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Smooth Surjective Toric Quotients are Categorical

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Abstract

We prove a criterion for the existence of a categorical quotient for the action of a subtorus on a smooth non–degenerate toric variety in the category of smooth algebraic varieties.

Introduction

In this note we give a criterion for the existence of categorical quotients for subtorus actions on toric varieties. Recall that a categorical quotient for a regular action of an algebraic group G on a variety X is a G–invariant morphism $p: X \to Y$ that is universal with respect to G–invariant morphisms from X to varieties Z. In general, the problem of existence of such quotients is quite delicate.

In order to obtain existence statements it is often useful, to treat the problem in a modified category. For proper actions the category of algebraic spaces is reasonable. Another approach is the category of dense constructible subsets, proposed by A. Białynicki–Birula in [BB]. In both cases the category of algebraic varieties is enlarged.

Sometimes it is also suitable to consider smaller categories: for example, in [AC,Ha;2], the problem of existence of categorical quotients for subtorus actions on toric varieties is solved for the category of quasi–projective varieties. In the present article we consider the analogue in the category of smooth varieties: Let X be a smooth non–degenerate toric variety and let H be an algebraic subgroup of the acting torus of X. Let $p: X \to Y$ be the toric quotient constructed in [AC,Ha;1]. Then we obtain (see Corollary 3.3):

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Theorem. *If p is surjective and Y is smooth, then p is a categorical quotient in the category of smooth algebraic varieties.*

If we drop surjectivity of p in the above statement, then we obtain existence of categorical quotients in the category of smooth dense constructible subsets (see Corollary 3.2). In the proof of these results, toric *prevarieties* play a central role:

In order to factorize a given H–invariant regular map $f\colon X \to Z$ we embed Z into a certain toric prevariety Z' and consider the situation in the framework of toric prevarieties. Besides the machinery developped in [AC,Ha;3] and Section 1, the technical heart of the proof consists of a lifting result for f (see Section 2) and the decomposition theorem for f into an H–invariant toric part and a non–toric part (see Section 3).

1 Toric Prevarieties as Quotients

In many situations it is useful to represent a given toric variety as a quotient space of an open toric subvariety of some $\mathbb{C}^r$ by the action of an algebraic subgroup of $(\mathbb{C}^*)^r$. The most basic example of such a representation is writing the projective space $\mathbb{P}_n$ as the quotient space of $\mathbb{C}^{n+1} \setminus \{0\}$ by the diagonal action of $\mathbb{C}^*$. For an arbitrary toric variety different variations of representing it as a quotient space are to be found in [De], [Au], [Co] and [Br,Ve].

For later use, we generalize in this section the construction of [Co] to certain toric prevarieties. Our construction will differ slightly from the representation given in [AC,Ha;3]. We begin with recalling [Co], Section 1: Let $X = X_\Delta$ denote the toric variety corresponding to a fan Δ in a lattice N and assume that the rays of Δ generate $N_{\mathbb{R}}$. That means that the variety does not contain a torus as a proper factor, and we will call such a toric variety *non–degenerate.*

For every ray $\varrho \in \Delta^{(1)}$ let v_ϱ denote the primitive vector in $\varrho \cap N$, and define a lattice homomorphism $Q\colon \widetilde{N} := \mathbb{Z}^{\Delta^{(1)}} \to N$ by $e_\varrho \mapsto v_\varrho$. For every maximal cone $\sigma \in \Delta^{\max}$ define a strictly convex cone in $\widetilde{N}$ by setting

$$\widetilde{\sigma} := \operatorname{cone}(e_\varrho;\ \varrho \in \Delta^{(1)})$$

Then the cones $\widetilde{\sigma}$, $\sigma \in \Delta$, are the maximal cones of a fan $\widetilde{\Delta}$ in $\widetilde{N}$, and Q induces a map of fans from $\widetilde{\Delta}$ to Δ. Denote the open toric subvariety of $\mathbb{C}^{\Delta(1)}$ corresponding to $\widetilde{\Delta}$ by $\mathrm{C}(X)$. The toric morphism $q\colon \mathrm{C}(X) \to X$ induced by Q is a good quotient with respect to the kernel $q^{-1}(x_0) \subset (\mathbb{C}^*)^{\Delta^{(1)}}$. Here x_0 denotes the base point of X. Summarizing we have:

1.1 Proposition. *For every non–degenerate toric variety X there is an open toric subvariety $\mathrm{C}(X)$ of some $\mathbb{C}^r$ such that $\mathbb{C}^r \setminus C(X)$ is of dimension at most $r-2$, and a toric morphism $q\colon \mathrm{C}(X) \to X$ that is a good quotient with respect to an algebraic subgroup of $(\mathbb{C}^*)^r$.*

In order to generalize this result to toric prevarieties we have to work with the notions introduced in [AC,Ha;3]. Let X be a complex algebraic prevariety and assume that a reductive group G acts on X by means of a regular map $G \times X \to X$. A G–invariant regular map $p\colon X \to Y$ onto a prevariety Y is called a *good prequotient* for the action of G on X if

i) p is an affine map, i.e., for every affine open subspace V of Y the open subspace $U := p^{-1}(V)$ of X is affine,

ii) $\mathcal{O}_Y$ is the sheaf $(p_*\mathcal{O}_X)^G$ of invariants, i.e., for every open set $V \subset Y$ we have $\mathcal{O}_Y(V) = \mathcal{O}_X(p^{-1}(V))^G$.

This notion generalizes the concept of a good quotient in the following sense: if a good prequotient exists and if both, X and Y, are separated, then the good prequotient is nothing but the good quotient. But in general, if X is separated, the action of G may admit a good prequotient but no good quotient. A simple example is the $\mathbb{C}^*$–action on $\mathbb{C}^2 \setminus \{0\}$ given by

$$t\cdot(z,w) := (tz, t^{-1}w).$$

Now we come to the case of a toric prevariety X. Recall that X is by definition a normal irreducible prevariety together with an effective regular action of an algebraic torus T that has an open orbit. The notion of a toric morphism is defined as in the separated case (see [AC,Ha;3], Section 3). We obtain the following criterion for good prequotients (compare [AC,Ha;3], Corollary 6.10:

1.2 Lemma. *Let $p\colon X \to X'$ be a surjective affine toric morphism of prevarieties and let H denote the kernel of the homomorphism of the acting tori associated to p. Then p is a good prequotient for the action of H on X.* □

In the sequel we use the language of systems of fans introduced in [AC,Ha;3], Section 2. To perform our analogue of Cox's construction, let $X_{\mathcal{S}}$ be a toric prevariety arising from an affine system of fans $\mathcal{S} = (\Delta_{ij})_{i,j\in I}$ in a lattice N. The most important facts are that each Δ_{ii} is a fan of faces of a cone σ_{ii} in N, the Δ_{ij} are subfans of $\Delta_{ii} \cap \Delta_{jj}$, and $X_{\mathcal{S}}$ is obtained as the glueing of the affine toric varieties $X_{\Delta_{ii}}$ along the open toric subvarieties $X_{\Delta_{ij}}$.

Let R denote the set of equivalence classes $[\varrho, i] \in \Omega(\mathcal{S})$ where ϱ is one–dimensional. Then any $[\varrho, i] \in R$ defines a distinguished point $x_{[\varrho,i]} \in X_{\mathcal{S}}$ and the orbit of the acting torus T of $X_{\mathcal{S}}$ through $x_{[\varrho,i]}$ is of codimension one in $X_{\mathcal{S}}$. This sets up a one–to–one correspondence between the elements of R and the T–orbits of codimension one in $X_{\mathcal{S}}$. The closure of such an orbit is the T–stable Weil divisor

$$D_{[\varrho,i]} := \overline{T\cdot x_{[\varrho,i]}} = \bigcup_{[\varrho,i]\prec[\tau,k]} T\cdot x_{[\tau,k]}.$$

Our representation of $X_{\mathcal{S}}$ as a quotient space will work only for special toric prevarieties. We will have to assume that $X_{\mathcal{S}}$ is of *affine intersection*, i.e., for any two maximal affine

T–stable charts $X_i, X_j \subset X_{\mathcal{S}}$ their intersection $X_i \cap X_j$ is again affine. In our setting this means that for any two $i, j \in I$ the fan Δ_{ij} is the fan of faces of a single cone σ_{ij}. The following property of such toric prevarieties will be used later:

1.3 Remark. *If $X := X_{\mathcal{S}}$ is of affine intersection, then for any maximal T–stable affine chart $X_i := X_{\Delta_{ii}}$, $i \in I$, we have*

$$X \backslash X_i \;=\; \bigcup_{[\varrho,j] \not\prec [\sigma_{ii},i]} D_{[\varrho,j]}.$$

Proof. First recall, that by [AC,Ha;3], Remark 2.8, the orbit $T \cdot x_{[\varrho,j]}$ is contained in X_i if and only if $[\varrho, j] \prec [\sigma_{ii}, i]$ holds. Hence we have

$$[\varrho, j] \not\prec [\sigma_{ii}, i] \Leftrightarrow D_{[\varrho,j]} \subset X \setminus X_i.$$

In particular, the inclusion "$\supset$" follows. As to "$\subset$", let $x \in X \setminus X_i$. Then, for some $j \in I$, we have $x \in X_j \setminus X_i$. Since $X_j \cap X_i$ is affine, the set $X_j \setminus X_i$ is a T–stable Weil divisor D in X_j. The closure of D in X contains x, lies in $X \setminus X_i$ and is a union of certain $D_{[\varrho,j]}$ with $[\varrho, j] \not\prec [\sigma_{ii}, i]$ □

As before, we consider the case when $X_{\mathcal{S}}$ is non–degenerate, i.e., it does not contain a proper torus factor. Then the rays of the Δ_{ii} generate N. Set $\widetilde{N} := \mathbb{Z}^R$. As before, for $\varrho \in \bigcup \Delta_{ij}^{(1)}$ denote by v_ϱ the primitive lattice vector contained in ϱ. Define a lattice homomorphism by

$$Q\colon \widetilde{N} \to N, \qquad Q(e_{[\varrho,i]}) := v_\varrho.$$

For every $i \in I$ define a strictly convex cone in $\widetilde{N}$ by setting

$$\widetilde{\sigma}_i := \operatorname{cone}(e_{[\varrho,i]};\ \varrho \in \Delta_{ii}^{(1)})$$

Then the cones $\widetilde{\sigma}_i$, $i \in I$ are the maximal cones of a fan $\widetilde{\Delta}$ in $\widetilde{N}$. Let $\widetilde{\mathcal{S}}$ denote the affine system of fans associated to $\widetilde{\Delta}$. Since $X_{\mathcal{S}}$ was assumed to be of affine intersection, we can use [AC,Ha;3], Lemma 5.6 to obtain

1.4 Remark. *The homomorphism Q together with $\mu := \mathrm{id}_I$ determines a map of systems of fans $(Q, \mathfrak{q})$ from $\widetilde{\mathcal{S}}$ to $\mathcal{S}$.*

By construction, the toric morphism $q\colon X_{\widetilde{\mathcal{S}}} \to X_{\mathcal{S}}$ corresponding to q is affine and surjective. Let H be the kernel of the homomorphism of the acting tori associated to q. The above criterion for good prequotients implies that the map q is a good prequotient for the action of H on $\mathrm{C}(X) := X_{\widetilde{\mathcal{S}}}$.

For a general toric prevariety of affine intersection we first choose a decomposition of the form $X = X' \times T''$, where T'' is a torus and X' a toric prevariety that does not contain a torus as a proper factor. Then as above we set $\mathrm{C}(X) := \mathrm{C}(X') \times T''$ and $q := q' \times \mathrm{id}_{T''}\colon \mathrm{C}(X) \to X$. Summarizing we obtain the following result:

1.5 Proposition. *Let X be a toric prevariety of affine intersection. Then there is an open toric subvariety $\mathrm{C}(X)$ of some $\mathbb{C}^r$ and a toric morphism $q\colon \mathrm{C}(X) \to X$, that is a good prequotient with respect to the action of an algebraic subgroup of $(\mathbb{C}^*)^r$ on $\mathrm{C}(X)$. If X is non–degenerate, one can achieve that $\mathbb{C}^r \setminus \mathrm{C}(X)$ is of dimension at most $r - 2$.*

2 Lifting of Morphisms

The aim of this section is to prove that a given (not necessarily toric) morphism from a non–degenerate toric variety X to a smooth toric prevariety X' of affine intersection can be lifted to a morphism of the spaces $\mathrm{C}(X) \subset \mathbb{C}^r$ and $\mathrm{C}(X') \subset \mathbb{C}^s$ constructed in the previous section.

2.1 Theorem. *Let X be a non–degenerate toric variety and let $f\colon X \to X'$ be a morphism to a smooth toric prevariety X' of affine intersection such that $f(X)$ intersects the open orbit of X'. Then there is a morphism $F\colon \mathrm{C}(X) \to \mathrm{C}(X')$ such that the diagram*

$$\begin{array}{ccc} \mathrm{C}(X) & \xrightarrow{F} & \mathrm{C}(X') \\ {\scriptstyle q_X}\downarrow & & \downarrow{\scriptstyle q_{X'}} \\ X & \xrightarrow{f} & X' \end{array}$$

is commutative. If moreover f is constant on H–orbits for an algebraic subgroup H of the acting torus T of X, then every component of F is a polynomial that is homogeneous with respect to some character of the group $H' := q_X^{-1}(H)$.

In order to prepare the proof of this theorem we consider the following special case: Assume that $X = \mathrm{C}(X)$, in other words that X is an open toric subvariety of some $\mathbb{C}^r$ such that $\mathbb{C}^r \setminus X$ is of dimension at most 2. Moreover, assume that X' is non–degenerate, i.e., it does not contain a torus as a proper factor.

Since X' is a toric prevariety it arises from an affine system of fans $\mathcal{S} = (\Delta_{ij})_{i,j \in I}$ in a lattice N. Let T' denote the acting torus of X'. As in Section 1, we consider the set R of equivalence classes $[\varrho, i] \in \Omega(\mathcal{S})$ with ϱ one–dimensional. To simplify notation we fix an enumeration $R = \{1, \ldots, s\}$.

Consider the T'–stable Weil divisors D_k, $k = 1, \ldots, s$, corresponding to the elements of R. Since X' is smooth, each of these divisors is in fact Cartier. Since moreover $f(X)$ intersects the open T'–orbit we obtain for every k a pullback $f^*(D_k) \in \mathrm{CDiv}(X)$. Now, $\mathbb{C}^r \setminus X$ is of codimension at least two. Thus we find polynomials $h_k \in \mathbb{C}[x_1, \ldots, x_r]$ with

$$f^*(D_k) = \mathrm{div}(h_k).$$

2.2 Lemma. *If f is H–invariant for some algebraic subgroup H of $(\mathbb{C}^*)^r$, then each h_k is homogeneous with respect to a character $\chi_k\colon H \to \mathbb{C}^*$.*

Proof. Since the divisor $f^*(D_k)$ is locally defined by H–invariant regular functions, it is invariant under the canonical action of H on $\mathrm{WDiv}(X)$. Hence, for any $g \in H$, we have

$$\mathrm{div}(g \cdot h_k) = g \cdot \mathrm{div}(h_k) = \mathrm{div}(h_k).$$

Thus $g \cdot h_k = \alpha_k(g) h_k$ holds for some $\alpha_k(g) \in \mathbb{C}^*$. The function $\alpha_k\colon H \to \mathbb{C}^*$ is in fact a character, and h_k is homogeneous with respect to α_k^{-1}. □

In order to obtain the desired lifting of the map f to $\mathrm{C}(X')$, we consider for every $t \in (\mathbb{C}^*)^s$ the morphism $F_t \colon X \to \mathbb{C}^s$, defined by

$$F_t(x) := t\cdot(h_1(x), \ldots, h_s(x)) = (t_1 h_1(x), \ldots, t_s h_s(x)).$$

2.3 Lemma. *The morphism F_t has the following properties*

i) $f^{-1}(D_k) = V(X; h_k) = F_t^{-1}(V(\mathbb{C}^s; x_k))$.

ii) $F_t(X) \subset \mathrm{C}(X')$.

iii) *For every T'–stable maximal affine chart $X'_i = X'_{[\sigma_{ii}, i]}$ of X' we have*

$$F_t(f^{-1}(X'_i)) \subset q_{X'}^{-1}(X'_i).$$

Proof. Assertion i) is clear by definition. We verify ii): Let $x \in X$ be any point. Consider the set

$$J := \{k \in R;\ f(x) \in D_k\}.$$

Let $[\tau, j] \in \Omega(\mathcal{S})$ be the element with $f(x) \in T'\cdot x_{[\tau,j]}$. By definition of the set J, one has

$$k = [\varrho, i] \prec [\tau, j] \ \Leftrightarrow\ k \in J.$$

As in Section 1 let $\widetilde{\Delta}$ denote the fan describing $\mathrm{C}(X')$. The cone above $[\tau, j]$ is

$$\sigma := \mathrm{cone}(e_{[\varrho,i]};\ [\varrho, i] \in J) \in \widetilde{\Delta}.$$

Note that the coordinates of the distinguished point $x_\sigma \in \mathrm{C}(X')$ satisfy

$$(x_\sigma)_k = 0 \ \Leftrightarrow\ k \in J.$$

On the other hand by i), $F_t(x)_k = 0$ if and only if $k \in J$. This implies $F_t(x) \in (\mathbb{C}^*)^s\cdot x_\sigma \in \mathrm{C}(X')$.

In order to check iii), assume that there exists a point $x \in f^{-1}(X'_i)$ with $F_t(x) \notin q_{X'}^{-1}(X'_i)$. Then Remark 1.3 implies that $q_{X'}(F_t(x)) \in D_k$ for some $k = [\varrho, j] \not\prec [\sigma_{ii}, i]$. Therefore

$$F_t(x) \in q_{X'}^{-1}(D_k) = |\operatorname{div}(x_k)|$$

holds, or in other words, $F_t(x)_k = 0$, and hence by i), $x \in f^{-1}(D_k)$. This in turn implies $f(x) \in X' \setminus X'_i$, and we arrive at a contradiction. □

Proof of Theorem 2.1. Of course, we may assume that $X = \mathrm{C}(X)$. If $X' = X'' \times T''$ holds for some torus T'' and a toric prevariety X'', then by definition, $\mathrm{C}(X') = \mathrm{C}(X'') \times T''$ and $q_{X'} = q_{X''} \times id_{T''}$. Therefore any lifting $F' \colon X \to \mathrm{C}(X'')$ of $f' := \mathrm{pr}_1 \circ f$ yields a lifting $F = (F', \mathrm{pr}_2 \circ f) \colon X \to \mathrm{C}(X')$ of f. Hence it suffices to treat the case that X' is non–degenerate.

In this situation we have the maps F_t defined above, and Lemma 2.3 shows that we can restrict the morphisms f and $q_{X'} \circ F_t$ to obtain well–defined morphisms from $f^{-1}(X'_i)$ to

X_i' for every $i \in I$. Because X_i' is separated, it suffices to prove that for a suitable choice of $t \in (\mathbb{C}^*)^s$ these morphisms coincide on the dense open subset $f^{-1}(T' \cdot x_0)$ of $f^{-1}(X_i')$. To this end it suffices to find a $t \in (\mathbb{C}^*)^s$ such that the diagram

$$\begin{array}{ccc} \mathbb{C}(X) & \xleftarrow{F_t^*} & \mathbb{C}(\mathrm{C}(X')) \\ f^* \nwarrow & & \nearrow q^* \\ & \mathcal{O}(T' \cdot x_0) & \end{array}$$

is commutative. Recall that $\mathbb{C}(\mathrm{C}(X')) = \mathbb{C}[x_1, \ldots, x_s]$ holds and that $\mathcal{O}(T' x_0)$ is generated by the characters χ^u, $u \in M := \mathrm{Hom}(N; \mathbb{Z})$ of T'. For every $u \in M$ we have

$$\begin{aligned} F_t^*(q_{X'}^*(\chi^u)) &= F_t^*\left(\prod_{k=1}^s x_k^{\langle u, v_k \rangle}\right) \\ &= \prod_{k=1}^s (t_k h_k)^{\langle v_k, u \rangle} \\ &= \left(\prod_{k=1}^s t_k^{\langle v_k, u \rangle}\right)\left(\prod_{k=1}^s h_k^{\langle v_k, u \rangle}\right). \end{aligned}$$

On the other hand, the character χ^u defines a T'–stable divisor on X', namely $\sum_k \langle v_k, u \rangle D_k$. Pulling back this divisor to X we obtain

$$\mathrm{div}(f^*(\chi^u)) = f^*(\mathrm{div}(\chi^u)) = \sum_{k=1}^s \langle v_k, u \rangle f^*(D_k) = \sum_{k=1}^s \langle v_k, u \rangle \, \mathrm{div}(h_k) = \mathrm{div}\left(\prod_{k=1}^s h_k^{\langle u, v_k \rangle}\right).$$

This calculation shows that $\mathrm{div}(f^*(\chi^u) = \mathrm{div}(F_t^*(q^*(\chi^u)))$, and hence $f^*(\chi^u)$ can only differ from $F_t^*(q^*(\chi^u))$ by some constant.

Now, let $u_1, \ldots, u_n$ be a $\mathbb{Z}$–basis of M. Then for every u_j there is a constant c_j with $f^*(\chi^{u_j}) = c_j \prod_{k=1}^s h_k^{\langle v_k, u_j \rangle}$. We consider the following system of equations:

$$\prod_{k=1}^s t_k{}^{\langle v_k, u_j \rangle} = c_j, \quad j = 1, \ldots, n.$$

We claim that there is a solution $t = (t_1, \ldots, t_s)$ of this system. To check this, recall that the homomorphism from $(\mathbb{C}^*)^s$ to $T' = (\mathbb{C}^*)^n$ associated to the toric morphism $q_{X'}$ is given by

$$\varphi\colon (t_1, \ldots, t_s) \mapsto \prod_{k=1}^s t_k{}^{\langle v_k, u_j \rangle}.$$

Thus the existence of a solution t follows from the surjectivity of φ, and for this t the above diagram is commutative. □

3 Decomposition of Morphisms

As before let X denote a non–degenerate toric variety. Let T be the acting torus of X and let $H \subset T$ be a closed subgroup. We prove the following decomposition result for regular maps:

3.1 Theorem. *Let $f: X \to Z$ be an H–invariant regular map to a smooth algebraic variety Z. Then there exists a dominant H–invariant toric morphism $g: X \to X'$, an open subset $U \subset X'$ with $g(X) \subset U$ and a regular map $h: U \to Z$ such that $f = h \circ g$.*

We apply this result to investigate quotients. We formulate our statement in the category of so–called *dc–subsets*, introduced by A. Białynicki–Birula: The objects of this category are pairs (U, X), where U is a dense constructible subset of an algebraic variety X. A morphism from a pair (U, X) to (U', X') is given by a rational map $\varphi: X \to X'$ such that U is contained in the domain of definition of φ and that $\varphi(U) \subset U'$.

Now suppose that we are given an algebraic action of an algebraic group G on a variety X, and let U be a G–stable dense constructible subset of X. Then a categorical quotient for the action of G on (U, X) in the category of dc–subsets is defined to be a G–invariant morphism p from (U, X) such that every G–invariant morphism from (U, X) factors uniquely through p.

We turn back to the action of the algebraic subgroup H of the acting torus on X. A *toric quotient* for the action of H on X is a toric morphism $p: X \to X \, /_{\mathrm{tor}} \, H$ that is universal with respect to H–invariant toric morphisms. Such a toric quotient does always exist:

According to [AC,Ha;1] there is a toric quotient $p_0: X \to X \, /_{\mathrm{tor}} \, H^0$ for the action of the identity component H^0 of H. The quotient space $X \, /_{\mathrm{tor}} \, H^0$ is again a toric variety with an acting torus T_0 that is a quotient of T/H^0. Note that the action of a finite subgroup of the acting torus on a toric variety always admits a geometric quotient that is automatically a toric quotient. So we can divide the space $X \, /_{\mathrm{tor}} \, H^0$ by the induced action of the finite group H/H^0 to obtain a toric quotient $p: X \to X \, /_{\mathrm{tor}} \, H$ for the action of H on X.

Now, assume that X is smooth. As an immediate consequence of the above decomposition theorem we obtain the following existence results for categorical quotients:

3.2 Corollary. *If the toric quotient space $X \, /_{\mathrm{tor}} \, H$ is smooth, then the map $X \to p(X)$ induced by the toric quotient p of X by H is a categorical quotient in the category of smooth dc–subsets.* □

3.3 Corollary. *If moreover the toric quotient p is surjective, then it is a categorical quotient in the category of smooth algebraic varieties.* □

Proof of Theorem 3.1. First w.l.o.g. we can make some assumptions on the situation. Consider Cox's construction for X to obtain a toric morphism $p: \mathrm{C}(X) \to X$, where $\mathrm{C}(X)$ is an open toric subvariety of some $\mathbb{C}^n$ with at least 2–codimensional boundary and p is a good (and hence also toric) quotient for the action of $H_1 := p^{-1}(x_0)$ on $\mathrm{C}(X)$.

Now suppose that we can decompose the $H' := p^{-1}(H)$–invariant morphism $f_1 := f \circ p$ into $f_1 = h_1 \circ g_1$ where g_1 is an H'–invariant toric morphism, and where h_1 is a regular map that is defined on an open neighbourhood U_1 of the image of g_1. Since p is a toric quotient for the action of $H_1 \subset H'$, the toric morphism g_1 factors through p. In other words there is a dominant H–invariant toric morphism g with $g_1 = g \circ p$. Moreover, since p is surjective, the image of g is contained in U_1. So $f = h_1 \circ g$ is the desired decomposition. Therefore we can assume $X = \mathrm{C}(X)$ in this proof.

As a further simplification we note that it suffices to give a proof for the case that H is connected: Suppose that X', g, h and U satisfy the assertion for the identity component H^0 of H. Then g induces an action of $\Gamma := H/H^0$ on X'. Let $r\colon X' \to X''$ be the geometric quotient for this action.

By appropriate shrinking, we can achieve that U is Γ–invariant. Since r is geometric, this means that $r^{-1}(r(U)) = U$ and hence $r(U)$ is open in X'' and the restriction $r\colon U \to r(U)$ is again a geometric quotient for the action of Γ. The regular map h is necessarily Γ–invariant, and therefore factors through r. So one has $h = h' \circ r$ for some regular map $h'\colon r(U) \to Z$. It follows that $f = h' \circ (r \circ g)$ is the desired decomposition. Consequently we may assume in our proof that H is connected.

After these preparations we can begin the proof of the theorem. By the modification of Włodarczyk's embedding theorem given in [Ha], there is a closed embedding $\iota\colon Z \to Z'$ of Z into a toric prevariety Z' of affine intersection. As [Wł] and [Ha], Proof of Theorem 9.1 and Proposition 1.1 iv) show, we can choose Z' to be smooth.

In order to obtain the desired factorization of f, we can view f as map from X to the closure Y of $f(X)$ in Z. Though Y is possibly singular, we still have an embedding $\iota\colon Y \to Z'$ into the smooth toric prevariety Z' of affine intersection. Let $Y' \subset Z'$ the minimal orbit closure of the acting torus of Z' such that $\iota(Y) \subset Y'$. Note that Y' is again a smooth toric prevariety of affine intersection and that $\iota(f(X))$ intersects the open orbit of Y'.

As explained in Section 1, we can present Y' as a good prequotient $q\colon \mathrm{C}(Y') \to Y'$ of an open toric subvariety $\mathrm{C}(Y')$ of some $\mathbb{C}^s$. Let $F\colon X \to \mathrm{C}(Y')$ be the lifting of $\iota \circ f$ with respect to q as it was constructed in Theorem 2.1. Recall that each component of F is in fact a polynomial that is homogeneous with respect to some character of H.

By writing the components of F as sums of monomials, we obtain a decomposition of F in the form $F = S \circ g'$, with a toric morphism $g'\colon X \to \mathbb{C}^r$ and a linear map $S\colon \mathbb{C}^r \to \mathbb{C}^s$. Note that S is H–equivariant with respect to the actions of H induced on $\mathbb{C}^r$ by g and on $\mathbb{C}^s$ by F.

Let W denote the normalization of the closure of $g'(X)$ in $\mathbb{C}^r$. Then W is an affine toric variety with big torus $T_W := g'(T_X)$. We can lift the morphism g' to a dominant toric morphism $\widehat{g}\colon X \to W$, and pull back S to a regular map $\widehat{S}\colon W \to \mathbb{C}^s$. Both, $\widehat{g}$ and $\widehat{S}$, are again equivariant for the induced H–action on W. So far we are in the following situation:

$$
\begin{array}{ccc}
W & \xrightarrow{\widehat{S}} & \mathbb{C}^s \\
{\scriptstyle \widehat{g}}\uparrow & & \cup \\
X & \xrightarrow{F} & \mathrm{C}(Y') \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle q} \\
Y & \xrightarrow{\iota} & Y'
\end{array} .
$$

The subset $V := \widehat{S}^{-1}(\mathrm{C}(Y'))$ is open in W and H–stable. Since the above diagram is commutative, we have $\widehat{g}(X) \subset V$. Note that $\widehat{g}(X)$ is dense in V. Thus, setting $Y := q^{-1}(\iota(Y))$ we obtain

$$\overline{\widehat{S}(V)} = \overline{\widehat{S}(\widehat{g}(X))} = \overline{F(X)} \subset Y,$$

where the closures are taken in $\mathrm{C}(Y')$. In particular it follows that $\widehat{S}(V) \subset Y$. We claim that the restriction $\widehat{S}\colon V \to Y$ is an affine map. To check this, note first, that $\widehat{S}\colon W \to \mathbb{C}^r \to \mathbb{C}^s$ as a composition of affine maps is affine. Thus $\widehat{S}\colon V \to \mathrm{C}(Y')$ is affine. Since Y is closed in $\mathrm{C}(Y')$ our claim follows.

Moreover, the restriction of q to Y yields an affine H–invariant morphism $q\colon Y \to i(Y)$. So $q \circ \widehat{S}\colon V \to i(Y)$ is an affine H–invariant regular map. With [Ra], Lemma 4.1 (see also [Ne], Prop. 3.12), we can conclude that V admits a good quotient $o\colon V \to V/\!/H$ for the action of H. So we obtain the following commutative diagram of regular maps:

$$\begin{array}{ccc} V & \xrightarrow{\widehat{S}} & Y \\ {\scriptstyle o}\downarrow & & \downarrow{\scriptstyle q} \\ V/\!/H & \xrightarrow{h'} & \iota(Y) \end{array}$$

By a result of J. Swiecicka (see [Sw]), there is an open toric subvariety V_1 of W containing V that has a good toric quotient $o_1\colon V_1 \to V_1/\!/H$ such that the map $V/\!/H \to V_1/\!/H$ induced by $V \subset V_1$ is an open inclusion.

As the final diagram shows, the data for the desired decomposition of f are the toric variety $X' := V_1/\!/H$, the open subset $U := V/\!/H$, the toric morphism $g := o_1 \circ \widehat{g}$ and the regular map $h := \iota^{-1} \circ h'$:

$$\begin{array}{ccccc} X & \xrightarrow{\widehat{g}} & V & \subset & V_1 \\ & {\scriptstyle g}\searrow & {\scriptstyle o}\downarrow & & \downarrow{\scriptstyle o_1} \\ {\scriptstyle f}\downarrow & & V/\!/H & \subset & V_1/\!/H \\ & {\scriptstyle h}\swarrow & {\scriptstyle h'}\downarrow & & \\ Y & \cong & \iota(Y) & & \end{array} \quad . \quad \square$$

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