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Abstract. We consider parameter-elliptic boundary value problems and uniform *a priori* estimates in L^p-Sobolev spaces of Bessel potential and Besov type. The problems considered are systems of uniform order and mixed-order systems (Douglis-Nirenberg systems). It is shown that compatibility conditions on the data are necessary for such estimates to hold. In particular, we consider the realization of the boundary value problem as an unbounded operator with the ground space being a closed subspace of a Sobolev space and give necessary and sufficient conditions for the realization to generate an analytic semigroup.

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1 Introduction

The aim of this paper is to establish resolvent estimates for parameter-elliptic boundary value problems in L^p-Sobolev spaces of higher order. *A priori* estimates involving parameter-dependent norms for parameter-elliptic or parabolic systems are known since long; classical works are, e.g., Agmon [1], Agranovich-Vishik [2] for scalar equations, and Geymonat-Grisvard [9], Roitberg-Sheftel [13] for systems. Further results on the L^p-theory for mixed-order systems were obtained, e.g., by Faierman [7]. For pseudodifferential boundary value problems, we refer to the parameter-dependent calculus developed by Grubb [10].

Parameter-dependent *a priori* estimates are motivated by their connection to operator theory: In the ground space L^p, the estimate immediately implies a uniform resolvent estimate for the L^p-realization of the boundary value problem. In particular, if the sector of parameter-ellipticity is large enough, i.e., if the problem is parabolic in the sense of Petrovskii, then the operator generates an analytic semigroup in L^p. Moreover, spectral properties and completeness of eigenfunctions can be obtained, see Denk-Faierman-Möller [4] and Faierman-Möller [8]. If the equation is given in the whole space, we obtain the generation of an analytic semigroup in the whole scale of Sobolev spaces. In fact, the operator even admits a bounded H^∞-calculus which was shown for general mixed-order systems of pseudodifferential operators in Denk-Saal-Seiler [5].

Consider the boundary value problem

$$\begin{cases} (A-\lambda)u = f, & \text{in } \Omega, \\ \quad B_j u = g_j, & \text{on } \partial\Omega, \quad j = 1,\dots,M, \end{cases} \tag{1.1}$$

in a bounded smooth domain $\Omega \subset \mathbb{R}^d$. Here A is a system of differential operators, and B_j is a vector of differential operators, and the number M of boundary conditions is determined by the order and the dimension of the system A (see below for details). In the present paper we study the question under which additional (compatibility) assumptions on the right-hand side this boundary value problem has a unique solution satisfying uniform (in λ) *a priori* estimates. In particular, for $s \geq 0$ and $1 < p < \infty$ let us consider a closed linear subspace Y of the Sobolev space $W_p^s(\Omega)$ as a ground space and define the realization of (1.1) as an unbounded operator $\mathcal{A}$ in Y with domain $D(\mathcal{A}) := \{v \in Y : Av \in Y,\ B_j v = 0,\ j = 1,\dots,M\}$. In the particular case $s = 0$, the parameter-elliptic theory mentioned above yields the generation of an analytic semigroup in $L^p(\Omega)$, provided the sector of parameter-ellipticity is large enough. For $s > 0$, however, the situation is more complicated. As an example, one may consider the Dirichlet-Laplacian $\triangle_D$ in $Y = W_p^1(\Omega)$ with domain $D(\triangle_D) = \{u \in W_p^3(\Omega)\colon u|_{\partial\Omega} = 0\}$. This operator does not generate an analytic semigroup in Y; in fact, its resolvent decays as $|\lambda|^{-1/2-1/2p}$ as $|\lambda| \to \infty$ (see Nesensohn [12]). Roughly speaking, additional compatibility conditions have to be incorporated into the basic space Y in order to obtain a decay of $|\lambda|^{-1}$.

Therefore, the question is to find equivalent conditions on Y for which $\mathcal{A}$ generates an analytic semigroup on Y. This question is fully answered by Theorem 3.5 below. We also study compatibility conditions for which the problem with inhomogeneous boundary data (1.1) is uniquely solvable with appropriate *a priori* estimate for the solution. As a ground space, we consider subspaces of integer or non-integer Sobolev spaces both of Besov type and of Bessel potential type.

The question of generation of an analytic semigroup for parabolic equations was also studied by Guidetti [11] where higher order scalar equations are considered. Writing such an equation as a first order system, in [11] necessary and sufficient conditions for the unique solvability of the non-stationary problem are given. Roughly speaking, in [11] the author observes that the order of the boundary operators has to be sufficiently large. This coincides with our conditions as in this case the trace conditions given in Theorem 3.5 are empty. Whereas the equations in [11] have more general coefficients, the mixed-order system is of special structure (arising from a higher order equation), and the basic space is fixed. Our paper considers general mixed-order systems and the whole scale of Sobolev spaces.

2 Notation and Auxiliary Results

Let Ω be a bounded domain in $\mathbb{R}^d$, $d \geq 2$, with boundary $\Gamma = \partial\Omega \in C^\infty$. The Besov spaces are denoted by $B^s_{p,q}(\Omega)$, for $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$, and the Bessel potential spaces are called $H^s_p(\Omega)$, for $s \in \mathbb{R}$ and $1 < p < \infty$. Then the Sobolev(–Slobodecky) spaces are

$$W^s_p = \begin{cases} H^s_p & : s \in \mathbb{N}_0, \\ B^s_{pp} & : s \notin \mathbb{N}_0, \end{cases}$$

with $s \in [0, \infty)$ and $1 < p < \infty$.

In this paper, $\mathcal{K}^s_p(\Omega)$ shall mean everywhere either the Bessel potential space $H^s_p(\Omega)$, or one of the Besov spaces $B^s_{p,q}(\Omega)$, $1 < q < \infty$. Here $s \in \mathbb{R}$ and $1 < p < \infty$. For $s > 1/p$, we define the space $\mathcal{K}^{s-1/p}_{p;\Gamma}$ of traces of functions from $\mathcal{K}^s_p(\Omega)$ at the boundary $\Gamma = \partial\Omega$:

$$\mathcal{K}^{s-1/p}_{p,\Gamma} := \begin{cases} B^{s-1/p}_{p,q}(\partial\Omega) & : \mathcal{K}^s_p(\Omega) = B^s_{p,q}(\Omega), \\ B^{s-1/p}_{p,p}(\partial\Omega) & : K^s_p(\Omega) = H^s_p(\Omega). \end{cases}$$

To simplify later formulae, we set $\mathcal{K}^0_{p,\Gamma} := L^p(\Omega)$, although this is not the space of traces of functions from $H^{1/p}_p(\Omega)$ or $B^{1/p}_{p,q}(\Omega)$, except when $q = 1$. The trace operator on $\partial\Omega$, mapping functions from $C^\infty(\overline{\Omega})$ to their boundary values, is called γ_0.

We will write $[\cdot,\cdot]_\theta$ for the complex interpolation method, and $(\cdot,\cdot)_{\theta,q}$ for the real interpolation method, where $0 \leq \theta \leq 1$ and $1 \leq q \leq \infty$. Then ∂^α_x maps continuously from $\mathcal{K}^s_p(\Omega)$ into $\mathcal{K}^{s-|\alpha|}_p(\Omega)$, for all $s \in \mathbb{R}$ and all $p \in (1, \infty)$, and $\{\mathcal{K}^s_p(\Omega)\}_{s\in\mathbb{R}}$ forms an interpolation scale with respect to the complex interpolation method:

$$\left[\mathcal{K}^{s_0}_p(\Omega), \mathcal{K}^{s_1}_p(\Omega)\right]_\theta = \mathcal{K}^{s_\theta}_p(\Omega), \qquad s_\theta = (1-\theta)s_0 + \theta s_1, \qquad 0 \leq \theta \leq 1.$$

We will also make free use of the following: if a Banach space X_θ is an interpolation space of the pair (X_0, X_1) of order θ, then

$$\varrho^{1-\theta} \|f\|_\theta \leq C(\varrho \|f\|_0)^{1-\theta} \|f\|_1^\theta \leq C(\|f\|_1 + \varrho \|f\|_0), \qquad \varrho \in \mathbb{R}_+, \qquad f \in X_0 \cap X_1.$$

For detailed representations of the theory of function spaces, we refer the reader to [3] and [14].

Lemma 2.1. *Suppose $0 \le \sigma_0 < 1/p < \sigma_1$. Then we have the estimates*

$$\varrho^{1-\theta} \|\gamma_0 u\|_{L^p(\partial\Omega)} \le C \left(\|u\|_{\mathcal{K}_p^{\sigma_1}(\Omega)} + \varrho \|u\|_{\mathcal{K}_p^{\sigma_0}(\Omega)} \right), \qquad \theta = \frac{1/p - \sigma_0}{\sigma_1 - \sigma_0}, \tag{2.1}$$

$$\varrho^{1-\theta} \|\gamma_0 u\|_{L^p(\partial\Omega)} \le C \left(\|u\|_{B_{p,q}^{\sigma_1}(\Omega)} + \varrho \|u\|_{L^p(\Omega)} \right), \qquad \theta = \frac{1}{p\sigma_1}, \tag{2.2}$$

for all $u \in \mathcal{K}_p^{\sigma_1}(\Omega)$ and all $\varrho \in [1, \infty)$.

Proof. In [14, Theorem 4.7.1], we find $\gamma_0 \in \mathcal{L}(B_{p,1}^{1/p}(\Omega), L^p(\partial\Omega))$, hence we conclude that

$$\|\gamma_0 u\|_{L^p(\partial\Omega)} \le C \|u\|_{B_{p,1}^{1/p}(\Omega)}.$$

Now we have, for the above σ_0, σ_1,

$$\left(B_{p,q}^{\sigma_0}(\Omega), B_{p,q}^{\sigma_1}(\Omega)\right)_{\theta,1} = \left(H_p^{\sigma_0}(\Omega), H_p^{\sigma_1}(\Omega)\right)_{\theta,1} = B_{p,1}^{1/p}(\Omega), \qquad \theta = \frac{1/p - \sigma_0}{\sigma_1 - \sigma_0},$$

which brings us (2.1). And (2.2) follows from

$$B^0_{p,\min(2,p)}(\Omega) \hookrightarrow L^p(\Omega) \hookrightarrow B^0_{p,\max(2,p)}(\Omega)$$

and the interpolation identity $(B^0_{p,r}(\Omega), B^{\sigma_1}_{p,q}(\Omega))_{\theta,1} = B^{1/p}_{p,1}(\Omega)$ for $r \in \{2, p\}$. □

3 Main Results

3.1 Systems of Uniform Order

First, let $A = (a_{jk}(x, D_x))_{j,k=1,\dots,N}$ be a matrix differential operator with $\operatorname{ord} a_{jk} \le m$ for all j, k. The coefficients of a_{jk} are smooth on a neighborhood of $\overline{\Omega}$. If A is a parameter-elliptic matrix differential operator in Ω, then $mN \in 2\mathbb{N}$, see [2]. Then let us be given differential operators $B_j = B_j(x, D_x)$ for $j = 1, \dots, mN/2$, with $\operatorname{ord} B_j = r_j \le m - 1$. For λ from a sector $\mathcal{L} \subset \mathbb{C}$ with vertex at the origin, we consider the boundary value problem

$$\begin{cases} (A - \lambda)u = f, & \text{in } \Omega, \\ \quad \gamma_0 B_j u = g_j, & \text{on } \partial\Omega, \quad j = 1, \dots, mN/2, \end{cases} \tag{3.1}$$

and its variant with homogeneous boundary data:

$$\begin{cases} (A - \lambda)u = f, & \text{in } \Omega, \\ \quad \gamma_0 B_j u = 0, & \text{on } \partial\Omega, \quad j = 1, \dots, mN/2. \end{cases} \tag{3.2}$$

We suppose that the operators $(A, B_1, \dots, B_{mN/2})$ constitute a parameter-elliptic boundary value problem on Ω in an open sector $\mathcal{L}$ of $\mathbb{C}$.

Proposition 3.1. *Let u be any function from $\mathcal{K}_p^{s+m}(\Omega)$ with $s \in [0, \infty)$ but $s \notin \mathbb{N} + 1/p$, and take $\lambda \in \mathbb{C}$ arbitrarily. Define f and g_j by the right-hand sides of (3.1).*

Then we have the inequality

$$\begin{aligned} &\|f\|_{\mathcal{K}_p^s(\Omega)} + \sum_{j=1}^{mN/2} \left(\|g_j\|_{\mathcal{K}_{p,\Gamma}^{s+m-r_j-1/p}} + |\lambda|^{1+\frac{1}{m}\min(s-r_j-1/p,0)} \|g_j\|_{\mathcal{K}_{p,\Gamma}^{\max(s-r_j-1/p,0)}} \right) \\ &\qquad \le C \left(\|u\|_{\mathcal{K}_p^{s+m}(\Omega)} + |\lambda| \|u\|_{\mathcal{K}_p^s(\Omega)} \right), \end{aligned}$$

with some constant C independent of u and λ.

Proof. It suffices to establish the inequalities

$$\begin{aligned} \|f\|_{\mathcal{K}^s_p(\Omega)} &\le C\left(\|u\|_{\mathcal{K}^{s+m}_p(\Omega)} + |\lambda|\,\|u\|_{\mathcal{K}^s_p(\Omega)}\right), \\ \|g_j\|_{\mathcal{K}^{s+m-r_j-1/p}_{p,\Gamma}} &\le C\,\|u\|_{\mathcal{K}^{s+m}_p(\Omega)}, \\ |\lambda|\,\|g_j\|_{\mathcal{K}^{s-r_j-1/p}_{p,\Gamma}} &\le C|\lambda|\,\|u\|_{\mathcal{K}^s_p(\Omega)}, \qquad (s-r_j-1/p>0), \qquad (3.3) \\ |\lambda|^{1+\frac{1}{m}(s-r_j-1/p)}\,\|g_j\|_{L^p(\partial\Omega)} &\le C\left(\|u\|_{\mathcal{K}^{s+m}_p(\Omega)} + |\lambda|\,\|u\|_{\mathcal{K}^s_p(\Omega)}\right), \qquad (s-r_j-1/p<0), \qquad (3.4) \end{aligned}$$

of which we will only discuss the last two. For $0 < s - r_j - 1/p \notin \mathbb{N}$, we have

$$\|g_j\|_{\mathcal{K}^{s-r_j-1/p}_{p,\Gamma}} = \|\gamma_0 B_j u\|_{\mathcal{K}^{s-r_j-1/p}_{p,\Gamma}} \le C\,\|B_j u\|_{\mathcal{K}^{s-r_j}_p(\Omega)} \le C\,\|u\|_{\mathcal{K}^s_p(\Omega)},$$

as claimed in (3.3). Concerning (3.4) in the case of $s \le r_j$, we write

$$1 + \frac{1}{m}\left(s - r_j - \frac{1}{p}\right) = \frac{s+m-r_j}{m}\cdot\left(1 - \frac{1}{p(s+m-r_j)}\right), \qquad \varrho := |\lambda|^{\frac{s+m-r_j}{m}},$$

and make use of Lemma 2.1:

$$\begin{aligned} |\lambda|^{1+\frac{1}{m}(s-r_j-1/p)}\,\|g_j\|_{L^p(\partial\Omega)} &= \varrho^{1-(p(s+m-r_j))^{-1}}\,\|\gamma_0 B_j u\|_{L^p(\partial\Omega)} \\ &\le C\left(\|B_j u\|_{\mathcal{K}^{s+m-r_j}_p(\Omega)} + \varrho\,\|B_j u\|_{\mathcal{K}^0_p(\Omega)}\right) \le C\left(\|u\|_{\mathcal{K}^{s+m}_p(\Omega)} + |\lambda|^{\frac{s+m-r_j}{m}}\,\|u\|_{\mathcal{K}^{r_j}_p(\Omega)}\right). \end{aligned}$$

Exploiting now $s \le r_j$, we can interpolate:

$$|\lambda|^{\frac{s+m-r_j}{m}}\,\|u\|_{\mathcal{K}^{r_j}_p(\Omega)} \le C\left(\|u\|_{\mathcal{K}^{s+m}_p(\Omega)} + |\lambda|\,\|u\|_{\mathcal{K}^s_p(\Omega)}\right),$$

which is what we wanted to show. And for (3.4) in the case of $r_j < s < r_j + 1/p$, we take $\sigma_1 = s + m - r_j$, $\sigma_0 = s - r_j < 1/p$, $\varrho = |\lambda|$, and then θ from (2.1) becomes $\theta = -(s - r_j - 1/p)/m$, which brings us to

$$|\lambda|^{1-\theta}\,\|\gamma_0 B_j u\|_{L^p(\partial\Omega)} \le C\left(\|B_j u\|_{\mathcal{K}^{s+m-r_j}_p(\Omega)} + |\lambda|\,\|B_j u\|_{\mathcal{K}^{s-r_j}_p(\Omega)}\right).$$

Then (3.4) quickly follows. □

Consequently, the norms of the given functions f and g_j appearing in the next result are the natural ones, and also the exponents of $|\lambda|$ are natural.

Theorem 3.2. *Let* (3.1) *be parameter-elliptic in* $\mathcal{L}$*, and suppose that* f *and the* g_j *are such that all solutions* u *to* (3.1) *enjoy the following estimate for all* $\lambda \in \mathcal{L}$ *with large* $|\lambda|$*:*

$$\begin{aligned} \|u\|_{\mathcal{K}^{s+m}_p(\Omega)} + |\lambda|\,\|u\|_{\mathcal{K}^s_p(\Omega)} &\le C\,\|f\|_{\mathcal{K}^s_p(\Omega)} \\ &+ C\sum_{j=1}^{mN/2}\left(\|g_j\|_{\mathcal{K}^{s+m-r_j-1/p}_{p,\Gamma}} + |\lambda|^{1+\frac{1}{m}\min(s-r_j-1/p,0)}\,\|g_j\|_{\mathcal{K}^{\max(s-r_j-1/p,0)}_{p,\Gamma}}\right), \end{aligned}$$

with some $s \in [0,\infty)$ *and* $1 < p < \infty$*.*

Then $g_j \equiv 0$ *for all* j *with* $r_j < s - 1/p$*.*

Proof. From $u \in \mathcal{K}^{s+m}_p(\Omega)$ we get $B_j A u \in \mathcal{K}^{s-r_j}_p(\Omega)$, which admits traces on $\partial\Omega$. We then have from Lemma 2.1

$$\begin{aligned} |\lambda|\cdot|\lambda|^{\frac{s-r_j}{m}(1-\frac{1}{p(s-r_j)})}\,\|g_j\|_{L^p(\partial\Omega)} &= |\lambda|^{\frac{s-r_j}{m}(1-\frac{1}{p(s-r_j)})}\,\|\gamma_0 B_j(Au-f)\|_{L^p(\partial\Omega)} \\ &\le C\left(\|B_j(Au-f)\|_{\mathcal{K}^{s-r_j}_p(\Omega)} + |\lambda|^{\frac{s-r_j}{m}}\,\|B_j(Au-f)\|_{\mathcal{K}^0_p(\Omega)}\right) \\ &\le C\left(\|Au-f\|_{\mathcal{K}^s_p(\Omega)} + |\lambda|^{\frac{s-r_j}{m}}\,\|Au-f\|_{\mathcal{K}^{r_j}_p(\Omega)}\right) \\ &\le C\left(\|u\|_{\mathcal{K}^{s+m}_p(\Omega)} + |\lambda|^{\frac{s-r_j}{m}}\,\|u\|_{\mathcal{K}^{m+r_j}_p(\Omega)} + \|f\|_{\mathcal{K}^s_p(\Omega)} + |\lambda|^{\frac{s-r_j}{m}}\,\|f\|_{\mathcal{K}^{r_j}_p(\Omega)}\right) \\ &\le C\left(\|u\|_{\mathcal{K}^{s+m}_p(\Omega)} + |\lambda|\,\|u\|_{\mathcal{K}^s_p(\Omega)} + \|f\|_{\mathcal{K}^s_p(\Omega)} + |\lambda|^{\frac{s-r_j}{m}}\,\|f\|_{\mathcal{K}^{r_j}_p(\Omega)}\right), \end{aligned}$$

the last step by interpolation. Then we can bring the assumed inequality into play:

$$
\begin{aligned}
&|\lambda| \cdot |\lambda|^{\frac{s-r_j}{m}(1-\frac{1}{p(s-r_j)})} \|g_j\|_{L^p(\partial\Omega)} \\
&\quad \le C \left(\|f\|_{\mathcal{K}_p^s(\Omega)} + |\lambda|^{\frac{s-r_j}{m}} \|f\|_{\mathcal{K}_p^{r_j}(\Omega)} \right) \\
&\quad\quad + C \sum_{l=1}^{mN/2} \left(\|g_l\|_{\mathcal{K}_{p,\Gamma}^{s+m-r_l-1/p}} + |\lambda|^{1+\frac{1}{m}\min(s-r_l-1/p,0)} \|g_l\|_{\mathcal{K}_{p,\Gamma}^{\max(s-r_l-1/p,0)}} \right).
\end{aligned}
$$

If $g_j \not\equiv 0$ then the exponent of $|\lambda|$ on the left-hand side is greater than each exponent of $|\lambda|$ on the right-hand side, giving a contradiction for large $|\lambda|$. □

Theorem 3.3. *Let* (3.2) *be parameter-elliptic in* $\mathcal{L}$. *Fix* $p \in (1,\infty)$ *and* $s \in [0,m]$. *Then the following two statements are equivalent for* $f \in \mathcal{K}_p^s(\Omega)$.

1. *there are positive constants* λ_0 *and* C_0 *such that all solutions* u *to* (3.2) *with* $\lambda \in \mathcal{L}$, $|\lambda| \ge \lambda_0$ *enjoy the following estimate:*

$$
\|u\|_{\mathcal{K}_p^{s+m}(\Omega)} + |\lambda| \, \|u\|_{\mathcal{K}_p^s(\Omega)} \le C_0 \|f\|_{\mathcal{K}_p^s(\Omega)},
$$

2. $\gamma_0 B_j f \equiv 0$ *for all* j *with* $s - r_j > 1/p$.

Proof. Suppose statement no.1 and choose j with $s - r_j > 1/p$. Then $B_j f \in \mathcal{K}_p^{s-r_j}(\Omega)$, which admits traces on $\partial\Omega$, and we can argue as in the proof of Theorem 3.2:

$$
\begin{aligned}
|\lambda|^{\frac{s-r_j}{m}(1-\frac{1}{p(s-r_j)})} \|\gamma_0 B_j f\|_{L^p(\partial\Omega)} &= |\lambda|^{\frac{s-r_j}{m}(1-\frac{1}{p(s-r_j)})} \|\gamma_0 B_j(Au - \lambda u)\|_{L^p(\partial\Omega)} \\
&= |\lambda|^{\frac{s-r_j}{m}(1-\frac{1}{p(s-r_j)})} \|\gamma_0 B_j Au\|_{L^p(\partial\Omega)} \\
&\le C \left(\|B_j Au\|_{\mathcal{K}_p^{s-r_j}(\Omega)} + |\lambda|^{\frac{s-r_j}{m}} \|B_j Au\|_{\mathcal{K}_p^0(\Omega)} \right) \\
&\le C \left(\|u\|_{\mathcal{K}_p^{s+m}(\Omega)} + |\lambda|^{\frac{s-r_j}{m}} \|u\|_{\mathcal{K}_p^{m+r_j}(\Omega)} \right) \\
&\le C \left(\|u\|_{\mathcal{K}_p^{s+m}(\Omega)} + |\lambda| \, \|u\|_{\mathcal{K}_p^s(\Omega)} \right) \\
&\le C \|f\|_{\mathcal{K}_p^s(\Omega)}.
\end{aligned}
$$

Now send $\lambda \to \infty$ in $\mathcal{L}$.

Conversely, suppose statement no.2. Define $X = L^p(\Omega)$ and an operator $\mathcal{A}\colon D(\mathcal{A}) \to X$ by

$$
\begin{aligned}
D(\mathcal{A}) &:= \{u \in W_p^m(\Omega) \colon \gamma_0 B_j u = 0, \quad j = 1, \dots, mN/2\}, \\
\mathcal{A}u &:= Au,
\end{aligned}
$$

and set

$$
\begin{aligned}
Y_s &:= \begin{cases} [L^p(\Omega), D(\mathcal{A})]_{s/m} & : \mathcal{K}_p^\bullet(\Omega) = H_p^\bullet(\Omega), \\ (L^p(\Omega), D(\mathcal{A}))_{s/m,q} & : \mathcal{K}_p^\bullet(\Omega) = B_{p,q}^\bullet(\Omega) \end{cases} \\
&= \left\{ u \in \mathcal{K}_p^s(\Omega) \colon \gamma_0 B_j u \equiv 0, \quad \forall\, j \text{ with } s - r_j > 1/p \right\}
\end{aligned}
$$

Then $D(\mathcal{A}) \hookrightarrow Y_s \hookrightarrow X$ with dense embeddings. From [9] we quote the estimate

$$
\|u\|_{W_p^m(\Omega)} + |\lambda| \, \|u\|_{L^p(\Omega)} \le C \|f\|_{L^p(\Omega)}, \qquad f \in X,
$$

for $u = (\mathcal{A} - \lambda)^{-1} f$ and $\lambda \in \mathcal{L}$, $|\lambda| \ge \lambda_0$. And for $f \in D(\mathcal{A})$, we have $u = (\mathcal{A} - \lambda)^{-1} f \in D(\mathcal{A}^2)$, hence

$$
\|u\|_{W_p^{2m}(\Omega)} + |\lambda| \, \|u\|_{W_p^m(\Omega)} \le C \|f\|_{W_p^m(\Omega)}, \qquad f \in D(\mathcal{A}).
$$

Interpolating between these two estimates then implies

$$
\|u\|_{\mathcal{K}_p^{s+m}(\Omega)} + |\lambda| \, \|u\|_{\mathcal{K}_p^s(\Omega)} \le C \|f\|_{\mathcal{K}_p^s(\Omega)}, \qquad f \in Y_s,
$$

for $s \in [0,m]$. □

Theorem 3.4. *Let* (3.2) *be parameter-elliptic in the sector* $\mathcal{L}$, *and fix* $p \in (1,\infty)$ *and* $s_{\max} \in [0,\infty)$. *Then the following two statements are equivalent, for* $f \in \mathcal{K}_p^{s_{\max}}(\Omega)$.

1. *there are positive constants* λ_0 *and* C_0 *such that all solutions* u *to* (3.2) *with* $\lambda \in \mathcal{L}$, $|\lambda| \geq \lambda_0$ *satisfy the collection of estimates*

$$\|u\|_{\mathcal{K}_p^{s+m}(\Omega)} + |\lambda| \, \|u\|_{\mathcal{K}_p^s(\Omega)} \leq C_0 \, \|f\|_{\mathcal{K}_p^s(\Omega)},$$

   for all $s \in [0, s_{\max}]$.

2. *for each pair* $(j,k) \in \{1,2,\ldots,m\} \times \mathbb{N}_0$ *with* $s_{\max} - r_j > mk + 1/p$, *we have* $\gamma_0 B_j A^k f \equiv 0$.

Proof. A proof for the case $s_{\max} \in [0,m]$ was given in Theorem 3.3, whose notations we adopt here. And the proof of the first statement from the second is very similar to the proof of Theorem 3.3, so we skip it. Therefore we may assume $s_{\max} \geq m$. We suppose now the statement no.1, and proceed by induction on $s_{\max}$ of step size m.

Choosing $s = m$, we find $\gamma_0 B_j f \equiv 0$ for all j, hence $f \in D(\mathcal{A})$, and then also $\mathcal{A}u \in D(\mathcal{A})$. By parameter-ellipticity in $\mathcal{L}$, there is a number $\lambda_* \in \mathcal{L}$ with $|\lambda_*| \geq \lambda_0 + 1$ such that $\mathcal{A} - \lambda_* \colon D(\mathcal{A}) \cap W_p^{\sigma+m}(\Omega) \to W_p^{\sigma}(\Omega)$ is a continuous isomorphism, for all $\sigma \in \mathbb{N}_0$. Fix this λ_*. Put $\tilde{u} := (\mathcal{A} - \lambda_*)u$, $\tilde{f} := (\mathcal{A} - \lambda_*)f$, and note that

$$\begin{cases} (A-\lambda)\tilde{u} = \tilde{f}, & \text{in } \Omega, \\ \gamma_0 B_j \tilde{u} = 0, & \text{on } \partial\Omega, \end{cases}$$

with $\tilde{f} \in \mathcal{K}_p^{s_{\max}-m}(\Omega)$. For $0 \leq s \leq s_{\max} - m$ and $\lambda \in \mathcal{L}$, $|\lambda| \geq \lambda_0$, we then have

$$\begin{aligned} &\|\tilde{u}\|_{\mathcal{K}_p^{s+m}(\Omega)} + |\lambda| \, \|\tilde{u}\|_{\mathcal{K}_p^s(\Omega)} \\ &\quad \leq C \left(\|u\|_{\mathcal{K}_p^{m+(s+m)}(\Omega)} + |\lambda| \, \|u\|_{\mathcal{K}_p^{s+m}(\Omega)} \right) \\ &\quad \leq C \, \|f\|_{\mathcal{K}_p^{s+m}(\Omega)} = C \left\| (\mathcal{A}-\lambda)^{-1} \tilde{f} \right\|_{\mathcal{K}_p^{s+m}(\Omega)} \\ &\quad \leq \tilde{C}_0 \left\| \tilde{f} \right\|_{\mathcal{K}_p^s(\Omega)}. \end{aligned}$$

By induction, we know that $\gamma_0 B_j A^k \tilde{f} \equiv 0$ for all pairs $(j,k) \in \{1,\ldots,m\} \times \mathbb{N}_0$ with $(s_{\max}-m) - r_j > mk + 1/p$. The definition of $\tilde{f}$ then brings us to $\gamma_0 B_j A^k f \equiv 0$ for all (j,k) with $s_{\max} - r_j > mk + 1/p$. □

Theorem 3.5. *Let* (3.2) *be parameter-elliptic in a sector* $\mathcal{L}$ *that is greater than the right half-plane. For* $s \geq 0$ *and* $1 < p < \infty$, *let* Y *be a closed linear subspace of* $\mathcal{K}_p^s(\Omega)$, *equipped with the norm of* $\mathcal{K}_p^s(\Omega)$. *Define an operator* $\mathcal{A}$ *in the ground space* Y *by* $\mathcal{A}u := Au$ *for*

$$u \in D(\mathcal{A}) := \{ v \in Y \colon Av \in Y, \quad \gamma_0 B_j v \equiv 0 \quad \forall\, j \}.$$

Then the following are equivalent:

1. *The operator* $\mathcal{A}$ *generates an analytic semigroup on* Y,
2. *The embedding* $D(\mathcal{A}) \hookrightarrow Y$ *is dense,* $(\mathcal{A}-\lambda)^{-1} \in \mathcal{L}(Y)$ *for all* $\lambda \in \mathcal{L}$ *of large modulus, and* $\gamma_0 B_j A^k f \equiv 0$ *for all* $f \in Y$ *and all pairs* (j,k) *with* $s - r_j > mk + 1/p$.

Proof. The domain of a generator of a C_0 semigroup is always dense in the ground space. Under the assumptions on $\mathcal{L}$, Y and $D(\mathcal{A})$, the analyticity of the semigroup is equivalent to the resolvent estimate

$$\left\| (\mathcal{A}-\lambda)^{-1} \right\|_{\mathcal{L}(Y)} \leq \frac{C}{|\lambda|}$$

for all $\lambda \in \mathcal{L}$ of large modulus. Now apply Theorem 3.4. □

Theorem 3.6. *Let (3.2) be parameter-elliptic in $\mathcal{L}$. Fix $p \in (1, \infty)$, $s \in [0, \infty)$, and $\gamma \in (-\infty, 1]$. Choose a function $f \in \mathcal{K}_p^s(\Omega)$.*

Assume that there are positive constants λ_0 and C_0 such that all solutions u to (3.2) with $\lambda \in \mathcal{L}$, $|\lambda| \geq \lambda_0$ enjoy the following estimate:

$$|\lambda|^\gamma \|u\|_{\mathcal{K}_p^s(\Omega)} \leq C_0 \|f\|_{\mathcal{K}_p^s(\Omega)} .$$

Then $\gamma_0 B_j f \equiv 0$ for all j with

$$\gamma > \frac{m + r_j + 1/p - s}{m}. \tag{3.5}$$

Remark 3.7. *In case of the Dirichlet Laplacian $\triangle_D$, considered in the space $W_p^1(\Omega)$, the condition (3.5) turns into*

$$\gamma > \frac{p+1}{2p},$$

which matches the result of [12], where the resolvent estimate from below,

$$\left\|(\triangle_D - \lambda)^{-1}\right\|_{\mathcal{L}(W_p^1(\mathbb{R}_+^n), W_p^1(\mathbb{R}_+^n))} \geq \frac{C}{|\lambda|^{(p+1)/(2p)}}, \qquad C > 0,$$

was proved.

Proof of Theorem 3.6. Choose such a j. By $\gamma \leq 1$ and the condition (3.5), we get $s - r_j > 1/p$, and therefore $\gamma_0 B_j f \in \mathcal{K}_{p,\Gamma}^{s-r_j-1/p}$ exists. Now we can take a frozen λ_* as in the proof of Theorem 3.4, and write

$$u = (\mathcal{A} - \lambda_*)^{-1}(f + (\lambda - \lambda_*)u),$$

and consequently $\|u\|_{\mathcal{K}_p^{s+m}(\Omega)} \leq C(\|f\|_{\mathcal{K}_p^s(\Omega)} + |\lambda| \|u\|_{\mathcal{K}_p^s(\Omega)}) \leq C|\lambda|^{1-\gamma} \|f\|_{\mathcal{K}_p^s(\Omega)}$, due to $|\lambda| \geq 1$. Now we can compute:

$$\begin{aligned}
|\lambda|^{\frac{s-r_j}{m}(1-\frac{1}{p(s-r_j)})} \|\gamma_0 B_j f\|_{L^p(\partial\Omega)} &= |\lambda|^{\frac{s-r_j}{m}(1-\frac{1}{p(s-r_j)})} \|\gamma_0 B_j A u\|_{L^p(\partial\Omega)} \\
&\leq C \left(\|B_j A u\|_{\mathcal{K}_p^{s-r_j}(\Omega)} + |\lambda|^{\frac{s-r_j}{m}} \|B_j A u\|_{\mathcal{K}_p^0(\Omega)} \right) \\
&\leq C \left(\|u\|_{\mathcal{K}_p^{s+m}(\Omega)} + |\lambda| \|u\|_{\mathcal{K}_p^s(\Omega)} \right) \\
&\leq C|\lambda|^{1-\gamma} \|f\|_{\mathcal{K}_p^s(\Omega)} .
\end{aligned}$$

Per (3.5), the left-hand side has a higher power of $|\lambda|$ than the right-hand side. □

3.2 Systems of Mixed Order

In this section, A shall be a matrix differential operator of mixed order:

$$A = (a_{jk}(x, D_x))_{j,k=1,\dots,N}, \qquad \operatorname{ord} a_{jk} \leq s_j + m_k,$$

for integers s_j and m_k. The orders on the diagonal of A shall be equal,

$$s_1 + m_1 = \ldots = s_N + m_N =: m,$$

and without loss of generality, we can set $\min_j m_j = 0$.

The principal part a_{jk}^0 of a_{jk} is that part with degree exactly equal to $s_j + m_k$ (if such a part exists, otherwise $a_{jk}^0 := 0$). Then we put $A^0 := (a_{jk}^0)_{j,k=1,\dots,N}$, and the operator A is called parameter-elliptic in the sector

$\mathcal{L} \subset \mathbb{C}$ if $\det(A^0(x,\xi)-\lambda) \neq 0$ for all $(x,\xi,\lambda) \in \overline{\Omega} \times \mathbb{R}^d \times \mathcal{L}$ with $|\xi|+|\lambda|>0$. Then (see [2]) $mN \in 2\mathbb{N}$, and we can consider a matrix of boundary differential operators,

$$B = (b_{j,k}(x,D_x))_{j,k}, \qquad j=1,\dots,mN/2, \qquad k=1,\dots,N, \qquad \operatorname{ord} b_{jk} \le r_j + m_k,$$

with integers $r_j \le m-1$. We define the principal part B^0 of B in the same way as A^0 was defined. We say that the Shapiro–Lopatinskii condition is satisfied if at each $x^* \in \partial\Omega$, after introducing a new frame of Cartesian coordinates with center at x^* and the x_d–axis pointing along the inner normal vector at x^*, the system of ordinary differential equations

$$\begin{cases} (A^0(x^*,\xi',D_{x_d})-\lambda)v(x_d)=0, & 0 \le x_d < \infty, \\ B^0(x^*,\xi',D_{x_d})v(x_d)=0, & x_d=0, \\ \lim_{x_d\to\infty} v(x_d)=0 & \end{cases}$$

possesses only the trivial solution, for all $(\xi',\lambda) \in \mathbb{R}^{n-1}\times\mathcal{L}$ with $|\xi'|+|\lambda|>0$.

Then the system (A,B) is called a parameter-elliptic boundary value problem in the sector $\mathcal{L} \subset \mathbb{C}$ if A is parameter-elliptic in $\mathcal{L}$, and the Shapiro–Lopatinskii condition holds.

Write $B = (B_1,\dots,B_{mN/2})^\top$ as a column of rows, and consider the boundary value problem

$$\begin{cases} (A-\lambda)u = f, & \text{in } \Omega, \\ \gamma_0 B_j u = 0, & \text{on } \partial\Omega, \quad j=1,\dots,mN/2. \end{cases} \tag{3.6}$$

In [7], it has been shown that a number λ_0 exists such that, for all λ from $\mathcal{L}$ with $|\lambda| \ge \lambda_0$, and for all $f \in W_p^{m_1}(\Omega)\times\ldots\times W_p^{m_N}(\Omega)$, a unique solution $u \in W_p^{m+m_1}(\Omega)\times\ldots\times W_p^{m+m_N}(\Omega)$ to (3.6) exists, and the estimate

$$\sum_{k=1}^N \left(\|u_k\|_{W_p^{m+m_k}(\Omega)} + |\lambda|^{1+m_k/m} \|u_k\|_{L^p(\Omega)} \right) \le C \sum_{k=1}^N \left(\|f_k\|_{W_p^{m_k}(\Omega)} + |\lambda|^{m_k/m} \|f_k\|_{L^p(\Omega)} \right)$$

holds, with C depending only on (A,B).

Having secured the existence of u for large $|\lambda|$, we can now ask under which conditions resolvent estimates for A might exist. Sufficient conditions were presented in Dreher [6].

Theorem 3.8. *If f is such that for all λ of large modulus the inequality*

$$\sum_{j=1}^N \left(\|u_j\|_{W_p^{m+m_j}(\Omega)} + |\lambda| \|u_j\|_{W_p^{m_j}(\Omega)} \right) \le C \sum_{j=1}^N \|f_j\|_{W_p^{m_j}(\Omega)}$$

holds with a constant C independent of λ, then $\gamma_0 B_j f \equiv 0$ for all j with $r_j \le -1$.

Proof. From $f_k \in W_p^{m_k}(\Omega)$ and $\operatorname{ord} b_{jk} \le r_j + m_k$, we deduce that $B_j f \in W_p^{-r_j}(\Omega)$, and this has a trace at the boundary for $r_j \le -1$. Pick such an index j.

Now we can estimate as usual:

$$\begin{aligned} |\lambda|^{\frac{1}{m}(1-\frac{1}{p})} \|\gamma_0 B_j f\|_{L^p(\partial\Omega)} &= |\lambda|^{\frac{1}{m}(1-\frac{1}{p})} \|\gamma_0 B_j A u\|_{L^p(\partial\Omega)} \\ &\le C \left(\|B_j A u\|_{W_p^1(\Omega)} + |\lambda|^{\frac{1}{m}} \|B_j A u\|_{L^p(\Omega)} \right) \\ &\le C \sum_{k=1}^N \left(\|u_k\|_{W_p^{m+m_k+r_j+1}(\Omega)} + |\lambda|^{\frac{1}{m}} \|u_k\|_{W_p^{m+m_k+r_j}(\Omega)} \right) \\ &\le C \sum_{k=1}^N \left(\|u_k\|_{W_p^{m+m_k}(\Omega)} + |\lambda|^{\frac{1}{m}} \|u_k\|_{W_p^{m+m_k-1}(\Omega)} \right) \\ &\le C \sum_{k=1}^N \left(\|u_k\|_{W_p^{m+m_k}(\Omega)} + |\lambda| \|u_k\|_{W_p^{m_k}(\Omega)} \right) \\ &\le C \sum_{k=1}^N \|f_k\|_{W_p^{m_k}(\Omega)}. \end{aligned}$$

Sending λ to infinity in Ω then implies $\gamma_0 B_j f \equiv 0$. □

References

[1] S. Agmon. On the eigenfunctions and on the eigenvalues of general elliptic boundary value problems. *Comm. Pure Appl. Math.*, 15:119–147, 1962.

[2] M. S. Agranovich and M. I. Vishik. Elliptic problems with a parameter and parabolic systems of general form. *Russ. Math. Surv.*, 19:53–157, 1964.

[3] J. Bergh and J. Löfström. *Interpolation Spaces. An Introduction.* Grundlehren math. Wiss. 223. Springer, 1976.

[4] R. Denk, M. Faierman, and M. Möller. An elliptic boundary problem for a system involving a discontinuous weight. *Manuscripta Math.*, 108(3):289–317, 2002.

[5] R. Denk, J. Saal, and J. Seiler. Bounded H_∞-calculus for pseudo-differential Douglis-Nirenberg systems of mild regularity. *Math. Nachr.*, 282(3):386–407, 2009.

[6] M. Dreher. Resolvent estimates for Douglis–Nirenberg systems. *Journ. Evol. Equ.*, 9:829–844, 2009.

[7] M. Faierman. Eigenvalue asymptotics for a boundary problem involving an elliptic system. *Math. Nachr.*, 279(11):1159–1184, 2006.

[8] M. Faierman and M. Möller. Eigenvalue asymptotics for an elliptic boundary problem. *Proc. Roy. Soc. Edinburgh Sect. A*, 137(2):281–302, 2007.

[9] G. Geymonat and P. Grisvard. Alcuni risultati di teoria spettrale per i problemi ai limiti lineari ellittici. *Rend. Sem. Mat. Univ. Padova*, 38:121–173, 1967.

[10] G. Grubb. *Functional calculus of pseudodifferential boundary problems*, volume 65 of *Progress in Mathematics.* Birkhäuser Boston Inc., Boston, MA, second edition, 1996.

[11] D. Guidetti. On boundary value problems for parabolic equations of higher order in time. *J. Differential Equations*, 124(1):1–26, 1996.

[12] M. Nesensohn. Randwertprobleme in Sobolevräumen höherer Ordnung. Diploma Thesis, University of Konstanz, 2009.

[13] J. A. Roitberg and Z. G. Sheftel. Boundary value problems with a parameter for systems elliptic in the sense of Douglis-Nirenberg. *Ukrain. Mat. Ž.*, 19(1):115–120, 1967.

[14] H. Triebel. *Interpolation Theory, Function Spaces, Differential Operators.* Deutscher Verlag der Wissenschaften, 1978.

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