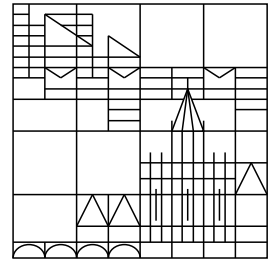


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A HILBERT-MUMFORD-CRITERION FOR SL_2 -ACTIONS

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ABSTRACT. Let the special linear group $G := \mathrm{SL}_2$ act regularly on a \mathbb{Q} -factorial variety X . Consider a maximal torus $T \subset G$ and its normalizer $N \subset G$. We prove: If $U \subset X$ is a maximal open N -invariant subset admitting a good quotient $U \rightarrow U//N$ with a divisorial quotient space, then the intersection $W(U)$ of all translates $g \cdot U$ is open in X and admits a good quotient $W(U) \rightarrow W(U)//G$ with a divisorial quotient space. Conversely, we obtain that every maximal open G -invariant subset $W \subset X$ admitting a good quotient $W \rightarrow W//G$ with a divisorial quotient space is of the form $W = W(U)$ for some maximal open N -invariant U as above.

INTRODUCTION

Given an action of a reductive group G on a variety X , the task of Geometric Invariant Theory is the construction of open G -invariant subsets $W \subset X$ that admit reasonable quotients. We ask here for *good quotients*, that means G -invariant affine regular maps $p: W \rightarrow W//G$ of prevarieties such that the structure sheaf of $W//G$ equals the sheaf of invariants $p_*(\mathcal{O}_W)^G$, compare [9]. Note that we allow nonseparated quotient spaces.

The purpose of Hilbert-Mumford-Criteria is to reduce the construction of quotients $W \rightarrow W//G$ to the construction of quotients $U \rightarrow U//T$ for some maximal torus $T \subset G$. More precisely, one considers the following problem, compare [1], [2], [3] and [4]: Suppose that the open set $U \subset X$ is invariant under the normalizer $N \subset G$ of T and admits a good quotient $U \rightarrow U//T$. When the intersection W of all translates $g \cdot U$, $g \in G$, is open in X and admits a good quotient $W \rightarrow W//G$?

In this note we continue the study of the case $G = \mathrm{SL}_2$ started in [2] and [3], where the above problem was solved for complete and for quasiprojective quotient spaces. Generalizing the latter setting, we focus on *divisorial* quotient spaces, i.e., prevarieties Y such that each $y \in Y$ has an affine neighbourhood of the form $Y \setminus \mathrm{Supp}(D)$ with an effective Cartier divisor D on Y , compare [6].

We shall use the following notions of maximality for open subsets admitting a good quotient: Let $H \subset G$ be any reductive subgroup. By an *H-subset* we mean H -invariant open subset $U \subset X$ admitting a good quotient $U \rightarrow U//H$. We say that an H -subset $U \subset X$ is

- *d-maximal* if $U//H$ is divisorial, and U does not occur as a saturated subset of some properly larger H -subset $U' \subset X$ with $U'//H$ divisorial,
- *s-d-maximal* if $U//H$ is separated and divisorial, and U is not a saturated subset of some properly larger H -subset $U' \subset X$ with $U'//H$ separated and divisorial.

As above, fix a maximal torus $T \subset G$, and let $N \subset G$ be its normalizer. The main result of this note provides a recipe to construct all d -maximal and all s - d -maximal G -subsets of a G -variety X from the collection of all d -maximal N -subsets of X , see Theorem 2.2:

Theorem. *Let $G = \mathrm{SL}_2(\mathbb{K})$ act regularly on a \mathbb{Q} -factorial variety X . For any open subset $U \subset X$ let*

$$W(U) := \bigcap_{g \in G} g \cdot U.$$

- (i) *If $U \subset X$ is a d -maximal N -subset then $W(U)$ is open in X and admits a good quotient with a divisorial prevariety $W(U) // G$.*
- (ii) *Every d -maximal G -subset $W \subset X$ is of the form $W = W(U)$ for some d -maximal open N -subset $U \subset X$.*
- (iii) *Every s - d -maximal G -subset $W \subset X$ is of the form $W = W(V)$ for some s - d -maximal open N -subset $V \subset X$.*

Note that in contrast to the setting of [2] and [3], we do not obtain algebraic spaces as quotient spaces. So, our result may also be viewed as an algebraicity criterion for quotient spaces. The proof of the above statement relies on the generalization of Mumford's Geometric Invariant Theory developed in [7]. In Section 1, we give brief a introduction to this theory and also provide some slight extensions. The proof of the above result is performed in Section 2.

1. GENERALIZED SEMISTABILITY

In [7] we generalized the construction of good quotients of [8] by replacing Mumford's single G -linearized line bundle with a certain group of Cartier divisors. The result is a theory producing all divisorial quotient spaces instead of only the quasiprojective ones. We recall here the basic results and adapt them to our actual purposes.

Let X be an irreducible prevariety over an algebraically closed field \mathbb{K} of characteristic zero. Fix a finitely generated free group $\Lambda \subset \mathrm{CDiv}(X)$ of Cartier divisors on X . Then one has an associated Λ -graded \mathcal{O}_X -algebra

$$\mathcal{A} := \bigoplus_{D \in \Lambda} \mathcal{A}_D := \bigoplus_{D \in \Lambda} \mathcal{O}_X(D).$$

The \mathcal{O}_X -algebra \mathcal{A} gives rise to a prevariety $\widehat{X} := \mathrm{Spec}(\mathcal{A})$ and a canonical map $q: \widehat{X} \rightarrow X$ such that $q_*(\mathcal{O}_{\widehat{X}})$ equals \mathcal{A} . For a homogeneous local section $f \in \mathcal{A}_D(U)$, one defines its set of zeroes to be

$$Z(f) := \mathrm{Supp}(D|_U + \mathrm{div}(f)).$$

We call the group Λ *ample* on an open subset $U \subset X$, if every $x \in U$ has an affine neighbourhood of the form $U \setminus Z(f)$ for a homogeneous $f \in \mathcal{A}(U)$. If Λ is ample on U , then $\widehat{U} := q^{-1}(U)$ is a quas affine variety. Note that there exists a $\Lambda \subset \mathrm{CDiv}(X)$ which is ample on X if and only if X is *divisorial* in the sense of [6].

Now, let $G \times X \rightarrow X$ be a regular action of a reductive group on X . As in [7], we mean by a *G -linearization* of the group Λ a graded G -sheaf structure on the \mathcal{O}_X -algebra \mathcal{A} such that for every G -invariant open $U \subset X$ the induced representation of G on $\mathcal{A}(U)$ is rational.

Definition 1.1. Let $\Lambda \subset \text{CDiv}(X)$ be G -linearized, and let $U \subset X$ be a G -invariant open subset. A point $x \in U$ is called U -semistable, if x has an affine neighbourhood $U' = U \setminus Z(f)$ with some G -invariant $f \in \mathcal{A}_D(U)$ such that the $D' \in \Lambda$ admitting a G -invariant $f' \in \mathcal{A}_{D'}(U')$ which is invertible in $\mathcal{A}(U')$ form a subgroup of finite index in Λ .

For a given G -linearized $\Lambda \subset \text{CDiv}(X)$ and a G -invariant open $U \subset X$, the associated set of U -semistable points will be denoted by $U^{ss}(\Lambda)$, or $U^{ss}(\Lambda, G)$ if we want to specify the group G . Note that for $U = X$, the above definition specializes to the notion of semistability introduced in [7].

Following the lines of [7, Section 2], we show now that every set of U -semistable points admits a good quotient. First let us recall the precise definition of the latter notion:

Definition 1.2. A *good quotient* for a regular action of a reductive group G on a prevariety X is a G -invariant affine regular map $p: X \rightarrow X//G$ of prevarieties such that the canonical map $\mathcal{O}_{X//G} \rightarrow p_*(\mathcal{O}_X)^G$ is an isomorphism. A *geometric quotient* is a good quotient that separates orbits. Geometric quotients are denoted by $p: X \rightarrow X/G$.

Suppose now that $\Lambda \subset \text{CDiv}(X)$ is G -linearized, and let $U \subset X$ be any G -invariant open subset. We shall need a geometric interpretation of U -semistability similar to the case $U = X$ treated in [7, Proposition 2.3]. For this we have to assume that Λ is ample on U . Let \mathcal{A} and $\widehat{X} := \text{Spec}(\mathcal{A})$ be as above.

Recall from [7, Section 1] that the map $q: \widehat{X} \rightarrow X$ is a geometric quotient for the action of the algebraic torus $H := \text{Spec}(\mathbb{K}[\Lambda])$ on \widehat{X} defined by the Λ -grading of \mathcal{A} . Moreover, the quasiaffine variety $\widehat{U} = q^{-1}(U)$ carries a regular G -action that commutes with the H -action and makes $q: \widehat{U} \rightarrow U$ equivariant.

Let $f_1, \dots, f_r \in \mathcal{A}(U)$ be homogeneous and G -invariant such that the sets $U \setminus Z(f_i)$ are as in Definition 1.1 and cover $U^{ss}(\Lambda)$. Choose a $(G \times H)$ -equivariant affine closure \overline{U} of \widehat{U} such that the functions $f_i \in \mathcal{O}(\widehat{U})$ extend regularly to \overline{U} and $\overline{U}_{f_i} = \widehat{U}_{f_i}$ holds for each $i = 1, \dots, r$. Then we have a good quotient

$$\overline{p}: \overline{U} \rightarrow \overline{U}//G := \text{Spec}(\mathcal{O}(\overline{U}))^G.$$

The quotient variety $\overline{U}//G$ inherits a regular action of the torus H such that the map $\overline{p}: \overline{U} \rightarrow \overline{U}//G$ becomes H -equivariant. Similar to [7, Proposition 2.3], we obtain:

Lemma 1.3. *Let $V_0 := \overline{U}//G \setminus \overline{p}(\overline{U} \setminus \widehat{U})$, and let $V_1 \subset \overline{U}//G$ be the union of all H -orbits with finite isotropy. Then*

$$q^{-1}(U^{ss}(\Lambda)) = \overline{p}^{-1}(V_0 \cap V_1).$$

Proof. Set $W := U^{ss}(\Lambda)$ and $\widehat{W} := q^{-1}(W)$. We begin with the inclusion “ \subset ”. By [7, Remark 1.6] the set $q^{-1}(U \setminus Z(f_i))$ equals \widehat{U}_{f_i} . Since each of the latter sets is \overline{p} -saturated and \widehat{W} is covered by the \widehat{U}_{f_i} , we see that \widehat{W} is \overline{p} -saturated. In particular, we obtain $\overline{p}(\widehat{W}) \subset V_0$.

To verify $\overline{p}(\widehat{W}) \subset V_1$, let $z \in \widehat{W}$. Take one of the f_i with $z \in \widehat{U}_{f_i}$. As it is G -invariant, f_i descends to an H -homogeneous function $h \in \mathcal{O}(\overline{U}//G)$. By the properties of f_i , the function h satisfies the condition of [7, Lemma 2.4] for the point $\overline{p}(z)$. Hence $H_{\overline{p}(z)}$ is finite, which means $\overline{p}(z) \in V_1$.

We come to the inclusion “ \supset ” of the assertion. Let $y \in V_0 \cap V_1$. Then [7, Lemma 2.4] provides an $h \in \mathcal{O}(\overline{U} // G)$, homogeneous with respect to some $\chi^D \in \text{Char}(H)$, such that $y \in V := (\overline{U} // G)_h$ holds and the $D' \in \Lambda$ admitting an invertible $\chi^{D'}$ -homogeneous function on V form a subgroup of finite index in Λ . Suitably modifying h , we achieve additionally $V \subset V_0 \cap V_1$.

Now, consider a point $z \in \overline{p}^{-1}(y)$. Since $y \in V_0$, we have $z \in \widehat{U}$. We have to show that $q(z)$ is U -semistable. For this, consider the G -invariant homogeneous section $f := \overline{p}^*(h)|_{\widehat{U}}$ of $\mathcal{A}_D(U)$. By the choice of h , this f fulfills the conditions of Definition 1.1 and thus the point $q(z)$ is in fact U -semistable. \square

As a consequence of this geometric description, we obtain existence of a good quotient for the set of U -semistable points for any G -invariant open subset $U \subset X$. The result generalizes [7, Theorem 3.1]:

Proposition 1.4. *Let $\Lambda \subset \text{CDiv}(X)$ be G -linearized, and let $U \subset X$ be an open G -invariant subset. Then there is a good quotient*

$$p: U^{ss}(\Lambda) \rightarrow U^{ss}(\Lambda) // G$$

and the quotient space $U^{ss}(\Lambda) // G$ is a divisorial prevariety. Moreover, for every G -invariant homogeneous $f \in \mathcal{A}(U^{ss}(\Lambda))$, the set $Z(f)$ is p -saturated.

Proof. We may assume that $U = U^{ss}(\Lambda)$ holds. Then Λ is ample on U , and we are in the setting of Lemma 1.3. Since the set \widehat{U} is saturated with respect to the good quotient $\overline{p}: \overline{U} \rightarrow \overline{U} // G$, restricting \overline{p} to \widehat{U} yields a good quotient $\widehat{p}: \widehat{U} \rightarrow \widehat{U} // G$.

Moreover, Lemma 1.3 tells us that H acts with at most finite isotropy groups on $\widehat{U} // G$. Thus, there is a geometric quotient $\widehat{U} // G \rightarrow (\widehat{U} // G) / H$. By [7, Lemma 3.3], the latter quotient space is a divisorial prevariety. Since good quotients are categorical, we obtain a commutative diagram

$$\begin{array}{ccc} \widehat{U} & \xrightarrow{\widehat{p}} & \widehat{U} // G \\ \downarrow /H & & \downarrow /H \\ U & \longrightarrow & (\widehat{U} // G) / H \end{array}$$

Now it is straightforward to check that the induced map $U \rightarrow (\widehat{U} // G) / H$ is the desired good quotient for the action of G on U . This proves the first part of the assertion.

For the supplement, consider a G -invariant homogeneous $f \in \mathcal{A}(U^{ss}(\Lambda))$. By [7, Remark 1.6], the set A of zeroes of $f \in \widehat{U}$ equals $q^{-1}(Z(f))$. Now, A is \widehat{p} -saturated, and $\widehat{p}(A)$ is saturated with respect to the geometric quotient $\widehat{U} // G \rightarrow (\widehat{U} // G) / H$. Thus surjectivity of the involved maps gives the claim. \square

Similar to [8], we have also a converse of Proposition 1.4. Recall from [7, Section 4] the a group $\Lambda \subset \text{CDiv}(X)$ is said to be *canonically G -linearized*, if on every homogeneous component \mathcal{A}_D the G -sheaf structure arises from the action

$$(g \cdot f)(x) := f(g^{-1} \cdot x)$$

on the function field $\mathbb{K}(X)$. Now, assume that X is a \mathbb{Q} -factorial G -variety, and let $U \subset X$ be an open G -invariant subset with a good quotient $U \rightarrow U // G$. The proof of [7, Theorem 4.1] gives:

Theorem 1.5. *If $U//G$ is divisorial, then there exists a canonically G -linearized $\Lambda \subset \text{CDiv}(X)$ such that U is contained in $X^{ss}(\Lambda)$ and is saturated with respect to the quotient map $X^{ss}(\Lambda) \rightarrow X^{ss}(\Lambda)//G$.*

2. PROOF OF THE MAIN RESULT

In this section, we prove our main result. First we have to introduce the following notions of maximality for open sets with quotient, compare also [1]:

Definition 2.1. Let the reductive group H act regularly on a variety Y . We say that $V \subset Y$ is an H -subset if it is open, H -invariant and admits a good quotient $V \rightarrow V//H$. We say that an H -subset $V \subset Y$ is

- (i) *d-maximal* if $V//H$ is divisorial, and V does not occur as a saturated subset of some properly larger H -subset $V' \subset X$ with $V'//H$ divisorial,
- (ii) *s-d-maximal* if $V//H$ is separated and divisorial, and V is not a saturated subset of some properly larger H -subset $V' \subset X$ with $V'//H$ separated and divisorial.

Here a *saturated* subset of an H -subset V' is a subset that is saturated with respect to the quotient map $V' \rightarrow V'//H$.

Now, consider the special linear group $G := \text{SL}_2(\mathbb{K})$. Fix a maximal torus $T \subset G$, and denote by N its normalizer in G . For example:

$$T := \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}; t \in \mathbb{K}^* \right\}, \quad N = T \cup nT, \quad \text{with } n := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The main result of this note generalizes and enhances [3, Theorem 9]. In the above notation, it reads as:

Theorem 2.2. *Let $G = \text{SL}_2(\mathbb{K})$ act regularly on a \mathbb{Q} -factorial variety X . For any open subset $U \subset X$ let*

$$W(U) := \bigcap_{g \in G} g \cdot U.$$

- (i) *If $U \subset X$ is a d-maximal N -subset then $W(U)$ is open in X and admits a good quotient with a divisorial prevariety $W(U)//G$.*
- (ii) *Every d-maximal G -subset $W \subset X$ is of the form $W = W(U)$ with a d-maximal N -subset $U \subset X$.*
- (iii) *Every s-d-maximal G -subset $W \subset X$ is of the form $W = W(V)$ with an s-d-maximal N -subset $V \subset X$.*

In the proof we shall use the techniques presented in Section 1. Let $\Lambda \subset \text{CDiv}(X)$ be a G -linearized group. Then Λ is also linearized with respect to every subgroup of G . In particular, we obtain a set $U^{ss}(\Lambda, H)$ of semistable points for G -invariant open $U \subset X$ and every reductive subgroup $H \subset G$.

Remark 2.3. Let $U \subset X$ be a G -invariant open subset, let $H \subset G$ be a reductive subgroup, and let $g \in G$. Then a point $x \in U$ is U -semistable with respect to H if and only if $g \cdot x$ is U -semistable with respect to gHg^{-1} .

The crucial step in the proof of Theorem 2.2 is to express generalized semistability in terms of maximal tori. For this, let $\text{MT}(G)$ denote the set of maximal tori of the group G , and fix an element $T \in \text{MT}(G)$.

Lemma 2.4. *Let $G = \mathrm{SL}_2(\mathbb{K})$ act regularly on a variety X , let $\Lambda \subset \mathrm{CDiv}(X)$ be a G -linearized group, and let $U \subset X$ be any G -invariant subset such that Λ is ample on U . Then we have:*

$$U^{ss}(\Lambda, G) = \bigcap_{S \in \mathrm{MT}(G)} U^{ss}(\Lambda, S) = \bigcap_{g \in G} g \cdot U^{ss}(\Lambda, T).$$

Moreover, the set $U^{ss}(\Lambda, G)$ is saturated in $U^{ss}(\Lambda, T)$ with respect to the quotient map $U^{ss}(\Lambda, T) \rightarrow U^{ss}(\Lambda, T) // T$.

Proof. The supplement follows from Proposition 1.4. The second equality is clear by Remark 2.3. Moreover, the inclusion “ \subset ” of the first equality is valid by the definition of semistability. Thus we are left with proving the inclusion “ \supset ” of the first equality.

For this, let \mathcal{A} be the graded \mathcal{O}_X -algebra associated to Λ , and set $\widehat{X} := \mathrm{Spec}(\mathcal{A})$. Moreover, let $q: \widehat{X} \rightarrow X$ be the canonical map and denote by $H := \mathrm{Spec}(\mathbb{K}[\Lambda])$ the algebraic torus acting on \widehat{X} . Finally, let $\widehat{U} := q^{-1}(U)$.

Choose G -invariant homogeneous $f_1, \dots, f_r \in \mathcal{A}(U)$ and T -invariant homogeneous $h_1, \dots, h_s \in \mathcal{A}(U)$ such that the complements $U \setminus Z(f_i)$ and $U \setminus Z(h_j)$ satisfy the conditions of Definition 1.1 and

$$\begin{aligned} U^{ss}(\Lambda, G) &= (U \setminus Z(f_1)) \cup \dots \cup (U \setminus Z(f_r)), \\ U^{ss}(\Lambda, T) &= (U \setminus Z(h_1)) \cup \dots \cup (U \setminus Z(h_s)). \end{aligned}$$

Since Λ is ample on U , there is a $(G \times H)$ -equivariant affine closure \overline{U} of \widehat{U} such that the f_i and h_j extend to regular functions on \overline{U} satisfying $\overline{U}_{f_i} = \widehat{U}_{f_i}$ and $\overline{U}_{h_j} = \widehat{U}_{h_j}$. Moreover, we obtain a commutative diagram of H -equivariant maps:

$$\begin{array}{ccc} \overline{U} & \xrightarrow{\overline{p}_G} & \overline{U} // G \\ & \searrow \parallel T & \nearrow \\ & \overline{U} // T & \end{array}$$

In the further proof, we shall apply the geometric characterization of semistability given in Lemma 1.3 to this diagram. A first step is to verify the following inclusion:

$$\bigcap_{S \in \mathrm{MT}(G)} q^{-1}(U^{ss}(\Lambda, S)) \subset \widehat{U} \setminus \overline{p}_G^{-1}(\overline{p}_G(\overline{U} \setminus \widehat{U})).$$

Let $x \in U^{ss}(\Lambda, S)$ for all maximal tori $S \subset G$, and let $z \in q^{-1}(x)$. Set $y := \overline{p}_G(z)$. Suppose $y \in \overline{p}_G(\overline{U} \setminus \widehat{U})$. Let Gz' be the closed orbit in $\overline{p}_G^{-1}(y)$. Then Gz' is contained in $\overline{U} \setminus \widehat{U}$. Moreover, the Hilbert-Mumford-Birkhoff Lemma [5], provides a maximal torus $S \subset G$ such that the closure of $S \cdot z$ intersects $G \cdot z'$.

Let $g \in G$ with $gSg^{-1} = T$. Then the closure of $T \cdot g \cdot z$ contains a point $z'' \in G \cdot z'$. Surely, $\overline{p}_T(g \cdot z)$ equals $\overline{p}_T(z'')$. Thus, since $z'' \in \overline{U} \setminus \widehat{U}$, Lemma 1.3 tells us that $g \cdot x = q(g \cdot z)$ is not semistable with respect to T . By Remark 2.3, the point x is not semistable with respect to S . A contradiction. So the first step is done.

Now assume that there is a point $x \in X$ that is semistable with respect to all maximal tori of G but not with respect to G itself. By the above inclusion, we have $x = q(z)$ for some $z \in \widehat{U} \setminus \overline{p}_G^{-1}(\overline{p}_G(\overline{U} \setminus \widehat{U}))$. Lemma 1.3 tells us that for $y := \overline{p}_G(z)$

the isotropy group H_y is infinite. Let H_0 be the connected component of the neutral element of H_y , and let $G \cdot z_0$ be the closed G -orbit in the fibre $\overline{p_G^{-1}}(y)$.

Then H_0 acts freely on the fibre $\overline{p_G^{-1}}(y) \subset \widehat{U}$. Since $G \cdot z_0$ is the only closed G -orbit in this fibre, it is invariant under the action of H_0 . Let G_0 denote the stabilizer of $H_0 \cdot z_0$. We claim that $G_0 \cdot z_0 = H_0 \cdot z_0$ holds. Indeed, let $h \in H_0$. Since $H_0 \cdot z_0 \subset G \cdot z_0$, there is a $g \in G$ with $g \cdot z_0 = h \cdot z_0$. In fact, $g \in G_0$ because

$$g \cdot (H \cdot z_0) = H \cdot (g \cdot z_0) = H \cdot (h \cdot z_0) = H \cdot z_0.$$

Denoting by $\mu: G_0 \rightarrow G_0 \cdot z_0$ and $\tau: H_0 \rightarrow H_0 \cdot z_0$ the orbit maps, we obtain an epimorphism $G_0 \rightarrow H_0$, $g \mapsto \tau^{-1}(\mu(g))$ of algebraic groups. In particular, for every maximal torus $S_0 \subset G_0$ we have $S_0 \cdot z_0 = H_0 \cdot z_0$. In particular, the maximal tori of G_0 are nontrivial, and hence of dimension one. So, applying Lemma 1.3 to a maximal torus of G_0 yields

$$q(z_0) \notin \bigcap_{S \in \text{MT}(G)} U^{ss}(\Lambda, S).$$

By the choice of z , this implies that $G \cdot z \neq G \cdot z_0$. Hence $G \cdot z_0$ is of smaller dimension than $G \cdot z$. In other words, the isotropy group G_{z_0} is infinite. Note that $G_{z_0} \subset G_0$. Since $G \cdot z_0$ is affine, G_{z_0} is reductive and hence contains a nontrivial torus S_0 of G_0 . Since S_0 is by dimension reasons already a maximal torus of G_0 , this contradicts $S_0 \cdot z_0 = H_0 \cdot z_0$. \square

Proof of Theorem 2.2. We begin with assertion (i). According to Proposition 1.5, there is a canonically N -linearized group $\Lambda \subset \text{CDiv}(X)$ such that $U = X^{ss}(\Lambda, N)$ holds. We show that on a subgroup $\Lambda'' \subset \Lambda$ of finite index, the canonical N -linearization of Λ'' extends to a G -linearization of Λ'' , compare also [7, Proof of Theorem 5.1]:

Using Sumihiro's Equivariant Completion Theorem [10, Theorem 3] and equivariant normalization, we find a complete normal G -variety Y containing X as an open subset. After removing all singular points from Y that lie in $Y \setminus X$, we can extend Λ by closing components to a group $\Gamma \subset \text{CDiv}(Y)$. Then Γ is also canonically N -linearized.

On the other hand, some subgroup $\Gamma' \subset \Gamma$ of finite index admits a G -linearization. By construction, we have $\mathcal{O}(Y) = \mathbb{K}$. Hence on some further subgroup $\Gamma'' \subset \Gamma$ of finite index the N -linearization inherited by the G -linearization and the canonical N -linearization coincide, see [7, Proposition 1.5]. Thus, restricting Γ'' to X gives the desired subgroup $\Lambda'' \subset \Lambda$.

We replace Λ with the above Λ'' . Let $V \subset X$ be the maximal open subset such that Λ is ample on V . Note that V is G -invariant and that $U \subset V$ holds. Since U is a d-maximal N -set, we have $U = V^{ss}(\Lambda, N)$. So, Lemma 2.4 yields

$$W = \bigcap_{g \in G} g \cdot U = \bigcap_{g \in G} g \cdot V^{ss}(\Lambda, N) \subset \bigcap_{g \in G} g \cdot V^{ss}(\Lambda, T) = V^{ss}(\Lambda, G).$$

Since $V^{ss}(\Lambda, G)$ is contained in $V^{ss}(\Lambda, N)$, we have in fact equality. In particular, W is open, and Proposition 1.4 yields a good quotient $W \rightarrow W//G$ with a divisorial quotient space $X//G$. This proves assertion (i).

We prove (ii). If $W \subset X$ is a d-maximal G -set, then Proposition 1.5 tells us that $W = X^{ss}(\Lambda, G)$ holds for some G -linearized group $\Lambda \subset \text{CDiv}(X)$. According to (i), any d-maximal N -set $U \subset X$ containing $X^{ss}(\Lambda, N)$ as a saturated open subset satisfies $W = W(U)$.

We come to (iii). Let $W \subset X$ be an s-d-maximal G -subset. Then W admits a good quotient $W \rightarrow W//N$ with a separated divisorial quotient space. Hence there is an s-d-maximal N -subset $V \subset X$ containing W as a saturated subset. Let $V' \subset X$ be a d-maximal subset containing V as a saturated subset.

By assertion (i), the set $W' := W(V')$ is open in X and admits a good quotient $W' \rightarrow W'//G$ with $W'//G$ divisorial. We claim that $W = W' \cap V$ holds. The inclusion $W \subset W' \cap V$ is obvious. The reverse inclusion follows from s-d-maximality of W as a G -subset of X :

Since $W \subset V \subset V'$ and $W' \subset V'$ are saturated inclusions of N -sets, we can conclude that $W \subset W' \cap V \subset W'$ are saturated inclusions of N -sets. Using the classical Hilbert-Mumford Lemma [5], one obtains that $W \subset W'$ is even a saturated inclusion of G -sets. Thus, W is a s-d-maximal G -subset of W' .

Now consider the maximal separated open subsets V_1, \dots, V_k of the quotient space $W'//G$, see [1, Theorem I], and the following commutative diagram:

$$\begin{array}{ccc} W' & \xrightarrow{p} & W'//G \\ & \searrow q & \nearrow r \\ & & W'//N \end{array}$$

Let $W'_i := p^{-1}(V_i)$. Then $W = W'_{i_0}$ for some i_0 . Since r is affine, every s-d-maximal N -subset of W' is covered by some W'_i . Thus, the s-d-maximal N -subsets of W' are G -invariant. Hence [7, Theorem 5.1] tells us that they are exactly the W'_i . In particular, W is a s-d-maximal N -subset of W' .

Since the action of N on $V \cap W'$ admits a good quotient with a separated divisorial quotient space and $W \subset V \cap W' \subset W'$ are saturated inclusions of N -subsets, we obtain $W = W' \cap V$ and our claim is verified. Assertion (iii) then follows from

$$W(V) = \bigcap_{g \in G} g \cdot V = W' \cap \bigcap_{g \in G} g \cdot V = \bigcap_{g \in G} g \cdot (W' \cap V) = \bigcap_{g \in G} g \cdot W = W. \quad \square$$

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