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Abstract

Conformal mesh refinement has gained much attention as a necessary preprocessing step for the finite element method in the computer-aided design of machines, vehicles, and many other technical devices. For many applications, such as torsion problems and crash simulations, it is important to have mesh refinements into quadrilaterals. In this paper, we consider the problem of constructing a minimum-cardinality conformal mesh refinement into quadrilaterals. However, this problem is $\mathcal{NP}$–hard, which motivates the search for good approximations. The previously best known performance guarantee has been achieved by a linear-time algorithm with a factor of 4. We give improved approximation algorithms. In particular, for meshes without so-called folding edges, we now present a 2–approximation algorithm. This algorithm requires $\mathcal{O}(n^2 \log n)$ time, where n is the number of polygons in the mesh. The asymptotic complexity of the latter algorithm is dominated by solving a minimum–cost perfect b–matching problem in a certain variant of the dual graph of the mesh.

1 Introduction

In recent years, the conformal refinement of *finite element meshes* has gained much attention as a necessary preprocessing step for the finite element method in the computer-aided design of machines, vehicles, and many other technical devices. Much work has been done on decompositions into *triangles*; see [Ho88] for a survey. However, for many applications, such as torsion problems and crash simulations, it is important to have mesh refinements into quadrilaterals [ZT89]. See also [Tou95] for a systematic survey on quadrangulations.

A *polygon* is a closed and connected region in the plane or, more generally, of a smooth surface in the three–dimensional space, bounded by a finite, closed sequence of straight line segments (*edges*). The endpoints of the line segments or curves are the *vertices*. A polygon

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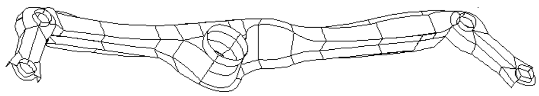

Figure 1: A coarse mesh modeling a chassis of a car. This mesh has been constructed by a German car company.

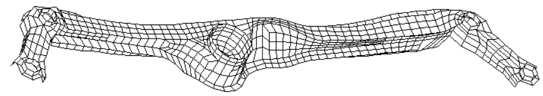

Figure 2: The conformal refinement produced by the algorithm in [MMW95].

is *convex* if the internal angle at each vertex is at most π. A *mesh* is a set of openly disjoint, convex polygons (Fig. 1). A mesh may contain *folding edges*, that is, edges incident to more than two polygons (Fig. 5). We call a mesh *homogeneous* if it does not contain folding edges.

In a *conformal refinement* of a mesh, each polygon is decomposed into strictly convex quadrilaterals, and if two quadrilaterals share more than a corner, they share exactly one edge as a whole (Fig. 2).

Workpieces are modeled interactively as meshes; see Fig. 1 for an example of an instance taken from practice. However, such meshes are usually very coarse and not conformal. To be suitable for the finite element method, the mesh has to be refined into a conformal mesh in a preprocessing step. Previous work puts emphasis on the shape of the quadrilaterals (angles should neither be too small nor too large; the *aspect ratio*, i.e. the ratio between the largest and the smallest side of a quadrilateral, should be small). This is important for the numerical accuracy in the later iterations of the cyclic design process, when the model has become mature and exact results are required for fine–tuning.

In this paper now, we focus on the early stages of this process, where the model is designed only roughly, and the numerical accuracy must only suffice to indicate the general tendency. Hence, the development time is crucial, which in turn is determined by the run time of the finite element method. This raises the following problem: Given a mesh, find a conformal refinement with a *minimum* number of quadrilaterals.

Until recently, work on this problem (cf. [MMW95] and [TA93]) has considered the number of quadrilaterals only heuristically or not at all. Usually, a *template model* is used, which restricts the possibilities of decomposing a single polygon to a few classes of *templates*. These templates are designed to achieve good angles and aspect ratios heuristically. For example, the most important template for quadrangular polygons is a $(p \times q)$–grid, where p and q are variable. However, this template uses $p \cdot q$ quadrilaterals, which is quadratic in size compared with minimal quadrangulations of size $\mathcal{O}(p+q)$ (easy to see). Therefore, algorithms often refine workpieces into too many quadrilaterals, which makes the finite element method very costly or even infeasible.

Unfortunately, it is hard to find conformal decompositions into a minimum number of quadrilaterals:

Theorem 1.1 [MW96] *The minimum cardinality conformal mesh refinement problem is $\mathcal{NP}$–hard even for homogeneous meshes.*

For single polygons, however, this problem is efficiently solvable. More precisely, two variants of the problem can be solved in linear time, namely the case which allows to insert additional vertices to arbitrary positions and the case which allows additional vertices only into the *interior* of the polygon, but not on its boundary, i.e. it forbids to subdivide edges.

In the mesh refinement problem, the polygons cannot be refined independently since we have to ensure that the mesh is conformal. Hence, we carefully distinguish between

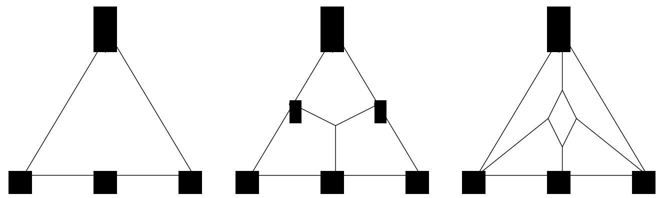

Figure 3: A triangular–shaped polygon with four vertices (left); an optimal *refinement*, which places two extra vertices on the boundary (middle); and an optimal *decomposition,* where no additional vertices on the boundary are allowed (right).

conformal refinements, where vertices can be inserted at arbitrary positions, and conformal decompositions (see Fig. 3 for an example). By a *conformal decomposition* of a single polygon we will always mean the variant which does not allow to subdivide edges but to place vertices into the *interior* of the polygon.

The following theorem holds:

Theorem 1.2 [MW96] *There is a linear–time algorithm which constructs a minimal conformal decomposition of a polygon into strictly convex quadrilaterals.*

There is also a well-known (see, for example [Joe95]), but important characterization of those polygons which can be decomposed into strictly convex quadrilaterals:

Lemma 1.3 *A simple polygon P admits a conformal decomposition if and only if the number of vertices of P is even.*

Lemma 1.3 and Theorem 1.2 give rise to the following two-stage approach: First, subdivide a couple of edges such that each polygon achieves an even number of vertices; second, refine each polygon separately according to the algorithm mentioned in Theorem 1.2. Clearly, the first stage determines the approximation factor. In [MW96], each edge of the mesh is subdivided exactly once, which trivially makes all polygons even. It is also proved in [MW96] that this simple strategy already yields a 4–approximation. The analysis of this simple strategy is tight: for a conformal mesh of quadrilaterals, this algorithm obviously takes four times as many quadrilaterals as the optimum. To improve upon this performance guarantee, we apply a more sophisticated strategy.

A few related problems have found some attention. Note, for example, that it is important that polygons are by definition convex polygons, as Lubiw [Lub85] has shown that both problems, minimum refinement and minimum decomposition, are $\mathcal{NP}$–hard for single, but *non-convex* polygons with holes. To the best of our knowledge, the complexity status of the refinement problem for non-convex polygons *without* holes is still open. Everett et al. [ELOSU92] give lower and upper bounds on the number of quadrilaterals in a conformal refinement of simple, not necessarily convex polygons (with and without holes), but not on decompositions. Refs. [Sac82, ST81] investigate perfect decompositions of (star-shaped) rectilinear polygons into *non–strictly* convex quadrilaterals, and [Lub85] considers perfect decompositions of non-convex polygons but even allows overlapping internal edges. See [Tou95] for a systematic survey.

In Sect. 4, we will present the main results of this paper:

- There is a linear–time approximation algorithm which exceeds ratio 2 by an additional term of at most $\Delta(\mathcal{M})$. This parameter $\Delta(\mathcal{M})$ (to be defined below in Def. 2.1) depends on the mesh structure, but for all practical instances that we know of, $\Delta(\mathcal{M})$ is significantly smaller than the minimal number of quadrilaterals in a conformal refinement. Hence, for such instances, this yields a 3–approximation. (For general instances, this algorithm always guarantees a 4–approximation.)

- As an immediate consequence, this yields a linear-time 3–approximation for homogeneous meshes. (This is not true for the algorithm in [MW96].)
- For homogeneous meshes, we can even do better, namely, we get a 2–approximation algorithm which runs in $\mathcal{O}(n^2 \log n)$ time, where n is the number of polygons in the mesh.

The asymptotic complexity of the latter algorithm, $\mathcal{O}(n^2 \log n)$, is dominated by solving a *minimum–cost perfect b–matching problem* (see the monograph by Derigs [Der88] for matching problems) in a certain variant on the dual graph of the mesh. In our application, the algorithm from [Gab83] requires $\mathcal{O}(n^2 \log n)$ time.

All our results also carry directly over to the following, slightly more general variant on the minimum mesh refinement problem. Suppose that the given mesh is too coarse to expect reasonable results from the finite element method, but a finite element error estimation gives lower bounds on the mesh density which should be achieved. More precisely, suppose that these lower bounds on the mesh density are expressed as lower bounds on the number of vertices which have to be placed on the original edges in a feasible refinement. (There are CAD packages which pursue this strategy.) The general problem is to find a conformal refinement which respects these lower bounds, but minimizes the number of quadrilaterals.

The rest of the paper is organized as follows. In Section 2, we start with some preliminaries and introduce further terminology. Then, in Section 3 we present a combinatorial result (cf. Lemma 3.1). Roughly speaking, this result means that the minimum number of quadrilaterals needed for a decomposition of a polygon does not increase exorbitantly, if each edge is subdivided at most once. The proof of Lemma 3.1 is quite involved and somewhat technical. Therefore, it is omitted, but we give all details in Appendix B. Finally, in Section 4, we present the new approximation algorithms and prove their performance guarantees. Appendix A reviews a characterization of minimal decompositions of polygons, on which the proof of Lemma 3.1 in the following section is based.

2 Preliminaries and Further Definitions

Let $\mathcal{P} = \{P_1, P_2, \ldots, P_n\}$ be the set of polygons forming the mesh. These polygons are convex, but not necessarily strictly convex. Two polygons are *neighbored* if they have an interval of the boundary in common which has strictly positive length in common. These neighborhood relationships induce an undirected graph $G = (V, E)$, which is embedded on the surface approximated by the mesh and whose faces are the polygons. More precisely, V consists of the corners of the polygons. If a corner of a polygon also belongs to the interior of a side of another polygon, it subdivides this side. Hence, we may identify common intervals of neighbored sides of polygons with each other, and E consists of these intervals after identification.

Note that the graph G of a mesh need not be planar; for example, a mesh approximating the surface of a torus has genus one. The set of all folding edges that are incident to exactly the same homogeneous components is called a *folding*.

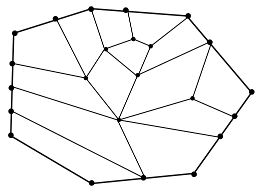

Figure 4: A convex polygon with 7 corners and 16 vertices and a conformal decomposition with 7 additional, internal vertices. (The decomposition is not minimal.)

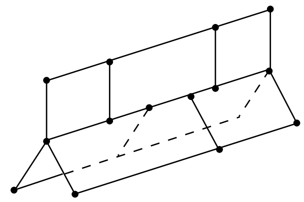

Figure 5: A small mesh with three homogeneous components and one folding, which consists of five folding edges. The corresponding hypergraph has 13 edges of degree one (boundary edges), 4 hyperedges of degree 2, and 5 hyperedges of degree 3.

For an edge $e_i \in E$, let E_i be the set of all those polygons which are incident to e_i. A *combinatorial description* of a mesh consists of the graph G and the hypergraph $H = (\mathcal{P}, \{E_1, \ldots, E_m\})$ with vertex set $\mathcal{P}$ and edge set $\{E_1, \ldots, E_m\}$. We will often identify a mesh with its combinatorial description.

A *non-folding path* in H is a path between two polygons $P_1, P_2 \in \mathcal{P}$ which contains only hyperedges of cardinality two, i.e. only such hyperedges which belong to exactly two polygons. Being connected by a non-folding path is an equivalence relation on the set of polygons. Its equivalence classes are exactly the *homogeneous components* of a mesh. For a mesh $G = (V, E)$ let $G_1, \ldots, G_{c(\mathcal{P})}$ denote the homogeneous components, and $c(\mathcal{P})$ the number of components. The *degree* of an edge in E is the number of incident polygons. The *boundary* of a mesh is the set of all hyperedges with degree one (the *boundary edges*).

Since all folding edges within a folding are incident to exactly the same homogeneous components, the degree of a folding is well defined. Let $D(\mathcal{M})$ denote the total sum of the degrees of all foldings that consist of an *odd* number of folding edges each. This allows us to define the parameter $\Delta(\mathcal{M})$ which appears in the performance guarantee we can achieve for meshes, in general:

Definition 2.1 $\Delta(\mathcal{M}) := D(\mathcal{M}) + |\mathcal{P}| - c(\mathcal{P})$.

Empirically, the mesh parameter $\Delta(\mathcal{M})$ is fairly small. In a whole bunch of real-world examples, which stem from the German car industry, the average number of odd foldings per homogeneous component is less than three, and always smaller than the minimum number of quadrilaterals needed for that component. (This means that we guarantee a 3–approximation for such instances from practice.) In fact, it seems hard to imagine a non-pathological instance, where $\Delta(\mathcal{M})$ is larger than the minimum number of quadrilaterals in an optimal mesh refinement.

A vertex of a convex polygon is a *corner* if its internal angle is strictly less than π. An *interval* of a polygon P is a path of edges on its boundary. A *segment* S is an interval between two successive corners of P.

A conformal decomposition of P is usually identified with the planar, embedded graph $G_P = (V, E)$ whose outer face is P and whose internal faces are the quadrilaterals. Let $q(G)$ denote the number of internal, quadrangular faces. Let $G^* = (V^*, E^*)$ be the variant on the *dual graph* which arises by removing the vertex corresponding to the outer face of G_P. We call a conformal decomposition of a convex polygon P *perfect* if it has no vertices other than P.

We will denote a polygon by the counterclockwise sequence of the lengths of its segments. For example, $(1,1,1,1)$ denotes the strictly convex quadrilateral, $(1,1,2) = (1,2,1) = (2,1,1)$ the quadrilateral degenerated to a triangle, and $(4,1,2,3,2,2,2) = (1,2,3,2,2,2,4) = \cdots$ the polygon in Fig. 4. This is justified by the following observation (cf. Lemma 3.4 in [MW96]): If two polygons P_1 and P_2 have the same such sequence (up to cyclic shifts) Then every graph of a conformal decomposition for P_1 is also the graph of some conformal decomposition for P_2 and vice versa.

3 Subdivisions of Polygons

In this section, we present a combinatorial result which relates the optimal conformal refinements of a polygon P to optimal refinements of those polygons, which arise if some of the edges of P are subdivided by one additional vertex. This result will be useful for conformal refinements of meshes in Sect. 4. In fact, Lemma 3.1 is the most difficult part of the proofs of Theorems 4.3 and 4.5.

For a convex polygon P with an even number of vertices, $\min(P)$ denotes the minimum number of quadrilaterals required by any conformal decomposition of P. For an arbitrary convex polygon P with edge set E_P, a mapping $X : E_P \to \mathbb{N}_0$ is called *feasible* if $\sum_{e \in E_P} X(e)$ has the same parity as E_P. In particular, if $|E_P|$ is even, $X \equiv 0$ is possible, too. The polygon P_X is constructed from P by subdividing each edge $e \in E_P$ exactly $X(e)$ times. Hence, X feasible means that P_X admits a conformal decomposition. Moreover, $\mathrm{Min}(P)$ denotes the minimum number of quadrilaterals in any conformal decomposition of any polygon P_X, $\mathrm{Min}(P) := \min\{q(G) \mid G \text{ conformal decomposition of } P_X,\ X : E_P \to \mathbb{N}_0 \text{ feasible}\}$.
If $X(e) \in \{0,1\}$ for all e, we denote $|X| = |\{e \in E_P : X(e) = 1\}|$.

Lemma 3.1 *For a polygon P and $X : E_P \to \{0,1\}$ we have* $\min(P_X) \leq 2 \cdot \mathrm{Min}(P) + |X| - 1$, *and we even have* $\min(P_X) \leq 2 \cdot \mathrm{Min}(P) + |X| - 2$ *except for the following cases:*

1. $P = (1,1,1,1)$ *and* $P_X = (1,1,1,1)$;
2. $P = (1,1,1,1)$ *and* $P_X = (2,2,1,1)$;
3. $P = (2,1,1,1)$ *and* $P_X = (3,1,1,1)$;
4. $P = (2,1,2,1)$ *and* $P_X = (4,1,2,1)$;
5. $P = (2,1,1,1,1)$ *and* $P_X = (4,1,1,1,1)$;
6. $P = (2,1,1)$ *and* $P_X = (2,1,1)$.

Proof: Omitted, the detailed proof can be found in Appendix B. □

4 Approximation of Minimal Conformal Mesh Refinements

In this section, we describe the improved approximation algorithms. In the following, we will need a certain variant $G^d = (V^d, E^d)$ on the dual graph of the graph $G = (V, E)$ of a homogeneous mesh $\mathcal{M}$.

Definition 4.1 *For a homogeneous mesh $\mathcal{M}$ and its corresponding graph $G = (V, E)$, the graph $G^d = (V^d, E^d)$ (multiple edges allowed) has a vertex for each polygon of $\mathcal{M}$. If the surface approximated by the homogeneous mesh $\mathcal{M}$ has a non-empty boundary, exactly one more vertex is added to V^d. Two dual vertices that correspond to polygons of $\mathcal{M}$ are connected by an edge of E^d if and only if they share an edge in E. For each boundary edge (if existing) the additional vertex is connected with the dual vertex whose corresponding polygon is incident to the boundary.*

We will need the next fact, Lemma 4.2, for all algorithmic results to follow. Let $G = (V, E)$ be an undirected graph, and let each vertex be either labeled *odd* or labeled *even*. This *odd/even* labeling is called *feasible* if the number of vertices labeled *odd* is even. A subgraph G' of a graph G is called *feasible* if the following holds: The degree of a vertex is odd in G' if and only if its label is "*odd*."

Lemma 4.2 *There is a linear–time algorithm that produces a feasible acyclic subgraph F of a connected graph G with respect to a feasible odd/even labeling.*

Proof: Let T be a spanning tree of G and let F be the forest comprising all edges e of T that divide $T - e$ into two subtrees, each with an odd number of vertices labeled *odd*. It is easy to see that F is feasible. □

Theorem 4.3 *There is a linear–time algorithm that constructs a conformal refinement of a mesh G such that the number of quadrilaterals exceeds twice the optimum by at most $\Delta(\mathcal{M})$.*

Proof: First we describe the algorithm and prove that it constructs a conformal refinement. Recall from Lemma 1.3 that we have to ensure that every polygon becomes even. For each folding that consists of an odd number of folding edges, we select one of these edges and subdivide it once. We will see that this suffices to refine all homogeneous components separately. So let $G_i = (V_i, E_i)$ be a homogeneous component. First consider the edges in E_i that have degree one but are not folding edges of the original mesh. In other words, consider the edges on the boundary of the surface approximated by G_i. If the number of these edges is odd, we select exactly one of these edges and subdivide it once, too. After this procedure, the number of edges of degree one in E_i is even. (Note that all other edges have degree 2, because G_i is homogeneous.)

Let $G_i^d = (V_i^d, E_i^d)$ denote the variant on the dual graph of G_i as in Definition 4.1. Then the vertex of G_i^d added for the boundary edges has even degree in G_i^d. Therefore, the number of odd vertices that correspond to polygons (and hence the number of odd polygons themselves) in G_i is even. Consequently, we may apply the algorithm of Lemma 4.2 to construct a feasible acyclic subgraph F_i of G_i^d, where a vertex is labeled *odd/even* according to the parity of its degree. Next each edge of E_i that corresponds to an edge in F_i is subdivided exactly once. Obviously, every polygon is now even, and we apply the algorithm from [MW96] to decompose each polygon separately.

It remains to show that this construction leads to a refinement that exceeds twice the optimum by at most $\Delta(\mathcal{M})$.

For a polygon P of the homogeneous component G_i, $i = 1, \ldots, k$, let $X_P : E_P \to \{0, 1\}$ be defined such that $X_P(e) = 1$ means that e corresponds to a dual edge of F_i. Analogously, let $Y_P : E_P \to \{0, 1\}$ attain 1 exactly on the edges of P that are selected in the algorithm to make all foldings even. Moreover, let $Z_P : E_P \to \{0,1\}$ attain 1 on an edge if and only if this edge is selected to make the number of degree–one edges of G_i even. Let $P' := P_{(X_P+Y_P+Z_P)}$. Of course, we have $X_P(e) + Y_P(e) + Z_P(e) \leq 1$ for each edge e. Therefore, Lemma 3.1 gives

$$\min(P') \leq 2 \cdot \mathrm{Min}(P) + |X_P| + |Y_P| + |Z_P| - 1 \ . \tag{1}$$

Let $\mathcal{P}_i$ denote the set of polygons in G_i. Since the feasible subgraph F_i of G_i^d constructed by the algorithm is acyclic, we have $\sum_{P\in\mathcal{P}_i} |X_P| \leq 2\cdot(|\mathcal{P}_i| - 1)$, and since $\sum_{P\in\mathcal{P}_i} |Z_P| \leq 1$, Ineq. (1) sums up to

$$\sum_{P\in\mathcal{P}_i} \min(P') \leq \sum_{P\in\mathcal{P}_i} \Big[2\cdot \mathrm{Min}(P) + |Y_P|\Big] + |\mathcal{P}_i| - 1 \ . \tag{2}$$

Note that $\sum_{i=1}^{k} \sum_{P\in\mathcal{P}_i} |Y_P| = D(\mathcal{M})$. Hence, Ineq. (2) sums up to

$$\sum_{P\in\mathcal{P}} \min(P') \ \leq \ 2\cdot \sum_{P\in\mathcal{P}} \mathrm{Min}(P) + D(\mathcal{M}) + |\mathcal{P}| - k \ = \ 2\cdot \sum_{P\in\mathcal{P}} \mathrm{Min}(P) + \Delta(\mathcal{M}) \ .$$

□

As an immediate consequence, we obtain for the special cases of meshes without foldings of odd degree, i.e. where $D(\mathcal{M})$ vanishes, the following corollary:

Corollary 4.4 *There is a linear-time algorithm that yields a 3–approximation for the minimum conformal refinement problem for the special cases where $D(\mathcal{M}) = 0$. This includes, in particular, the homogeneous meshes.*

For meshes without foldings, we can even find significantly better approximations using a nice application of matching techniques:

Theorem 4.5 *For homogeneous meshes, there is an $\mathcal{O}(n^2 \log n)$ algorithm that constructs a conformal refinement such that the number of quadrilaterals is at most twice the optimum.*

Like in the proof of Theorem 4.3, we construct a feasible acyclic subgraph F, but now we use edge weights to find subgraphs which allow for a better analysis. The idea is to choose edge weights in such a way that we get an improved lower bound if some of the expensive edges are chosen in a feasible subgraph of minimum weight. Note that, in general, it is not true that a feasible subgraph with a smaller number of edges gives a better result.

We determine F in an auxiliary graph $G^d_{\mathrm{aux}} = (V^d_{\mathrm{aux}}, E^d_{\mathrm{aux}})$. This is necessary to cope with the exceptions 2. - 5. in Lemma 3.1. For these four types of polygons, we try to avoid a subdivision which leads to the exception. This is in contrast to the first and last exceptions. In particular, for the last case $P = (2,1,1)$ a solution with a subdivision of edges is favorable (see again Fig. 3). The analysis below will show that we do not have to treat polygons of type $P = (2,1,1)$ in a special way.

The graph G^d_{aux} is constructed from G^d as follows. Each polygon $v^d \in V^d$ of type $(1,1,1,1)$, $(2,1,1,1)$, $(2,1,2,1)$, or $(2,1,1,1,1)$ is replaced by a couple of vertices and edges, which respectively form subgraphs as shown in Fig. 6. Each edge $e \in E^d_{\mathrm{aux}}$ is assigned a weight $w(e^d)$. The weights $w(e^d)$ of the edges in Fig. 6 are introduced there, too. If a polygon P is perfect, then all edges e^d, for which e belongs to P, get a weight of $w(e^d) = 1/2$. (Recall that we can test for perfectness of a polygon in linear time.) All other edges $e^d \in E^d_{\mathrm{aux}}$ have weight $w(e^d) = 0$. We say that a subgraph F_{aux} of G^d_{aux} is feasible if the degree of every vertex outside these four kinds of subgraphs has the same parity in G^d_{aux} and F_{aux}, and all vertices inside these subgraphs have even degree in F_{aux} except for the vertex indicated by an arrow in Fig. 6, which must have odd degree in F_{aux}.

Lemma 4.6 *Let F_{aux} be a feasible subgraph of G^d_{aux} such that the sum of all weights $w(\cdot)$ of edges in F_{aux} is minimum. Let F be a feasible acyclic subgraph of G^d such that F is constructed from F_{aux} by shrinking all subgraphs in Fig. 6 and removing all cycles from the shrunken F_{aux}. Then subdividing once all edges of G that correspond to dual edges in F yields a 2–approximation.*

Proof: The following facts are easy to see from Fig. 6:

- There are no negative cycles in G^d_{aux} with respect to the edge weights $w(\cdot)$.
- For an optimal F_{aux}, the weights $w(\cdot)$ inside a polygon in Fig. 6 sum up to zero if and only if none of the exceptional situations in Lemma 3.1 occurs.
- If such an exceptional situation occurs for a polygon in Fig. 6, the edges inside this polygon contribute exactly one unit of weight in total for an optimal F_{aux}.

Note that any perfect polygon is even. Hence, the number of edges incident to a perfect polygon in a feasible subgraph is always even, too. Moreover, for a perfect polygon P, the subdivision of an edge always means that one needs more than $\text{Min}(P)$ quadrilaterals (easy to see).

In summary, the total weight of an optimal F_{aux} in G^d_{aux} is less or equal to the number r, say, of exceptional situations that occur in the corresponding conformal refinement plus the number of perfect polygons where one of its corresponding dual edges is in the optimal F_{aux}.

Hence, r is minimum among all those conformal refinements of G wherein no edge in E is subdivided more than once, and $\sum_{P\in\mathcal{P}} \text{Min}(P) + r$ is a lower bound on the number of quadrilaterals in any such refinement.

Next we show that $\sum_{P\in\mathcal{P}} \text{Min}(P) + r$ is a lower bound on the number of quadrilaterals in *every* conformal refinement. To see this, take an arbitrary conformal refinement. For $e \in E$, let $Y(e)$ denote how often e is subdivided in this refinement. Let $Y'(e) \in \{0,1\}$ be the remainder of $Y(e)/2$. Then subdividing each edge $e \in E$ exactly $Y'(e)$ times makes all polygons even. However, it is easy to see that a polygon P that is exceptional in the refinement induced by Y' cannot be refined with $\text{Min}(P)$ quadrilaterals in the refinement induced by Y. In addition, a perfect polygon P which contributes a weight of $s > 0$ units to the minimum F_{aux} needs at least $\text{Min}(P) + s$ quadrilaterals in the refinement induced by Y.

This shows that $\sum_{P\in\mathcal{P}} \text{Min}(P) + r$ is a lower bound on the number of quadrilaterals in the refinement induced by Y, too.

For a polygon $P \in \mathcal{P}$, let $X_P : E_P \to \{0,1\}$ denote which edges are subdivided in the refinement induced by F. Now it suffices to show

$$\sum_{P\in\mathcal{P}} \min(P_{X_P}) \leq 2 \cdot \sum_{P\in\mathcal{P}} \text{Min}(P) + r \ .$$

For each $i \in \mathbb{N}_0$ let n_i denote the number of vertices in F having degree i. Recall that a polygon of type $T = (2,1,1)$ belongs to the exceptions of Lemma 3.1 only if none of its edges is subdivided. If $X_T \equiv 0$, then our solution needs $\min(T) = 5 \leq 2 \cdot \text{Min}(T) = 6$, and is therefore good enough to achieve a factor of 2 for such a polygon.

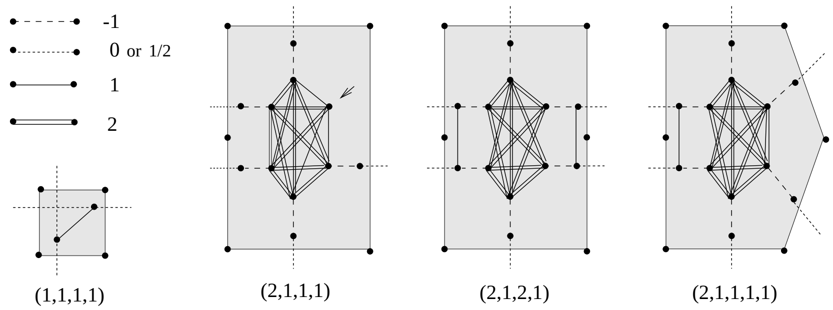

Figure 6: The subgraphs of G^d_{aux} for polygons v^d of type $(1,1,1,1)$, $(2,1,1,1)$, $(2,1,2,1)$, and $(2,1,1,1,1)$. The weight of an edge is indicated by its line style. With the exception of $(2,1,1,1)$, these polygons are perfect. Hence, the dual edges have weight 0 only for the polygon $(2,1,1,1)$, and weight $1/2$ otherwise.

Thus, Lemma 3.1 implies

$$\sum_{P\in\mathcal{P}} \min(P_{X_P}) \leq 2 \cdot \sum_{P\in\mathcal{P}} \mathrm{Min}(P) - n_1 + \sum_{i\geq 3}(i-2)\cdot n_i + r \ \ .$$

Hence, it suffices to show $\sum_{i\geq 3}(i-2)\cdot n_i \leq n_1$. To see this, we first replace each vertex with degree $i > 3$ by a path of $i-2$ vertices each with degree 3. In this modified graph, the claim follows from $n_3 \leq n_1$, which is well known for binary trees (and all the more true for binary forests). Obviously, the backward transformation preserves validity of the claim. □

Proof of Theorem 4.5: Because of Lemma 4.6, it remains to show how to construct an optimal feasible graph F_{aux}. We solve this problem by a reduction to a *capacitated minimum–cost perfect b–matching problem* [Der88]. In order to introduce this reduction, we first state the problem we want to reduce in more general terms, also known in the literature as the *T–join problem* (see, for example [Ger95]). So let $G = (V, E)$ be an undirected graph, let $w(\cdot)$ be a weighting of E, and for $v \in V$ let $\mathit{equal}(v)$ be a logical flag. We call a subgraph F of G feasible if the following holds: The degree of each $v \in V$ in F has the same parity as the degree in G if and only if $\mathit{equal}(v)$ is true. The problem is to find a feasible subgraph that minimizes the sum of the edge weights $w(\cdot)$.

The reduction is as follows (and was first proposed by Edmonds and Johnson [EJ73]). For $v \in V$, let $b(v)$ equal the degree of v if $\mathit{equal}(v)$ is true, otherwise let $b(v)$ equal the degree plus one. Let $\bar{G} = (V, \bar{E})$ denote G with all loops $\{v, v\}$, $v \in V$, added to E. The weight of such a loop is $w(\{v, v\}) := 0$. Moreover, we set $\ell(e) := 0$ for all $e \in \bar{E}$, $u(e) := 1$ for $e \in E$, and $u(\{v, v\}) := \lfloor b(v)/2 \rfloor$ for $\{v, v\} \in \bar{E} \setminus E$.

There is a straightforward one–to–one correspondence between feasible subgraphs of G and perfect b–matchings in $\bar{G}$ with lower bounds $\ell(\cdot)$ and upper bounds $u(\cdot)$. Moreover, the cost of a b–matching with respect to $w(\cdot)$ equals the sum of edge weights of the corresponding feasible subgraph. Note that the graph of the b–matching instance is sparse, i.e. it has $\mathcal{O}(n)$ edges, although it is not planar, in general. This establishes the reduction, and by an application of Gabow's [Gab83] algorithm the time bound claimed in the introduction follows. □

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Appendix

A The Structure of Minimal Decompositions of Polygons

This section reviews a characterization of the structure of minimal decompositions of polygons. The proof of Lemma 3.1 is based on it.

We need some additional terminology. For a conformal decomposition $G = G_P = (V, E)$ of polygon P, recall the definition of G^* from Section 2. Each degree–one vertex v^* of G^* points to a trivial segment of P. We will sometimes identify such a vertex with this trivial segment.

Let $i(G)$ denote the number of internal vertices, that is, the members of V that do not lie on P. With the help of Euler's formula it is easy to see, that $q(G)$ and $i(G)$ are related via $q(G) = i(G) + |E_1|/2 - 1$, where E_1 is the number of edges of P.

The graph $K_{1,3}$ is the complete bipartite graph on $1 + 3$ vertices. We use the term *subdivision* of $K_{1,3}$ when each edge of the $K_{1,3}$ is replaced by a path of arbitrary length.

An *interval* on a polygon P is a path on its boundary. An interval is *trivial* if it consists of exactly one edge of P. A *segment* S is an interval between two successive corners of P. Let e_1 and e_2 be two different edges of P. Then $I[e_1, e_2]$ denotes the interval counterclockwise from e_1 to e_2, including neither e_1 nor e_2. The length $L(I)$ of an interval I is the number of its edges. Moreover, $K(I)$ denotes the maximum size of a choice of strictly convex internal vertices of I such that no two of them are neighbored on P. We often denote $(L-2K)(I) := L(I)-2\cdot K(I)$. Note that $(L - 2K)(I)$ is always nonnegative.

Lemmas A.1 and A.3 first characterize minimal decompositions of perfect polygons, whereas Lemma A.4 treats the general case.

Lemma A.1 [MW96] *Let P be a polygon with exactly two trivial segments e_1 and e_2, and let $I_1 := I[e_1, e_2]$ and $I_2 := I[e_2, e_1]$. W.l.o.g. we have $L(I_1) \geq L(I_2)$. Then P is perfectly decomposable if and only if $(L-2K)(I_1) \leq L(I_2)$. The dual graph G^* of a perfect decomposition is a path with leaves e_1 and e_2.*

In Lemma A.3, we assume the following scenario.

Scenario A.2 *Let P be a polygon with at least three trivial segments. Let e_1, e_2, and e_3 be three trivial segments such that the counterclockwise order around P is $e_1 \prec e_2 \prec e_3 \prec e_1$. Let $I_1 := I[e_1, e_2]$, $I_2 := I[e_2, e_3]$, $I_3 := I[e_3, e_1]$, and w.l.o.g. $L(I_1) \geq L(I_2)$ and $L(I_1) \geq L(I_3)$. Assume that $L(I_1)$ is minimum subject to all these conditions.*

Lemma A.3 [MW96] *In Scenario A.2, P is perfectly decomposable if and only if $(L - 2K)(I_1) \leq L(I_2) + L(I_3) + 1$. In this case, there is a perfect decomposition such that either G^* is a path from e_1 to e_2, or G^* is a subdivision of $K_{1,3}$ with leaves e_1, e_2, and e_3.*

Let $G = (V, E)$ be an undirected planar, embedded graph. An *area component* of G is a subgraph G' induced by a connected component of G^*. More precisely, G' consists of all vertices and edges incident to the polygons that correspond to this component of G^*. An *area decomposition* of G is a collection of area components such that the inducing components of G^*

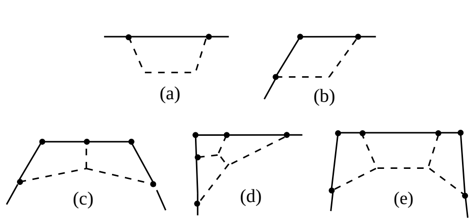

Figure 7: The first class of cut components in Lemma A.4(2). The solid lines belong to P, and the dashed lines are internal edges. Only the structure of the graph matters; the concrete lengths and angles are only exemplary.

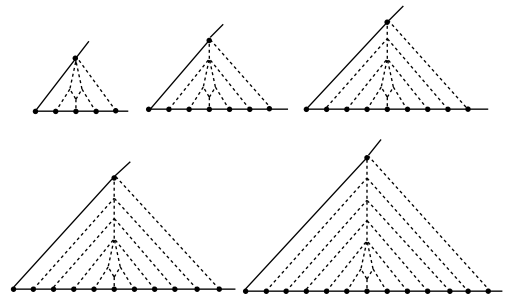

Figure 8: The five smallest cut components of the second class. The definition of the whole (infinite) class might be obvious.

partition all vertices in V^*. Intuitively, this means that the internal faces of G are partitioned and covered by closed, but openly disjoint, connected areas.

Lemma A.4 [**MW96**] *For $P \notin \{(2,1,1),\ (4,2,2),\ (4,3,3),\ (3,3,3,3)\}$, there is a conformal decomposition G with minimum $q(G)$ such that there is an area decomposition of G with the following properties:*

1. *The area decomposition consists of at most four area components.*
2. *All area components except one are isomorphic to one of the components depicted in Figs. 7 and 8. These area components are henceforth called the* cut components.
3. *The remaining area component is outerplanar. This area component is henceforth called the* core component.
4. *No two cut components share an edge.*
5. *All cut components except at most one are of type (c), (d), or (e) in Fig. 7.*
6. *If a cut component of type (a) or (b) in Fig. 7 or a cut component in Fig. 8 occurs, the core component admits a path decomposition.*
7. *If a cut component of the type in Fig. 8 occurs, the area decomposition contains at most two cut components; if a cut component of type (a) or (b) in Fig. 7 occurs, this is the only cut component.*

B Detailed Proof of Lemma 3.1

Lemma 3.1 is easy to see for the five exceptional cases. Hence, we have to show

$$\min(P_X) \leq 2 \cdot \mathrm{Min}(P) + |X| - 2. \tag{3}$$

for all other situations. The proof is divided into several steps. We first use the characterizations from Lemma A.1 and prove in Lemma B.2 and B.3 the correctness of Lemma 3.1 for the case that an minimal decomposition of the polygon admits a perfect decomposition such that G^* is a path or a subdivision of $K_{1,3}$, respectively. Finally, we treat the non-perfect case in Lemma B.4.

Assumption B.1 *Let $Y : E_P \to \mathbb{N}_0$ be optimal, that is,* $\mathrm{Min}(P) = \min(P_Y)$.

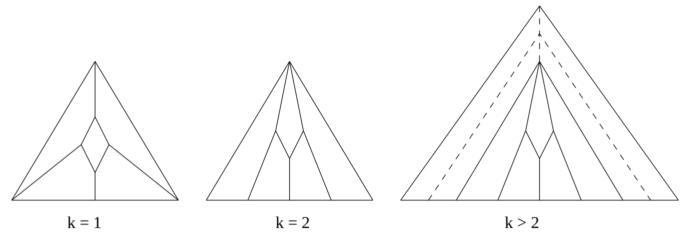

Figure 9: Illustration of optimal conformal decompositions of $(2k, 1, 1)$

Lemma B.2 *Lemma 3.1 is true for P and X if P_Y admits a perfect decomposition such that G^* is a path.*

Proof: Let G' be such a decomposition of P_Y, and let e_1 and e_2 be the leaves of the corresponding variant G^* on the dual graph where the vertex corresponding to the outer face is removed. Clearly, e_1 and e_2 are trivial segments of P, too, but not necessarily of P_X.

For P, let $I_1 := I[e_1, e_2]$ and $I_2 = I[e_2, e_1]$. Let I'_1 and I'_2 denote the corresponding intervals of P_Y and I''_1 and I''_2 the corresponding intervals of P_X. Let $|Y| := \sum_{e \in E_P} Y(e)$. As G' is perfect, we have

$$q(G') = \frac{L(I'_1) + L(I'_2)}{2} = \frac{L(I_1) + L(I_2) + |Y|}{2}. \tag{4}$$

The decomposition we construct for P_X is denoted by G''. To prove Ineq. (3), it suffices to show

$$q(G'') \leq L(I_1) + L(I_2) + |Y| + |X| - 2. \tag{5}$$

W.l.o.g., we have $L(I''_1) \geq L(I''_2)$. Let

$$\delta := \min\left\{K(I''_1), \left\lceil \frac{1}{2}[L(I''_1) - L(I''_2)] \right\rceil \right\}.$$

Fig. 10 shows the different cases for G'' provided $\delta = 0$ (explanations below).

For $\delta > 0$, we modify the procedure as follows. Let $\mathcal{K}$ be an arbitrary set of internal corners of I''_1 such that $|\mathcal{K}| = \delta$ and no two vertices in $\mathcal{K}$ are neighbored on P_X. Then we construct P'_X from P_X by shrinking each edge that is incident to a vertex in $\mathcal{K}$; in other words, each vertex in $\mathcal{K}$ is identified with its two neighbors. Each such "supervertex" is treated as a non–corner, so that it is incident to an internal edge in any conformal decomposition of P'_X.

Let $G''' = (V''', E''')$ be the conformal decomposition of P'_X according to Fig. 10. Then the decomposition G'' for P_X is constructed from G''' as follows: Let $v \in \mathcal{K}$ and let v_1 and v_2 be the neighbors of v on I''_1. Then we choose an arbitrary internal edge $\{v, w\} \in E'''$ and replace it by $\{v_1, w\}$ and $\{v_2, w\}$. This yields δ additional quadrilaterals.

Let $X_1 := X \cap I_1$, $X_2 := X \cap I_2$ and $\bar{X} := X \cap \{e_1, e_2\}$. Then we have $X = X_1 \cup X_2 \cup \bar{X}$ and $0 \leq |\bar{X}| \leq 2$. Now we are going to consider the individual cases in Fig. 10.

Case I: $(L - 2K)(I''_1) - L(I''_2) \leq 0$.

The following equation is easy to see for $|\bar{X}| = 0, 1, 2$:

$$q(G'') = \frac{L(I''_1) + L(I''_2) + |\bar{X}|}{2} + |\bar{X}|. \tag{6}$$

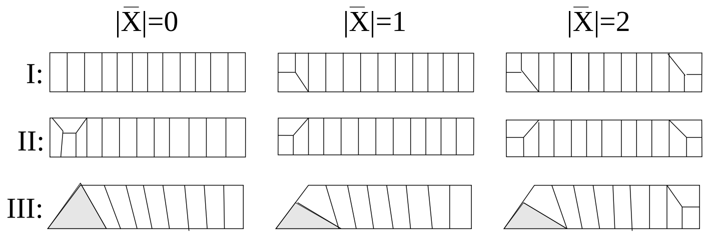

Figure 10: G'' in the different cases in the proof of Lemma 3.1 for $\delta = 0$. In each case, I_1'' is the vertical line below. The grey triangles in Case III indicate decompositions according to Fig. 9, respectively.

Therefore, we have to show

$$\frac{L(I_1'') + L(I_2'') + |\bar{X}|}{2} + |\bar{X}| \leq \; L(I_1') + L(I_2') + |X| - 2 \; . \tag{7}$$

However, since $L(I_1'')+L(I_2'')+|\bar{X}| = L(I_1)+L(I_2)+|X|$ and $L(I_1')+L(I_2') = L(I_1)+L(I_2)+|Y|$, this is fulfilled whenever

$$|\bar{X}| + 4 \leq L(I_1) + L(I_2) + |X_1| + |X_2| + 2|Y| \, . \tag{8}$$

We next consider all cases for which Ineq. (8) is not immediate. (In particular, this means $L(I_1) + L(I_2) < 6$.)

The case $L(I_1) + L(I_2) = 2$ (i.e., $P = (1, 1, 1, 1)$) is easily checked "by hand." (Note that this includes the first and second exceptions of Lemma 3.1). Next consider the case $L(I_1) + L(I_2) = 3$, that is, $P \in \{(2, 1, 1, 1), (1, 1, 1, 1, 1)\}$. Then we have $|Y| = 1$, so Ineq. (8) reduces to $|\bar{X}| + 2 \leq L(I_1) + L(I_2) + |X_1| + |X_2|$. This is fulfilled if $|\bar{X}| < 2$. However, $|X|$ is odd. Hence, $|\bar{X}| = 2$ implies $|X_1| + |X_2| > 0$, and Ineq. (8) is fulfilled again.

Now assume $L(I_1) + L(I_2) = 4$. Then Ineq. (8) is true unless $|X_1| = |X_2| = |Y| = 0$, because $|\bar{X}| \leq 2$, and $|\bar{X}|$ has the same parity as $|X_1| + |X_2|$. If $L(I_1) = 3$ and $L(I_2) = 1$ or vice versa, the dual path G^* does not end with e_1 and e_2. However, the case $L(I_1) = L(I_2) = 2$ and $|X_1| = |X_2| = |Y| = 0$ is easily checked by hand again.

Finally assume $L(I_1) + L(I_2) = 5$. Then $|Y|$ is odd, and Ineq. (8) is fulfilled.

Case II: $1 \leq (L - 2K)(I_1'') - L(I_2'') \leq 2$.

Let $\Delta := (L - 2K)(I_1'') - L(I_2'')$. Now we easily obtain for $|\bar{X}| = 0, 1, 2$:

$$q(G'') = \frac{L(I_1'') + L(I_2'') + |\bar{X}|}{2} + \Delta \, , \tag{9}$$

Therefore, Ineq. (5) is fulfilled whenever

$$\Delta + 2 \leq \left(\frac{L(I_1)}{2} + \frac{L(I_2)}{2} + \frac{|Y|}{2} \right) + \frac{|Y|}{2} + \frac{|X|}{2} \, . \tag{10}$$

Recall that $q(G') = [L(I_1) + L(I_2) + |Y|]/2$. As $\Delta \leq 2$ in Case II, Ineq. (10) is fulfilled whenever

$$q(G') \geq 4 \, . \tag{11}$$

If one of $|X|$ or $|Y|$ is strictly positive, we have $(|X| + |Y|)/2 \geq 1$, because $|X| = 1$ implies $|Y| > 0$ and vice versa. Hence, Ineq. (11) can be strengthened to $q(G') \geq 3$ except for the trivial case $|X| = |Y| = 0$.

Obviously, $q(G') = 1$ is impossible in Case II. The remaining case $q(G') = 2$ is easily checked by hand. (Note that this includes the last three exceptions of Lemma 3.1.)

Case III: $(L - 2K)(I_1'') - L(I_2'') \geq 3$ and $P \neq (2,1,1,1,1)$.

In the third row of Fig. 10, the number of quadrilaterals in the white area is $L(I_2'')$ for $|\bar{X}| \leq 1$ and $L(I_2'')+2$ for $|\bar{X}| = 2$. On the other hand, the grey area is decomposed according to Fig. 9. Like in Fig. 9, let $2k$ denote the number of horizontal edges below the grey area. Then we have $2k = L(I_1'') - L(I_2'')$, if $|\bar{X}|$ is even, and $2k = L(I_1'') - L(I_2'') + 1$, if $|\bar{X}|$ is odd. So the number of quadrilaterals in the grey area is $L(I_1'') - L(I_2'') + 1$ and $L(I_1'') - L(I_2'') + 2$, respectively. Recall that $\delta = 0$ is assumed in Fig. 10. If $\delta > 0$, restoring the shrunken edges yields $K(I_1'')$ additional quadrilaterals, but now we have $2k = L(I_1'') - L(I_2'') - 2K(I_1'')$ and $2k = L(I_1'') - L(I_2'') - 2K(I_1'') + 1$, respectively. In any case, we obtain for $|\bar{X}| = 0,1,2$:

$$q(G'') = L(I_1'') - K(I_1'') + |\bar{X}| + 1\,. \tag{12}$$

Hence, Ineq. (5) is fulfilled whenever

$$3 \leq L(I_2) + |Y| + |X_2| + K(I_1'')\,. \tag{13}$$

So assume $L(I_2) + |Y| + |X_2| + K(I_1'') \leq 2$ in the remainder. Note that $K(I_1'') \geq K(I_1)$. Hence, we have $K(I_1) \leq 1$, because otherwise we had $L(I_2) = 0$, and the dual path G^* would not point to e_1 and e_2. From the proof of Lemma A.1, it is easy to see that $(L - 2K)(I_1) \leq L(I_2) + |Y|$. Therefore, $K(I_1) = 0$ implies $L(I_1) \leq 2$, which is impossible in Case III.

So consider the case $K(I_1) = 1$. Then we also have $L(I_2) + |Y| = 1$, that is, $L(I_2) = 1$ and $|Y| = 0$. Therefore, $L(I_1)$ is odd. Since $L(I_1) = 1$ is impossible in Case III, this means $L(I_1) = 3$. However, this means $P = (2,1,1,1,1)$, and again G^* would not point to e_1 and e_2. □

For Lemmas B.3 and B.4, we need some more terminology. Note that constructing P_Y from P may be seen as replacing each edge $e \in E_P$ by a segment $S(e)$ of length $Y(e) + 1$. We extend X from P to P_Y as follows: For $e \in E_P$, we choose an arbitrary $e' \in S(e)$ and define $X(e') := X(e)$. For all other $e' \in S(e)$, we set $X(e') = 0$.

Lemma B.3 *Lemma 3.1 is true if P_Y is perfect.*

Proof: Because of Lemmas A.1, A.3, and B.2, it suffices to consider the case that P_Y admits a perfect decomposition G such that G^* is a subdivision of $K_{1,3}$. Let v^* be the branching vertex of G^*, let e_1^*, e_2^*, and let e_3^* be the edges of G^* incident to v^*. Note that removing one of the edges e_1^*, e_2^*, and e_3^* decomposes G^* into two paths. The basic idea of the proof is to apply Lemma B.2 to both paths.

For $j = 1,2,3$, let e_j be the primal edge of G corresponding to e_j. The primal operation corresponding to the removal of e_j^* is cutting G along e_j and inserting e_j in both connected components. Moreover, let X_j be the subset of edges in X that are incident to quadrilaterals in the jth branch. Next we apply a case distinction.

Case I: at least one $|X_j|$ is even.

Let G' and G'' denote the subgraphs of G resulting from cutting the jth branch as described above. In particular, let G' correspond to this branch and G'' to the rest. Let P' and P'' denote the outer faces of G' and G'', respectively. It is easy to see that $\min(P_Y) = \min(P') + \min(P'') = \mathrm{Min}(P') + \mathrm{Min}(P'')$. By Lemma B.2, we further have $\min(P'_{X_j}) \leq 2 \cdot \mathrm{Min}(P') + |X_j| - 1$ and $\min(P''_{X \setminus X_j}) \leq 2 \cdot \mathrm{Min}(P'') + |X \setminus X_j| - 1$. In summary, we obtain $\min(P_X) \leq \min(P'_X) + \min(P''_X) \leq 2 \cdot \mathrm{Min}(P') + 2 \cdot \mathrm{Min}(P'') + |X| - 2 = 2 \cdot \mathrm{Min}(P) + |X| - 2$.

Case II: all $|X_j|$ are odd.

Now we choose $j \in \{1, 2, 3\}$ arbitrarily and construct G', G'', P'_Y, and P'' by cutting the jth branch as described above. Let $X' := X_j \cup \{v_j, w_j\}$ and $X'' := X \setminus X_j \cup \{v_j, w_j\}$. From Lemma B.2, we conclude $\min(P''_{X''}) \leq 2 \cdot \mathrm{Min}(P'') + |X''| - 2$, because the trivial segment $\{v_j, w_j\}$ of P'' belongs to X'' and hence none of the exceptions in Lemma 3.1 applies to P'' and X'' (not even the second one, since obviously $P'' \neq (1,1,1,1)$). Analogously, the second exception in Lemma 3.1 is the only one that may apply to P' and X'. Hence, if the second exception does not apply either, we further conclude $\min(P'_{X'}) \leq 2 \cdot \mathrm{Min}(P') + |X'| - 2$ from Lemma B.2, which gives $\min(P_X) \leq 2 \cdot \mathrm{Min}(P) + |X'| + |X''| - 4 = 2 \cdot \mathrm{Min}(P) + |X| - 2$.

Now assume that the second exception of Lemma 3.1 does apply. Then we can only conclude $\min(P_X) \leq 2 \cdot \mathrm{Min}(P) + |X| - 1$ at this point of the argumentation. Let w_1^* be the (unique) quadrilateral in the jth branch, and for some optimal decomposition of P'', let $w_2^*, \ldots, w_i^*$ be the quadrilaterals incident to the vertex between the two copies of e_j in P''. Let P''' be the polygon comprising $w_1^*, \ldots, w_i^*$. It is easy to see that the current decomposition of P''' can be replaced by another decomposition such that at least one quadrilateral is saved by that. This proves the claim for Case II, too. □

Lemma B.4 *Lemma 3.1 is true if P_Y is not perfect.*

Proof: The cases $P \in \{(2,1,1),\ (4,2,2),\ (4,3,3),\ (3,3,3,3)\}$ can be checked by hand. For all other cases, we can apply Lemma A.4. Again let G' be an optimal conformal decomposition of P_Y, and G'' an optimal conformal decomposition of P_X. In other words, we have $q(G') = \min(P_Y) = \mathrm{Min}(P)$ and $q(G'') = \min(P_X)$. Let $i(G')$ and $i(G'')$ denote the number of internal vertices of G' and G'', respectively, and let $|Y| := \sum_{e \in E_P} [Y(e) - 1]$. Hence, $|Y|$ is the number of additional vertices of P_Y resulting from subdividing each edge of E_P. The relation between the number of quadrilaterals and internal vertices writes now

$$q(G') = \frac{|E_P| + |Y|}{2} - 1 + i(G') \tag{14}$$

and

$$q(G'') = \frac{|E_P| + |X|}{2} - 1 + i(G'')\ . \tag{15}$$

Hence, Ineq. (3) may be rewritten as

$$i(G'') \leq \frac{|E_P| + |X|}{2} + |Y| + 2 \cdot i(G') - 3\ . \tag{16}$$

Let C be an internal component of G''. Then let $i(C)$ denote the number of internal vertices in C, and let $e(C)$ denote the number of edges in E_{P_X} cut off by C. Let $\delta(C) := i(C) - e(C)/2$. The concrete values are given in the following table.

	$i(C)$	$e(C)$	$\delta(C)$
Fig. 7(a)	2	1	3/2
Fig. 7(b)	1	2	0
Fig. 7(c)	1	4	−1
Fig. 7(d)	2	4	0
Fig. 7(e)	2	5	−1/2
Fig. 8	—	$2 \cdot (i(C) - 1)$	1

Let $C_1, \ldots, C_k$ be all internal components of G'', and let $E'_{P_X} \subseteq E_{P_X}$ denote the set of all edges not cut off by any internal component. Clearly, we have

$$\sum_{i=1}^{k} \delta(C_i) - \frac{|E'_{P_X}|}{2} = i(G'') - \frac{|E_P| + |X|}{2} \quad . \tag{17}$$

Next we apply a case distinction.

Case I: There is an internal component of type Fig. 7(a) in G''.

By Lemma A.4(5), this is the only internal component of G''. Hence, we have $\sum \delta(C_i) = 3/2$. Since, obviously, $|E_{P_X}| = |E_P| + |X| \geq 6$, and since $i(G') \geq 1$ in Lemma B.4, Ineq. (16) is immediate.

Case II: complement to Case I.

Lemma A.4(4) implies $\delta(C_i) \leq 0$ for all but at most one component C_i. Since type Fig. 7(a) does not occur in G', we have $\sum_{i=1}^{k} \delta(C_i) \leq 1$. Putting this together with Eq. (17) we obtain

$$i(G'') - \frac{|E_P| + |X|}{2} \leq \sum_{i=1}^{k} \delta(C_i) \leq 1 \quad . \tag{18}$$

Hence, Ineq. (16) is trivially fulfilled unless $i(G') = 1$ and $|Y| < 2$. (Note that the precondition of Lemma B.4 means $i(G') > 0$.) So assume $i(G') = 1$ and $|Y| < 2$. We apply yet another case distinction.

Case II(a): $|Y| = 1$.

Then Ineq. (16) reduces to

$$i(G'') \leq \frac{|E_P| + |X|}{2} \quad . \tag{19}$$

In Case II, no internal component of type Fig. 7(a) occurs in G''. If no internal component of the type in Fig. 8 occurs either, Ineq. (19) follows immediately from Ineq. (18), because then we have $\sum \delta(C_i) \leq 0$. So assume that G'' contains an internal component C of the type in Fig. 8.

If C is the only internal component of G'', we have $|E'_{P_X}| \geq 2$, and hence Eq. (17) gives Ineq. (19). So assume for the remainder of Case II(a) that there is another internal component C'. By Lemma A.4(4), C' is the only further internal component, and C' is of type (c), (d), or (e) in Fig. 7. If C' is of type (c), we have $\delta(C') = -1$, and Ineq. (19) follows again. On the other hand, if C' is of type (e), we have $\delta(C') = -1/2$. However, in this case $|E'_{P_X}|$ is odd. In particular, $|E'_{P_X}|$ is positive, which again implies Ineq. (19).

Finally, if C' is of type (d), Ineq. (19) is fulfilled unless $|E'_{P_X}| \leq 1$. Since $|E'_{P_X}|$ is even in this case, this means $|E'_{P_X}| = 0$. As G'' is optimal, G'' arises from these two internal components by gluing them together. In other words, the two edges that separate C from the rest are identified with the two edges that separate C' from the rest. It is easy to see that Ineq. (19) is fulfilled in this special case, too (as an equality in fact).

Case II(b): $|Y| = 0$.

Since $i(G') = 1$, G' contains exactly one internal component C, and this component is either of type (b) or of type (c) in Fig. 7.

We decompose P into two polygons, P' and P''. The polygon P' embraces the quadrilaterals of G' cut off by C. Hence, we have $P' = (1,1,1,1)$ if C is of type (b), and $L(P') = 6$ if C is of type (c). More precisely, in the latter case we say that the internal vertex is not strongly convex relative to P', which means $P' = (2,1,2,1)$. The polygon P'' embraces all other quadrilaterals of G'', and if C is of type (c), the internal vertex is strongly convex relative to P''.

Next we construct two further polygons, P'_X and P''_X, from P' and P'', respectively. If $|X \cap E_{P'}|$ is even, P'_X and P''_X are constructed from P' and P'' by subdividing once all edges e with $X(e) = 1$. Conformal decompositions for P'_X and for P''_X together form a conformal decomposition for P_X. Obviously, P' is perfect, and since C is the only internal component of G', P'' is perfect, too. Hence, Lemma B.3 yields $\min(P'_X) \leq 2\mathrm{Min}(P') + |X \cap E_{P'}| - 1$ and $\min(P''_X) \leq 2\mathrm{Min}(P'') + |X \cap E_{P''}| - 1$. Since G'' is optimal for P_X, optimal conformal decompositions for P' and P'' together form an optimal conformal decomposition for $P = P_Y$. Therefore, these two inequalities may be combined into Ineq. (3).

Now assume that $|X \cap E_{P'}|$ is odd. Then P'_X and P''_X are constructed from P' and P'' slightly differently, namely exactly one edge shared by P' and P'' is once subdivided, too. If C is of type (b), this is the edge of P' opposite to the edge e with $X(e) = 1$; if C is of type (c), the subdivided edge is arbitrary. Now it is easy to see that neither P'_X nor P''_X is one of the five exceptional cases stated in Lemma 3.1, regardless whether C is of type (b) or (c). Therefore, we obtain $\min(P'_X) \leq 2\mathrm{Min}(P') + (|X \cap E_{P'}| + 1) - 2$ and $\min(P''_X) \leq 2\mathrm{Min}(P'') + (|X \cap E_{P''}| + 1) - 2$. Again this yields Lemma 3. □