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Filter Functions with Exponential Convergence Order

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Abstract: Oversampled functions can be evaluated using generalized sinc-series and filter functions connected with these series. A standard filter function is given by $\exp((\xi^2-1)^{-1})$. We show that the Fourier transform of this filter posseses the convergence order $O(\exp(-\sqrt{x}))$, improving an estimation given in [10]. Moreover, we define a family of filter functions with convergence order $O(x\cdot\exp(-x^\sigma))$ with σ arbitrary close to 1.

1 Introduction

We call a function f bandlimited with bandwidth b, if its Fourier-transform

$$\hat{f}(\xi) := \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x)e^{-ix\xi}\,dx$$

vanishes for all $|\xi| > b$. According to Shannon's sampling theorem, for a b-bandlimited function f and for every $h \le \frac{\pi}{b}$ the equation

$$(1)\qquad f(x) = \sum_{k\in\mathbb{Z}} f(hk)\,\mathrm{sinc}(\tfrac{\pi}{h}(x-hk))$$

holds, where $\mathrm{sinc}(x) := \frac{\sin x}{x}$. (A survey of the sampling theorem and some extensions can be found in [4], for an application in computerized tomography see [11].)

If we have $h < \frac{\pi}{b}$, the function f is called oversampled. In this case the convergence of the series (1) can be improved using filter functions. Here some function $\gamma \in C_c^\infty$ is called a filter function, if the support of γ is contained in [-1,1] and $\tilde{\gamma}(0) = 1$. ($\tilde{\gamma}$ denotes the inverse Fourier-transform of γ.) Let f be a bandlimited function with bandwidth b and γ a filter function. Then it is well known that for every $h < \frac{\pi}{b}$ the function f can be reconstructed by generalized sinc-series:

$$(2)\qquad f(x) = \sum_{k\in\mathbb{Z}} f(hk)\,\mathrm{sinc}(\tfrac{\pi}{h}(x-hk))\,\tilde{\gamma}((\tfrac{\pi}{h}-b)(x-hk))\,.$$

The series (2) converges much faster than (1), because for every C_c^∞-function the (inverse) Fourier-transform converges faster to 0 for $|x|\to\infty$ than every power of $\frac{1}{x}$. For an estimation of the truncation error of series (2) the convergence of the function $\tilde{\gamma}(x)$ for large x is essential, cf. [10].

It follows easily from the theory of entire functions that there cannot exist a filter function whose Fourier transform possesses the convergence order $O(\exp(-|x|))$. On the other hand, for every positive function w that satisfies $\int_{\mathbb{R}} \frac{w(x)}{1+x^2}\,dx < \infty$ and some regularity properties, there exists an entire function $\tilde{\gamma}$ of exponential type with convergence order $O(\exp[-w(|x|)])$ for $|x| \to \infty$, cf. [3], [8]. In [5] an explicit expression for a function $\tilde{\gamma}$ with almost optimal convergence properties is given in the form of an infinite product of sinc-terms.

In this paper we make a different approach: The weight function $\tilde{\gamma}$ in (2) is defined as the Fourier transform of an explicitly given filter function. In section 2 we show that a very good exponential order can already be achieved by modifying the standard example of a filter function which is given by

$$\gamma_0(\xi) := \begin{cases} c\, e^{\frac{1}{\xi^2-1}} & \text{if } |\xi| < 1\,, \\ 0 & \text{if } |\xi| \geq 1\,. \end{cases}$$

(Here the constant c is determined by $\tilde{\gamma}_0(0) = 1$.) Theorem 1 shows that the Fourier transform of this function has convergence order $O(\exp(-\sqrt{|x|}))$, improving an estimation given in [10]. In Theorem 2 we see that a generalization of the filter γ_0 leads to a convergence order $O(|x| \exp(-|x|^\sigma))$ where σ is arbitrary close to 1.

2 A family of filter functions with exponential convergence order

For the filter function γ_0 F. Natterer proved in [10] the following estimation:

$$(3) \qquad |\tilde{\gamma}_0(x)| \leq \frac{8c}{\sqrt{\pi}}\, x\, e^{1+\frac{4}{\sqrt{x}}-\sqrt{x}}\,.$$

The estimation (3), however, remains valid if the factor x on the right side is omitted:

Theorem 1: *For $x \geq 1$ we have:*

$$|\tilde{\gamma}_0(x)| \leq \frac{8c}{\sqrt{\pi}}\, e^{1+\frac{4}{\sqrt{x}}-\sqrt{x}}\,.$$

Proof: This estimation can be shown by a modification of the proof of Theorem 4 in [10]:
If we write

$$\tilde{\gamma}_0(x) = \frac{c}{\sqrt{2\pi}} \int_{-1}^{1} e^{\frac{1}{\xi^2-1}+ix\xi}\, d\xi$$

and integrate by parts k times we get for $x \geq 1$:

$$|\tilde{\gamma}_0(x)| \leq \frac{c}{\sqrt{2\pi}} \frac{1}{x^k} \int_{-1}^{1} \left| \left(\frac{d}{d\xi}\right)^k e^{\frac{1}{\xi^2-1}} \right| d\xi\,.$$

According to Cauchy's integral formula for $|\xi| < 1$ we have:

$$\left(\frac{d}{d\xi}\right)^k e^{\frac{1}{\xi^2-1}} = \frac{k!}{2\pi i} \int\limits_C e^{\frac{1}{z^2-1}} \frac{1}{(z-\xi)^{k+1}} \, dz \, ,$$

where C is the closed path in the complex plane that connects the points 1, i, -1, -i by straight lines. Let C_1 be the part of C connecting 1 and i. Then the following estimation holds (cf. [10]):

$$\left| \int\limits_{C_1} e^{\frac{1}{z^2-1}} \frac{1}{(z-\xi)^{k+1}} \, dz \right| \leq \sqrt{2} \int\limits_0^1 e^{\frac{1}{4(t-1)}} \frac{1}{(1-t)^{k+1}} \, dt \, .$$

Substituting $v = \frac{1}{1-t}$, the right side is equal to

$$\sqrt{2} \int_1^\infty e^{-\frac{1}{4}v} \, v^{k-1} \, dv \leq \sqrt{2} \int_0^\infty e^{-\frac{1}{4}v} \, v^{k-1} \, dv =$$

$$= \sqrt{2} \, 4^k \, \Gamma(k) = \sqrt{2} \, 4^k \, (k-1)!$$

(In [10] the integral over C_1 was estimated by $\sqrt{2} \, 4^{k+1} \, (k+1)!$). The rest of the proof uses the same steps as the proof in [10] and leads to the desired estimation given in the lemma. □

Let the filter function γ_α $(\alpha \geq 1)$ be defined by

$$\gamma_\alpha(\xi) := \begin{cases} c_\alpha \, e^{f(\alpha;\xi)} & \text{if } |\xi| < 1 \, , \\ 0 & \text{if } |\xi| \geq 1 \, , \end{cases}$$

where

$$f(\alpha;\xi) := -\beta \left(\frac{1}{1-\xi^2}\right)^\alpha \; , \; \beta := 2^{\alpha+\frac{3}{2}} \cdot \alpha^{\alpha-1} \, ,$$

and c_α is to be chosen such that $\tilde{\gamma}_\alpha(0) = 1$.
To achieve an estimation of the convergence order of $\tilde{\gamma}_\alpha$ we need the following result:

Lemma 1: *Let* $z = t + \frac{1-t}{\alpha} i$, $\alpha \geq 1$, $0 \leq t < 1$. *Then*

$$\operatorname{Re} f(\alpha;z) \leq -\frac{1}{\alpha} \left(\frac{\alpha}{1-t}\right)^\alpha .$$

Proof: For z like above, we have

$$\frac{1}{1-z^2} = \frac{1}{1-t} \cdot \frac{[1+\frac{1}{\alpha^2}+(1-\frac{1}{\alpha^2})t] + 2\,\frac{t}{\alpha}\,i}{[1+\frac{1}{\alpha^2}+(1-\frac{1}{\alpha^2})t]^2 + 4\,\frac{t^2}{\alpha^2}} \, .$$

Writing $\frac{1}{1-z^2}$ in polar coordinates $r\,e^{i\varphi}$, we get:

$$r^2 = \left|\frac{1}{1-z^2}\right|^2 = \frac{1}{(1-t)^2}\,\frac{1}{[1+\frac{1}{\alpha^2}+(1-\frac{1}{\alpha^2})t]^2+4\,\frac{t^2}{\alpha^2}}\,.$$

Here we see that for every fixed $\alpha \geq 1$, the denominator of the second quotient is a monotone increasing function of $t \in [0,1)$, and therefore

$$r \geq \frac{1}{1-t}\cdot\frac{1}{2\sqrt{1+\frac{1}{\alpha^2}}}\,.$$

For the angle φ we have $\varphi = \arctan g(t)$ with

$$g(t) := \frac{2\frac{t}{\alpha}}{1+\frac{1}{\alpha^2}+(1-\frac{1}{\alpha^2})t}\,.$$

Because of $g'(t) > 0$ for all $t \in [0,1]$, φ can be estimated by

$$\varphi \leq \arctan g(1) = \arctan\frac{1}{\alpha} \leq \frac{1}{\alpha}\,.$$

Therefore,

$$\begin{aligned}
&\mathrm{Re}\,f(\alpha;z) = \mathrm{Re}\,(-\beta\,r^\alpha\,e^{i\alpha\varphi}) = -\beta\,r^\alpha\,\cos(\alpha\varphi)\\
&\leq\ -\beta\left(\frac{1}{1-t}\right)^\alpha\frac{1}{2^\alpha}\,\frac{1}{(1+\frac{1}{\alpha^2})^{\frac{\alpha}{2}}}\ \cos 1\\
&\leq\ -\beta\,2^{-\alpha-\frac{3}{2}}\left(\frac{1}{1-t}\right)^\alpha,
\end{aligned}$$

where we used $\cos 1 \geq \frac{1}{2}$ and $(1+\frac{1}{\alpha^2})^{\alpha/2} \leq \sqrt{2}$. ($(1+\frac{1}{\alpha^2})^{\alpha/2}$ is a monotone decreasing function of $\alpha \geq 1$.)

With $\beta = 2^{\alpha+\frac{3}{2}}\cdot\alpha^{\alpha-1}$ this means

$$\mathrm{Re}\,f(\alpha;z) \leq -\alpha^{\alpha-1}\left(\frac{1}{1-t}\right)^\alpha = -\frac{1}{\alpha}\left(\frac{\alpha}{1-t}\right)^\alpha,$$

what was to be proved. □

Now the following estimation for the convergence order of $\tilde{\gamma}_\alpha$ can be established:

Theorem 2: *For $x \geq 1$ and $\alpha \geq 1$ the following inequality holds:*

$$|\tilde{\gamma}_\alpha(x)|\ \leq d_\alpha\,x\,\exp\left(\frac{\alpha+1}{\alpha}\,\frac{1}{x^\sigma} - \frac{\alpha+1}{\alpha}\,x^\sigma\right),$$

where the constants d_α and σ are defined by $d_\alpha := \frac{4\sqrt{2}\,e}{\pi}\,c_\alpha$ and $\sigma := \frac{\alpha}{\alpha+1}$.

Proof: Let C_α be the closed path consisting of four straight lines $C_{\alpha,j}$ $(1 \leq j \leq 4)$ that connect the points 1, $\frac{1}{\alpha}i$, -1 and $-\frac{1}{\alpha}i$ (see Fig. 1).

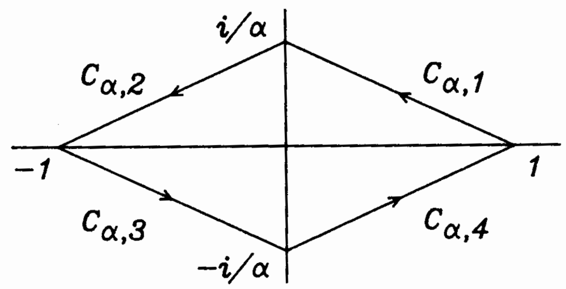

Fig. 1: The closed path C_α.

Like in the proof of Theorem 1 we get

$$(4) \qquad |\tilde{\gamma}_\alpha(x)| \leq \frac{c_\alpha}{\sqrt{2\pi}}\, x^{-k} \int_{-1}^{1} \Big| \Big(\frac{d}{d\xi}\Big)^k e^{f(\alpha;\xi)} \Big|\, d\xi$$

and

$$\Big(\frac{d}{d\xi}\Big)^k e^{f(\alpha;\xi)} = \frac{k!}{2\pi i} \int_{C_\alpha} e^{f(\alpha;z)}\, \frac{1}{(z-\xi)^{k+1}}\, dz\ .$$

Setting $z = t + \frac{1-t}{\alpha} i$ $(0 \leq t < 1)$, we obtain according to Lemma 1:

$$\begin{aligned} &\Big| \int_{C_{\alpha,1}} e^{f(\alpha;z)}\, \frac{1}{(z-\xi)^{k+1}}\, dz \Big| \leq \\ &\leq \ \sqrt{2} \int_0^1 e^{-\frac{1}{\alpha}(\frac{\alpha}{1-t})^\alpha}\, \frac{1}{|t + \frac{1-t}{\alpha}\, i - \xi|^{k+1}}\, dt \\ &\leq \ \sqrt{2} \int_0^1 e^{-\frac{1}{\alpha}(\frac{\alpha}{1-t})^\alpha} \Big(\frac{\alpha}{1-t}\Big)^{k+1} dt \\ &\leq \ \sqrt{2} \sup_{0<u\leq 1}\ e^{-\frac{1}{\alpha}(\frac{\alpha}{u})^\alpha} \Big(\frac{\alpha}{u}\Big)^{k+1} . \end{aligned}$$

Now we consider the inequality $a^b e^{-a} \leq b^b e^{-b}$, valid for all $a, b > 0$. Setting $a := \frac{1}{\alpha}\,(\frac{\alpha}{u})^\alpha$ and $b := \frac{k+1}{\alpha}$, we see that

$$e^{-\frac{1}{\alpha}(\frac{\alpha}{u})^\alpha} \leq (k+1)^{\frac{k+1}{\alpha}}\, e^{-\frac{k+1}{\alpha}} \Big(\frac{u}{\alpha}\Big)^{k+1} .$$

Therefore, the integral over $C_{\alpha,1}$ can be estimated by

$$\left| \int_{C_{\alpha,1}} e^{f(\alpha;z)} \frac{1}{(z-\xi)^{k+1}} \, dz \right| \leq \sqrt{2}\,(k+1)^{\frac{k+1}{\alpha}} \, e^{-\frac{k+1}{\alpha}} \, .$$

For the remaining three parts of C_α the same inequality is valid, and so

$$\left| \left(\frac{d}{d\xi}\right)^k e^{f(\alpha;\xi)} \right| \leq \frac{1}{2\pi} \, 4\sqrt{2}\, k!\,(k+1)^{\frac{k+1}{\alpha}} \, e^{-\frac{k+1}{\alpha}} \, .$$

Now we make use of the inequality

$$k! \leq \sqrt{2\pi}\, k^{k+\frac{1}{2}} \, e^{-k} \qquad [1,\ 6.1.38]$$

and get

$$\left| \left(\frac{d}{d\xi}\right)^k e^{f(\alpha;\xi)} \right| \leq \frac{4\sqrt{e}}{\sqrt{\pi}} \left(\frac{k+1}{e}\right)^{k+\frac{k+1}{\alpha}+\frac{1}{2}} \, .$$

Substituting this into (4) yields

$$|\tilde{\gamma}_\alpha(x)| \leq \frac{4\sqrt{2e}}{\pi} \, c_\alpha \, x^{-k} \left(\frac{k+1}{e}\right)^{k+\frac{k+1}{\alpha}+\frac{1}{2}} \, .$$

Differentiating the right side as a function of k, we see that a minimum is achieved if k is close to x^σ , $\sigma := \frac{\alpha}{\alpha+1}$. So we set $k+1 := x^\sigma + u$ with $u \in [0,1)$:

$$\begin{aligned} |\tilde{\gamma}_\alpha(x)| &\leq \frac{4\sqrt{2e}}{\pi} \, c_\alpha \, x^{-x^\sigma - u + 1} \left(\frac{x^\sigma + u}{e}\right)^{\frac{\alpha+1}{\alpha}(x^\sigma+u)-\frac{1}{2}} \\ &\leq \quad \frac{4\sqrt{2}\,e}{\pi} \, c_\alpha \, x \left(1 + \frac{u}{x^\sigma}\right)^{\frac{\alpha+1}{\alpha}(x^\sigma+u)-\frac{1}{2}} e^{-\frac{\alpha+1}{\alpha}(x^\sigma+u)} \, . \end{aligned}$$

If we set $a := u \cdot x^{-\sigma}$ and $b := \frac{\alpha+1}{\alpha}(x^\sigma + u) - \frac{1}{2}$ in $(1+a)^b \leq e^{ab}$ (valid for all $a, b \geq 0$), we see that

$$|\tilde{\gamma}_\alpha(x)| \leq \frac{4\sqrt{2}\,e}{\pi} \, c_\alpha \, x \, \exp\left(\frac{\alpha+1}{\alpha} \frac{u^2}{x^\sigma} - \frac{u}{2x^\sigma} - \frac{\alpha+1}{\alpha} x^\sigma \right) \, .$$

Because of $\frac{\alpha+1}{\alpha}u^2 - \frac{u}{2} \leq \frac{\alpha+1}{\alpha}$ for all $u \in [0,1)$, the following inequality holds:

$$|\tilde{\gamma}_\alpha(x)| \leq \frac{4\sqrt{2}e}{\pi} \, c_\alpha \, x \, \exp\left(\frac{\alpha+1}{\alpha} \frac{1}{x^\sigma} - \frac{\alpha+1}{\alpha} x^\sigma \right) \, ,$$

and the proof is completed. □

Theorem 2 enables us to estimate the convergence of the partial sums in the generalized sinc-series (2):

Lemma 2: *Let f be b-bandlimited and $h < \frac{\pi}{b}$. We set for $\alpha \geq 1$ and $K \geq 1$:*

$$f_{K,\alpha}(x) := \sum_{|x-hk|<hK} f(hk)\,\mathrm{sinc}(\tfrac{\pi}{h}(x-hk))\,\tilde{\gamma}_\alpha((\tfrac{\pi}{h}-b)(x-hk))\,.$$

If we write $w := K(\pi - bh)$, then

$$|(f - f_{K,\alpha})(x)| \leq \frac{1}{\sqrt{h}}\, d_\alpha\, w\, \exp\left(\frac{\alpha+1}{\alpha} w^{-\frac{\alpha}{\alpha+1}} - \frac{\alpha+1}{\alpha} w^{\frac{\alpha}{\alpha+1}}\right) \cdot \|f\|_{L^2}$$

holds if $w \geq 1$.

Proof: This follows immediately from Theorem 3 in [10] and Theorem 2.

□

According to Lemma 2, it is possible to achieve an accuracy of order $w \cdot \exp(-\frac{\alpha+1}{\alpha} w^{\frac{\alpha}{\alpha+1}})$ by evaluating $2K+1$ terms (where $w = K(\pi - bh)$) in the generalized sinc-series (2) with γ_α instead of γ. For numerical applications some piecewice rational approximation of $\tilde{\gamma}_\alpha$ can be used, as was proposed in [10] for $\tilde{\gamma}_0$.

Figures 2 and 3 show the Fourier transforms of γ_1 and γ_2 in logarithmic vertical scale. We see that the higher convergence order of $\tilde{\gamma}_2$ is fundamental for calculation with high accuracy. However, for lower values of x, $\tilde{\gamma}_1(x)$ decreases faster than $\tilde{\gamma}_2(x)$. This effect of increasing main-lobe width and decreasing side-lobe level is discussed in [6] for some classical filter functions. For optimality criterions of filter functions, see e.g. [2], [7].

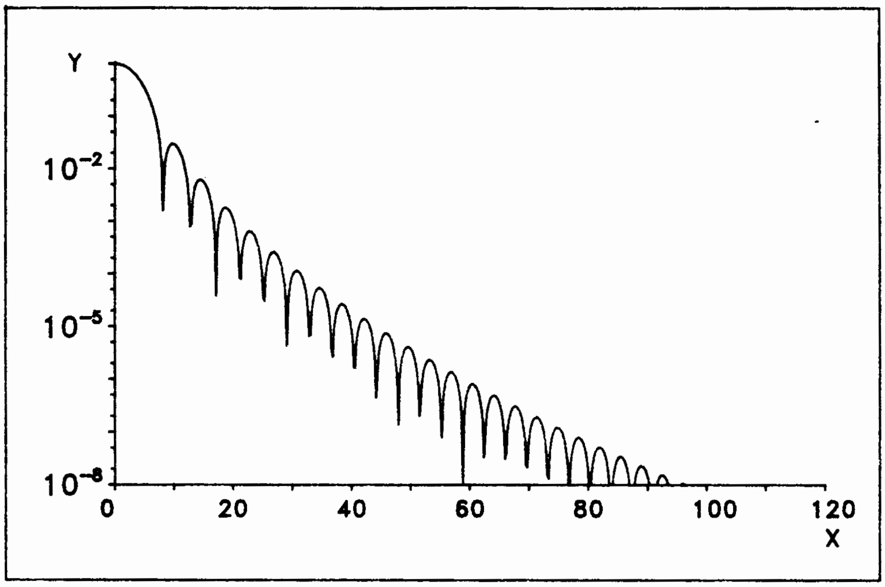

Fig. 2: The Fourier transform of γ_1, $y = \tilde{\gamma}_1(x)$.

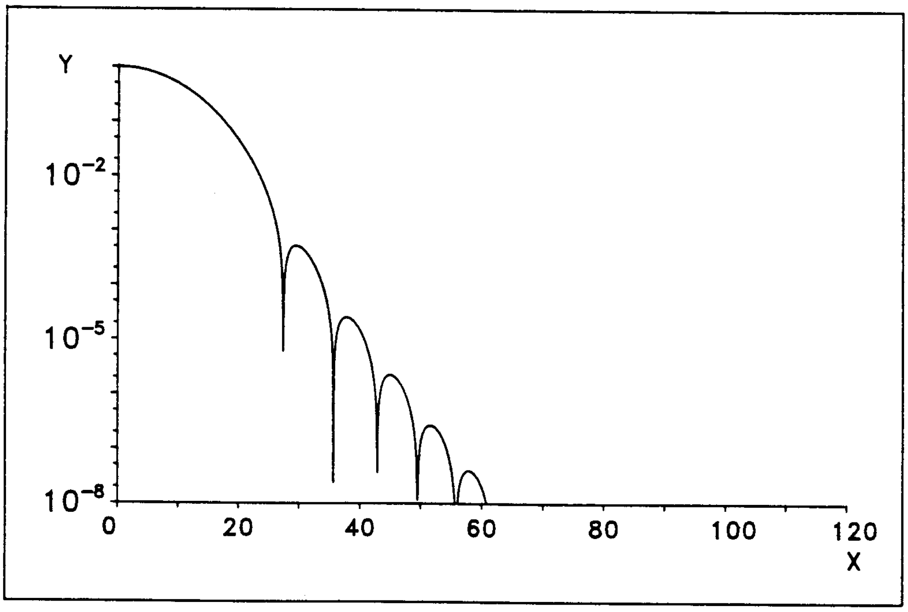

Fig. 3: The Fourier transform of γ_2, $y = \tilde{\gamma}_2(x)$.

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