

then we have

Theorem 4.1. *Assume $a \in L^\infty(0,1)$, satisfy (4.1), then there is $\tau > 0$ such that if $\| a - \bar{a} \|_{L^2} < \tau$, the equation (2.1)–(2.2) is exponentially stable, that is, the energy*

$$E := \int_0^1 (u_{xx}^2 + u_t^2)dx$$

satisfies

$$\exists \;\; \alpha_0 > 0, \quad C > 0, \quad \forall t > 0 : \quad E(t) \leq Ce^{-2\alpha_0 t}E(0).$$

Proof. Recalling [9], it suffices to show that for sufficiently small $\tau > 0$ and for λ with $Re\lambda \geq \omega_\sigma + \varepsilon$, for some $\varepsilon > 0$ such that $\omega_\sigma + \varepsilon < 0$, $(\lambda - \mathcal{A})U = F$ is uniquely solvable for any $F = (f_1, f_2) \in \mathcal{H}$ and $\| U \|_{\mathcal{H}} \leq C \| F \|_{\mathcal{H}}$ with a constant $C > 0$ may depending on τ and ε. We shall apply a fixed point argument to prove this.

The equation

$$(\lambda - \mathcal{A})U = F$$

is equivalent to

$$(\lambda - A)U = F + (\mathcal{A} - A)U. \tag{4.2}$$

Let $U = (u, v)$ be defined as solution to (4.2), then u, v satisfy

$$\lambda u - v = f_1, \tag{4.3}$$

$$\lambda v + \bar{a}v - \kappa u_{xx} + u_{xxxx} = f_2 + (\bar{a} - a)v. \tag{4.4}$$

(4.3), (4.4) can be written as

$$u_{xxxx} - \kappa u_{xx} + (\lambda^2 + \bar{a}\lambda)u = f_2 + (\lambda + \bar{a})f_1 - (\bar{a} - a(x))f_1 + (\bar{a} - a(x))\lambda u. \tag{4.5}$$

If we define $\hat{f} := f_2 + (\lambda + \bar{a})f_1 - (\bar{a} - a(x))f_1$, (4.5) turns into

$$u_{xxxx} - \kappa u_{xx} + (\lambda^2 + \bar{a}\lambda)u = \hat{f} + (\bar{a} - a(x))\lambda u \tag{4.6}$$

For simple, we let $\alpha := \lambda^2 + \bar{a}\lambda$, then denote $\mathcal{N}_{\kappa,\alpha}$ as the solution to the problem

$$u_{xxxx} - \kappa u_{xx} + \alpha u = g,$$

$$u \mid_{x=0,1} = u_{xx} \mid_{x=0,1} = 0.$$

Thus the solution u to (4.6) can be written as

$$u = \mathcal{N}_{\kappa,\alpha}(\hat{f} + (\bar{a} - a(x))\lambda u).$$

Hence (4.6) turns into

$$u_{xxxx} - \kappa u_{xx} + (\lambda^2 + \bar{a}\lambda)u = \hat{f} + (\bar{a} - a(x))\lambda \cdot \mathcal{N}_{\kappa,\alpha}(\hat{f} + (\bar{a} - a(x))\lambda u).$$

If we define

$$G(\omega) := \mathcal{N}_{\kappa,\alpha}(\hat{f} + (\bar{a} - a(x))\lambda\omega)$$

and consider the mapping

$$P: \quad H^2 \cap H_0^1 \to H^2 \cap H_0^1$$

$$\omega \quad \to \quad u$$

defined as solution u to

$$u_{xxxx} - \kappa u_{xx} + (\lambda^2 + \bar{a}\lambda)u = \hat{f} + (\bar{a} - a(x))\lambda \cdot G(\omega), \tag{4.7}$$

$$u \mid_{x=0,1} = u_{xx} \mid_{x=0,1} = 0. \tag{4.8}$$

In the following we try to prove P has a fixed point u provided $\| \ a(x) - \bar{a} \ \|_{L^2}$ small enough.

Let u be this fixed point, and let

$$\hat{u} := G(u) = \mathcal{N}_{\kappa,\alpha}(\hat{f} + (\bar{a} - a(x))\lambda u),$$

hence

$$\hat{u}_{xxxx} - \kappa \hat{u}_{xx} + (\lambda^2 + \lambda\bar{a})\hat{u} = \hat{f} + (\bar{a} - a(x))\lambda u,$$

$$\hat{u} \mid_{x=0,1} = \hat{u}_{xx} \mid_{x=0,1} = 0$$

Since u is a fixed point of P, we have

$$u_{xxxx} - \kappa u_{xx} + (\lambda^2 + \bar{a}\lambda)u = \hat{f} + (\bar{a} - a(x))\lambda\hat{u}.$$

If we define $\Psi := \hat{u} - u$, then we have

$$\Psi_{xxxx} - \kappa\Psi_{xx} + \alpha\Psi = (\bar{a} - a)\lambda\Psi, \tag{4.9}$$

and

$$\Psi = \mathcal{N}_{\alpha,\kappa}((\bar{a} - a(x))\lambda\Psi). \tag{4.10}$$

In the following we estimate the operator $\mathcal{N}_{\alpha,\kappa}$. In fact, we can rewrite equation

$$u_{xxxx} - \kappa u_{xx} + \alpha u = g$$

into two separate equations as

$$u_{xx} - \kappa_1 u = v, \tag{4.11}$$

$$u \mid_{x=0,1} = 0, \tag{4.12}$$

and

$$v_{xx} - \kappa_2 v = 0, \tag{4.13}$$

$$v \mid_{x=0,1} = 0, \tag{4.14}$$

which are well defined if $\kappa_i \neq j^2\pi^2$ for $i = 1, 2$ and $j \in N$.

We take $v = u_{xx} - \kappa_1 u$ into equation (4.11) to obtain

$$u_{xxxx} - (\kappa_1 + \kappa_2)u_{xx} + \kappa_1\kappa_2 u = g.$$

From the relations

$$\kappa_1 + \kappa_2 = \kappa, \quad \kappa_1\kappa_2 = \alpha,$$

we deduce

$$\kappa_1 = \frac{\kappa + \sqrt{\kappa^2 - 4\alpha}}{2}, \quad \kappa_2 = \frac{\kappa - \sqrt{\kappa^2 - 4\alpha}}{2}$$

If we denote $\mathcal{N}_{\kappa_1}(v)$, $\mathcal{N}_{\kappa_2}(g)$ the solution to (4.11)–(4.12), (4.13)–(4.14) respectively. Then we can obtain the estimate of $\mathcal{N}_{\alpha,\kappa}$ from the estimates of $\mathcal{N}_{\kappa_1}$ and $\mathcal{N}_{\kappa_2}$. Since $v = \mathcal{N}_{\kappa_2}(g)$ and $u = \mathcal{N}_{\kappa_1}(v)$, in a similar manner to Remark 2.1, we have

$$\mid u \mid \leq C_1 \parallel v \parallel_{L^1} \leq C_1 \mid v \mid \leq C_1C_2 \parallel g \parallel_{L^1} \tag{4.15}$$

with positive constants C_1, C_2.

Returning to (4.9), we have

$$\mid \Psi \mid \leq C_1C_2 \parallel (\bar{a} - a(x)\lambda\Psi) \parallel_{L^1} \leq \tilde{C} \parallel \bar{a} - a(x) \parallel_{L^2} \cdot \parallel \Psi \parallel_{L^2} . \tag{4.16}$$

Hence

$$\parallel \Psi \parallel_{L^2} \leq \tilde{C} \parallel \bar{a} - a(x) \parallel_{L^2} \cdot \parallel \Psi \parallel_{L^2}, \tag{4.17}$$

implying $\Psi = 0$ if $\parallel a - \bar{a} \parallel_{L^2} \leq \frac{1}{\tilde{C}}$.

Next we shall prove that P is a contraction mapping provided $\| a-\bar{a} \|_{L^2}$ small enough. For this purpose let

$$u^1 = P\omega^1, \quad u^2 = P\omega^2 \tag{4.18}$$

and

$$u = u^1 - u^2, \quad \omega = \omega^1 - \omega^2. \tag{4.19}$$

Then u, ω satisfy

$$u_{xxxx} - \kappa u_{xx} + (\lambda^2 + \lambda\bar{a})u = (\bar{a} - a(x))\lambda G(\omega), \tag{4.20}$$

with $G(\omega) = \mathcal{N}_{\kappa,\alpha}((\bar{a} - a(x))\lambda\omega)$.

Multiplying (4.20) by $\overline{\lambda u}$ and integrating the result with respect to x yields

$$\bar{\lambda}\int_0^1 | u_{xx} |^2 \, dx + \kappa\bar{\lambda}\int_0^1 | u_x |^2 \, dx + (\lambda + \bar{a})\int_0^1 | \lambda u |^2 \, dx = \int_0^1 (\bar{a} - a(x))\lambda G(\omega)\overline{\lambda u} dx. \tag{4.21}$$

Then we have

$$\begin{aligned} &Re\lambda \cdot (\int_0^1 | u_{xx} |^2 \, dx + \kappa\int_0^1 | u_x |^2 \, dx + \int_0^1 | \lambda u |^2 \, dx) + \bar{a}\int_0^1 | \lambda u |^2 \, dx \\ &= Re\left\{\int_0^1 (\bar{a} - a(x)) \mid \lambda \mid^2 G(\omega)\bar{u} dx\right\} \end{aligned} \tag{4.22}$$

Multiplying (4.20) by $\bar{u}$ and integrating with respect t yields

$$\int_0^1 | u_{xx} |^2 \, dx + \kappa\int_0^1 | u_x |^2 \, dx + (\lambda^2 + \bar{a}\lambda)\int_0^1 | u |^2 \, dx = \int_0^1 (\bar{a} - a(x))\lambda G(\omega)\bar{u} dx. \tag{4.23}$$

Since $0 \in \rho(A)$,

$$\exists\lambda_0 > 0, \quad c_1 > 0, \quad \forall\lambda, \quad | \lambda | \leq \lambda_0 : \quad \lambda \in \rho(\mathcal{A}) \wedge \| (\lambda - \mathcal{A})^{-1} \| \leq c_1$$

hence w.l.o.g we assume $| \lambda | \geq \lambda_0$. Then

$$\int_0^1 | u |^2 \, dx \leq \frac{1}{\lambda_0^2}\int_0^1 | \lambda u |^2 \, dx. \tag{4.24}$$

Combining (4.23) with (4.24) to obtain

$$\int_0^1 | u_{xx} |^2 \, dx + \kappa\int_0^1 | u_x |^2 \, dx \leq \frac{| \lambda |^2 + | \lambda | \bar{a}}{\lambda_0^2}\int_0^1 | \lambda u |^2 \, dx + \int_0^1 (\bar{a} - a(x))\lambda G(\omega)\bar{u} dx. \tag{4.25}$$

Multiplying (4.25) by $\frac{\lambda_0^2}{|\lambda|^2 + |\lambda|\bar{a}} \cdot \frac{\bar{a}}{2}$ and adding up with (4.22) yields there exists γ_0 such that

$$(Re\lambda + \gamma_0)(\int_0^1 | u_{xx} |^2 \, dx + \kappa\int_0^1 | u_x |^2 \, dx + \int_0^1 | \lambda u |^2 \, dx)$$

$$\leq | \lambda |^2 \cdot | \int_0^1 (\bar{a} - a(x))G(\omega)\bar{u}dx | + | \lambda | \cdot | \int_0^1 (\bar{a} - a(x))G(\omega)\bar{u}dx | \quad (4.26)$$

In the following we estimate $G(\omega)$.

Since $G(\omega) = \mathcal{N}_{\kappa,\alpha}((\bar{a} - a(x))\lambda\omega)$, as we have done before, we can decompose $\mathcal{N}_{\kappa,\alpha}$ as $\mathcal{N}_{\kappa_1} \cdot \mathcal{N}_{\kappa_2}$, then we have

$$G(\omega) = \mathcal{N}_{\kappa,\alpha}((\bar{a} - a(x))\lambda\omega) = \mathcal{N}_{\kappa_1}(\mathcal{N}_{\kappa_2}((\bar{a} - a(x))\lambda\omega)) \quad (4.27)$$

with

$$\kappa_i = \frac{\kappa \pm \sqrt{\kappa^2 - 4\alpha}}{2}, \quad i = 1, 2 \;\; and \;\; \alpha = \lambda^2 + \lambda\bar{a} \quad (4.28)$$

Similarly to [9], if we decompose $\lambda = \gamma + i\eta$, $\sqrt{\kappa_1} \cdot \sqrt{\kappa_2} = a_1 + ia_2$ into its real and imaginary part, respectively, we have

$$\exists \;\; \beta > 0, \;\; \forall \;\; \lambda, \;\; | a_1 | \geq \beta, \;\; | a_2 | = o(\eta), \;\; (\eta \to \infty) \quad (4.29)$$

with some negative d_0 and some sufficiently large, but fixed d_1.

From the estimate in Remark 2.1, we conclude

$$\begin{aligned} | \lambda G(\omega) | &= | \lambda | \cdot | \mathcal{N}_{\kappa,\alpha}((\bar{a} - a(x))\lambda\omega) | \\ &\leq C \parallel (\bar{a} - a(x))\lambda\omega) \parallel_{L^1} \\ &\leq C \parallel \bar{a} - a(x) \parallel_{L^2} \cdot \parallel \lambda\omega \parallel_{L^2} \end{aligned}$$

with C being a positive constant.

Thus, the right hand of (4.26) satisfy

$$R \leq C \parallel \bar{a} - a(x) \parallel_{L^2} \cdot \parallel \lambda\omega \parallel_{L^2} \cdot \parallel \lambda u \parallel_{L^2} . \quad (4.30)$$

If we define

$$\parallel u \parallel_\lambda := \int_0^1 | u_{xx} |^2 \, dx + \int_0^1 | \lambda u |^2 \, dx,$$

from (4.26), (4.30), we conclude for $Re\lambda > -\gamma_0$

$$\parallel u \parallel_\lambda \leq C \parallel \bar{a} - a(x) \parallel_{L^2} \cdot \parallel \omega \parallel_\lambda \leq d \parallel \omega \parallel_\lambda \quad (4.31)$$

for some $d < 1$ provided $\parallel \bar{a} - a(x) \parallel_{L^2}$ is small enough. Thus, P is a contraction mapping.

Finally, we prove there exists unique fixed point U of P which is the unique solution $U = (u, v)$ to $(\lambda - \mathcal{A})U = F$.

Combining the definition of the norm $\parallel \cdot \parallel_{\mathcal{H}}$, and $\parallel \cdot \parallel_\lambda$ with the relation form $(\lambda - \mathcal{A})U = F$, i.e., $\lambda u - v = f_1$ implies

$$\parallel (u, v)^T \parallel_{\mathcal{H}} \leq \parallel u \parallel_\lambda + \parallel F \parallel_{\mathcal{H}} \quad (4.32)$$

and

$$\| u \|_\lambda \leq \| (u,v)^T \|_{\mathcal{H}} + \| F \|_{\mathcal{H}} \tag{4.33}$$

Let $\tilde{u}$ to be the solution with $\omega = 0$ in (4.7), i.e.,

$$\tilde{u}_{xxxx} - \kappa \tilde{u}_{xx} + (\lambda^2 + \bar{a}\lambda)\tilde{u} = \hat{f} + (\bar{a} - a(x))\lambda \cdot G(0), \tag{4.34}$$

$$\tilde{u} \mid_{x=0,1} = \tilde{u}_{xx} \mid_{x=0,1} = 0. \tag{4.35}$$

Similarly as the above estimates, we obtain

$$\| \tilde{u} \|_\lambda \leq C \| F \|_{\mathcal{H}} .$$

Since

$$\begin{aligned} \| u \|_\lambda - \| \tilde{u} \|_\lambda &\leq \| u - \tilde{u} \|_\lambda = \| Pu - P\tilde{u} \|_\lambda \\ &\leq d \| u - \tilde{u} \|_\lambda \\ &\leq d \| u \|_\lambda + C \| F \|_{\mathcal{H}} . \end{aligned}$$

Thus

$$\begin{aligned} \| u \|_\lambda &\leq \frac{1}{1-d} \| \tilde{u} \| + \frac{c}{1-d} \| F \|_{\mathcal{H}} \\ &\leq C \| F \|_{\mathcal{H}} \end{aligned} \tag{4.36}$$

Finally, we have

$$\| (u,v)^T \|_{\mathcal{H}} \leq C \| F \|_{\mathcal{H}} \tag{4.37}$$

Thus we have prove that for $Re\lambda > -\gamma_0$ and $\kappa_i \neq j^2\pi^2$, we have $\lambda \in \rho(\mathcal{A})$ and the norm of $(\lambda - \mathcal{A})^{-1}$ is uniformly bounded. The proof is complete.

□

5 Global existence for the nonlinear equation

We now return to the nonlinear equation (1.1)–(1.3) assuming again the positivity of the mean value (1.4). The local well-posedness can be obtained easily, in the following we focus on the global existence of the solution. We assume that $\| a - \bar{a} \|_{L^2}$ small enough which assured the exponential stability of the linear equation as given in Theorem 4.1.

The local existence can be obtained by applying the contraction mapping theorem, we can omit the details here.

Theorem 5.1. *Suppose $(u_0, u_1) \in H^4(0,1) \times H^2(0,1)$ are given functions that satisfy the compatibility conditions, then there is $t^* = t^*(\| u_0 \|_{H^4}, \| u_1 \|_{H^2}) > 0$, such that (1.1)–(1.3) has a unique local solution*

$$u \in C^2([0,t^*], L^2(0,1)) \cap C^1([0,t^*], H^2(0,1)) \cap C([0,t^*], H^4(0,1)). \tag{5.1}$$

In the following we shall prove the uniform boundedness of the solution.

Lemma 5.1. *For any $t \in [0,T]$, the following estimates hold.*

$$\| u_t \|_{L^2} \leq C_T, \quad \| u_{xx} \|_{L^2} \leq C_T, \quad \| u_x \|_{L^\infty} \leq C_T. \tag{5.2}$$

Here C_T is a positive constant which may depend on T and the initial data.

Proof. Multiplying (1.1) by u_t and integrating with respect x yields

$$\frac{1}{2}\frac{d}{dt}\int_0^1 u_t^2 dx + \frac{1}{2}\frac{d}{dt}\int_0^1 u_{xx}^2 dx + \int_0^1 a(x) u_t^2 dx + \frac{d}{dt}\int_0^1 F(u_x) dx = 0 \tag{5.3}$$

with $F'(u_x) = f(u_x)$.

Using Young's inequality, we have

$$F(u_x) \geq C_1 u_x^6 - C_2.$$

We obtain from (5.3) that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_0^1 u_t^2 dx + \frac{1}{2}\frac{d}{dt}\int_0^1 u_{xx}^2 dx + \frac{d}{dt}\int_0^1 F(u_x) dx &= -\int_0^1 a(x) u_t^2 dx \\ &\leq \| a(x) \|_{L^\infty} \int_0^1 u_t^2 dx. \end{aligned}$$

Using Gronwall's inequality, we have

$$\| u_t \|_{L^2} \leq C_T, \quad \| u_{xx} \|_{L^2} \leq C_T, \quad \| u_x \|_{L^6} \leq C_T.$$

Applying Poincare's inequality and the boundary conditions yields

$$\| u_x \|_{L^\infty} \leq C_T.$$

The proof is complete.

□

Now we turn to the higher-order norm estimates of the solution.

Lemma 5.2. *For any $t \in [0,T]$, the following estimates hold.*

$$\| u_t \|_{H^2} \leq C_T, \quad \| u_{xx} \|_{H^2} \leq C_T. \tag{5.4}$$

Proof. First we rewrite (1.1)–(1.3) as a first-order system for

$$U := (u, u_t)^T.$$

Then U satisfies

$$U_t - \mathcal{A}U = F(u_x, u_{xx}),$$

$$U(t=0) = U_0,$$

with

$$\mathcal{A}U = \begin{pmatrix} v \\ -a(x)v + \kappa u_{xx} - u_{xxxx} \end{pmatrix}$$

as the above, and

$$F(u_x, u_{xx}) := (0, f(u_x)_x - \kappa u_{xx})^T,$$

$$U_0 := (u_0, u_1)^T.$$

The defined operator $\mathcal{A}$ generates a C_0-semigroup, for $F = 0$, i.e., the linear system as we have studied in Section 2, the solution U is given by

$$U(t) = e^{\mathcal{A}t} U_0$$

and the solution to (1.1)–(1.3) satisfies

$$U(t) = e^{\mathcal{A}t} U_0 + \int_0^t e^{(t-r)\mathcal{A}} F(u_x, u_{xx})(r) dr \tag{5.5}$$

From Section 4 we obtain that U as solution of the linear system satisfies

$$\| U(t) \|_{\mathcal{H}} \leq C e^{-\alpha_0 t} \| U_0 \|_{\mathcal{H}} .$$

Differentiating (2.1) with respect to t one and then twice, we derive higher norm estimates as

$$\| U \|_{H^s} \leq C_s e^{-\alpha_0 t} \| U_0 \|_{H^s}, \quad s = 1, 2.$$

Using the representation (5.5), we can estimate

$$\begin{aligned} \| U \|_{H^2} &\leq \| e^{t\mathcal{A}} U_0 \|_{H^2} + \int_0^t \| e^{(t-r)\mathcal{A}} F(u_x, u_{xx}) \|_{H^2} \, dr \\ &\leq C_2 e^{-\alpha_0 t} \| U_0 \|_{H^2} + C_2 \int_0^t e^{-\alpha_0 (t-r)} \| F(u_x, u_{xx}) \|_{H^2} \, dr \end{aligned}$$

$$\leq \quad C_2 e^{-\alpha_0 t} \parallel U_0 \parallel_{H^2} + C_2 \int_0^t e^{-\alpha_0 (t-r)} \parallel (f'(u_x) - \kappa) u_{xx} \parallel_{H^2} dr.$$

Using the estimate $\parallel u_x \parallel_{L^\infty} \leq C_T$ obtained in lemma 5.1, we derive

$$e^{\alpha_0 t} \parallel U \parallel_{H^2} \leq C_2 \parallel U_0 \parallel_{H^2} + C_T \int_0^t e^{\alpha_0 r} \parallel U \parallel_{H^2} dr.$$

Using Gronwall's inequality again, we finally have

$$e^{\alpha_0 t} \parallel U \parallel_{H^2} \leq C_2 \parallel U_0 \parallel_{H^2} e^{C^T t}$$

and

$$\parallel U \parallel_{H^2} \leq C_2 \parallel U_0 \parallel_{H^2} e^{(C_T - \alpha_0) t} \leq C_T$$

Combing with the equation and the boundary conditions, the estimates (5.4) following immediately.

□

The global existence follows by the usual continuation argument.

Theorem 5.2. *Let the assumptions (1.4) satisfied and* $\parallel a - \bar{a} \parallel_{L^2} \leq \tau$ *small enough. Then there is a unique solution to (1.1)–(1.3) satisfying*

$$u \in C^2([0, \infty), L^2(0,1)) \cap C^1([0, \infty), H^2(0,1)) \cap C([0, \infty), H^4(0,1)). \qquad (5.6)$$

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