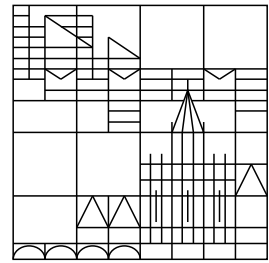


Universität Konstanz



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Konstanzer Schriften in Mathematik und Informatik

Nr. 151, Juli 2001

ISSN 1430-3558

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WWW: <http://www.informatik.uni-konstanz.de/Schriften/>

ORBIT SPACES OF SMALL TORI

ANNETTE A'CAMPO-NEUEN AND JÜRGEN HAUSEN

ABSTRACT. Consider an algebraic torus of small dimension acting on an open subset of \mathbb{C}^n , or more generally on a quas affine variety such that a separated orbit space exists. We discuss under which conditions this orbit space is quasiprojective. One of our counterexamples provides a toric variety with enough effective invariant Cartier divisors that is not embeddable into a smooth toric variety.

INTRODUCTION

Suppose a reductive group G acts on a quasiprojective variety X with a geometric quotient $X \rightarrow X/G$. The question when the quotient space X/G is again quasiprojective is studied by several authors, see e.g. [5] and the classical counterexample [4]. In the present note, we complement known partial answers for actions of low-dimensional tori on quas affine varieties.

A case of particular interest are diagonal actions of tori T on \mathbb{C}^n . Any maximal open subset $X \subset \mathbb{C}^n$ admitting a geometric quotient $X \rightarrow X/T$ is in fact invariant under the big torus $\mathbb{T}^n := (\mathbb{C}^*)^n$ and the orbit space X/T is a simplicial toric variety with at most $n - \dim(T)$ invariant prime divisors. We obtain:

Theorem 1. *Let T be a torus acting diagonally on \mathbb{C}^n , and let $X \subset \mathbb{C}^n$ be a \mathbb{T}^n -invariant subset admitting a geometric quotient $X \rightarrow X/T$. The following table indicates when X/T is quasiprojective:*

	<i>X/T complete, T acts freely</i>	<i>X/T complete</i>	<i>X/T arbitrary</i>
$\dim(T) = 1$	+	+	+, [5, Example 5A]
$\dim(T) = 2$	+, [14, Theorem 2]	+, <i>Proposition 2.1</i>	-, [5, Example 5B]
$\dim(T) = 3$	+, [15, Theorem 1]	-, <i>Proposition 1.2</i>	-
$\dim(T) = 4$	-, [18, Prop. 9.4]	-	-

Here the top row and the left column specify the assumptions on the action, “+” stands for “the quotient space X/T is quasiprojective”, and “-” means that X/T is not necessarily quasiprojective.

Let us turn to torus actions on arbitrary quas affine varieties X . If the torus T is onedimensional and X/T is complete, then the action defines a w.l.o.g. positive grading of the algebra $\mathcal{O}(X)$ and the quotient space is nothing but $\text{Proj}(\mathcal{O}(X))$. As soon as we drop either of these two assumptions, the orbit space is in general no longer quasiprojective, see Propositions 4.3 and 2.4:

1991 *Mathematics Subject Classification.* 14L30, 14L24, 14M25, 14C20, 14E25.

Theorem 2. i) *There is an action of a 2-dimensional torus T on a 5-dimensional quasiaffine toric variety X with a complete nonprojective orbit space X/T .*
 ii) *There is an action of a 1-dimensional torus T on a 5-dimensional quasiaffine toric variety X with a nonquasiprojective orbit space X/T .*

The orbit space X/T we construct to prove part ii) serves also as a subtle example in the context of embeddings into toric varieties, compare [22] and [12]: We ask, which varieties can be embedded into smooth toric varieties. Note that embeddability into a smooth toric variety requires existence of “many” Cartier divisors.

T. Kajiwara [13] says that a toric variety has *enough effective invariant Cartier divisors*, if the complements of these divisors provide an affine cover. By [12], such a toric variety always admits an embedding into a smooth toric prevariety with affine diagonal, even by means of a toric morphism. However, we have:

Theorem 3. *There exists a toric with enough effective invariant Cartier divisors that admits no embedding into a separated smooth toric variety.*

This note is organized as follows: In Section 1 we consider actions of 3-dimensional tori, and Section 2 is devoted to 2-dimensional torus actions. Finally, in Sections 3 and 4, we construct the example of Theorem 2 ii) and Theorem 3.

1. QUOTIENTS OF THREEDIMENSIONAL TORUS ACTIONS

Throughout the whole note, we work over the field \mathbb{C} of complex numbers. First we recall the definition of a geometric quotient. Consider an algebraic torus action $T \times X \rightarrow X$ on a complex algebraic variety X .

A *good quotient* for this T -action is an affine T -invariant regular map $p: X \rightarrow Y$ onto a variety Y such that the canonical map $\mathcal{O}_Y \rightarrow p_*(\mathcal{O}_X)^T$ is an isomorphism. A good quotient $p: X \rightarrow Y$ of the T -action is called *geometric* if the fibres $p^{-1}(y)$, $y \in Y$, are precisely the T -orbits of X .

Given such a geometric quotient $X \rightarrow Y$, the variety Y is called the *orbit space* and is denoted by X/T . Note that by [20, Corollary 3] and [17, Propositions 0.7 and 0.8], an effective algebraic torus action admits a geometric quotient if and only if it is proper.

In this section, we consider the standard action of $\mathbb{T}^6 = (\mathbb{C}^*)^6$ on \mathbb{C}^6 and present a 3-dimensional subtorus $T \subset \mathbb{T}^6$ and an open \mathbb{T}^6 -invariant subset $X \subset \mathbb{C}^n$ with a complete but nonprojective orbit space X/T .

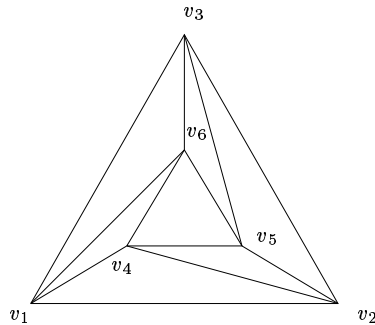
We shall use some basic notions of the theory of toric varieties; standard references are [9] and [18]. All we need is the following variant of a well-known example [18, Proposition 9.4]:

Example 1.1. Let e_1, e_2 and e_3 denote the canonical basis vectors of the lattice \mathbb{Z}^3 . Consider the vectors

$$\begin{aligned} v_1 &:= (-1, 0, 0), & v_4 &:= (0, 1, 1), \\ v_2 &:= (0, -1, 0), & v_5 &:= (1, 0, 1), \\ v_3 &:= (0, 0, -1), & v_6 &:= (1, 1, 0). \end{aligned}$$

Let Δ be the fan in \mathbb{Z}^3 with eight maximal cones, namely $\sigma_{i,j,k} := \text{cone}(v_i, v_j, v_k)$, where the triple (i, j, k) runs through the list

$$(1, 4, 6), (1, 3, 6), (3, 5, 6), (2, 3, 5), (2, 4, 5), (1, 2, 4), (1, 2, 3), (4, 5, 6).$$



Then the toric variety X corresponding to the fan Δ is simplicial and complete, but not projective.

Proposition 1.2. *There exists a threedimensional subtorus $T \subset \mathbb{T}^6$ and an open \mathbb{T}^6 -invariant subset $X \subset \mathbb{C}^6$ admitting a geometric quotient with a nonprojective complete orbit space X/T .*

Proof. Using Cox's Construction [7, Theorem 2.1], we represent the toric variety X of Example 1.1 as a geometric quotient of an open toric subvariety $\tilde{X} \subset \mathbb{C}^6$: Define a lattice homomorphism $Q: \mathbb{Z}^6 \rightarrow \mathbb{Z}^3$ by $Q(e_i) := v_i$, and for $\sigma \in \tilde{\Delta}$ consider

$$\tilde{\sigma} := \text{cone}(e_i; v_i \in \sigma).$$

These cones form a fan $\tilde{\Delta}$ in \mathbb{Z}^6 . The associated toric variety \tilde{X} is an open toric subvariety of \mathbb{C}^6 and the toric morphism $\tilde{X} \rightarrow X$ defined by $Q: \mathbb{Z}^6 \rightarrow \mathbb{Z}^3$ is a geometric quotient for the action of the subtorus $T \subset \mathbb{T}^6$ defined by the sublattice $\ker(Q) \subset \mathbb{Z}^6$. \square

2. QUOTIENTS FOR TWODIMENSIONAL TORUS ACTIONS

In this section we prove the statements on actions of twodimensional tori made in Theorems 1 and 2. The first result settles the case of actions on open subsets of \mathbb{C}^n . As before, let $\mathbb{T}^n := (\mathbb{C}^*)^n$, and endow \mathbb{C}^n with the standard \mathbb{T}^n -action.

Proposition 2.1. *Let $T \subset \mathbb{T}^n$ be a subtorus of dimension two, and suppose that $X \subset \mathbb{C}^n$ is a \mathbb{T}^n -invariant open subset with a geometric quotient $X \rightarrow X/T$. If X/T is complete, then X/T is projective.*

Note that in the setting of this proposition, the orbit space X/T is a complete simplicial toric variety of dimension $n - 2$ with at most n invariant prime divisors. Thus the statement is an immediate consequence of the following result:

Proposition 2.2. *Let N be an n -dimensional lattice, and let Δ be a complete simplicial fan in N having at most $n + 2$ rays. Then Δ is strongly polytopal.*

Recall that P. Kleinschmidt proved this for the case that Δ is a regular fan, see [14]. As in [14], we shall use the following projectivity criterion in terms of Gale transforms, first stated by Shephard and later employed by Ewald, compare [8] and [19]:

Suppose that R denotes a list of generators of the d rays of a complete fan Δ in \mathbb{Z}^n . A linear Gale transform \overline{R} of R consists of the columns of a matrix whose rows form a basis for the linear relations of R . The coface of a cone $\sigma \in \Delta$ generated by v_{i_1}, \dots, v_{i_r} is

$$\overline{\sigma} := \text{cone}(\overline{R} \setminus \{\overline{v_{i_1}}, \dots, \overline{v_{i_r}}\}) \subset \mathbb{R}^{d-n}.$$

Lemma 2.3. *The fan Δ is strongly polytopal if and only if the intersection of the relative interiors of all $\bar{\sigma}$, $\sigma \in \Delta^{\max}$, is nonempty.*

Proof of Proposition 2.2. If Δ has $n+1$ rays, then the primitive vectors in the rays generate a simplex and there is nothing to show. So let us assume that Δ has $n+2$ rays. First we reduce to the case that there is a regular maximal cone $\sigma_0 \in \Delta$:

Fix any maximal cone σ_0 of Δ and let $N_0 \subset N$ be the sublattice spanned by the primitive generators of σ_0 . The cones of Δ also form a fan Δ_0 in N_0 . Clearly, if Δ_0 is strongly polytopal, then so is Δ . Thus, replacing Δ with Δ_0 , we may assume $\sigma_0 \in \Delta$ is regular.

Choosing the primitive generators of σ_0 as a basis, we may assume that $N = \mathbb{Z}^n$ and that σ_0 is generated by the canonical base vectors e_1, \dots, e_n . We denote the primitive vectors of the remaining two rays of Δ by u and v . By the combinatorial classification of spherical complexes, see [16] and [10], the list

$$R := (e_1, \dots, e_n, u, v)$$

can be partitioned into two complementary subsets U and V with $u \in U$, $v \in V$ and $|U|, |V| \geq 2$ such that

$$\Delta^{\max} = \{\text{cone}(R \setminus \{w, z\}); w \in U, z \in V\}.$$

After renumbering, we may assume that U equals $\{e_1, \dots, e_r, u\}$ and V equals $\{e_{r+1}, \dots, e_n, v\}$ for some $1 \leq r < n$. Then, in addition to σ_0 , we have the following three types of maximal cones in Δ :

$$\begin{aligned} \sigma_i &:= \text{cone}(u, e_k; k \neq i) && \text{for } 1 \leq i \leq r, \\ \sigma_j &:= \text{cone}(v, e_k; k \neq j) && \text{for } r < j \leq n, \\ \sigma_{ij} &:= \text{cone}(u, v, e_k; k \neq i, j) && \text{for } 1 \leq i \leq r \text{ and } r < j \leq n. \end{aligned}$$

Note that the linear form e_i^* separates the pair σ_0, σ_i , i.e. the two cones lie on different sides of the hyperplane defined by e_i^* . Similarly, e_j separates σ_0, σ_j . Moreover, σ_i and σ_{ij} are separated by $u_i e_j^* - u_j e_i^*$. Applying these linear forms to the generators of the respective cones successively yields

$$\begin{aligned} u_i &< 0 && \text{for } 1 \leq i \leq r, \\ v_j &< 0 && \text{for } r < j \leq n, \\ u_i v_j - u_j v_i &> 0 && \text{for } 1 \leq i \leq r \text{ and } r < j \leq n. \end{aligned}$$

We shall combine these inequalities with the above mentioned projectivity criterion. Consider the following linear Gale transform of R :

$$\bar{R} = ((u_1, v_1), \dots, (u_n, v_n), (-1, 0), (0, -1)).$$

Then the cofaces associated to the maximal cones of Δ are given by

$$\begin{aligned} \bar{\sigma}_0 &:= \text{cone}((-1, 0), (0, -1)), \\ \bar{\sigma}_i &:= \text{cone}((u_i, v_i), (0, -1)) && \text{for } 1 \leq i \leq r, \\ \bar{\sigma}_j &:= \text{cone}((-1, 0), (u_j, v_j)) && \text{for } j < r \leq n, \\ \bar{\sigma}_{ij} &:= \text{cone}((u_i, v_i), (u_j, v_j)) && \text{for } 1 \leq i \leq r \text{ and } r < j \leq n. \end{aligned}$$

The above inequalities imply that all these cones are twodimensional, and by Shephard's criterion we only have to show their intersection is again twodimensional. There are $1 \leq i_0 \leq r$ and $r < j_0 \leq n$ with

$$\bar{\sigma}_{i_0} = \bigcap_{i \leq r} \bar{\sigma}_i, \quad \bar{\sigma}_{j_0} = \bigcap_{j > r} \bar{\sigma}_j$$

and the intersection over all cones $\overline{\sigma_{i,j}}$ equals $\overline{\sigma_{i_0,j_0}}$. Consequently we obtain for the intersection of all cofaces of maximal cones:

$$\bigcap_{\sigma \in \Delta^{\max}} \overline{\sigma} = \text{cone}((-1, 0), (0, -1)) \cap \overline{\sigma_{i_0,j_0}}.$$

Using the above inequalities for i_0 and j_0 , we conclude that this cone is in fact of dimension two. \square

Now we turn to an arbitrary quasiaffine variety X . We show by means of an example that the orbit space X/T in general may not be quasiprojective:

Proposition 2.4. *There exists a five dimensional quasiaffine toric variety X with big torus T_X and a twodimensional subtorus $T \subset T_X$ acting with geometric quotient on X such that X/T is a complete nonprojective variety.*

Proof. We will represent the three-dimensional toric variety X introduced in Example 1.1 as an orbit space of the action of a twodimensional torus on a quasiaffine toric variety \widehat{X} . Consider the following vectors in \mathbb{Z}^5 :

$$\begin{aligned} w_1 &:= (1, 0, 0, 0, 0), & w_4 &:= (0, 0, 1, 0, 0), \\ w_2 &:= (0, 1, 0, 0, 0), & w_5 &:= (0, 0, 0, 1, 0), \\ w_3 &:= (-1, 0, 0, -1, 2), & w_6 &:= (0, -2, -1, 1, 1). \end{aligned}$$

Let $v_i \in \mathbb{Z}^3$ as in Example 1.1. Then there is a lattice homomorphism $P: \mathbb{Z}^5 \rightarrow \mathbb{Z}^3$ with $P(w_i) = v_i$ for all i , namely the homomorphism defined by the matrix

$$\begin{bmatrix} -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

One directly checks that the vectors w_i generate a strictly convex cone σ_0 in \mathbb{Z}^5 . Moreover, for every maximal cone $\sigma \in \Delta$, one obtains a face $\widehat{\sigma}$ of σ_0 by setting

$$\widehat{\sigma} := \text{cone}(w_i; v_i \in \sigma).$$

The fan $\widehat{\Delta}$ generated by these $\widehat{\sigma}$ corresponds to a quasiaffine toric variety \widehat{X} . Since $\dim \widehat{\sigma} = \dim \sigma$ for all $\sigma \in \Delta$, the toric morphism $p: \widehat{X} \rightarrow X$ defined by the lattice homomorphism $P: \mathbb{Z}^5 \rightarrow \mathbb{Z}^3$ is the desired geometric quotient, use e.g., [11, Theorem 5.1]. \square

3. WHEN IS A TORIC VARIETY k -DIVISORIAL?

In this section we give a criterion for k -divisoriality, a notion that comes up naturally in the context of toric embeddings. The criterion will be used in the following section for the construction of a toric variety with enough effective invariant Cartier divisors that cannot be embedded into a smooth toric variety.

As in [12], we call an irreducible variety X k -divisorial if any k -points x_1, \dots, x_k admit a common affine neighbourhood of the form $X \setminus \text{Supp}(D)$ with an effective Cartier divisor D on X . For $k = 1$ this gives back the usual notion of a divisorial variety, see e.g. [6], [3].

In the whole section we assume that our toric variety X is non degenerate, i.e., there exists no toric decomposition $X \cong X' \times \mathbb{K}^*$. Our criterion reads as follows:

Proposition 3.1. *A toric variety X is k -divisorial if and only if for any k closed orbits $B_1, \dots, B_k \subset X$ of the big torus $T_X \subset X$ there exist T_X -invariant effective Cartier divisors D_1, \dots, D_k on X such that*

- i) any two D_i, D_j are linearly equivalent to each other,
- ii) $X \setminus \text{Supp}(D_i)$ is an affine neighbourhood of B_i .

Note that for $k = 1$, this is just [2, Proposition 1.2] for toric varieties. Moreover, if we choose the number k in this proposition to be the number of all closed T_X -orbits, then we obtain the following well known result, compare [20], Lemma 8:

Corollary 3.2. *A toric variety X with at most k closed T_X -orbits is quasiprojective if and only if it is k -divisorial. \square*

The proof of Proposition 3.1 relies on a more explicit formulation of the result which we state below in Proposition 3.3. For this, we use Cox's construction [7, Theorem 2.1] to obtain an open toric subset \tilde{X} of some \mathbb{C}^n with complement $\mathbb{C}^n \setminus \tilde{X}$ of dimension at most $n - 2$ and a closed subgroup $H \subset \mathbb{T}^n$ such that there is good quotient

$$q: \tilde{X} \rightarrow \tilde{X} // H = X.$$

Note that this quotient map is a toric morphism. Suppose now that X is divisorial. Then Kajiwara's construction [13, Theorem 1.9] gives rise to a commutative diagram of toric morphisms

$$\begin{array}{ccc} \tilde{X} & \xrightarrow[\substack{\parallel H_1 \\ q_1}]{} & \hat{X} \\ & \searrow q & \swarrow q_2 \\ & & X \\ & \parallel H & \swarrow / H_2 \end{array}$$

where \hat{X} is a quasiaffine toric variety, H_1, H_2 are closed subgroups of the respective big tori $T_{\tilde{X}}$ and $T_{\hat{X}}$ and the map $q_2: \hat{X} \rightarrow X$ is a geometric quotient. In our criterion, we use the notion of a *distinguished point*, see [9, p. 28]. The distinguished points of \mathbb{C}^n are just the points having only coordinates 0 or 1.

Proposition 3.3. *The toric variety X is k -divisorial if and only if for any k distinguished points $\tilde{x}_1, \dots, \tilde{x}_k \in \tilde{X}$ with $\mathbb{T}^n \cdot \tilde{x}_i$ closed in \tilde{X} there exist monomials $f_1, \dots, f_k \in \mathbb{C}[z_1, \dots, z_n]$ such that*

- i) the polynomial $f := f_1 + \dots + f_k$ is H -homogeneous and H_1 -invariant,
- ii) $f_i(\tilde{x}_i) = 1$ and $f_i(\tilde{x}) = 0$ for every $\tilde{x} \in \overline{\mathbb{T}^n \cdot \tilde{x}_i} \setminus \mathbb{T}^n \cdot \tilde{x}_i \subset \mathbb{C}^n$.

Proof. Assume first that X is k -divisorial and let $\tilde{x}_1, \dots, \tilde{x}_k \in \tilde{X}$ be distinguished points with closed \mathbb{T}^n -orbit in \tilde{X} . Then there exists an effective Cartier divisor $D \in \text{CDiv}(X)$ such that $X \setminus \text{Supp}(D)$ is affine and contains the points $q(\tilde{x}_1), \dots, q(\tilde{x}_k)$.

We claim that the pullback $q_2^*(D) \in \text{CDiv}(\hat{X})$ is principal. To see this note first that D is of the form $E + \text{div}(h)$ with some T_X -invariant Cartier divisor $E \in \text{CDiv}(X)$ and some function $h \in \mathbb{C}(X)$, see e.g. [9, p. 63]. Thus we obtain

$$q_2^*(D) = q_2^*(E) + \text{div}(q_2^*(h)).$$

By [1, Proposition 2.6], we have $q_2^*(E) = \text{div}(\hat{g})$ with some character function \hat{g} of the big torus $T_{\hat{X}} \subset \hat{X}$. Setting $\hat{h} := \hat{g}q_2^*(h)$, we have $q_2^*(D) = \text{div}(\hat{h})$. Note that \hat{h} is H_2 -homogeneous and, since $q_2^*(D)$ is effective, \hat{h} is a regular function on \hat{X} . In particular, our claim is verified.

To proceed, consider $\tilde{h} := q_1^*(\hat{h})$. This is a regular function on \tilde{X} and hence it is a polynomial. Moreover, \tilde{h} is H_1 -invariant and H -homogeneous. Finally, on \tilde{X} , we

have $\operatorname{div}(\tilde{h}) = q^*(D)$. Since $q^{-1}(X \setminus \operatorname{Supp}(D))$ is affine and $\mathbb{C}^n \setminus \tilde{X}$ is small, this implies $\tilde{X}_{\tilde{h}} = \mathbb{C}_{\tilde{h}}^n$.

Now, consider one of the distinguished points \tilde{x}_i . By construction, we have $\tilde{h}(\tilde{x}_i) \neq 0$. Consequently, there appears a monomial f_i in \tilde{h} with $f_i(\tilde{x}_i) \neq 0$. Surely, also f_i is H_1 -invariant and it is H -homogeneous with respect to the same weight as \tilde{h} . Let

$$f_i(z_1, \dots, z_n) = z_1^{u_1} \dots z_n^{u_n}, \quad \tilde{x}_i = (\tilde{x}_{i1}, \dots, \tilde{x}_{in}).$$

Then clearly $\tilde{x}_{ij} = 0$ implies $u_j = 0$. We claim that also the converse is true. Indeed, suppose that $u_j = 0$ but $x_{ij} \neq 0$ holds for some j . Then, replacing in x_i the coordinate x_{ij} with zero yields a point

$$x'_i \in \mathbb{C}^n \setminus \tilde{X}.$$

with $f_i(x'_i) \neq 0$. This implies that the restriction of \tilde{h} to the orbit $\mathbb{T}^n \cdot x'_i$ is not the zero function. But this contradicts the fact that $\tilde{X}_{\tilde{h}}$ equals $\mathbb{C}_{\tilde{h}}^n$. So our claim is verified.

Now, take for each distinguished point \tilde{x}_i a monomial f_i as above. Then these monomials fulfill the desired conditions, and one implication of the proposition is proved.

For the reverse direction, suppose that conditions i) and ii) of the assertion hold. First we consider k distinguished points $x_1, \dots, x_k \in X$ with $T_X \cdot x_i$ closed in X . Choose distinguished point $\tilde{x}_1, \dots, \tilde{x}_k \in \tilde{X}$ such that $x_i = q(\tilde{x}_i)$ holds and $\mathbb{T}^n \cdot \tilde{x}_i$ is closed in \tilde{X} .

Let f_1, \dots, f_k be polynomials satisfying conditions i) and ii), and let $f := f_1 + \dots + f_k$. Since f is H_1 -invariant, it is of the form $f = q_2^*(\hat{h})$ for some $\hat{h} \in \mathcal{O}(\hat{X})$. It follows as in [2, Proof of Proposition 1.3] that there is an effective Cartier divisor D on X with

$$\operatorname{Supp}(D) = q_2(\operatorname{Supp}(\operatorname{div}(\hat{h}))) = q(\operatorname{Supp}(\operatorname{div}(f))).$$

By the properties of the f_i , we have $f(\tilde{x}_i) = 1$. Hence the above equation yields $x_1, \dots, x_k \in X \setminus \operatorname{Supp}(D)$. This settles the case of k distinguished points x_1, \dots, x_k with closed T_X -orbits.

If $x'_1, \dots, x'_k \in X$ are arbitrary, then we can choose distinguished points x_i in the closure of $T_X \cdot x'_i$ with $T_X \cdot x_i$ closed in X . The above consideration provides an effective Cartier divisor D on X such that $U := X \setminus \operatorname{Supp}(D)$ is an affine neighbourhood of x_1, \dots, x_k . Let $t \in T_X$ with $t \cdot x'_i \in U$. Then $t^{-1} \cdot D$ is the desired Cartier divisor. \square

Proof of Proposition 3.1. In the setting of Proposition 3.3, the monomials f_i correspond to Cartier divisors D_i on X and vice versa, see e.g. [1, Proposition 2.6]. The properties of the f_i translate directly to the desired properties of the D_i . Here the fact that the D_i are pairwise linearly equivalent, corresponds to the fact that all f_i are H -homogeneous with respect to the same weight. \square

4. A DIVISORIAL TORIC VARIETY THAT IS NOT 2-DIVISORIAL

In this section we present an example of a fourdimensional divisorial toric variety X that is not 2-divisorial. In order to define the fan of X , consider the following

lattice vectors in \mathbb{Z}^4 :

$$\begin{aligned} v_1 &:= (1, 0, 0, 0), & v_2 &:= (0, -2, 1, 0), & v_3 &:= (0, -1, 1, 1), \\ v_4 &:= (0, 0, 0, 1), & v_5 &:= (0, 1, 0, 2), & v_6 &:= (-1, -1, -1, 2), \\ v_7 &:= (1, -1, 0, -1), & v_8 &:= (1, 1, -1, 0), & v_9 &:= (0, 0, -2, 1). \end{aligned}$$

The toric variety X is defined by the fan Δ in \mathbb{Z}^4 having the following five cones as its maximal cones:

$$\begin{aligned} \sigma_1 &:= \text{cone}(v_2, v_3, v_4, v_5, v_6), & \sigma_2 &:= \text{cone}(v_1, v_2, v_3, v_7), & \sigma_3 &:= \text{cone}(v_4, v_5, v_8), \\ \sigma_4 &:= \text{cone}(v_1, v_5, v_8), & \sigma_5 &:= \text{cone}(v_1, v_7, v_9). \end{aligned}$$

Proposition 4.1. *The toric variety X defined by the fan Δ is divisorial but not 2-divisorial.*

Proof. In order to apply Proposition 3.3, we have to determine the quotient presentations of X due to Cox and Kajiwara. To obtain Cox's construction, consider the fan $\tilde{\Delta}$ in \mathbb{Z}^9 generated by the cones

$$\tilde{\sigma}_i := \text{cone}(e_j; v_j \in \sigma_i), \quad i = 1, \dots, 5.$$

The associated toric variety \tilde{X} is an open toric subvariety of \mathbb{C}^9 with 7-dimensional complement. Moreover, the lattice homomorphism $Q: \mathbb{Z}^9 \rightarrow \mathbb{Z}^4$ sending the canonical base vector e_i to v_i , induces Cox's quotient presentation $q: \tilde{X} \rightarrow X$.

To obtain Kajiwara's quotient presentation, we have to determine the invariant Cartier divisors of X . Let D_i denote the invariant Weil divisor on X corresponding to the ray through v_i . An explicit calculation shows that

$$D_6, D_8, D_9, D_1 + D_7, D_2 + D_3, D_4 + D_5, D_2 + D_4 + D_7$$

form a basis for the group of invariant Cartier divisors of X . Thus we have to consider the lattice homomorphism $Q_1: \mathbb{Z}^9 \rightarrow \mathbb{Z}^7$ given by the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

One directly checks that the vectors $Q_1(e_1), \dots, Q_1(e_9)$ generate the extremal rays of a strictly convex cone $\hat{\sigma} \subset \mathbb{Q}^7$ and that the cones

$$\hat{\sigma}_i := Q_1(\tilde{\sigma}_i), \quad i = 1, \dots, 5$$

are faces of $\hat{\sigma}$. Consequently, the fan $\hat{\Delta}$ in \mathbb{Z}^7 generated by $\hat{\sigma}_1, \dots, \hat{\sigma}_5$ defines a quasiprojective toric variety \hat{X} . Moreover, we have a commutative diagram of toric morphisms

$$\begin{array}{ccc} \tilde{X} & \xrightarrow[\text{\scriptsize } q_1]{\text{\scriptsize } // H_1} & \hat{X} \\ & \searrow \text{\scriptsize } q & \swarrow \text{\scriptsize } q_2 \\ & & X \\ & \swarrow \text{\scriptsize } // H & \searrow \text{\scriptsize } / H_2 \end{array}$$

where $q_1: \tilde{X} \rightarrow \hat{X}$ arises from the lattice homomorphism $Q_1: \mathbb{Z}^9 \rightarrow \mathbb{Z}^7$. In particular, X is divisorial and the toric morphism $q_2: \hat{X} \rightarrow X$ is Kajiwara's quotient presentation.

Now assume that X were even 2-divisorial. The distinguished points $\tilde{x}_3, \tilde{x}_5 \in \tilde{X}$ corresponding to the maximal cones $\tilde{\sigma}_3$ and $\tilde{\sigma}_5$ of $\tilde{\Delta}$ are given by

$$\tilde{x}_3 = (1, 1, 1, 0, 0, 1, 1, 0, 1), \quad \tilde{x}_5 = (0, 1, 1, 1, 1, 1, 0, 1, 0).$$

Let $H \subset \mathbb{T}^9$ and $H_1 \subset \mathbb{T}^9$ denote the subtori corresponding to $\ker(Q)$ and $\ker(Q_1)$ respectively. Proposition 3.3 yields monomials of the form

$$f_3(z_1, \dots, z_9) = z_1^{a_1} z_2^{a_2} z_3^{a_3} z_6^{a_6} z_7^{a_7} z_9^{a_9}, \quad f_5(z_1, \dots, z_9) = z_2^{b_2} z_3^{b_3} z_4^{b_4} z_5^{b_5} z_6^{b_6} z_8^{b_8}$$

where $a_i, b_i > 0$, such that f_3, f_5 are invariant with respect to H_1 and both are H -homogeneous with respect to the same weight. The kernel of Q_1 is generated by the vectors

$$(-1, -1, 1, 0, 0, 0, 1, 0, 0) \quad (-1, 0, 0, -1, 1, 0, 1, 0, 0).$$

Thus, H_1 -invariance of the monomials f_3 and f_5 implies

$$a_1 = a_7, \quad a_2 = a_3, \quad b_4 = b_5, \quad b_2 = b_3.$$

Now consider the one parameter subgroup $\mathbb{C}^* \rightarrow H$ corresponding to the following lattice vector

$$(3, -1, 1, -3, 0, 0, -1, -2, 1) \in \ker(Q).$$

Since both f_3 and f_5 have the same weight with respect to H , they also have the same weight with respect to the above one parameter subgroup, and we obtain the relation

$$2a_1 + a_9 = -3b_4 - 2b_8.$$

This contradicts the assumption that all the exponents a_i and b_i are positive. Consequently, X cannot be 2-divisorial. \square

In [12], 1- and 2-divisoriality are characterized in terms of toric embeddings. Using [12, Theorems 3.1 and 3.2], we obtain:

Theorem 4.2. *The toric variety X of Proposition 4.1 admits a closed toric embedding into a smooth toric prevariety with affine diagonal but it cannot be embedded into a smooth toric variety.*

We turn back to quasiprojectivity of orbit spaces. Our example serves also to show that for a \mathbb{C}^* -action on a quas affine variety with geometric quotient, the resulting orbit space in general need not be quasiprojective:

Proposition 4.3. *There exists a five-dimensional affine toric variety X with \mathbb{C}^* -action and a toric open subset $U \subset X$ admitting a geometric quotient $U \rightarrow U/\mathbb{C}^*$ such that U/\mathbb{C}^* is not 2-divisorial.*

Proof. The affine toric variety X in question arises from the five-dimensional cone τ in \mathbb{Z}^5 generated by the following 9 lattice vectors:

$$\begin{aligned} w_1 &:= (0, 0, 0, 1, 0), & w_2 &:= (0, 0, 1, 0, -1), & w_3 &:= (0, 0, 1, 0, 0), \\ w_4 &:= (0, 1, 0, 0, 0), & w_5 &:= (0, 1, 0, 0, 1), & w_6 &:= (1, 0, 1, -1, 0), \\ w_7 &:= (0, 0, 0, 1, -1), & w_8 &:= (0, 1, -1, 1, 0), & w_9 &:= (1, 0, 0, 0, 0). \end{aligned}$$

We consider the action of the one parameter subgroup $\mathbb{C}^* \rightarrow \mathbb{T}^5$ on X corresponding to the lattice vector

$$w := (1, -5, 2, 0, 2).$$

The open toric subset $U \subset X$ is given by the fan Σ in \mathbb{Z}^5 with the following maximal cones:

$$\begin{aligned}\tau_1 &:= \text{cone}(w_2, w_3, w_4, w_5, w_6), & \tau_2 &:= \text{cone}(w_1, w_2, w_3, w_7), \\ \tau_3 &:= \text{cone}(w_4, w_5, w_8), & \tau_4 &:= \text{cone}(w_1, w_5, w_8), \\ \tau_5 &:= \text{cone}(w_1, w_7, w_9).\end{aligned}$$

Note that τ_1, \dots, τ_5 are in fact faces of the cone τ . Let $P: \mathbb{Z}^5 \rightarrow \mathbb{Z}^4$ be the projection defined by the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ -2 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

The kernel of P is generated by the lattice vector $w \in \mathbb{Z}^5$. Projecting the cones of Σ via P , we obtain just the fan Δ of the toric variety presented in Proposition 4.1.

Since P is injective on all the cones τ_i , it defines in fact a geometric quotient $U \rightarrow U/C^*$ for the C^* -action on U corresponding to $w \in \mathbb{Z}^5$. As we have seen in Proposition 4.1, the quotient variety U/C^* is not 2-divisorial. \square

Acknowledgement. In the search for explicit examples we used the Maple-Package `convex` developed by M. Franz. The package contains several basic functions on convex geometry. We would like to thank M. Franz for making his package available under <http://www.mathe.uni-konstanz.de/~franz/convex/>.

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