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# Mildly dissipative nonlinear Timoshenko systems — global existence and exponential stability*

Jaime E. Muñoz Rivera and Reinhard Racke

Abstract: We consider nonlinear systems of Timoshenko type in a one-dimensional bounded domain. The system has a dissipative mechanism being present only in the equation for the rotation angle; it is a damping effect through heat conduction. The global existence of small, smooth solutions as well as the exponential stability are investigated.

## 1 Introduction

We consider a nonlinear Timoshenko type system

$$\rho_1\varphi_{tt} - \sigma(\varphi_x, \psi)_x = 0, \tag{1.1}$$

$$\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \gamma\theta_x = 0, \tag{1.2}$$

$$\rho_3\theta_t - \kappa\theta_{xx} + \gamma\psi_{tx} = 0, \tag{1.3}$$

where the functions $\varphi, \psi$ and $\theta$ depending on $(t,x) \in (0,\infty) \times (0,L)$ model the transverse displacement of a beam with reference configuration $(0,L) \subset \mathbb{R}$, the rotation angle of a filament, and the temperature difference, respectively.
By $\rho_1, \rho_2, \rho_3, b, k, \gamma, \kappa$ we are denoting positive constants, and $\sigma$ will be assumed to satisfy

$$\sigma_{\varphi_x}(0,0) = \sigma_\psi(0,0) = k \tag{1.4}$$

and

$$\sigma_{\varphi_x\varphi_x}(0,0) = \sigma_{\varphi_x\psi}(0,0) = \sigma_{\psi\psi}(0,0) = 0. \tag{1.5}$$

A simple example with essential nonlinearity in the first variable is given by

$$\sigma(r,s) = k\frac{r}{\sqrt{1+r^2}} + ks,$$

the nonlinear part corresponding to a vibrating string. The linearized system then consists of (1.2), (1.3) and

$$\rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0, \tag{1.6}$$

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the usual linear Timoshenko part for $\varphi$, cp. [3], [10].
Boundary conditions both for the linear and the nonlinear system will be given for $t \geq 0$ by

$$\varphi(t,0) = \varphi(t,L) = \psi_x(t,0) = \psi_x(t,L) = \theta(t,0) = \theta(t,L) = 0, \tag{1.7}$$

for the linear system we shall also consider

$$\varphi(t,0) = \varphi(t,L) = \psi(t,0) = \psi(t,L) = \theta_x(t,0) = \theta_x(t,L) = 0 \tag{1.8}$$

or

$$\varphi_x(t,0) = \varphi_x(t,L) = \psi(t,0) = \psi(t,L) = \theta(t,0) = \theta(t,L) = 0. \tag{1.9}$$

Additionally one has initial conditions

$$\varphi(0,\cdot) = \varphi_0, \;\; \varphi_t(0,\cdot) = \varphi_1, \;\; \psi(0,\cdot) = \psi_0, \;\; \psi_t(0,\cdot) = \psi_1, \;\; \theta(0,\cdot) = \theta_0 \text{ in } (0,L). \tag{1.10}$$

If $\gamma = 0$ then (1.6), (1.2) build a purely hyperbolic system where the energy is conserved and solutions, respectively the energy, does not decay at all, of course. Moreover, the system (1.1), (1.2) is expected to develop singularities in finite time because of its typical nonlinear hyperbolic character in the equation (1.1).
If the term $\gamma\theta_x$ in (1.2) is replaced by a control function $\bar{b}(x)\psi_t$, $\bar{b} > 0$, then Soufyane [10] proved the exponential stability of the linearized system if and only if

$$\frac{\rho_1}{k} = \frac{\rho_2}{b} \tag{1.11}$$

holds, that is, if and only if the wave speeds associated to (1.6), (1.2), respectively, are equal.
A weaker type of dissipation, also being presented only in the equation (1.3) for $\psi$, was considered in [3] replacing $\gamma\theta_x$ by a memory term $\int\limits_0^t g(t-s)\psi_{xx}(s,x)ds$. For exponential type kernels $g$ the exponential stability follows again if and only if (1.11) holds.
Here we consider a dissipation through a coupling to a heat equation. The coupling is direct only for the rotation angle $\psi$ in (1.3) while the coupling to $\varphi$ is only given indirectly in (1.2). The equations (1.2), (1.3) are for $k = 0$ those of linear thermoelasticity, for which the exponential stability for various boundary conditions has been proved, see [5]. The questions here will be whether the dissipation is strong enough to stabilize the whole linear system exponentially under various boundary conditions, and if there are small, smooth solutions to the nonlinear system that decay exponentially. In other words: The a priori open problem if there is a control of the Timoshenko system through a heat equation system in only one equation (for $\psi$), that exponentially stabilizes, will be considered and solved. The positive answers to the questions under condition (1.11) will be given in Sections 2 and 3 for the linearized system and different boundary conditions, and in Section 5 for the nonlinear system, respectively. In Section 4 we prove for the linearized case and for one of the boundary conditions that the condition is also sufficient.
Energy methods and pertubation arguments will be used that will have to combine methods previously used for pure thermoelastic systems [5], for Timoshenko systems as in [3], and for nonlinear systems as described for Cauchy problems in [9]. For the proof of the necessity of

(1.11) for the exponential stability we use an approach from [4], which is simpler than that from [7] used in [3] for a hyperbolic situation, while we consider a more complex hyperbolic-parabolic system. Standard notations for function spaces are used.
We remark that it might be possible to use the decoupling technique from [2]. This would require a careful analysis of domains of operators. We, instead, use the energy method, which in addition yields the possibility to get estimates for the constants appearing, like the decay rates.

## 2 Linear exponential stability — $\varphi = \psi = \theta_x = 0$

First we consider the linear system (1.6), (1.2), (1.3), (1.8), (1.10), i.e.

$$\rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0 \quad \text{in } (0,\infty) \times (0, L), \tag{2.1}$$

$$\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \gamma\theta_x = 0 \quad \text{in } (0,\infty) \times (0, L), \tag{2.2}$$

$$\rho_3\theta_t - \kappa\theta_{xx} + \gamma\psi_{tx} = 0 \quad \text{in } (0,\infty) \times (0, L) \tag{2.3}$$

with positive constants $\rho_1, k, \rho_2, b, \gamma, \rho_3, \kappa$ together with initial conditions

$$\varphi(0,\cdot) = \varphi_0, \quad \varphi_t(0,\cdot) = \varphi_1, \quad \psi(0,\cdot) = \psi_0, \quad \psi_t(0,\cdot) = \psi_1, \quad \theta(0,\cdot) = \theta_0 \quad \text{in } (0, L) \tag{2.4}$$

and boundary conditions

$$\varphi(t, 0) = \varphi(t, L) = \psi(t, 0) = \psi(t, L) = \theta_x(t, 0) = \theta_x(t, 0) = 0 \quad \text{in } (0,\infty). \tag{2.5}$$

The well-posedness of (2.1)–(2.5) is standard, see also Section 5, hence we assume that the solution will have the regularity we work with in this section. We look for the exponential decay of the energy

$$\begin{aligned} E(t) \; &:= \; \frac{1}{2}\int_0^L (\rho_1\varphi_t^2 + \rho_2\psi_t^2 + b\psi_x^2 + k|\varphi_x + \psi|^2 + \rho_3\theta^2)(t, x)dx, \\ &\equiv \; E(t, \varphi, \psi, \theta), \end{aligned}$$

mostly dropping $(t, x)$ in the sequel. The main idea is to construct a Lyapunov functional $\mathcal{L}$ satisfying

$$\beta_1 E(t) \le \mathcal{L}(t) \le \beta_2 E(t)$$

for $t \ge 0$ and positive constants $\beta_1, \beta_2$, and

$$\frac{d}{dt}\mathcal{L}(t) \le -\alpha\mathcal{L}(t)$$

for some $\alpha > 0$. A careful choice of multipliers and the sequence of estimates in the energy method will give the desired result.
Multiplying (2.1) by $\varphi_t$, (2.2) by $\psi$, and (2.3) by $\theta$, one easily concludes

$$\frac{d}{dt}E(t) = -\kappa\int_0^L \theta_x^2 dx. \tag{2.6}$$

Let $w$ be defined as solution to

$$-w_{xx} = \psi_x, \qquad w(0) = w(L) = 0,$$

and let

$$I_1 := \int_0^L \rho_2 \psi_t \psi + \rho_1 \varphi_t w dx.$$

Then

$$\begin{aligned}
\frac{d}{dt}\int_0^L \rho_2 \psi_t \psi \, dx &= \rho_2 \int_0^L \psi_t^2 dx - b\int_0^L \psi_x^2 dx - k\int_0^L \psi^2 dx - k\int_0^L \varphi_x \psi \, dx + \gamma \int_0^L \theta_x \psi \, dx. \\
\frac{d}{dt}\int_0^L \rho_1 \varphi_t w \, dx &= k\int (\varphi_x + \psi)_x w \, dx + \rho_1 \int_0^L \varphi_t w_t dx \\
&= -k\int_0^L \varphi \psi_x dx + k\int_0^L w_x^2 dx + \rho_1 \int_0^L \varphi_t w_t dx.
\end{aligned}$$

Hence

$$\frac{d}{dt} I_1 = \rho_2 \int_0^L \psi_t^2 dx - b\int_0^L \psi_x^2 dx - k\int_0^L \psi^2 + k\int_0^L w_x^2 dx + \gamma \int_0^L \theta_x \psi \, dx.$$

Observing

$$\int_0^L w_x^2 dx \leq \int_0^L \psi^2 dx \leq c\int_0^L \psi_x^2 dx, \tag{2.7}$$

with the Poincaré constant $c > 0$, we conclude that there is a constant $C > 0$, and for any $\varepsilon_1 > 0$ there is a positive constant $C_{\varepsilon_1} > 0$ such that

$$\frac{d}{dt} I_1 \leq -\frac{b}{2}\int_0^L \psi_x^2 dx + \varepsilon_1 \int_0^L \varphi_t^2 dx + C_{\varepsilon_1} \int_0^L \psi_t^2 dx + C\int_0^L \theta_x^2 dx. \tag{2.8}$$

Let $\overline{\theta}(t,x) := \theta(t,x) - \frac{1}{L}\int_0^L \theta_0(x) dx$. From (2.3) we conclude

$$\frac{d}{dt}\int_0^L \theta(t,x) dx = 0,$$

hence

$$\int_0^L \overline{\theta}(t,x) dx = 0$$

for all $t \geq 0$, making the first Poincaré inequality for $\overline{\theta}$ applicable. On the other hand $(\varphi, \psi, \overline{\theta})$ satisfy the same differential equations (2.1)–(2.3) and boundary conditions (2.5) as $(\varphi, \psi, \theta)$ —

only the initial condition $\overline{\theta}(0,x)$ differs from $\theta(0,x)$ in general. In the sequel we shall look at $(\varphi,\psi,\overline{\theta})$ and the associated energy

$$\overline{E}(t) \equiv E(t,\varphi,\psi,\overline{\theta}).$$

rather than at $(\varphi,\psi,\theta)$ with $E(t)$, but for simplicity we keep the notation $\theta$ instead of $\overline{\theta}$, and $E$ instead of $\overline{E}$ until the end of this section.

Let

$$I_2(t) := \rho_2\rho_3 \int_0^L \int_0^x \theta(t,y)dy\, \psi_t(t,x)dx.$$

Then

$$\begin{aligned}
\frac{d}{dt}I_2 &= \int_0^L \int_0^x \rho_3\theta_t dy\, \rho_2\psi_t dx + \int_0^L \int_0^x \rho_3\theta\, dy\, \rho_2\psi_{tt}dx \\
&= \int_0^L \int_0^x \kappa\theta_{xx} - \gamma\psi_{tx}dy\, \rho_2\psi_t dx + \int_0^L \int_0^x \rho_3\theta\, dy(b\psi_{xx} + k(\varphi_x+\psi) + \gamma\theta_x)dx \\
&= \int_0^L (\kappa\theta_x - \gamma\psi_t)\rho_2\psi_t dx - \int_0^L \rho_3\theta b\psi_x dx + k\rho_3 \int_0^L \theta\varphi\, dx + \rho_3 k \int_0^L \int_0^x \theta\, dy\, \psi\, dx - \int_0^L \gamma\rho_3\theta^2 dx
\end{aligned}$$

which implies that for any $\varepsilon_2 > 0$ there exists a positive constant $C_{\varepsilon_2}$ such that

$$\frac{d}{dt}I_2 \le -\frac{\gamma\rho_2}{2}\int \psi_t^2 dx + \varepsilon_2 \int_0^L \psi_x^2 dx + \varepsilon_2 \int_0^L \varphi_x^2 dx + C_{\varepsilon_2}\int_0^L \theta_x^2 dx. \tag{2.9}$$

From (2.8) and (2.9) we get for $N_2 > 0$

$$\begin{aligned}
\frac{d}{dt}(I_1 + N_2 I_2) \le &-\left(\frac{N_2\gamma\rho_2}{2} - C_{\varepsilon_1}\right)\int_0^L \psi_t^2 dx - \left(\frac{b}{2} - N_2\varepsilon_2\right)\int_0^L \psi_x^2 dx \\
&+ (C_{\varepsilon_1} + C_{\varepsilon_2})\int_0^L \theta_x^2 dx + \varepsilon_1 \int_0^L \varphi_t^2 dx + N_2\varepsilon_2 \int_0^L \varphi_x^2 dx.
\end{aligned} \tag{2.10}$$

At the end we shall choose $\varepsilon_1$ small enough, then $N_2$ large enough and afterwards $\varepsilon_2$ small enough.

Let, for $N = N(\varepsilon_1,\varepsilon_2) > 0$,

$$I_3 := NE + I_1 + N_2 I_2,$$

then, from (2.6) and (2.10), it follows

$$\begin{aligned}
\frac{d}{dt}I_3 \le\; &-(N\kappa - (C_{\varepsilon_1} + C_{\varepsilon_2}))\int_0^L \theta_x^2 dx - \left(\frac{N_2\gamma\rho_2}{2} - C_{\varepsilon_1}\right)\int_0^L \psi_t^2 dx \\
&- \left(\frac{b}{2} - N_2\varepsilon_2\right)\int_0^L \psi_x^2 dx + \varepsilon_1 \int_0^L \varphi_t^2 dx + N_2\varepsilon_2 \int_0^L \varphi_x^2 dx.
\end{aligned} \tag{2.11}$$

On the other hand let us define

$$I_4 := \int_0^L \rho_2\psi_t(\varphi_x+\psi)dx + \int_0^L \rho_2\psi_x\varphi_t dx.$$

Then we have (cp. Lemma 2.4 in [3])

**Lemma 2.1** *Assume (1.11), i.e.*

$$\frac{\rho_1}{k} = \frac{\rho_2}{b}.$$

*Then there exists for any $\varepsilon_3 > 0$ a constant $C_{\varepsilon_3} > 0$ such that for $t \geq 0$:*

$$\frac{d}{dt}I_4(t) \leq \left[b\psi_x\varphi_x\right]_{x=0}^{x=L} - k\int_0^L |\varphi_x+\psi|^2dx + \varepsilon_3\int_0^L \varphi_x^2dx + \varepsilon_3\int_0^L \psi_x^2dx + \rho_2\int_0^L \psi_t^2dx + C_{\varepsilon_3}\int_0^L \theta_x^2dx.$$

PROOF:

$$\begin{aligned}
\frac{d}{dt}\int_0^L \rho_2\psi_t(\varphi_x+\psi)dx &= b\int_0^L \psi_{xx}(\varphi_x+\psi)dx - k\int_0^L |\varphi_x+\psi|^2dx \\
&\quad +\gamma\int_0^L \theta_x(\varphi_x+\psi)dx + \rho_2\int_0^L \psi_t(\varphi_x+\psi)_t dx \\
&= \left[b\psi_x\varphi_x\right]_{x=0}^{x-L} - b\int_0^L \psi_x\varphi_{xx}dx - b\int_0^L \psi_x^2dx - k\int_0^L |\varphi_x+\psi|^2dx \\
&\quad +\gamma\int_0^L \theta_x(\varphi_x+\psi)dx + \rho_2\int_0^L \psi_t^2dx + \rho_2\int_0^L \psi_t\varphi_{xt}dx \\
&= \left[b\psi_x\varphi_x\right]_{x=0}^{x=L} - \frac{b\rho_1}{k}\int_0^L \psi_x\varphi_{tt}dx - k\int_0^L |\varphi_x+\psi|^2dx \\
&\quad +\gamma\int_0^L \theta_x(\varphi_x+\psi)dx + \rho_2\int_0^L \psi_t^2dx + \rho_2\int_0^L \psi_t\varphi_{xt}dx \\
&= \left[b\psi_x\varphi_x\right]_{x=0}^{x=L} - k\int_0^L |\varphi_x+\psi|^2dx \\
&\quad +\gamma\int_0^L \theta_x(\varphi_x+\psi)dx + \rho_2\int_0^L \psi_t^2dx - \frac{d}{dt}\rho_2\int_0^L \psi_x\varphi_t dx, \qquad (2.12)
\end{aligned}$$

where the last equality follows, using the assumption (1.11), from

$$\begin{aligned}
\rho_2\int_0^L \psi_t\varphi_{xt}dx &= \frac{d}{dt}\rho_2\int_0^L \psi\varphi_{xt}dx - \rho_2\int_0^L \psi\varphi_{xtt}dx \\
&= -\frac{d}{dt}\rho_2\int_0^L \psi_x\varphi_t dx + \rho_2\int_0^L \psi_x\varphi_{tt}dx.
\end{aligned}$$

The assertion of the lemma now follows from (2.12).

Q.E.D.

In order to deal with the boundary terms appearing in Lemma 2.1, we prove (cp. Lemma 2.5 in [3])

**Lemma 2.2** *Let $q \in C^1([0,L])$ satisfy $q(0) = -q(L) = 2$, e.g. $q(x) = 2 - \frac{4}{L}x$. Then there exists $C_1 > 0$ and for any $\tilde{\varepsilon} > 0$ a positive constant $C_{\tilde{\varepsilon}}$ such that for $t \geq 0$ we have*

*(i)*

$$\begin{aligned} \frac{d}{dt}\int_0^L \rho_2\psi_t q b\psi_x dx &\leq -\{b^2|\psi_x(t,L)|^2 + b^2|\psi_x(t,0)|^2\} + \tilde{\varepsilon}\int_0^L \varphi_x^2 dx + C_1\int_0^L \psi_t^2 dx \\ &\quad + C_{\tilde{\varepsilon}}\int_0^L \theta_x^2 + \psi_x^2 dx \end{aligned}$$

*(ii)*

$$\frac{d}{dt}\int_0^L \rho_1\varphi_t q\varphi_x dx \leq -k\{|\varphi_x(t,L)|^2 + |\varphi_x(t,0)|^2\} + C_1\int_0^L \varphi_t^2 + \varphi_x^2 + \psi_x^2 dx.$$

PROOF:

$$\begin{aligned} \frac{d}{dt}\int_0^L \rho_2\psi_t q b\psi_x dx &= b^2\int_0^L \psi_{xx} q\psi_x dx - kb\int_0^L (\varphi_x + \psi) q\psi_x dx \\ &\quad -\gamma b\int_0^L \theta_x q\psi_x dx + \rho_2 b\int_0^L \psi_t q b\psi_{xt} dx \\ &= \frac{b^2}{2}[q\psi_x^2]_{x=0}^{x=L} - \frac{1}{2}\int_0^L q_x b^2\psi_x^2 dx - kb\int_0^L (\varphi_x + \psi) q\psi_x dx \\ &\quad -\gamma b\int_0^L \theta_x q\psi_x dx - \frac{1}{2}\rho_2 b\int_0^L q_x\psi_t^2 dx \end{aligned}$$

which proves (i). The assertion (ii) follows from

$$\begin{aligned} \frac{d}{dt}\int_0^L \rho_1\varphi_t q\varphi_x dx &= k\int_0^L q\varphi_{xx}\varphi_x dx + k\int_0^L q\psi\varphi_x dx + \rho_1\int_0^L \varphi_t q\varphi_{xt} dx \\ &= \frac{1}{2}k[q\varphi_x^2]_{x=0}^{x=L} + k\int_0^L q\psi\varphi_x dx - \frac{1}{2}\rho_1\int_0^L q_x\varphi_t^2 dx. \end{aligned}$$

Q.e.d.

Let, for $\delta > 0$

$$I_5 := I_4 + \tilde{N}\int_0^L \rho_2\psi_t q\psi_x dx + \delta\int_0^L \rho_1\varphi_t\varphi_x dx.$$

From Lemma 2.1 and Lemma 2.2 we conclude, observing

$$-\frac{k}{2}\int_0^L |\varphi_x+\psi|^2 dx \le -\frac{k}{4}\int_0^L \varphi_x^2 dx + C\int_0^L \psi_x^2 dx$$

for some constant $C>0$, that for sufficiently small $\tilde{\varepsilon}$, $\varepsilon_3$, $\delta>0$, large $\tilde{N}$, we have for $0<\tau<1$ and some $C_\tau>0$ and $C_2>0$ that

$$\frac{d}{dt}I_5 \le -\frac{k}{2}\int_0^L |\varphi_x+\psi|^2 + C_2\tau\int_0^L \varphi_t^2 dx + C_\tau\int_0^L \psi_x^2+\psi_t^2+\theta_x^2 dx. \tag{2.13}$$

(Choose first $\delta$ of order $\tau$, then $\varepsilon_3$ small enough, then $\tilde{N}$ large enough, finally $\tilde{\varepsilon}$ small enough.) With

$$I_6 := -\int_0^L \rho_1\varphi_t\varphi dx - \int_0^L \rho_2\psi_t\psi dx$$

it easily follows

$$\frac{d}{dt}I_6 \le -\rho_1\int_0^L \varphi_t^2 dx - \rho_2\int_0^L \psi_t^2 dx + k\int_0^L |\varphi_x+\psi|^2 dx + C\int_0^L \psi_x^2+\theta_x^2 dx \tag{2.14}$$

for some constant $C>0$.
The inequalities (2.13) and (2.14) imply, choosing $\tau$ small enough,

$$\begin{aligned}\frac{d}{dt}\left\{I_5+\frac{2C_2\tau}{\rho_1}I_6\right\} \;\le\; & -\frac{k}{4}\int_0^L |\varphi_x+\psi|^2 dx - C_2\tau\int_0^L \varphi_t^2 dx \\ & +C_\tau\int_0^L \psi_t^2+\psi_x^2+\theta_x^2 dx. \end{aligned}\tag{2.15}$$

The Lyapunov functional $\mathcal{L}$ is now defined by

$$\mathcal{L} := I_3 + \mu\left(I_5+\frac{2C_2\tau}{\rho_1}I_6\right), \tag{2.16}$$

for $\mu>0$, and we obtain from (2.11) and (2.15), choosing $\mu$, $\varepsilon_1$, $\varepsilon_2$ sufficiently small, and $N_2$, $N$ sufficiently large (order: $\mu$, $\varepsilon_1$, $N_2, \varepsilon_2$, $N$),

$$\frac{d}{dt}\mathcal{L}(t) \le -\beta_0 E(t)$$

for some $\beta_0\ge 0$. Moreover, there are positive constants $\beta_1$, $\beta_2>0$ such that for $t\ge 0$

$$\beta_1 E(t) \le \mathcal{L}(t) \le \beta_2 E(t) \tag{2.17}$$

whence

$$\frac{d}{dt}\mathcal{L}(t) \le -2\alpha\mathcal{L}(t)$$

for $\alpha := \frac{\beta_0}{2\beta_2}$ follows. Using (2.17) again we have thus proved the main result of this section:

**Theorem 2.3** *Let the initial data satisfy*

$$\varphi_0,\, \psi_0,\, \theta_{0,x} \in \; H_0^1((0,L)),\;\; \varphi_1,\, \psi_1 \in L^2((0,L)),$$

*and assume for the coefficients the equality (1.11), i.e.*

$$\frac{\rho_1}{k} = \frac{\rho_2}{b}.$$

*Then the energy* $\overline{E}$ *of the solution* $(\varphi, \psi, \theta)$ *to (2.1)–(2.5),*

$$\overline{E}(t) = \frac{1}{2}\int_0^L (\rho_1\varphi_t^2 + \rho_2\psi_t^2 + b\psi_x^2 + k|\varphi_x + \psi|^2 + \rho_3 \left(\theta - \frac{1}{L}\int_0^L \theta_0(y)dy\right)^2 (t,x)dx,$$

*decays exponentially, that is, there exist constants* $C > 0$ *and* $\alpha > 0$, *being independent of the initial data, such that for all* $t \geq 0$*:*

$$\overline{E}(t) \leq C\overline{E}(0)e^{-2\alpha t}.$$

**Remark:** The expectation is that the condition (1.11) on the coefficients is also necessary for exponential decay but this has not been proved yet, cp. Section 4 for the boundary condition (1.7).

## 3 Linear exponential stability — $\varphi = \psi_x = \theta = 0$

The second set of boundary conditions will be

$$\varphi(t,0) = \varphi(t,L) = \psi_x(t,0) = \psi_x(t,L) = \theta(t,0) = \theta(t,L) = 0, \quad t \geq 0 \tag{3.1}$$

which consider for the linear system (2.1)–(2.4). These boundary conditions will be also considered for the corresponding nonlinear system in Section 5. In section 5 we shall benefit from the linear exponential stability to be proved now. The proof first is simple after the previous sections, but we shall also introduce a semigroup formulation and estimates for higher derivatives that we shall need in Section 5.
First, note that from (2.2) we conlude that

$$\rho_2 g''(t) + kg(t) = 0,$$

where

$$g(t) := \int_0^L \psi\, dx,$$

that is, if

$$\int_0^L \psi_0(x)dx = \int_0^L \psi_1(x)dx = 0 \tag{3.2}$$

holds, then $g = 0$ follows, hence

$$\int_0^L \psi(t,x)dx = 0, \quad t \geq 0. \tag{3.3}$$

We shall assume (3.2) in the sequel and thus can exploit (3.3) in the usual Poincaré estimate for $\psi$.
Let the energy $E$ be defined as in Section 2, i.e.

$$E(t) = \frac{1}{2}\int_0^L (\rho_1\varphi_t^2 + \rho_2\psi_t^2 + b\psi_x^2 + k|\varphi_x + \psi|^2 + \rho_3\theta^2)(t,x)dx.$$

Then we have again as in (2.6)

$$\frac{d}{dt}E = -\kappa \int_0^L \theta_x^2 dx. \tag{3.4}$$

Using $I_1$ as in Section 2,

$$I_1 = \int_0^L \rho_2\psi_t\psi + \rho_1\varphi_t w\, dx,$$

where $w$ satisfies

$$-w_{xx} = \psi_x, \qquad w(0) = w(L) = 0,$$

we find again (cp. (2.8)) the estimate,

$$\frac{d}{dt}I_1 \leq -\frac{b}{2}\int_0^L \psi_x^2 dx + \varepsilon_1 \int_0^L \varphi_t^2 dx + C_{\varepsilon_1}\int_0^L \psi_t^2 dx + C\int_0^L \theta_x^2 dx, \tag{3.5}$$

where $\varepsilon_1 > 0$ and $C_{\varepsilon_1}, C > 0$ are positive constants.
Instead of $I_2$ from Section 2 we now define

$$\tilde{I}_{2(t)} := -\rho_2\rho_3 \int_0^L \theta(t,x) \int_0^x \psi_t(t,y)dy\, dx$$

and obtain, using (3.3),

$$\begin{aligned}
\frac{d}{dt}\tilde{I}_2 &= -\rho_2\rho_3 \int_0^L \theta_t \int_0^x \psi_t dy\, dx - \rho_2\rho_3 \int_0^L \theta \int_0^x \psi_{tt} dy\, dx \\
&= -\rho_2 \int_0^L (\kappa\theta_{xx} - \gamma\psi_{tx}) \int_0^x \psi_t dy\, dx - \rho_3 \int_0^L \theta \int_0^x b\psi_{xx} - k(\varphi_x + \psi) - \gamma\theta_x dy\, dx \\
&= \rho_2 \int_0^L \kappa\theta_x\psi_t - \gamma\psi_t^2 dx - \rho_3 \int_0^L \theta(b\psi_x - k\varphi - k\int_0^x \psi\, dy - \gamma\theta)dx
\end{aligned}$$

which implies (cp. (2.9))

$$\frac{d}{dt}\tilde{I}_2 \le -\frac{\gamma\rho_2}{2}\int_0^L \psi_t^2 dx + \varepsilon_2 \int_0^L \varphi_x^2 dx + \varepsilon_2 \int_0^L \psi_x^2 dx + C_{\varepsilon_2}\int_0^L \theta_x^2 dx, \tag{3.6}$$

where $\varepsilon_2 > 0$ and $C_{\varepsilon_2} > 0$ is a constant.
Combining (3.4)–(3.6) we conclude for $N = N(\varepsilon_1, \varepsilon_2) > 0$, $N_2 > 0$ and

$$\tilde{I}_3 := NE + I_1 + N_2\tilde{I}_2,$$

that

$$\begin{aligned}\frac{d}{dt}\tilde{I}_3 \quad &\le \quad -N\kappa\int_0^L \theta_x^2 dx - \left(\frac{N_2\gamma\rho_2}{2} - C_{\varepsilon_1}\right)\int_0^L \psi_t^2 dx - \left(\frac{b}{2} - N_2\varepsilon_2\right)\int_0^L \psi_x^2 dx \\ &\quad + (C_{\varepsilon_1} + C_{\varepsilon_2})\int_0^L \theta_x^2 dx + \varepsilon_1\int_0^L \varphi_t^2 dx + N_2\varepsilon_2\int_0^L \varphi_x^2 dx. \end{aligned} \tag{3.7}$$

Lemma 2.1 holds again, now with

$$\Big[b\psi_x\varphi_x\Big]_{x=0}^{x=L} = 0,$$

hence we have, if (1.11) holds, i.e. if

$$\frac{\rho_1}{k} = \frac{\rho_2}{b},$$

then for

$$I_4 = \int_0^L \rho_2\psi_t(\varphi_x + \psi)dx + \int_0^L \rho_2\psi_x\varphi_t dx$$

we have

$$\frac{d}{dt}I_4 \le -k\int_0^L |\varphi_x + \psi|^2 dx + \varepsilon_3\int_0^L \varphi_x^2 dx + \varepsilon_3\int_0^L \psi_x^2 dx + \rho_2\int_0^L \psi_t^2 dx + C_{\varepsilon_3}\int_0^L \theta_x^2 dx,$$

where $\varepsilon_3 > 0$ and $C_{\varepsilon_3} > 0$ is a constant, and choosing $\varepsilon_3$ small, we get

$$\frac{d}{dt}I_4 \le -\frac{k}{2}\int_0^L |\varphi_x + \psi|^2 dx + \tilde{C}_1\int_0^L (\psi_x^2 + \psi_t^2 + \theta_x^2)dx. \tag{3.8}$$

with a constant $\tilde{C}_1 > 0$.
Letting

$$I_6 = -\int_0^L \rho_1\varphi_t\varphi\, dx - \int_0^L \rho_2\psi_t\psi\, dx$$

again, we obtain as in (2.14)

$$\frac{d}{dt}I_6 \le -\rho_1\int_0^L \varphi_t^2 dx - \rho_2\int_0^L \psi_t^2 dx + k\int_0^L |\varphi_x + \psi|^2 dx + \tilde{C}_2\int_0^L \psi_x^2 + \theta_x^2 dx \tag{3.9}$$

for a constant $\tilde{C}_2 > 0$.
Now choosing as Lyapunov functional

$$\tilde{\mathcal{L}} := \tilde{I}_3 + \mu(I_4 + I_6)$$

we conclude, choosing $\mu, \varepsilon_1, \varepsilon_2$ sufficiently small and $N_2, N$ sufficiently large,

$$\frac{d}{dt}\tilde{\mathcal{L}}(t) \leq -\alpha \mathcal{L}(t)$$

for some $\alpha > 0$, and we get as in Section 2.3 the exponential stability.

**Theorem 3.1** *Let the initial data satisfy*

$$\varphi_0, \psi_{0,x}, \theta_0 \in H_0^1((0,L)), \quad \varphi_1, \psi_1 \in L^2((0,L)),$$

*as well as (3.2), and assume for the coefficients the equality (1.11), i.e.*

$$\frac{\rho_1}{k} = \frac{\rho_2}{b}.$$

*Then the energy $E$ of the solution $(\varphi, \psi, \theta)$ to (2.1)–(2.4), (3.1),*

$$E(t) = \frac{1}{2}\int_0^L (\rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + b\psi_x^2 + k|\varphi_x + \psi|^2 + \rho_3 \theta^2)(t,x)dx,$$

*decays exponentially, that is, there exist constants $C > 0$ and $\alpha > 0$, being independent of the initial data, such that for all $t \geq 0$:*

$$E(t) \leq CE(0)e^{-2\alpha t}.$$

We now present a semigroup formulation and estimates on higher derivatives which we shall need for the discussion of the nonlinear problem in Section 5.
Let $V := (\varphi, \varphi_t, \psi, \psi_t, \theta)'$, then $V$ formally satisfies

$$V_t = AV, \qquad V(t=0) = V_0 \tag{3.10}$$

where $V_0 := (\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0)'$ and $A$ is the (formal) differential operator

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ k/\rho_1 \partial_x^2 & 0 & k/\rho_1 \partial_x & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -k/\rho_2 \partial_x & 0 & b/\rho_2 \partial_x^2 - k/\rho_2 & 0 & -\gamma/\rho_2 \partial_x \\ 0 & 0 & 0 & -\gamma/\rho_3 \partial_x & \kappa/\rho_3 \partial_{xx} \end{pmatrix}. \tag{3.11}$$

Let

$$\mathcal{H} := H_0^1((0,L)) \times L^2((0,L)) \times H_*^1((0,L)) \times L_*^2((0,L)) \times L^2((0,L))$$

be the Hilbert space with

$$\begin{aligned} H_*^1((0,L)) &:= \{w \in H^1((0,L)) | \int_0^L w(x)dx = 0\}, \\ L_*^2((0,L)) &:= \{w \in L^2((0,L)) | \int_0^L w(x)dx = 0\}, \end{aligned}$$

and norm given by

$$\begin{aligned}\|V\|_{\mathcal{H}}^2 &= \|(V^1,\dots,V^5)'\|_{\mathcal{H}}^2\\ &\equiv \|V_x^1\|_{L^2}^2+\rho_1\|V^2\|_{L^2}^2+b\|V_x^3\|_{L^2}^2+k\|V_x^1+V^3\|^2+\rho_2\|V^4\|_{L^2}^2+\rho_3\|V^5\|_{L^2}^2.\end{aligned}$$

Then $A$, formally given in (3.11), with domain

$$\begin{aligned}D(A) \;:=\; &\{V\in\mathcal{H}\mid V^1\in H^2((0,L)),\;\; V^2\in H_0^1((0,L)),\;\; V^3\in H^2((0,L)),\\ &V_x^3\in H_0^1((0,L)),\;\; V^4\in H_*^1((0,L)),\;\; V^5\in H^2((0,L))\cap H_0^1((0,L))\},\end{aligned}$$

generates a contraction semigroup $\{e^{tA}\}_{t\geq 0}$. We observe that for a solution $(\varphi,\psi,\theta)$ to (2.1)–(2.4), (3.1) and the corresponding $V$ the energy $E(t)$ of $(\varphi,\psi,\theta)$ equals $\frac{1}{2}\|V(t)\|_{\mathcal{H}}^2$, independent of $t$. Theorem 3.1 now implies the exponential stability of the semigroup as usual:

$$\exists c_1>0\quad \forall t\geq 0\quad \forall V_0\in\mathcal{H}: \|e^{tA}V_0\|_{\mathcal{H}}\leq c_1e^{-\alpha t}\|V_0\|_{\mathcal{H}}. \tag{3.12}$$

For the nonlinear part we shall need estimates for higher derivatives of $(\varphi,\psi,\theta)$ and $V$, respectively.
Observe that if $V_0\in D(A)$ then we can estimate $AV(t)$ in the same way as $V(t)$ is estimated in (3.12), which in turn, using the structure of $A$ in (3.11), implies that $(V_x^1,V_x^2,V_x^3,V_x^4,V_{xx}^5)'$ can be estimated in the $\|\cdot\|_{\mathcal{H}}$-norm, hence we may estimate $((\varphi_x)_x,(\varphi_t)_x,(\psi_x)_x,(\psi_t)_x,\theta_{xx})'$ in $L^2$.
Let for $s\in\mathbb{N}$

$$\mathcal{H}_s := (H^s\times H^{s-1}\times H^s\times H^{s-1}\times H^{2(s-1)})((0,L)).$$

with natural norms $\|\cdot\|_{H^s}$ for the components.
For $V_0\in D(A^{s-1})$, we thus can estimate

$$\|V(t)\|_{\mathcal{H}_s}\leq c_s\|V_0\|_{\mathcal{H}_s}e^{-\alpha t}. \tag{3.13}$$

with $c_s$ being a positive constant, independent of $V_0$ and $t$.

# 4 Non-exponential decay if $\rho_1/k\neq\rho_2/b$

For the system considered in the last section we shall now prove that the condition (1.11) is also necessary for exponential stability, i.e. we shall prove

**Theorem 4.1** *If*

$$\frac{\rho_1}{k}\neq\frac{\rho_2}{b} \tag{4.1}$$

*then the system associated to the initial boundary value problem (2.1)–(2.4), (3.1) is not exponentially stable.*

PROOF: We use the same approach as in [4]. Recalling from the previous section that $V=(\varphi,\varphi_t,\psi,\psi_t,\theta)'$ satisfies

$$V_t=AV,\qquad V(t=0)=V_0$$

where $V_0=(\varphi_0,\varphi_1,\psi_0,\psi_1,\theta_0)'$ and $A$ is given as differential operator in (3.11), we notice that $A$ generates a contraction semigroup. By well-known results from semigroup theory (see e.g. [6, Theorem 1.3.2]) it suffices to show the existence of sequences $(\lambda_n)_n \subset \mathbb{R}$ with $\lim_{n\to\infty} |\lambda_n| = \infty$, and $(V_n)_n \subset D(A)$, $(F_n)_n \subset \mathcal{H}$, such that $(i\lambda_n - A)V_n = F_n$ is bounded and

$$\lim_{n\to\infty} \|V_n\|_{\mathcal{H}} = \infty.$$

As $F \equiv F_n$ we choose

$$F = (0, f^2, 0, f^4, 0)'$$

with

$$f^2(x) := \sin(\delta\lambda x), \qquad f^4(x) := \cos(\delta\lambda x)$$

where

$$\delta := \sqrt{\frac{\rho_2}{b}}, \qquad \lambda \equiv \lambda_n := \frac{n\pi}{\delta L} \quad (n \in \mathbb{N}).$$

The solution $V = (v^1, v^2, v^3, v^4, v^5)'$ to $(i\lambda - A)V = F$, which will be determined below, has to satisfy

$$i\lambda v^1 - v^2 = 0, \tag{4.2}$$
$$i\lambda v^2 - \frac{k}{\rho_1} v^1_{xx} - \frac{k}{\rho_1} v^3_x = f^2, \tag{4.3}$$
$$i\lambda v^3 - v^4 = 0, \tag{4.4}$$
$$i\lambda v^4 - \frac{b}{\rho_2} v^3_{xx} + \frac{k}{\rho_2} v^1_x + \frac{k}{\rho_2} v^3 + \frac{\gamma}{\rho_2} v^5_x = f^4, \tag{4.5}$$
$$i\lambda v^5 - \frac{\kappa}{\rho_3} v^5_{xx} + \frac{\gamma}{\rho_3} v^4_x = 0. \tag{4.6}$$
$$\tag{4.7}$$

Eliminating $v^2, v^4$ we obtain for $v^1, v^3, v^5$:

$$-\lambda^2 v^1 - \frac{k}{\rho_1} v^1_{xx} - \frac{k}{\rho_1} v^3_x = f^2, \tag{4.8}$$
$$-\lambda^2 v^3 - \frac{b}{\rho_2} v^3_{xx} + \frac{k}{\rho_2} v^1_x + \frac{k}{\rho_2} v^3 + \frac{\gamma}{\rho_2} v^5_x = f^4, \tag{4.9}$$
$$i\lambda v^5 - \frac{\kappa}{\rho_3} v^5_{xx} + \frac{\gamma}{\rho_3} v^3_x = 0. \tag{4.10}$$
$$\tag{4.11}$$

This can be solved by the ansatz

$$v^1(x) = A\sin(\delta\lambda x), \quad v^3(x) = B\cos(\delta\lambda x), \quad v^5(x) = C\sin(\delta\lambda x),$$

where $A, B, C$ depend on $\lambda$ and will be determined explicitly in the sequel. We remark that this choice is just compatible with the boundary conditions and the chosen right-hand sides $f^2, f^4$.

In order to satisfy (4.8)–(4.10) with this ansatz, it is necessary and sufficient that the coefficients $A, B, C$ satisfy

$$-\lambda^2 A + \frac{k}{\rho_1}\delta^2\lambda^2 A + \frac{k}{\rho_1}\delta\lambda B = 1, \tag{4.12}$$

$$-\lambda^2 B + \frac{b}{\rho_2}\delta^2\lambda^2 B + \frac{k}{\rho_2}\delta\lambda A + \frac{k}{\rho_2} B + \frac{\gamma}{\rho_2}\delta\lambda C = 1, \tag{4.13}$$

$$i\lambda C + \frac{\kappa}{\rho_3}\delta^2\lambda^2 C - \frac{i\gamma}{\rho_3}\delta\lambda^2 B = 0. \tag{4.14}$$

The last identity gives

$$C = Q_1(\lambda)B \tag{4.15}$$

where

$$Q_1(\lambda) := \frac{i(\gamma/\rho_3)\delta\lambda}{i + (\kappa/\rho_3)\delta^2\lambda}$$

satisfies

$$\lim_{\lambda\to\infty} Q_1(\lambda) = \frac{i\gamma}{\kappa\delta}. \tag{4.16}$$

Observing $\delta = \sqrt{\rho_2/b}$ we conclude from (4.13)

$$\frac{k\sqrt{\rho_2}}{\rho_2\sqrt{b}}\lambda A + \frac{k}{\rho_2}B + \frac{\gamma\sqrt{\rho_2}}{\rho_2\sqrt{b}}\lambda C = 1$$

implying

$$A = -\sqrt{\frac{b}{\rho_2}}\frac{1}{\lambda}B - \frac{\gamma}{k}Q_1(\lambda)B + \frac{\sqrt{\rho_2 b}}{k}\frac{1}{\lambda}. \tag{4.17}$$

Combining (4.12) and (4.17) we obtain

$$-\alpha_0\sqrt{\frac{b}{\rho_2}}\lambda B - \frac{\alpha_0\gamma}{k}Q_1(\lambda)\lambda^2 B + \frac{\alpha_0\sqrt{\rho_2 b}}{k}\lambda + \frac{k\delta}{\rho_1}\lambda B = 1,$$

where $\alpha_0 := \frac{k\rho_2}{\rho_1 b} - 1$. This implies

$$\lambda(\sqrt{\frac{b}{\rho_2}} - \frac{\alpha_0\gamma}{k}Q_1(\lambda)\lambda)B = 1 - \frac{\alpha_0\sqrt{\rho_2 b}}{k}\lambda$$

or

$$\begin{aligned} B &= \frac{1}{\lambda}\frac{1 - \frac{\alpha_0\sqrt{\rho_2 b}}{k}\lambda}{\sqrt{\frac{b}{\rho_2}} - \frac{\alpha_0\gamma}{k}Q_1(\lambda)\lambda} \\ &\equiv \frac{1}{\lambda}Q_2(\lambda). \end{aligned} \tag{4.18}$$

Then $Q_2$ satisfies

$$\lim_{\lambda\to\infty} Q_2(\lambda) = i\frac{\kappa\rho_2}{\gamma^2}. \tag{4.19}$$

We conclude combining (4.17), (4.18) and (4.15)

$$A = -\sqrt{\frac{b}{\rho_2}}\frac{Q_2(\lambda)}{\lambda^2} - \frac{\gamma}{k}\frac{Q_1(\lambda)Q_2(\lambda)}{\lambda} + \frac{\sqrt{\rho_2 b}}{k}\frac{1}{\lambda}, \tag{4.20}$$

$$C = \frac{Q_1(\lambda)Q_2(\lambda)}{\lambda}. \tag{4.21}$$

As a consequence we get from

$$\int_0^L |v_x^1|^2 dx = \frac{L\delta^2}{2}\lambda^2|A|^2$$

the uniform (in $\lambda$) boundedness of the $H^1$-norm of $v^1$, and also of the $L^2$-norm of $\lambda v^1$. Analogously we obtain the uniform boundedness of the $H^1$-norm of $v^3, v^5$ as well as of the $L^2$-norm of $\lambda v^3, \lambda v^5$.

Multiplying equation (4.5) by $\overline{v_x^1+v^3}$ and integrating we obtain

$$i\lambda\int_0^L v^4(\overline{v_x^1+v^3})dx+\frac{b}{\rho_2}\int_0^L v_x^3(\overline{v_{xx}^1+v_x^3})dx+\frac{k}{\rho_2}\int_0^L|\overline{v_x^1+v^3}|^2dx+\frac{\gamma}{\rho_2}\int_0^L v_x^5(\overline{v_x^1+v^3})dx$$
$$=\int_0^L f^4(\overline{v_x^1+v^3})dx. \tag{4.22}$$

Let

$$I:=i\lambda\int_0^L v^4\overline{v_x^1}dx+\frac{b}{\rho_2}\int_0^L v_x^3(\overline{v_{xx}^1+v_x^3})dx. \tag{4.23}$$

By equation (4.3) we get

$$I=i\lambda\int_0^L v^4\overline{v_x^1}dx-\frac{b}{\rho_2}i\lambda\frac{\rho_1}{k}\int_0^L v_x^3\overline{v^2}dx-\frac{b\rho_1}{\rho_2 k}\int_0^L v_x^3\overline{f^2}dx.$$

Using the equations (4.2) and (4.4) we conclude

$$i\lambda\int_0^L v^4\overline{v_x^1}dx=i\lambda\int_0^L v_x^3\overline{v^2}dx,$$

hence

$$I=i\lambda(1-\frac{b\rho_1}{\rho_2 k})\int_0^L v_x^3\overline{v^2}dx-\frac{b\rho_1}{\rho_2 k}\int_0^L v_x^3\overline{f^2}dx. \tag{4.24}$$

A combination of (4.22)–(4.24) yields

$$\begin{aligned} i\lambda(1-\frac{b\rho_1}{\rho_2 k})\int_0^L v_x^3\overline{v^2}dx &= \frac{b\rho_1}{\rho_2 k}\int_0^L v_x^3\overline{f^2}dx-\frac{k}{\rho_2}\int_0^L|\overline{v_x^1+v^3}|^2dx-i\lambda\int_0^L v^4\overline{v^3}dx\\ &\quad -\frac{\gamma}{\rho_2}\int_0^L v_x^5(\overline{v_x^1+v^3})dx+\int_0^L f^4(\overline{v_x^1+v^3})dx. \end{aligned}\tag{4.25}$$

We shall now show that the *assumption*, that $\|V\|_{\mathcal{H}}$ remains bounded as $\lambda\to\infty$, leads to a contradiction. If $\|V\|_{\mathcal{H}}$ remains bounded, then we conclude from (4.25), using the basic assumption (4.1), that

$$|\lambda\int_0^L v_x^3\overline{v^2}dx| \text{ remains bounded as } \lambda\to\infty. \tag{4.26}$$

On the other hand we have, using (4.2),

$$i\lambda\int_0^L v_x^3\overline{v^2}dx=-i\lambda\int_0^L B\delta\lambda\sin(\delta\lambda x)(-i\lambda\overline{A}\sin(\delta\lambda x)dx=-\frac{\delta L}{2}\lambda^3\overline{A}B. \tag{4.27}$$

Observing

$$\begin{aligned}\lim_{\lambda\to\infty}(-\frac{\delta L}{2}\lambda^2\overline{A}B) &= -\frac{\delta L}{2}\left(-\sqrt{\frac{b}{\rho_2}}\lim_{\lambda\to\infty}\frac{|Q_2(\lambda)|^2}{\lambda}-\frac{\gamma}{k}\lim_{\lambda\to\infty}\overline{Q_1(\lambda)}|Q_2(\lambda)|^2+\frac{\sqrt{\rho_2 b}}{k}\lim_{\lambda\to\infty}\overline{Q_2(\lambda)}\right)\\ &= -i\frac{\delta L\rho_2\kappa}{2k\gamma^2}(\frac{\rho_2}{\delta}+\sqrt{\rho_2 b}),\end{aligned}\tag{4.28}$$

a combination of (4.27) and (4.28) yields the contradiction to (4.26), and the proof is completed.

Q.E.D.

# 5 Global stability for the nonlinear system

Now we return to the nonlinear system (1.1)–(1.3), (1.7), (1.10), i.e. we consider

$$\rho_1\varphi_{tt} - \sigma(\varphi_x, \psi)_x = 0, \tag{5.1}$$

$$\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \gamma\theta_x = 0, \tag{5.2}$$

$$\rho_3\theta_t - \kappa\theta_{xx} + \gamma\psi_{tx} = 0, \tag{5.3}$$

with positive constants $\rho_1, \rho_2, b, k, \gamma, \rho_3$ and $\kappa$, together with the boundary conditions

$$\varphi(t,0) = \varphi(t,L) = \psi_x(t,0) = \psi_x(t,L) = \theta(t,0) = \theta(t,L) = 0, \quad t \geq 0, \tag{5.4}$$

and the initial conditions

$$\varphi(0,\cdot) = \varphi_0,\;\; \varphi_t(0,\cdot) = \varphi_1,\;\; \psi(0,\cdot) = \psi_0,\;\; \psi_t(0,\cdot) = \psi_1,\;\; \theta(0,\cdot) = \theta_0, \text{ in } (0,L). \tag{5.5}$$

We remark that we can also deal with other boundary conditions in the same way. The nonlinear function $\sigma$ is assumed to be smooth and to satisfy

$$\sigma_{\varphi_x}(0,0) = \sigma_\psi(0,0) = k \tag{5.6}$$

and

$$\sigma_{\varphi_x\varphi_x}(0,0) = \sigma_{\varphi_x\psi}(0,0) = \sigma_{\psi\psi}(0,0) = 0. \tag{5.7}$$

The local well-posedness is standard by now and can be obtained as that for the system of thermoelasticity, cp. [5].
Let for $m \geq 2$, $j \geq 1$

$$\varphi_m(\cdot) := (\partial_t^m\varphi)(0,\cdot), \quad \psi_m(\cdot) := (\partial_t^m\psi)(0,\cdot), \quad \theta_j(\cdot) := (\partial_t^j\theta)(0,\cdot)$$

be defined through the differential equations (5.1)–(5.3) and the initial conditions (5.5). For $s \geq 3$ assume

$$\varphi_m \in H^{s-m}((0,L)) \cap H_0^1((0,L)), \quad 2 \leq m \leq s-1, \quad \varphi_s, \psi_s \in L^2((0,L)), \tag{5.8}$$

$$\psi_{m,x} \in H^{s-m-1}((0,L)) \cap H_0^1((0,L)), \quad 2 \leq m \leq s-1, \quad \psi_s \in L^2((0,L)), \tag{5.9}$$

$$\theta_j \in H^{s-j}((0,L)) \cap H_0^1((0,L)), \quad 1 \leq j \leq s-2, \quad \theta_{s-1} \in L^2((0,L)). \tag{5.10}$$

**Theorem 5.1** *Let $s = 3$ and assume the compatibility conditions (5.8)–(5.10). Then there is $T = T\Big(\Big\|(\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0)\Big\|_{\mathcal{H}_3}\Big) > 0$ such that (5.1)–(5.5) has a unique local solution $(\varphi, \psi, \theta)$ satisfying*

$$(\varphi,\psi) \in \bigcap_{k=0}^{3} C^k\Big([0,T], H^{3-k}((0,L))\Big),$$

$$\theta \in \bigcap_{j=0}^{1} C^j\Big([0,T], H^{3-j}((0,L))\Big) \cap C^2\Big([0,T]), L^2((0,L))\Big),$$

$$\partial_t^2\theta_x \in L^2\Big([0,T], L^2((0,L))\Big),$$

*and $(\varphi, \psi, \theta)$ satisfy the boundary conditions (5.4).*

As in Section 3 we rewrite everything as a first-order system for $V = (\varphi, \varphi_t, \psi, \psi_t, \theta)$ and obtain as in (3.10)

$$V_t = AV + F(V, V_x), \quad V(t=0) = V_0 \tag{5.11}$$

where $A$ is defined in Section 3 and

$$\begin{aligned} F(V, V_x) &\equiv (0, \sigma_{\varphi_x}(\varphi_x, \psi)\varphi_{xx} - k\varphi_{xx} + \sigma_\psi(\varphi_x, \psi)\psi_x - k\psi_x, 0, 0, 0) \\ &= (0, \sigma_{\varphi_x}(V_x^1, V_3)V_{xx}^1 - kV_{xx}^1 + \sigma_\psi(V_x^1, V_3)V_x^3 - kV_x^3, 0, 0, 0). \end{aligned} \tag{5.12}$$

As in the linear part in Section 3 we observe that the condition

$$\int_0^L \psi_0(x) = \int_0^L \psi_1(x)dx = 0 \tag{5.13}$$

implies

$$\forall t \geq 0 : \int_0^L \psi(t, x)dx = 0, \tag{5.14}$$

and we shall assume (5.13) resp. (5.14) in the sequel. Moreover we observe for each $t$

$$\int_0^L \mathcal{F}(V, V_x)_{3.,4.}(t, x)dx = \int_0^L 0\, dx = 0,$$

hence $\mathcal{F}(V, V_x)(t, \cdot)$ will be in the space $\mathcal{H}$ included in Section 3.
The (local) solution satisfies

$$V(t) = e^{tA}V_0 + \int_0^t e^{(t-r)A}F(V, V_x)(r)dr. \tag{5.15}$$

The technique that we use is an adaption of one known suitable for Cauchy problems, see [9]. We point out that the energy method, which could be applied also to the nonlinear thermoelastic problem, see [5] seems not to work here because it seems to be not possible to exploit the condition $\rho_1/k = \rho_2/b$ for a nonlinear system. Therefore the perturbation arguments of the method used here are more appropriate.
Here we will assume that the solution is small in the $\mathcal{H}_2$ norm. That is

$$\|V_0\|_{\mathcal{H}_2} < \delta$$

But it can be large in the $\mathcal{H}_3$ norm. That is, let us fix a number $\mu > 1$ such that

$$\|V_0\|_{\mathcal{H}_3} < \mu.$$

By the continuity of the solution we have that there exist intervals $[0, T_0]$ and $[0, T_1]$, respectively, for which we have

$$\|V(t)\|_{\mathcal{H}_2} \leq \delta, \qquad \forall t \in [0, T_0],$$
$$\|V(t)\|_{\mathcal{H}_3} \leq \mu, \qquad \forall t \in [0, T_1].$$

Let us take $d > 1$ to be fixed later, and let the positive numbers $T_m^1$ and $T_m^0$, respectively, be the maximal values for which we have that the local solution satisfies

$$\|V(t)\|_{\mathcal{H}_2} \leq 2c_1\delta, \qquad \forall t \in [0, T_m^0],$$

and, respectively,

$$\|V(t)\|_{\mathcal{H}_3} \leq d\mu, \qquad \forall t \in [0, T_m^1],$$

where $c_1$ from (3.12) is such that

$$\|e^{At}V_0\|_{\mathcal{H}_2} \leq c_1\|V\|_{\mathcal{H}_2}.$$

Under these conditions we have the following high energy estimate:

**Lemma 5.2** *There are constants $c_2, c_3 > 0$, neither depending on $V_0$ nor on $T$, such that the local solution given in Theorem 5.1 satisfies for $t \in [0, T_m^1]$:*

$$\|V(t)\|^2_{\mathcal{H}_3} \leq c_2\|V_0\|^2_{\mathcal{H}_3} e^{c_3\sqrt{d\mu}\int_0^t \|V(r)\|^{1/2}_{\mathcal{H}_2}} dr.$$

PROOF: We rewrite (5.1) as

$$\rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = \tilde{b}(\varphi_x, \psi)\varphi_{xx} + \tilde{d}(\varphi_x, \psi)\psi_x \tag{5.16}$$

with

$$\tilde{b}(\varphi_x, \psi) := \sigma_{\varphi_x}(\varphi_x, \psi) - \sigma_{\varphi_x}(0,0), \quad \tilde{d}(\varphi_x, \psi) := \sigma_\psi(\varphi_x, \psi) - \sigma_\psi(0,0). \tag{5.17}$$

Multiplying (5.16), (5.2), (5.3) in $L^2$ by $\varphi_t$, $\psi_t$, and $\theta$, respectively, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^L \rho_1\varphi_t^2 + \rho_2\psi_t^2 + k|\varphi_x + \psi|^2 + b\psi_x^2 dx + \kappa\int_0^L \theta_x^2 dx$$

$$\begin{aligned}
&= \int_0^L \tilde{b}(\varphi_x, \psi)\varphi_{xx}\varphi_t + \tilde{d}(\varphi_x, \psi)\psi_x\varphi_t dx dx \\
&= \int_0^L \tilde{d}(\varphi_x, \psi)\psi_x\varphi_t dx dx - \int_0^L (\partial_x\tilde{b}(\varphi_x, \psi))\varphi_x\varphi_t dx - \int_0^L \tilde{b}(\varphi_x, \psi)\varphi_x\varphi_{xt} dx \\
&\equiv I.1 + I.2 + I.3.
\end{aligned} \tag{5.18}$$

Without loss of generality we neglect in the sequel lower order terms (l.o.t.) and write $\tilde{b}(\varphi_x)$ instead of $\tilde{b}(\varphi_x, \psi)$; then $\tilde{b}'$ will denote the derivative with respect to the first variable.

$$|I.2| \leq \frac{1}{2}\|\tilde{b}'(\varphi_x)\varphi_{xx}\|_{L^\infty}\int_0^L \varphi_x^2 + \varphi_t^2 dx$$

$$
\begin{aligned}
&\leq c\|\varphi_{xx}\|_{L^2}^{1/2}\|\varphi_{xx}\|_{H^1}^{1/2}\int_0^L \varphi_x^2+\varphi_t^2 dx \\
&\leq c\|V\|_{\mathcal{H}_2}^{1/2}\|V\|_{\mathcal{H}_3}^{1/2}\int_0^L \varphi_x^2+\varphi_t^2 dx,
\end{aligned} \tag{5.19}
$$

where $c$ will denote a constant not depending on $V^0$ or on $T$.

$$
\begin{aligned}
I.3 &= -\frac{1}{2}\frac{d}{dt}\int_0^L \tilde{b}(\varphi_x)\varphi_x^2 dx+\frac{1}{2}\int_0^L(\partial_t\tilde{b}(\varphi_x))\varphi_x^2 \\
&\equiv I.3.1+I.3.2.
\end{aligned} \tag{5.20}
$$

The term I.3.2 can be estimated in the same way as I.2 in (5.19):

$$
|I.3.2|\leq c\|V\|_{\mathcal{H}_2}^{1/2}\|V\|_{\mathcal{H}_3}^{1/2}\int_0^L \varphi_x^2 dx. \tag{5.21}
$$

The term I.3.1 can be incorporated into and be dominated by the left-hand side of inequality (5.18) after an integration with respect to $t$ later on, since

$$
\int_0^t I.3.1(r)dr=-\frac{1}{2}\int_0^L \tilde{b}(\varphi_x)\varphi_x^2 dx+\frac{1}{2}\int_0^L \tilde{b}(\varphi_x(t=0))\varphi_x^2(t=0)dx. \tag{5.22}
$$

Summarizing (5.18)–(5.22) we have proved

$$
\|V(t)\|_{\mathcal{H}_1}^2\leq c\|V_0\|_{\mathcal{H}_1}^2+\int_0^t c\|V(r)\|_{\mathcal{H}_2}^{1/2}\|V(r)\|_{\mathcal{H}_3}^{1/2}\|V(r)\|_{\mathcal{H}_1}^2 dr. \tag{5.23}
$$

In order to get estimates for the higher-order derivatives of $V$ we differentiate the differential equation (5.16) with respect to $t$ and obtain, again neglecting some l.o.t.:

$$
\rho_1\varphi_{ttt}-k(\varphi_{xt}+\psi_t)_x=\tilde{b}'(\varphi_x)\varphi_{xt}\varphi_{xx}+\tilde{b}(\varphi_x)\varphi_{txx}. \tag{5.24}
$$

Differentiating also the differential equations (5.2), (5.3) with respect to $t$ and multiplying the differentiated equations by $\varphi_{tt}$, $\psi_{tt}$ and $\theta_t$, respectively, in $L^2$ we get

$$
\frac{1}{2}\frac{d}{dt}\int_0^L \rho_1\varphi_{tt}^2+\rho_2\psi_{tt}^2+k|\varphi_{xt}+\psi_t|^2+b\psi_{xt}^2 dx+\kappa\int_0^L \theta_{xt}^2 dx\leq \tag{5.25}
$$

$$
\int_0^L \tilde{b}(\varphi_x)\varphi_{txx}\varphi_{tt}dx+\int_0^L \tilde{b}'(\varphi_x)\varphi_{xt}\varphi_{xx}\varphi_{tt}dx+\text{ l.o.t.}\equiv\ I.4+I.5.+\text{ l.o.t.} \tag{5.26}
$$

The term I.4 can be treated like the term I.2 + I.3.

$$
|I.5|\leq c\|V\|_{\mathcal{H}_2}^{1/2}\|V\|_{\mathcal{H}_3}^{1/2}\int_0^L \varphi_{tt}^2+\varphi_{tx}^2 dx. \tag{5.27}
$$

Using the differential equations we obtain from (5.23), (5.26), (5.27)

$$\|V(t)\|_{\mathcal{H}_2}^2 \leq c\|V_0\|_{\mathcal{H}_2}^2 + \int_0^t c\|V(r)\|_{\mathcal{H}_2}^{1/2}\|V(r)\|_{\mathcal{H}_3}^{1/2}\|V(r)\|_{\mathcal{H}_2}^2 dr. \tag{5.28}$$

Differentiating the differential equation (5.24) once more with respect to $t$ we get, neglecting some l.o.t.,

$$\varphi_{tttt} - k(\varphi_{xtt} + \psi_{tt})_x = \tilde{b}''(\varphi_x)\varphi_{xt}^2\varphi_{xx} + \tilde{b}'(\varphi_x)\varphi_{xtt}\varphi_{xx} + 2\tilde{b}'(\varphi_x)\varphi_{xt}\varphi_{xxt} + \tilde{b}(\varphi_x)\varphi_{ttxx}. \tag{5.29}$$

Differentiating also the differential equations (5.2), (5.3) once more with respect to $t$ and multiplying the differentiated equations by $\varphi_{ttt}$, $\psi_{ttt}$ and $\theta_{tt}$, respectively, in $L^2$ we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^L \rho_1\varphi_{ttt}^2 + \rho_2\psi_{ttt}^2 + k|\varphi_{xtt} + \psi_{tt}|^2 + b\psi_{xtt}^2 dx + \kappa\int_0^L \theta_{xtt}^2 dx \\ &\leq \int_0^L \tilde{b}''(\varphi_x)\varphi_{xt}^2\varphi_{xx}\varphi_{ttt}dx + \int_0^L \tilde{b}'(\varphi_x)\varphi_{xtt}\varphi_{xx}\varphi_{ttt}dx \\ &\quad +2\int_0^L \tilde{b}'(\varphi_x)\varphi_{xt}\varphi_{xxt}\varphi_{ttt}dx + \int_0^L \tilde{b}(\varphi_x)\varphi_{ttxx}\varphi_{ttt}dx \\ &\equiv I.6 + I.7 + I.8 + I.9. \end{aligned} \tag{5.30}$$

The term I.9 is again dealt with like I.2 + I.3.

$$\begin{aligned} |I.6| + |I.7| + |I.8| \leq\ & c(\|V\|_{\mathcal{H}_2}\|V\|_{\mathcal{H}_3} + \|V\|_{\mathcal{H}_2}^{1/2}\|V\|_{\mathcal{H}_3}^{1/2})\int_0^L \varphi_{xt}^2 + \varphi_{ttt}^2 + \\ & \varphi_{ttt}^2 + \varphi_{xtt}^2 + \varphi_{xxt}^2 dx. \end{aligned} \tag{5.31}$$

Hence we obtain from (5.28), (5.31),

$$\|V(t)\|_{\mathcal{H}_3}^2 \leq c\|V_0\|_{\mathcal{H}_3}^2 + \int_0^t c\|V(r)\|_{\mathcal{H}_2}^{1/2}\|V(r)\|_{\mathcal{H}_3}^{1/2}\|V(r)\|_{\mathcal{H}_3}^2 dr. \tag{5.32}$$

which yields our conclusion. Q.E.D.

Next we want to prove a weighted a priori estimate for $\|V(t)\|_{\mathcal{H}_2}$. Using the representation (5.15) and (3.13) — if $\rho_1/k = \rho_2/b$ is satisfied! — we can estimate

$$\begin{aligned} \|V(t)\|_{\mathcal{H}_2} &\leq \|e^{At}V_0\|_{\mathcal{H}_2} + \int_0^t \|e^{(t-r)A}\mathcal{F}(V, V_x)(r)\|_{\mathcal{H}_2} dr \\ &\leq c_1 e^{-\alpha t}\|V_0\|_{\mathcal{H}_2} + c_1\int_0^t e^{-\alpha(t-r)}\|F(V, V_x)\|_{\mathcal{H}_2} dr. \end{aligned} \tag{5.33}$$

We observe that $\mathcal{F}$ satisfies

$$\mathcal{F}(V,V_x)(r)\in D(A), \quad r\geq 0,$$

hence (3.13) can be applied for $s=2$. By the assumptions (5.6), (5.7) on the nonlinearity $\sigma$ the functions $\tilde{b}$ and $\tilde{d}$ defined in (5.17) we can estimate the nonlinearity as follows.

**Lemma 5.3** *There exists a positive constant c such that for all $W\in\mathcal{H}_3$ we have*

$$\|F(W,W_x)\|_{\mathcal{H}_2}\leq c\|W\|^2_{\mathcal{H}_2}\|W\|_{\mathcal{H}_3}.$$

PROOF: (cp. [9] in $\mathbb{R}^n$). Let $\varphi:=W^1$ and $\psi:=W^1$. Then

$$F(W,W_x)=(0,\tilde{b}(\varphi_x)\varphi_{xx}+\tilde{d}(\varphi_x)\psi_x,0,0,0)'$$

and

$$\begin{aligned}\|\tilde{b}(\varphi_x)\varphi_{xx}\|_{H^1} &\leq c\|\tilde{b}(\varphi_x)\varphi_{xx}\|_{L^2}+\|\tilde{b}'(\varphi_x)\varphi_x\varphi_{xx}\|_{L^2}+\|\tilde{b}(\varphi_x)\varphi_{xxx}\|_{L^2}\\ &\leq c\|\varphi_x\|^2_{L^\infty}\|\varphi\|_{H^3}\\ &\leq c\|W\|^2_{\mathcal{H}_2}\|W\|_{\mathcal{H}_3}.\end{aligned}$$

Analogously,

$$\|\tilde{d}(\varphi_x)\psi_x\|_{H^1}\leq c\|W\|^2_{\mathcal{H}_2}\|W\|_{\mathcal{H}_3}.$$

Q.E.D.

Using Lemma 5.3 we conclude from (5.33)

$$\|V(t)\|_{\mathcal{H}_2}\leq c_1e^{-\alpha t}\|V_0\|_{\mathcal{H}_2}+c\int_0^t e^{-\alpha(t-r)}\|V(r)\|^2_{\mathcal{H}_2}\|V(r)\|_{\mathcal{H}_3}dr, \tag{5.34}$$

which is the starting point to prove the following weighted a priori estimate.

**Lemma 5.4** *For $0\leq t\leq T_m^1$ let*

$$M_2(t):=\sup_{0\leq r\leq t}\left(e^{\alpha r}\|V(r)\|_{\mathcal{H}_2}\right),$$

*and let*

$$\frac{\rho_1}{k}=\frac{\rho_2}{b}.$$

*Then there are $M_0>0$ and $\delta>0$ such that if $\|V_0\|_{\mathcal{H}_3}<\mu$ and $\|V_0\|_{\mathcal{H}_2}<\delta$ we have for all $0\leq t\leq T_m^1$:*

$$M_2(t)\leq M_0<\infty$$

*$M_0$ is independent of $T$ (and of $V_0$).*

PROOF: From (5.34) and the energy estimate in Lemma 5.2 we conclude

$$\|V(t)\|_{\mathcal{H}_2}\leq c_1\|V_0\|_{\mathcal{H}_2}e^{-\alpha_t}+c\int_0^t e^{-\alpha(t-r)}\|V(r)\|^2_{\mathcal{H}_2}\|V_0\|_{\mathcal{H}_3}e^{c\sqrt{d\mu}\int_0^r\|V(\tau)\|^{1/2}_{\mathcal{H}_2}dr}.$$

If $\|V_0\|_{\mathcal{H}_2} \leq \delta \quad$ ($\delta$ to be determined) we get for $t \leq \min\{T_m^0, T_m^1\}$

$$\begin{aligned}\|V(t)\|_{\mathcal{H}_2} &\leq c_1\delta e^{-\alpha t} + c\delta^{1/2}\mu e^{c\sqrt{d\mu}\int_0^t \|V(\tau)\|_{\mathcal{H}_2}^{1/2} d\tau} \int_0^t e^{-\alpha(t-r)}\|V(r)\|_{\mathcal{H}_2}^{3/2} dr \\ &\leq c_1\delta e^{-\alpha t} + c\delta^{1/2}\mu e^{c\sqrt{d\mu}\sqrt{M_2(t)}} M_2(t)^{3/2} \int_0^t e^{-\alpha(t-r)} e^{-3\alpha r/2} dr\end{aligned}$$

which implies

$$M_2(t) \leq c_1\delta + c\delta^{1/2}\mu e^{c\sqrt{d\mu}\sqrt{M_2(t)}} M_2(t)^{3/2} \sup_{0\leq t<\infty} e^{\alpha t} \int_0^t e^{-\alpha(t-r)} e^{-3\alpha r/2} dr. \tag{5.35}$$

Since it easily follows that

$$\sup_{0\leq t<\infty} e^{\alpha t} \int_0^t e^{-\alpha(t-r)} e^{-3\alpha r/2} dr \leq c < \infty,$$

we obtain from (5.35)

$$M_2(t) \leq c_1\delta + c\delta^{1/2}\mu M_2(t) e^{c\sqrt{d\mu}\sqrt{M_2(t)}}. \tag{5.36}$$

By standard arguments (cp. e.g. [9]), considering the function

$$f(x) := c_1\delta + c\delta^{1/2}x^{3/2}e^{c\sqrt{x}} - x$$

it follows that $M_2(t)$ is uniformly bounded by the first zero $M_0$ of $f$ if $\delta$ and $M_2(0)$ are sufficiently small, up to now for $t \leq \min\{T_m^0, T_m^1\}$. If $T_m^0 \geq T_m^1$ holds, the claim of the lemma follows immediately.

Finally, suppose that $T_m^0 < T_m^1$. Note that for $\delta$ small enough we have that $M_0 < 2c_1\delta$. In fact

$$f(2c_1\delta) = 2cc_1\mu\delta^{3/2}e^{c\sqrt{2c_1 d\mu\delta}} - c_1\delta < 0$$

for small values of $\delta$. This means that

$$\|V(t)\|_{\mathcal{H}_2} \leq M_0 < 2c_1\delta.$$

But this inequality contradicts to the maximality of $T_m^0$, therefore we must have $T_m^0 \geq T_m^1$, which completes the proof.

Q.E.D.

Now we can formulate and prove the main theorem on global existence and exponential decay.

**Theorem 5.5** *Let $s = 3$ and assume the conditions (5.8)–(5.10) on the initial data as well as $\int_0^L \psi_0(x)dx = \int_0^L \psi_1(x)dx = 0$. Let $\mu > 1$ be arbitrary but fixed. If*

$$\frac{\rho_1}{k} = \frac{\rho_2}{b}$$

*and (5.6) hold, there is a $\delta > 0$ such that if $\|V_0\|_{\mathcal{H}_2} < \delta$ and $\|V_0\|_{\mathcal{H}_3} < \mu$ there exists a unique global solution $(\varphi, \psi, \theta)$ to (5.1)–(5.5) satisfying*

$$(\varphi, \psi) \in \bigcap_{k=0}^{3} C^k\Big([0,\infty), H^{3-k}((0,L))\Big),$$

$$\theta \in \bigcap_{j=0}^{1} C^j\Big([0,\infty), H^{3-j}((0,L))\Big) \cap C^2\Big([0,\infty), L^2((0,L))\Big)$$

*and $(\varphi, \psi, \theta)$ satisfy the boundary conditions (5.4).*
*Moreover, there is a constant $C_0(V_0) > 0$ such that for $t \geq 0$:*

$$\|V(t)\|_{\mathcal{H}_2} \leq C_0 e^{-\alpha t},$$

*with $\alpha$ from Theorem 3.1.*

Proof: From Lemma 5.2 and Lemma 5.4 we conclude for the local solution on $[0, T_m^1]$

$$\begin{aligned} \|V(t)\|_{\mathcal{H}_3} &\leq c\|V_0\|_{\mathcal{H}_3} e^{\tilde{c}\sqrt{d\mu}\int_0^t \|V(r)\|_{\mathcal{H}_2}^{1/2} dr} \\ &\leq c\|V_0\|_{\mathcal{H}_3} e^{\tilde{c}\sqrt{d\mu M_0}} \\ &\leq ce\|V_0\|_{\mathcal{H}_3}, \end{aligned}$$

where $\tilde{c} > 0$ and where we have chosen $\delta$ small enough such that

$$\tilde{c}\sqrt{d\mu M_0} < 1.$$

Taking $d := 2ce$ we conclude that $T_m^1 = T$ otherwise we arrive at a contradiction. Since $c$ is independent of $t$ or $V_0$, we conclude the global existence result by the usual continuation argument. The claim on exponential decay of $\|V(t)\|_{\mathcal{H}_2}$ now is a consequence of Lemma 5.4.

Q.E.D.

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