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Asymptotic Behaviour of Radially Symmetric Solutions in Thermoelasticity

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Abstract: In this paper we consider equations of thermoelasticity in the linear and nonlinear cases with various boundary conditions. We assume radial symmetry to prove exponential decay and to show the global existence of solutions of the nonlinear problem with small initial data.

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[1]This paper summarizes results of a diploma thesis written at the University of Constance

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Notation

Throughout this paper we use the following notation:

$\mathbb{R}^{n\times n}$	Set of all $n \times n$–matrices on $\mathbb{R}$.
$O(n)$	Group of the orthogonal matrices
A^T	Transposed matrix of A
$\mathcal{C}^m(G,\mathbb{R})$	Set of m times continously differentiable functions on $G \subset \mathbb{R}^n$ (G open domain)
$\mathcal{C}_0^\infty(G,\mathbb{R})$	Infinitely differentiable functions with compact support.
$\mathcal{L}^2(G,\mathbb{R})$	Square integrable functions
$\mathcal{L}^\infty(G,\mathbb{R})$	Bounded functions
$H^m(G,\mathbb{R})$	Sobolev space of all m times weakly differentiable functions in $\mathcal{L}^2$
$\|\|\cdot\|\| := \|\|\cdot\|\|_{\mathcal{L}^2(G)}$	$:= \int_G \|\cdot\|^2$ ($\mathcal{L}^2$–Norm)
$\langle\cdot,\cdot\rangle$	$\mathcal{L}^2$–scalar product
$\vec{n} := \vec{n}(x)$	Normal vector on the boundary of G in $x \in \partial G$
∇_t	Derivative in the direction(s) of the tangent at the boundary
$\mathcal{O}(f), o(f)$	Landau symbols
$\mathcal{B}(x,r)$	Open ball with center x and radius r

Furthermore we often use the Einstein convention, i.e. we sum up all terms with the same index from 1 to n, where n is the dimension (in most cases $n = 2$ or $n = 3$). The operator ∇ is also used on vector fields, where we define $\nabla v := (\partial_i v_j)_{i,j=1\ldots n}$. Differential operators (like ∂_t or ∇) always operate on the very next function if not explicitly marked in a different way.

Chapter 1

Introduction

If we consider bodies reacting elastically on mechanical deformations and if we neglect other dependences of the elasticity (e.g. the dependence on temperature), a physical investigation leads to a model described by an elasticity equation. In this model we describe the unstressed body as a subset G of R^n (the reference configuration). The deformation of the body is described by a vector field $u(x,t)$, where $t \geq 0$ is the time and $x \in G$ the correspondig material point in the reference configuration. At time t the point x lies at $x + u(x,t)$. Therefore we call $u(x,t)$ the *displacement vector*.

The equation of elasticity is now (neglecting external forces):

$$\rho(x)u_{tt}(x,t) - \operatorname{div} S(\nabla u(x,t)) \quad = \quad 0 \tag{1.1}$$

S is a function describing the special behaviour of the material, ρ is the density of the medium. The linearized equation takes (in the case of homogenous media) the following form:

$$u_{tt} - \mu\Delta u - (\mu + \lambda)\nabla \operatorname{div} u \quad = \quad 0 \tag{1.2}$$

In this equation the coefficients μ and λ have to satisfy $2\mu + n\lambda > 0$. The equation is hyperbolic, its behaviour is quite similar to that of the wave equation. Especially there is no decay in bounded domains with the usual boundary conditions. For the nonlinear case (1.2) and the Cauchy problem we mention the proof of global existence for small initial data in [10].

Now we want to study bodies with elastic behaviour, but non–negligible temperature gradients. In this case we have to consider the interaction of the equations of elasticity and of thermodynamics. We eventually arrive at the *equations of thermoelasticity*. (For a short summary of the physical derivation see [11] and for a more complete one [2].) At the moment we only want to give the linearized form in the homogenous case:

$$u_{tt} - \mu\Delta u - (\mu + \lambda)\nabla \operatorname{div} u + \beta\nabla\theta \quad = \quad 0 \tag{1.3}$$

$$c\theta_t - \kappa\Delta\theta + \beta \operatorname{div} u_t \quad = \quad 0 \tag{1.4}$$

This system consists of the linearized equation of elasticity (1.3) and a heat equation (1.4), both are coupled with each other.

Again u is the displacement vector and now θ describes the temperature. The various other variables are material constants and will be defined later.

The "classical" theory of this system leads to a proof of existence and uniqueness in suitable Sobolev spaces. For this aim one can use semigroup theory. For details see [6]. To get results for the nonlinear case we have to know something about the asymptotic behaviour of (1.3) and (1.4). A first guess would be that the heat equation should lead to an exponential decay in bounded domains, but the elastic part would have no decay. So we come to an interesting problem: Which part of the system will dominate? The somewhat surprising answer is: This depends on the dimension!

In the case of $n = 1$ the decay is dominated by the heat equation, in higher dimensions (in most cases) the elasticity equation is the dominating one: E.g. there are bounded domains with the Dirichlet boundary condition, where θ decays exponentially but $u \to \tilde{u} \neq 0$. Furthermore one can prove by calculating the eigenvalues of the differential operator, that $\tilde{u} = 0$ iff the system

$$\begin{aligned} \Delta v + \lambda v &= 0, \\ \operatorname{div} v &= 0, \\ v\,|_{\partial G} &= 0 \end{aligned} \tag{1.5}$$

has no non–trivial solution. This is possible if e.g. the following scalar eigenvalue equation allows only single eigenvalues:

$$\begin{aligned} \Delta v + \lambda v &= 0, \\ v\,|_{\partial G} &= 0. \end{aligned} \tag{1.6}$$

For simple domains like discs and balls this is not correct. At least these domains are "rare" in a certain sense. (To be more exact: They are 1. Baire category in a certain C^3-metric. For more details see [6].) Recently H. Koch proved that in every domain containing a line which intersects in right angles with the boundary initial data can be constructed such that the decay is arbitrarily slow, see [5]. Of course in this category there are all convex domains with smooth boundary.

Nevertheless for special situations we are able to prove rates of decay: In the recent work of Jiang, Muñoz Rivera and Racke [4] it is proved, that *rotation free* solutions of the linear problem in the case of the Dirichlet boundary condidtion decay exponentially. This result can be used to obtain global solutions for the nonlinear problem.

In this paper we extend the work of Jiang, Muñoz Rivera and Racke to other boundary conditions, including the often neglected Robin condition. It turns out that some new ideas are necessary but that the results more or less still hold in these cases. In chapter 2 we treat the linear case for various boundary conditions, in chapter 3 we will study the nonlinear case for a typical Neumann condition. The problem we have to solve here is based in the nonlinearity of the Neumann condition. — Remember that the Dirichlet condition is linear also if the thermoelasticity equations are nonlinear. — This leads to interesting technical problems.

This work is based upon a diploma thesis (in German) at the University of Constance [11] which is more detailed and also sums up the results of [4], a derivation of the equations of thermoelasticity and the existence theory in the linearized case.

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Chapter 2

Linear Thermoelasticity

In this chapter we consider the system of linear thermoelasticity for a homogenous, isotropic medium (with appropriate initial conditions):

$$u_{tt} - \mu\Delta u - (\mu+\lambda)\nabla \mathrm{div}\, u + \beta\nabla\theta = 0 \quad (2.1)$$

$$c\theta_t - \kappa\Delta\theta + \beta \mathrm{div}\, u_t = 0 \quad (2.2)$$

For a physical derivation of these equations see [2] and the references therein.
Throughout this paper we always consider initial boundary value problems in bounded subdomains of $\mathbb{R}^n$, where $n = 2$ or $n = 3$ in most cases. (The results of this chapter may also be extended to other dimensions.) Our goal will be to describe the asymptotic behaviour of special solutions for various boundary conditions. For u and θ we consider Dirichlet and Neumann conditions and for θ we consider also a mixed boundary condition, the so–called Robin condition. The physical meaning of these boundary condition is shown in the following table:

	Dirichlet	Neumann	Robin boundary condition
u	body fixed on the boundary	free boundary	—
θ	temperature fixed on the boundary	perfect isolated boundary	heat flow on the boundary

To proof the existence of solutions to this problems we can use semigroup theory. We only want to state the result, a proof can be found for example in [6].

Remark 2.1 *There exists a unique solution* (u, θ) *of the initial boundary value problems with* $u \in \mathcal{C}^2([0,\infty), \mathcal{L}^2) \cap \mathcal{C}^1([0,\infty), H^1), \theta \in \mathcal{C}^1([0,\infty), \mathcal{L}^2), \Delta\theta \in \mathcal{C}([0,\infty), \mathcal{L}^2), \mathcal{D}'\mathcal{S}\mathcal{D}u \in \mathcal{C}([0,\infty), \mathcal{L}^2)$.[1]

We will assume, that for all $x \in G$, $t \geq 0$: *rot* $u(x,t) = 0$, sometimes we will assume explicit radially symmetric initial data and therefore explicit radially symmetric solutions (u, θ).
The boundary condition $u\mid_{\partial G}= 0$, $\theta\mid_{\partial G}= 0$ was considered by Jiang, Muñoz Rivera and Racke.

[1] For a definition of $\mathcal{S}$ and $\mathcal{D}$ see the next section.

They proved exponential decay in the case of rotation–free solutions [4]. In the next two sections we try to find similiar results for the Neumann boundary condition in u resp. the Robin boundary condition in θ. The case $u \mid_{\partial G}= 0$, $\frac{\partial\theta}{\partial\vec{n}} \mid_{\partial G}= 0$ is omitted because it is the easiest case, for a proof see [11].

2.1 The Neumann boundary conditions for u

To investigate the Neumann boundary conditions for u we assume explicitly radial symmetry, so we only consider discs, balls and annular discs and spheres as domain G with radially symmetric initial values. The resulting solutions $u(x,t)$ and $\theta(x,t)$ can be written as:

$$u(x,t) = w(|x|,t)x, \quad \theta(x,t) = \Theta(|x|,t) \tag{2.3}$$

We notice:

Lemma 2.2 *Under these assumptions $\theta(\cdot,t)$ is locally constant on ∂G, i.e. it is constant on all components of ∂G.*

We now define some auxiliary operators to formulate the Neumann boundary condition in the cases $n = 2$ and $n = 3$.

Definition 2.3 *Let τ, λ, μ be the Lamé moduli then define for $\vec{n} = (n_1, \ldots, n_n)$:*

$$\underline{n=2}: \quad \mathcal{D} = \begin{pmatrix} \partial_1 & 0 \\ 0 & \partial_2 \\ \partial_2 & \partial_1 \end{pmatrix}, \quad \mathcal{N} = \begin{pmatrix} n_1 & 0 \\ 0 & n_2 \\ n_2 & n_1 \end{pmatrix}, \quad \mathcal{S} = \begin{pmatrix} \tau & \lambda & 0 \\ \lambda & \tau & 0 \\ 0 & 0 & \mu \end{pmatrix}$$

$$\underline{n=3}: \quad \mathcal{D} = \begin{pmatrix} \partial_1 & 0 & 0 \\ 0 & \partial_2 & 0 \\ 0 & 0 & \partial_3 \\ 0 & \partial_3 & \partial_2 \\ \partial_3 & 0 & \partial_1 \\ \partial_2 & \partial_1 & 0 \end{pmatrix}, \quad \mathcal{N} = \begin{pmatrix} n_1 & 0 & 0 \\ 0 & n_2 & 0 \\ 0 & 0 & n_3 \\ 0 & n_3 & n_2 \\ n_3 & 0 & n_1 \\ n_2 & n_1 & 0 \end{pmatrix}, \quad \mathcal{S} = \begin{pmatrix} \tau & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \tau & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \tau & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{pmatrix}$$

As a convention we denote in this context the transposed matrix $\mathcal{D}^T$ with $\mathcal{D}'$. The equations of thermoelasticity take the form:

$$u_{tt} - \mathcal{D}'\mathcal{S}\mathcal{D}u + \mathcal{D}'\vec{\beta}\theta = 0, \tag{2.4}$$

$$\theta_t + c\Delta\theta - \vec{\beta}'\mathcal{D}u_t = 0, \tag{2.5}$$

where we have defined for $n = 2$: $\vec{\beta} := (\beta, \beta, 0)^T$ and for $n = 3$: $\vec{\beta} := (\beta, \beta, \beta, 0, 0, 0)^T$. We are now able to formulate the boundary conditions we want to study:

$$\mathcal{N}'\mathcal{S}\mathcal{D}u \mid_{\partial G}= 0, \quad \theta \mid_{\partial G}= 0 \tag{2.6}$$

$$\mathcal{N}'\mathcal{S}\mathcal{D}u - \mathcal{N}'\vec{\beta}\theta \mid_{\partial G}= 0, \quad \frac{\partial\theta}{\partial\vec{n}} \mid_{\partial G}= 0 \tag{2.7}$$

Before we start our energy estimates we need some other lemmata concerning the dependencies of the different differential operators.

Lemma 2.4 *For $v \in H^1$ we have:* $|\mathcal{D}v(x)|^2 \geq \frac{1}{2}|div\, v(x)|^2$

Lemma 2.5 *For $v \in H^1$ and $rot\, v = 0$ we have:* $|\mathcal{D}v(x)|^2 \geq |\nabla v(x)|^2 \geq \frac{1}{2}|\mathcal{D}v(x)|^2$

Lemma 2.6 *For $v \in H^1$ radially symmetric we have:* $||div\, v||^2 \geq ||\nabla v||^2 \geq \frac{1}{2}||div\, v||^2$

Proof: You can easily check the lemmata 2.4 and 2.5 using the Young inequality. The same is true for the first inequality in 2.6. For the proof of the second inequality it is necessary to use explicit radial symmetry to show that the boundary terms appearing by the partial integration fit together, that means you get:

$$\int_{\partial G} (\nabla v\vec{n} - \text{div}\, v\vec{n})v \quad = \quad -\int_{\partial G} (n-1)w(|x|)^2|x|$$

Now we are ready to state the main result of this chapter and to prove it:

Theorem 2.7 (Exponential decay) *Let (u, θ) be a solution of (2.4)-(2.5) with respect to the boundary condition (2.6) or (2.7) with radially symmetric initial values (u^0, u^1, θ^0), and let*

$$E(t) := e^{\gamma t}\left\{\sum_{k=0}^{2} ||\partial_t^k u(t)||^2_{H^{2-k}} + ||\theta_t(t)||^2 + ||\theta(t)||^2_{H^2}\right\}. \tag{2.8}$$

Then there are constants $\Gamma \geq 1$ and $\gamma > 0$ with

$$E(t) + \int_0^t e^{\gamma s}||\nabla\theta_t(s)||^2 \leq \Gamma E(0),\ t \geq 0. \tag{2.9}$$

Proof: To apply the energy method we define the "energies":

$$\begin{aligned}
F_1(t) &:= \frac{1}{2}\left(||u_t||^2 + \langle\mathcal{D}u, \mathcal{S}\mathcal{D}u\rangle + c||\theta||^2\right)(t)\\
F_2(t) &:= \frac{1}{2}\left(||u_{tt}||^2 + \langle\mathcal{D}u_t, \mathcal{S}\mathcal{D}u_t\rangle + c||\theta_t||^2\right)(t)\\
F_3(t) &:= \frac{1}{2}\left(||\Delta u||^2 + \langle\mathcal{D}u_t, \mathcal{S}\mathcal{D}u_t\rangle + c||\nabla\theta||^2\right)(t)
\end{aligned}$$

For F_1 and F_2 we derive by multiplicating the differential equations with suitable terms and integrating by parts:

$$\frac{d}{dt}F_1(t) = -\kappa||\nabla\theta||^2, \quad \frac{d}{dt}F_2(t) = -\kappa||\nabla\theta_t||^2 \tag{2.10}$$

For F_3 we get:

$$\frac{d}{dt}F_3(t) = -\kappa||\Delta\theta||^2 + \beta\int_{\partial G}\frac{\partial\theta}{\partial\vec{n}}\text{div}\, u_t - \frac{1}{\tau}\int_{\partial G} u_{tt}\mathcal{N}'\vec{\beta}\theta_t \tag{2.11}$$

As an important tool we need the Poincaré inequality and the Korn inequality for u in the same form as in the Dirichlet case. We have two possible attempts stated in the two following lemmata, where we have $\mathcal{D}_0 := \{v \in (H^1(G))^n | \mathcal{D}v = 0\}$:

Lemma 2.8 (Korn inequality for u) *Let u_0 and $u_1 \in \mathcal{D}_0^\perp$, then:*
$u(t) \in \mathcal{D}_0^\perp$ for all $t \geq 0$, and there exists $C > 0$ with: $||u|| \leq C||\mathcal{D}u||$

Remark 2.9 *This is the "classical" attempt resulting from the nullspace of the differential operator. With lemma 2.5 we get the intended type of the Poincaré inequality for u (see lemma 2.10).*

Proof of the lemma: It follows from the (normal) Korn inequality (see e.g. [9]):

$$||u||^2 \leq C\left(||\mathcal{D}u||^2 + \sup_{v\in\mathcal{D}_0, ||v||=1} |\langle u, v\rangle|\right)$$

On the other hand we have $u \in \mathcal{D}_0^\perp$, because for $v \in \mathcal{D}_0$ we have:

$$\langle u_{tt}, v\rangle = \langle \mathcal{D}'\mathcal{S}\mathcal{D}u - \mathcal{D}'\vec{\beta}\theta, v\rangle = -\langle \mathcal{S}\mathcal{D}u - \vec{\beta}\theta, \underbrace{\mathcal{D}v}_{=0}\rangle + \int_{\partial G} \underbrace{\mathcal{N}'\mathcal{S}\mathcal{D}u - \mathcal{N}'\vec{\beta}\theta}_{=0} = 0$$

□

Lemma 2.10 (Poincaré inequality for u) *Let $\int_G u_0 = 0$ and $\int_G u_1 = 0$ (which we can assume after appropriate normalization), then there exists a $C \geq 0$, such that:*

$$||u|| \leq C||\nabla u||$$

Proof: We start with the (normal) Poincaré inequality:

$$||u|| \leq C||\nabla u|| + \int_G u$$

Under the given assumptions we have $\int_G u = 0$ (using the differential equation, the divergence theorem and the boundary condition).

With the help of lemma 2.5 we again arrive at the Korn inequality in the form of lemma 2.8. □

Of course this leads to the question whether or not the assumptions made for the initial data are satisfied by the physics. Furthermore it is important to find any *radially symmetric* functions satisfying the assumptions to avoid an "empty" result. In the first case we can show that for *simply connected* domains (i.e. balls and discs) *all* radially symmetric functions are automatically in $\mathcal{D}_0^\perp$ (for the simple proof consider [11]). In the second case we even have a stronger result: For radially symmetric functions we obviously have $u(-x) = -u(x)$, so it follows (independently of the topology of G) that $\int_G u = 0$. There is also a physical interpretation for this: We choose our coordinates such that the centre of gravity S defined by $\int_G (u(x) - S)dx = 0$ lies in $x = 0$ for $t = 0$, and that the velocity of S for $t = 0$ is zero. The proof of the lemma shows that the velocity is constant and therefore zero. Of course this is an immediate consequence of Newton's laws that have been used in the physical derivation of the thermoelasticity equations.

Now we have to consider the boundary conditions for θ separately:

(a) The boundary condition (2.6) with $\theta \mid_{\partial G} = 0$:

Here we get using (2.11):

$$\frac{d}{dt}F_3(t) = -\kappa||\Delta\theta||^2 + \beta\int_{\partial G}\frac{\partial\theta}{\partial\vec{n}}\text{div }u_t \tag{2.12}$$

We decompose the boundary integral with the Young inequality to:

$$\beta\int_{\partial G}\frac{\partial\theta}{\partial\vec{n}}\text{div }u_t \leq \underbrace{\beta\frac{C}{\varepsilon}\int_{\partial G}\left|\frac{\partial\theta}{\partial\vec{n}}\right|^2}_{:=I_1} + \underbrace{\beta\varepsilon\int_{\partial G}|\text{div }u_t|^2}_{:=I_2} \tag{2.13}$$

(b) The boundary condition (2.7) with $\frac{\partial\theta}{\partial\vec{n}} \mid_{\partial G} = 0$:

Here we get using (2.11):

$$\frac{d}{dt}F_3(t) = -\kappa||\Delta\theta||^2 + \frac{1}{\tau}\int_{\partial G} u_{tt}\mathcal{N}'\vec{\beta}\theta_t$$

Using the Young inequality again leads to:

$$\frac{1}{\tau}\int_{\partial G} u_{tt}\mathcal{N}'\vec{\beta}\theta_t \leq \underbrace{\frac{1}{\tau}\varepsilon\int_{\partial G}|u_{tt}|^2}_{:=I_3} + \underbrace{\frac{C}{\varepsilon}\int_{\partial G}|\mathcal{N}'\vec{\beta}\theta_t|^2}_{:=I_4}$$

Using a Sobolev trace theorem (see e.g. [7]) we get for I_4:

$$\frac{C}{\varepsilon}\int_{\partial G}|\mathcal{N}'\vec{\beta}\theta_t|^2 \leq \frac{C}{\varepsilon^2}||\theta_t||^2 + C||\nabla\theta_t||^2$$

We mention that in the case of the boundary condition (2.7) we have *no* appropriate form of the Poincaré inequality for θ. To estimate I_1 and I_2 resp. I_3 we need two theorems. We will also apply these theorems in the next chapter for the nonlinear case, for this purpose it makes sense to extend the results needed in this chapter slightly and not to use the boundary condition.

Theorem 2.11 *Assume that θ satisfies the differential equation:*

$$c\theta_t + \kappa\Delta\theta = h_2 \tag{2.14}$$

Furthermore let θ be locally constant on ∂G and $\sigma \in (\mathcal{C}^1(\bar{G}))^n$ with $\sigma = \vec{n}$ on ∂G. (For the existence of such a σ see appendix A.)

Then we have:

$$\kappa\int_{\partial G}\left|\frac{\partial\theta}{\partial\vec{n}}\right|^2 = 2c\int_G \theta_t\sigma\nabla\theta + 2\kappa\int_G \nabla\theta\nabla\sigma_k\partial_k\theta - \kappa\int_G div\,\sigma|\nabla\theta|^2 - \int_G h_2\sigma\nabla\theta \tag{2.15}$$

Proof: We start with (2.14), multiply with $\sigma_k\partial_k v$ and integrate over G. θ is locally constant on ∂G, so we have: $\frac{\partial\theta}{\partial\vec{n}} = \nabla\theta$. If we apply this, an elementary calculation and partial integration leads to the theorem's statement. □

Theorem 2.12 *Let $v(x) = (v_1, \dots, v_n)(x) = w(|x|)x$ be a radially symmetric solution of:*

$$v_{tt} - \tau\Delta u = h_1 \tag{2.16}$$

Let $\sigma \in (\mathcal{C}^1(\bar{G}))^n$ with $\sigma = \vec{n}$ on ∂G.

Then we have:

$$\begin{aligned}\tau \int_{\partial G} |div\, v|^2 + \int_{\partial G} |v_t|^2 - \tau \int_{\partial G} (n-1)w(|x|)\, div\, v \\ = 2\frac{d}{dt}\int_G v_t\sigma_k\partial_k v + \int_G div\,\sigma|v_t|^2 - \tau\int_G div\,\sigma|div\, v|^2 \\ +\tau\int_G div\, v\nabla\sigma_k\partial_k v - 2\int_G h_1\sigma_k\partial_k v\end{aligned} \tag{2.17}$$

Proof: We start with equation (2.1), multiply with $\sigma_k\partial_k v$ and integrate over G:

$$\int_G v_{tt}\sigma_k\partial_k v - \tau\int_G \nabla \mathrm{div}\, v\sigma_k\partial_k v \tag{2.18}$$

Using integration by parts we get:

$$\begin{aligned}\frac{d}{dt}\int_G v_t\sigma_k\partial_k v + \frac{1}{2}\int_G |v_t|^2\partial_k\sigma_k - \frac{1}{2}\int_{\partial G}|v_t|^2 \\ +\tau\int_G \mathrm{div}\, v\nabla\sigma_k\partial_k v - \frac{1}{2}\tau\int_G \mathrm{div}\,\sigma|\mathrm{div}\, v|^2 \\ +\frac{1}{2}\tau\int_{\partial G}|\mathrm{div}\, v|^2 - \tau\int_{\partial G}\partial_i v_i\sigma_j\sigma_k\partial_k v_j \quad = \quad \int_G h_1\sigma_k\partial_k v\end{aligned}$$

Now we explicitly use the radial symmetry of v to "sum up" the boundary integrals. After an elementary calculation where we use that $\sigma\,|_{\partial G} = \pm\vec{n}$, we arrive at (2.17). □

We can apply this theorem for thermoelasticity. It is useful to extend the equations slightly for reasons we will see later, so we reach the following corollary:

Corollary 2.13 *Let u and θ be a smooth radially symmetric vector field respective a smooth radially symmetric function. For a smooth vector field f we assume the differential equation:*

$$u_{tt} - \tau\Delta u + \beta\nabla\theta = f.$$

Furthermore we assume the Poincaré inequality:

$$||u||^2 \leq C||\nabla u||^2$$

Then we have:

$$\int_{\partial G}|div\, u|^2 + \int_{\partial G}|u_t|^2 \quad \leq \quad C\frac{d}{dt}\int_G u_t\sigma_k\partial_k u + C||(u_t, \nabla u, \nabla\theta)||^2 + C\int_G f\sigma_k\partial_k u$$

Proof: The proof is an immediate consequence of the theorem (2.12) that we get using the Young inequality, the Sobolev trace theorem and the assumed Poincaré inequality. □

Remark 2.14 *We get other useful inequalities of the same type by differentiating with respect to t. For example we get:*

$$\int_{\partial G} |div\, u_t|^2 + \int_{\partial G} |u_{tt}|^2 \ \leq\ C\frac{d}{dt}\int_G u_{tt}\sigma_k\partial_k u_t + C||(u_{tt},\nabla u_t,\nabla\theta_t)||^2 + C\int_G f_t\sigma_k\partial_k u_t$$

Now we want to use corollary 2.13 and theorem 2.11 to estimate the terms I_1 and I_2.

For the boundary condition (2.6) we get:

$$\frac{d}{dt}F_3(t) \leq -\kappa||\Delta\theta||^2 + K\frac{d}{dt}\int_G u_{tt}\sigma_k\partial_k u_t + C\varepsilon||(u_{tt},\nabla u_t,\nabla\theta_t)||^2 + \frac{C}{\varepsilon^3}||(\nabla\theta,\nabla\theta_t)||^2 \qquad (2.19)$$

For the other boundary condition (2.7) we get estimating I_3 by using corolary 2.13:

$$\begin{aligned}\frac{d}{dt}F_3(t) \ &\leq\ -\kappa||\Delta\theta||^2 + K\frac{d}{dt}\int_G u_{tt}\sigma_k\partial_k u_t + C\varepsilon^2||\theta_t||^2 + C\varepsilon||(u_{tt},\nabla u_t)||^2 \\ &\quad +C||\nabla\theta_t||^2 + \frac{C}{\varepsilon^3}||(\nabla\theta,\nabla\theta_t)||^2 \end{aligned} \qquad (2.20)$$

We are now able to prove three auxiliary estimates we will need below:

$$\frac{d}{dt}\int_G u_t u \ \leq\ -\frac{1}{2}\tau s_1||\nabla u||^2 + C||(\theta,u_t)||^2, \qquad (2.21)$$

$$\begin{aligned}\frac{d}{dt}\int_G \text{div}\, u_t \text{div}\, u \ &\leq\ -\frac{\tau}{2}||\Delta u||^2 + C||\nabla\theta||^2 \\ &\quad +K_1\alpha\left\{\frac{d}{dt}\int_G u_{tt}\sigma_k\partial_k u_t + ||(u_{tt},\nabla u_t,\nabla\theta_t)||^2\right\} \\ &\quad +\frac{K_2}{\alpha}\left\{\frac{d}{dt}\int_G u_t\sigma_k\partial_k u + ||(u_t,\nabla u,\nabla\theta)||^2\right\} + ||\nabla u_t||^2 \end{aligned} \qquad (2.22)$$

and

$$-\frac{\tau}{2}||\Delta u||^2 \leq -\frac{\tau}{4}||\Delta u||^2 - \frac{1}{8\tau}||u_{tt}||^2 + \frac{\beta^2}{4\tau}||\nabla\theta||^2 \qquad (2.23)$$

To prove these statements we have to use the differential equation in the form (2.4) resp. (2.1), the lemmata 2.5 and 2.6 and the corollary 2.13.

We now define the auxiliary energy

$$\begin{aligned} H(t) \ &:=\ \eta F_1 + \eta F_2 + F_3 - (K+K_1\alpha)\int_G u_{tt}\sigma_k\partial_k u_t - \frac{K_2}{\alpha}\int_G u_t\sigma_k\partial_k u \\ &\quad +\varepsilon^{1/4}\int_G u_t u + \varepsilon^{1/2}\int_G \text{div}\, u \text{div}\, u_t. \end{aligned}$$

For the Neumann–Neumann boundary condition we have the following remark:

Remark 2.15 *There is no Poincaré inequality of the type $||\theta||^2 \leq C||\nabla\theta||^2$ for θ if we have boundary condition (2.7). The ansatz to assume $\int_G \theta_0 = 0$ and to calculate $d/dt\int_G\theta$ is failing: Although if we got $\int_G\theta_0 = 0$ with Dirichlet boundary condition for u, we now (i.e. in the case of the Neumann boundary condition for u) get:*

$$\int_G \theta(t) = -\frac{\beta}{c}\int_{\partial G} u(t)\vec{n}$$

This equation leads to an interesting physical interpretation: The (normalized) temperature in the body depends linearly on the volume of the body. If we expand the body its temperature will decrease, if we shrink it its temperatre will increase — a physically sensible result.

Instead of a Poincaré inequality we can derive the following estimates:

Lemma 2.16 *If* (u, θ) *satisfies (2.7), then there exists a* $C > 0$ *with:*

$$\begin{aligned} ||\theta||^2 &\leq C||(\nabla\theta, \nabla u)||^2 && (2.24) \\ ||\theta_t||^2 &\leq C||(\nabla\theta, \nabla\theta_t)||^2 + \varepsilon K \frac{d}{dt}\int_G u_t \sigma_k \partial_k u + C\varepsilon\left\{\frac{d}{dt}\langle u_t, t\rangle + ||\nabla u||^2\right\} && (2.25) \end{aligned}$$

Proof: First we prove (2.24):

We take the Poincaré inequality of the type:

$$||\theta||^2 \leq C||\nabla\theta||^2 + C\left|\int_G \theta - \widetilde{\theta}\right|^2,$$

where $\widetilde{\theta} := \frac{1}{|G|}\int_G \theta$.

Now we use the differential equation (2.2), integrate by parts, and estimate the resulting boundary integral with the Sobolev trace theorem. So we arrive at:

$$||\theta||^2 \leq C||\nabla\theta||^2 + C||\nabla u||^2 dx$$

For the second statement we use (2.4), multiply it with u and integrate by parts. So we get:

$$||u_t||^2 - \frac{d}{dt}\langle u_t, u\rangle \leq C\left(||\nabla u||^2 + ||\theta||^2\right) \tag{2.26}$$

On the other hand if we multiply (2.2) with θ_t and again integrate by parts we have:

$$\begin{aligned} ||\theta_t||^2 &= \langle\Delta\theta, \theta_t\rangle + \langle \operatorname{div} u_t, \theta_t\rangle \\ &= -\langle\nabla\theta, \nabla\theta_t\rangle - \langle u_t, \nabla\theta_t\rangle + \int_{\partial G} u_t \vec{n}\theta_t \end{aligned}$$

Using the Young inequality we get:

$$||\theta_t||^2 \leq \frac{C}{\varepsilon}||(\nabla\theta, \nabla\theta_t)||^2 + C\varepsilon||u_t||^2 + \varepsilon\int_{\partial G}|u_t\vec{n}|^2 + \frac{C}{\varepsilon}\int_{\partial G}|\theta_t|^2$$

Now we can use corollary 2.13 and a special kind of Sobolev trace theorem stated e.g. in [7] to get for $\delta > 0$:

$$\begin{aligned} ||\theta_t||^2 \leq\ & C(\varepsilon)||(\nabla\theta, \nabla\theta_t)||^2 + \varepsilon K\frac{d}{dt}\int_G u_t\sigma_k\partial_k u + C\varepsilon||(u_t, \nabla u, \nabla\theta)||^2 \\ & + \frac{C}{\varepsilon}\delta||\theta_t||^2 + \frac{C}{\varepsilon\delta}||\nabla\theta_t||^2 \end{aligned}$$

For suitable small $\delta > 0$ we can bring $\frac{C}{\varepsilon}\delta||\theta_t||^2$ to the left side of the inequality. Applying (2.26) and (2.24) we arrive at:

$$||\theta_t||^2 \leq C(\varepsilon)||(\nabla\theta, \nabla\theta_t)||^2 + \varepsilon K\frac{d}{dt}\int_G u_t\sigma_k\partial_k u + C\varepsilon\left\{||\nabla u||^2 + \frac{d}{dt}\langle u_t, u\rangle\right\}$$

That is the second statement.

The third auxiliary inequality follows immediately by using the differential equations. □

Now our goal is to show the exponential decay of the auxiliary energy $H(t)$ by using the Gronwall inequality. For a suitably big η we estimate $\frac{d}{dt}H(t)$ for the first of our two boundary conditions, i.e. the condition (2.6):

$$\begin{aligned}
\frac{d}{dt}H(t) &\leq -C\eta||(\nabla\theta,\nabla\theta_t)||^2 - C||\Delta\theta||^2 + \frac{d}{dt}\left\{\varepsilon^{1/4}\int_G u_t u + \varepsilon^{1/2}\int_G \operatorname{div} u \operatorname{div} u_t\right\} \\
&\quad + C\varepsilon||(u_{tt},\nabla u_t,\nabla\theta_t)||^2 - K_1\alpha\left\{\frac{d}{dt}\int_G u_{tt}\sigma_k\partial_k u_t\right\} \\
&\quad - \frac{K_2}{\alpha}\left\{\frac{d}{dt}\int_G u_t\sigma_k\partial_k u\right\} \qquad \text{using (2.10) and (2.19)} \\
&\leq -C\eta||(\nabla\theta,\theta,\nabla\theta_t,\theta_t)||^2 - C||(\Delta\theta,\nabla u_t,u_t||^2 \\
&\quad + \frac{d}{dt}\left\{\varepsilon^{1/4}\int_G u_t u + \varepsilon^{1/2}\int_G \text{div udiv } u_t\right\} + C\varepsilon||u_{tt}||^2 \\
&\quad - K_1\alpha\left\{2\frac{d}{dt}\int_G u_{tt}\sigma_k\partial_k u_t\right\} \\
&\quad - \frac{K_2}{\alpha}\left\{2\frac{d}{dt}\int_G u_t\sigma_k\partial_k u\right\} \text{ using Poincaré for } \theta\text{, (2.5) and Poincaré for } u_t \\
&\leq -C_1(\eta)||(\nabla\theta,\theta,\nabla\theta_t,\theta_t,\Delta\theta,\nabla u_t,u_t)||^2 + \varepsilon^{1/4}\left\{-\frac{\tau s_1}{2}||\nabla u||^2 + C||(\theta,u_t)||^2\right\} \\
&\quad + \varepsilon^{1/2}\left\{-\frac{\tau}{2}||\Delta u||^2 + \frac{C}{\alpha}||(u_t,\nabla u,\nabla\theta)||^2 + C\alpha||(u_{tt},\nabla u_t,\nabla\theta_t)||^2 + ||\nabla u_t||^2\right\} \\
&\quad + C\varepsilon||u_{tt}||^2 \qquad \text{using (2.21) and (2.22)} \\
&\leq -C_1(\eta)||(\nabla\theta,\theta,\nabla\theta_t,\theta_t,\Delta\theta,\nabla u_t,u_t)||^2 + \varepsilon^{1/4}\left\{-\frac{\tau s_1}{2}||\nabla u||^2 + C||(\theta,u_t)||^2\right\} \\
&\quad + \varepsilon^{1/2}\left\{-\frac{\tau}{4}||\Delta u||^2 - \frac{1}{8\tau}||u_{tt}||^2 + \frac{\beta^2}{4\tau}||\nabla\theta||^2 + \frac{C}{\alpha}||(u_t,\nabla u,\nabla\theta)||^2\right. \\
&\quad \left. + C\alpha||(u_{tt},\nabla u_t,\nabla\theta_t)||^2 + ||\nabla u_t||^2\right\} + C\varepsilon||u_{tt}||^2 \qquad \text{using (2.23)}
\end{aligned}$$

Now we define: $\alpha := \varepsilon^{1/8}$, and we choose ε small enough. Then using the Poincaré inequality we get after some calculations:

$$\begin{aligned}
\frac{d}{dt}H(t) &\leq -C_1||(\nabla\theta,\theta,\nabla\theta_t,\theta_t,\Delta\theta,\nabla u_t,u_t)||^2 - C\varepsilon^{1/4}||\nabla u||^2 - C\varepsilon^{1/4}||u||^2 - C\varepsilon^{1/2}||\Delta u||^2 \\
&\quad - C\varepsilon||u_{tt}||^2
\end{aligned}$$

We now try to immitate the same steps for the boundary condition (2.7). In this case we use the lemma 2.16 instead of the Poincarè inequality for θ or θ_t. We get:

$$\begin{aligned}
\frac{d}{dt}H(t) &\leq -C\eta||(\nabla\theta,\nabla\theta_t)||^2 - C||\Delta\theta||^2 + \frac{d}{dt}\left\{\varepsilon^{1/4}\int_G u_t u + \varepsilon^{1/2}\int_G \text{div udiv } u_t\right\} \\
&\quad + C\varepsilon||(u_{tt},\nabla u_t)||^2 + C\varepsilon^2||\theta_t||^2 + \frac{C}{\varepsilon^4}||\nabla\theta_t||^2 - K_1\alpha\left\{\frac{d}{dt}\int_G u_{tt}\sigma_k\partial_k u_t\right\} \\
&\quad - \frac{K_2}{\alpha}\left\{\frac{d}{dt}\int_G u_t\sigma_k\partial_k u\right\} \qquad \text{using (2.10) and (2.20)} \\
&\leq -C\eta||(\nabla\theta,\nabla\theta_t,\theta_t)||^2 - C||(\Delta\theta,\nabla u_t,u_t)||^2 \\
&\quad + \frac{d}{dt}\left\{\varepsilon^{1/4}\int_G u_t u + \varepsilon^{1/2}\int_G \text{div udiv } u_t\right\} + C\varepsilon||u_{tt}||^2 \\
&\quad - K_1\alpha\frac{d}{dt}\int_G u_{tt}\sigma_k\partial_k u_t
\end{aligned}$$

$$
\begin{aligned}
&+\left(\varepsilon K-\frac{K_2}{\alpha}\right)\frac{d}{dt}\int_G u_t\sigma_k\partial_k u \quad \text{using (2.25), (2.5) and Poincaré for } u_t \\
\leq\ & -C_1(\eta)||(\nabla\theta,\nabla\theta_t,\theta_t,\Delta\theta,\nabla u_t,u_t)||^2+\varepsilon^{1/4}\left\{-\frac{\tau s_1}{2}||\nabla u||^2+C||(\theta,u_t)||^2\right\} \\
&+\varepsilon^{1/2}\left\{-\frac{\tau}{2}||\Delta u||^2+\frac{C}{\alpha}||(u_t,\nabla u,\nabla\theta)||^2+C\alpha||(u_{tt},\nabla u_t,\nabla\theta_t)||^2+||\nabla u_t||^2\right\} \\
&+C\varepsilon||u_{tt}||^2 \qquad\qquad \text{using (2.21) and (2.22)} \\
\leq\ & -C_1(\eta)||(\nabla\theta,\nabla\theta_t,\theta_t,\Delta\theta,\nabla u_t,u_t)||^2+\varepsilon^{1/4}\left\{-\frac{\tau s_1}{2}||\nabla u||^2+C||(\theta,u_t)||^2\right\} \\
&+\varepsilon^{1/2}\left\{-\frac{\tau}{4}||\Delta u||^2-\frac{1}{8\tau}||u_{tt}||^2+\frac{\beta^2}{4\tau}||\nabla\theta||^2+\frac{C}{\alpha}||(u_t,\nabla u,\nabla\theta)||^2\right. \\
&\left.+C\alpha||(u_{tt},\nabla u_t,\nabla\theta_t)||^2+||\nabla u_t||^2\right\}+C\varepsilon||u_{tt}||^2 \qquad\qquad \text{using (2.23)}
\end{aligned}
$$

With $\alpha := \varepsilon^{1/8}$ and a small $\varepsilon > 0$ we get using the Poincaré inequality and (2.24):

$$
\begin{aligned}
\frac{d}{dt}H(t) \leq\ & -C_1||(\nabla\theta,\nabla\theta_t,\theta_t,\Delta\theta,\nabla u_t,u_t)||^2-C\varepsilon^{1/4}||\nabla u||^2-C\varepsilon^{1/4}||\theta||^2-C\varepsilon^{1/4}||u||^2 \\
& -C\varepsilon^{1/2}||\Delta u||^2-C\varepsilon||u_{tt}||^2
\end{aligned}
$$

So besides different constants we have the same result for both boundary conditions.
Using the elliptic regularity property $||u||^2_{H^2} \leq C||\Delta u||^2_{\mathcal{L}^2}$ (see appendix B), we have for big η:

There are constants C_1, $C_2 > 0$, such that:

$$
C_1E(t) \leq H(t)e^{\gamma t} \leq C_2E(t) \tag{2.27}
$$

This is an immediate consequence of the definitions of $E(t)$ and $H(t)$. Hence we have:

$$
\frac{d}{dt}H(t) \leq -CE(t)-C||\nabla\theta_t||^2 \leq -CH(t)-C||\nabla\theta_t||^2
$$

Now we can use the Gronwall inequality and arrive at:

$$
H(t) \leq e^{-\gamma t}\left\{H(0)-\int_0^t ||\nabla\theta_t||^2e^{Cr}dr\right\}
$$

Applying (2.27) we get:

$$
E(t) \leq e^{-\gamma t}\left\{E(0)-\int_0^t ||\nabla\theta_t||^2e^{Cr}dr\right\} \tag{2.28}
$$

This is the statement of theorem 2.7. □

2.2 The Robin boundary condition for θ

In this section we want to consider the Robin boundary condition (sometimes called "third kind boundary condition") for θ combined with the Dirichlet boundary condition for u. (It should be possible to modify these arguments slightly for the Neumann boundary condition in u.) The physical interpretation to this problem is a thermoelastic body fixed on its boundary with heat flow through its boundary. We normalize the constant temperature of the environment to zero. This leads to

$$a(x)\frac{\partial\theta}{\partial\vec{n}} = -b(x)\theta \text{ on } \partial G, \tag{2.29}$$

where a and b describe the heat flow in $x \in \partial G$.
We assume:

$$\begin{aligned} \partial G &= \Gamma_D \cup \Gamma_N \cup \Gamma_R, \\ a(x) = 0,\ b(x) &= 1 \text{ for } x \in \Gamma_D, \\ a(x) = 1,\ b(x) &= 0 \text{ for } x \in \Gamma_N, \\ \frac{b(x)}{a(x)} > 0, \quad \left|\frac{b(x)}{a(x)}\right| &\text{ bounded for } x \in \Gamma_R \end{aligned} \tag{2.30}$$

Furthermore we assume in this section that (u, θ) is a radially symmetric solution. Our goal is to prove the following theorem:

Theorem 2.17 (Exponential decay for Robin boundary condition) *If Γ_D or Γ_R have positive $(n-1)$–measure, we have under the assumptions (2.30) for a radially symmetric solution (u, θ) of the equations (2.1), (2.2) together with appropriate initial conditions and the boundary condition (2.29):*
There exist constants $\Gamma \geq 1$ and $\gamma > 0$ with

$$E(t) + \int_0^t e^{\gamma s}||\nabla\theta_t(s)||^2\, ds \leq \Gamma E(0), \tag{2.31}$$

where

$$E(t) := e^{\gamma t}\left\{\sum_{k=0}^{2} ||\partial_t^k u(t)||^2_{H^{2-k}} + ||\theta_t(t)||^2 + ||\theta(t)||^2_{H^2}\right\}. \tag{2.32}$$

To prove this we need again a certain form of Poincaré inequality for θ. We therefore quote the following lemma (see [12]):

Lemma 2.18 (Poincaré inequality for θ) *Let $\partial G = \Gamma_D \cup \Gamma_N \cup \Gamma_R$; let Γ_D or Γ_R have positive $(n-1)$–measure. Furthermore let $\theta \in H^2(G)$, and θ satisfy (2.29) and the assumptions (2.30). Then there is a $C > 0$ with: $||\theta||^2 \leq C||\nabla\theta||^2$*

Remark 2.19 *So we have up to three different types of boundary conditions on ∂G: On Γ_D we have the Dirichlet, on Γ_N the Neumann and on Γ_R the Robin boundary condition.*

Proof of the theorem:

To prove the exponential decay we use a modification of the proofs for the other boundary conditions. The only thing that has to be modified are the estimates of the boundary integrals. We estimate the integral $\int_{\partial G} \frac{\partial\theta}{\partial\vec{n}}\theta$ as follows:

$$\begin{aligned}\int_{\partial G} \frac{\partial\theta}{\partial\vec{n}}\theta &= \int_{\Gamma_D} \frac{\partial\theta}{\partial\vec{n}} \underbrace{\theta}_{=0} + \int_{\Gamma_N} \underbrace{\frac{\partial\theta}{\partial\vec{n}}}_{=0} \theta + \int_{\Gamma_R} \frac{\partial\theta}{\partial\vec{n}}\theta \\ &= -\int_{\Gamma_R} \frac{b(x)}{a(x)}|\theta|^2\end{aligned}$$

We put this into the equation for $\frac{d}{dt}F_1(t)$ and arrive at:

$$\frac{d}{dt}F_1(t) = -\kappa||\nabla\theta||^2 - \int_{\Gamma_R} \underbrace{\frac{b(x)}{a(x)}}_{>0} |\theta|^2 \leq -\kappa||\nabla\theta||^2$$

Similiarly we can estimate the other integrals on the boundary, e.g.:

$$\begin{aligned}\int_{\partial G} \frac{\partial\theta}{\partial\vec{n}}\theta_t &= -\int_{\Gamma_R} \frac{b(x)}{a(x)}\theta_t\theta \\ &\leq C\int_{\Gamma_R} \left|\frac{b(x)}{a(x)}\right|^2 |\theta|^2 + C\int_{\Gamma_R} |\theta_t|^2 \\ &\leq C\int_{\partial G} |\theta|^2 + C\int_{\partial G} |\theta_t|^2 \\ &\leq C||\nabla\theta||^2 + C||\nabla\theta_t||^2\end{aligned}$$

(Such a term is small if η (compare with the previous section) is big enough.)

The estimate of

$$\int_{\partial G} \text{div } u_t \frac{\partial\theta}{\partial\vec{n}} \leq \varepsilon\int_{\partial G} |\text{div } u_t|^2 + \frac{C}{\varepsilon}\int_{\partial G} \left|\frac{\partial\theta}{\partial\vec{n}}\right|^2 \tag{2.33}$$

is the same as in section 2.1. — Remember that the theorems 2.11 and 2.12 did not depend on the boundary condition as long as u and θ are radially symmetric! □

Remark 2.20 *In the radially symmetric case with G not simply connected this theorem allows us to consider different boundary conditions on the interior and exterior boundary. A physical example would be a sphere isolated to the interior but with a heat flow to the exterior or perhaps a constant temperature on one of these boundaries.*

Chapter 3

The nonlinear equations

3.1 Formulation of the problem

After studying the linearized equations of thermoelasticity we want to consider the nonlinear case. Here normally we have no global existence of solutions. In this chapter however we will show global existence for radially symmetric initial values and radially symmetric boundary conditions for small initial data.

First we formulate the initial boundary value problem. We start with the nonlinear equations in the case $n = 3$. The case $n = 2$ is similiar, the condition $n < 4$ however is neccessary as we will see later.

We define (starting with a smooth Helmholtz potential ψ):

$$\begin{aligned} C_{i\alpha j\beta}(\nabla u, \theta) &:= \frac{\partial^2 \psi(\nabla u, \theta)}{\partial(\partial u_i/\partial x_\alpha)\partial(\partial u_j/\partial x_\beta)}, \\ \tilde{C}_{i\alpha} &:= \frac{\partial^2 \psi(\nabla u, \theta)}{\partial(\partial u_i/\partial x_\alpha)\partial\theta}, \\ a(\nabla u, \theta) &:= -\frac{\partial^2 \psi(\nabla u, \theta)}{\partial\theta^2}. \end{aligned}$$

This leads to the following differential equations:

$$\begin{aligned} \frac{\partial^2 u_i}{\partial t^2} &= \operatorname{div} S(\nabla u, \theta) \\ &= C_{i\alpha j\beta}(\nabla u, \theta)\frac{\partial^2 u_j}{\partial x_\alpha \partial x_\beta} + \tilde{C}_{i\alpha}(\nabla u, \theta)\frac{\partial\theta}{\partial x_\alpha}, \quad i = 1,2,3 \end{aligned} \tag{3.1}$$

$$a(\nabla u, \theta)\theta_t = \frac{1}{b(\theta)}\operatorname{div} q(\nabla u, \theta, \nabla\theta) + \tilde{C}_{i\alpha}(\nabla u, \theta)\frac{\partial^2 u_i}{\partial x_\alpha \partial t} \tag{3.2}$$

Here we have used the Einstein summation convention again.

For the functions a and b we assume: $a \geq a_0 > 0$, $b \in \mathcal{C}^\infty(\mathbb{R})$, $b(\theta) = \theta + T_0$ for $|\theta| \leq T_0/2$, $0 < b_1 \leq b(\theta) \leq b_2 < \infty$, $-\infty < \theta < \infty$, $T_0 > 0$ reference temperature.

For $\nabla u = 0,\ \theta = 0$ the medium should be isotropic, so we assume:

$$\begin{aligned} C_{i\alpha j\beta}(0,0) &= \lambda\delta_{i\alpha}\delta_{j\beta} + \mu(\delta_{ij}\delta_{\alpha\beta} + \delta_{\alpha j}\delta_{i\beta}) \\ \tilde{C}_{i\alpha}(0,0) &= -\beta\delta_{i\alpha} \\ \frac{\partial q_i(0,0,0)}{\partial(\partial\theta/\partial x_j)} &= \kappa\delta_{ij} \\ a(0,0) &= c \end{aligned} \tag{3.3}$$

Furthermore we assume:

$$\begin{aligned} C_{i\alpha j\beta}(\nabla u,\theta) &= C_{j\beta i\alpha}(\nabla u,\theta) \\ \frac{\partial q_i(\nabla u,\theta,\nabla\theta)}{\partial(\partial\theta/\partial x_j)} &= \frac{\partial q_j(\nabla u,\theta,\nabla\theta)}{\partial(\partial\theta/\partial x_i)} \\ \frac{\partial q_i(\nabla u,\theta,\nabla\theta)}{\partial\theta} &= 0 \\ \frac{\partial q_i(\nabla u,\theta,\nabla\theta)}{\partial(\partial u_\alpha/\partial x_\beta)} &= 0 \quad, 1 \le i,j,\alpha,\beta \le 3 \end{aligned} \tag{3.4}$$

The initial conditions are:

$$u(0) = u_0,\ u_t(0) = u_1,\ \theta(0) = \theta_0$$

To formulate the boundary conditions we have to start with (3.1) and (3.2).

The easiest boundary condition (Dirichlet—Dirichlet) was considered in [4]. In this chapter we want to look at one of the more delicate boundary conditions: the Dirichlet condition for θ together with the Neumann condition for u. The difficulties we have to handle are mainly based on the nonlinearity of this boundary condition: Starting with (3.1) we get for u:

$$\vec{n}S(\nabla u,\theta)\mid_{\partial G} = 0. \tag{3.5}$$

This difference to the Dirichlet case leads to some interesting technical problems.

Some ideas to handle the other possible boundary conditions are sketched in [11].

Now we want to reformulate the problem (3.1)–(3.5) to apply the methods of the last chapter. For this aim we define the nonlinearities f and g as follows:

$$f_i := (C_{i\alpha j\beta}(\nabla u,\theta) - C_{i\alpha j\beta}(0,0))\frac{\partial^2 u_j}{\partial x_\alpha x_\beta} + (\tilde{C}_{i\alpha}(\nabla u,\theta) - \tilde{C}_{i\alpha}(0,0))\frac{\partial\theta}{\partial x_\alpha} \tag{3.6}$$

$$\begin{aligned} g &:= c\left\{\frac{1}{a(\nabla u,\theta)b(\theta)}\frac{\partial q_i(\nabla u,\theta,\nabla\theta)}{\partial(\partial\theta/\partial x_j)} - \frac{1}{a(0,0)b(0)}\frac{\partial q_i(0,0,0)}{\partial(\partial\theta/\partial x_j)}\right\}\frac{\partial^2\theta}{\partial x_i\partial x_j} \\ &+c\left\{\frac{\tilde{C}_{i\alpha}(\nabla u,\theta)}{a(\nabla u,\theta)} - \frac{\tilde{C}_{i\alpha}(0,0)}{a(0,0)}\right\}\frac{\partial^2 u_u}{\partial x_\alpha\partial t} + \frac{c}{a(\nabla u,\theta)b(\theta)}\frac{\partial q_i(\nabla u,\theta,\nabla\theta)}{\partial\theta}\frac{\partial\theta}{\partial x_i} \\ &+\frac{c}{a(\nabla u,\theta)b(\theta)}\frac{\partial q_i(\nabla u,\theta,\nabla\theta)}{\partial(\partial u_\alpha/\partial x_\beta)}\frac{\partial^2 u_\alpha}{\partial x_i\partial x_\beta} \end{aligned} \tag{3.7}$$

We can verify by some calculations that with these definitions the equations (3.1) and (3.2) become:

$$u_{tt} - \mu\Delta u - (\mu+\lambda)\nabla\mathrm{div}\, u + \beta\nabla\theta = f(\nabla u,\theta,\nabla^2 u,\nabla\theta) \tag{3.8}$$

$$c\theta_t - \kappa\Delta\theta + \beta\mathrm{div}\, u_t = g(\nabla u,\theta,\nabla^2 u,\nabla^2\theta,\nabla u_t) \tag{3.9}$$

(For (3.8) we will sometimes use the form $u_{tt} - \mathcal{D}'\mathcal{S}\mathcal{D}u + \beta\nabla\theta = f$.)

For technical reasons we have to restrict the tensor $C_{i\alpha j\beta}$ to a special form. We assume:

$$C_{i\alpha j\beta}\frac{\partial^2 u_j}{\partial x_\alpha \partial x_\beta} = A_{ij}(\nabla u, \theta)\Delta u_j, \quad i = 1, \cdots, n \tag{3.10}$$

3.2 A Neumann boundary condition

We want to prove global existence of our problem for radially symmetric initial datas and boundary conditions.

Therefore we use a local existence result guaranteeing the global existence if the solution remains small in a certain norm. To show this we will study the decay of local solutions by comparing it to some linearized equations.

Before stating the local existence theorem we will collect a few auxiliary results.

First we need the Poincaré and the Korn inequality for u, which we get in the same way as in the last chapter:

Lemma 3.1 (Poincaré and Korn inequality for u) *Assume for u_0, u_1: $\langle u_0, 1\rangle = \langle u_1, 1\rangle = 0$, then we have:*

$$||u|| \leq C||\mathcal{D}u||, \quad ||u|| \leq C||\nabla u|| \tag{3.11}$$

Lemma 3.2 (Estimate for the nonlinear boundary condition) *For $m = 0, 1, 2, 3$ we have, if $\operatorname{rot} u = 0$:*

$$\left|\frac{d^m}{dt^m}\left((\mathcal{N}'\mathcal{S}\mathcal{D}u - \mathcal{N}'\vec{\beta}\theta) - \vec{n}S(\nabla u, \theta)\right)\right| \leq C\sum_{i=1}^{m}|(\partial_t^i \nabla u, \partial_t^i \theta)|^2$$

Proof: We start with $m = 0$ and expand $S(\nabla u, \theta) = (S_{ij}(\nabla u, \theta))_{ij}$ around $(0, 0)$ in a Taylor expansion. With $S(0, 0) = 0$ we get:

$$\begin{aligned} S(\nabla u, \theta) &= S_{\nabla u}(0,0)\nabla u + S_\theta(0,0)\theta + \mathcal{O}(|\nabla u|^2 + |\theta|^2) \\ &= (C_{i\alpha j\beta}(0,0)\partial_\alpha u_\beta + \tilde{C}_{ij}(0,0)\theta)_{ij} + \mathcal{O}(|\nabla u|^2 + |\theta|^2) \end{aligned}$$

Using the isotropy in $(0, 0)$ — see (3.3) — we get:

$$|(\mathcal{N}'\mathcal{S}\mathcal{D}u - \mathcal{N}'\vec{\beta}\theta) - \vec{n}S(\nabla u, \theta)| = |(n_j \underbrace{(C_{i\alpha j\beta}(0,0)\partial_\alpha u_\beta - \tilde{C}_{ij}(0,0)\theta - S_{ij}(\nabla u, \theta))_i}_{=\mathcal{O}(|\nabla u|^2 + |\theta|^2)}| \tag{3.12}$$

This leads to the stated estimate.

In the case $m = 1$ this calculations is quite similiar, so we only write down briefly the case $m = 2$

(the case $m = 3$ indeed is again analogous). — The arguments of the functions are omitted:

$$\begin{aligned}
& \left| \frac{d^2}{d^2t} \left((\mathcal{N}'\mathcal{S}\mathcal{D}u - \mathcal{N}'\vec{\beta}\theta) - \vec{n}S(\nabla u, \theta) \right) \right| \\
= & \ |\mathcal{N}'\mathcal{S}\mathcal{D}u_{tt} - \mathcal{N}'\vec{\beta}\theta_{tt} - \vec{n}\,(S_{\nabla u}\nabla u_{tt} + S_{\nabla u \nabla u}\nabla u_t \nabla u_t + S_{\nabla u \theta}\nabla u_t \theta_t + S_\theta \theta_{tt} \\
& + S_{\nabla u \theta}\nabla u_t \theta_t + S_{\theta\theta}\theta_t\theta_t)\,| \\
= & \ |n_j C_{i\alpha j\beta}(0,0)\partial_\alpha \partial_t^2 u_\beta - n_j \tilde{C}_{ij}(0,0)\theta_{tt} - n_j C_{i\alpha j\beta}(\nabla u, \theta)\partial_\alpha \partial_t^2 u_\beta - n_j \tilde{C}_{ij}(\nabla u, \theta)\theta_{tt} \\
& - \vec{n}\,(S_{\nabla u \nabla u}\nabla u_t \nabla u_t + S_{\nabla u \theta}\nabla u_t \theta_t + S_{\nabla u \theta}\nabla u_t \theta_t + S_{\theta\theta}\theta_t\theta_t)\,| \\
\leq & \ C(|\nabla u|^2 + |\theta|^2 + |\nabla u_t|^2 + |\theta_t|^2 + |\nabla u_{tt}|^2 + |\theta_{tt}|^2)
\end{aligned}$$

We just mention that the second order derivatives of $S(\nabla u, \theta)$ exist and are bounded because we have assumed that S is smooth, and we have small ∇u and small θ. □

We now cite the local existence theorem. It is a special application of the theorem given in [3].

Theorem 3.3 (Local existence theorem) *Let* $u_j \in H^{4-j}$, $j = 0, \ldots, 4$, $\theta_j \in H^{4-j}$, $j = 0, 1, 2$, $\theta_3 \in \mathcal{L}^2$, *where* $u_j := \partial_t^j u\,|_{t=0}$, $\theta_j := \partial_t^j \theta\,|_{t=0}$. *Furthermore there is a* $K_0 \leq \min\{1, T_0/2\}$ *with* $|\nabla u_0(x)|, |u_1(x)|, |\theta_0(x)|, |\nabla\theta_0(x)| < K_0$ *for all* $x \in \bar{G}$. *Then we have a unique solution* (u, θ) *of (3.8),(3.9),(3.5) defined on a maximal existence interval* $[0, T)$, $T \leq \infty$, *such that for all* $\hat{t} \in [0, T)$:

$$\begin{aligned}
u &\in \mathcal{C}^j([0, \hat{t}], H^{4-j}), \quad j = 0, \ldots, 4 \\
\theta &\in \mathcal{C}^j([0, \hat{t}], H^{4-j}), \quad j = 0, 1, 2 \\
\theta_{ttt} &\in \mathcal{C}^0([0, \hat{t}], \mathcal{L}^2) \cap \mathcal{L}^2([0, \hat{t}], H^1)
\end{aligned} \tag{3.13}$$

Furthermore for all $(x, t) \in \bar{G} \times [0, T)$ *we have:*

$$|\nabla u(x,t)|, |u_t(x,t)|, |\theta(x,t)|, |\nabla\theta(x,t)| < K_0 \tag{3.14}$$

Furtheron there is $T = \infty$, *i.e. the solution is a global solution, if:*

$$\sup_{t \in [0,T)} \left(\sum_{j=0}^{4} ||\partial_t^j u||^2_{H^{4-j}} + \sum_{j=0}^{2} ||\partial_t^j \theta||^2_{H^{4-j}} + ||\theta_{ttt}||^2 \right)(t) + \int_0^T ||\nabla\theta_{ttt}(s)||^2 ds < \infty \tag{3.15}$$

and

$$\sup_{x \in G, t \in [0,T)} (|\nabla u(x,t)|, |u_t(x,t)|, |\theta(x,t)|, |\nabla\theta(x,t)|) < K_0. \tag{3.16}$$

We are now able to formulate and to prove our global existence theorem:

Theorem 3.4 (Global existence theorem) *Let* (u, θ) *be a local solution of theorem 3.3 beeing also radially symmetric. Then there exists an* $\delta > 0$ *with*

$$\sum_{j=0}^{4} ||u_j||^2_{H^{4-j}} + \sum_{j=0}^{2} ||\theta_j||^2_{H^{4-j}} + ||\theta_3||^2 \leq \delta^2, \tag{3.17}$$

such that a global solution exists, which satisfies:

$$||u(t)||_{H^4}, ||\theta||_{H^4} \to 0 \quad \textit{exponentially}$$

Remark 3.5 *To get radially symmetric solutions we have to assume that for all $\Omega \in O(n), W \in \mathbb{R}^{n\times n}$ and all $x \in G$:*

$$\left.\begin{array}{r} S(\Omega^T W\Omega, \cdot) = \Omega^T S(W, \cdot)\Omega \\ g(\nabla u, \theta, \nabla^2 u, \nabla^2\theta, \nabla u_t)(\Omega x) = g(\nabla u, \theta, \nabla^2 u, \nabla^2\theta, \nabla u_t)(x) \end{array}\right\} \tag{3.18}$$

In addition all initial values must be radially symmetric. It follows the radial symmetry of the solution (u, θ). To see this let (u, θ) be a solution and $v(x) := \Omega^T u(\Omega x), \varphi(x) := \theta(\Omega x)$, then (v, φ) satisfies the differential equations (3.8) and (3.9):

$$f(\nabla u, \theta, \nabla^2 u, \nabla\theta) \;=\; div\, S(\nabla u, \theta)_{|\nabla u=0,\theta=0} - div\, S(\nabla u, \theta)$$

means that for all $\Omega \in O(n), W \in \mathbb{R}^{n\times n}$ and all $x \in G$ we have:

$$f(\nabla u, \theta, \nabla^2 u, \nabla\theta)(\Omega x) \;=\; f(\nabla u, \theta, \nabla^2 u, \nabla\theta)(x)$$

But (v, φ) satisfies the same initial values like (u, θ), because they are radially symmetric as we have assumed. Furthermore we have:

$$\begin{aligned} \vec{n}S(\nabla v, \psi)(x) &= n_r S_{sr}(\Omega_{kj}\partial_m u_k(\Omega x)\Omega_{mi}, \theta(\Omega x)) \\ &= n_r S_{sr}(((\nabla u\Omega)^T_{jm}\Omega_{mi})(\Omega x), \theta(\Omega x)) \\ &= n_r \Omega^T_{is} S_{sr}((\nabla u)(\Omega x), \theta(\Omega x))\Omega_{rj} \\ &= \Omega^T \vec{n}S(\nabla u, \theta)(\Omega x)\Omega = 0 \end{aligned}$$

So (v, φ) satisfies the same boundary conditions as (u, θ). Using the uniqueness we have: $(u, \theta) = (v, \varphi)$, but that means especially that (u, θ) is radially symmetric.
There exist indeed functions satisfying our assumptions, for an example see [11].

Now we prove the global existence theorem:

Proof of the theorem: First we define:

$$\begin{aligned} M(t) &:= e^{\gamma t}\left\{\sum_{j=0}^{4} ||\partial_t^j u||^2_{H^{4-j}} + \sum_{j=0}^{2} ||\partial_t^j \theta||^2_{H^{4-j}} + ||\theta_{ttt}||^2\right\}(t) \\ &\quad + \int_0^t e^{\gamma s}||\nabla\theta_{ttt}(s)||^2 ds \end{aligned} \tag{3.19}$$

$$\Lambda := 258[(\beta^2 + c^2 + 1)\tilde{\Gamma}]^3\Gamma(1 + \kappa^{-2})\sum_{j=0}^{2}\tau^{-2j} \tag{3.20}$$

Here Γ is the constant of the exponential decay theorem in chapter 2. $\tilde{\Gamma} > 1$ is given by the inequalities:

$$\begin{aligned} ||h||^2_{H^{j+2}} &\le \tilde{\Gamma}||\Delta h||^2_{H^j}, \\ (\text{where } h \in H_0^{j+2}(G),\ \Delta h \in \mathcal{L}^2,\ j = 0,1,2), \\ ||v||^2_{H^{j+2}} &\le \tilde{\Gamma}||\Delta v||^2_{H^j} + C\delta^3, \\ (\text{where } v \in (H^{j+1}(G))^n,\ \vec{n}S(v,0) = 0 \text{ on } \partial G,\ \langle v, 1\rangle = 0,\ \Delta v \in (H^j)^n, \\ j = 0,1,2 \text{ and } v = u \text{ or } j = 0,1 \text{ and } v = u_t.) \end{aligned} \tag{3.21}$$

(For the second one of these elliptic regularity properties compare with appendix B.)
Then there exists a $t_0 \in (0, T]$, such that $M(t) \le \Lambda\delta^2$ for all $t \in [0, t_0)$, because we have assumed $M(0) \le \delta^2$, it is $\Lambda > 1$ (because $\tilde{\Gamma} \ge 1$), and M is continous.
Now let

$$T^* := \sup\{t_1 > 0 \mid M(t) \le \Lambda\delta^2 \text{ für } t \in [0, t_1)\}, \tag{3.22}$$

then obviously we have $0 < T^* \le T$.
We consider the cases $T^* < T$ and $T^* = T$:
If $T = T^*$ the solution lies in $\mathcal{L}^\infty(\bar{G})$, because we can use the Sobolev imbedding theorem $H^4(G) \hookrightarrow C_b(G)$ having $4 > n/2$. Using lemma 3.3 we therefore have $T = \infty$, i.e. the solution is a global solution. So it is sufficent to find a contradiction to the second case.
So let $T^* < T$. For $t \in [0, T^*)$ we now using (3.22), that:

$$u(t), \theta(t) \in H^4(G),\ u_t(t), \theta_t(t) \in H^3(G),\ u_{tt}(t), \theta_{tt}(t) \in H^2(G)$$

Applying the Sobolev imbedding theorem[1] $H^2(G) \hookrightarrow \mathcal{L}^\infty(G)$ we get (for all $t \in [0, T^*)$):
There are $C, \gamma > 0$, satisfying:

$$||u(t)||_{W^{2,\infty}}, ||\theta(t)||_{W^{2,\infty}}, ||u_t(t)||_{W^{1,\infty}}, ||\theta_t(t)||_{W^{1,\infty}}, ||u_{tt}(t)||_{\mathcal{L}^\infty}, ||\theta_{tt}(t)||_{\mathcal{L}^\infty} \le C\delta e^{-\gamma\frac{t}{2}} \tag{3.23}$$

We now define an auxiliary energy similiar to the energy $E(t)$ in chapter 2:

$$\mathcal{E}(t; u, \theta) := e^{\gamma t}\left(\sum_{j=0}^{2} ||\partial_t^j u||^2_{H^{2-j}} + ||\theta_t||^2 + ||\theta||^2_{H^2}\right)(t) + \int_G e^{\gamma s}||\nabla\theta_t(s)||^2 ds \tag{3.24}$$

In contrast to the linear case we will not be able to prove exponential decay for arbitrarily large initial data. Nevertheless we apply the methods we have used in the linear case to get energy estimates. But these energy estimates contain certain terms with f and g. Utilizing the smallness of the initial data we can estimate these terms to get exponential decay at least for

[1] Here we use explicitly $n < 4$.

small initial data.

First we prove:

$$\mathcal{E}(t;u,\theta)+\int_0^t e^{\gamma s}\int_{\partial G}\left|\frac{\partial u_t}{\partial\vec{n}}\right|^2 dx\,ds \le \Gamma\mathcal{E}(0;u,\theta)+Ce^{\gamma t}||g(t)||^2$$
$$+C\int_0^t e^{\gamma s}\{||f||(||u||+||u_t||+||\Delta u||+||f||)+||g||(||\theta||+||\nabla\theta||+||\Delta\theta||+||g||)ds\}$$
$$+C\Big\{|\int_0^t e^{\gamma s}\langle f_t,u_{tt}\rangle ds|+|\int_0^t e^{\gamma s}\langle f,\Delta u_t\rangle ds|+|\int_0^t e^{\gamma s}\langle g_t,\theta_t\rangle ds|+|\int_0^t e^{\gamma s}\langle f_t,\sigma_k\partial_k u_t\rangle ds|\Big\}$$
$$+\int_0^t e^{\gamma s}\int_{\partial G}|\nabla u|^4+\int_0^t e^{\gamma s}\int_{\partial G}|\nabla u_t|^4$$
$$:=\Gamma\mathcal{E}(0;u,\theta)+\mathcal{P}(t;u,\theta,f,g),\quad t\in[0,T^*) \qquad (3.25)$$

For the proof we use the energy method in the same manner as in the linear case. This is a rather long but straight forward calculation, so we only want to mention that we can take advantage of the generalized nature of corollary 2.11. (For a complete proof we cite again [11].)

Using the auxiliary energies $F_i(t)$ defined in the previous chapter we finally get:

$$\begin{aligned}\frac{d}{dt}(\eta F_1+\eta F_2+F_3) \le\ & -\kappa\eta(||\nabla\theta||^2+||\nabla\theta_t||^2)-\kappa||\Delta\theta||^2+\eta(\int_G fu_t+\int_G g\theta+\int_G f_t u_{tt}+\int_G g_t\theta_t)\\ & -\int_G g\Delta\theta-\int_G f\Delta u_t+\frac{C}{\eta}||(\nabla u,u_t,\nabla\theta,\nabla u_t,u_{tt},\nabla\theta_t,f)||^2\\ & +\frac{C}{\eta}\int_G f_t\sigma_k\partial_k u_t+\frac{K_2}{\eta}\frac{d}{dt}\int_G u_t\sigma_k\partial_k u+\frac{K_3}{\eta}\frac{d}{dt}\int_G u_{tt}\sigma_k\partial_k u_t\\ & +\frac{C}{\varepsilon^3}||(\nabla\theta,\theta_t)||^2+\varepsilon||\text{div }u_t||^2+C\varepsilon||(\nabla u_t,u_t t,\nabla\theta_t,g)||^2\\ & +\frac{C}{\varepsilon}||g||\,||\nabla\theta||+\frac{C\eta^2}{\tau}\int_{\partial G}\left(|\nabla u_t|^4+|\nabla u|^4\right)+C\varepsilon\int_G f_t\sigma_k\partial_k u_t\\ & +K_1\frac{d}{dt}\int_G u_{tt}\sigma_k\partial_k u_t-C\varepsilon\int_{\partial G}\left|\frac{\partial u_t}{\partial\vec{n}}\right|^2,\end{aligned} \qquad (3.26)$$

where we have used this lemma:

Lemma 3.6 *Let $v=w(|x|)x$ be radially symmetric, then we have:*

$$\frac{1}{2}\tau\int_{\partial G}\left|\frac{\partial v}{\partial\vec{n}}\right|^2\le\frac{1}{2}\tau\int_{\partial G}|div\,v|^2+C||(v,\nabla v)||^2$$

Proof: The idea is similiar to the proof of the theorems 2.11 and 2.12: Here we multiply $\Delta v=\text{div}\nabla v$ with $\sigma_k\partial_k v$, integrate by parts and calculate the boundary integrals using explicitly the radial symmetry. □

We now define similiar to the linear case the energy

$$\begin{aligned}H(t) := \ & \eta F_1+\eta F_2+F_3-\left(\frac{K_2}{\eta^2}+\frac{K_5}{\alpha}\right)\int_G u_{tt}\sigma_k\partial_k u_t-\left(K_1\varepsilon+\frac{K_3}{\eta^2}+K_4\alpha\right)\int_G u_t\sigma_k\partial_k u\\ & +\varepsilon^{1/4}\int_G u_t u+\varepsilon^{1/2}\int_G\text{div }u\,\text{div }u_t.\end{aligned}$$

The constants K_4 and K_5 will be given later.

Again like in the linear case we show three auxiliary estimates:

Lemma 3.7 *Under the given assumptions we have:*

$$\frac{d}{dt}\int_G u_t\, u \;\leq\; -\frac{\tau s_1}{2}||\nabla u||^2 + C||(\theta,u_t)||^2 + C\int_{\partial G}|\nabla u|^4 + ||f||\,||u|| \qquad (3.27)$$

$$\begin{aligned}\frac{d}{dt}\int_G div\, u_t\, div\, u \;\leq\; & -\frac{\tau}{2}||\Delta u||^2 + C||\nabla\theta||^2 + ||\Delta u||\,||f|| + ||\nabla u_t||^2 \\ & +\alpha\left\{K_4\frac{d}{dt}\int_G u_{tt}\sigma_k\partial_k u_t + ||(u_{tt},\nabla u_t,\nabla\theta_t)||^2 + \int_G f_t\sigma_k\partial_k u_t\right\} \\ & +\frac{C}{\alpha}\left\{K_5\frac{d}{dt}\int_G u_t\sigma_k\partial_k u + ||(u_t,\nabla u,\nabla\theta,f)||^2\right\} \qquad (3.28)\end{aligned}$$

$$-\frac{\tau}{2}||\Delta u||^2 \;\leq\; -\frac{\tau}{4}||\Delta u||^2 - \frac{1}{8\tau}||u_{tt}||^2 + \frac{\beta^2}{4\tau}||\nabla\theta||^2 + \frac{1}{8\tau}||f||^2 \qquad (3.29)$$

Proof: The proofs correspond to the linear case in the previous chapter. □

Now we can derive an energy estimate for $H(t)$. For this purpose we use (3.26), (3.27), (3.28) and (3.29) to arrive at:

$$\begin{aligned}\frac{d}{dt}H(t) \;\leq\; & -\kappa\eta(||\nabla\theta||^2 + ||\nabla\theta_t||^2) - \kappa||\Delta\theta||^2 + \eta\left\{\int_G f u_t + \int_G g\theta + \int_G f_t u_{tt} + \int_G g_t\theta_t\right\} \\ & -\int_G g\Delta\theta - \int_G f\Delta u_t + \frac{C}{\varepsilon^3}||(\nabla\theta,\theta_t)||^2 + C\varepsilon||(\nabla u_t,u_{tt},\nabla\theta,f,g)||^2 \\ & +\frac{C}{\varepsilon}||g||\,||\nabla\theta|| + \frac{C}{\eta}||(u_t,\nabla u,\nabla\theta,f,u_{tt},\nabla u_t,\nabla\theta_t)||^2 + \frac{C}{\eta}\int_G f_t\sigma_k\partial_k u_t \\ & +C\eta^2\int_{\partial G}|\nabla u|^4 + C\eta^2\int_{\partial G}|\nabla u_t|^4 + C\varepsilon\int_G f_t\sigma_k\partial_k u_t \\ & +\varepsilon^{1/4}\left\{C||(\theta,u_t)||^2 + ||f||\,||u|| - \frac{\tau s_1}{2}||\nabla u||^2\right\} + \varepsilon^{1/2}\left\{C||(\nabla\theta,\nabla u_t)||^2 + ||\Delta u||\,||f||\right\} \\ & +\varepsilon^{1/2}\varepsilon^{1/8}\left\{C||(u_{tt},\nabla u_t,\nabla\theta_t)||^2 + \int_G f_t\sigma_k\partial_k u_t\right\} + \varepsilon^{1/2}\frac{C}{\varepsilon^{1/8}}||(u_t,\nabla u_t,\nabla\theta,f)||^2 \\ & -\varepsilon^{1/2}\frac{\tau}{4}||\Delta u||^2 - \varepsilon^{1/2}\frac{1}{8\tau}||u_{tt}||^2 + \varepsilon^{1/2}\frac{\beta^2}{4\tau}||\nabla\theta||^2 + \varepsilon^{1/2}\frac{1}{8\tau}||f||^2 \\ & -C\varepsilon\int_{\partial G}\left|\frac{\partial u_t}{\partial\vec{n}}\right|^2 \qquad (3.30)\end{aligned}$$

Using the Poincaré inequality, equation (3.9), and lemma 2.6 we get:

$$-\kappa\eta(||\nabla\theta||^2 + ||\nabla\theta_t||^2) - \kappa||\Delta\theta||^2 \;\leq\; -C\eta||(\nabla\theta,\theta,\nabla\theta_t,\theta_t)||^2 - C||(\Delta\theta,\nabla u_t,u_t)||^2 + C||g||^2 \qquad (3.31)$$

We insert (3.31) into (3.30), choose a small $\varepsilon > 0$ and a big η, such that we arrive at:

$$\begin{aligned}\frac{d}{dt}H(t) \;\leq\; & -C||(u_t,u_{tt},\nabla u,\nabla u_t,\Delta u,\theta,\theta_t,\nabla\theta,\nabla\theta_t)||^2 + C||f||^2 + C||g||^2 \\ & +C\left\{\int_G f u_t + \int_G g\theta + \int_G f_t u_{tt} + \int_G g_t\theta_t\right\} \\ & -\int_G g\Delta\theta - \int_G f\Delta u_t + C||g||\,||\nabla\theta|| + C\int_G f_t\sigma_k\partial_k u_t \\ & +C\int_{\partial G}|\nabla u|^4 + C\int_{\partial G}|\nabla u_t|^4 + C\int_G f_t\sigma_k\partial_k u_t \\ & +C||f||\,||u|| + C||\Delta u||\,||f|| + C\int_G f_t\sigma_k\partial_k u_t - C\int_{\partial G}\left|\frac{\partial u_t}{\partial\vec{n}}\right|^2\end{aligned}$$

Now we can use the Gronwall inequality. To show the equivalence of $\mathcal{E}(t;u,\theta)$ and $H(t)$ for big η we have to use some kind of elliptic regularity property. For a nonlinear Neumann boundary condition however there is no such analogon. Nevertheless for the radially symmetric case we have a similiar result proved in appendix B:

$$||u||_{H^2}^2 \le C||\Delta u||^2 + C\int_{\partial G} |\nabla u|^4 \tag{3.32}$$

Taking this all together we can prove (3.25).

If we consider the differential equations (3.8) and (3.9), we see that they are fulfiled by $\partial_t^l u$ and $\partial_t^l \theta$ if the nonlinearities f and g are replaced with $\partial_t^l f$ resp. $\partial_t^l g$. Using this and (3.25) we get:

$$\begin{aligned} &\mathcal{E}(t;\partial_t^l u,\partial_t^l\theta) + \int_0^t e^{\gamma t}\int_{\partial G}\left|\partial_t^{l+1}\frac{\partial u}{\partial\vec{n}}\right|^2 dxds \\ &\le \Gamma\mathcal{E}(0;\partial_t^l u,\partial_t^l\theta) + \mathcal{P}(t;\partial_t^l u,\partial_t^l\theta,\partial_t^l f,\partial_t^l g), \quad l=0,1,2 \end{aligned} \tag{3.33}$$

We now want to estimate $\mathcal{P}(t;\partial_t^l u,\partial_t^l\theta,\partial_t^l f,\partial_t^l g)$ to get a priori estimates for all derivatives of u and θ up to order four with not more than order two in space. It is useful to divide $\mathcal{P}(t;\partial_t^l u,\partial_t^l\theta,\partial_t^l f,\partial_t^l g)$ in six terms T_0–T_5:

$$\begin{aligned} \mathcal{P}(t;\partial_t^l u,\partial_t^l\theta,\partial_t^l f,\partial_t^l g) = {}& \underbrace{C\int_0^t e^{\gamma s}\{||\partial_t^l f||(||\partial_t^l u|| + ||\partial_t^l u_t|| + ||\Delta\partial_t^l u|| + ||\partial_t^l f||)}_{} \\ & + \underbrace{||\partial_t^l g||(||\partial_t^l\theta|| + ||\nabla\partial_t^l\theta|| + ||\Delta\partial_t^l\theta|| + ||\partial_t^l g||)ds\}}_{=:T_0} + C\Bigg\{\underbrace{\left|\int_0^t e^{\gamma s}\langle\partial_t^l f,\Delta\partial_t^l u_t\rangle ds\right|}_{=:T_1} \\ & + \underbrace{\left|\int_0^t e^{\gamma s}\langle\partial_t^l f,\Delta\partial_t^l u_t\rangle ds\right|}_{=:T_2} + \underbrace{\left|\int_0^t e^{\gamma s}\langle\partial_t^l f_t,\sigma_k\partial_k\partial_t^l u_t\rangle ds\right|}_{=:T_3} + \underbrace{\left|\int_0^t e^{\gamma s}\langle\partial_t^l g_t,\partial_t^l\theta_t\rangle ds\right|}_{=:T_4} \\ & + \sum_{i=1}^{l+1}\underbrace{\left|\int_0^t e^{\gamma s}\int_{\partial G}|\nabla\partial_t^i u|^4 ds\right|}_{=:T_5}\Bigg\} \end{aligned} \tag{3.34}$$

We have $T_0 \le C\varepsilon^3 + C\varepsilon^4$ because using (3.10) and denoting

$$\begin{aligned} A := A(\nabla u,\theta) &:= (A_{ij}(\nabla u,\theta) - A_{ij}(0,0))_{ij}, \\ \tilde{C} := \tilde{C}(\nabla u,\theta) &:= (\tilde{C}_{ij}(\nabla u,\theta) - \tilde{C}_{ij}(0,0))_{ij} \end{aligned}$$

we get:

$$\begin{aligned} f_t &= \partial_t(A(\nabla u,\theta))\Delta u + A(\nabla u,\theta)\Delta u_t + \partial_t(\tilde{C}(\nabla u,\theta))\nabla\theta + \tilde{C}(\nabla u,\theta)\nabla\theta_t \\ &= A_{\nabla u}(\nabla u,\theta)\nabla u_t\Delta u + A_\theta(\nabla u,\theta)\theta_t\Delta u + A(\nabla u,\theta)\Delta u_t \\ &\quad + \tilde{C}_{\nabla u}(\nabla u,\theta)\nabla u_t\nabla\theta + \tilde{C}_\theta(\nabla u,\theta)\theta_t\nabla\theta + \tilde{C}(\nabla u,\theta)\nabla\theta_t \end{aligned}$$

And if we differentiate this with respect to t we get:

$$\begin{aligned} f_{tt} &= \partial_t(A_{\nabla u})\nabla u_t\Delta u + A_{\nabla u}\nabla u_{tt}\Delta u + A_{\nabla}u\nabla u_t\Delta u_t \\ &+\partial_t(A_\theta)\theta_t\Delta u + A_\theta\theta_{tt}\Delta u + A_\theta\theta_t\Delta u_t \\ &+A_{\nabla u}\nabla u_t\Delta u_t + A_\theta\theta_t\Delta u_t + A\Delta u_{tt} \\ &+\partial_t(\tilde{C}_{\nabla}u)\nabla u_t\nabla\theta + \tilde{C}_{\nabla u}\nabla u_{tt}\nabla\theta + \tilde{C}_{\nabla u}\nabla u_t\nabla\theta_t \\ &+\partial_t(\tilde{C}_\theta)\theta_t\nabla\theta + \tilde{C}_\theta\theta_{tt}\nabla\theta + \tilde{C}_\theta\theta_t\nabla\theta_t \\ &+\tilde{C}_{\nabla u}\nabla u_t\nabla\theta_t + \tilde{C}_\theta\theta_t\nabla\theta_t + \tilde{C}\nabla\theta_{tt} \end{aligned} \tag{3.35}$$

Using that f is continuous and applying (3.23) we get: $||A(\nabla u,\theta)||_\infty \le C(||\nabla u||_\infty + ||\theta||_\infty) \le C\delta e^{-\frac{\gamma}{2}s}$ and also $||\tilde{C}(\nabla u,\theta)||_\infty \le C(||\nabla u||_\infty + ||\theta||_\infty) \le C\delta e^{-\frac{\gamma}{2}s}$. We therefore are able to estimate in each of the terms all functions but one with respect to the $\mathcal{L}_\infty$–norm. So we can prove:

$$||f_{tt}|| \le C\delta^2 e^{-\gamma s}$$

A similiar calculation for g leads to:

$$T_0 \le C\delta^3$$

We now consider T_1. Here we only want to discuss the case $l = 2$. The terms of lower order make fewer difficulties.

Using the special form (3.23) of f, the symmetry of $A(\nabla u,\theta)$, the Leibniz formula, the mean theorem of differentiation and the Cauchy–Schwarz inequality we have:

$$\begin{aligned} T_1 &\le C\delta^3 + \left|\int_0^t e^{\gamma s}\langle f_{tt}, \Delta u_{ttt}\rangle ds\right| \\ &\le C\delta^3 + \left|\int_0^t e^{\gamma s}\langle A(\nabla u,\theta)\partial_t^2\Delta u, \partial_t^3\Delta u\rangle ds\right| \\ &\le C\delta^3 + \frac{1}{2}\left|\int_0^t e^{\gamma s}\frac{d}{dt}\langle A(\nabla u,\theta)\partial_t^2\Delta u, \partial_t^2\Delta u\rangle ds\right| \\ &\le \frac{1}{2}e^{\gamma t}\left|\langle A(\nabla u,\theta)\partial_t^2\Delta u, \partial_t^2\Delta u\rangle\right| \\ &\quad +\frac{\gamma}{2}\left|\int_0^t e^{\gamma s}\langle A(\nabla u,\theta)\partial_t^2\Delta u, \partial_t^2\Delta u\rangle ds\right| \\ &\le C\delta^3 \end{aligned}$$

To estimate T_2 we need the following lemma for radially symmetric functions:

Lemma 3.8 *Let $v = w(|x|)x$ be a radially symmetric function, then we have:*

$$\int_{\partial G}|\nabla v|^2 \le \int_{\partial G}\left|\frac{\partial v}{\partial\vec{n}}\right|^2 + C||v||_{H^1}^2$$

Proof: Direct calculation and using the Sobolev trace theorem. □

Now we can argue similiarly to the estimate of T_1. Again we use the special form of f, integrate by parts, but now we use the lemma 3.8 to estimate the boundary integrals. We arrive at:

$$T_2 \leq C\delta^3 + C\delta \int_0^t e^{\gamma s} \int_{\partial G} \left| \frac{\partial u_{ttt}}{\partial \vec{n}} \right|^2 dx ds \tag{3.36}$$

In a similiar way we also estimate T_3 and get:

$$\begin{aligned} T_3 &\leq C\delta^3 + \left| \int_0^t e^{\gamma s} \langle A(\nabla u, \theta) \partial_t^3 \nabla u, \partial_t^4 \nabla u \rangle ds \right| \\ &+ C\delta \left| \int_0^t e^{\frac{\gamma}{2} s} \int_{\partial G} \partial_t^3 \nabla u \partial_t^4 u \vec{n} ds \right| \end{aligned} \tag{3.37}$$

The boundary integral we have to estimate, using lemma 2.6 and corollary 2.13, as follows:

$$\begin{aligned} \int_{\partial G} \partial_t^3 \nabla u \partial_t^4 u \vec{n} &\leq C \left\{ \int_{\partial G} |\text{div } u_{ttt}|^2 + \int_{\partial G} |u_{tttt} \vec{n}|^2 \right\} + C ||(\nabla u_{ttt}, u_{tttt})||^2 \\ &\leq C \frac{d}{dt} \int_G u_{tttt} \sigma_k \partial_k u_{ttt} + C ||(u_{tttt}, \nabla u_{ttt}, \nabla \theta_{ttt})||^2 \\ &+ C \int_G f_{ttt} \sigma_k \partial_k u_{ttt} \end{aligned} \tag{3.38}$$

Here we can use the estimate for T_2 and get, using Leibniz formula and the mean theorem of differentiation:

$$T_3 \leq C\delta^3 + C\delta^2 \int_0^t e^{\gamma s} \int_{\partial G} \left| \frac{\partial}{\partial \vec{n}} u_{ttt} \right|^2 dx ds$$

To estimate T_4 we use a similiar method utilizing that θ vanishes on the boundary and get:

$$T_4 \leq C\delta^3$$

To estimate the boundary terms of order four summarized in T_5 we have to use a very different method. The idea is to show that in the linear and radially symmetric case we can estimate the derivatives of a function on the boundary by the function itself. Then we show that the nonlinear boundary condition for small data approximates somehow the linear one, so we can get a similiar estimate using the implicit function theorem.

For the proof we want to consider an easier case, so let $G := \mathcal{B}(0,1) \subset \mathbb{R}^2$. Let $v := \partial_t^m u$ be radially symmetric with $m = 0, 1, 2, 3$ and $v(x) = w(|x|)x$. Then on ∂G we have:

$$\begin{aligned} \partial_1 v_1 &= w' x_1^2 + w, & \partial_1 v_2 &= w' x_1 x_2, \\ \partial_2 v_1 &= w' x_1 x_2, & \partial_2 v_2 &= w' x_2^2 + w, \end{aligned}$$

where we have defined $w' := w'(1)$ and $w := w(1)$.

We now consider the linear boundary condition $\mathcal{N}'\mathcal{S}\mathcal{D}u = 0$ on ∂G, e.g. in $x = (0,1)$, where we have:

$$\mathcal{N}'\mathcal{S}\mathcal{D}v = \begin{pmatrix} n_1(\tau \partial_1 v_1 + \lambda \partial_2 v_2) + n_2(\mu \partial_1 v_2 + \mu \partial_2 v_1) \\ n_2(\lambda \partial_1 v_1 + \tau \partial_2 v_2) + n_1(\mu \partial_1 v_2 + \mu \partial_2 v_1) \end{pmatrix} = \begin{pmatrix} \tau(w' + w) + \lambda w \\ 0 \end{pmatrix}$$

If v satisfies the linear boundary condition it follows:

$$\tau w' + (\tau + \lambda) w = 0 \tag{3.39}$$

So we could estimate w' by w. We now want to transfer this for the nonlinear case. Therefore we define the following functions for $x = (x_1, x_2) \in \partial G$:

$$g : \mathbb{R}^2 \to \mathbb{R}^4, \quad g(a,b) := \left(ax_1^2 + b, ax_1x_2, ax_1x_2, ax_2^2 + b\right),$$
$$f_i : \mathbb{R}^2 \to \mathbb{R}, \quad f_i(a,b) := (\vec{n}S(g(a,b),0)_{i=1,2},$$

where $\vec{n}$ is the normal in x.

Obviously g is in $\mathcal{C}^\infty$ and f_i is also smooth, if we assume S to be smooth. We have g defined such that $g(w', w) = \nabla v$. Remembering $M(t) \leq \Lambda\delta^2$ we have: $||v(t)||^2_{H^1} = ||\partial_t^m u(t)||^2_{H^1} \leq \Lambda\delta^2$ für $t \in [0, T^*)$. Using the Sobolev trace theorem we get: $\int_{\partial G} |v|^2 \leq C\Lambda\delta^2$

Using the radial symmetry of u we have $|v|^2$ locally constant on ∂G (and constant for $G = \mathcal{B}(0,1)$), so we have:

$$\sup_{x \in \partial G} |v(x,t)|^2 = ||\partial_t^m u(t)||^2_{\mathcal{L}^\infty(\partial G)} \leq C\Lambda\delta^2 =: K\delta^2 \quad \text{für} \quad t \in [0, T^*)$$

Now we consider the nonlinear boundary condition $f_i(a,b) = 0$ in $x = (1,0)$. We know:

$$|\mathcal{N}'\mathcal{S}\mathcal{D}v - \vec{n}S(v,0)| = \mathcal{O}(|\nabla v|^2) \quad \text{for } |\nabla v| \to 0 \text{ and } \theta = 0$$

Together with (3.39) we get:

$$\tau a + (\tau + \lambda)b - f_1(a,b) = \mathcal{O}(a^2 + b^2)$$

We calculate the derivative of f_1 with respect to the first argument in $(0,0)$. Let $\psi(a) := f_1(a,0)$, then we have:

$$\begin{aligned} \partial_1 f_1(a,b)\,|_{a=0,\, b=0} &= \psi'(0) = \lim_{h\to 0} \frac{\psi(h) - \psi(0)}{h} \\ &= \lim_{h \to 0} \frac{\tau h + \mathcal{O}(h^2)}{h} = \tau > 0 \end{aligned}$$

Using $f_1 \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R})$ we know that there exists an $U(0) \subset \mathbb{R}$, such that: $\partial_1 f_1(0,b) \neq 0$ for $b \in U(0)$. Furthermore we have: $f_1(0,0) = 0$. So we can use the implicit function theorem for f_1, and therefore it exists a $\phi \in \mathcal{C}^1(U(0), \mathbb{R}) : \quad \phi(b) = a$ with $f_1(a,b) = 0$ for $b \in U(0)$. If we choose δ such that $[-K\delta^2, +K\delta^2] \subset U(0)$, then we can estimate T_5 as follows:

$$\begin{aligned} \int_{\partial G} |\partial_t^m \nabla u|^4 &= \left|\int_{\partial G} |w'|^2 + 2w'w + |2w|^2\right|^2 \leq C \left|\int_{\partial G} |\underbrace{w'}_{=\phi(w)}|^2\right|^2 + C\left|\int_{\partial G} |w|^2\right|^2 \\ &\leq C\left|\int_{\partial G} |w|^2\right|^2 + C\left|\int_{\partial G} |w|^2\right|^2 \leq C\left|||\partial_t^m u||^2_{H^1}\right|^2 \leq C||\partial_t^m u||^4_{H^1} \leq C\delta^4 \end{aligned}$$

As we can see this proof is also applicable for other radially symmetric domains.
Summarizing the estimates for T_0–T_5 we get using (3.17):

$$\begin{aligned}&\mathcal{E}(t;\partial_t^l u,\partial_t^l\theta)+\int_0^t e^{\gamma s}\int_{\partial G}\left|\frac{\partial}{\partial\vec{n}}\partial_t^{l+1}u\right|^2 dxds\\ &\leq \Gamma\delta^2+C\delta^3+(C\delta+C\delta^2)\int_0^t e^{\gamma s}\int_{\partial G}\left|\frac{\partial}{\partial\vec{n}}\partial_t^{l+1}u\right|^2 dxds\end{aligned}$$

For small δ we get with $t\in[0,T^*)$:

$$\sum_{l=0}^{2}\mathcal{E}(t;\partial_t^l u,\partial_t^l\theta)\leq 3\Gamma\delta^2+C\delta^3 \tag{3.40}$$

So we have estimated all derivatives up to order four with order one or two in space. To conclude we have to estimate the missing derivatives of order three and four in space.
Using the ellipctic regularity property (see (3.21)), the differential equation (3.8) and (3.40) we get:

$$\begin{aligned}||u_t||_{H^3} &\leq \tilde{\Gamma}||\Delta u_t(t)||_{H^1}^2+C\delta^3\\ &\leq \frac{3(\beta^2+1)\tilde{\Gamma}}{\tau^2}(||u_{ttt}||_{H^1}^2+||\nabla\theta_t||_{H^2}^2+||f_t||_{H^1}^2)+C\delta^3\\ &\leq \frac{3}{86}258\Gamma(\beta^2+c^2+1)^3\tilde{\Gamma}\sum_{j=0}2\tau^{-2j}\delta^2=\frac{3}{86}\Lambda\delta^2\end{aligned} \tag{3.41}$$

Here we have also used that $f(\nabla u,\theta,\nabla^2 u,\nabla\theta)=\mathcal{O}(|(\nabla u,\theta,\nabla^2 u,\nabla\theta)|^2)$. Using (3.9) we get in a similar manner:

$$\begin{aligned}||\theta||_{H^4}^2+||\theta_t||_{H^3}^2 &\leq \tilde{\Gamma}(||\Delta\theta||_{H^2}^2+||\Delta\theta_t||_{H^1}^2)\\ &\leq \frac{3\tilde{\Gamma}(c^2+\beta^2+1)}{\kappa^2}(||u_t|_{H^3}^2+3\Gamma\delta^2+C\delta^3)\end{aligned}$$

Here we have used, that $g(\nabla u,\theta,\nabla^2 u,\nabla^2\theta,\nabla u_t)=\mathcal{O}((\nabla u,\theta,\nabla^2 u,\nabla^2\theta,\nabla u_t)^2)$. We now use the inequality (3.41) and the definition of Λ to derive:

$$\begin{aligned}||\theta||_{H^4}^2+||\theta_t||_{H^3}^2 &\leq \frac{3\tilde{\Gamma}(c^2+\beta^2+1)}{\kappa^2}\left(3\Gamma\delta^2\frac{3(\beta^2+1)\tilde{\Gamma}}{\tau^2}+3\Gamma\delta^2\right)+C\delta^3\\ &\leq \frac{9}{86}\Lambda\delta^2+C\delta^3\end{aligned} \tag{3.42}$$

For the last term we use (3.21), (3.8) and (3.42) to get:

$$||u||_{H^4}^2 \leq \frac{27}{86}\Lambda\delta^2+C\delta^3 \tag{3.43}$$

Summarizing (3.40), (3.41), (3.42) and (3.43) we arrive at:

$$\begin{aligned}M(t) &\leq \frac{39}{86}\Lambda\delta^2+3\Gamma\delta^2+C\delta^3\\ &\leq \frac{1}{2}\Lambda\delta^2+C\delta^3\end{aligned}$$

Choosing $\delta > 0$ small, we finally get:

$$M(t) \leq \frac{5}{6}\Lambda\delta^2 < \Lambda\delta^2 \quad \text{for all } t \in [0, T^*)$$

Using the continuity of M, we get the inequality $M(T^*) < \Lambda\delta^2$ by taking the limes $t \to T^*$. This is a contradiction to the definition of T^* in (3.22). So we have proved 3.4. □

There however are no known solutions for the other possible boundary conditions, although it would seem very likely that the same methods could work. For at least key points where problems may arise consider chapter 4.2 in [11].

Appendix A

Continuing the normal from the boundary into the interior

To prove the existence of the auxiliary function σ taking the value of $\vec{n}$ on the boundary and being smooth throughout G, which we have used in the chapters 2 and 3, we use two theorems taken from [8]. In the radially symmetric case such a σ of course is easy to find, but we can extend some results to rotation–free solutions if we have these theorems.

Theorem A.1 *Let $G \subset \mathbb{R}^n$ bounded*[1]*, ∂G be a $\mathcal{C}^2$–boundary, then there exists a $\sigma = (\sigma_1, \ldots, \sigma_n) \in (\mathcal{C}^1(\bar{G}))^n$ with $\sigma(x) = \vec{n}(x)$ on ∂G.*

Proof: Using the regularity of ∂G we have for all $x \in \partial G$ a neighbourhood $V_x \subset \mathbb{R}^n$ and a function $\sigma_x \in \mathcal{C}^2(V_x, \mathbb{R})$ satisfying:

$$\nabla \sigma_x(z) \neq 0 \quad \forall z \in V_x \tag{A.1}$$

$$\sigma_x(z) = 0 \iff z \in V_x \cap \partial G \tag{A.2}$$

$$\vec{n}(z) = \frac{\nabla \sigma_x(z)}{|\nabla \sigma_x(z)|} \quad \forall z \in V_x \cap \partial G \tag{A.3}$$

We can construct such a function by transforming V_x with a $\mathcal{C}^2$–transformation Ψ, such that $V_x \cap \partial G$ becomes a piece of a hyper plane $\Gamma_x \subset \mathbb{R}^{n-1}$. For $y = (y_1, \ldots y_n) \in V_x \cap \partial G$ we define: $\sigma_x(z) := (\Psi(x))_n$. It is easy to see that σ_x satisfies indeed (A.1) and (A.3) and therefore fulfils (A.3) because the tangential derivatives of σ_x vanish on the boundary.
Using ∂G compact we have a finite set $\{x_j\}_{1 \le j \le l} \subset \Gamma$, such that $\{V_j\} := \{V_{x_j}\}$ covers ∂G. Furthermore there is a $V_0 \subset G$ with $\bar{V_0} \subset G$, such that $G \subset \bigcup_{j=1}^{l} v_j$.
We now consider a partition of unity $(e_j)_{0 \le j \le l}$ subordinate to $\{V_j\}$, i.e. we have:

$$e_j \in \mathcal{C}_0^\infty(V_j, [0,1]) \tag{A.4}$$

$$\sum e_j = 1 \quad \text{in } \bar{G} \tag{A.5}$$

[1] As the proof shows it is here and in theorem A.2 sufficent to assume that ∂G is compact.

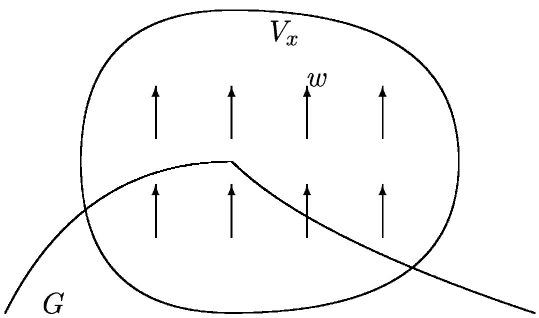
With the help of this partition we can define using the auxiliary functions $\sigma_j := \sigma_{x_j}$:

$$\sigma(x) := \sum_{j=1}^{l} e_j(x) \frac{\nabla \sigma_j(x)}{|\nabla \sigma_j(x)|}$$

We have $\sigma \in (\mathcal{C}^1(\bar{G}))^n$, because $\sigma_j \in \mathcal{C}^2(V_j)$. Furthermore we have $\sigma(x) = \vec{n}(x)$ for $x \in \partial G$ (see (A.3)). □

Modifying the proofs in the text we can show that a $\mathcal{C}^2$–boundary is not neccessary (except for the regularity of the solution to the linear problem!) and instead a Lipschitz–boundary is sufficent if we use the following theorem:

Theorem A.2 *Let $G \subset \mathbb{R}^n$ bounded, ∂G Lipschitz–boundary, then there is a $\delta > 0$ and a $\sigma = (\sigma_1, \ldots, \sigma_n) \in \left(\mathcal{C}^1(\bar{G})\right)^n$ with $\sigma(x) \cdot \vec{n}(x) \geq \delta$ on ∂G.*



Proof: For every $x \in \partial G$ there exists a $V_x \subset \mathbb{R}^n$, such that using an affine transformation $\Psi_x$the piece of the boundary $\partial G \cap V_x$ can be written as $(y, \phi(y))$ with a Lipschitz–continuous function

$$\phi := \phi_x : \; U \subset \mathbb{R}^{n-1} \to \mathbb{R}.$$

For our x we can assume $\Psi_x = Id$.

Now let $w := (\underbrace{0, \ldots, 0}_{(n-1)\text{times}}, 1)$ (see. Fig.). Then we have for $(y, \phi(y)) \in \partial G$

$$w \cdot \vec{n}(y, \phi(y)) \geq \delta > 0,$$

because otherwise we would have a sequence $(y_m)_m$ with $w \cdot \vec{n}(y_m, \phi(y_m)) \to 0$ for $m \to \infty$. Because of $w := (0, \ldots, 0, 1)$, this would mean: $\vec{n}_n(y_m, \phi(y_m)) \to 0$ resp. $|\nabla \phi(y_m)| \to \infty$ in contrary to ϕ Lipschitz–continuous. (Remember that for Lipschitz–continuous functions the derivatives are defined almost everywhere.)

Using again a partition of unity we can construct σ defining

$$\sigma(x) := \sum_{j=0}^{l} e_j(x) \frac{\Psi_j^{-1}(w)}{|\Psi_j^{-1}(w)|},$$

where $\Psi_j := \Psi_{x_j}$. □

We can extend this result to the following remark we do not need in this paper:

Remark A.3 *The σ constructed above lies in $(\mathcal{C}^\infty(\bar{G}))^n$.*

Appendix B

An elliptic regularity property

To use the elliptic regularity property for the Neumann boundary condition in (3.21) we have to use theorem 4.4 of [13]. Applying this to our situation we get for $j = 0, 1, 2$: For all $v \in H^{j+2}(G)$ there exist $\tilde{\Gamma} > 0$ and $C > 0$ with:

$$||v||^2_{H^{j+2}} \leq \tilde{\Gamma}||\Delta v||^2_{H^j} + C||\mathcal{N}'\mathcal{S}\mathcal{D}v||^2_{H^{j+1/2}(\partial G)} \tag{B.1}$$

In the nonlinear case we have to estimate the boundary term. For this purpose we remember on (3.12) in lemma 3.2, where we had:

$$|(\mathcal{N}'\mathcal{S}\mathcal{D}u - \mathcal{N}'\vec{\beta}\theta) - \vec{n}S(\nabla u, \theta)| \;=\; |n_j \underbrace{(C_{ij\alpha\beta}(0,0)\partial_\alpha u_\beta - \tilde{C}_{ij}(0,0)\theta - S_{ij}(\nabla u, \theta)}_{=\mathcal{O}(|\nabla u|^2+|\theta|^2)}|$$

Furthermore we need a special trace theorem proved e.g. in [1]:

Theorem B.1 (Trace theorem) *Let G be a bounded domain with smooth boundary then we have $H^{m+1}(G) \hookrightarrow H^{m+1/2}(\partial G)$.*

We now want to use after applying this theorem the equation 3.2 in G:

$$\begin{aligned}
||\mathcal{N}'\mathcal{S}\mathcal{D}v||^2_{H^{j+1/2}(\partial G)} &= ||\mathcal{N}'\mathcal{S}\mathcal{D}v - \mathcal{N}'\vec{\beta}\theta||^2_{H^{j+1/2}(\partial G)} \\
&\leq ||n_j(C_{ij\alpha\beta}(0,0)\partial_\alpha u_\beta - \tilde{C}_{ij}(0,0)\theta - S_{ij}(\nabla u, \theta)||^2_{H^{j+1/2}(\partial G)} \\
&\leq C||\underbrace{(C_{ij\alpha\beta}(0,0)\partial_\alpha u_\beta - \tilde{C}_{ij}(0,0)\theta - S_{ij}(\nabla u, \theta)}_{=:\phi(\nabla u,\theta)=\mathcal{O}(|\nabla u|^2+|\theta|^2)}||^2_{H^{j+1/2}(\partial G)} \\
&\leq C||\phi(\nabla u, \theta)||^2_{H^{j+1}(G)} \\
&= C\sum_{m=0}^{j+1}\int_G |\nabla^m \phi(\nabla u, \theta)|^2
\end{aligned} \tag{B.2}$$

Using $|\phi(\nabla u, \theta)| = \mathcal{O}(|\nabla u|^2 + |\theta|^2)$ we know that $\phi(0,0) = \phi'(0,0) = 0$. Now we calculate the terms of (B.2)utilizing (3.19) and (3.23):

For the term with $m = 0$ we get:

$$\int_G |\phi(\nabla u, \theta)|^2 \;\;\leq\;\; C\int_G |\nabla u|^4 + C\int_G |\theta|^4$$

$$
\begin{aligned}
&\leq C||\nabla u||_\infty^2 ||\nabla u||^2 + C||\theta||_\infty^2 ||\theta||^2 \\
&\leq C||u||_{H^3}^2 ||\nabla u||^2 + C||\theta||_{H^2}^2 ||\theta||^2 \\
&\leq C\delta^4
\end{aligned}
$$

The cases $m = 1$ and $m = 2$ are very similiar. For $m = 3$ we have:

$$
\begin{aligned}
\int_G |\nabla^3 \phi(\nabla u, \theta)|^2 &\leq C \int_G |\phi'''(\nabla u, \theta)\nabla^2 u \nabla^2 u \nabla^2 u|^2 + C \int_G |3\phi''(\nabla u, \theta)\nabla^3 u \nabla^2 u|^2 \\
&\quad + C \int_G |\phi'(\nabla u, \theta)\nabla^4 u|^2 + C \int_G |\phi'''(\nabla u, \theta)\nabla\theta\nabla\theta\nabla\theta|^2 \\
&\quad + C \int_G |3\phi''(\nabla u, \theta)\nabla^2\theta\nabla\theta|^2 + C \int_G |\phi'(\nabla u, \theta)\nabla^3\theta|^2 \\
&\leq C||\nabla^2 u||_\infty^4 ||\nabla^2 u||^2 + C||\nabla^2 u||_\infty^2 ||\nabla^3 u||^2 + C||\phi'||_\infty^2 ||\nabla^4 u||^2 \\
&\quad + C||\nabla\theta||_\infty^4 ||\nabla\theta||^2 + C||\nabla\theta||_\infty^2 ||\nabla^2\theta||^2 + C||\phi'||_\infty^2 ||\nabla^3\theta||^2 \\
&\leq C||u||_{H^4}^4 ||u||_{H^2}^2 + C||u||_{H^4}^2 ||u||_{H^3}^2 + C||u||_{H^1}^2 ||u||_{H^4}^2 + C||\theta||^2 ||u||_{H^4}^2 \\
&\quad + C||\theta||_{H^3}^4 ||\theta||_{H^1}^2 + C||\theta||_{H^3}^2 ||\theta||_{H^2}^2 + C||u||_{H^1}^2 ||\theta||_{H^3}^2 + C||\theta||^2 ||\theta||_{H^3}^2 \\
&\leq C\delta^4 + C\delta^6
\end{aligned}
$$

Summarizing these estimates (for small $\delta > 0$) we arrive at the following elliptic regularity property:

$$
||u||_{H^{j+2}}^2 \leq \tilde{\Gamma}_1 ||\Delta u||_{H^j}^2 + C\delta^4, \quad j = 0, 1, 2
$$

For u_t we derive in the same way:

$$
||u_t||_{H^{j+2}}^2 \leq \tilde{\Gamma}_2 ||\Delta u_t||_{H^j}^2 + C\delta^4, \quad j = 0, 1
$$

We now define $\tilde{\Gamma} := \max\{\tilde{\Gamma}_1, \tilde{\Gamma}_2\}$ to get the estimates used in chapter 3.
Now we prove (3.32), see chapter 3. Starting with (B.1) we only have to show that in the radially symmetric case it is:

$$
||\mathcal{N}'\mathcal{S}\mathcal{D}u||_{H^{1/2}(\partial G)}^2 \leq C \int_{\partial G} |\nabla u|^4
$$

We show this for the simplest case, where $G = \mathcal{B}(0, 1) \subset \mathbb{R}^2$. We parametrize ∂G with:

$$
x_1 = \sin\phi, \quad x_2 = \cos\phi, \quad \phi \in [0, 2\pi)
$$

Now we estimate the $H^{1/2}(\partial G)$–norm to the $H^1(\partial G)$–norm using the trace theorem:

$$
\begin{aligned}
||\mathcal{N}'\mathcal{S}\mathcal{D}u||_{H^{1/2}(\partial G)}^2 &\leq ||\mathcal{N}'\mathcal{S}\mathcal{D}u||_{H^1(\partial G)}^2 \\
&= \int_{\partial G} \sum_{|m|=0}^{1} |\nabla_t^m (\mathcal{N}'\mathcal{S}\mathcal{D}u)|^2
\end{aligned}
$$

Estimating the terms by caluclating $\mathcal{N}'\mathcal{SD}u$ on ∂G we get denoting $u(x) = w(|x|)x$ (u radially symmetric!), w for $w(1)$ and w' for $w'(1)$:

$$\begin{aligned}
\mathcal{N}'\mathcal{SD}u &= \begin{pmatrix} \tau x_1\partial_1 u_1 + \lambda x_1\partial_2 u_2 + \mu x_2\partial_2 u_1 + \mu x_2\partial_1 u_2 \\ \lambda x_2\partial_1 u_1 + \tau x_2\partial_2 u_2 + \mu x_1\partial_2 u_1 + \mu x_1\partial_1 u_2 \end{pmatrix} \\
&= \begin{pmatrix} \tau x_1^3 w' + \tau x_1 w + \lambda x_1 x_2^2 w' + \lambda x_1 w + 2\mu x_1 x_2^2 w' \\ \lambda x_1^2 x_2 w' + \lambda x_2 w + \tau x_2^3 w' + \tau x_2 w + 2\mu x_1^2 x_2 w' \end{pmatrix} \\
&= \begin{pmatrix} \tau \sin^3\phi w' + \tau\sin\phi w + \lambda\sin\phi\cos^2\phi w' + \lambda\sin\phi w + 2\mu\cos^2\phi\sin\phi w' \\ \lambda\sin^2\phi\cos\phi + \lambda\cos\phi w + \tau\cos^3\phi w' + \tau\cos\phi w + 2\mu\sin^2\phi\cos\phi w' \end{pmatrix} \\
&= \begin{pmatrix} w'\tau\sin(\phi) + (\lambda+\tau)w\sin(\phi) \\ w'\tau\cos(\phi) + (\lambda+\tau)w\cos(\phi) \end{pmatrix}
\end{aligned}$$

So for $|m| = 0$ we get:

$$\begin{aligned}
\int_{\partial G} |\mathcal{N}'\mathcal{SD}u|^2 &= \int_0^{2\pi} \left| \begin{pmatrix} w'(\tau\sin(\phi) + (\lambda+\tau)w\sin(\phi) \\ w'(\tau\cos(\phi) + (\lambda+\tau)w\cos(\phi) \end{pmatrix} \right|^2 d\phi \\
&= \int \left|(w'\tau + w(\lambda+\tau))\right|^2
\end{aligned}$$

And for $|m| = 1$:

$$\int_{\partial G} |\nabla_t^m \mathcal{N}'\mathcal{SD}u|^2 = \int \left|(w'\tau + w(\lambda+\tau))\right|^2$$

So for $|m| = 1$ we have: $||\nabla_t^m \mathcal{N}'\mathcal{SD}u||_{L^2(\partial G)} = ||\mathcal{N}'\mathcal{SD}u||_{L^2(\partial G)}$. Therefore we can use lemma 3.2 and get:

$$||\mathcal{N}'\mathcal{SD}u||^2_{H^{1/2}(\partial G)} \leq C \int_{\partial G} |\nabla u|^4$$

Together with (B.1) we arrive at:

$$||u||^2_{H^2} \leq C||\Delta u||^2 + C\int_{\partial G} |\nabla u|^4 \tag{B.3}$$

It is possible to extend this proof to other radially symmetric domains, so we finally have proved (3.32).

Bibliography

[1] R.A.Adams, *Sobolev Spaces*, Academic Press, San Diego, 1978

[2] D.E.Carlson, *Linear thermoelasticity*, Handbuch der Physik VIa/2, Springer-Verlag, Berlin, 1972, 297–346

[3] W.Dan, *On a Local in Time Solvability of the Neumann Problem of Quasilinear Hyperbolic Parabolic Coupled Systems*, Math.Meth.Appl.Sci. 18, 1053-1082, 1995

[4] S.Jiang, J.E.Muñoz Rivera, R.Racke, *Asymptotic stability and global existence in thermoelasticity with symmetry*, Quart.Appl.Math., LVI, 259, 1998

[5] H.Koch, *Slow decay in linear thermoelasticity*, Quart.Appl.Math., to appear

[6] R.Leis, *Initial boundary value problems in mathematical physics*, Teubner-Verlag, Stuttgart; John Wiley & Sons, Chichester et al., 1986

[7] S.Mizohata, *The theory of partial differential equations*, Cambridge Univ.Press, 1973

[8] J.–L.Lions, *Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués*, tome 1, Masson, Paris, 1988

[9] R.Racke, *Exponential Decay for a Class of Initial Boundary Value Problems in Thermoelasticity*, Comp.Appl.Math. 12, 67-80, 1993

[10] R.Racke, *Lectures on nonlinear evolution equations. Initial value problems*, Vieweg, Braunschweig/Wiesbaden, 1992

[11] M.O.Rieger, *Asymptotisches Verhalten radialsymmetrischer Lösungen von Thermoelastizitätsgleichungen*, diploma thesis, University of Constance, 1998

[12] J.Smoller, *Shock Waves and Reaction–Diffusion Equations*, Springer Verlag, New York et al., 1994

[13] Y.Shibata, G.Nakamura, *On a Local Existence Theorem of Neumann Problem for Some Quasilinear Hyperbolic Systems of 2nd Order*, Mathematische Zeitschrift 202, 1–64, 1989