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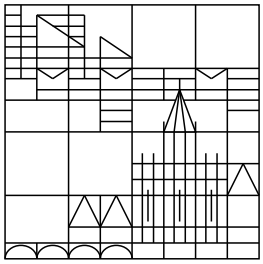

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A Linear Time Algorithm for the Arc Disjoint Menger Problem in Planar Directed Graphs

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Abstract

Given a graph $G = (V, E)$ and two vertices $s, t \in V$, $s \neq t$, the Menger problem is to find a maximum number of disjoint paths connecting s and t. Depending on whether the input graph is directed or not, and what kind of disjointness criterion is demanded, this general formulation is specialized to the directed or undirected vertex, and the edge or arc disjoint Menger problem, respectively.

For planar graphs the edge disjoint Menger problem has been solved to optimality [Wei94a], while the fastest algorithm for the arc disjoint version is Weihe's general maximum flow algorithm for planar networks [Wei94b], which has running time $\mathcal{O}(|V| \log |V|)$. Here we present a linear time, i.e. asymptotically optimal, algorithm for the arc disjoint version in planar directed graphs.

1 Introduction

Due to their importance, in their own right as well as in bottleneck routines of other algorithms, disjoint path problems have been studied extensively. The famous Menger Theorems [Men27] are structural in nature. However, they have not only been generalized to capacitated versions like the max-flow/min-cut theorem, but also extended to algorithms actually constructing disjoint paths, separators, or cuts.

A generic formulation of Menger's problem is the following: Given a graph $G = (V, E)$ and two distinct vertices $s, t \in V$, find a maximum cardinality set of disjoint paths connecting s and t. This leads to four concrete versions of the problem. The instances are either directed or undirected, and the (s, t)–paths have to be vertex or edge (arc) disjoint.

For planar undirected graphs, linear time algorithms exist for both the vertex [RLWW97] and edge disjoint case [Wei94a]. However, there is no such algorithm for either case when the planar input graphs are directed. In any graph the arc disjoint Menger problem obviously corresponds to a maximum flow problem with unit capacities [AMO93]. The first algorithm tailored to solve the maximum flow problem with arbitrary capacities especially in planar graphs was presented in [IS79]. Faster algorithms have subsequently been developed, e.g. [JV82] and [KRRHS94]. By now, the fastest algorithm is that of [Wei94b] yielding a running time of $\mathcal{O}(n \log n)$, where $n = |V|$. Here we concentrate on the more special Menger problem and present a linear time solution. Our algorithm is not only faster than the max-flow algorithm, but also considerably simpler.

Our approach is based on right-first-search, which appears to be extremely suitable for path problems in planar graphs (cf. [RLWW95]). In particular, the optimal algorithms for the Menger problem in undirected graphs [RLWW97, Wei94a] are based on this variant of depth-first-search. In a right-first-search, arcs are chosen according to a *right-hand-rule*, i.e. the continuation arc is the counterclockwise next in the adjacency list of the vertex entered by the current arc. One of the main difficulties encountered by this strategy is the treatment of right cycles. Similar to [Wei94a], we therefore use an observation of [KNK93] to restrict the set of input instances to graphs without right cycles.

Roughly speaking, the algorithm successively occupies arcs in order to build a set of (s,t)–paths. The paths in this set are frequently reorganized, such that the determination of consecutive arcs on the same path becomes intricate. Another problem is the efficient choice of an arc to backtrack with when the path that is currently build can no longer be extended. Together, these problems make a linear time implementation rather difficult. The obstacles are overcome by a careful analysis of partial solutions which leads to local characterizations resolving both problems.

In Section 2, we introduce our basic terminology and show how to restrict the problem to certain input instances. Section 3 gives a description of the algorithm on an abstract level, providing a better understanding of the underlying ideas. Its correctness is proved in Section 4. It is worth mentioning that the proof results in an algorithmic checker that can be used to verify the results of a concrete implementation. In Section 5 properties of partial solutions are examined. Based on these properties, a linear time implementation of the algorithm is described in Section 6. We conclude in Section 7 with a short discussion on related problems.

2 Preliminaries

Let us first introduce our basic assumptions and terminology. We are given an embedded planar graph $G = (V, A)$ with distinct vertices $s \neq t$. The *adjacency list* of a vertex $v \in V$ is a cyclic list of all arcs incident to v, arranged in the order in which they appear around v in the embedding. We will often make use of this ordering, and say that an arc a is the *first arc after* b in (counter)clockwise order around v, if b is an immediate successor of a when the adjacency list of v is traversed in a (counter)clockwise fashion.

With the assumption of a fixed embedding we can make heavy use of spatially descriptive terms, like left and right, inside and outside, etc. For example, the right side of a directed path is its right-hand-side when following its arcs directions. A directed cycle divides the plane into two disjoint regions, its left-hand-region and its right-hand-region. The region containing the outer face is called its *exterior*, the other is called *interior*. A cycle is called a *left (right) cycle*, if its interior equals its left-hand-region (right-hand-region). Cycles with s in their interior are called *orbits*.

Note that a maximum collection of arc disjoint directed (s,t)–paths in G corresponds to a maximum flow from s to t, if all arcs have unit capacity. Conversely, it is easy to construct a maximum collection of (s,t)–paths from an integral maximum flow. Also, given a maximum integral flow, a partition of the vertices inducing a minimum cut can always be found in linear time by a simple labelling algorithm.

Moreover, the maximum flow value does not change when a set of right cycles is replaced by left cycles which are obtained by simply altering arc orientations. Therefore, let $\mathcal{C}$ be a set

of right cycles. Then $G_{\mathcal{C}} = (V, A_{\mathcal{C}})$ is called the *residual graph*, where $A_{\mathcal{C}}$ is the set of all arcs (v, w), with $(v, w) \in A$ and (v, w) does not belong to a cycle in $\mathcal{C}$, or $(w, v) \in A$ and (w, v) does belong to a cycle in $\mathcal{C}$. Note that reversion of arcs may introduce multiple arcs, which makes $A_{\mathcal{C}}$ a multiset. If $f_{\mathcal{C}} : A_{\mathcal{C}} \to \{0, 1\}$ is a maximum integral flow in the residual graph $G_{\mathcal{C}}$, then a maximum integral flow $f : A \to \{0, 1\}$ in G is obtained by setting $f(v, w) = f_{\mathcal{C}}(v, w)$, if $(v, w) \in A$ and $(v, w) \in A_{\mathcal{C}}$, and $f(v, w) = 1 - f_{\mathcal{C}}(w, v)$, if (w, v) is the replacement of (v, w) in $A_{\mathcal{C}}$. From [KNK93] it can be seen that there always exists a set $\mathcal{C}$ of right cycles such that the residual graph $G_{\mathcal{C}}$ contains left cycles only[1]. Moreover, this special set $\mathcal{C}$ can always be found in linear time using a breadth-first-search in the planar dual of G, which can also be obtained in linear time. Thus, a linear time algorithm solving the arc disjoint Menger problem in $G_{\mathcal{C}}$ is sufficient to provide a linear time solution for the problem in G. This technique was also used in [Wei94a], where further details can be found.

In the remainder we assume that we are given a planar directed graph $G = (V, A)$ that is embedded in the plane, such that t is on the boundary of the outer (i.e. the infinite) face[2] and contains no right cycle. We may further assume that there are no arcs entering s, and no arcs leaving t, since these obviously do not affect the maximum number of arc disjoint directed (s, t)–paths.

3 The Algorithm

In this section, we present an algorithm that determines a maximum set of (possibly non-simple) arc disjoint (s, t)–paths in a planar directed graph that contains no right cycle, and is embedded (at least combinatorially) such that t is on the boundary of the outer face. We have outlined in the previous section that the arc disjoint Menger problem can be solved for any planar directed graph with linear overhead, if it can be solved for instances of this particular class of graphs.

For convenience, we here describe the algorithm on an abstract level, which both facilitates understanding and displays the basic simplicity of our approach. Nonetheless, it is not at all obvious how a linear worst case complexity can be obtained.

The algorithm applies a special variant of depth-first-search, namely right-first-search, which is suitable for many problems involving paths in planar graphs [RLWW95]. As a by-product, the resulting solution is rightmost in the sense that no path can be routed further to the right without changing others.

All paths and cycles in this section are directed. After each step, the partial solution consists of a *search path*, which starts at s and ends at some vertex $v \neq t$, and a set of (s, t)–paths and left cycles, such that every arc belongs to at most one path or cycle. Given such a set of arc disjoint directed paths and cycles, each path (cycle) induces a straightforward (cyclic) *traversal order* on its arcs. For a directed (sub-)path, its *first* and *last* arc are well defined, then. We say that two arcs are *consecutive*, if they are immediate successors in the traversal order of a path or cycle, respectively. The last arc

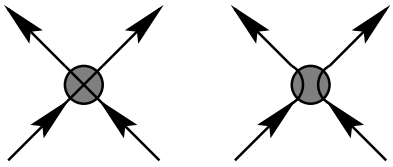

Figure 1: Crossing (left) and non-crossing (right) pairs of consecutive arcs

[1] This set $\mathcal{C}$ corresponds to the *bottom element* of the distributive lattice formed by all circulations of G with appropriately defined operations for *meet* and *join* [KNK93].

[2] Such an embedding can always be obtained in linear time [HT74].

of the search path is called the *leading arc*, and its head is called the *leading vertex*. We say that two pairs of consecutive arcs form a *crossing*, if they share their middle vertex v, and their arcs are encountered alternatingly when traversing the cyclic order of arcs incident to v. See Fig. 1. A set of arc disjoint paths and cycles is said to be *non-crossing*, if no two pairs of consecutive arcs form a crossing.

The algorithm uses only three basic operations: *search steps*, *backtracking steps*, and *realignments*.

search step An unsearched arc leaving the leading vertex is added to the search path. Among all unsearched arcs, the counterclockwise first after the current leading arc in the adjacency list of the leading vertex is chosen (right-hand-rule).

backtracking step Some arc of the search path entering the leading vertex is removed from the graph. Which arc exactly need not be specified in this general version of the algorithm. In the implementation presented in Section 6, this choice is subject to certain configurations and the stage of the algorithm. If a non-simple search path has more than one arc entering the leading vertex, the removal may split the search path into the new search path starting at s and ending at the removed arc's tail, and a left-over subpath starting and ending at the leading vertex, say v. We then modify the traversal order with respect to the arcs of the cut-off end of the search path that are incident to v, such that the subpath is transformed into a set of left cycles that do not cross at v. Each of these cycles is constraint to have exactly two arcs incident to v. See Fig. 2 and note that there is a unique reassignment of consecutive arcs satisfying these conditions.

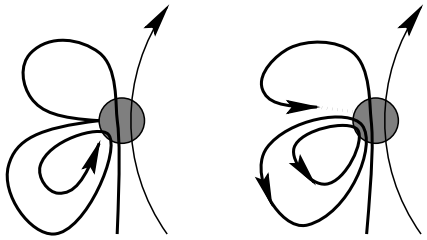

Figure 2: After backtracking with the second arc of the search path entering the leading vertex, there are two resulting left cycles (containing four irrelevant arcs incident to the formerly leading vertex). The removed arc is indicated by the dotted line.

We refer to the reassignment of consecutive arcs during a backtracking step as *closing left cycles*, and to the arcs that are reassigned as *irrelevant*. Then, every arc that belongs to a path or cycle and is not irrelevant is called *relevant*[3].

In order to introduce the third operation, some more terminology is needed. For a vertex $v \in V \setminus \{s, t\}$, we define a *passage through* v, or *v–passage* for short, of a path or cycle to be an (inclusion-)maximal subpath with the following properties: Its first arc is a relevant arc entering v and its last arc is a relevant arc leaving v. Moreover, if it is non-simple, then s is in the exterior of every cycle formed by the subpath. Fig. 3 (a) gives an example. If existent, the last v–passage of the search path is called *leading v–passage*. An arc (u, v) is said to *hit* some v–passage p from the right (left), if it is on the v–passage's right (left) side, and p and (u, v) are on the same sides of every other v–passage. An arc (v, w) is said to *leave* a v–passage to the right (left) in the analogous situation. Two v–passages *touch* at v, if they do not cross at v, and each of them contains an arc hitting or leaving the other. Paths and cycles are hit, left or touched at a vertex v, if one of their v–passages is. Fig. 3 (b) summarizes these definitions.

[3]Actually, arcs ought to be called relevant or irrelevant *with respect to* one of their incident vertices. We omit this distinction, because from context it should always be clear, with respect to which vertex an arc is relevant or irrelevant, respectively.

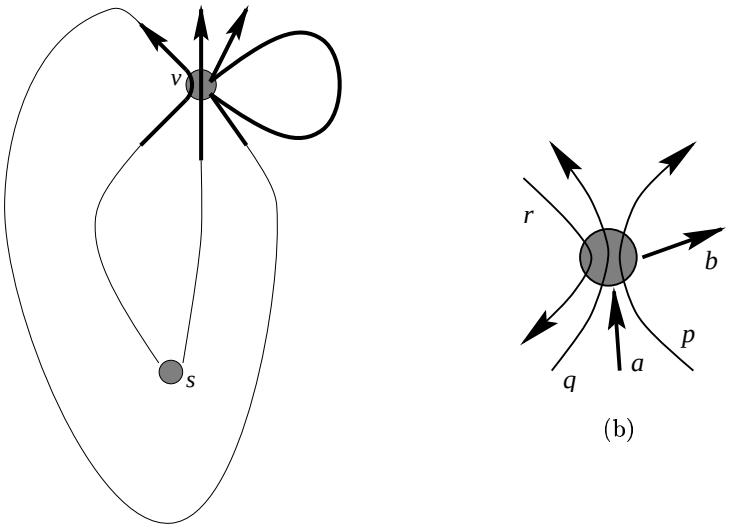

Figure 3: (a) Two paths making up three v–passages. (b) Three passages p, q, r, and two arcs a, b incident to the same vertex. Passage p touches passage q on the right, while q touches p on the left side. On the other hand, the p and r do not touch at all. Arc a hits q from the right, and p from the left. However, it does not hit r. Arc b leaves p to the right, but neither q nor r. Since they do not lie on a common path or cycle, a and b are not consecutive.

realignment Let v be the leading vertex, and let some v–passage be hit from the right by the leading arc. Since there are no right cycles, there must be consecutive arcs (u, v) and (v, w) of the v–passage hit, such that the leading arc appears between (u, v) and (v, w) in the counterclockwise cyclic order of arcs incident to v. The search path is said to be realigned with the corresponding path or cycle, if the leading arc is made consecutive with (v, w), such that (u, v) becomes the new leading arc. See Fig. 4.

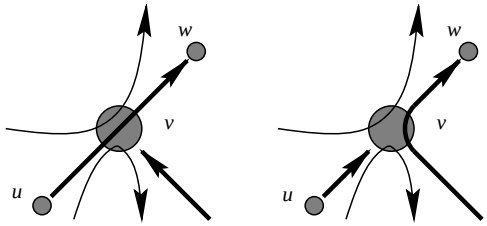

Figure 4: Realignment of a passage that is hit from the right.

We are now ready to state our algorithm in simple terms. The bottom line is that we always try to go as far to the right as possible. The contribution of realignments is two-fold: on one hand, they prevent crossings, and on the other hand, they ensure that the search path is in some sense leftmost at the leading vertex when backtracking has to be performed.

Algorithm 1 Menger algorithm

for each *outgoing arc a of s* **do**
  let the search path consist of arc a
  while *the leading vertex is neither s nor t* **do**
    if *the leading arc hits some passage from the right* **then**
      realign the search path with the corresponding path or cycle
    else
      if *there is an unsearched arc leaving the leading vertex* **then**
        perform a search step
      else
        perform a backtracking step

4 Correctness

In this section, we prove that after termination of Algorithm 1 the set of (s,t)–paths generated is maximum. A set of arcs whose removal disconnects s and t is called an (s,t)–*cut*. By Menger's theorem, a set of (s,t)–paths is maximum, if and only if its cardinality equals the cardinality of a minimum (s,t)–cut. Such (s,t)–cuts are called *saturated*. There are two easy cases, for which the following lemma yields correctness of Algorithm 1.

Lemma 1 *If either all or none of the arcs leaving s have been removed by Algorithm 1, the arcs leaving s that have not been removed form a saturated directed* (s,t)*–cut.*

Proof: If none of the arcs leaving s has been removed, each of them is part of a distinct (s,t)–path, because no arc is entering s and Algorithm 1 does only produce (s,t)–paths and left cycles. For the other extreme, note that the number of (s,t)–paths found does never decrease during execution of Algorithm 1. Since a right-first-search eventually finds a path to every vertex reachable from s, there is no (s,t)–path, if every arc leaving s is removed. □

Three more observations are trivial and stated without proof:

Lemma 2 *During the execution of Algorithm 1, the following statements hold:*

a) Just before a search or backtracking step, the leading arc does not hit any passage through the leading vertex from the right.

b) Just before a search step, there is no irrelevant arc incident to the leading vertex and no removed arc entering the leading vertex.

c) Just before a backtracking step, every arc leaving the leading vertex has already been searched.

Correctness is based on the fact that the solution produced by Algorithm 1 is maximal and rightmost, i.e. no (s,t)–path can be routed further to the right without changing others, too. This notion of rightmost is characterized by the following four invariants:

Lemma 3 *During the execution of Algorithm 1, the following properties remain invariant:*

(P1) All paths and left cycles are arc disjoint and non-crossing.

(P2) No unsearched arc leaves a passage to the right.

(P3) No removed arc hits a passage from the right.

(P4) No two passages mutually touch their right sides.

Proof: Initially, the set of paths and cycles is empty and every condition is met. It is then sufficient to show that (P1)–(P4) are invariants of the *while*–loop, i.e. they remain satisfied after a single realignment, or a single search or backtracking step.

By (P1) and the fact that the search path is realigned only when it hits a passage through the leading vertex from the right, a realignment does not affect any of (P1)–(P4).

(P1) Clearly, no arc is assigned to two different paths or cycles at the same time. Searching does not cause a crossing because of Lemma 2a and (P2). By Lemma 2a and 2b, and the way we close left cycles, no crossing is produced when an arc of the search path is removed.

(P2) By our choice of the new arc, (P2) remains valid after a search step. We need not consider backtracking because of Lemma 2c.

(P3) Lemma 2b shows that this property remains satisfied after a search step. According to Lemma 2a, backtracking does not violate (P3), since all reassigned arcs become irrelevant arcs.

(P4) By Lemma 2a, we can safely add a new arc to the search path. All changes caused by backtracking involve only irrelevant arcs. □

Algorithm 2 Saturated cut algorithm

```
if all or none of the arcs leaving s are removed then
  let the search path consist of s only
else
  let the search path consist of a removed arc leaving s
  repeat
    if there is an unsearched candidate arc then
      search the clockwise next candidate arc
    else
      discard the leading arc from the search path
  until the search path consists of s only
        and the clockwise next candidate arc has been searched
    or the clockwise next candidate arc is part of the search path
        and s does not lie in the exterior of the resulting cycle
```

Based on the above observations, Algorithm 2 determines a saturated directed (s,t)–cut in the output of Algorithm 1. This cut is induced by the exterior of a cycle enclosing s. This obviously suffies to prove correctness. From the discussion in Section 6, it is easy to see that Algorithm 2 can also be implemented with linear running time. It is based on the analogous variant of depth-first-search, left-first-search. Arcs removed or not searched by Algorithm 1

are searched in forward direction, while arcs belonging to paths or cycles are searched in backward direction.[4] When Algorithm 2 backtracks, the backtrack arc is said to be *discarded* from the search path. Algorithm 2 terminates, when the search path hits itself from the left, such that the resulting cycle is a right cycle[5] surrounding s. The arcs on (s,t)–paths having their tail on this cycle and their head in the exterior then form the desired cut. Fig. 5 may serve to build an intuition.

To avoid confusion, an arc is denoted *ignored*, if it was not searched by Algorithm 1. An arc is then called a *candidate arc*, if it leaves the leading vertex and was either removed or ignored, or if it enters the leading vertex and is part of an (s,t)–path or left cycle produced by Algorithm 1.

Lemma 4 *Algorithm 2 never discards an arc from the search path that belongs to an (s,t)–path produced by Algorithm 1.*

Proof: If the leading arc of Algorithm 2 belongs to an (s,t)–path produced by Algorithm 1, there either is a preceding arc on this path, or the leading vertex is s. In the latter case, the clockwise next candidate arc is searched next, or the algorithm terminates. □

Lemma 5 *If an ignored arc is searched by Algorithm 2, it lies in the interior of one of the left cycles produced by Algorithm 1.*

Proof: By Lemma 2c, the head of a removed arc is never the tail of an ignored arc. Hence, the immediate predecessor of an ignored arc on the search path of Algorithm 2 must be an ignored arc, or the arc of an (s,t)–path or a left cycle produced by Algorithm 1. Therefore, Lemma 4 and (P2) prove the claim. □

Lemma 6 *The leading arc of Algorithm 2 never hits an (s,t)–path determined by Algorithm 1 from the right.*

Proof: Assume that the leading arc hits an (s,t)–path of Algorithm 1 from the right. Then, it is either an ignored or removed arc hitting the (s,t)–path, or a path or cycle arc leaving it. Because, by Lemma 4, arcs of (s,t)–paths are never discarded, (P1), (P4), and the clockwise selection of candidate arcs imply that the leading arc does not belong to a passage through the leading vertex. From Lemma 2a, it follows that irrelevant arcs appear only on the left side of passages through the respective vertex, i.e. inside of a left cycle or to the left of an (s,t)–path. The leading arc is therefore neither a path nor a cycle arc because of (P1). Removed arcs are excluded by (P3), and since Lemma 5 states that ignored arcs are searched inside of left cycles only, (P1) and (P4) rule them out, too. □

After termination of Algorithm 2, the search path either consists of s alone, or it hits itself from the left while surrounding s. In the second case, let $C = (v_1, a_1, v_2, a_2, \ldots, v_k = v_1)$ be the sequence of vertices and arcs of the search path beginning with the leading vertex and the clockwise next arc on the search path, and ending with the leading arc and the leading vertex. See Fig. 5. In the first case, let C be the trivial cycle s.

[4] Basically, the algorithm tries to find an augmenting path.

[5] Since some arcs are searched in backward direction, right cycles are possible.

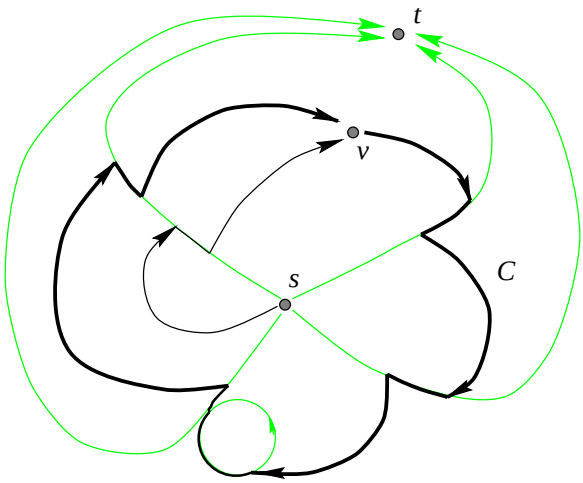

Figure 5: Algorithm 1 has produced four (s,t)–paths and one left cycle, indicated by grey lines. In its output, Algorithm 2 terminates after returning to v. The final search path is depicted by black lines, with the resulting right cycle, C, drawn thicker.

Lemma 7 *If C does not equal s, it is a right cycle with s in its interior or on its boundary.*

Proof: Clearly, s does not lie in the exterior of C. If only some of the arcs leaving s are removed arcs, Algorithm 1 has found at least one (s,t)–path. (Simply verify that every arc that was neither removed nor ignored lies on an (s,t)–path or a left cycle and no arc is entering s.) Thus, Lemma 6 implies that C cannot be a left cycle. □

Lemma 8 *The set of arcs on (s,t)–paths produced by Algorithm 1 having their tail on C and their head in its exterior form a saturated directed (s,t)–cut.*

Proof: If the precondition in the first line of Algorithm 2 is true, we may apply Lemma 1. Therefore assume it is false.

Because no arc is leaving vertex t and no incoming arc of t is removed, Algorithm 2 can enter t on ignored arcs only. However, t lies on the outer face, and from Lemma 5 we can thus deduce that it is never reached. Using Lemma 7, we have that C is a right cycle with s in its interior or on its boundary. To verify that the arcs leaving C on (s,t)–paths form a directed cut, consider an arbitrary simple path $(s = v'_1, a'_1, \ldots, v'_l = t)$ from s to t. Let $v'_i = v_j$ be its last vertex on the boundary of C, and assume that a'_i is a removed or ignored arc. a'_i must have been the clockwise next candidate arc of Algorithm 2 some time that v_j was the leading vertex. Then, a'_i was indeed searched and later discarded by Algorithm 2. Also, every arc reachable from a'_i was searched and discarded. By Lemma 4, no (s,t)–path of Algorithm 1 was hit. (Note that it doesn't matter that left cycles produced by Algorithm 1 are searched the other way round.) This is a contradiction, because t was not reached. Neither can a'_i belong to a left cycle produced by Algorithm 1, because candidate arcs are chosen in clockwise order. Hence, a'_i must be the arc of an (s,t)–path. Since by Lemma 6 Algorithm 2 does not hit (s,t)–paths from the right, the clockwise choice of candidate arcs implies that every (s,t)–path has exactly one arc leaving the right cycle formed by C. □

The appropriate version of the Menger Theorems now yields correctness of our algorithm.

Corollary 1 *The set of arc disjoint (s,t)–paths determined by Algorithm 1 is maximum.*

5 Properties of Partial Solutions

In Section 3, an algorithm solving the arc disjoint Menger problem in planar directed graphs was described. In this section now, we prove a number of invariants that are used to efficiently implement this algorithm.

Since our goal is to achieve linear running time, the possibly more than linear number of realignments cannot actually be performed. While the arc to be searched next is still computed easily, it can be difficult to identify an arc on the search path that may be used for backtracking, when it is not known which arcs are consecutive. We subsequently analyze the structure of partial solutions. This will permit an implementation that does not need to explicitly represent which arcs are consecutive.

A first structural insight is the relative orientation of passages. Two passages p, q through the same vertex are said to be *oriented likewise*, if p is completely to the left of q, while q is completely to the right of p. In Fig. 3 (b), passages p, q are oriented likewise, while passages p, r and q, r are oriented differently. The following lemma shows that at most the last v–passage of the search path can be oriented different than other v–passages.

Lemma 9 *During the execution of Algorithm 1, the following property remains invariant:*

(P5) For all $v \in V \setminus \{s,t\}$, all v–passages are oriented likewise, possibly except for the leading v–passage.

Proof: Just like in the proof of Lemma 3, we only have to consider a single realignment, or a single search or backtracking step.

The search path is either realigned with an (s,t)–path, with a cycle, or with itself. First, let the search path be realigned with an (s,t)–path and assume that (P5) is not satisfied afterwards. By induction, there has to be a vertex v such that before the realignment its leading v–passage is oriented different than another v–passage it touches, while after the realignment it belongs to the (s,t)–path hit. By (P4) these two v–passages touch on their left sides, since differently oriented passages mutually touch on the same side. Therefore, by (P1), this leading v–passage does not belong to a left cycle, but to an (s,t)–path or orbit. But then, again by (P1), the search path did not hit an (s,t)–path from the right (recall that t is on the outer face and note that there is no (s,t)–path, if there is an orbit). Fig. 6 illustrates the situation of a touched (s,t)–path.

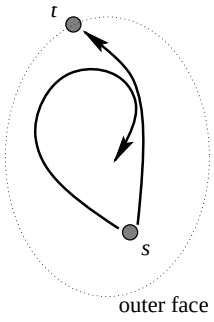

Figure 6: If the search path touches a differently oriented (s,t)–path from the left, it does not hit an (s,t)–path from the right, because of (P1).

If the search path is realigned with itself (because it surrounds s), the considerations are almost the same.

If the search path is realigned with a cycle, the cycle does not surround s (i.e. it is not an orbit), because of (P1). Similar arguments thus show that any leading v–passage oriented different than another v–passage still is a leading v–passage after the realignment.

Obviously, search steps do not affect the invariant, since, except for the added arc, the leading passages remain unchanged. Note that the orientation of the new and augmented leading passage of the leading arc is the same as before in case the search path is extended after hitting itself from the left.

The only possibility for a backtracking step to violate this invariant is to create a left cycle which has, for some $v \in V \setminus \{s,t\}$, a v–passage that is oriented different than another v–passage. In this case, every arc of the search path that is incident to the leading vertex either remains on the same passage or becomes irrelevant. Hence v is not the leading vertex. Just like above the search path then has to touch a v–passage of an (s,t)–path or orbit on the left side. Because of (P1) and t being on the outer face, this is impossible. □

By the above property, all but at most one specific v–passage (which then is the leading v–passage) of a vertex $v \in V \setminus \{s,t\}$ are oriented likewise. We define the *leftmost* v–passage to be the unique leftmost of these, and *lastLeft*(v) to be its last arc. As an immediate, yet crucial, consequence of (P5) the following corollary states that knowledge of *lastLeft*(v) is sufficient to identify an arc that may be used in a backtracking step (i.e. an arbitrary arc of the search path entering the leading vertex).

Corollary 2 *During the execution of Algorithm 1, the following property remains invariant:*

(P6) If $v \in V \setminus \{s,t\}$ is the leading vertex and the search path does not hit a v–passage from the right, then the counterclockwise next relevant arc after lastLeft(v) *is an incoming arc of the search path.*

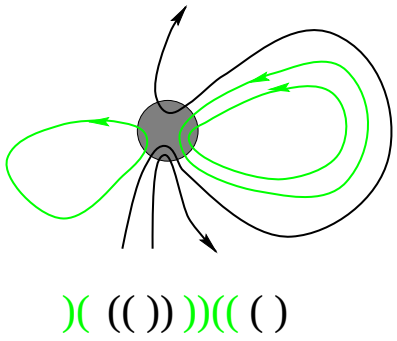

Figure 7: A vertex v and its associated parenthesis expressions. $M(v)$ is depicted darker than $m(v)$, *lastLeft*(v) is the arc leaving v upwards.

We can gain even further knowledge from *lastLeft*(v): If the relevant arcs incident to v are traversed in counterclockwise order such that *lastLeft*(v) is encountered last, define $M(v)$ to be a string of parentheses, one opening for each incoming relevant arc, and one closing for each outgoing relevant arc of v. If v is the leading vertex, the leading arc does not contribute to $M(v)$. Analogously, let $m(v)$ be the string of parentheses for the irrelevant arcs. We say that a string of parentheses has *parenthesis structure*, if the number of closing parentheses does not exceed the number of opening parentheses in any prefix of the string. Likewise, it has *inverse parenthesis structure*, if the number of closing parentheses in a prefix is never less than the number of opening parentheses. Two strings of (possibly inverse) parenthesis structure are called *interleaved* (in a common superstring), if no parenthesis of one string is positioned between a pair of matching parentheses in the other.

The following property is not only useful to recompute pairs of consecutive arcs from *lastLeft*(v), but interesting in its own right.

Lemma 10 *During the execution of Algorithm 1, the following property remains invariant:*

(P7) For $v \in V \setminus \{s,t\}$, $M(v)$ has parenthesis structure, $m(v)$ has inverse parenthesis structure, $M(v)$ and $m(v)$ are interleaved, and every pair of consecutive relevant or irrelevant arcs incident to v corresponds to a pair of matching parentheses in $M(v)$ or $m(v)$, respectively.

Proof: Again we treat the three basic operations of Algorithm 1. In the following, let $v \in V \setminus \{s,t\}$ be the leading vertex.

If the search path is realigned, the position corresponding to the leading arc is inside of a pair of matching parentheses of $M(v)$ (corresponding to the pair of consecutive arcs of the v–passage that is hit from the right). After the realignment the roles of the arc corresponding to the pair's opening parenthesis and of the leading arc are switched, so that $M(v)$ still has parenthesis structure. No other statement of (P7) is affected.

Before a search step, $m(v)$ is empty because of Lemma 2b, and the leading arc corresponds to a position outside of every matching pair in $M(v)$ because of Lemma 2a. By (P2), the newly searched arc is not positioned inside any matching pair of $M(v)$, so that all statements remain valid.

For a backtracking step, first note that the arcs of the search path correspond to parentheses outside of every pair of parentheses that represent consecutive arcs not on the search path (Lemma 2a and (P1)). Let (u,v) be the arc that is removed. If it is not the leading arc, it corresponds to an opening parenthesis in $M(v)$ or $m(v)$, respectively. The numbers of opening and closing parentheses remain equal, because after the removal the leading arc is either removed (if it equals (u,v)) or corresponds to a newly inserted opening parenthesis replacing the removed one. The removal splits the set of arcs on the search path incident to v into those becoming irrelevant and those unchanged. Those becoming irrelevant are positioned outside of those unchanged, since there are no right cycles. Hence, $M(v)$ and $m(v)$ remain interleaved. By definition, the closing of left cycles respects the inverse parenthesis structure of $m(v)$ (resulting left cycles do not cross).

Finally, consider the tail $u \in V \setminus \{s,t\}$ of the arc (u,v) that is removed (if $u = s$ we are done). The arc corresponds to a closing parenthesis in $M(u)$ or $m(u)$. By induction, the new leading arc will correspond to its matching opening parenthesis, so that (P7) remains valid for vertex u. No other vertex needs to be considered. □

Even though *lastLeft*(v) provides all the information needed to implement Algorithm 1 efficiently, we are not yet done, because it is not always possible to update *lastLeft*(v) correctly based on local knowledge only (an example is given in Fig. 9).

Define *lastLeading*(v) to be the last arc of the search path leaving v. We now give a simple rule to mark an arc *last*(v), which equals at least one of *lastLeft*(v) and *lastLeading*(v). Consider, for a vertex $v \in V \setminus \{s,t\}$ that is not the leading vertex, the arc that has most recently been searched or removed. If it has been searched, it is outgoing and we let *last*(v) be just this arc. If it is removed, it is incoming, and we let *last*(v) be the clockwise next relevant outgoing arc. Now, if v is the leading vertex, it will be clear from context, whether *last*(v) refers to the current or next arc with the above properties. Furthermore we define *first*(v) to be the incoming relevant arc corresponding to the opening parenthesis immediately after *last*(v) in a cyclic traversal of $M(v)$.

The next lemma states that *last*(v) is the last arc of the v–passage that is to the left of all likewise oriented v–passages. Recall from (P5) that at most the leading v–passage is oriented differently, and observe that *lastLeft*(v) equals *lastLeading*(v), if and only if v has a leading v–passage that is to the left of every other v–passage and oriented likewise.

Lemma 11 *During the execution of Algorithm 1, the following property remains invariant:*

(P8) For every $v \in V \setminus \{s,t\}$, last$(v)$ equals lastLeft(v) or lastLeading(v). If last(v) equals lastLeading(v), the leading v–passage is to the left of every other v–passage.

Proof: Again, we only consider a single operation. Let $v \in V \setminus \{s,t\}$ be the leading vertex.

If the search path is realigned, we first consider those vertices $u \in V \setminus \{s,v,t\}$, for which $last(u) = lastLeading(u)$. If the search path is realigned with an (s,t)–path, the leading u–passage afterwards belongs to the resulting (s,t)–path. But since by induction the leading u–passage is to the left of every other u–passage, it cannot be oriented differently because of (P1). Hence, $last(u)$ equals $lastLeft(u)$. In case the search path is realigned with itself, similar arguments hold. If the leading u–passage is to the left of every other u–passage, a left cycle hit by the search path does not have a u–passage because of (P1) and (P4). Therefore, the leading u–passage remains leading when the search path is realigned with a left cycle. Now, consider the leading vertex v. If the search path is realigned, it is not to the left of every other v–passage. Therefore, $last(v) = lastLeft(v)$ by induction and neither of them is altered.

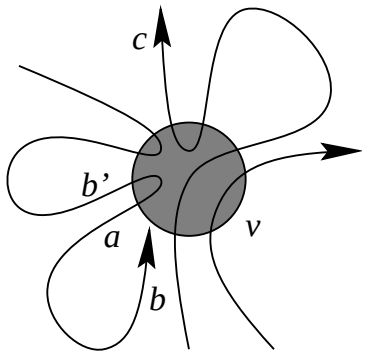

Figure 8: Which arc becomes $last(v)$ after a backtracking step? By (P5), b is the leading arc, $a = last(v) = lastLeading(v)$, and $c = lastLeft(v)$. After backtracking with b, these equalities remain valid. After backtracking with b', there is no $lastLeading(v)$ (unless c is on the search path), and $last(v) = c = lastLeft(v)$.

For a search step that does not make t the new leading vertex, we only have to consider the current leading vertex v. By definition, $last(v)$ becomes $lastLeading(v)$. By Lemma 2a, the new leading v–passage is to the left of every other v–passage, and no other vertex is affected. If, in turn, the new leading vertex is t, the situation compares to a realignment with an (s,t)–path. Then it follows just like above that $last(u)$ also equals $lastLeft(u)$ for every vertex $u \in V \setminus \{s,t\}$ with $last(u) = lastLeading(u)$.

In case of a backtracking step, we need to treat the leading vertex v and those vertices u on the search path that lie on a closed left cycle after the step. If the leading passage of such a vertex u is to the left of every other u–passage, the leading passage is not oriented different than the other u–passages because of (P1) and (P4). Thus, $lastLeading(u)$ equals $lastLeft(u)$ and $last(u)$ equals $lastLeft(u)$ by induction. Now consider the leading vertex. By Lemma 2c, the arc to be removed is to the left of every v–passage. Thus, (P7) implies that the clockwise next relevant arc after the removed arc is outgoing and will be the new $last(v)$, which is thus also to the left of every v–passage. It follows from (P1) and the absence of right cycles, that $last(v)$ equals $lastLeft(v)$ or $lastLeading(v)$, since the removed arc is on the search path (see Fig. 8). In any case, a leading v–passage remains left of every other v–passage. □

Our final observation gives a sufficient condition for $last(v)$ to equal $lastLeft(v)$.

Lemma 12 *Let $v \in V \setminus \{s,t\}$ be the leading vertex. If the leading arc is positioned between $first(v)$ and $last(v)$ in a cyclic traversal of $M(v)$, then $last(v) = lastLeft(v)$.*

Proof: Suppose, the condition is satisfied, but $last(v) \neq lastLeft(v)$. Then (P8) implies that there is a leading v–passage to the left of every other v–passage. From (P5) it follows that the leading v–passage is oriented different than every other v–passage. But because of the leading arc's position and the absence of right cycles the search path hits some v–passage from the right. This is a contradiction to (P1). □

6 Linear Time Implementation

In this section, we show that Algorithm 1 can be realized with linear running time. Since the number of changes caused by realignments can be more than linear in the number of arcs, it is crucial to avoid an explicit maintainance of consecutive arcs, at least in general. Fortunately, due to the highly structured partial solutions generated by the algorithm, realignments need not be performed explicitly. We show that the arc to be searched next can be determined from the current leading arc, while an arc to backtrack with can be determined from *first*(v) and *last*(v).

In each iteration of the main loop, the search path is initialized with the particular arc leaving s, and the inner loop is executed. The implementation of the inner loop consists of search and backtracking steps only. For a search step, consider the next arc that is searched. By (P2), it is the counterclockwise next unsearched arc after the leading arc, even if no realignment is performed. For a backtracking step, consider the step immediately afterwards. Let (u, v) be the arc that is removed, then the next step is either a search or a backtracking step with leading vertex u. If it is a search step, we again have that the counterclockwise next unsearched arc after (u, v) is the next unsearched arc chosen by Algorithm 1 (possibly after a number of realignments). If it is a backtracking step, an arc of the search path must be determined that is removed next. When it is known that *last*(u) equals *lastLeft*(v), this is done according to Corollary 2, otherwise the preceding arc of (u, v) is retrieved from a temporary assignment of consecutive arcs. According to (P7), this can easily be computed the first time that there is no outgoing arc at u. By Lemma 12 it is only needed for a certain transition phase, in which there is no (not even implicit) realignment at u.

The main loop and the implementations of the search and backtracking steps are stated separately. Each formal description is accompanied by a commenting paragraph.

A formal description of the main loop is given by Algorithm 3. A variable `global_mode` determines, whether the next step to be performed is a search or a backtracking step. Variable `leading_arc` stores the leading arc. Vertices and arcs are represented by records containing fields of data. Field `v.mode` is used to store a *local mode* for vertex v, which is either `FORWARD` (v still has outgoing unsearched arcs), `TRANSITION` (all outgoing arcs are searched, but *lastLeft*(v) is not yet identified), `SKIP` (all outgoing arcs are searched and *lastLeft*(v) = *last*(v)), or `DONE` (all outgoing arcs of the vertex are removed). Every arc a that is not removed from the graph has a field `a.flow` containing either `0` or `1`. It has value `1`, if and only if the arc is occupied and hence belongs to a path or cycle.

The search step is formally described by Algorithm 4. An additional field `v.last` is used to store *last*(v) for every vertex $v \in V \setminus \{s, t\}$. It was argued above, that the new leading arc is the counterclockwise next unsearched arc after the current leading arc. If there is no unsearched arc leaving the leading vertex, we must prepare for a backtracking step. The necessary actions are dictated by the local mode of leading vertex v.

If the local mode still equals `FORWARD`, this indicates that we need to backtrack from v for the first time. But until now, we are not able to decide, whether *last*(v) equals *lastLeft*(v) (cf. Fig. 9). Thus, the current pairs of consecutive arcs are computed by a subroutine `match_all`, taking a vertex v, and computing the predecessor `a.pred` of every outgoing arc a of v. Since, by Lemma 2a, *last*(v) corresponds to a toplevel closing parenthesis in $M(v)$, predecessors are computed easily. Another field `v.first` is introduced to store *first*(v). Note that, if `v.last` equals *lastLeft*(v), `v.first` corresponds to the first parenthesis in $M(v)$.

Algorithm 3 Menger implementation

```
for each a ∈ A do a.flow := 0
for each v ∈ V \ {s,t} do
   if v has no outgoing arcs then
      v.mode := DONE
   else
      v.mode := FORWARD

for each outgoing arc a of s do
   leading_arc := a
   leading_arc.flow := 1
   global_mode := SEARCH
   while (global_mode=SEARCH and head(leading_arc)≠t) or
         (global_mode=BACKTRACK and tail(leading_arc)≠s) do
      case global_mode of
         SEARCH:    search
         BACKTRACK: backtrack
   if global_mode=BACKTRACK then
      remove leading_arc from graph
```

If the leading arc hits every passage through the leading vertex from the left, local mode `TRANSITION` simply invokes a backtracking step. Otherwise the transition phase ends for this vertex, since we can deduce from Lemma 12, that *last*(v) now equals *lastLeft*(v) for sure.

During local mode `SKIP`, `v.last` is updated such that it always equals *lastLeft*(v). Thus the leading arc is to the left of every v–passage, if it is to the left of the pair `v.first` and `v.last`. In this case we simply backtrack with the leading arc. Otherwise, `v.first` may be used for backtracking according to Corollary 2. `v.first` is then updated to be the first remaining parenthesis of $M(v)$. Every arc between the former and the new `v.first` must be irrelevant. Hence, a subroutine `inverse_match_left` can realize the update of `v.first` by matching a maximal inverse parenthesis expression beginning just behind the former `v.first`.

If every outgoing arc of the leading vertex is already removed (local mode `DONE`), we simply invoke a backtracking step.

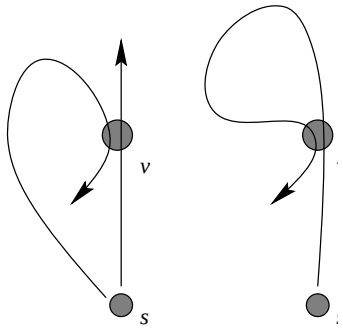

Figure 9: In the left situation there is an (s,t)–path touched on the left by the search path. Here, *lastLeft*(v) is the arc leaving v on the (s,t)–path, and hence *lastLeft*(v) $\neq$ *lastLeading*(v) $=$ *last*(v). In the right situation, the search path returns to v, and we have *last*(v) $=$ *lastLeading*(v) $=$ *lastLeft*(v). Note that these situations cannot be distinguished solely based on the arcs incident to v and the order in which they were searched.

Algorithm 5 implements the backtracking step. Since, in general, there is no information available, whether arcs are consecutive or not, the arc that is removed is used to determine the new leading arc. Consequently, Algorithm 5 operates on the tail of the leading arc, say u.

Algorithm 4 Procedure `search`

```
v := head(leading_arc)
if there is an outgoing arc of v with flow 0 then
   leading_arc := first arc after leading_arc
                  in counterclockwise order around v
                  that is outgoing and has flow 0
   leading_arc.flow := 1
   v.last := leading_arc
else
   case v.mode of
      FORWARD:
         match_all(v)
         v.first :=  first arc after v.last
                     in counterclockwise order around v
                     that is incoming with flow 1
                     and does not equal leading_arc
         v.mode := TRANSITION
      TRANSITION:
         if leading_arc is to the right of {v.first,v.last} then
            v.mode := SKIP
         else
            global_mode := BACKTRACK
      SKIP:
         if leading_arc is to the right of {v.first,v.last} then
            leading_arc := v.first
            inverse_match_left(v)
         global_mode := BACKTRACK
      DONE:
         global_mode := BACKTRACK
```

The necessary actions for local mode `FORWARD` are the same as in the search step. During local mode `TRANSITION`, the pairing of consecutive arcs at u is computed and valid. In case the arc that is removed equals $last(u)$, `u.first` and `u.last` are updated to be the arcs corresponding to neighboring positions in $M(u)$. Otherwise, a realignment occured or a new iteration of the main loop was started since the last update of `u.last`. Similar to Lemma 12 we may conclude that $last(u)$ equals $lastLeft(u)$ and switch to local mode `SKIP`.

Just as in the search step, local mode `SKIP` skips a maximal subpath of the search path that is starting and ending at u and backtracks with the first arc of $M(u)$, i.e. the first arc of the search path incident to the leading vertex. If the arc that is removed equals $lastLeft(u)$, the end of $M(u)$ is adjusted in a subroutine `inverse_match_right`, which is similar to `inverse_match_left`.

Observe that the arc that is removed during a backtracking step has `flow 1`. Thus, its tail u cannot be in local mode `DONE`.

Algorithm 5 Procedure `backtrack`

```
u := tail(leading_arc)
dummy := leading_arc
if leading_arc is the only outgoing arc of u then
   leading_arc := the incoming arc with flow 1
   u.mode := DONE
else if there is an outgoing arc of u with flow 0 then
   leading_arc := first arc after leading_arc
                  in counterclockwise order around u
                  that is outgoing and has flow 0
   leading_arc.flow := 1
   u.last := leading_arc
   global_mode := SEARCH
else
   case u.mode of
      FORWARD:
         match_all(u)
         u.first :=  first arc after u.last
                     in counterclockwise order around u
                     that is incoming with flow 1
                     and does not equal leading_arc
         u.mode := TRANSITION
      TRANSITION:
         if leading_arc = u.last then
            leading_arc := leading_arc.pred
            u.last :=   first arc after leading_arc
                        in clockwise order around u
                        that is outgoing and has flow 1
            u.first :=  first arc after leading_arc
                        in counterclockwise order around u
                        that is incoming and has flow 1
         else
            u.mode := SKIP
      SKIP:
         if leading_arc is to the left of {u.first,u.last} then
            leading_arc := leading_arc.pred
         else
            if leading_arc = u.last then
               inverse_match_right(u)
            leading_arc := u.first
            inverse_match_left(u)
remove dummy from graph
```

Theorem 1 *Algorithm 3 determines a maximum set of arc disjoint (non-crossing) (s,t)–paths of G in linear time.*

Proof: From the discussion above we see that Algorithm 3 is indeed an implementation of Algorithm 1 and therefore computes a maximum solution (Corollary 1).

For a linear time realization, observe that every time an arc is used, its state is altered from unsearched (`flow 0`) to searched (`flow 1`), or from searched to removed (no longer present). The predecessor of an arc is computed at most twice, and because of (P7) a simple stack algorithm matches consecutive arcs in linear time. Thus it is sufficient to show that, computation of consecutive arcs not accounted for, a single search or backtracking step can be implemented with constant (amortized) running time. The only critical operation of a search step is the determination of a counterclockwise next arc after the current leading arc. It was shown in [WW95] how Gabow and Tarjan's technique for the efficient implementation of certain *union-find*-structures [GT85] can be adapted to determine this arc in constant (amortized) time. The corresponding operation needed during a backtracking step can be performed on the same data structure. One easily verifies that during the update of fields `first` and `last` in modes `FORWARD` and `TRANSITION` every incident arc of a vertex v needs to be traversed at most once. □

7 Discussion

The Menger Problem has four basic variants: edge disjoint (s,t)–paths in undirected graphs, arc disjoint (s,t)–paths directed graphs, vertex disjoint (s,t)–paths in undirected graphs, and vertex disjoint (s,t)–paths in directed graph. For planar graphs, three of these four cases have been solved to optimality by the algorithms in [Wei94a, RLWW97], and in this paper.

The linear time algorithm for the edge disjoint Menger problem in planar undirected graphs transforms an undirected input graph into a directed graph [Wei94a]. Each undirected edge is replaced by two arcs, one for either direction. Even though right cycles are eliminated as described in Section 2, this results in a very special directed graph, i.e. an Eulerian graph. A right-first-search without backtracking is used to find a maximum number of (s,t)–paths. Backtracking is never needed, since every time a vertex is entered by the search path, there must be an unsearched outgoing arc. It is precisely the potential need to backtrack, which makes the directed version much more difficult.

By now, in planar graphs only for the directed vertex disjoint Menger Problem no linear time solution is known. From the undirected versions of the problem one may draw the conclusion that a linear algorithm for this problem might be more difficult to find (assuming there is one at all), since approaches using right-first-search run into difficulties when right cycles are present in the graph. In [KNK93] it was argued that in the case of vertex capacities the set of maximum flows does not have a lattice structure. But it was precisely this structure that allowed an easy restriction to planar graphs without right cycles. In other terms, it appears to be more difficult to resolve the problems caused by right cycles in the case of vertex disjointness.

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