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OPTIMAL L_p-L_q-REGULARITY FOR PARABOLIC PROBLEMS WITH INHOMOGENEOUS BOUNDARY DATA

ROBERT DENK, MATTHIAS HIEBER, AND JAN PRÜSS

Abstract. In this paper we investigate vector-valued parabolic initial boundary value problems $(\mathcal{A}(t,x,D), \mathcal{B}_j(t,x,D))$ subject to general boundary conditions in domains G in $\mathbb{R}^n$ with compact C^{2m}-boundary. The top-order coefficients of $\mathcal{A}$ are assumed to be continuous. We characterize optimal L^p-L^q-regularity for the solution of such problems in terms of the data. We also prove that the normal ellipticity condition on $\mathcal{A}$ and the Lopatinskii–Shapiro condition on $(\mathcal{A}, \mathcal{B}_1, \ldots, \mathcal{B}_m)$ are necessary for these L^p-L^q-estimates. As a byproduct of the techniques being introduced we obtain new trace and extension results for Sobolev spaces of mixed order and a characterization of Triebel-Lizorkin spaces by boundary data.

1. Introduction

One might think that the theory of linear parbolic boundary value problems subject to general boundary conditions was completed by the fundamental work of Agmon, Douglas, Nirenberg [2], [1] and Solonnikov [30], [3] about 40 years ago. Note, however, that recent developments in free boundary value problems and other nonlinear parabolic problems ask for optimal L^p-L^q space-time estimates for the corresponding linearized equation, see [13]. Observe that these estimates are not provided by the results mentioned above. In fact, consider the boundary value problem

$$\begin{aligned} \partial_t u + \mathcal{A}(t,x,D)u &= f(t,x),\ t \in J,\ x \in G, \\ \mathcal{B}_j(t,x,D)u &= g_j(t,x),\ t \in J,\ x \in \partial G, \\ u(0,x) &= u_0(x),\ x \in G. \end{aligned} \tag{1.1}$$

Here, J is a finite time interval, $G \subset \mathbb{R}^n$ is a domain and the coefficents of $\mathcal{A}$ and $\mathcal{B}_j$ are $\mathcal{B}(E)$-valued, where E is a so-called UMD-Banach space. Then by the results in [2] or [30] we obtain a-priori estimates for the solution of the corresponding autonomous *elliptic* problem. These estimates yield also results for the associated semigroups but space-time estimates for the solution of the non-autonmous *parabolic* problem (1.1) of the form

$$\int_0^T \|u'(t)\|_q^p \,\mathrm{dt} + \int_0^T \|A_B\, u(t)\|_q^p \,\mathrm{dt} \ \le C \int_0^T \|f(t)\|_q^p \,\mathrm{dt} \tag{1.2}$$

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for $g_j = u_0 = 0$ or even estimates for the inhomogeneous case, as needed in many applications, are not implied. Here A_B denotes the realization of the boundary value problem.

In a first step towards this direction, the authors proved in [11] optimal L^p-L^q estimates for autonomous boundary value problems with homogenous boundary conditions. It was even shown in [12] that the operators associated to such boundary value problems admit a bounded H^∞-calculus on $L^p(\Omega)$. For previous results dealing with the situation of $\mathbb{R}^n$ or with special boundary conditions, mainly Dirichlet conditions, we refer to [14], [28], [6], [17], [22], [15], [10], [16], [18], [4], [5], [7], [34] and the references in [11].

In this paper we give a complete answer to the above problem, i.e. we consider the non-autonomous and parabolic situation and characterize optimal L^p-L^q estimates for this problem with general boundary conditions by its data. More precisely, we characterize optimal L^p-L^q estimates for boundary value problems of the form (1.1) in terms of the data f, g_i and u_0. Here optimal L^p-L^q-estimates mean that the problem (1.1) is uniquely solvable with solution

$$u \in H^1_p(J; L_q(G; E)) \cap L_p(J; H^{2m}_q(G; E)),$$

depending continuously on the data f, g_j and u_0. Note that so far only very few results are know for the case of inhomogeneous boundary conditions. We also remark that is was proved by Kalton and Lancien [23] that for arbitrary operators estimates of the above kind do not hold, in general. It is an open problem to decide whether the above estimate (1.2) is true for any differential operator.

As in [11] and [12] we choose a rather general setting, i.e. $\mathcal{A}$ and $\mathcal{B}_j$ are differential operators with operator-valued coefficients, i.e. including in particular the case of $N \times N$ systems. For a precise formulation ot the result we refer to Theorem 2.1 and Theorem 2.3.

The proofs use Sobolev spaces of mixed order with respect to t and x and related trace and extension results. These results show in particular that the assumptions on the data given in Theorem 2.1 are necessary. Moreover, our result on optimal L^p-L^q-estimates gives a new characterization of Triebel-Lizorkin spaces by boundary values.

The second main result of the paper, Theorem 2.2 shows that the ellipticity conditions used above are necessary for optimal L^p-L^q estimates. In particular, we show that the generalized Lopatinskii-Shapiro condition introduced in [11] is necessary for optimal L^p-L^q estimates and cannot be replaced by a weaker condition. It seems that a result of this kind is not even known in the classical case of a scalar equation or $N \times N$ systems.

This paper is organized as follows. In Section 2 we introduce basic notation and give the precise formulations of our assumptions and main results. In Section 3 we derive trace and extension results for vector-valued mixed order Sobolev spaces and consider pointwise multipliers in these spaces. The proof of optimal regularity, i.e.

of unique solvability of (1.1) in the space stated above, is contained in Section 4. In Section 5 we prove the necessity of the ellipticity conditions. The final Section 6 deals with the case $p \neq q$.

2. Preliminaries and main results

In this section we first introduce the notation used throughout this paper. In particular, we introduce sectorial operators, the concept of $\mathcal{R}$-boundedness and operator-valued Fourier multipliers. Then we state the precise ellipticity and smoothness assumptions needed for the formulation of the main results of this work.

For Banach spaces X and Y we denote the space of all bounded linear operators from X to Y by $\mathcal{B}(X, Y)$; as usual, $\mathcal{B}(X) := \mathcal{B}(X, X)$. The spectrum of a linear (possibly unbounded) operator A is denoted by $\sigma(A)$, its resolvent set by $\rho(A)$. The domain, range and kernel of an operator A are denoted by $D(A)$, $R(A)$ and $N(A)$, respectively. By C and c we denote unspecified constants which may differ from line to line but which are independent of the free variables.

A closed linear operator A in a complex Banach space X is called *sectorial* if $\overline{D(A)} = X$, $\overline{R(A)} = X$, $(-\infty, 0) \subset \rho(A)$ and

$$t|(t + A)^{-1}| \leq C, \quad t > 0,$$

for some constant $C < \infty$. The class of sectorial operators in X is denoted by $\mathcal{S}(X)$. If $A \in \mathcal{S}(X)$, then $\rho(-A) \supset \Sigma_\theta$ for some $\theta > 0$, where

$$\Sigma_\theta := \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| < \theta\}.$$

The *spectral angle* ϕ_A of a sectorial operator A is defined by

$$\phi_A := \inf\{\phi : \rho(-A) \supset \Sigma_{\pi-\phi}, \sup_{\lambda \in \Sigma_{\pi-\phi}} |\lambda(\lambda + A)^{-1}| < \infty\}.$$

Obviously, we have $\phi_A \in [0, \pi)$ and $\phi_A \geq \sup\{|\arg \lambda| : \lambda \in \sigma(A)\}$.

An important subclass of $\mathcal{S}(X)$ is the class $\mathcal{BIP}(X)$ of all operators with bounded imaginary power. Here, a sectorial operator A in X belongs to $\mathcal{BIP}(X)$ if $A^{is} \in \mathcal{B}(X)$ for every $s \in \mathbb{R}$ and there is a constant $C < \infty$ such that $|A^{is}| \leq C$ for all $|s| \leq 1$. An operator A admits bounded imaginary powers if and only if $\{A^{is} : s \in \mathbb{R}\}$ forms a strongly continuous group of bounded linear operators in X. In this case, the growth bound of this group, defined by

$$\theta_A = \limsup_{|s| \to \infty} \frac{1}{|s|} \log |A^{is}|,$$

is called the *power angle* of the operator A. For the power angle the inequality $\theta_A \geq \phi_A$ holds.

In the following, we will make use of the concept of $\mathcal{R}$-boundedness which plays a prominent rôle in the theory of Fourier multipliers. For complex Banach spaces X and Y, a family of operators $\mathcal{T} \subset \mathcal{B}(X, Y)$ is called *$\mathcal{R}$-bounded* if there is a constant $C > 0$ and $p \in [1, \infty)$ such that for each $N \in \mathbb{N}$, $T_j \in \mathcal{T}$, $x_j \in X$ and for all independent, symmetric, $\{+1,-1\}$-valued random variables ε_j on a probability space $(\Omega, \mathcal{M}, \mu)$ the inequality

$$\Big|\sum_{j=1}^{N} \varepsilon_j T_j x_j\Big|_{L_p(\Omega;Y)} \le C \Big|\sum_{j=1}^{N} \varepsilon_j x_j\Big|_{L_p(\Omega;X)}$$

holds. The smallest such C is called the $\mathcal{R}$-bound of $\mathcal{T}$ and is denoted by $\mathcal{R}(\mathcal{T})$. The notion of $\mathcal{R}$-boundedness does not depend on p, but the $\mathcal{R}$-bound does. As a general reference for $\mathcal{R}$-boundedness, we refer to [35], [11] or [26].

The concept of $\mathcal{R}$-boundedness in connection with sectorial operators immediately leads to the notion of $\mathcal{R}$-sectorial operators. It was shown in [9] that an operator $A \in \mathcal{BIP}(X)$ is $\mathcal{R}$-sectorial. The class of $\mathcal{R}$-sectorial operators will be denoted by $\mathcal{R}S(X)$, and if $A \in \mathcal{R}S(X)$ then ϕ_A^R denotes its $\mathcal{R}$-angle.

The concept of $\mathcal{R}$-boundedness has shown its importance in particular in view of the relation to vector-valued Fourier multiplier theorems in Banach spaces of class $\mathcal{HT}$. Here a complex Banach space X is said to be of class $\mathcal{HT}$ if the Hilbert transform is bounded on $L_p(\mathbb{R};X)$ for some (and then all) $p \in (1,\infty)$. The Hilbert transform H is defined by

$$Hf := \frac{1}{\pi} PV\Big(\frac{1}{t}\Big) * f$$

for $f \in \mathcal{S}(\mathbb{R};X)$, the Schwartz space of rapidly decreasing X-valued functions. A well-known theorem states that the set of all Banach spaces of class $\mathcal{HT}$ coincides with the class of all Banach spaces with the *unconditional martingale difference property* (UMD spaces), see e.g. [8] and [26].

Let X and Y be Banach spaces. We denote by $\mathcal{D}(\mathbb{R};X)$ the space of X-valued C^∞-functions with compact support and by $\mathcal{D}'(\mathbb{R};X) := \mathcal{B}(\mathcal{D}(\mathbb{R}), X)$ the space of X-valued distributions. The space $\mathcal{S}'(\mathbb{R};X)$ of X-valued tempered distributions is defined in an analogous way. For a function $M \in L_{1,loc}(\mathbb{R};\mathcal{B}(X,Y))$ we define the operator $T_M : \mathcal{F}^{-1}\mathcal{D}(\mathbb{R};X) \to \mathcal{S}'(\mathbb{R};X)$ by

$$T_M\phi := \mathcal{F}^{-1} M F\phi \quad \text{for all } F\phi \in \mathcal{D}(\mathbb{R};X). \tag{2.1}$$

Here $\mathcal{F}$ stands for the Fourier transform. Since $\mathcal{F}^{-1}\mathcal{D}(\mathbb{R};X)$ is dense in $L_p(\mathbb{R};X)$, the operator T_M acting on $L_p(\mathbb{R};X)$ is linear and densely defined. Conditions on M which guarantee the boundedness of T_M as an operator in $L_p(\mathbb{R};X)$ are given by a result of Weis [35] which uses $\mathcal{R}$-boundedness, see Theorem 4.1 below. This result generalizes Mikhlin's classical Fourier multiplier theorem to the vector-valued case. For vector-valued Fourier multiplier theorems in several variables, see also [20], [31], [11], [26].

Let us now turn attention to vector-valued parabolic initial value problems of the form

$$\begin{aligned} \partial_t u + \mathcal{A}(t,x,D)u &= f(t,x), \quad t \in J,\ x \in G, \\ \mathcal{B}_j(t,x,D)u &= g_j(t,x), \quad t \in J,\ x \in \partial G,\ j = 1,\dots,m, \\ u(0,x) &= u_0(x), \quad x \in G. \end{aligned} \tag{2.2}$$

Here $J = [0,T]$ for some $T > 0$, and $G \subset \mathbb{R}^n$ is an open connected set with boundary ∂G. The operator $\mathcal{A}(t,x,D)$ is a partial differential operator of order $2m$ and $\mathcal{B}_j(t,x,D)$ are partial boundary differential operators of order $m_j < 2m$. More precisely, let E be a Banach space and $m, m_1, \dots, m_m$ be natural numbers with $m_j < 2m$ for $j = 1, \dots, m$, and let

$$\mathcal{A}(t,x,D) = \sum_{|\alpha|\le 2m} a_\alpha(t,x) D^\alpha,$$
$$\mathcal{B}_j(t,x,D) = \sum_{|\beta|\le m_j} b_{j\beta}(t,x) D^\beta$$

where a_α and $b_{j\beta}$ are $\mathcal{B}(E)$-valued variable coefficients. Here and in the following, we use the standard multi-index notation $D^\alpha = (-i)^{|\alpha|}\partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n}$ with $\partial_{x_i} = \frac{\partial}{\partial x_i}$ and, later on, $\xi^\alpha = \xi_1^{\alpha_1}\cdot\ldots\cdot\xi_n^{\alpha_n}$. Note that the boundary operators have to be interpreted as $\mathcal{B}_j(t,x,D)u = \sum_{|\beta|\le m_j} b_{j\beta}(t,x)\gamma_0 D^\beta u$ where $\gamma_0 v$ denotes the trace of the function v on the boundary ∂G (given sufficient smoothness of v and ∂G); we will often omit γ_0 in the notation. In particular, the coefficients $b_{j\beta}$ are defined on ∂G only.

We are interested in maximal L_p-regularity of (2.2) for $1 < p < \infty$ which means that we are looking for solutions in the class

$$u \in H_p^1(J; L_p(G;E)) \cap L_p(J; H_p^{2m}(G;E)). \tag{2.3}$$

Here H_p^k stands for the standard (vector-valued) Sobolev space of integer order. Trace theorems stated below will show that the given data have to satisfy the following conditions:

(D) Assumptions on the Data

- **(i)** $f \in L_p(J \times G; E)$,
- **(ii)** $g_j \in W_p^{\kappa_j}(J; L_p(\partial G;E)) \cap L_p(J; W_p^{2m\kappa_j}(\partial G;E))$ with $\kappa_j := (2m - m_j - 1/p)/(2m)$
- **(iii)** $u_0 \in B_{pp}^{2m(1-1/p)}(G;E)$,
- **(iv)** If $\kappa_j > 1/p$ then $B_j(0,x)u_0(x) = g_j(0,x)$ for $x \in \partial G$.

In (ii), $W_p^{\kappa_j}$ denotes the vector-valued Sobolev–Slobodeckii space of non-integer order κ_j, and in (iii) the notation B_{pq}^κ stands for the Besov space of order κ with indices p, q. For more information on scalar-valued functions spaces we refer to [32] and [33]. There is few information about vector-valued versions of these spaces. Note that $B_{pp}^s = W_p^s$ in case s is non-integer. Condition (iv) is a compatibility condition for the data.

One aim of the present paper is to find conditions on the operators $\mathcal{A}$ and $\mathcal{B}_j$ that guarantee – given (D) – that (2.2) has a unique solution u in the space (2.3). Such conditions naturally can be divided into two types:

• smoothness assumptions on the coefficients and

• ellipticity assumptions.

We start with the ellipticity assumptions. To this end, we denote the *principal part* of a partial differential operator $\mathcal{A}$ by $\mathcal{A}_\#$. The outer normal of ∂G at $x \in \partial G$ is denoted by $\nu(x)$, and we set $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. We then introduce the conditions (E) and (LS) as follows.

(E) (Ellipticity of the interior symbol)
For all $t \in J$, $x \in \overline{G}$ and $\xi \in \mathbb{R}^n$, $|\xi| = 1$, we have

$$\sigma(\mathcal{A}_\#(t, x, \xi)) \subset \mathbb{C}_+,$$

i.e. $\mathcal{A}(t, x, D)$ is *normally elliptic*. If G is unbounded this condition is also imposed at $x = \infty$.

(LS) (Lopatinskii–Shapiro condition)
For all $t \in J$, $x \in \partial G$, all $\xi \in \mathbb{R}^n$ with $\xi \cdot \nu(x) = 0$, all $h \in E^m$ and all $\lambda \in \overline{\mathbb{C}_+}$ with $|\xi| + |\lambda| \neq 0$, the ordinary differential equation system in $\mathbb{R}_+ := (0, \infty)$

$$\begin{aligned} \lambda v(y) + \mathcal{A}_\#(t, x, \xi + i\nu(x)\partial_y)v(y) = 0, \quad y > 0, \\ \mathcal{B}_{j,\#}(t, x, \xi + i\nu(x)\partial_y)v(0) = h_j \quad j = 1, \dots, m \end{aligned} \tag{2.4}$$

admits a unique solution $v \in C_0(\mathbb{R}_+; E)$.

Note that for fixed $t \in J$ and $x \in \overline{G}$ the symbol $\mathcal{A}_\#(t, x, \xi)$ is a homogeneous $\mathcal{B}(E)$-valued polynomial in ξ of degree $2m$. As usual, this symbol is called *elliptic*, too, if condition (E) holds. The notion of ellipticity in (E) can trivially be generalized to closed sectors in the complex plane instead of $\overline{\mathbb{C}_+}$ and is then called parameter-ellipticity. For operator-valued symbols, parameter-ellipticity was introduced by Amann in [5] for the case of the whole space and by the authors in [11] for boundary value problems.

Observe also that in the scalar case the Lopatinskii–Shapiro condition essentially is of algebraic nature, as it can be reformulated as a condition on the roots of a homogeneous polynomial. The same is true if the functions have values in a finite-dimensional vector space. In the case of an infinite-dimensional state space as considered here, (LS) is a condition on the spectrum of operator pencils and is no longer of algebraic nature, as will be made precise at the beginning of Section 4.

Now we turn to smoothness assumptions on the coefficients of $\mathcal{A}$ and $\mathcal{B}_j$.

(SD) There are $r_k, s_k \geq p$ with $\frac{1}{s_k} + \frac{n}{2mr_k} < 1 - \frac{k}{2m}$ such that

$$\begin{aligned} &a_\alpha \in L_{s_k}(J; (L_{r_k} + L_\infty)(G; \mathcal{B}(E))), \quad |\alpha| = k < 2m, \\ &a_\alpha \in C(J \times \overline{G}; \mathcal{B}(E)), \quad |\alpha| = 2m. \end{aligned}$$

If G is unbounded, the limits $a_\alpha(t, \infty) := \lim_{|x|\to\infty, x\in G} a_\alpha(t, x)$ exists uniformly w.r.t. $t \in J$, $|\alpha| = 2m$.

(SB) There are $s_{jk}, r_{jk} \geq p$ with $\frac{1}{s_{jk}} + \frac{n-1}{2mr_{jk}} < \kappa_j + \frac{m_j - k}{2m}$ such that

$$b_{j\beta} \in W^{\kappa_j}_{s_{jk}}(J; L_{r_{jk}}(\partial G; \mathcal{B}(E))) \cap L_{s_{jk}}(J; W^{2m\kappa_j}_{r_{jk}}(\partial G; E))), \ |\beta| = k \leq m_j.$$

The first main result of this paper shows that under the assumptions made so far the initial boundary value problem (2.2) admits maximal regularity. More precisely, the following holds.

Theorem 2.1. *Let $G \subset \mathbb{R}^n$ be open and connected with compact boundary ∂G of class C^{2m}. Let the Banach space E be of class $\mathcal{HT}$, suppose assumptions* (E), (LS), (SD) *and* (SB) *are satisfied, and let $1 < p < \infty$. Then problem* (2.2) *admits a unique solution*

$$u \in H^1_p(J; L_p(G; E)) \cap L_p(J; H^{2m}_p(G; E))$$

if and only if the data are subject to conditions (D).

Now let us take a closer look on the ellipticity conditions. As we have seen, they guarantee maximal L_p-regularity. But in fact they are necessary, too, as we will show in Section 4. In particular, this means that for maximal regularity conditions (E) and (LS) are the "correct" conditions. More precisely, our second main result reads as follows.

Theorem 2.2. *Let E be of class $\mathcal{HT}$ and G be an open connected domain in $\mathbb{R}^n$ with boundary of class C^{2m}. Let* (SD) *and* (SB) *be satisfied, and let $1 < p < \infty$. Assume that there exists a constant $C > 0$ such that for all data f, g_j and u_0 satisfying* (D) *the initial boundary value problem* (2.2) *has a unique solution*

$$u \in H^1_p(J; L_p(G; E)) \cap L_p(J; H^{2m}_p(G; E)),$$

and the inequality

$$\begin{aligned} &|\partial_t u|_{L_p(J_a \times G; E)} + |D^{2m} u|_{L_p(J_a \times G; E)} \\ &\leq C\Big[|f|_{L_p(J_a \times G; E)} + \sum_{j=1}^{m} (|g_j|_{W^{\kappa_j}_p(J_a; L_p(\partial G; E))} + |g_j|_{L_p(J_a; W^{2m\kappa_j}_p(\partial G; E))})\Big] \end{aligned} \tag{2.5}$$

holds for every interval $J_a = [0, a] \subset J$. Then $\mathcal{A}(t, x, D)$ is normally elliptic in $J \times \overline{G}$, and the Lopatinskii–Shapiro condition (LS) *holds in $J \times \partial G$.*

The third main result of this paper considers the case where $p \neq q$. To this end, we have to slightly modify our smoothness assumptions on the coefficients. More precisely, we introduce the following two conditions

(SD1) There are $s_k \geq p$, $r_k \geq q$ with $\frac{1}{s_k} + \frac{n}{2mr_k} < 1 - \frac{k}{2m}$ such that

$$\begin{aligned} &a_\alpha \in L_{s_k}(J; (L_{r_k} + L_\infty)(G; \mathcal{B}(E))), \quad |\alpha| = k < 2m, \\ &a_\alpha \in C(J \times \overline{G}; \mathcal{B}(E)), \quad |\alpha| = 2m. \end{aligned}$$

If G is unbounded, the limits $a_\alpha(t,\infty) := \lim_{|x|\to\infty, x\in G} a_\alpha(t,x)$ exists uniformly w.r.t. $t \in J$, $|\alpha| = 2m$.

(SB1) There are $s_{jk} \geq p$, $r_{jk} \geq q$ with $\frac{1}{s_{jk}} + \frac{n-1}{2mr_{jk}} < \kappa_j + \frac{m_j-k}{2m}$ such that

$$b_{j\beta} \in W^{\kappa_j}_{s_{jk}}(J; L_{r_{jk}}(\partial G; \mathcal{B}(E))) \cap L_{s_{jk}}(J; W^{2m\kappa_j}_{r_{jk}}(\partial G; E))), \ |\beta| = k \leq m_j.$$

Then the following holds where F^{κ}_{pq} stands for the vector-valued Triebel-Lizorkin spaces.

Theorem 2.3. *Let $G \subset \mathbb{R}^n$ be open and connected with compact boundary ∂G of class C^{2m}. Let the Banach space E be of class $\mathcal{HT}$, suppose assumptions* (E), (LS), (SD1) *and* (SB1) *are satisfied, and let $1 < p, q < \infty$. Then problem* (2.2) *admits a unique solution*

$$u \in H^1_p(J; L_q(G; E)) \cap L_p(J; H^{2m}_q(G; E))$$

if and only if the data are subject to the following conditions

- **(i)** $f \in L_p(J; L_q(G; E))$,
- **(ii)** $g_j \in F^{\kappa_j}_{pq}(J; L_q(\partial G; E)) \cap L_p(J; B^{2m\kappa_j}_{qq}(\partial G; E))$ *with* $\kappa_j := (2m - m_j - 1/q)/(2m)$,
- **(iii)** $u_0 \in B^{2m(1-1/p)}_{qp}(G; E)$,
- **(iv)** *If $\kappa_j > 1/q$ then $B_j(0,x)u_0(x) = g_j(0,x)$ for $x \in \partial G$.*

The proofs of the above Theorems 2.1, 2.2 and 2.3 are fairly involved and use several ingredients which are of independent interest.

First we investigate vector-valued Sobolev spaces of mixed order with respect to time and space derivatives. In particular, we discuss questions of traces and extensions in such Sobolev spaces. Results of this type seem to be new and will be proved in the following section 3.

We also need explicit representations for the solution of (2.2) in the case of model problems, i.e. problems in the whole space $\mathbb{R}^n$ and in the half-space $\mathbb{R}^n_+ := \{x = (x', x_n) \in \mathbb{R}^n : x_n > 0\}$ with constant coefficients and without lower-order terms. These representations follow from [11]. In particular, we make use of the kernel estimates derived in [11].

Finally, localization in time leads to autonomous problems and the techniques of localization and coordinate transformation in space, together with perturbation arguments, are employed to obtain the stated results in full generality.

3. Mixed order Sobolev spaces: the vector-valued case

In this section we derive results on vector-valued Sobolev spaces with mixed order of differentiation. The spaces we will consider consist of distributions $u(t,x)$ depending on $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$, and the order of differentiability will be by a factor $2m$ larger with respect to x than with respect to t. Such spaces naturally appear in the context of parabolic problems. The aim of this section is to obtain results on traces, extension operators and pointwise multipliers in such spaces. Being of interest for themselves, the facts collected in this section will be used later.

Our approach is mostly semigroup based; therefore, we first state some known facts on sectorial operators. In the following, the real interpolation spaces between two Banach spaces X and Z will be denoted by $(X, Z)_{\alpha,p}$. For a closed operator A in X with domain $D(A)$ we set $D_A(\alpha, p) = (X, D_A)_{\alpha,p}$ where D_A means the Banach space $D(A)$ equipped with the graph norm of A. We start with a result on the real interpolation spaces due to Grisvard [19].

Lemma 3.1. *Let A and B be sectorial operators in a Banach space X which commute in the resolvent sense and let $\alpha \in (0, 1)$ and $p \in [1,\infty]$. Then*

$$(X, D(A) \cap D(B))_{\alpha,p} = (X, D(A))_{\alpha,p} \cap (X, D(B))_{\alpha,p}.$$

The next result is due to Sobolevskii [29] and is known as the *mixed derivative theorem.*

Proposition 3.2. *Suppose A and B are sectorial linear operators in a Banach space X with spectral angles $\phi_A + \phi_B < \pi$, which commute and are coercively positive, i.e. $A+tB$ with natural domain $D(A+tB) = D(A)\cap D(B)$ is closed for each $t > 0$ and there is a constant $M > 0$ such that*

$$|Ax|_X + t|Bx|_X \le M|Ax + tBx|_X \quad \text{for all } x \in D(A) \cap D(B),\ t > 0.$$

Then there is a constant $C > 0$ such that

$$|A^\alpha B^{1-\alpha}x|_X \le C|Ax + Bx|_X \quad \text{for all } x \in D(A) \cap D(B),\ \alpha \in [0, 1].$$

Consider next an abstract evolution equation

$$\partial_t u + Au = f,\ t > 0, \quad u(0) = u_0 \tag{3.1}$$

in a Banach space X. The following well-known result deals with maximal L_p-regularity of (3.1).

Proposition 3.3. *Suppose X is a Banach space of class $\mathcal{HT}$, $1 < p < \infty$, and let $A \in \mathcal{RS}(X)$ be invertible with $\mathcal{R}$-angle $\phi^R_A < \pi/2$. Then (3.1) has precisely one solution in $H^1_p(\mathbb{R}_+; X) \cap L_p(\mathbb{R}_+; D_A)$ if and only if*

$$f \in L_p(\mathbb{R}_+; X) \text{ and } u_0 \in D_A(1 - 1/p, p).$$

Now we turn to Sobolev spaces. Let $J = [0, T]$ for some $T > 0$, $G \subset \mathbb{R}^n$ be open and E be a Banach space. For $p, q > 1$ and $\tau, \mu \in [0,\infty)$ we set

$$W^{\tau,\mu}_{p,q}(J \times G; E) := W^\tau_p(J; L_q(G; E)) \cap L_p(J; W^\mu_q(G; E)).$$

For $\tau = \mu = 0$ we also write $L_{p,q}(J \times G; E) = L_p(J; L_q(G; E))$ and denote the norm in $L_{p,q}(J \times G; E)$ by $|\cdot|_{L_{p,q}(J\times G;E)}$ or simply $|\cdot|_{p,q}$ if J, G and E are clear from the context. In an analogous way, Sobolev spaces of mixed order are defined if G is replaced by a sufficiently smooth closed manifold; in particular, we will consider Sobolev spaces on the boundary ∂G. Recall that the trace of a (sufficiently smooth) function u defined in G onto the boundary ∂G is denoted by $\gamma_0 u$.

The trace and extension results will be formulated in the half-space $\mathbb{R}^n_+$ with boundary $\mathbb{R}^{n-1}$. Note that by standard techniques of localization, these results carry over to domains $G \subset \mathbb{R}^n$ with sufficiently smooth boundary ∂G. The following extension result is a generalization from the scalar to the vector-valued case. The proof given in [32] carries over to the vector-valued case.

Lemma 3.4. *Let $p, q > 1$ and $s \geq 0$. There exist bounded linear extension operators*

$$\begin{aligned} &E_0 : B^s_{pq}(\mathbb{R}^n_+; E) \to B^s_{pq}(\mathbb{R}^n; E) \quad \text{and} \\ &E_0 : W^s_p(\mathbb{R}^n_+; E) \to W^s_p(\mathbb{R}^n; E). \end{aligned}$$

We will also need extension operators as in Lemma 3.4 both with respect to $x \in \mathbb{R}^n_+$ and to $t \in \mathbb{R}_+$. In a trivial way, the operator from the lemma above induces a bounded linear extension operator

$$E_x : W^{\tau,2m\tau}_{pp}(\mathbb{R}_+ \times \mathbb{R}^n_+; E) \to W^{\tau,2m\tau}_{pp}(\mathbb{R}_+ \times \mathbb{R}^n; E),$$

being a right-inverse of the restriction R_x onto $x \in \mathbb{R}^n_+$, and a bounded linear extension operator

$$E_t : W^{\tau,2m\tau}_{pp}(\mathbb{R}_+ \times \mathbb{R}^n_+; E) \to W^{\tau,2m\tau}_{pp}(\mathbb{R} \times \mathbb{R}^n_+; E),$$

being a right-inverse of the restriction R_t onto $t \in \mathbb{R}_+$. For simplicity of notation, we do not indicate the underlying Sobolev or Besov space for E_y and E_t. The spaces in which they act will be clear from the context. There are several possible choices of E_0 and, consequently, of E_x and E_t. Throughout the following, we will consider arbitrary but fixed extension operators E_x and E_t.

Next, consider the operator

$$L = \mu + \partial_t + (-\Delta_{n-1})^m,$$

where Δ_{n-1} stands for the Laplacian in $\mathbb{R}^{n-1}$, and $\mu > 0$. This operator in $L_p(\mathbb{R} \times \mathbb{R}^{n-1}; E)$, endowed with its natural domain $D(L) = W^{1,2m}_{pp}(\mathbb{R} \times \mathbb{R}^{n-1}; E)$ and modifications of it will be important in the following as a shifting operator in the scale of Sobolev spaces. The parameter-dependent version of L is given by $L_\lambda = \lambda + (-\Delta_{n-1})^m$ in $L_p(\mathbb{R}^{n-1}; E)$ with domain $D(L_\lambda) = W^{2m}_p(\mathbb{R}^{n-1}; E)$. Note that $L \in \mathcal{BIP}(L_p(\mathbb{R} \times \mathbb{R}^{n-1}; E))$ and $L_\lambda \in \mathcal{BIP}(L_p(\mathbb{R}^{n-1}; E))$ for $\lambda \in \mathbb{C} \setminus (-\infty, 0]$. By Lemma 3.1, we have $D_L(\theta, p) = W^{\theta,2m\theta}_{pp}(\mathbb{R} \times \mathbb{R}^{n-1}; E)$ for $\theta \in [0, 1]$.

Lemma 3.5. *Let E be of class $\mathcal{HT}$, $p > 1$ and $2m\tau \in \{1, \ldots, 2m\}$. Then the trace with respect to x,*

$$\gamma_0 : W^{\tau,2m\tau}_{pp}(\mathbb{R}_+ \times \mathbb{R}^n_+; E) \to W^{\tau-1/(2mp),2m\tau-1/p}_{pp}(\mathbb{R}_+ \times \mathbb{R}^{n-1}; E)$$

is continuous and surjective. Moreover, the operator $(F_0 g)(t,x,y) := e^{-yL^{1/2m}} g(t,x)$ *with* L *being defined above, gives rise to a bounded linear operator*

$$F_0 \colon W_{pp}^{\tau-1/(2mp),2m\tau-1/p}(\mathbb{R}_+ \times \mathbb{R}^{n-1}; E) \to W_{pp}^{\tau,2m\tau}(\mathbb{R}_+ \times \mathbb{R}^n_+; E)$$

which is a right inverse to γ_0.

Proof. Let $u \in W_{pp}^{\tau,2m\tau}(\mathbb{R}_+ \times \mathbb{R}^n_+; E)$. By Lemma 3.4, we may assume that u is defined for all $t \in \mathbb{R}$. Consider u as a function in $y \in \mathbb{R}_+$ whose values are functions in (t,x). Then, setting $k = 2m\tau$,

$$u \in W_p^k(\mathbb{R}_+; L_p(\mathbb{R} \times \mathbb{R}^{n-1})) \cap L_p\big(\mathbb{R}_+; W_p^\tau(\mathbb{R}; L_p(\mathbb{R}^{n-1})) \cap L_p(\mathbb{R}; W_p^k(\mathbb{R}^{n-1}))\big)$$

where now the first variable is y. This implies

$$u \in W_p^k(\mathbb{R}_+; L_p(\mathbb{R} \times \mathbb{R}^{n-1})) \cap L_p(\mathbb{R}_+; D_{L^{k/2m}}).$$

Set

$$v := L^{(k-1)/(2m)} u \in V := W_p^1(\mathbb{R}_+; L_p(\mathbb{R} \times \mathbb{R}^{n-1}) \cap L_p(\mathbb{R}_+; D_{L^{1/2m}});$$

then $f := (\partial_y + L^{1/2m})v \in L_p(\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^{n-1})$. L is sectorial, invertible and admits bounded imaginary powers with power angle $\pi/2$, hence $L^{1/2m}$ has the same properties with power angle $\pi/4m < \pi/2$. Therefore by Theorem 3.3 the Cauchy problem

$$\begin{aligned} (\partial_y + L^{1/2m})w &= f \quad (y > 0), \\ w|_{y=0} &= 0 \end{aligned}$$

has a unique solution $w \in V$. Thus, $z = v - w \in V$ is a solution of

$$\begin{aligned} (\partial_y + L^{1/2m})z &= 0 \quad (y > 0), \\ z|_{y=0} &= v|_{y=0}, \end{aligned}$$

which yields

$$z(y) = e^{-L^{1/2m} y} v|_{y=0}.$$

By Theorem 3.3, we have $v|_{y=0} = (L^{(k-1)/(2m)} u)|_{y=0} \in D_{L^{1/2m}}(1-1/p, p)$ which implies

$$\begin{aligned} u|_{y=0} \in D_{L^{1/2m}}(k-1/p,p) &= D_L((k-1/p)/(2m),p) \\ &= W_p^{(k-1/p)/(2m)}(\mathbb{R}; L_p(\mathbb{R}^{n-1})) \cap L_p(\mathbb{R}; W_p^{k-1/p}(\mathbb{R}^{n-1})). \end{aligned}$$

Restricting $u|_{y=0}$ on $t > 0$, we obtain the statement on γ_0 of the Lemma. Moreover, the above considerations show that an extension of $g \in D_L(k-1/p,p)$ is given by

$$e^{-yL^{1/2m}} g \in W_p^k(\mathbb{R}_+; L_p(\mathbb{R} \times \mathbb{R}^{n-1})) \cap L_p(\mathbb{R}_+; D_L(k,p)).$$

Obviously $\gamma_0 R_x e^{-yL^{1/2m}} E_x g = g$. □

Remark 3.6. *In the situation of Lemma 3.5, assume that*

$$g \in W_{pp}^{\tau-1/(2mp),2m\tau-1/p}(\mathbb{R}_+ \times \mathbb{R}^{n-1}; E)$$

satisfies $g|_{t=0} = 0$ *for* $\tau > 1/p$. *Then the extension of* g *by zero belongs to the space* $W_{pp}^{\tau-1/(2mp),2m\tau-1/p}(\mathbb{R} \times \mathbb{R}^{n-1}; E)$. *Thus* $e^{-yL^{1/2m}}E_0 g = e^{-yL_0^{1/2m}} g$ *in* $\mathbb{R}_+ \times \mathbb{R}^{n-1}$ *where the operator* L_0 *in* $L_p(\mathbb{R}_+ \times \mathbb{R}^{n-1})$ *is defined by*

$$\begin{aligned} D(L_0) &:= \{u \in W_{pp}^{1,2m}(\mathbb{R}_+ \times \mathbb{R}^{n-1}; E) : u|_{t=0} = 0\}, \\ L_0 &:= \mu + \partial_t + (-\Delta_{n-1})^m. \end{aligned}$$

Lemma 3.7. *Let* E *be of class* $\mathcal{HT}$ *and* $p \in (1, \infty)$. *Then the trace with respect to* t, $u(t,x) \mapsto u(0,x)$, *is continuous and surjective from*

$$W_{pp}^{1,2m}(\mathbb{R}_+ \times \mathbb{R}_+^n; E) \text{ onto } B_{pp}^{2m(1-1/p)}(\mathbb{R}_+^n; E),$$

and there exists a bounded linear right-inverse of this trace.

Proof. In an analogous way as in the proof of Lemma 3.5, we first extend $u \in W_{pp}^{1,2m}(\mathbb{R}_+ \times \mathbb{R}_+^n; E)$ to a function $u \in W_{pp}^{1,2m}(\mathbb{R}_+ \times \mathbb{R}^n; E)$. With $\mu > 0$ define the operator $A := \mu + (-\Delta)^m$ with domain $D(A) := W_p^{2m}(\mathbb{R}^n; E)$. Then $A \in \mathcal{BIP}(L_p(\mathbb{R}^n; E))$ is invertible. From Theorem 3.3 we know that

$$(\partial_t + A)v = f \quad (t > 0), \qquad v|_{t=0} = 0$$

has a unique solution $v \in W_{pp}^{1,2m}(\mathbb{R}_+ \times \mathbb{R}^n; E)$, where we have set $f := (\partial_t + A)u$. Thus $z = u - v$ is a solution of

$$(\partial_t + A)z = 0 \quad (t > 0), \; z|_{t=0} = u|_{t=0}.$$

By Theorem 3.3 we have

$$u|_{t=0} \in D_A(1 - 1/p, p) = (L_p(\mathbb{R}^n; E), W_p^{2m}(\mathbb{R}^n; E))_{1-1/p,p} = B_{pp}^{2m(1-1/p)}(\mathbb{R}^n; E).$$

In the same way as before, the right inverse of the trace with respect to t is given by $u_0 \mapsto R_y e^{-tA} E_y u_0$. □

Lemma 3.8. *Let* $p > 1$ *and* $2m\kappa \in \{0, \dots, 2m\}$. *Then each of the operators* $\partial_t^{1/2m}$ *and* ∂_{x_j}, $j = 1, \dots, m$, *are continuous from*

$$W_{pp}^{\kappa,2m\kappa}(\mathbb{R}_+ \times \mathbb{R}_+^n; E) \text{ to } W_{pp}^{\kappa-1/2m,2m\kappa-1}(\mathbb{R}_+ \times \mathbb{R}_+^n; E).$$

Proof. After extension to $\mathbb{R} \times \mathbb{R}^n$, this follows from the fact that $\partial_t^{1/2m}(\mu + \partial_t + (-\Delta_n)^m)^{-1/2m}$ and $\partial_{x_j}(\mu + \partial_t + (-\Delta_n)^m)^{-1/2m}$ are bounded operators due to the mixed derivative theorem and to boundedness of the Riesz transforms. □

Lemma 3.9. *Let* $p, r, s > 1$ *and* $\kappa > \frac{1}{s} + \frac{n}{2mr}$. *Then*

$$W_{sr}^{\kappa,2m\kappa}(\mathbb{R}_+ \times \mathbb{R}^n; E) \hookrightarrow BUC(\overline{\mathbb{R}}_+ \times \mathbb{R}^n; E).$$

Proof. By the mixed derivative theorem we have

$$W^{\kappa,2m\kappa}_{sr}(\mathbb{R}_+ \times \mathbb{R}^n; E) \subset W^{\kappa(1-\theta)}_{s}(\mathbb{R}_+; W^{2m\kappa\theta}_{r}(\mathbb{R}^n)),$$

where $\theta \in [0,1]$ is arbitrary. Choosing θ in such a way that

$$\kappa(1-\theta) > 1/s, \quad 2m\kappa\theta > n/r,$$

which means $1 - 1/s\kappa > \theta > n/2m\kappa r$, we obtain the desired embedding. □

The following inequality of Gagliardo–Nirenberg type will be useful to treat lower order perturbations both for function defined in G and on ∂G.

Lemma 3.10. *Let $p > 1$, $r, x \geq p$ and $\sigma > \kappa > 0$. Define r' and s' by $\frac{1}{r} + \frac{1}{r'} = \frac{1}{s} + \frac{1}{s'} = \frac{1}{p}$. If $\kappa + \frac{n}{2mr} + \frac{1}{s} < \sigma$, then*

$$|u|_{L_{s'}(\mathbb{R}; W^{2m\kappa}_{r'}(\mathbb{R}^n;E))} \leq C|u|^{\mu}_{L_p(\mathbb{R}; W^{2m\sigma}_p(\mathbb{R}^n;E))} \cdot |u|^{1/s\sigma}_{W^{\sigma}_p(\mathbb{R}; L_p(\mathbb{R}^n;E))} \cdot |u|^{1-\mu-1/s\sigma}_{L_p(\mathbb{R}\times\mathbb{R}^n;E)}$$

holds with a constant $C > 0$ where $\mu := (\kappa + \frac{n}{2mr})/\sigma$. In particular, for every $\varepsilon > 0$ there exists a $C_\varepsilon > 0$ such that

$$|u|_{L_{s'}(\mathbb{R}; W^{2m\kappa}_{r'}(\mathbb{R}^n;E))} \leq \varepsilon |u|_{W^{\sigma,2m\sigma}_{pp}(\mathbb{R}\times\mathbb{R}^n;E)} + C_\varepsilon |u|_{L_p(\mathbb{R}\times\mathbb{R}^n;E)}.$$

Proof. For fixed t, we use the Gagliardo–Nirenberg inequality to see that

$$|u(t,\cdot)|_{W^{2m\kappa}_{r'}(\mathbb{R}^n;E)} \leq C|u(t,\cdot)|^{\mu}_{W^{2m\sigma}_p(\mathbb{R}^n;E)} |u(t,\cdot)|^{1-\mu}_{L_p(\mathbb{R}^n;E)}.$$

Taking the $L_{s'}$-norm with respect to t and applying Hölder's inequality, we get

$$\begin{aligned} |u|_{L_{s'}(\mathbb{R}; W^{2m\kappa}_{r'}(\mathbb{R}^n;E))} &\leq C \Big| |u(t,\cdot)|^{\mu}_{W^{2m\sigma}_p(\mathbb{R}^n;E)} \Big|_{L_{p/\mu}(\mathbb{R})} \cdot \Big| |u(t,\cdot)|^{1-\mu}_{L_p(\mathbb{R}^n;E)} \Big|_{L_{s''}(\mathbb{R})} \\ &= C|u|^{\mu}_{L_p(\mathbb{R}; W^{2m\sigma}_p(\mathbb{R}^n;E))} \cdot |u|^{1-\mu}_{L_{(1-\mu)s}(L_p(\mathbb{R}^n;E))} \end{aligned} \tag{3.2}$$

with $\frac{1}{s''} + \frac{\mu}{p} = \frac{1}{s'}$. The Gagliardo–Nirenberg inequality with respect to t now gives

$$|u|_{L_{(1-\mu)s}(L_p(\mathbb{R}^n;E))} \leq C|u|^{\nu}_{W^{\sigma}_p(\mathbb{R}; L_p(\mathbb{R}^n;E))} \cdot |u|^{1-\nu}_{L_p(\mathbb{R}; L_p(\mathbb{R}^n;E))}$$

with

$$\nu = \Big(\frac{1}{p} - \frac{1}{(1-\mu)s''}\Big) = \frac{1}{\sigma s(1-\mu)}\ .$$

Inserting the last estimate into (3.2) leads to the first statement of the Lemma. The second inequality of the Lemma is an immediate consequence of the fact that all exponents are in $(0,1)$ due to the condition $\kappa + \frac{n}{2mr} + \frac{1}{s} < \sigma$. □

We finally add some remarks on coordinate transformations, assuming that G is a domain in $\mathbb{R}^n$ with boundary ∂G of class C^{2m}. Let $x_0 \in \partial G$ and choose local coordinates corresponding to x_0. Recall that local coordinates are obtained from the original ones by a shift and rotation such that $x_0 = 0$ and the positive x_n-axis has the direction of the interior normal to ∂G at x_0. Then there exists an open neighbourhood $U = U_1 \times U_2$ of x_0 with $U_1 \subset \mathbb{R}^{n-1}$ and $U_2 \subset \mathbb{R}$ open and a function $h \in C^{2m}(\overline{U_1})$ such that $G \cap U = \{x \in U : x_n > h(x')\}$. Here we have set $x' = (x_1, \ldots, x_{n-1})$.

Defining

$$g(x) := \begin{pmatrix} x' \\ x_n - h(x') \end{pmatrix},$$

we obtain a C^{2m}-diffeomorphism $g\colon U \to \mathbb{R}^n$ such that $G \cap U = \{x \in U : g_n(x) > 0\}$ and $G \cap \partial U = \{x \in \mathbb{R}^n : g(x_n) = 0\}$. We set $\widetilde{U} = g(U)$. Extending $h \in C^{2m}(\overline{U_1})$ to a function $\widetilde{h} \in C^{2m}(\mathbb{R}^{n-1})$ with compact support, we obtain an extension of g to a C^{2m}-diffeomorphism $\widetilde{g} : G_{x_0} \to \mathbb{R}^n_+$ where

$$G_{x_0} := \{x \in \mathbb{R}^n : x_n > \widetilde{h}(x')\}.$$

It is easily seen that we may assume that the Jacobian $Dg(x_0)$ equals the unit matrix. For a function $u\colon U \cap G \to E$ we define the push-forward $\widetilde{u} = \mathcal{G}u$ defined on $\widetilde{U} \cap \mathbb{R}^n_+$ by $\widetilde{u} := u \circ g^{-1}$. The linear transformation $\mathcal{G}$ induces isomorphisms $\mathcal{G}^{(p)}\colon H^j_p(U \cap G; E) \to H^j_p(\widetilde{U} \cap \mathbb{R}^n_+; E)$ for $p \in [1, \infty]$. Extending $\mathcal{G}$ in the trivial way to functions depending on (t, x), we obtain isomorphisms

$$\mathcal{G}^{(p,p,\kappa)}\colon W^{\kappa,2m\kappa}_{p,p}(J \times (U \cap G); E) \to W^{\kappa,2m\kappa}_{p,p}(J \times (\widetilde{U} \cap \mathbb{R}^n_+); E)$$

for $p \in (1, \infty)$ and $2m\kappa \in \{0, \ldots, 2m\}$. Moreover, taking the trace on the boundary ∂G, we obtain isomorphisms

$$\mathcal{G}^{(p,p,\kappa,\partial G)}\colon W^{\kappa-1/2mp,\kappa-1/p}_{p,p}(J \times (U \cap \partial G); E) \to W^{\kappa-1/2mp,\kappa-1/p}_{p,p}(J \times (\widetilde{U} \cap \mathbb{R}^{n-1}); E)$$

for $p \in (1, \infty)$ and $2m\kappa \in \{1, \ldots, 2m\}$.

The differential operator $\mathcal{A}(t, x, D)$ is transformed into the operator $\mathcal{A}^g(y, D) := \mathcal{G}\mathcal{A}(t, x, D)\mathcal{G}^{-1}$. In the same way we obtain the transformed boundary operators $\mathcal{B}^g_j(t, y, D)$.

Note that the smoothness assumptions (SD) and (SB) are satisfied for $\mathcal{A}$ and $\mathcal{B}_j$ iff they are satisfied for the transformed operators $\mathcal{A}^g$ and $\mathcal{B}^g_j$, respectively. Moreover, the ellipticity conditions (E) and (LS) for $\mathcal{A}$ and $\mathcal{B}_j$ at some point $(t_0, x_0) \in J \times \overline{G}$ are satisfied iff conditions (E) and (LS) hold for the transformed problem $(\mathcal{A}^g, \mathcal{B}^g_1, \ldots, \mathcal{B}^g_m)$ at the point $(t_0, g(x_0))$. Note that $Dg(x_0) = I$ implies that for the principal part we have

$$\mathcal{A}_\#(t, x_0, \xi) = \mathcal{A}^g_\#(t, g(x_0), \xi) \quad (t \in J, \xi \in \mathbb{R}^n).$$

Finally, let us remark that the coefficients $\widetilde{a}_\alpha$ and $\widetilde{b}_{j\beta}$ of the transformed operators $\mathcal{A}^g$ and $\mathcal{B}^g_j$, respectively, can be extended to the whole half-space $\mathbb{R}^n_+$ and $\mathbb{R}^{n-1}$, respectively. It is possible to define an extension which preserves smoothness properties (SD) and (SB). We refer to [11], Section 8.2, for details.

4. Proof of maximal regularity

In this section we prove Theorem 2.1. As already mentioned, the proof will be based on kernel estimates for the solution operators and on the vector-valued version of Mikhlin's multiplier theorem. We therefore start with a formulation of the multiplier result due to [35]; see also [11], Theorem 3.19.

Theorem 4.1 (Weis)**.** *Suppose that X and Y are Banach spaces of class $\mathcal{HT}$ and let $1 < p < \infty$. Assume that $M \in C^1(\mathbb{R} \setminus \{0\}; \mathcal{B}(X,Y))$ satisfies the following conditions*

(i) $\mathcal{R}(\{M(\rho) : \rho \in \mathbb{R} \setminus \{0\}\}) =: \kappa_0 < \infty$;
(ii) $\mathcal{R}(\{\rho M'(\rho) : \rho \in \mathbb{R} \setminus \{0\}\}) =: \kappa_1 < \infty$.

Then the operator T_M defined above is bounded from $L_p(\mathbb{R}; X)$ to $L_p(\mathbb{R}; Y)$ with norm

$$|T|_{\mathcal{B}(L_p(\mathbb{R};X),L_p(\mathbb{R};Y))} \le C(\kappa_0 + \kappa_1),$$

where the constant $C > 0$ depends only on p, X, Y.

Following the standard approach in elliptic theory, we start with model problems in the whole space $\mathbb{R}^n$ and in the half space $\mathbb{R}^n_+$. Throughout this section, let E be a Banach space of class $\mathcal{HT}$ and $p \in (1,\infty)$.

Lemma 4.2. *Let $\mathcal{A}(D) = \sum_{|\alpha|=2m} a_\alpha D^\alpha$ be normally elliptic and $\mu > 0$. Then the Cauchy problem*

$$\begin{aligned} (\mu + \partial_t + \mathcal{A}(D))u &= f \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^n, \\ u|_{t=0} &= u_0 \quad \text{in } \mathbb{R}^n \end{aligned} \tag{4.1}$$

has a unique solution $u \in W^{1,2m}_{pp}(\mathbb{R}_+ \times \mathbb{R}^n; E)$ if and only if

$$f \in L_p(\mathbb{R}_+ \times \mathbb{R}^n; E) \text{ and } u_0 \in B^{2m(1-1/p)}_{pp}(\mathbb{R}^n; E).$$

The solution depends continuously on f and u_0.

Proof. Define the operator A by $D(A) = W^{2m}_p(\mathbb{R}^n; E)$ and $Au = \mu u + \mathcal{A}(D)u$. By [11], Theorem 5.5, the operator A admits a H_∞-calculus and therefore $A \in \mathcal{BIP}(L_p(\mathbb{R}^n; E))$, and it is invertible since $\mu > 0$. Applying Theorem 3.3 we see that (4.1) has a unique solution $u \in W^{1,2m}_{pp}(\mathbb{R}_+ \times \mathbb{R}^n_+; E)$ if and only if $f \in L_p(\mathbb{R}_+ \times \mathbb{R}^n_+; E)$ and $u_0 \in D_A(1-1/p,p)$. Because of

$$D_A(1-1/p,p) = (L_p(\mathbb{R}^n; E), W^{2m}_p(\mathbb{R}^n; E))_{1-1/p,p} = B^{2m(1-1/p)}_{pp}(\mathbb{R}^n; E),$$

the proof is complete. □

Next, we investigate the model problem in the half space. The main part of the proof of Theorem 2.1 is to solve the following initial boundary value problem.

$$\begin{aligned} (\mu+\partial_t+\mathcal{A}(D))u &= 0 \quad \text{in } \mathbb{R}_+\times\mathbb{R}^n_+,\\ \mathcal{B}_j(D)u &= g_j \quad \text{in } \mathbb{R}_+\times\mathbb{R}^{n-1},\ j=1,\dots,m, \\ u(0,x) &= 0 \quad \text{in } \mathbb{R}^n_+. \end{aligned} \tag{4.2}$$

Here $\mathcal{A}(D) = \sum_{|\alpha|=2m} a_\alpha D^\alpha$ and $\mathcal{B}_j(D) = \sum_{|\beta|=m_j} b_j^\beta D^\beta$. The corresponding parameter-dependent boundary value problem is given by

$$\begin{aligned} (\lambda+\mathcal{A}(D))u &= 0 \quad \text{in } \mathbb{R}^n_+,\\ \mathcal{B}_j(D)u &= g_j \quad \text{in } \mathbb{R}^{n-1}.\ j=1,\dots,m, \end{aligned} \tag{4.3}$$

Throughout the following, we will write $x=(x',y)\in\mathbb{R}^n_+$ with $x'\in\mathbb{R}^{n-1}$ and $y>0$.

Lemma 4.3. *Let $(\mathcal{A}(D),\mathcal{B}_1(D),\dots,\mathcal{B}_m(D))$ satisfy* (E) *and* (LS). *Let $\phi>\phi_A$. Then for every $g_j\in W_p^{2m\kappa_j}(\mathbb{R}^{n-1})$, $j=1,\dots,m$, and $\lambda\in\overline{\mathbb{C}_+}\setminus\{0\}$ the problem* (4.3) *has a unique solution $v\in W_p^{2m}(\mathbb{R}^n_+)$ which is given by*

$$v=\sum_{j=1}^m T_\lambda^j h_j$$

with $h_j = L_\lambda^{(2m-m_j)/(2m)} e^{-yL_\lambda^{1/2m}} g_j$. The operator T_λ^j in $L_p(\mathbb{R}^n_+)$ has the form

$$(T_\lambda^j h_j)(x',y) = \int_0^\infty\int_{\mathbb{R}^{n-1}} T_j(\lambda,x'-\tilde{x}',y+\tilde{y})h_j(\tilde{x}',\tilde{y})d\tilde{x}'d\tilde{y}.$$

Here the kernel $(T_1(\lambda,x,y),\dots,T_m(\lambda,x,y))$ is the first row of the $2m\times m$-dimensional $\mathcal{B}(E)$-valued matrix

$$T(\lambda,x,y) = -\frac{1}{(2\pi)^{n-1}}\int_{\mathbb{R}^{n-1}} e^{ix'\xi'}\frac{1+iA_0(b,\sigma)}{\rho^{2m-1}}e^{i\rho A_0(b,\sigma)y}M(b,\sigma)d\xi' \tag{4.4}$$

where $\rho=(\lambda+|\xi'|^{2m})^{1/2m}$, $b=\xi'/\rho$ and $\sigma=\lambda/\rho$. Moreover, $A_0(b,\sigma): E^{2m}\to E^{2m}$ and $M_0(b,\sigma): E^m\to E^{2m}$ are holomorphic jointly in b and σ.

Proof. The explicit description of the solution operator is essentially contained in [11], Section 6.4. We only note that in [11] the scaling parameter ρ is defined as $\rho=(|\lambda|^{1/m}+|\xi'|^2)^{1/2}$, a difference which does not affect the following considerations. Moreover, in [11] the solution is given in the form

$$\begin{aligned}(T_\lambda^j h_j)(x',y) = \int_0^\infty\int_{\mathbb{R}^{n-1}} \partial_{\tilde{y}}\Big[&K_\lambda^j(x'-\tilde{x}',y+\tilde{y})h_j(\tilde{x}',\tilde{y})\\ &+K_\lambda^j(x'-\tilde{x}',y+\tilde{y})h_j(\tilde{x}',\tilde{y})\Big]d\tilde{x}'d\tilde{y}.\end{aligned}$$

Here $(K_\lambda^1,\dots,K_\lambda^m)$ is the first row of the matrix

$$K(\lambda,x,y) = -\frac{1}{(2\pi)^{n-1}}\int_{\mathbb{R}^{n-1}} e^{ix'\xi'}\rho^{-2m}e^{i\rho A_0(b,\sigma)y}M(b,\sigma)d\xi'.$$

In [11], the function h_j was an arbitrary extension of $L_\lambda^{(2m-m_j)/(2m)} g_j$. But now we can make use of our advantageous choice $e^{-yL_\lambda^{1/2m}}$ of the extension operator which gives

$$\partial_{\tilde{y}} h(\tilde{x}', \tilde{y}) = -L_\lambda^{1/2m} h(\tilde{x}', \tilde{y}).$$

As $L_\lambda^{1/2m}$ has the symbol $(\lambda + |\xi'|^{2m})^{1/2m} = \rho$, we obtain the above representation with kernel $T(\lambda, x, y)$. □

Lemma 4.4. *Let* $(\mathcal{A}(D), \mathcal{B}_1(D), \dots, \mathcal{B}_m(D))$ *satisfy* (E) *and* (LS). *Then*

$$\mathcal{R}\big\{\lambda^{1-\frac{|\alpha|}{2m}} D^\alpha T_\lambda^j : \lambda \in \overline{\mathbb{C}_+} \setminus \{0\}, |\alpha| \le 2m, j = 1, \dots, m\big\} < \infty,$$
$$\mathcal{R}\big\{\lambda^{2-\frac{|\alpha|}{2m}} D^\alpha \partial_\lambda T_\lambda^j : \lambda \in \overline{\mathbb{C}_+} \setminus \{0\}, |\alpha| \le 2m, j = 1, \dots, m\big\} < \infty.$$

Proof. The proof follows the lines of the proof of [11], Proposition 7.6. We start from the representation (4.4) and will estimate $\lambda^{1-|\alpha|/2m} D^\alpha T(\lambda, x', y)(|x'| + y)^n$. For this, we choose a rotation Q in $\mathbb{R}^{n-1}$ such that $Qx = (|x'|, 0, \dots, 0)$ and write $Q\xi' = (a, r\varphi)$ with $a \in \mathbb{R}$, $r > 0$ and $\varphi \in \mathbb{R}^{n-2}$ with $|\varphi| = 1$. For $\alpha = (\alpha', \alpha_n)$ we obtain

$$\begin{aligned}
&\lambda^{1-\frac{|\alpha|}{2m}} D^\alpha T(\lambda, x', y)(|x'| + y)^n \\
&= -\frac{1}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} e^{ix'\xi'} \frac{1 + A_0(b,\sigma)}{\rho^{2m-1}} e^{i\rho A_0(b,\sigma)} M(b,\sigma) \\
&\qquad\qquad \times \lambda^{1-\frac{|\alpha|}{2m}} (\xi')^{\alpha'} (\rho A_0(b,\sigma))^{\alpha_n} (|x'| + y)^n d\xi' \qquad (4.5) \\
&= -\frac{1}{(2\pi)^{n-1}} \int_{S^{n-2}} \int_0^\infty r^{n-3} \int_{\mathbb{R}} e^{i|x'|a} \frac{1 + A_0(b,\sigma)}{\rho^{2m-1}} e^{i\rho A_0(b,\sigma)} M(b,\sigma) \\
&\qquad\qquad \times \lambda^{1-\frac{|\alpha|}{2m}} (\xi')^{\alpha'} (\rho A_0(b,\sigma))^{\alpha_n} (|x'| + y)^n da\,dr\,d\varphi
\end{aligned}$$

where now ρ, b, σ are functions of a, r. For small $\varepsilon > 0$ we shift the path of integration for a to the contour $\tau \mapsto \tau + i\varepsilon(r + |\tau| + |\lambda|^{1/2m})$, $\tau \in \mathbb{R}$. Then the last integral above can be written in the form

$$\int_M F(r, \tau, \varphi, \lambda, x', y) d\mu(r, \tau, \varphi)$$

where $M = S^{n-2} \times \mathbb{R}_+ \times \mathbb{R}_+$ and the measure μ is given by

$$d\mu(r, \tau, \varphi) = r^{n-2}(|x'| + y)^{n-1} e^{-(\varepsilon|x'| + cy)(r+\tau)/2} dr\,d\tau\,d\varphi.$$

The function F has the form

$$\begin{aligned}
F(r, \tau, \varphi, \lambda, x', y) = {}& \frac{1 + iA_0(b,\sigma)}{\rho^{2m-1}} e^{cy(r+\tau+|\lambda|^{1/2m})} M(b,\sigma) \lambda^{1-\frac{|\alpha|}{2m}} (\xi')^{\alpha'} (\rho A_0(b,\sigma))^{\alpha_n} \\
& \times (|\xi'| + y) e^{-(\varepsilon|x'| + cy)|\lambda|^{1/2m}} e^{-(\varepsilon|x'| + cy)(r+\tau)/2}
\end{aligned}$$

with a suitably chosen constant $c > 0$. It is shown in [11] that we have

$$\mathcal{R}\big\{F(r, \tau, \varphi, \lambda, x', y) : \lambda \in \overline{\mathbb{C}_+} \setminus \{0\}, r, \tau > 0, x' \in \mathbb{R}^{n-1}, \phi \in S^{n-3}\big\} < \infty.$$

Moreover, we have $\int_M d\mu \le C$ for some constant $C > 0$. By the contraction principle (see [11], Lemma 3.5), we obtain

$$\mathcal{R}\big\{\lambda^{1-\frac{|\alpha|}{2m}} D^\alpha T(\lambda, x', y) : \lambda \in \overline{\mathbb{C}_+} \setminus \{0\}, |\alpha| \le 2m, j = 1, \dots, m\big\} \le \frac{C}{(|x'| + y)^n}.$$

Now we can Lemma 7.1 of [11] to see that

$$\mathcal{R}\big\{\lambda^{1-\frac{|\alpha|}{2m}} D^\alpha T^j_\lambda : \lambda \in \overline{\mathbb{C}_+} \setminus \{0\}, |\alpha| \le 2m, j = 1, \dots, m\big\} < \infty.$$

To prove the second statement of the Lemma, we have to replace T^j_λ by $\lambda\partial_\lambda T^j_\lambda$. Thus we have to replace the integrand in (4.5) by

$$\lambda\partial_\lambda\Big[e^{ix'\xi'}\frac{1 + A_0(b,\sigma)}{\rho^{2m-1}} e^{i\rho A_0(b,\sigma)} M(b,\sigma)\lambda^{1-\frac{|\alpha|}{2m}}(\xi')^{\alpha'}(\rho A_0(b,\sigma))^{\alpha_n}(|x'| + y)\Big]. \tag{4.6}$$

For this it suffices to remark that

$$\begin{aligned}
\lambda\partial_\lambda\rho &= \frac{1}{2m}\,\frac{\lambda}{\lambda + |\xi'|^{2m}}\,\rho,\\
\lambda\partial_\lambda b = \lambda\partial_\lambda\Big(\frac{\xi'}{\lambda}\Big) &= -\frac{1}{2m}\,\frac{\lambda}{\lambda + |\xi'|^{2m}}\,b,\\
\lambda\partial_\lambda\sigma = \lambda\partial_\lambda\Big(\frac{\lambda}{\rho^{2m}}\Big) &= \frac{|\xi'|^{2m}}{\lambda + |\xi'|^{2m}}\,\sigma.
\end{aligned}$$

Inserting these expressions into (4.6), we see that we obtain a sum of products of the same form as in (4.5) multiplied with a bounded function. The only additional point to mention here is

$$\lambda\partial_\lambda e^{i\rho A_0(b,\sigma)y} = e^{i\rho A_0(b,\sigma)y}\lambda\partial_\lambda(i\rho A_0(b,\sigma)y)$$

for which we use an estimate of the form

$$|\rho y| \le C(\tau + r + |\lambda|^{1/2m})y \le Ce^{cy(r+\tau+|\lambda|^{1/2m})/2}.$$

Thus, the same proof as above with a modified expression for $F(r, \tau, \lambda, x', y)$ leads to

$$\mathcal{R}\big\{\lambda^{2-\frac{|\alpha|}{2m}} D^\alpha\partial_\lambda T^j_\lambda : \lambda \in \overline{\mathbb{C}_+} \setminus \{0\}, |\alpha| \le 2m, j = 1, \dots, m\big\} < \infty.$$

□

Proposition 4.5. *Let* $(\mathcal{A}(D), \mathcal{B}_1(D), \dots, \mathcal{B}_m(D))$ *satisfy* (E) *and* (LS). *Let* $g_j \in W^{\kappa_j, 2m\kappa_j}_{pp}(\mathbb{R}_+ \times \mathbb{R}^{n-1}; E)$ *for* $j = 1, \dots, m$ *with* $g_j(0, x) = 0$ *if* $\kappa_j > 1/p$. *Then* (4.2) *has a unique solution* $u \in W^{1,2m}_{pp}(\mathbb{R}_+ \times \mathbb{R}^n_+; E)$ *depending continuously on* g_j.

Proof. As $g_j(0, x) = 0$ if $\kappa_j > 1/p$, we can extend g_j by zero to a function $g_j \in W^{\kappa_j, 2m\kappa_j}_{pp}(\mathbb{R} \times \mathbb{R}^{n-1}; E)$. We set

$$h_j(t, x', y) := L^{(2m-m_j)/(2m)}e^{-yL^{1/2m}}g_j(t, x').$$

Then $h_j \in L_p(\mathbb{R} \times \mathbb{R}^n_+; E)$. Taking Fourier transform $F_{t\to\lambda}$ with respect to t and applying Lemma 4.3, we obtain

$$u = \sum_{j=1}^m (F_{t\to\lambda})^{-1} T^j_{i\lambda} F_{t\to\lambda} h_j.$$

Now Theorem 4.1 and Lemma 4.4 immediately show

$$(\mu + \partial_t + (-\Delta)^m)u \in L_p(\mathbb{R}_+ \times \mathbb{R}^n_+; E).$$

Moreover, as $g_j = 0$ for $t = 0$, we get $u|_{t=0} = 0$. □

The last model problem we consider is the inhomogeneous initial boundary value problem in the half space:

$$\begin{aligned} (\mu + \partial_t + \mathcal{A}(D))u = f \quad & \text{in } \mathbb{R}_+ \times \mathbb{R}^n_+, \\ \mathcal{B}_j(D)u = g_j \quad & \text{in } \mathbb{R}_+ \times \mathbb{R}^{n-1},\ j = 1, \dots, m, \\ u(0,x) = u_0 \quad & \text{in } \mathbb{R}^n_+. \end{aligned} \tag{4.7}$$

The following result holds true.

Proposition 4.6. *Let* $(\mathcal{A}(D), \mathcal{B}_1(D), \dots, \mathcal{B}_m(D))$ *satisfy* (E) *and* (LS). *Then for any* $\mu > 0$, $f \in L_p(\mathbb{R}_+ \times \mathbb{R}^n_+; E)$, $g_j \in W^{\kappa_j, 2m\kappa_j}_{pp}(\mathbb{R}_+ \times \mathbb{R}^{n-1}; E)$ *and any* $u_0 \in B^{2m(1-1/p)}_{pp}(\mathbb{R}^n_+; E)$ *satisfying the compatibility conditions*

$$\mathcal{B}_j(D)u_0|_{y=0} = g_j|_{t=0}, \quad \kappa_j > 1/p,\ j = 1, \dots, m,$$

there exists a unique solution $u \in W^{1,2m}_{pp}(\mathbb{R}_+ \times \mathbb{R}^n_+; E)$ *of* (4.7).

Proof. We extend f to $\mathbb{R}_+ \times \mathbb{R}^n$ by zero and choose an extension $u_0 \in B^{2m(1-1/p)}_{pp}(\mathbb{R}^n; E)$. Define $u_1 \in W^{1,2m}_{pp}(\mathbb{R}_+ \times \mathbb{R}^n; E)$ as the unique solution of

$$\begin{aligned} (\mu + \partial_t + \mathcal{A}(D))u_1 = f \quad & \text{in } \mathbb{R}_+ \times \mathbb{R}^n, \\ u_1(0,x) = u_0 \quad & \text{in } \mathbb{R}^n. \end{aligned}$$

Here we use Lemma 4.2. Next, consider

$$\begin{aligned} (\mu + \partial_t + \mathcal{A}(D))u_2 = 0 \quad & \text{in } \mathbb{R}_+ \times \mathbb{R}^n_+, \\ \mathcal{B}_j(D)u_2 = \tilde{g}_j \quad & \text{in } \mathbb{R}_+ \times \mathbb{R}^{n-1},\ j = 1, \dots, m, \\ u_2(0,x) = 0 \quad & \text{in } \mathbb{R}^n_+ \end{aligned} \tag{4.8}$$

with

$$\tilde{g}_j := g_j - \mathcal{B}_j(D)u_1 \in W^{\kappa_j, 2m\kappa_j}_{pp}(\mathbb{R}_+ \times \mathbb{R}^{n-1}; E)$$

following to the trace theorems. Because of the compatibility condition, we have $\tilde{g}_j|_{t=0} = 0$. Applying Proposition 4.5, we obtain unique solvability of (4.8) with $u_2 \in W^{1,2m}_{pp}(\mathbb{R}_+ \times \mathbb{R}^n_+; E)$. Finally, $u = u_1 + u_2$ is the unique solution of (4.7), depending continuously on the data. □

We are finally prepared for the proof of our first main result, Theorem 2.1. Note that if $u \in W^{1,2m}_{pp}(J \times G; E)$ is a solution of (2.2), then the data are subject to conditions (D) by the trace results stated above. Assume now that (E), (LS), (SD), (SB) and (D) are satisfied. In order to prove unique solvability of (2.2), we use the method of localization. We do not give all details as in [11], Section 8, however, we state the main step of localization as the following perturbation result.

Lemma 4.7. *Consider the initial boundary value problem*

$$\begin{aligned} (\mu + \partial_t + \mathcal{A}(D))u &= f + \mathcal{A}^{sm}(t,x,D)u \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^n_+, \\ \mathcal{B}_j(D)u &= g_j + \mathcal{B}_j^{sm}(t,x,D)u \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^{n-1}, \\ u(0,x) &= u_0 \quad \text{in } \mathbb{R}^n_+. \end{aligned} \tag{4.9}$$

Assume that $\mathcal{A}^{sm}$ is a small perturbation in the sense that

$$\mathcal{A}^{sm}(t,x,D) = \mathcal{A}^{sm}_{\#}(t,x,D) + \mathcal{A}^{sm}_1(t,x,D)$$

where

$$|\mathcal{A}^{sm}_{\#}(t,x,D)u|_{p,p} \le \varepsilon |u|_{W^{1,2m}_{pp}(\mathbb{R}_+ \times \mathbb{R}^n_+;E)}$$

and where $\mathcal{A}^{sm}_1(t,x,D)$ is a lower order operator in the sense that

$$\mathcal{A}^{sm}_1(t,x,D) \in \mathcal{B}\big(W^{1-\frac{1}{2m},2m-1}_{pp}(\mathbb{R}_+ \times \mathbb{R}^n_+;E), L_p(\mathbb{R}_+ \times \mathbb{R}^n_+;E)\big).$$

Assume also that $\mathcal{B}^{sm}_j$ is a small perturbation, $\mathcal{B}^{sm}_j = \mathcal{B}^{sm}_{j,\#} + \mathcal{B}^{sm}_{j,1}$, with

$$|\mathcal{B}^{sm}_{j,\#}(t,x,D)u|_{W^{\kappa_j,2m\kappa_j}_{pp}(\mathbb{R}_+ \times \mathbb{R}^{n-1};E)} \le \varepsilon |u|_{W^{1,2m}_{pp}(\mathbb{R}_+ \times \mathbb{R}^n_+;E)}$$

and

$$\mathcal{B}^{sm}_{j,1}(t,x,D) \in \mathcal{B}\big(W^{1-\frac{1}{2m},2m-1}_{pp}(\mathbb{R}_+ \times \mathbb{R}^n_+;E), W^{\kappa_j,2m\kappa_j}_{pp}(\mathbb{R}_+ \times \mathbb{R}^{n-1};E)\big).$$

Let $(\mathcal{A}(D), \mathcal{B}_1(D), \ldots, \mathcal{B}_m(D))$ satisfy (E) *and* (LS) *and let $\varepsilon > 0$ be sufficiently small. Then* (4.9) *has a unique solution $u \in W^{1,2m}_{pp}(\mathbb{R}_+ \times \mathbb{R}^n_+;E)$ depending continuously on f, g_j and u_0.*

The proof of Theorem 2.1 follows finally by the localization and perturbation procedure (see [11], Section 8) from Propositions 4.6 and 4.7. We omit the details at this point.

5. Necessity of ellipticity conditions

In this section we show that the conditions of ellipticity are also necessary for optimal regularity. Throughout this section, we fix a Banach space E of class $\mathcal{HT}$ and $J = [0,T]$. We will consider a partial differential operator

$$\mathcal{A}(t,x,D) = \sum_{|\alpha| \le 2m} a_\alpha(t,x) D^\alpha$$

in $J \times G$ with a domain $G \subset \mathbb{R}^n$ and boundary differential operators $\mathcal{B}_j$ of the form

$$\mathcal{B}_j(t,x,D) = \sum_{|\beta| \le m_j} b_{j\beta}(t,x) D^\beta$$

for $j = 1, \dots, m$. Precise assumptions on the regularity of G and the coefficients will be made below.

We start with an elementary remark on linear operator pencils. Let X be a Banach space and T an operator pencil of the form $T(\lambda) = T_0 + \lambda J$, where $J \in \mathcal{B}(X)$ with $|J| \le 1$ and T_0 is a closed densely defined operator. Recall that the resolvent set $\rho(T)$ consists of all $\lambda \in \mathbb{C}$ for which $T(\lambda)$ is boundedly invertible and $\sigma(T) := \mathbb{C} \setminus \rho(T)$. The approximative spectrum $\sigma_{\rm app}(T)$ is defined as the set of all $\lambda \in \mathbb{C}$ for which there exists a sequence $(x_n) \subset D(T_0)$ with $|x_n| = 1$ and $|T(\lambda)x_n| \to 0$ for $n \to \infty$. Note that $\sigma_p(T) \subset \sigma_{\rm app}(T) \subset \sigma(T)$ where $\sigma_p(T)$ stands for the point spectrum of T (the set of all λ for which $\ker T(\lambda)$ is nontrivial).

Lemma 5.1. *Let $T(\cdot)$ be an operator pencil as above. Then $\partial\sigma(T) \subset \sigma_{\rm app}(T)$.*

Proof. Let $\lambda \in \partial\sigma(T)$ and $\lambda_n \in \rho(T)$ with $\lambda_n \to \lambda$. Then $|T(\lambda_n)^{-1}| \to \infty$ for $n \to \infty$. Indeed, in the other case there would exist (after changing to a subsequence) a constant $M > 0$ with $|T(\lambda_n)^{-1}| \le M$ for all n. Let $n \in \mathbb{N}$ with $|\lambda - \lambda_n| < M^{-1}$. Then

$$|T(\lambda_n)^{-1}(\lambda - \lambda_n)J| \le M|\lambda - \lambda_n| < 1,$$

and $T(\lambda)$ would be invertible with inverse

$$T(\lambda)^{-1} = (I + T(\lambda_n)^{-1}(\lambda - \lambda_n)J)^{-1} T(\lambda_n)^{-1}.$$

Choose $y_n \in X$ with $|y_n| = 1$ and $|T(\lambda_n)^{-1}y_n| \to \infty$ and set $x_n := T(\lambda_n)^{-1}y_n / |T(\lambda_n)^{-1}y_n|$. Then $|x_n| = 1$, $x_n \in D(T_0)$ and

$$\begin{aligned} |T(\lambda)x_n| &\le |(T(\lambda) - T(\lambda_n))x_n| + |T(\lambda_n)x_n| \\ &\le |\lambda - \lambda_n| + |T(\lambda_n)^{-1}y_n|^{-1}, \end{aligned}$$

which tends to zero for $n \to \infty$. □

Proposition 5.2. *Let $G \subset \mathbb{R}^n$ be open with compact boundary ∂G, let $J = [0,T]$, and suppose smoothness assumption* **(SD)** *is valid.*

Assume that there exists a constant $C > 0$ such that for every $u \in W^{1,2m}_{p,p}(J \times G; E)$ with compact support $\operatorname{supp} u \subset (0,T) \times G$ *the inequality*

$$|\partial_t u|_{L_p(J_a \times G;E)} + |D^{2m}u|_{L_p(J_a \times G;E)} \le C|\partial_t u + \mathcal{A}(x,t,D)u|_{L_p(J_a \times G;E)} \tag{5.1}$$

holds for every interval $J_a = [0,a] \subset J$. Then $\mathcal{A}(t,x,D)$ is normally elliptic for every $(t,x) \in J \times \overline{G}$ and also for every $t \in J$, $x = \infty$, in case G is unbounded.

Proof. The proof is subdivided into three steps.

(i) In the first step we show that it is sufficient to consider the principal part of $\mathcal{A}$. For this, let

$$\mathcal{A}_1(t,x,D) = \mathcal{A}(t,x,D) - \mathcal{A}_\#(t,x,D) = \sum_{|\alpha|<2m} a_\alpha(t,x)D^\alpha.$$

For $u \in W^{1,2m}_{p,p}(J\times G;E)$ with $\operatorname{supp} u \subset J\times G$ we can apply Lemma 3.10. In fact, we may consider u as a distribution in $W^{1,2m}_{p,p}(J\times \widetilde{G};E)$ where $\widetilde{G}$ is a domain with smooth boundary satisfying $\operatorname{supp} u \subset \widetilde{G} \subset G$. We obtain that for any $\varepsilon_1>0$ there exists a constant $C_{\varepsilon_1}>0$ with

$$|\mathcal{A}_1(t,x,D)u|_{p,p} \le \varepsilon_1(|D^{2m}u|_{p,p} + |\partial_t u|_{p,p}) + C_{\varepsilon_1}|u|_{p,p}. \tag{5.2}$$

¿From this (with J_a instead of J) and (5.1) we obtain the inequality

$$\begin{aligned} &|\partial_t u|_{L_p(J_a\times G;E)} + |D^{2m}u|_{L_p(J_a\times G;E)} \\ &\qquad \le C\Big[|\partial_t u + \mathcal{A}_\#(t,x,D)u|_{L_p(J_a\times G;E)} + |u|_{L_p(J_a\times G;E)}\Big] \end{aligned} \tag{5.3}$$

for every $a\in(0,T]$.

(ii) From now on we fix $(t_0,x_0)\in J\times\overline{G}$ and show that we may replace $\mathcal{A}_\#(t,x,D)$ in (5.3) by $\mathcal{A}_\#(t_0,x_0,D)$. Due to the continuity of a_α for $|\alpha|=2m$, for $\varepsilon>0$ there exist t_1,x_1 and $\delta>0$ with

$$|a_\alpha(t,x)-a_\alpha(t_0,x_0)|_{\mathcal{B}(E)} \le \varepsilon \quad (|\alpha|=2m, t\in\overline{B_\delta(t_1)}, x\in\overline{B_\delta(x_1)})$$

and $\overline{B_\delta(t_1)}\subset(0,T)$, $\overline{B_\delta(x_1)}\subset G$. For all $u\in W^{1,2m}_{p,p}(J\times G;E)$ with $\operatorname{supp} u\subset \overline{B_\delta(t_1)}\times\overline{B_\delta(x_1)}$ we have

$$\begin{aligned} &|\partial_t u + \mathcal{A}_\#(t,x,D)u|_{L_p(J_a\times G;E)} \\ &\qquad \le |\partial_t u + \mathcal{A}_\#(t_0,x_0,D)u|_{L_p(J_a\times G;E)} + \varepsilon|D^{2m}u|_{L_p(J_a\times G;E)} \end{aligned}$$

and therefore (5.3) holds with $\mathcal{A}_\#(t_0,x_0,D)$ instead of $\mathcal{A}_\#(t,x,D)$. In particular,

$$\begin{aligned} |(-\Delta)^{2m}u|_{L_p(J_a\times G;E)} &\le |D^{2m}u|_{L_p(J_a\times G;E)} \\ &\le C\Big[|\partial_t u + \mathcal{A}_\#(t_0,x_0,D)u|_{L_p(J_a\times G;E)} + |u|_{L_p(J_a\times G;E)}\Big] \end{aligned} \tag{5.4}$$

(iii) Assume that for some $\xi_0\in\mathbb{R}^n$, $|\xi_0|=1$, we have

$$\sigma(\mathcal{A}_\#(t_0,x_0,\xi_0)) \cap (\mathbb{C}\setminus\mathbb{C}_+) \ne \emptyset.$$

By Lemma 5.1 (with $T(\lambda)=\mathcal{A}_\#(t_0,x_0,\xi_0)-\lambda$) there exists $\lambda_0\in\sigma_{\text{app}}(\mathcal{A}_\#(t_0,x_0,\xi_0))$ with $\operatorname{Re}\lambda_0\le 0$. Given $\eta>0$ we choose $v_\eta\in E$ with $|v_\eta|=1$ and

$$|(\mathcal{A}_\#(t_0,x_0,\xi_0)-\lambda_0)v_\eta| < \eta.$$

We fix $\psi\in C^\infty(\mathbb{R}^n)$ and $\chi\in C^\infty(\mathbb{R})$ with $0\le\psi,\chi\le 1$ and $\operatorname{supp}\psi\subset\overline{B_\delta(x_1)}$, $\operatorname{supp}\chi\subset[t_1-\delta,t_1]$. Here x_1 and t_1 are given in part (ii) of the proof. We set

$$u(t,x) = \chi(t)\psi(x)e^{-\mu t}e^{i\zeta x}v_\eta$$

with

$$\mu = r^{2m}\lambda_0,\ \zeta = r\xi_0$$

for large $r > 0$. We want to apply (5.4) with $J_a = [0, t_1]$. For this we estimate

$$|(-\Delta)^m u|_{L_p([0,t_1];L_p(G;E))} = |\psi e^{-\mu t}|_{L_p([0,t_1])} |(-\Delta)^m \psi e^{i\zeta x} v_\eta|_{L_p(G;E)},$$
$$|(-\Delta)^m \psi e^{i\zeta x} v_\eta|_{L_p(G;E)} \geq r^{2m} |\psi|_{L_p(G)} |v_\eta| - \sum_{k<2m} r^k c_k(\psi)$$
$$\geq \frac{1}{2} r^{2m} |\psi|_{L_p(G)} \quad \text{for } r \geq r_0$$

where $c_k(\psi)$ depends on $D^\beta \psi$, $1 \leq |\beta| \leq 2m$, and r_0 is sufficiently large.

For the right-hand side of (5.4) we have

$$\partial_t u = \dot{\chi}(t)\psi(x) e^{-\mu t} e^{i\zeta x} v_\eta - \mu u,$$
$$\mathcal{A}_\#(t_0, x_0, D) u = \chi(t) e^{-\mu t} \sum_{|\alpha|=2m} \Big[a_\alpha(t_0, x_0) \psi(x) \zeta^\alpha e^{i\zeta x} v_\eta + \sum_{\beta<\alpha} c_{\alpha\beta} D^{\alpha-\beta} \psi(x) \zeta^\beta e^{i\zeta x} \Big]$$

and consequently

$$|\partial_t u + A_\#(t_0, x_0, D) u|_{L_p([0,t_1];L_p(G;E))} \leq |\dot{\chi} e^{-\mu t}|_{L_p([0,t_1])} |\psi|_{L_p(\mathbb{R}^n)} + r^{2m} |\chi e^{-\mu t}|_{L_p([0,t_1])} |\psi|_{L_p(\mathbb{R}^n)} |A_\#(t_0, x_0, \xi_0) - \lambda) v_\eta| + \sum_{k=0}^{2m-1} c'_k(\psi) r^k.$$

The last term on the right-hand side of (5.4) equals

$$|u|_{p,p} \leq |\chi e^{-\mu t}|_{L_p([0,t_1])} |\psi|_{L_p(\mathbb{R}^n)}.$$

Inserting this into (5.4), we obtain for sufficiently large $r > 0$:

$$r^{2m} |\chi e^{-\mu t}|_{L_p([0,t_1])} |\psi|_{L_p(\mathbb{R}^n)} \leq C \Big[|\dot{\chi} e^{-\mu t}|_{L_p([0,t_1])} |\psi|_{L_p(\mathbb{R}^n)} + (\eta r^{2m} + 1) |\chi e^{-\mu t}|_{L_p([0,t_1])} |\psi|_{L_p(\mathbb{R}^n)}.$$

Therefore,

$$\frac{|\dot{\chi} e^{-\mu t}|_{L_p([0,t_1])}}{r^{2m} |\chi e^{-\mu t}|_{L_p([0,t_1])}} + \eta + r^{-2m} \geq C \tag{5.5}$$

with a constant C independent of χ, r and η. Now we choose χ such that

$$\chi(t) = \begin{cases} 0, & t \leq t_1 - \frac{2}{3}\delta, \\ 1, & t \geq t_1 - \frac{1}{3}\delta. \end{cases}$$

Then

$$|\dot{\chi} e^{-\mu t}|_{L_p([0,t_1])} \leq \sup_{t \in \mathbb{R}} |\dot{\chi}(t)| (\delta/3)^{1/p} e^{\mathrm{Re}(-\zeta)(t_1 - \delta/3)}$$

and

$$|\chi e^{-\mu t}|_{L_p([0,t_1])} \geq (\delta/3)^{1/p} e^{\mathrm{Re}(-\zeta)(t_1 - \delta/3)}$$

where we used $\operatorname{Re}\zeta = r^{2m}\operatorname{Re}\lambda_0 \le 0$. Therefore, the quotient

$$\frac{|\dot{\chi}e^{-\mu t}|_{L_p([0,t_1])}}{|\chi e^{-\mu t}|_{L_p([0,t_1])}}$$

is bounded for $r \to \infty$. From this we see that the left-hand side of (5.5) tends to η for $r \to \infty$. Choosing η sufficiently small, we obtain a contradiction. □

Proposition 5.3. *Let $G \subset \mathbb{R}^n$ be an open domain with C^{2m}-boundary ∂G. Let* (SD) *and* (SB) *be satisfied, and set $\kappa_j = (2m - m_j - 1/p)/(2m)$. Assume that there exists a constant $C > 0$ such that for all $u \in W^{1,2m}_{p,p}(J \times G; E)$ satisfying $u(x,0) = 0$ in G the inequality*

$$|\partial_t u|_{L_p(J_a\times G;E)} + |D^{2m}u|_{L_p(J_a\times G;E)} \le C\Big[|\partial_t u + \mathcal{A}(t,x,D)u|_{L_p(J_a\times G;E)} + \sum_{j=1}^m |\mathcal{B}_j(t,x,D)u|_{W^{\kappa_j,2m\kappa_j}_{p,p}(J_a\times\partial G)}\Big] \tag{5.6}$$

holds for every interval $J_a = [0,a] \subset J$. Then $\mathcal{A}(t,x,D$ is normally elliptic in $J\times\overline{G}$, and the Lopatinskii–Shapiro condition holds in $J \times \partial G$.

Proof. The proof is subdivided into five steps.

(i) Normal ellipticity for the interior symbol.

We already know from the proof of Propostion 5.2 that $\mathcal{A}(t,x,D)$ is normally elliptic in $J \times \overline{G}$. Therefore, we only have to show that the Lopatinskii-Shapiro condition holds in $J \times \partial G$.

(ii) Transformation to the half-space.

We fix $x_0 \in \partial G$ and choose C^{2m}-coordinates $g = g_{x_0}$ as described at the end of Section 3. Let $u \in W^{1,2m}_{p,p}(J \times G)$ with $\operatorname{supp} u \subset J \times (U \cap \overline{G})$. The transformed function $\widetilde{u} = \widetilde{u}(t,y)$ satisfies

$$\begin{aligned} \partial_t\widetilde{u} + \mathcal{A}^g(t,y,D)u &= \widetilde{f} \quad \text{in } J\times(\mathbb{R}^n_+\cap\widetilde{U}),\\ \mathcal{B}^g_j(t,y,D)\widetilde{u} &= \widetilde{g}_j \quad \text{on } J\times(\mathbb{R}^{n-1}\cap\widetilde{U}),\\ \widetilde{u}(0,y) &= 0 \quad \text{in } \mathbb{R}^n_+\cap\widetilde{U}. \end{aligned}$$

Here the transformed operators $\mathcal{A}^g$ and $\mathcal{B}^g_j$ are defined in Section 3, and we have set $f := \partial_t u + \mathcal{A}(t,x,D)u$, $g_j := \mathcal{B}_j(t,x,D)u$ and $\widetilde{f} = \mathcal{G}f$, $\widetilde{g}_j = \mathcal{G}g_j$.

From the fact that the transformation $\mathcal{G}$ induces isomorphisms of the corresponding Sobolev spaces and from the estimate (5.6) we see that the inequality

$$\begin{aligned} |\partial_t \widetilde{u}|_{L_p(J_a\times\mathbb{R}^n_+;E)} + |D^{2m}\widetilde{u}|_{L_p(J_a\times\mathbb{R}^n_+;E)} \le C\Big[|\partial_t\widetilde{u} + \mathcal{A}^g(t,y,D)u|_{L_p(J_a\times\mathbb{R}^n_+;E)} \\ + \sum_{j=1}^m |\mathcal{B}_j^g(t,y,D)\widetilde{u}|_{W^{\kappa_j,2m\kappa_j}_{p,p}(J_a\times\mathbb{R}^{n-1})}\Big] \end{aligned} \tag{5.7}$$

holds for all $\widetilde{u}\in W^{1,2m}_{p,p}(J\times\mathbb{R}^n_+)$ with $\operatorname{supp} u\subset J\times(\widetilde{U}\cap\overline{\mathbb{R}^n_+})$ and $u(0,y)=0$ in $\mathbb{R}^n_+$. As it was remarked in Section 3, the Lopatinskii–Shapiro condition for $(\mathcal{A},\mathcal{B})$ holds at (t_0,x_0) if and only if the Lopatinskii–Shapiro condition for $(\mathcal{A}^g,\mathcal{B}^g)$ holds at (t_0,y_0) with $y_0=g(x_0)$. Therefore, to finish the proof of the Theorem, we have to show that (LS) holds for the transformed problem at the point (t_0,y_0).

(iii) Reduction to the principal parts.

In the following, we will consider the transformed problem $(\mathcal{A}^g,\mathcal{B}^g_1,\dots,\mathcal{B}^g_m)$ where for simplicity of notation we will omit the tilde for the coefficients a_α and $b_{j\beta}$ as well as for the transformed functions u. Let $\mathcal{A}^g_1(t,y,D)=\mathcal{A}^g(t,y,D)-\mathcal{A}^g_\#(t,y,D)$ and $B^g_{j,1}(t,y,D)=\mathcal{B}^g_j(t,y,D)-\mathcal{B}^g_{j\#}(t,y,D)$.

For $u\in W^{1,2m}_{p,p}(J\times\mathbb{R}^n_+;E)$ with $\operatorname{supp} u\subset J\times(\widetilde{U}\cap\overline{\mathbb{R}^n_+})$, we apply Lemma 3.10 and 5.6 to see that for any $\varepsilon_1>0$ there exists a constant $C_{\varepsilon_1}>0$ such that

$$\begin{aligned} |\mathcal{A}^g_1(t,y,D)u|_{L_p(J_a\times\mathbb{R}^n_+;E)} + \sum_{j=1}^m |\mathcal{B}^g_{j,1}(t,y,D)u|_{W^{\kappa_j,2m\kappa_j}_{p,p}(J_a\times\mathbb{R}^{n-1};E)} \\ \le C\varepsilon_1(|D^{2m}u|_{L_p(J_a\times\mathbb{R}^n_+;E)} + |\partial_t u|_{L_p(J_a\times\mathbb{R}^n_+;E)} \\ + C\varepsilon_1|u|_{L_p(J_a\times\mathbb{R}^n_+;E)} \end{aligned}$$

where C is independent of u and ε_1. Choosing $C\varepsilon_1<1/2$, we may replace $\mathcal{A}^g$ and $\mathcal{B}^g_j$ in (5.7) by their principal parts if we add a $|u|_{p,p}$-term on the right-hand side.

(iv) Freezing the coefficients.

Now let $t_0\in J$ be fixed and let $y_0=g(x_0)=0$ as above. Due to (SD), the coefficients a_α are continuous in $J\times\mathbb{R}^n_+$ for $|\alpha|=2m$. Moreover, we know from Lemma 3.9 that $b_{j\beta}\in BUC(J\times\mathbb{R}^{n-1};\mathcal{B}(E))$. Thus for $\varepsilon>0$ there exist $t_1\in(0,T)$ and $\delta>0$ with

$$\begin{aligned} |a_\alpha(t,y)-a_\alpha(t_0,y_0)|_{\mathcal{B}(E)} &\le \varepsilon \quad (|\alpha|=2m), \\ |b_{j\beta}(t,y)-b_{j\beta}(t_0,y_0)|_{\mathcal{B}(E)} &\le \varepsilon \quad (|\beta|=m_j, j=1,\dots,m) \end{aligned}$$

for all $t\in\overline{B_\delta(t_1)}$ and $y\in\overline{B_\delta(y_0)}\cap\overline{\mathbb{R}^n_+}$, where additionally $\overline{B_\delta(t_1)}\subset(0,T)$.

From this we obtain

$$\begin{aligned} |b_{j\beta}D^\beta u|_{W^{\kappa_j,2m\kappa_j}_{p,p}(J_a\times\mathbb{R}^{n-1};E)} \le \varepsilon|D^\beta u|_{W^{\kappa_j,2m\kappa_j}_{p,p}(J_a\times\mathbb{R}^{n-1};E)} \\ + C|D^\beta u|_{W^{\kappa_j-1/2m,2m\kappa_j-1}_{p,p}(J_a\times\mathbb{R}^{n-1};E)}. \end{aligned} \tag{5.8}$$

The first term is not greater than

$$\varepsilon(|D^{2m}u|_{L_p(J_a\times\mathbb{R}^n_+;E)} + |\partial_t u|_{L_p(J_a\times\mathbb{R}^n_+;E)}).$$

From Lemma 3.10 we see that for any $\varepsilon_1 > 0$ the last term in (5.8) can be estimated by

$$C\varepsilon_1(|D^{2m}u|_{L_p(J_a\times\mathbb{R}^n_+;E)} + |\partial_t u|_{L_p(J_a\times\mathbb{R}^n_+;E)}) + CC_{\varepsilon_1}|u|_{L_p(J_a\times\mathbb{R}^n_+;E)}$$

with a constant $C_{\varepsilon_1} > 0$. We obtain for $u \in W^{1,2m}_{p,p}(J\times\mathbb{R}^n_+;E)$ with $\operatorname{supp} u \subset \overline{B_\delta(t_1)} \times (\overline{B_\delta(y_0)} \cap \overline{\mathbb{R}^n_+})$ the inequalities

$$\begin{aligned}
&|(\mathcal{A}^g_\#(t,y,D) - \mathcal{A}^g_\#(t_0,y_0,D))u|_{L_p(J_a\times\mathbb{R}^n_+;E)} \le C\varepsilon|D^{2m}u|_{L_p(J_a\times\mathbb{R}^n_+;E)},\\
&|(\mathcal{B}^g_{j\#}(t,y,D) - \mathcal{B}^g_{j\#}(t_0,y_0,D))u|_{W^{\kappa_j,2m\kappa_j}_{p,p}(J_a\times\mathbb{R}^{n-1};E)}\\
&\qquad \le (\varepsilon + C\varepsilon_1)(|D^{2m}u|_{L_p(J_a\times\mathbb{R}^n_+;E)} + |\partial_t u|_{L_p(J_a\times\mathbb{R}^n_+;E)}) + C|u|_{L_p(J_a\times\mathbb{R}^n_+;E)}.
\end{aligned}$$

Therefore, for such u we can replace $\mathcal{A}^g_\#(t,y,D)$ and $\mathcal{B}^g_\#(t,y,D)$ in (5.7) by $\mathcal{A}^g_\#(t_0,y_0,D)$ and $\mathcal{B}^g_{j\#}(t_0,y_0,D)$, respectively. In particular, we get

$$\begin{aligned}
|(-\Delta_{n-1})^m u|_{L_p(J_a\times\mathbb{R}^n_+;E)} \le C\Big[&|f|_{L_p(J_a\times R^n_+;E)}\\
&+\sum_{j=1}^m |g_j|_{W^{\kappa_j,2m\kappa_j}_{p,p}(J_a\times\mathbb{R}^{n-1};E)} + |u|_{L_p(J_a\times\mathbb{R}^n_+;E)}\Big].
\end{aligned} \tag{5.9}$$

Here Δ_{n-1} denotes the Laplacian in $\mathbb{R}^{n-1}$, and we have set $f := \partial_t u + \mathcal{A}^g_\#(t_0,y_0,D)u$ and $g_j := \mathcal{B}^g_{j\#}(t_0,y_0,D)u$.

(v) Proof of the Lopatinskii–Shapiro condition.

Assume that at (t_0,y_0) the Lopatinskii–Shapiro condition does not hold for the boundary value problem $(\mathcal{A}^g_\#(t_0,y_0,D), \mathcal{B}^g_{1\#}(t_0,y_0,D), \dots, \mathcal{B}^g_{m\#}(t_0,y_0,D))$. Define the linear operator pencil

$$T(\lambda)\colon L_p(\mathbb{R}_+;E) \to L_p(\mathbb{R}_+;E)\times E^m$$

by means of

$$T(\lambda)v = \begin{pmatrix} \mathcal{A}^g_\#(t_0,y_0,\xi_0',D_y)v + \lambda v\\ \mathcal{B}^g_{1\#}(t_0,y_0,\xi_0',D_y)v(0)\\ \vdots\\ \mathcal{B}^g_{m\#}(t_0,y_0,\xi_0',D_y)v(0)\end{pmatrix}.$$

From Lemma 5.1 we see that there exists a $\lambda_0 \in \sigma_{\mathrm{app}}(T)$ with $\operatorname{Re}\lambda_0 \le 0$. For any $\eta > 0$ there exists $v_\eta \in W^{2m}_p(\mathbb{R}_+;E)$ with $|v_\eta|_{L_p(\mathbb{R}_+;E)} = 1$ and $|T(\lambda_0)v_\eta|_{L_p(\mathbb{R}_+;E)} < \eta$. Similarly as in the proof of Proposition 5.2, we fix $\psi \in C^\infty(\mathbb{R}^{n-1})$, $\varphi \in C^\infty(\mathbb{R})$ and $\chi \in C^\infty(\mathbb{R})$ with $0 \le \psi,\varphi,\chi \le 1$ and $\operatorname{supp}\psi \subset \overline{B_\delta(y_0)} \cap \mathbb{R}^{n-1}$, $\operatorname{supp}\varphi \subset [0,\delta)$ and $\operatorname{supp}\chi \subset (t_1-\delta, t_1]$. Here t_1 is taken from part (iii) of the proof. We assume $\varphi = 1$ in $[0,\delta/2]$. Define

$$u(t,x',y) := \chi(t)e^{-\mu t}\psi(x')e^{i\zeta' x'}\varphi(y)v_\eta(ry)$$

with $\mu = r^{2m}\lambda_0$, $\zeta' = r\xi_0'$ and $r > 0$ large.

Then

$$\begin{aligned}|(-\Delta_{n-1})^m u|_{L_p([0,t_1]\times\mathbb{R}^n_+;E)} \\ &= |\chi e^{-\mu t}|_{L_p([0,t_1])}\big|\big((-\Delta_{n-1})^m \psi e^{i\zeta' x'}\big)\varphi(y)v_\eta(ry)|_{L_p(\mathbb{R}^n_+;E)} \\ &\geq C|\chi e^{-\mu t}|_{L_p([0,t_1])} r^{2m}|\psi e^{i\zeta' x'}|_{L_p(\mathbb{R}^{n-1})}|\varphi(y)v_\eta(ry)|_{L_p(\mathbb{R}_+)} \qquad (5.10)\end{aligned}$$

for sufficiently large r. Using the homogeneity of $\mathcal{A}^g_\#$, we see that

$$\mathcal{A}^g_\#(t_0,y_0,\xi'_0,D_y)\big(\varphi(y)v_\eta(ry)\big) = \varphi(y)r^{2m}\big[\mathcal{A}^g_\#(t_0,y_0,\xi'_0,D_y)v_\eta\big](ry) + O(r^{2m-1}).$$

Now we choose χ as in the proof of Proposition 5.2. In particular, $|\dot\chi e^{-\mu t}|_{L_p([0,t_1])} \leq C|\chi e^{-\mu t}|_{L_p([0,t_1])}$. We obtain

$$\begin{aligned}|(\partial_t + \mathcal{A}^g_\#(t_0,y_0,\xi'_0,D_y))u|_{p,p} \\ &\leq Cr^{2m}|\chi e^{-\mu t}|_{L_p([0,t_1])}|\psi e^{i\zeta' x'}|_{L_p(\mathbb{R}^{n-1})} \\ &\quad \cdot |\varphi(y)(\mathcal{A}^g_\#(t_0,x_0,\xi'_0,D_y) - \lambda_0)v_\eta(ry)|_{L_p(\mathbb{R}_+;E)} \\ &\quad + C|\dot\chi e^{-mut}|_{L_p[0,t_1])}|\psi e^{i\zeta' x'}|_{L_p(\mathbb{R}^{n-1})}|\varphi(y)v_\eta(ry)|_{L_p(\mathbb{R}_+;E)} \\ &\leq C|\chi e^{-\mu t}|_{L_p([0,t_1])}\cdot|\psi e^{i\zeta' x'}|_{L_p(\mathbb{R}^{n-1})} \\ &\quad \cdot \Big[r^{2m}\Big|[(\mathcal{A}^g_\#(t_0,x_0,\xi'_0,D_y) - \lambda_0)v_\eta](ry)\Big|_{L_p(\mathbb{R}_+;E)} + |v_\eta(ry)|_{L_p(\mathbb{R}_+;E)}\Big]. \qquad (5.11)\end{aligned}$$

Note that for sufficiently large r we have

$$\begin{aligned}|v_\eta(ry)|_{L_p(\mathbb{R}_+;E)} &\geq |\varphi v_\eta(ry)|_{L_p(\mathbb{R}_+;E)} \\ &\geq \frac{1}{2}|v_\eta(ry)|_{L_p(\mathbb{R}_+;E)} = \frac{1}{2}r^{-1/p}|v_\eta|_{L_p(\mathbb{R}_+;E)} = \frac{1}{2}\,r^{-1/p}.\end{aligned}$$

Therefore, we may replace $|\varphi(y)v_\eta(ry)|_{L_p}$ in (5.10) and $|v_\eta(ry)|_{L_p}$ in (5.11) by $r^{-1/p}$. In the same way, we use

$$\Big|[(\mathcal{A}^g_\#(t_0,x_0,\xi'_0,D_y) - \lambda_0)v_\eta](ry)\Big|_{L_p} = r^{-1/p}\Big|(\mathcal{A}^g_\#(t_0,x_0,\xi'_0,D_y) - \lambda_0)v_\eta\Big|_{L_p}.$$

Now let us consider the boundary operators. We have $\varphi(y) = 1$ in a neighbourhood of $y = 0$, and therefore

$$\begin{aligned}|\mathcal{B}^g_{j\#}(t_0,y_0,D)u|_{W_p^{\kappa_j}([0,t_1];L_p(\mathbb{R}^{n-1};E))} \\ &= |\chi e^{-\mu t}|_{W_p^{\kappa_j}([0,t_1])}\Big|\sum_{|\beta|=m_j} b_{j\beta}(t_0,y_0)D^{\beta'}(\psi(x')e^{i\zeta' x'})D_y^{\beta_n}v_\eta(ry)\Big|_{L_p} \\ &= |\chi e^{-\mu t}|_{W_p^{\kappa_j}([0,t_1])}\Big|\sum_{|\beta|=m_j} b_{j\beta}(t_0,y_0)r^{m_j}\psi(x')\xi_0^{\beta'}e^{i\zeta' x'}(D_y^{\beta_n}v_\eta)(0) + O(r^{m_j-1})\Big|_{L_p} \\ &\leq C|\chi e^{-\mu t}|_{W_p^{\kappa_j}([0,t_1])}r^{m_j}|\psi e^{i\zeta' x'}|_{L_p}|\mathcal{B}^g_{j\#}(t_0,y_0,\xi'_0,D_y)v_\eta(0)|_E. \qquad (5.12)\end{aligned}$$

Here we used the homogeneity of $\mathcal{B}^g_{j\#}(t_0, y_0, \xi)$. We further estimate

$$\begin{aligned}
|\chi e^{-\mu t}|_{W_p^{\kappa_j}([0,t_1])} &\le C|\chi e^{-\mu t}|^{\kappa_j}_{W_p^1([0,t_1])} \cdot |\chi e^{-\mu t}|^{1-\kappa_j}_{L_p([0,t_1])} \\
&\le C\Big[|\dot\chi e^{-\mu t}|^{\kappa_j}_{L_p([0,t_1])} \cdot |\chi e^{-\mu t}|^{1-\kappa_j}_{L_p([0,t_1])} + r^{2m\kappa_j}|\chi e^{-\mu t}|_{L_p([0,t_1])}\Big] \\
&\le C(r^{2m\kappa_j} + 1)|\chi e^{-\mu t}|_{L_p([0,t_1])} \\
&\le Cr^{2m\kappa_j}|\chi e^{-\mu t}|_{L_p([0,t_1])}
\end{aligned}$$

for large r. Here we again used $|\dot\chi e^{-\mu t}|_{L_p([0,t_1])} \le C|\chi e^{-\mu t}|_{L_p([0,t_1])}$. Inserting this into (5.12), we get

$$\begin{aligned}
&|\mathcal{B}^g_{j\#}(t_0, y_0, D)u|_{W_p^{\kappa_j}([0,t_1];L_p(\mathbb{R}^{n-1};E))} \\
&\qquad \le Cr^{2m-1/p}|\chi e^{-\mu t}|_{L_p([0,t_1])} \cdot |\psi e^{i\zeta' x'}|_{L_p(\mathbb{R}^{n-1})}|\mathcal{B}^g_{j\#}(t_0, y_0, \xi_0', D_y)v_\eta(0)|_E. \quad (5.13)
\end{aligned}$$

Combining these estimates we obtain a contradiction as $r \to \infty$. □

This also completes the proof of Theorem 2.2.

6. The case $p \neq q$.

In this section we consider the case $p \neq q$. We assume as before that the condition (E) of normal ellipticity as well as the Lopatinski-Shapiro condition (LS) are valid.

We start with the case of constant coefficients for $G = \mathbb{R}^n$.

Proposition 6.1. *Let $\mathcal{A}(D) = \sum_{|\alpha|=2m} a_\alpha D^\alpha$ be normally elliptic and $\mu > 0$. Then the Cauchy problem*

$$\begin{aligned}
(\mu + \partial_t + \mathcal{A}(D))u &= f \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^n, \\
u|_{t=0} &= u_0 \quad \text{in } \mathbb{R}^n
\end{aligned} \qquad (6.1)$$

has a unique solution $u \in W^{1,2m}_{pq}(\mathbb{R}_+ \times \mathbb{R}^n; E)$ if and only if

$$f \in L_p(\mathbb{R}; L_q(\mathbb{R}^n; E)) \text{ and } u_0 \in B^{2m(1-1/p)}_{qp}(\mathbb{R}^n; E).$$

The solution depends continuously on f and u_0.

Proof. Define the operator A by $D(A) = W^{2m}_q(\mathbb{R}^n; E)$ and $Au = \mu u + \mathcal{A}(D)u$. By [11], Theorem 5.5, the operator A admits an H_∞-calculus and therefore $A \in \mathcal{BIP}(L_q(\mathbb{R}^n; E))$ and it is invertible since $\mu > 0$. Applying Theorem 3.3 we see that (4.1) has a unique solution $u \in W^{1,2m}_{pq}(\mathbb{R}_+ \times \mathbb{R}^n_+; E)$ if and only if $f \in L_p(\mathbb{R}_+; L_q(\mathbb{R}^n_+; E))$ and $u_0 \in D_A(1 - 1/p, p)$. Because of

$$D_A(1 - 1/p, p) = (L_q(\mathbb{R}^n; E), W^{2m}_q(\mathbb{R}^n; E))_{1-1/p,p} = B^{2m(1-1/p)}_{qp}(\mathbb{R}^n; E),$$

this finishes the proof. □

Next we consider the inhomogeneous initial boundary value problem in the half space:

$$
\begin{aligned}
(\mu + \partial_t + \mathcal{A}(D))u &= f \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^n_+,\\
\mathcal{B}_j(D)u &= g_j \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^{n-1},\ j = 1, \dots, m,\\
u(0,x) &= u_0 \quad \text{in } \mathbb{R}^n_+.
\end{aligned} \tag{6.2}
$$

In the following, the spaces F^s_{pq} denote the vector-valued Triebel-Lizorkin spaces defined for $E = \mathbb{C}$ in [33], Chapter 2. The definition given there carries over the the vector-valued setting. Moreover, the spaces ${}_0F^s_{pq}$ are defined as in the situation of Soblev spaces.

For $\alpha \in (0,1)$ and $p, q \in (1,\infty)$, these spaces can be characterized as follows.

Lemma 6.2. *Let $1 < p, q < \infty$ and $\alpha \in (0,1)$. Suppose E is a Banach space of class $\mathcal{HT}$, and set $B = (\partial_t)^\alpha$ in $L_p(\mathbb{R}_+; L_q([0,1]; E))$ with domain $D(B) = {}_0W^\alpha_p(\mathbb{R}_+; L_q([0,1]; E))$.*

Then, for any $g \in L_p(\mathbb{R}_+; E)$,

$$w := Be^{-By}g \in L_p(\mathbb{R}_+; L_q([0,1]; E))$$

if and only if $g \in {}_0F^{\alpha(1-1/q)}_{pq}(\mathbb{R}_+; E)$.

For what follows we might have taken the assertion of Lemma 6.2 as a definition for the vector-valued spaces ${}_0F^{\alpha(1-1/q)}_{pq}(\mathbb{R}_+; E)$, nothing more is needed below. However, in order to draw the connection with the definition of the spaces F^s_{pq} given by Triebel in [33], we need to give a proof of Lemma 6.2.

Proof. For $E = \mathbb{C}$, Theorem 2.4.1 of [33] shows Lemma 6.2 with the choices $\phi(x) = (ix)^\alpha e^{-(ix)^\alpha}$ and $\phi_0(x) = 1$, $s_0 = 0$, $s_1 = \alpha$. The proof given there carries over to the vector-valued case since E is assumed to be of class $\mathcal{HT}$, provided $\alpha > a > 1/\min\{p,q\}$. For general $p, q \in (1,\infty)$, Theorem 2.4.1 of [33] does not apply since the moment condition (8) in that reference does not hold.

To see sufficiency of the condition in the general case, assume that $Bw_0 \in L_p(\mathbb{R}_+; L_q[0,1]; E))$ with $w_0 := e^{-By}g$. Using maximal regularity we solve successively the problems

$$\partial_y w_k + Bw_k = kBw_{k-1}, \quad w_k|_{y=0} = 0,$$

to obtain

$$Bw_k = y^k B^{k+1} e^{-yB} g \in L_p(\mathbb{R}_+; L_q([0,1]; E)), \quad k \in \mathbb{N}_0.$$

Now we have with the variable transformation $y = \tau^\alpha$

$$\begin{aligned}\int_0^1 |y^k B^{k+1} e^{-yB} g|_E^q dy &= \alpha \int_0^1 \tau^{-q\alpha(1-1/q)} |(\tau^\alpha B)^{k+1} e^{-(\tau^\alpha B)} g|_E^q \frac{d\tau}{\tau} \\ &= \alpha \int_0^1 \tau^{-q\alpha(1-1/q)} |\phi(\tau D) g|_E^q \frac{d\tau}{\tau},\end{aligned}$$

where we used the notation in [33], Section 2.4.1, with $\phi(\xi) = (i\xi)^{\alpha(k+1)} e^{-(i\xi)^\alpha}$. It is not difficult to check that the relevant conditions (7) and (9) are valid for all $k \in \mathbb{N}_0$ with $s_0 = 0$. On the other hand, (8) holds in case $\alpha k \geq 1$. In fact, the inverse Fourier-transform $p^{k+1}(t)$ of $\phi(\xi)$, with contour $\Gamma = e^{-i\theta}(\infty, 0] \cup e^{i\theta}[0, \infty)$, $\theta \in (\pi/2, \pi)$, $\alpha\theta < \pi/2$, becomes

$$p^{k+1}(t) = \frac{1}{2\pi i} \int_\Gamma z^{\alpha(k+1)} e^{-z^\alpha} e^{zt} dz, \quad t \geq 0.$$

Note that the support of p^{k+1} is contained in $\mathbb{R}_+$, thanks to holomorphy. This formula is valid for all $\alpha(k+1) > -1$, and it implies that $p^{k+1}(t)$ is bounded and behaves asymptotically like $t^{-(1+\alpha(k+1))}$ as $t \to \infty$. Therefore $(1 + t^a) p^{k+1} \in L_1(\mathbb{R}_+)$ if and only if $a < \alpha(k+1)$. Choosing $s_1 = \alpha$ and $1/\min\{p, q\} < a < 1$, and $k \geq 1/\alpha$, the vector-valued version of Theorem 2.4.1 of [33] implies $g \in F_{pq}^{\alpha(1-1/q)}(\mathbb{R}_+; E)$.

For the converse statement we definitely need to choose $k = 0$. Since the critical condition (8) does not hold, we have to modify Steps 1 and 4 of the proof of Theorem 2.4.1 of [33], the only places where (8) is used. We concentrate on the modification of Step 1, and employ the notation used there. Let $s = \alpha(1 - 1/q)$ and fix a resolution of unity $\{\rho_j\}_{j \in \mathbb{N}_0}$ in the sense of [33] Section 2.3.1. Then by definition, $g \in {}_0F_{pq}^s(R_+; E)$ if and only if

$$(2^{sj} \rho_j(D) g)_{j \in \mathbb{N}_0} \in L_p(\mathbb{R}_+; l_q(\mathbb{N}_0; E)).$$

Now we have as in [33], proof of Theorem 2.4.1, Step 1

$$2^{js} \mathcal{F}^{-1} \mathcal{L} p^1(2^{-j} i\xi) \mathcal{F} = \sum_{l=-\infty}^{\infty} 2^{js} \mathcal{F}^{-1} \mathcal{L} p^1(2^{-j} i\xi) \rho_{l+j}(\xi) \mathcal{F} g.$$

Here $\mathcal{L}$ denotes the Laplace transform. Splitting the sum into two parts, we have to estimate in Step 1 the part running from $l = -\infty$ to $l = k$. We write

$$\begin{aligned}&2^{js} \mathcal{F}^{-1} \mathcal{L} p^1(2^{-j} i\xi) \rho_{l+j}(\xi) \mathcal{F} g \\ &\quad = 2^{\alpha l/q} \mathcal{F}^{-1} \mathcal{L} p^0(2^{-j} i\xi) \cdot (2^{-(j+l)} i\xi)^\alpha \chi(2^{-(j+l)} \xi) \cdot 2^{s(j+l)} \rho_{j+l} \mathcal{F} g,\end{aligned}$$

where $\chi(r)$ denotes a cut off function which is 1 on $|r| \leq 2$. Since $\sum_{l=-\infty}^k 2^{\alpha l/q} < \infty$, it suffices to estimate

$$\mathcal{F}^{-1} \mathcal{L} p^0(2^{-j} i\xi) \cdot (2^{-(j+l)} i\xi)^\alpha \chi(2^{-(j+l)} \xi) \cdot 2^{s(j+l)} \rho_{j+l} \mathcal{F} g$$

in $L_p(\mathbb{R}_+; l_q(\mathbb{N}_0; E))$, uniformly w.r.t. l. By assumption we have

$$|(2^{s(j+l)} \mathcal{F}^{-1} \rho_{j+l} \mathcal{F} g)_{j \geq 0}|_{L_p(\mathbb{R}_+; l_q(\mathbb{N}_0; E))} \leq |g|_{F_{pq}^s(\mathbb{R}_+; E)},$$

hence its is enough to show that the sequences $(\mathcal{L}p^0(2^{-j}i\xi))_{j\in\mathbb{N}_0}$ and $((2^{-(j+l)}i\xi)^\alpha\chi(2^{-(j+l)}\xi))_{j\in\mathbb{N}_0}$ define Fourier multipliers for $L_p(\mathbb{R}_+;l_q(\mathbb{N}_0;E))$ with bounds independent of l.

Now we use the argument given in the proof of Theorem 4.8 of [11]. For the first sequence, observe that $\mathcal{L}p^0(\lambda)=e^{-\lambda^\alpha}$ is completely monotonic, hence $p^0(t)$ is non-negative and integrable with integral equal to 1, i.e. p^0 is a probability density. Therefore the operator defined by the first sequence is given by

$$(T_1f)_j(t)=2^jp^0(2^j\cdot)*f_j(t),\quad t>0,\ j\in\mathbb{N}_0.$$

Thus we obtain

$$|(T_1f)_j(t)|_E\le M|f_j|_E(t),\quad t>0,\ j\in\mathbb{N}_0,$$

where M denotes the usual maximal operator. Since M is bounded in $L_p(\mathbb{R}_+;l_q(\mathbb{N}_0))$, the assertion follows for the first sequence, i.e. T_1 is bounded in $L_p(\mathbb{R}_+;l_q(\mathbb{N}_0;E))$.

The second sequence is treated in a similar way. We write

$$(i\xi)^\alpha\chi(\xi)=\frac{(i\xi)^\alpha}{1+i\xi}+\frac{(i\xi)^\alpha}{1+i\xi}(\chi(\xi)-1)+\frac{(i\xi)^{1+\alpha}}{1+i\xi}\chi(\xi).$$

The first term belongs to the Hardy space $\mathcal{H}^\infty(\mathbb{C}_+)$ and its derivative belongs to $\mathcal{H}^1(\mathbb{C}_+)$, therfore by Hardy's inequality it is the Laplace transform of a function $k_1\in L_1(\mathbb{R}_+)$. The second and the third terms belong to $L_2(\mathbb{R})$ as well as their derivatives, hence by means of a theorem of Bernstein they are Fourier transforms of functions $k_j\in L_1(\mathbb{R})$, $j=2,3$. This shows that $(i\xi)^\alpha\chi(\xi)=\mathcal{F}k(\xi)$, for some $k\in L_1(\mathbb{R})$. Now we may argue as before to see that also the second sequence defines a bounded operator T_2 in $L_p(\mathbb{R}_+;l_q(\mathbb{N}_0;E))$, with bound independent of l. This completes the proof of Lemma 6.2. □

Remark 6.3. *It is not difficult to see that we may replace B by $B_\mu=(\mu+\partial_t)^{1/2m}$ and the interval $[0,1]$ by $\mathbb{R}_+$.*

By means of Lemma 6.2 we are able to consider (6.2).

Proposition 6.4. *Let $(\mathcal{A}(D),\mathcal{B}_1(D),\dots,\mathcal{B}_m(D))$ satisfy* (E) *and* (LS). *Then there exists a unique solution $u\in W^{1,2m}_{pq}(\mathbb{R}_+\times\mathbb{R}^n_+;E)$ of (6.2) if and only if $f\in L_p(\mathbb{R}_+;L_q(\mathbb{R}^n_+;E))$, $u_0\in B^{2m(1-1/p)}_{qp}(\mathbb{R}^n_+;E)$ and with $\kappa_j=(2m-m_j-2m/q)/(2m)$*

$$g_j\in F^{\kappa_j}_{pq}(\mathbb{R}_+;L_q(\mathbb{R}^{n-1};E))\cap L_p(\mathbb{R}_+;B^{2m\kappa_j}_{qq}(\mathbb{R}^{n-1};E)),$$

and the compatibility condition $B_j(D)u_0|_{y=0}=g_j|_{t=0}$ holds in case $\kappa_j>1/q$.

Proof. Necessity. Let $u\in W^{1,2m}_{p,q}(\mathbb{R}_+\times\mathbb{R}^n_+;E)$ be a given solution of (6.2). Then we obviously have

$$f=\mu u+\partial_tu+\mathcal{A}(D)u\in L_p(\mathbb{R}_+;L_q(\mathbb{R}^n_+;E)).$$

Extending u in y and applying Proposition 6.1 we obtain

$$u_0 = u|_{t=0} \in B_{qp}^{2m(1-1/p)}(\mathbb{R}_+^n; E),$$

and taking traces in y we get

$$u|_{y=0} \in L_p(\mathbb{R}_+; B_{qq}^{2m-1/q}(\mathbb{R}^{n-1}; E)),$$

which implies

$$g_j \in L_p(\mathbb{R}_+; B_{qq}^{2m-m_j-1/q}(\mathbb{R}^{n-1}; E)).$$

The time regularity of the boundary data is more involved. For this purpose, we may concentrate on the two variables t and y by considering u as an element of the space

$$u \in H_p^1(\mathbb{R}_+; L_q(\mathbb{R}_+; \tilde{E})) \cap L_p(\mathbb{R}_+; H_q^{2m}(\mathbb{R}_+; \tilde{E})),$$

where $\tilde{E} = L_q(\mathbb{R}^{n-1}; E)$. Note that $\tilde{E}$ also belongs to the class $\mathcal{HT}$ since E does by assumption and $q \in (1, \infty)$. W.l.og. we may assume $u_0 = 0$. Define an operator B in $X = L_p(\mathbb{R}_+; L_q(\mathbb{R}_+; \tilde{E}))$ by means of

$$Bu = (\partial_t)^{1/2m}, \quad D(B) = {}_0W_p^{1/2m}(\mathbb{R}_+; L_q(\mathbb{R}_+; \tilde{E})),$$

let A be given as

$$A = \partial_y, \quad D(A) = L_p(\mathbb{R}_+; {}_0H_q^1(\mathbb{R}_+; \tilde{E})),$$

and set $L = A + B$. By the Dore-Venni theorem, $1 + L$ with domain $D(A) \cap D(B)$ is invertible. Let $\alpha \in \mathbb{N}_0^n$ with $|\alpha| < 2m$ and set $u_\alpha = B^{2m-|\alpha|-1} D^\alpha u$; then

$$u_\alpha \in {}_0W_p^{1/2m}(\mathbb{R}_+; L_q(\mathbb{R}_+; \tilde{E})) \cap L_p(\mathbb{R}_+; H_q^1(\mathbb{R}_+; \tilde{E})).$$

Solve the problem

$$v + Lv = u_\alpha + Bu_\alpha + \partial_y u_\alpha \in L_p(\mathbb{R}_+; L_q(\mathbb{R}_+; \tilde{E})),$$

and set $w = e^y(u_\alpha - v)$. Then $w = e^{-By}(u_\alpha|_{y=0})$ has the same regularity as u_α, hence by Lemma 6.2 $u_\alpha|_{y=0} \in {}_0F_{pq}^{1/2m(1-1/q)}(\mathbb{R}_+; \tilde{E})$. This yields

$$D^\alpha u|_{y=0} \in {}_0F_{pq}^{1-|\alpha|/2m-1/2mq}(\mathbb{R}_+; \tilde{E}),$$

which implies $g_j \in {}_0F_{pq}^{\kappa_j}(\mathbb{R}_+; L_q(\mathbb{R}^{n-1}; E))$. This completes the necessity part of the proof.

Sufficiency. We extend f to $\mathbb{R}_+ \times \mathbb{R}^n$ by zero and choose an extension of $u_0 \in B_{qp}^{2m(1-1/p)}(\mathbb{R}^n; E)$. Define $u_1 \in W_{pq}^{1,2m}(\mathbb{R}_+ \times \mathbb{R}^n; E)$ as the unique solution of

$$\begin{aligned}(\mu + \partial_t + \mathcal{A}(D))u_1 &= f \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^n,\\ u_1(0, x) &= u_0 \quad \text{in } \mathbb{R}^n.\end{aligned}$$

(Here we use Proposition 6.1.) Then set $u_2 = u - u_1$; obviously, u_2 must satisfy

$$\begin{aligned}(\mu + \partial_t + \mathcal{A}(D))u_2 &= 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{R}_+^n,\\ \mathcal{B}_j(D)u_2 &= \tilde{g}_j \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^{n-1},\ j = 1, \ldots, m,\\ u_2(0, x) &= 0 \quad \text{in } \mathbb{R}_+^n\end{aligned} \tag{6.3}$$

with

$$\tilde{g}_j := g_j - \mathcal{B}_j(D)u_1 \in F^{\kappa_j}_{pq}(\mathbb{R}_+; L_q(\mathbb{R}^{n-1}; E)) \cap L_p(\mathbb{R}_+; B^{2m\kappa_j}_{qq}(\mathbb{R}^{n-1}; E)),$$

by the necessity part of this proof. Because of the compatibility condition, we have $\tilde{g}_j|_{t=0} = 0$. As in the proof of Proposition 4.5 we set

$$h_j(t, x', y) := L^{(2m-m_j)/2m} e^{-yL^{1/2m}} \tilde{g}_j(t, x'),$$

where $L = \mu + \partial_t + (-\Delta)^m$ with domain

$$D(L) = {}_0H^1_p(\mathbb{R}_+; L_q(\mathbb{R}^{n-1}; E)) \cap L_p(\mathbb{R}_+; H^{2m}_q(\mathbb{R}^{n-1}; E)).$$

The solution operators $T^j = F^{-1}_{t\to\lambda} T^j_\lambda F_{t\to\lambda}$ are also bounded from $L_p(\mathbb{R}_+; L_q(\mathbb{R}^n_+; E))$ to $W^{1,2}_{pq}(\mathbb{R}_+ \times \mathbb{R}^n_+)$, and $L^{(2m-m_j)/2m}$ maps

$${}_0W^{1-m_j/2m}_p(\mathbb{R}_+; L_q(\mathbb{R}^n_+; E)) \cap L_p(\mathbb{R}_+; H^{2m-m_j}_q(\mathbb{R}^n_+; E))$$

into $X := L_p(\mathbb{R}_+; L_q(\mathbb{R}^n_+; E))$. Thus we obtain $u_2 := \sum_{j=1}^m T^j h_j \in W^{1,2}_{pq}$, provided $v := L^{1/2m} e^{-yL^{1/2m}} g \in X := L_{pq}(\mathbb{R}_+ \times \mathbb{R}^n_+)$, whenever

$$g \in {}_0F^{1/2m-1/2mq}_{pq}(\mathbb{R}_+; E) \cap L_p(\mathbb{R}_+; B^{1-1/q}_{qq}(\mathbb{R}^{n-1}; E)).$$

So let g belong to this class. From Lemma 6.2 and the remark following it we know that $Be^{-yB}g$ belongs to X and with $C = (-\Delta)^{1/2}$ we also have $Ce^{-yC}g \in X$. Since $D(L^{1/2m}) = D(B) \cap D(C)$ we see by the perturbation theorem 1.5 of [11] that $L^{1/2m} - \eta(B + C + 1)$ is bounded in X, for $\eta > 0$ sufficiently small. By the Dore-Venni theorem, $\partial_y + L^{1/2m}$ has maximal regularity in X. Solve the problem

$$\partial_y w + L^{1/2m} w = (B + C + 1)e^{-\eta(B+C+1)y} g, \quad y > 0, \; w(0) = 0.$$

Observe that

$$(B + C + 1)e^{-\eta(B+C+1)y} g = e^{-\eta(C+1)y} Be^{-\eta By} g + e^{-\eta(B+1)y} Ce^{-\eta Cy} g + e^{-(B+C+1)y} g$$

belongs to X since B and C commute, and $e^{-\eta y(B+1/2)}$ and $e^{-\eta y(C+1/2)}$ are bounded in X. With w we obtain the following representation of $L^{1/2m} e^{-L^{1/2m} y} g$.

$$L^{1/2m} e^{-L^{1/2m} y} g = L^{1/2m} e^{-\eta(B+C+1)y} g + (\eta - L^{1/2m}(B + C + 1)^{-1}) L^{1/2m} w.$$

This representation shows that $L^{1/2m} e^{-L^{1/2m} y} g$ belongs to X.

Finally, $u = u_1 + u_2$ is the unique solution of (6.2), depending continuously on the data. □

The proof of Theorem 2.3 follows from Propositions 6.1 and 6.4 as in the case $p = q$ via perturbation and localization. We do not repeat these arguments here.

Remark 6.5. *We remark that the special case of $m = 1$, $E = \mathbb{C}$ and $q \le p$ has been obtained recently by Weidemaier* [34].

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