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GEOMETRIC INVARIANT THEORY BASED ON WEIL DIVISORS

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Abstract. Given an action of a reductive group on a normal variety, we construct all invariant open subsets admitting a good quotient with a quasiprojective or a divisorial quotient space. Our approach extends known constructions like Mumford's Geometric Invariant Theory. We obtain several new Hilbert-Mumford type theorems, and we extend a projectivity criterion of Białynicki-Birula and Święcicka for varieties with semisimple group action from the smooth to the singular case.

1. Introduction

This article is devoted to a central task of Geometric Invariant Theory formulated in Białynicki-Birula's recent survey article in the Encyclopedia of Mathematical Sciences [3]: Given an action of a reductive group G on a normal variety X, describe all G-invariant open subsets $U \subset G$ admitting a good quotient, i.e., a G-invariant affine morphism $U \to U /\!\!/ G$ such that the structure sheaf of $U /\!\!/ G$ equals the sheaf of invariants $p_*(\mathcal{O}_U)^G$.

We provide a complete and rounded picture for the collections of all $U \subset X$ admitting quasiprojective or divisorial quotient spaces. Our results comprise Mumford's Geometric Invariant Theory [22] as well as the generalization given in [16]. Within our framework, we prove Hilbert-Mumford type theorems conjectured in [3]. Moreover, we extend a projectivity criterion of Bialynicki-Birula and Święcicka from the smooth to the singular case, and thereby answer a question posed in [7].

Whereas the approaches [22] and [16] use line bundles (or Cartier divisors), we work here in terms of Weil divisors. Given a single Weil divisor D, or a finitely generated group Λ of Weil divisors, we introduce the concept of a G-linearization and define corresponding sets $X^{ss}(D)$ and $X^{ss}(\Lambda)$ of semistable points. A first result concerns quasiprojective quotient spaces, see Theorems 5.2 and 5.3 (below, *G-saturated* means saturated w.r. to the quotient map):

Theorem. *Let a reductive group G act morphically on a normal variety X.*

(i) *For any G-linearized Weil divisor D on X, there is a good quotient $X^{ss}(D) \to X^{ss}(D) /\!\!/ G$ with a quasiprojective variety $X^{ss}(D) /\!\!/ G$.*

(ii) *If $U \subset X$ is open, G-invariant, and has a good quotient $U \to U /\!\!/ G$ with $U /\!\!/ G$ quasiprojective, then U is a G-saturated subset of some $X^{ss}(D)$.*

(iii) *Let D be a G-linearized Weil divisor on X, and let $T \subset G$ be a maximal torus. Then we have*

$$X^{ss}(D, G) = \bigcap_{g \in G} g \cdot X^{ss}(D, T).$$

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This complements Mumford's results: The first statement is a direct generalization of [22, Thm. 1.10]. For smooth X, the second statement basically occurs in [22, Converse 1.13]; but in the singular case there are quasiprojective quotients that do not arise from Mumford's construction, see our example 2.2. The last result is in the spirit of the classical Hilbert-Mumford Criterion. It extends [22, Thm. 2.1], where a related statement for G-linearized line bundles is given under the additional hypothesis that X is projective and the bundle is ample. The fact one can now drop these two assumptions is crucial for our subsequent results.

In Section 6, we replace the single Weil divisor D with a finitely generated group Λ of Weil divisors. As mentioned, the concept of a G-linearization extends, and we also have a notion of semistability. We obtain a quotient construction providing all open subsets with divisorial quotient spaces — a prevariety Y is *divisorial* if every $y \in Y$ has an affine neighbourhood $Y \setminus \mathrm{Supp}(E)$ with an effective Cartier divisor E; this notion was introduced as a natural and far reaching generalization of quasiprojectivity, see [9] and [2]. The first two statements of the following theorem generalize corresponding results of [16], and the third one is a new Hilbert-Mumford type result, see Theorems 6.2 and 6.3:

Theorem. *Let a reductive group G act morphically on a normal variety X.*

(i) *For any G-linearized group $\Lambda \subset \mathrm{WDiv}(X)$, there is a good quotient $X^{ss}(\Lambda) \to X^{ss}(\Lambda)/\!\!/G$ with a divisorial prevariety $X^{ss}(\Lambda)/\!\!/G$.*
(ii) *If $U \subset X$ is open, G-invariant, and has a good quotient $U \to U/\!\!/G$ with $U/\!\!/G$ divisorial, then U is a G-saturated subset of some $X^{ss}(\Lambda)$.*
(iii) *Let $\Lambda \subset \mathrm{WDiv}(X)$ be a G-linearized group, and let $T \subset G$ be a maximal torus. Then we have*
$$X^{ss}(\Lambda, G) = \bigcap_{g \in G} g \cdot X^{ss}(\Lambda, T).$$

In Section 7, we apply our theory to the action of a semisimple group G, and complement the picture obtained so far. We look for maximal sets in the following sense: A *qp-maximal G-set* is an open set $U \subset X$ admitting a good quotient $U \to U/\!\!/G$ with a quasiprojective quotient space such that U does not occur as a G-saturated proper subset in some $U' \subset X$ with the same properties. Similarly, a *d-maximal G-set* is a subset having the analogous properties with respect to divisorial quotient spaces. As well, one can define qp- and d-maximal H-sets for any reductive subgroup $H \subset G$.

In view of the various approaches to quotients by torus actions, see e.g. [4] and [3, Chap. 11], a general strategy is the following: First construct the qp- and d-maximal T-sets for some maximal torus $T \subset G$, and then try to gain the respective G-sets out of these collections. The latter task, means to develop further Hilbert-Mumford type statements in the spirit of Białynicki-Birula's Conjecture [3, 12.1]. Given a maximal T-set $U \subset X$ which is invariant under the normalizer $N \subset G$ of T, one considers the intersection of all translates
$$W(U) := \bigcap_{g \in G} g \cdot U.$$

The question then is whether $W(U)$ is open and admits a good quotient by the action of G. For semisimple groups G, some positive results concerning qp- and d-maximal T-sets $U \subset X$ are known in special cases: The case of $G = \mathrm{SL}_2$ acting on a smooth X is settled in [6, Thm. 9] and [19, Thm. 2.2]. If $U/\!\!/T$ is projective and X

is smooth, then [7, Cor. 1] gives positive answer for a general connected semisimple group G. Moreover, the problem is solved in the case $U = X$, see [3, Thm. 12.4] and [16, Thm. 5.1]. Within our actual setup, we can give a comprehensive answer to Białynicki-Birula's Conjecture [3, 12.1], see Theorem 7.2 and Corollary 7.5:

Theorem. *Let G be a connected semisimple group, and let $T \subset G$ be a maximal torus with normalizer $N \subset G$. Let X be a normal G-variety, let $U \subset X$ be an N-invariant open subset of X, and let $W(U)$ denote the intersection of all translates $g \cdot U$, where $g \in G$.*

(i) *If U is a qp-maximal T-subset of X, then $W(U)$ is an open T-saturated subset of U, and there is a good quotient $W(U) \to W(U)/\!\!/G$ with $W(U)/\!\!/G$ quasiprojective.*
(ii) *If U admits a good quotient $U \to U/\!\!/T$ with $U/\!\!/T$ projective, then $W(U)$ is an open T-saturated subset of U, and there is a good quotient $W(U) \to W(U)/\!\!/G$ with $W(U)/\!\!/G$ projective.*
(iii) *If U is a d-maximal N-subset of X then $W(U)$ is an open T-saturated subset of U, and there is a good quotient $W(U) \to W(U)/\!\!/G$ with $W(U)/\!\!/G$ divisorial.*

Note that one obtains indeed all qp- and d-maximal G-sets in this way, because by [8, Cor. 7] any of these sets is contained in some qp-maximal T-set or a d-maximal N-set. In the setting of (ii), we can prove much more. In fact, it turns out that U and $W(U)$ are the sets of semistable points of an ordinary linearized ample line bundle, and — even more surprising — that X is projective. This extends the main result of [7] from the smooth to the normal case and thereby gives an answer to the problem discussed in [7, Remark p. 965]. More precisely, we obtain, see Theorem 7.4:

Theorem. *Let G be a connected semisimple group, let $T \subset G$ be a maximal torus with normalizer $N \subset G$, and let X be a normal G-variety. Suppose that $U \subset X$ is N-invariant, open and admits a good quotient $U \to U/\!\!/T$ with $U/\!\!/T$ projective. Then there is an ample G-linearized line bundle L over X such that $U = X^{ss}(L, T)$ holds. Moreover, we have $X = G \cdot U$, and X is a projective variety.*

The article is organized as follows: In Section 2, we recall the necessary background on good quotients and Mumford's construction. Sections 3 and 4 are devoted to G-linearizations of Weil divisors and, more generally, polyhedral semigroups of Weil divisors. Such semigroups are related to toric fibre spaces, which recently attracted some attention, see [14] and [25]. In Sections 5 and 6, we give the first main results concerning quasiprojective and divisorial quotient spaces. Finally, in Section 7, we treat semisimple group actions. Throughout the whole article, we shall adapt ideas and methods presented in [16] and [18] to our actual setting.

2. Some background and an example

In this section, we recall the notion of a good quotient, and briefly present Mumford's construction of good quotients with quasiprojective quotient spaces. Moreover, we show by means of an explicit example that for actions on singular varieties, Mumford's method does not yield all good quotients with quasiprojective quotient space.

In the sequel, we work over an algebraically closed field $\mathbb{K}$ of characteristic zero. By a *G-(pre-)variety* X we mean a (pre-)variety X together with a morphical action

$G \times X \to X$ of an algebraic group G. In this section, G is always reductive. The notion of a good quotient is the following, compare [22, p. 38] and [26, Def. 1.5]:

Definition 2.1. A *good quotient* for a G-prevariety X is an affine G-invariant morphism $p\colon X \to Y$ such that the canonical map $p_*(\mathcal{O}_X)^G \to \mathcal{O}_Y$ is an isomorphism. A good quotient is called *geometric* if its fibres are precisely the G-orbits.

Note that in our setting a separated G-variety may have a good quotient with a nonseparated quotient space. If a good quotient $X \to Y$ exists for a G-variety X, then it is categorical, i.e. any G-invariant morphism $X \to Z$ factors uniquely through $X \to Y$. In particular, good quotient spaces are unique up to isomorphism. As usual, we write $X \to X/\!\!/G$ for a good and $X \to X/G$ for a geometric quotient.

In general, a given G-variety X need not have a good quotient; but there frequently exist many G-invariant open subsets $U \subset X$ with good quotient $U \to U/\!\!/G$. It is one of the central tasks of Geometric Invariant Theory to describe or even to construct all these subsets.

As noted in [3, Sec. 7.2], the problem reduces to the construction of "maximal" subsets. More precisely, a subset U of an open G-invariant subset $U' \subset X$ with good quotient $p\colon U' \to U'/\!\!/G$ is called *G-saturated* in U' if U equals $p^{-1}(p(U))$. Then one asks for maximal subsets with respect to G-saturated open inclusion. In the course of this one often imposes additional conditions on the quotient spaces, like completeness and, as we shall do, quasiprojectivity and divisoriality.

Let us recall Mumford's construction. A *G-linearization* of a line bundle L over a G-variety X is a morphical action $G \times L \to L$ such that $L \to X$ is G-equivariant, and all induced maps $L_x \to L_{gx}$ of the fibres are linear. A G-linearization of a bundle L makes its $\mathcal{O}_X$-module $\mathcal{L}$ of sections to a G-sheaf: for a section f, define its translate $g\cdot f$ via

$$(g\cdot f)(x) := g\cdot f(g^{-1}\cdot x).$$

Since any tensor power of a G-linearized bundle L comes along with an induced G-linearization, we obtain a G-sheaf structure on any symmetric power $\mathcal{S}^n\mathcal{L}$. Together this gives a graded G-sheaf structure on the symmetric algebra $\mathcal{S}\mathcal{L}$. In particular, one obtains a representation of G on

$$\mathcal{S}\mathcal{L}(X) = \bigoplus_{n\in\mathbb{N}} \mathcal{S}^n\mathcal{L}(X).$$

Mumford calls a point $x \in X$ *semistable* with respect to L if there is a G-invariant section $f \in \mathcal{S}\mathcal{L}(X)$ which is homogeneous of positive degree such that removing the zero set $Z(f)$ from X gives an affine neighbourhood of x. The set of all semistable points of $L \to X$ is denoted by $X^{ss}(L)$. The crucial properties are, see [22, Thm. 1.10, Converse 1.13, Thm. 2.1], and [5]:

- For every G-linearized bundle L, the set $X^{ss}(L)$ is open, G-invariant, and has a good quotient $X^{ss}(L) \to X^{ss}(L)/\!\!/G$, with a quasiprojective variety $X^{ss}(L)/\!\!/G$.
- If X is smooth, and $U \subset X$ is open and G-invariant with good quotient $U \to U/\!\!/G$ such that $U/\!\!/G$ is quasiprojective, then U is a G-saturated subset of some $X^{ss}(L)$.
- If X is projective, L is an ample G-linearized bundle, and $T \subset G$ is a maximal torus, then we have

$$X^{ss}(L,G) = \bigcap_{g\in G} g\cdot X^{ss}(L,T).$$

In [16], we generalized Mumford's construction by replacing the G-linearized bundle L with a finitely generated free group Λ of Cartier divisors on X. The result is a theory providing the first two statements not only for quasiprojective quotient spaces but more generally for divisorial ones. Recall from [9] that an irreducible prevariety Y is *divisorial*, if every $y \in Y$ has an affine neighbourhood $Y \setminus \mathrm{Supp}(D)$ with an effective Cartier divisor D on Y.

If X is singular, then Mumford's method and the generalization given in [16] need no longer provide all open subsets with quasiprojective or divisorial quotient spaces. For a concrete example, consider the cone X over the image of $\mathbb{P}_1 \times \mathbb{P}_1$ in $\mathbb{P}_3$ under the Segre embedding, i.e.:

$$X = V(\mathbb{K}^4; z_1z_3 - z_2z_4).$$

Then X is a normal variety having precisely one singular point. Let $U := X_{z_2} \cup X_{z_4}$ be the set of points having nonvanishing 2nd or 4th coordinate. We consider the following action of the twodimensional torus $T := \mathbb{K}^* \times \mathbb{K}^*$ on X:

$$t \cdot x := (t_1^2 x_1, t_1 t_2^2 x_2, t_1 t_2 x_3, t_1^2 t_2^{-1} x_4).$$

Proposition 2.2. *The set $U \subset X$ has a geometric quotient $U \to U/T$ with $U/T \cong \mathbb{P}_1$, but U is not the set of semistable points of a T-linearized line bundle on X.*

Proof. The most convenient way is to view X as a toric variety, and to work in the language of lattice fans, see [12] for the basic notions. As a toric variety, X corresponds to the lattice cone σ in $\mathbb{Z}^3$ generated by the vectors

$$v_1 := (1,0,0), \qquad v_2 := (0,1,0), \qquad v_3 := (0,1,1), \qquad v_4 := (1,0,1).$$

The big torus of X is $T_X = (\mathbb{K}^*)^3$. The torus T acts on X by $(t,x) \mapsto \varphi(t) \cdot x$, where $\varphi \colon T \to T_X$ is the homomorphism of tori corresponding to the linear map

$$\mathbb{Z}^2 \to \mathbb{Z}^3, \qquad (1,0) \mapsto (2,1,1), \qquad (0,1) \mapsto (0,2,1).$$

Our open set $U \subset X$ is a union of three T_X-orbits: the big T_X-orbit, and the two twodimensional T_X-orbits corresponding to the rays $\varrho_1 := \mathbb{Q}_{\geq 0} v_1$ and $\varrho_3 := \mathbb{Q}_{\geq 0} v_3$ of the cone σ. The well known fan theoretical criterion [15, Thm. 5.1], tells us that there is a geometric quotient for the action of T on U; namely the toric morphism $p \colon U \to \mathbb{P}_1$ defined by the linear map

$$P \colon \mathbb{Z}^3 \to \mathbb{Z}, \qquad (w_1, w_2, w_3) \mapsto w_1 + 2w_2 - 4w_3.$$

We show now that there is no T-linearized line bundle on X having U as its set of semistable points. First note that as an affine toric variety, X has trivial Picard group. Thus we only have to consider T-linearizations of the trivial bundle. Since $\mathcal{O}^*(X) = \mathbb{K}^*$ holds, each T-linearization of the trivial bundle is given by a character χ of T in the following way:

$$t \cdot (x, z) = (t \cdot x, \chi(t) z).$$

Thus we only have to show that U is not a union of sets X_f, for a collection of functions $f \in \mathcal{O}(X)$ that are T-homogeneous with respect to a common character of the torus T.

Now, any T-homogeneous regular function on X is a sum of T-homogeneous character functions $\chi^u \in \mathcal{O}(X)$, where $u = (u_1, u_2, u_3)$ is a lattice vector of the dual cone $\sigma^\vee$ of σ. Recall that $u \in \sigma^\vee$ means that the linear form u is nonnegative on σ, i.e. we have

$$u_1 \geq 0, \qquad u_2 \geq 0, \qquad u_2 + u_3 \geq 0, \qquad u_1 + u_3 \geq 0.$$

For such a character function $\chi^u \in \mathcal{O}(X)$, we can determine its character of T-homogeneity by applying the dual of the embedding $\mathbb{Z}^2 \to \mathbb{Z}^3$ to the vector u. Thus, χ^u is T-homogeneous with respect to the character of T corresponding to the lattice vector

$$(2u_1 + u_2 + u_3, 2u_2 + u_3).$$

The conditions that a character function $\chi^u \in \mathcal{O}(X)$ does not vanish along the orbit $T_X \cdot x_i$ corresponding to one of the rays ϱ_i are $u_1 = 0$ for nonvanishing along $T_X \cdot x_1$, and $u_3 = -u_2$ for nonvanishing along $T_X \cdot x_3$.

Suppose that $\chi^u \in \mathcal{O}(X)$ does not vanish along $T_X \cdot x_1$ and that $\chi^{\widetilde{u}} \in \mathcal{O}(X)$ does not vanish along $T_X \cdot x_3$. Then their respective T-homogeneities are given by the vectors

$$(u_2 + u_3, 2u_2 + u_3), \qquad (2\widetilde{u}_1, \widetilde{u}_2).$$

If both are T-homogeneous with respect to the same character, then we must have $2\widetilde{u}_1 \le \widetilde{u}_2$. But then nonvanishing along $T_X \cdot x_3$ and the last regularity condition imply $\widetilde{u} = 0$.

In conclusion, we obtain that the only character of T-admitting homogeneous functions that do not vanish along $T_X \cdot x_1$ and functions that do not vanish along $T_X \cdot x_3$ is the trivial one. Since T acts with an attractive fixed point on X this means that we cannot obtain U as a union of sets X_f as needed. □

3. Linearization and toric bundles

In this section, G is an arbitrary linear algebraic group, and X is an irreducible normal G-prevariety. We introduce the notion of a G-linearization for a polyhedral semigroup of Weil divisors, and provide first basic statements concerning existence and uniqueness of such G-linearizations.

As we shall see, semigroups consisting of Cartier divisors correspond to certain toric bundles over X, i.e. bundles having a toric variety as typical fibre, and G-linearizations correspond to certain liftings of the G-action to the total space. Toric bundles have recently drawn some attention apart from our context, see [14], [25].

Let us fix the notation. By $\mathrm{WDiv}(X)$ we denote the group of Weil divisors of X, and $\mathrm{CDiv}(X) \subset \mathrm{WDiv}(X)$ is the subgroup of Cartier divisors. For a finitely generated subsemigroup $\Lambda \subset \mathrm{WDiv}(X)$, let $\Gamma(\Lambda) \subset \mathrm{WDiv}(X)$ denote the subgroup generated by Λ. We say that a subsemigroup $\Lambda \subset \mathrm{WDiv}(X)$ is *polyhedral*, if in $\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma(\Lambda)$ it is the intersection of $\Gamma(\Lambda)$ with a convex polyhedral cone.

Let $\Lambda \subset \mathrm{WDiv}(X)$ be a polyhedral semigroup. Since we assumed X to be normal, there is an associated $\mathcal{O}_X$-module $\mathcal{O}_X(D)$ of rational functions for any $D \in \Lambda$. In fact, multiplication in the function field $\mathbb{K}(X)$ gives even rise to a Λ-graded $\mathcal{O}_X$-algebra:

$$\mathcal{A} := \bigoplus_{D \in \Lambda} \mathcal{A}_D := \bigoplus_{D \in \Lambda} \mathcal{O}_X(D).$$

Recall that a *graded G-sheaf structure* on $\mathcal{A}$ is a collection of graded $\mathcal{O}(U)$-algebra homomorphisms $\mathcal{A}(U) \to \mathcal{A}(g \cdot U)$, which are compatible with group operations in G and with restriction and algebra operations in $\mathcal{A}$. Thereby G acts as usual on the structure sheaf $\mathcal{O}_X$ via $g \cdot f(x) := f(g^{-1} \cdot x)$.

Definition 3.1. By a *G-linearization* of a polyhedral semigroup $\Lambda \subset \mathrm{WDiv}(X)$ we mean a graded G-sheaf structure on the $\mathcal{O}_X$-algebra $\mathcal{A}$ such that for every G-invariant open $U \subset X$ the induced representation of G on $\mathcal{A}(U)$ is rational.

In the case of a semigroup $\Lambda = \mathbb{N}D$ with some $D \in \mathrm{WDiv}(X)$, we shall often speak of a G-linearization of the Weil divisor D and will not mention the semigroup $\mathbb{N}D$ behind it.

We give two existence statements for G-linearizations. The first one generalizes the corresponding result of Mumford, see [22, Cor. 1.6] and also [21, Prop. 2.4]. We use the following terminology: Given polyhedral semigroups $\Lambda' \subset \Lambda$, we say that Λ' *is of finite index* in Λ if there is an $n \in \mathbb{N}$ with $n\Lambda \subset \Lambda'$.

Proposition 3.2. *Suppose that X is separated and that G is connected. Then, for any polyhedral semigroup $\Lambda \subset \mathrm{WDiv}(X)$, some subsemigroup $\Lambda' \subset \Lambda$ of finite index admits a G-linearization.*

Proof. First we construct the G-linearization over the set $U \subset X$ of smooth points. Consider the group $\Gamma \subset \mathrm{WDiv}(X)$ generated by Λ. Then [16, Prop. 1.4], or more explicitely [17, Prop. 3.5] give us a subgroup $\Gamma' \subset \Lambda$ of finite index that is G-linearized over U. Thus $\Lambda' := \Gamma' \cap \Lambda$ has the desired properties over U.

The task now is to exend the G-sheaf structure of the $\mathcal{O}_X$-algebra $\mathcal{A}'$ corresponding to Λ' from U to X. But this is easy: First note that for any open $U' \subset X$ restricting from $\mathcal{A}'(U')$ to $\mathcal{A}'(U' \cap U)$ is a bijection, and that $g \cdot (U' \cap U)$ equals $g{\cdot}U' \cap U$. Then define the translate $g{\cdot}f$ of an $f \in \mathcal{A}'(U')$ to be the unique extension of $g \cdot f_{|U' \cap U}$ to $g{\cdot}U'$. □

The second existence statement provides *canonical G-linearizations*. We say that a Weil divisor $D = \sum n_E E$ is *G-tame* if $n_{gE} = n_E$ holds for any prime cycle E. The support of a G-tame Weil divisor is G-invariant, whereas its components may be permuted. Recall that G acts on the function field $\mathbb{K}(X)$ via $(g{\cdot}f)(x) := f(g^{-1}{\cdot}x)$.

Proposition 3.3. *Let $\Lambda \subset \mathrm{WDiv}(X)$ be polyhedral semigroup consisting of G-tame Weil divisors. Then the action of G on the function field $\mathbb{K}(X)$ induces a G-linearization of Λ.*

Proof. Compare [16, Lemma 4.2]. Let $\mathcal{A}$ denote the graded $\mathcal{O}_X$-algebra associated to Λ. Firstly, we have to show that for any local section $f \in \mathcal{O}(U)$, the translate $g{\cdot}f$ belongs to $\mathcal{A}(g{\cdot}U)$. But this follows from the fact that for any prime divisor E on X we have

$$\mathrm{ord}_E(g{\cdot}f) = \mathrm{ord}_{g^{-1}{\cdot}E}(f).$$

So, G acts on the sheaf $\mathcal{A}$. The remaining point is to verify that for any G-invariant open $U \subset X$ the representation of G on $\mathcal{A}(U)$ is rational. For any G-invariant separated open $V \subset X \setminus \mathrm{Supp}(D)$ this is clear by [21, Lemma 2.5]. The general case follows, because the restriction $\mathcal{A}(U) \to \mathcal{A}(V)$ is injective. □

We shall need a geometric understanding of G-linearizations. For this, we consider polyhedral semigroups consisting of Cartier divisors. Such semigroups correspond to fibre bundles having an affine toric variety as typical fibre:

Proposition 3.4. *Let $\Lambda \subset \mathrm{CDiv}(X)$ be a polyhedral semigroup with associated Λ-graded $\mathcal{O}_X$-algebra $\mathcal{A}$, and let $\widetilde{X} := \mathrm{Spec}(\mathcal{A})$.*

(i) *The Λ-grading of $\mathcal{A}$ defines an action of the torus $S := \mathrm{Spec}(\Gamma(\Lambda))$ on $\widetilde{X}$, and the canonical map $q\colon \widetilde{X} \to X$ is a good quotient for this action.*
(ii) *The map $q\colon \widetilde{X} \to X$ is locally trivial with typical fibre $\widetilde{X}_x \cong \mathrm{Spec}(\mathbb{K}[\Lambda])$. The open set $\widehat{X} \subset \widetilde{X}$ of free S-orbits is an S-principal bundle over X.*

(iii) *The inclusion* $\widehat{X} \subset \widetilde{X}$ *corresponds to the inclusion* $\mathcal{A} \subset \mathcal{B}$ *of the graded* $\mathcal{O}_X$*-algebras* $\mathcal{A}$ *and* $\mathcal{B}$ *associated to* Λ *and* $\Gamma(\Lambda)$.

(iv) *For any homogeneous section* $f \in \mathcal{A}(X)$*, its zero set as a function on* $\widehat{X}$ *equals* $\widehat{X} \cap q^{-1}(\mathrm{Supp}(\mathrm{div}(f) + D))$.

Proof. Consider the group $\Gamma(\Lambda)$ generated by Λ and its $\mathcal{O}_X$-algebra $\mathcal{B}$. Locally, $\mathcal{B}$ is a Laurent monomial algebra over $\mathcal{O}_X$, i.e. for small affine open $U \subset X$, we have a graded isomorphism over $\mathcal{O}(U)$:

$$\mathcal{B}(U) \;\cong\; \mathcal{O}(U) \otimes_{\mathbb{K}} \mathbb{K}[\Gamma(\Lambda)].$$

Cutting down this to the subsemigroup $\Lambda \subset \Gamma(\Lambda)$ and the associated subalgebra $\mathcal{A} \subset \mathcal{B}$, we obtain local triviality of $q\colon \widetilde{X} \to X$. The remaining statements follow then easily. □

In 3.4, the typical fibre $\widetilde{X}_x \cong \mathrm{Spec}(\mathbb{K}[\Lambda])$ is an affine toric variety with big torus S. Hence we speak of $\widetilde{X} \to X$ as the *toric bundle* associated to $\Lambda \subset \mathrm{CDiv}(X)$. As already done in (iii), we shall often skip between viewing a homogeneous $f \in \mathcal{A}(X)$ as a section of a divisor on X and regarding it as a function on $\widetilde{X}$.

In terms of the toric bundle associated to a polyhedral semigroup of Cartier divisors, the meaning of a G-linearization is the following:

Proposition 3.5. *Let* $\Lambda \subset \mathrm{CDiv}(X)$ *be a* G*-linearized polyhedral semigroup with associated graded* $\mathcal{O}_X$*-algebra* $\mathcal{A}$ *and toric bundle* $q\colon \widetilde{X} \to X$*. Then there is a unique set theoretical group action* $G \times \widetilde{X} \to \widetilde{X}$ *with the following properties:*

(i) *For fixed* $g \in G$*, the translation* $\widetilde{X} \to \widetilde{X}$*,* $z \mapsto g\cdot z$ *is a morphism and satisfies* $(g\cdot f)(z) = f(g^{-1}\cdot z)$ *for any local section* $f \in \mathcal{A}(U)$.

(ii) *The actions of* $S = \mathrm{Spec}(\Gamma(\Lambda))$ *and* G *on* $\widetilde{X}$ *commute. In particular* $q\colon \widetilde{X} \to X$ *is* G*-equivariant, and the maps* $\widetilde{X}_x \to \widetilde{X}_{gx}$ *are* S*-equivariant.*

Proof. For any open set $U \subset X$, we have $\mathcal{O}(q^{-1}(U)) = \mathcal{A}(U)$. Thus, for every affine open $U \subset X$ and every $g \in G$, we obtain a commutative diagram of algebra homomorphisms and a corresponding diagram of morphisms:

$$\begin{array}{ccc}
\mathcal{O}(q^{-1}(U)) & \xleftarrow{\;g^{-1}\cdot f \leftarrow f\;} & \mathcal{O}(q^{-1}(g\cdot U)) \\
{\scriptstyle q^*}\uparrow & & \uparrow{\scriptstyle q^*} \\
\mathcal{O}(U) & \xleftarrow[\;g^{-1}\cdot f \leftarrow f\;]{} & \mathcal{O}(g\cdot U)
\end{array}
\qquad
\begin{array}{ccc}
q^{-1}(U) & \xrightarrow{\;T_{U,g}\;} & q^{-1}(g\cdot U) \\
{\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\
U & \xrightarrow[\;x \to gx\;]{} & g\cdot U
\end{array}$$

The morphisms $T_{U,g}$ fit together to morphisms $T_g\colon \widetilde{X} \to \widetilde{X}$, and these in turn define an action $G \times \widetilde{X} \to \widetilde{X}$ as in (i). By the second diagram, the map $q\colon \widetilde{X} \to X$ is G-equivariant. Since all homomorphisms $\mathcal{O}(q^{-1}(g\cdot U)) \to \mathcal{O}(q^{-1}(U))$ are graded, the actions of S and G commute. This gives the second assertion. □

If $\Lambda = \mathbb{N}D$ holds with a Cartier divisor D, then the corresponding toric bundle $\widetilde{X} \to X$ is a line bundle. Thus, Mumford's G-linearized line bundles correspond to semigroups $\Lambda = \mathbb{N}D$ that are strongly G-linearized in the following sense:

Definition 3.6. We say that a G-linearization of a polyhedral semigroup $\Lambda \subset \mathrm{CDiv}(X)$ is *strong* if the corresponding action $G \times \widetilde{X} \to \widetilde{X}$ on the toric bundle space is a morphism.

Note that this definition differs from [16, Def. 1.2]. We shall need to know if a given G-linearization of a polyhedral semigroup of Cartier divisors is strong. Here comes a first couple of criteria; a further result will be given in Proposition 4.8.

Proposition 3.7. *A G-linearization of a polyhedral semigroup $\Lambda \subset \mathrm{CDiv}(X)$ is strong if one of the following statements holds:*

(i) *X is covered by affine G^0-invariant open subsets, e.g., G^0 is a torus.*
(ii) *The G-linearization of Λ is strong over the set of smooth points.*

Proof. As before, let $\widetilde{X} \to X$ be the toric bundle associated to Λ. Suppose that we are in the setting of (i). Surely it suffices to prove regularity of the induced action $G^0 \times \widetilde{X} \to \widetilde{X}$ of the unit component $G^0 \subset G$. By assumption, this action locally arises from rational G^0-representations. This readily implies regularity.

In the setting of (ii), we know that the G-action on $\widetilde{X}$ is regular over the smooth points of X. Moreover, by G-equivariance of $\widetilde{X} \to X$, we can cover $\widetilde{X}$ by affine open sets such that their inverse images in $G \times \widetilde{X}$ under the action map are open. Then regularity of $G \times \widetilde{X} \to \widetilde{X}$ follows from normality of $\widetilde{X}$. □

We turn to uniqueness properties of G-linearizations. Let $\mathrm{Char}(G)$ denote the group of characters of G, i.e. the group of all homomorphisms $G \to \mathbb{K}^*$. For groups G with few characters, we have the following two statements, compare [22, Prop. 1.4] and [16, Prop. 1.5]:

Proposition 3.8. *Let X be separated, and let $\Lambda \subset \mathrm{CDiv}(X)$ be a polyhedral semigroup.*

(i) *If $\mathrm{Char}(G)$ is trivial and G is connected, then any two strong G-linearizations of Λ coincide.*
(ii) *If $\mathrm{Char}(G)$ is finite and $\mathcal{O}^*(X) = \mathbb{K}^*$ holds, then any two strong G-linearizations of Λ induce the same G-linearization on some $\Lambda' \subset \Lambda$ of finite index.*

Proof. Consider the toric bundle $q\colon \widetilde{X} \to X$ associated to Λ and the action of the torus $S := \mathrm{Spec}(\mathbb{K}[\Gamma(\Lambda)])$ on $\widetilde{X}$. Given two strong G-linearizations of Λ, we have two morphical G-actions on $\widetilde{X}$. We denote them by $g{\cdot}z$ and $g{*}z$, and consider the morphism

$$\Phi\colon G \times \widetilde{X} \to \widetilde{X}, \qquad z \mapsto g^{-1}{*}g{\cdot}z.$$

For fixed g, the map $z \mapsto \Phi(g,z)$ is an S-equivariant bundle automorphism. Hence, on each fibre it is multiplication with an element of the torus S. Consequently, there is a morphism $\alpha\colon G \times X \to S$ such that Φ is of the form

$$\Phi(g,z) = \alpha(g, q(z)){\cdot}z$$

In the setting of (i), Rosenlicht's Lemma [11, Lemma 2.1] yields a decomposition $\alpha(g,z) = \chi(g)\beta(q(z))$ with a regular homomorphism $\chi\colon G \to S$ and a morphism $\beta\colon X \to S$. Since we assumed G to have only trivial characters, we can conclude that Φ is the identity map.

If we are in the situation of (i), then $\mathcal{O}^*(X) = \mathbb{K}^*$ implies that $\Phi(g,z) = \chi(g){\cdot}z$ holds with a regular homomorphism $\chi\colon G \to S$. Hence, after dividing $\widetilde{X}$ by the finite subgroup $\chi(G) \subset S$, the two induced G-actions coincide. But this process means replacing Λ with a subsemigroup of finite index. □

Let us remark that there are simple examples showing that for non connected G, one cannot omit the assumption $\mathcal{O}^*(X) = \mathbb{K}^*$ in the second statement.

4. The ample locus

We introduce the ample locus of a polyhedral semigroup of Weil divisors, and study its behaviour in the case of G-linearized semigroups. The considerations of this section prepare the proofs of the various Hilbert-Mumford type theorems given later.

Unless otherwise stated, X denotes in this section an irreducible normal prevariety. Given a polyhedral semigroup $\Lambda \subset \mathrm{WDiv}(X)$, and a homogeneous local section $f \in \mathcal{A}_D(U)$ of the graded $\mathcal{O}_X$-algebra $\mathcal{A}$ associated to Λ, we define the *zero set* of this section as

$$Z(f) := \mathrm{Supp}(\mathrm{div}(f) + D|_U).$$

Definition 4.1. Let $\Lambda \subset \mathrm{WDiv}(X)$ be a polyhedral semigroup with associated Λ-graded $\mathcal{O}_X$-algebra $\mathcal{A}$.

(i) The *Cartier locus* of Λ is the set of all points $x \in X$ such that every $D \in \Lambda$ is Cartier near x.

(ii) The *ample locus* of Λ is the set of all $x \in X$ admitting an affine neighbourhood $X \setminus Z(f)$ with a homogeneous section $f \in \mathcal{A}(X)$ such that $X \setminus Z(f)$ is contained in the Cartier locus of Λ.

We shall speak of an *ample* semigroup $\Lambda \subset \mathrm{WDiv}(X)$ if the ample locus of Λ equals X. Thus, ample semigroups consist by definition of Cartier divisors. The relations to the usual concepts of ample divisors [13] and more generally ample families [9], [2] are the following:

Remark 4.2. (i) A polyhedral semigroup of the form $\Lambda = \mathbb{N}D$ is ample if and only if D is an ample Cartier divisor in the usual sense.

(ii) A normal prevariety is divisorial if and only if it admits an ample group of Cartier divisors.

Let us explain the geometric meaning of the ample locus of a polyhedral semigroup $\Lambda \subset \mathrm{CDiv}(X)$ in terms of the corresponding toric bundle $q\colon \widetilde{X} \to X$. Recall from Section 3 that $\widetilde{X}$ comes along with an action of the torus $S := \mathrm{Spec}(\mathbb{K}[\Gamma(\Lambda)])$, and that the set $\widehat{X} \subset \widetilde{X}$ of free S-orbits is an S-principle bundle over X.

Proposition 4.3. *Let $\Lambda \subset \mathrm{CDiv}(X)$ be a polyhedral semigroup with associated toric bundle $q\colon \widetilde{X} \to X$ and ample locus $U \subset X$. Then $q^{-1}(U) \cap \widehat{X}$ is quasiaffine.*

Proof. Consider the subgroup $\Gamma(\Lambda) \subset \mathrm{CDiv}(X)$ generated by Λ, and denote the associated graded $\mathcal{O}_X$-algebra by $\mathcal{B}$. Then $\widehat{X}$ equals $\mathrm{Spec}(\mathcal{B})$, and for any homogeneous $f \in \mathcal{B}(X)$, its zero set as a function on $\widehat{X}$ is equal to the inverse image $q^{-1}(Z(f)) \cap \widehat{X}$. Consequently, the set $q^{-1}(U) \cap \widehat{X}$ is covered by affine open subsets of the form $\widehat{X}_f$ with $f \in \mathcal{O}(\widehat{X})$. This gives the assertion. □

We turn to the equivariant setting. Let G be a linear algebraic group, and suppose that G acts morphically on the normal prevariety X. A first observation is that the zero set $Z(f)$ of a homogeneous section f behaves natural with respect to the G-action:

Lemma 4.4. *Let $\Lambda \subset \mathrm{WDiv}(X)$ be a G-linearized polyhedral semigroup, and let $f \in \mathcal{A}_D(U)$ be a local section of the associated graded $\mathcal{O}_X$-algebra $\mathcal{A}$. Then we have $Z(g \cdot f) = g \cdot Z(f)$ for any $g \in G$.*

Proof. By normality of X, we may assume that U is smooth. The problem being local, we may moreover assume that D is principal on U, say $D = -\mathrm{div}(h)$. Then the section f is of the form $f = f'h$ with a regular function f', and $Z(f)$ is just the zero set $Z(f')$ of f'. Translating with $g \in G$ gives

$$Z(g\cdot f) = Z(g\cdot f'\, g\cdot h) = Z(g\cdot f') \cup Z(g\cdot h).$$

Since h is a generator of $\mathcal{A}(U)$, the translate $g\cdot h$ is a generator of $\mathcal{A}(g\cdot U)$. This means that $Z(g\cdot h)$ is empty. By the definition of a G-linearization, G acts canonically on the structure sheaf $\mathcal{O}_X$, that means that $g\cdot f'(x)$ equals $f'(g^{-1}\cdot x)$. This implies $Z(g\cdot f') = g\cdot Z(f')$, and the assertion follows. □

Proposition 4.5. *Let $\Lambda \subset \mathrm{WDiv}(X)$ be a G-linearized polyhedral semigroup. Then the Cartier locus and the ample locus of Λ are G-invariant.*

Proof. Let $\mathcal{A}$ denote the graded $\mathcal{O}_X$-algebra corresponding to Λ. The Cartier locus of Λ is the set of all points $x \in X$ such that for any $D \in \Lambda$ the stalk $\mathcal{A}_{D,x}$ is generated by a single element. Thus, using the G-sheaf structure of $\mathcal{A}$, we obtain that the Cartier locus is G-invariant. Invariance of the ample locus, then is a simple consequence of Lemma 4.4. □

As a direct application, we extend a fundamental observation of Sumihiro on actions of connected linear algebraic groups G on normal varieties X, see [27, Lemma 8], and [28, Thm. 3.8]: Every point $x \in X$ admits a G-invariant quasiprojective open neighbourhood. Our methods give more generally:

Proposition 4.6. *Let G be a connected linear algebraic group, let X be a normal G-variety, and let $U \subset X$ be an open subset.*

(i) *If U is quasiprojective, then $G\cdot U$ is quasiprojective.*
(ii) *If U is divisorial, then $G\cdot U$ is divisorial.*

In particular, the maximal quasiprojective and the maximal divisorial open subsets of X are G-invariant.

If X admits a normal completion for which the factor group of Weil divisors modulo $\mathbb{Q}$-Cartier divisors is of finite rank, then [29, Thm. A] says that X has only finitely many maximal open quasiprojective subvarieties. In particular, for such X, statement (i) even holds with an arbitrary connected algebraic group G, see [29, Thm. D]. A special case of the second statement is proved in [1, Lemma 1.7].

Proof of Proposition 4.6. Let $D_1, \ldots, D_r$ be the prime cycles contained in the complement $X\setminus U$. Consider a Cartier divisor D' on U, and a global section f' of $\mathcal{O}_U(D')$ such that $U \setminus Z(f')$ is affine. Then the complement of $U \setminus Z(f')$ in X is of pure codimension one, and thus we have

$$U \setminus Z(f') = X \setminus (D_1 \cup \ldots \cup D_r \cup \overline{Z(f')}).$$

Consequently, by closing the components of D' and adding a suitably big multiple of $D_1 + \ldots + D_r$, we obtain a Weil divisor D on X such that $D|_U$ equals D' and f' extends to a global section f of $\mathcal{O}_X(D)$ satisfying

$$X \setminus Z(f) = U \setminus Z(f').$$

Suppose we are in (i), and $\Lambda' = \mathbb{N}D'$ is an ample semigroup in $\mathrm{CDiv}(U)$. Let $f_1', \ldots, f_r' \in \mathcal{A}'(U)$ be homogeneous sections of the associated $\mathcal{O}_U$-algebra $\mathcal{A}'$ such that the sets $U \setminus Z(f_i')$ cover U. Using the above principle, we can extend Λ'

to a semigroup $\Lambda = \mathbb{N}D$ in $\mathrm{WDiv}(X)$ such that the f_i' extend to global sections $f_i \in \mathcal{A}(X)$ of the $\mathcal{O}_X$-algebra $\mathcal{A}$ associated to Λ and satisfy $X \setminus Z(f_i) = U \setminus Z(f_i')$.

Similarly, in the situation of (ii), we can fix a group $\Lambda' \subset \mathrm{CDiv}(U)$ and a collection $f_1', \ldots, f_r'$ of homogeneous sections such that the sets $U \setminus Z(f_i')$ are affine and cover U. By the same reasoning as before, we can extend these data to a group $\Lambda \subset \mathrm{WDiv}(X)$ with a collection of global homogeneous sections $f_1, \ldots, f_r$ satisfying $X \setminus Z(f_i) = U \setminus Z(f_i')$, and hence being as in 4.1 (ii).

In both cases U is contained by construction in the ample locus of the extension Λ. Since passing to subsemigroups of finite index does not shrink the ample locus, we can use Proposition 3.2, and endow Λ with a G-linearization. The assertion then follows from G-invariance of the ample locus of Λ and the fact that quasiprojectivity as well as divisoriality transfer to open subvarieties. □

We conclude this section with a further regularity criterion for the G-action on the toric bundle $\widetilde{X} \to X$ associated to a G-linearized polyhedral semigroup of Cartier divisors on X. A first step is the following equivariant and refined version of Proposition 4.3; again, we consider the subset $\widehat{X} \subset \widetilde{X}$ of free S-orbits:

Lemma 4.7. *Let $\Lambda \subset \mathrm{CDiv}(X)$ be a G-linearized polyhedral semigroup with associated toric bundle $q \colon \widetilde{X} \to X$. Let $U \subset X$ be the ample locus of Λ, and set $\widehat{U} := \widehat{X} \cap q^{-1}(U)$. Then there is a $(G \times S)$-equivariant open embedding $\widehat{U} \to Z$ into an affine $(G \times S)$-variety Z. Moreover,*

- (i) *one can achieve that the image of the pullback map $\mathcal{O}(Z) \to \mathcal{O}(\widehat{U})$ is contained in $\mathcal{O}(\widetilde{X})$,*
- (ii) *given $f_1, \ldots, f_k \in \mathcal{A}(X)$ as in 4.1 with $\widetilde{X}_{f_i} \subset \widehat{X}$, one can achieve that each f_i extends regularly to Z and satisfies $\widehat{U}_{f_i} = Z_{f_i}$.*

Proof. Let $f_1, \ldots, f_k \in \mathcal{A}(X)$ be as in (ii), and complement this collection by further homogeneous sections $f_{k+1}, \ldots, f_r \in \mathcal{A}(X)$ as in Definition 4.1 such that the affine sets $X_i := X \setminus Z(f_i)$ cover the ample locus $U \subset X$. Then each f_i, regarded as a regular function on $\widetilde{X}$, vanishes outside the affine open set $\widetilde{X}_i := q^{-1}(X_i)$ and has no zeroes inside $\widetilde{X}_i \cap \widehat{X}$.

For each index i, we choose finitely many homogeneous functions $h_{ij} \in \mathcal{O}(\widetilde{X})$ such that the affine algebra $\mathcal{O}(\widetilde{X})_{f_i}$ is generated by functions $h_{ij}/f^{l_{ij}}$. Since the G-representation on $\mathcal{O}(\widetilde{X})$ is rational, we find finite dimensional graded G-modules $M_i, M_{ij} \subset \mathcal{O}(\widetilde{X})$ such that $f_i \in M_i$ and $h_{ij} \in M_{ij}$ holds.

Let $R \subset \mathcal{O}(\widetilde{X})$ denote the subalgebra generated by the elements of the M_i and the M_{ij}. Then R is graded, G-invariant, and it is finitely generated. Thus it defines an affine $(G \times S)$-variety $Z := \mathrm{Spec}(R)$. By construction, we have $Z_{f_i} = \widetilde{X}_{f_i}$. In particular, Z contains $\widehat{U}$ as an open invariant subvariety and has the desired properties. □

Proposition 4.8. *Let $\Lambda \subset \mathrm{CDiv}(X)$ be a G-linearized polyhedral semigroup. Then the associated action $G \times \widetilde{X} \to \widetilde{X}$ on the toric bundle is regular over the ample locus of Λ.*

Proof. Consider once more the subset $\widehat{X} \subset \widetilde{X}$ consisting of free orbits with respect to the action of $S := \mathrm{Spec}(\mathbb{K}[\Gamma(\Lambda)])$ on $\widetilde{X}$. As noted earlier, $\widehat{X}$ is an S-principal bundle over X. Let $U \subset X$ be the ample locus of Λ, and set $\widetilde{U} := q^{-1}(U)$. According to Lemma 4.7, the group G acts morphically on the set $\widehat{U} := \widehat{X} \cap \widetilde{U}$.

Choose a cover of U by open subsets $U_i \subset U$ such that the toric bundle $\widetilde{U} \to U$ is trivial over the U_i. Let $\widetilde{U}_i := q^{-1}(U_i)$, and set $\widehat{U}_i := \widetilde{U}_i \cap \widehat{U}$. Moreover, for any two i, j, consider the sets

$$\begin{aligned} U_{(i,j)} &:= \{(g,x) \in G \times U_i;\ g\cdot x \in U_j\}, \\ \widetilde{U}_{(i,j)} &:= \{(g,z) \in G \times \widetilde{U}_i;\ g\cdot z \in \widetilde{U}_j\}, \\ \widehat{U}_{(i,j)} &:= \widetilde{U}_{(i,j)} \cap (G \times \widehat{U}_i). \end{aligned}$$

Then $\widehat{U}_{(i,j)}$ is isomorphic to the product $G \times U_{(i,j)} \times S$. Since the map $q\colon \widehat{U} \to U$ is G-equivariant, the G-action is on $\widehat{U}_{(i,j)}$ of the form

$$(g,x,s) \mapsto (g\cdot x, \alpha(g,x,s))$$

with a morphism $\alpha\colon \widehat{U}_{(i,j)} \to S$. Set $\beta(g,x) := \alpha(g,x,e_S)$. Then, by Rosenlicht's Lemma [11, Cor. 2.2], the scaled morphism

$$(g,x,s) \mapsto \alpha(g,x,s)\beta(g,x)^{-1}$$

defines for fixed (g,x) a homomorphism $S \to S$. By rigidity of tori, see e.g. [20, Prop. 16.3], the scaled morphism does not depend on (g,x). In other words, the map α is of the form

$$\alpha(g,x,s) = \beta(g,x)\varphi(s)$$

with a morphism $\varphi\colon S \to S$ of algebraic groups. Since each map $\widetilde{X}_x \to \widetilde{X}_{gx}$ is regular, φ extends to a morphism of the typical fibre. This shows that $\widehat{U}_{(i,j)} \to \widehat{U}_i$ extends to a morphism $\widetilde{U}_{(i,j)} \to \widetilde{U}_i$, and thus implies regularity $G \times \widetilde{U} \to \widetilde{U}$. □

5. Quasiprojective good quotient spaces

In this section, G is a reductive group, and X is a normal G-variety. We extend Mumford's concept of semistability to G-linearized semigroups $\mathbb{N}D$, where D is a Weil divisor on X. This gives an appropriate language to describe all open subsets of X that admit a quasiprojective good quotient space.

Fix a Weil divisor D on X, and a G-linearization of the semigroup $\Lambda := \mathbb{N}D$; recall that the latter is a certain G-sheaf structure on the associated Λ-graded $\mathcal{O}_X$-algebra $\mathcal{A}$. We shall speak in the sequel of the G-linearized Weil divisor D. Our definition of semistability is a straightforward generalization of [22, Def. 1.7 (b)]:

Definition 5.1. We call a point $x \in X$ *semistable* if there is an integer $n > 0$ and a G-invariant $f \in \mathcal{A}_{nD}(X)$ such that $X \setminus Z(f)$ is a affine neighbourhood of x and D is Cartier on $X \setminus Z(f)$.

Following Mumford's notation, we denote the set of semistable points of a G-linearized Weil divisor D on X by $X^{ss}(D)$, or by $X^{ss}(D,G)$ if we want to specify the group G. Our concept of semistability yields all open subsets admitting a quasiprojective good quotient space:

Theorem 5.2. *Let a reductive group G act morphically on a normal variety X.*

(i) *For any G-linearized Weil divisor D on X, there is a good quotient $X^{ss}(D) \to X^{ss}(D) /\!\!/ G$ with a quasiprojective variety $X^{ss}(D) /\!\!/ G$.*

(ii) *If $U \subset X$ is open, G-invariant, and has a good quotient $U \to U /\!\!/ G$ with $U /\!\!/ G$ quasiprojective, then U is a G-saturated subset of the set $X^{ss}(D)$ of semistable points of a canonically G-linearized Weil divisor D.*

Proof. For i), we can follow the lines of [22, Thm. 1.10]: Choose G-invariant homogeneous $f_1, \dots, f_r \in \mathcal{A}(X)$ as in Definition 5.1 such that $X^{ss}(D)$ is covered by the sets $X_i := X \setminus Z(f_i)$. Replacing the f_i with suitable powers, we may assume that all of them have the same degree. Consider the good quotients:

$$p_i \colon X_i \to X_i /\!\!/ G = \operatorname{Spec}(\mathcal{O}(X_i)^G).$$

Each $X_i \setminus X_j$ is the zero set of the G-invariant regular function f_j/f_i. Thus $X_i \cap X_j$ is saturated with respect to the quotient map $p_i \colon X_i \to X_i /\!\!/ G$. It follows that the p_i glue together to a good quotient $p \colon X^{ss}(D) \to X^{ss}(D) /\!\!/ G$. Moreover, for fixed i_0, the f_{i_0}/f_i are local equations for an ample Cartier divisor on $X^{ss}(D) /\!\!/ G$.

To prove (ii), let $Y := U /\!\!/ G$, and let $p \colon U \to Y$ denote the quotient map. Choose an ample Cartier divisor E on Y allowing homogeneous sections $h_1, \dots, h_r \in \mathcal{O}_Y(E)$ such that the sets $Y \setminus Z(h_i)$ form an affine cover of Y. Let $D_1, \dots, D_k$ be the prime cycles of X contained in $X \setminus U$. Similarly as in the proof of Proposition 4.6, we can consider the Weil divisor

$$D := mD_1 + \dots + mD_k + \overline{p^*(E)}.$$

Here the bar over the pullback $p^*(E)$ means closing components. By construction, D is a G-tame Weil divisor, and $D|_U$ equals $p^*(E)$. Moreover, if we choose $m \in \mathbb{N}$ big enough, then the pullbacks $p^*(h_i)$ extend to global sections f_i of $\mathcal{O}_X(D)$ and satisfy

$$X \setminus Z(f_i) = p^{-1}(Y \setminus Z(h_i)).$$

Let $\mathcal{A}$ be the graded $\mathcal{O}_X$-algebra associated to D, and consider the canonical G-linearization of D provided by Proposition 3.3. Then the sections $f_i \in \mathcal{A}(X)$ are G-invariant, and satisfy the conditions of 5.1. It follows that U is a saturated subset of $X^{ss}(D)$. $\square$

We come to the first Hilbert-Mumford type statement of the article. It allows us to express the set of G-semistable points in terms of the T-semistable points for a maximal torus $T \subset G$. In the case of an ample divisor D on a projective G-variety, our result gives back [22, Thm. 2.1].

Theorem 5.3. *Let a reductive group G act morphically on a normal variety X, let $T \subset G$ be a maximal torus, and let D be a G-linearized Weil divisor on X. Then we have:*

$$X^{ss}(D, G) = \bigcap_{g \in G} g \cdot X^{ss}(D, T).$$

The proof relies on a geometric analysis of instability and makes repeated use of the classical Hilbert Mumford Theorem [10, Thm. 4.2]. For later purposes, we formulate the basic steps separately. We study the following situation: G is a reductive group, Z is an affine G-variety, and $T \subset G$ is a maximal torus. Then we have good quotients

$$p_T \colon Z \to Z /\!\!/ T, \qquad p_G \colon Z \to Z /\!\!/ G.$$

Lemma 5.4. *Let $A \subset Z$ be G-invariant and closed, and let $z \in p_G^{-1}(p_G(A))$. Then there is a $g \in G$ with $g \cdot z \in p_T^{-1}(p_T(A))$.*

Proof. Since $p_G\colon Z \to Z/\!\!/G$ separates disjoint G-invariant closed sets, the closure of $G\cdot z$ intersects A. By [10, Thm. 4.2], there is a maximal torus $S \subset G$ such that the closure of $S\cdot z$ intersects A. Choose a $g \in G$ with $T = gSg^{-1}$. Then the closure of $T\cdot g\cdot z$ intersects A. This implies $p_T(g\cdot z) \in p_T^{-1}(p_T(A))$. □

Suppose that in addition to the G-action there is an action of $\mathbb{K}^*$ on Z such that these two actions commute. Then there are induced $\mathbb{K}^*$-actions on the quotient spaces $Z/\!\!/T$ and $Z/\!\!/G$ making the respective quotient maps equivariant. Let $B_T^0 \subset Z/\!\!/T$ and $B_G^0 \subset Z/\!\!/G$ denote the fixed point sets of these $\mathbb{K}^*$-actions.

Lemma 5.5. *Let $z \in Z$ with $p_G(z) \in B_G^0$. Then there is a $g \in G$ with $p_T(gz) \in B_T^0$.*

Proof. Let $G\cdot z_0$ be the closed G-orbit in the fibre $p_G^{-1}(p_G(z))$. If z_0 is a fixed point of the $\mathbb{K}^*$-action on Z, then the whole orbit $G\cdot z_0$ consists of $\mathbb{K}^*$-fixed points, and the assertion is a direct consequence of [10, Thm. 4.2]. So we may assume for this proof that the orbit $\mathbb{K}^*\cdot z_0$ is nontrivial.

By [10, Thm. 4.2] there are a onedimensional subtorus $S_0 \subset G$ and a $g_0 \in G$ such that z_0 lies in the closure of $S_0\cdot z'$, where $z' := g_0\cdot z$. Note that for any $t \in \mathbb{K}^*$, the point $t\cdot z_0$ lies in the closure of $S_0\cdot t\cdot z'$. This implies in particular that any point of $\mathbb{K}^*\cdot z_0$ is fixed by S_0. Consequently, S_0 is a subgroup of the stabilizer G_0 of $\mathbb{K}^*\cdot z_0$.

Let $n \in \mathbb{N}$ denote the order of the isotropy group of $\mathbb{K}^*$ in z_0. Then the orbit maps $\mu\colon g \mapsto g\cdot z_0$ of G_0 and $\nu\colon t \mapsto t\cdot z_0$ of $\mathbb{K}^*$ give rise to a well defined morphism of linear algebraic groups:

$$G_0 \to \mathbb{K}^*, \qquad g \mapsto (\nu^{-1}(\mu(g)))^n.$$

Clearly, S_0 is contained in the kernel of this homomorphism. By general properties of linear algebraic groups any maximal torus of G_0 is mapped onto $\mathbb{K}^*$, see e.g. [20, Cor. C, p. 136]. We choose a maximal torus $S_1 \subset G_0$ such that S_1 contains S_0.

Let $S \subset G$ be a maximal torus with $S_1 \subset S$. Then z_0 lies in the closure of $S\cdot z'$. Moreover, $\mathbb{K}^*\cdot z_0$ is contained in $S\cdot z_0$. Writing $S = g_1^{-1}Tg_1$ with a suitable $g_1 \in G$, we obtain that $g_1\cdot z_0$ lies in the closure of $T\cdot g_1\cdot z'$, and $\mathbb{K}^*\cdot g_1\cdot z_0$ is contained in $T\cdot g_1\cdot z_0$. Thus, $g := g_1g_0$ is as wanted. □

The next observation concerns limits with respect to the $\mathbb{K}^*$-action on the quotient spaces. For $H = T$ and $H = G$ we consider the sets:

$$B_H^- := \{y \in Z/\!\!/H;\ \lim_{t\to\infty} t\cdot y \text{ exists and differs from } y\}.$$

Lemma 5.6. *Let $z \in Z$ with $p_G(z) \in B_G^-$. Then there is a $g \in G$ such that $p_T(g\cdot z) \in B_T^-$ holds.*

Proof. Let $y_0 \in Z/\!\!/G$ be the limit point of $p_G(z)$, and choose $z_0 \in Z$ with $G\cdot z_0$ closed in Z and $p_G(z_0) = y_0$. Note that $G\cdot z_0$ is $\mathbb{K}^*$-invariant. Consider the quotient $q\colon Z \to Z/\!\!/\mathbb{K}^*$. Then $G\cdot q(z_0)$ is contained in the closure of $G\cdot q(z)$, because $q(G\cdot z_0)$ is closed, and we have

$$(Z/\!\!/G)/\!\!/\mathbb{K}^* = (Z/\!\!/\mathbb{K}^*)/\!\!/G.$$

Thus, according to [10, Thm. 4.2], there exist $g, g_0 \in G$ such that $g_0\cdot q(z_0)$ lies in the closure of $T\cdot g\cdot q(z)$. We can conclude that in $Z/\!\!/T$, the $\mathbb{K}^*$-orbit closures of the points $p_T(g\cdot z)$ and $p_T(g_0\cdot z_0)$ intersect nontrivially; this time we use

$$(Z/\!\!/T)/\!\!/\mathbb{K}^* = (Z/\!\!/\mathbb{K}^*)/\!\!/T.$$

Since we have a $\mathbb{K}^*$-equivariant map $Z/\!\!/T \to Z/\!\!/G$, and there is a G-invariant homogeneous function $f \in \mathcal{O}(Z)$ of negative weight with $f(z) \neq 0$ and $f(z_0) = 0$, it follows that $p_T(g \cdot z)$ belongs to B_T^-. □

Proof of Theorem 5.3. Let $\mathcal{A}$ be the graded $\mathcal{O}_X$-algebra associated to D. Consider the Cartier locus $X_0 \subset X$ of $\Lambda := \mathbb{N}D$. By Proposition 4.5, the set X_0 is G-invariant. Moreover, by normality of X, the complement $X \setminus X_0$ is of codimension at least two in X. Consequently, $X_0^{ss}(D,T)$ equals $X^{ss}(D,T)$, and $X_0^{ss}(D,G)$ equals $X^{ss}(D,G)$. Thus we may assume for this proof that $X = X_0$ holds.

Moreover, we may assume that D and hence Λ are nontrivial. Then the associated toric bundle $\widetilde{X} := \mathrm{Spec}(\mathcal{A})$ is a line bundle over X, and the torus acting on $\widetilde{X}$ is $\mathbb{K}^*$. Consider the G-action on $\widetilde{X}$ provided by Proposition 3.5. Removing the zero section gives the $(G \times \mathbb{K}^*)$-invariant open subvariety $\widehat{X} \subset \widetilde{X}$. Let $q\colon \widetilde{X} \to X$ be the canonical map, let $U \subset X$ be the ample locus of D, and set $\widehat{U} := q^{-1}(U) \cap \widehat{X}$.

Choose T-invariant homogeneous sections $f_1, \ldots, f_r \in \mathcal{A}(X)$ and G-invariant homogeneous sections $h_1, \ldots, h_s \in \mathcal{A}(X)$ as in Definition 5.1 such that the complements $X \setminus Z(f_i)$ and $X \setminus Z(h_j)$ cover the respective sets $X^{ss}(D,T)$ and $X^{ss}(D,G)$ of semistable points. Regarded as functions on $\widetilde{X}$, the f_i and the h_j vanish along the zero section $\widetilde{X} \setminus \widehat{X}$ because they are of positive degree.

According to Lemma 4.7, we can choose a G-equivariant open embedding $\widehat{U} \subset Z$ into an affine G-variety Z with the following two properties: Firstly, we have $\mathcal{O}(Z) \subset \mathcal{O}(\widetilde{X})$. Secondly, the functions $f_i, h_j \in \mathcal{O}(\widehat{U})$ extend regularly to Z and satisfy $\widehat{U}_{f_i} = Z_{f_i}$ and $\widehat{U}_{h_j} = Z_{h_j}$.

Now consider the induced $\mathbb{K}^*$-actions on the quotient spaces $Z/\!\!/T$ and $Z/\!\!/G$. As before, let B_T^0, B_G^0 be the fixed point sets of these $\mathbb{K}^*$-actions, and let B_T^-, B_G^- be the sets of non fixed points admitting a limit for $t \to \infty$. Then, setting $A := Z \setminus \widehat{U}$, we claim that for the respective sets of semistable points one has:

$$
\begin{aligned}
\widehat{X} \cap q^{-1}(X^{ss}(D,T)) &= Z \setminus p_T^{-1}(p_T(A) \cup B_T^0 \cup B_T^-), \\
\widehat{X} \cap q^{-1}(X^{ss}(D,G)) &= Z \setminus p_G^{-1}(p_G(A) \cup B_G^0 \cup B_G^-).
\end{aligned}
$$

Indeed, the inclusion "$\subset$" of the first equation is due to the facts that the intersection $\widehat{X} \cap q^{-1}(X \setminus Z(f_i))$ equals Z_{f_i}, and that each f_i by T-invariance and homogeneity of positive degree vanishes along the set $p_T^{-1}(p_T(A) \cup B_T^0 \cup B_T^-)$. Analogously one obtains the inclusion "$\subset$" for the second equality.

Conversely, to see the inclusions "$\supset$", we treat again exemplarily the first equation. First note that $\mathcal{O}(Z) \subset \mathcal{O}(\widehat{X})$ defines a morphism $\widehat{X} \to Z$. Since we have $\widehat{X}_{f_i} = \widehat{U}_{f_i} = Z_{f_i}$, this morphism is an isomorphism over $\widehat{U}$. In particular, for every $f \in \mathcal{O}(Z)$ vanishing along $Z \setminus \widehat{U}$, we have $\widehat{X}_f = Z_f$.

Next consider the ideal of $p_T(A) \cup B_T^0 \cup B_T^-$ in $\mathcal{O}(Z/\!\!/T)$. This ideal is generated by functions f' that are homogeneous of positive degree. Since $\mathcal{O}(Z) \subset \mathcal{O}(\widetilde{X})$ holds, each $f := p_T^*(f')$ is a T-invariant homogeneous section of positive degree in $\mathcal{A}(X)$. By the above consideration, we have

$$
Z_f \;=\; \widehat{X}_f \;=\; \widehat{X} \cap q^{-1}(X \setminus Z(f)).
$$

It follows that $X \setminus Z(f)$ is affine, and hence f is as in Definition 5.1. Consequently, Z_f lies over the set of T-semistable points of X. Since the functions f generate the ideal of $p_T^{-1}(p_T(A) \cup B_T^0 \cup B_T^-)$, we obtain the desired inclusion.

Now, Lemmas 5.4, 5.5, and 5.6 show that the inclusion "$\supset$" of the assertion is valid. The reverse inclusion is easy: Every translate $g \cdot X^{ss}(D,T)$ is the set of semistable points of gTg^{-1} and hence contains $X^{ss}(D,G)$. □

6. Divisorial good quotient spaces

In this section, we construct all divisorial good quotient spaces of a given normal variety with reductive group action. The results are analogous to those of the preceding section. The first theorem of this section generalizes the corresponding results of [16], and the second one is a new Hilbert-Mumford type statement.

Let G be a reductive group, and let X be a normal G-variety. We work with finitely generated subgroups $\Lambda \subset \mathrm{WDiv}(X)$; these are in particular polyhedral semigroups. Fix such a subgroup $\Lambda \subset \mathrm{WDiv}(X)$, and a G-linearization of Λ as introduced in Section 3. That means that we have a certain graded G-sheaf structure on the associated $\mathcal{O}_X$-algebra

$$\mathcal{A} = \bigoplus_{D \in \Lambda} \mathcal{A}_D = \bigoplus_{D \in \Lambda} \mathcal{O}_X(D).$$

Definition 6.1. We call a point $x \in X$ *semistable*, if x has an affine neighbourhood $U = X \setminus Z(f)$ with some G-invariant homogeneous $f \in \mathcal{A}(X)$ such that all $D \in \Lambda$ are Cartier on U, and the $D \in \Lambda$ admitting a G-invariant invertible $h \in \mathcal{A}_D(U)$ form a subgroup of finite index in Λ.

Similar to the preceding section, the set of semistable points is denoted by $X^{ss}(\Lambda)$, or $X^{ss}(\Lambda, G)$ if we want to specify the group G. Note that for G-linearized groups of Cartier divisors we get back the notion of semistability introduced in [16, Def. 2.1]. We obtain the following generalizations of [16, Thms. 3.1, 4.1]:

Theorem 6.2. *Let a reductive group G act morphically on a normal variety X.*

(i) *For any G-linearized group $\Lambda \subset \mathrm{WDiv}(X)$, there is a good quotient $X^{ss}(\Lambda) \to X^{ss}(\Lambda)/\!\!/G$ with a divisorial prevariety $X^{ss}(D)/\!\!/G$.*
(ii) *If $U \subset X$ is open, G-invariant, and admits a good quotient $U \to U/\!\!/G$ with $U/\!\!/G$ divisorial, then U is a G-saturated subset of the set $X^{ss}(\Lambda)$ of semistable points of a canonically G-linearized group $\Lambda \subset \mathrm{WDiv}(X)$.*

Proof. To prove (i), we reduce to the case that Λ consists of Cartier divisors. So, consider the Cartier locus $X_0 \subset X$ of Λ. By Proposition 4.5, the set X_0 is G-invariant. Since X is normal, $X \setminus X_0$ is of codimension at least two in X. Hence $X_0^{ss}(\Lambda)$ equals $X^{ss}(\Lambda)$, and we may assume that Λ consists of Cartier divisors. But then [16, Thm. 3.1] gives the assertion.

The proof of (ii) is analogous to the proof of [16, Thm. 4.1]: Since $Y := U/\!\!/G$ is a normal divisorial prevariety, we find effective $E_1, \ldots, E_r \in \mathrm{CDiv}(Y)$ such that the sets $V_i := Y \setminus \mathrm{Supp}(E_i)$ are affine and cover Y. By [16, Lemma 4.3], there are global sections h_{ij} of some $\mathcal{O}_Y(nE_i)$ such that each $V_{ij} := Y \setminus Z(h_{ij})$ is affine, admits an invertible section h_{ijk} in $\mathcal{O}_{V_{ij}}(E_k)$ for any $k = 1, \ldots, r$, and, finally, the V_{ij} cover Y.

Consider the quotient map $p\colon U \to Y$ and the pullback divisors $p^*(E_i)$ on U. As in the proof of Theorem 5.2, we can construct G-tame Weil divisors $D_i \in \mathrm{WDiv}(X)$ such that $D_i|_U = p^*(E_i)$ holds, and every $f_{ij} := p^*(h_{ij})$ extends to a global section of $\mathcal{O}_X(nD_i)$ and satisfies

$$X \setminus Z(f_{ij}) = p^{-1}(V_{ij}).$$

Let $\Lambda \subset \mathrm{WDiv}(X)$ be the group generated by $D_1, \ldots, D_r$, and let $\mathcal{A}$ be the associated Λ-graded $\mathcal{O}_X$-algebra. Proposition 3.3 tells us that Λ is canonically G-linearized. To conclude the proof, consider the sections

$$f_{ij} = p^*(h_{ij}) \in \mathcal{A}_{nD_i}(X), \qquad f_{ijk} := p^*(h_{ijk}) \in \mathcal{A}_{D_k}(U_{ij}).$$

By construction, the f_{ij} are as in Definition 6.1. Since $U \subset X$ is covered by the subsets $U_{ij} := p^{-1}(V_{ij})$, we obtain $U \subset X^{ss}(\Lambda)$. Since each U_{ij} is defined by the G-invariant section f_{ij}, it is G-saturated in $X^{ss}(\Lambda)$. Consequently, also U is G-saturated in $X^{ss}(\Lambda)$. □

As in the preceding section, we have a Hilbert-Mumford type statement which allows us to express the set of G-semistable points in terms of the T-semistable points for a maximal torus $T \subset G$:

Theorem 6.3. *Let a reductive group G act morphically on a normal variety X. Let $T \subset G$ be a maximal torus, and let $\Lambda \subset \mathrm{WDiv}(X)$ be a G-linearized group of Weil divisors on X. Then we have:*

$$X^{ss}(\Lambda, G) = \bigcap_{g \in G} g \cdot X^{ss}(\Lambda, T).$$

Proof. As in the proof of Theorem 6.2 (i), one reduces to the case that Λ consists of Cartier divisors. Then we can proceed analogously to the Proof of Theorem 5.3: Let $\mathcal{A}$ be the Λ-graded $\mathcal{O}_X$-algebra associated to Λ. Consider the variety $\widehat{X} := \mathrm{Spec}(\mathcal{A})$ with its actions of $S := \mathrm{Spec}(\mathbb{K}[\Lambda])$ and G, and the G-equivariant canonical map $q\colon \widehat{X} \to X$. Let $U \subset X$ denote the ample locus of Λ, and set $\widehat{U} := q^{-1}(U)$.

Choose T-invariant homogeneous sections $f_1, \ldots, f_r \in \mathcal{A}(X)$ and G-invariant homogeneous sections $h_1, \ldots, h_s \in \mathcal{A}(X)$ as in Definition 6.1 such that the sets $X \setminus Z(f_i)$ and $X \setminus Z(h_j)$ cover the respective sets $X^{ss}(\Lambda, T)$ and $X^{ss}(\Lambda, G)$ of semistable points. Then Lemma 4.7 provides a $(G \times S)$-equivariant open embedding $\widehat{U} \subset Z$ into an affine $(G \times S)$-variety Z such that $\mathcal{O}(Z) \subset \mathcal{O}(\widehat{X})$ holds, and the functions f_i, h_j extend regularly to Z and satisfy

$$\widehat{X}_{f_i} = \widehat{U}_{f_i} = Z_{f_i}, \qquad \widehat{X}_{h_j} = \widehat{U}_{h_j} = Z_{h_j}.$$

For $H = T, G$, we consider the induced action of S on the quotient spaces $Z /\!\!/ H$, and describe the sets $X^{ss}(\Lambda, H)$ of semistable points in terms of these actions: Let $p_H\colon Z \to Z /\!\!/ H$ be the quotient map, let $A := Z \setminus \widehat{U}$, and let $B_H^0 \subset Z /\!\!/ H$ be the set of all points $y \in Z /\!\!/ H$ with an infinite isotropy group S_y. We show now

$$q^{-1}(X^{ss}(\Lambda, H)) = Z \setminus p_H^{-1}(p_H(A) \cup B_H^0).$$

The inclusion "$\subset$" is [16, Prop. 2.3 (i)]. For the reverse inclusion, we use [16, Lemma 2.4]: It tells us that the ideal of $p_H(A) \cup B_H^0$ in $\mathcal{O}(Z /\!\!/ H)$ is generated by S-homogeneous elements f' such that $\mathcal{O}(Z /\!\!/ H)_{f'}$ admits homogeneous invertible elements for almost every character of the torus S.

The pullback $f := p_H^*(f')$ of such an f' is an H-invariant element of $\mathcal{O}(Z)$ and hence of $\mathcal{A}(X)$. As in the proof of 5.3, we have $\widehat{X}_f = Z_f$. Thus $q(\widehat{X}_f) = X \setminus Z(f)$ is affine, and we can conclude that f has the properties required in Definition 6.1. It follows that Z_h lies over the set $q^{-1}(X^{ss}(\Lambda, H))$. Since the right hand side of the above equation is covered by the Z_h, we obtain the inclusion "$\supset$".

We can describe the sets B_H^0 as well in terms of suitable one parameter subgroups of the torus S: Choose $\mu_1, \ldots, \mu_k\colon \mathbb{K}^* \to S$ such that B_T^0 is the union of the fixed point sets $B^0_T(\mu_i)$ of the $\mathbb{K}^*$-actions on $Z/\!\!/T$ determined by $\mu_i\colon \mathbb{K}^* \to S$. Analogously, let $\nu_1, \ldots, \nu_l\colon \mathbb{K}^* \to S$ describe the set $B^0_G \subset Z/\!\!/G$. Then we obtain for the respective sets of semistable points:

$$\begin{aligned} q^{-1}(X^{ss}(D,T)) &= Z \setminus p_T^{-1}(p_T(A) \cup B^0_T(\mu_1) \cup \ldots \cup B^0_T(\mu_k)), \\ q^{-1}(X^{ss}(D,G)) &= Z \setminus p_G^{-1}(p_G(A) \cup B^0_G(\mu_1) \cup \ldots \cup B^0_G(\mu_k)). \end{aligned}$$

This enables us to apply Lemmas 5.4 and 5.5. As in the proof of Theorem 5.3, they tell us that the inclusion "$\supset$" of the assertion is valid. The reverse inclusion is again easy: Every translate $g\cdot X^{ss}(\Lambda, T)$ is the set of semistable points of gTg^{-1} and hence contains $X^{ss}(\Lambda, G)$. ☐

7. Actions of semisimple groups

In this section, we apply our theory to actions of semisimple groups. This gives generalizations of several results presented in [6], [7], and [19]. In order to formulate the statements, we introduce the following notions of maximality, compare [6] and [19]:

Definition 7.1. Let G be a reductive group, let X be a G-variety, and let $U \subset X$ be a G-invariant open subset. We say that

(i) U is a *qp-maximal G-set* if there is a good quotient $U \to U/\!\!/G$ with $U/\!\!/G$ quasiprojective, and U is not a G-saturated subset of a properly larger $U' \subset X$ admitting a good quotient $U' \to U'/\!\!/G$ with $U'/\!\!/G$ quasiprojective,

(ii) U is a *d-maximal G-set* if there is a good quotient $U \to U/\!\!/G$ with $U/\!\!/G$ divisorial, and U is not a G-saturated subset of a properly larger $U' \subset X$ admitting a good quotient $U' \to U'/\!\!/G$ with $U'/\!\!/G$ divisorial.

In the sequel, G denotes a connected semisimple group, $T \subset G$ is a maximal torus, and $N \subset G$ denotes the normalizer of T in G. Moreover, X is a normal G-variety. The first result is a further Hilbert-Mumford type statement. It generalizes [7, Cor. 1], and the results on the case $G = \mathrm{SL}_2$ given in [6, Thm. 9] and [19, Thm. 2.2]:

Theorem 7.2. *Let $U \subset X$ be an N-invariant open subset of X, and let $W(U)$ denote the intersection of all translates $g\cdot U$, where $g \in G$.*

(i) *If U is a qp-maximal T-subset of X, then $W(U)$ is an open T-saturated subset of U, and there is a good quotient $W(U) \to W(U)/\!\!/G$ with $W(U)/\!\!/G$ quasiprojective.*

(ii) *If U is a d-maximal N-subset of X then $W(U)$ is an open T-saturated subset of U, and there is a good quotient $W(U) \to W(U)/\!\!/G$ with $W(U)/\!\!/G$ divisorial.*

The proof of this Theorem consists of combining the Hilbert-Mumford type Theorems 5.3 and 6.3 with the following observation:

Proposition 7.3. *Let $U \subset X$ be an N-invariant open subset.*

(i) *If U is a qp-maximal N-set, then there exists a G-linearized Weil divisor D on X with $U = X^{ss}(D, N)$.*

(ii) *If U is a d-maximal N-set, then there is a G-linearized group $\Lambda \subset \mathrm{WDiv}(X)$ with $U = X^{ss}(\Lambda, N)$.*

Proof. We only prove the first assertion; the second one is settled analogously. By Theorem 5.2 (ii), there is a canonically N-linearized Weil divisor D on X such that

U is an N-saturated subset of $X^{ss}(D,N)$. Since we assumed U to be N-maximal, we have in fact $U = X^{ss}(D,N)$. We show now that after possibly replacing D with a positive multiple, the N-linearization extends to a G-linearization.

Let Z be a G-equivariant completion of X, see [27, Theorem 3]. Applying equivariant normalization, we achieve that Z is normal. By closing the support, we can extend D to a Weil divisor E of Z. Note that E is still N-tame and hence, by Proposition 3.3, it is canonically N-linearized. According to Proposition 3.7 (i), the canonical N-linearization of E is even a strong one over the set of smooth points Z_{reg} of Z.

Proposition 3.2 tells us that after replacing E (and D) with suitable multiples, we can choose a G-linearization of E. Again by Proposition 3.7 (i), the resulting N-linearization of E is strong over Z_{reg}. Since we have $\mathcal{O}(Z_{\mathrm{reg}}) = \mathbb{K}$ and the character group of N is finite, Proposition 3.8 (ii) says that after possibly passing to a further multiple, the G-linearization of E induces the canonical N-linearization of E over Z_{reg} and hence over Z. Restricting to $X \subset Z$, we obtain the assertion. □

Note that this Proposition is the place, where semisimplicity of G came in: In the proof, we made essentially use of the fact that the character group of N is finite.

Proof of Theorem 7.2. Note first that in the setting of (i), the induced action of the Weyl group N/T on $U /\!\!/ T$ admits a geometric quotient with a quasiprojective quotient space. The composition of the quotients by T and N/T is a good quotient $U \to U /\!\!/ N$. It follows that U is a qp-maximal N-set.

Now, for (i), choose a G-linearized semigroup $\Lambda = \mathbb{N}D$, and for (ii) a G-linearized group $\Lambda \subset \mathrm{WDiv}(X)$ as provided by Proposition 7.3. By the definition of semistability, we have

$$X^{ss}(\Lambda, G) \;\subset\; \bigcap_{g \in G} g \cdot X^{ss}(\Lambda, N) \;\subset\; \bigcap_{g \in G} g \cdot X^{ss}(\Lambda, T).$$

From Theorems 5.3 and 6.3 we infer that also the reverse inclusions hold. This gives the assertion. □

In the case of complete quotient spaces, the approach via Weil divisors finally turns out to be a detour: Here everything can be done in terms of line bundles. More precisely, we have the following generalization of [7, Thm. 1], compare also [7, Remark, p. 965]:

Theorem 7.4. *Let $U \subset X$ be an N-invariant open subset admitting a good quotient $U \to U /\!\!/ T$ with $U /\!\!/ T$ projective. Then there is an ample G-linearized line bundle L over X such that $U = X^{ss}(L,T)$ holds. Moreover, we have $X = G \cdot U$, and X is a projective variety.*

Before giving the proof, we combine this result with [22, Thm. 2.1] to obtain the following supplement to the Hilbert-Mumford Theorem 7.2:

Corollary 7.5. *Let X and $U \subset X$ be as in Theorem 7.4. Then the intersection $W(U)$ of all translates $g \cdot U$, $g \in G$, is an open T-saturated subset of U, there is a good quotient $W(U) \to W(U) /\!\!/ G$, and $W(U) /\!\!/ G$ is projective.* □

We come the proof of Theorem 7.4. A first ingredient is a reformulation of an observation by Białynicki-Birula and Święcicka concerning semisimple group actions on the projective space, compare [7, Lemma, p. 963]:

Lemma 7.6. *Let G act morphically on $\mathbb{P}_n$. Then the translates $g\cdot\mathbb{P}_n^{ss}(\mathcal{O}(1),T)$, where $g\in G$, cover $\mathbb{P}_n$.*

Proof. The complement Y of the union of all translates $g\mathbb{P}_n^{ss}(\mathcal{O}(1),T)$, where $g\in G$, is empty, because otherwise [7, Lemma, p. 963] would provide a T-semistable point in some irreducible component of Y. □

The second ingredient of the proof is the following refinement of Sumihiro's Embedding Theorem, compare [27, Thm. 1] and [22, Prop. 1.7]:

Lemma 7.7. *Let D be a strongly G-linearized Cartier divisor, and suppose that X equals $G{\cdot}X^{ss}(D,T)$. Then there is a G-equivariant locally closed embedding $X\subset\mathbb{P}_n$ such that $X^{ss}(D,T)$ is T-saturated in $\overline{X}\cap\mathbb{P}_n^{ss}(\mathcal{O}(1),T)$, where $\overline{X}$ is the closure of X in $\mathbb{P}_n$.*

Proof. Let $\mathcal{A}$ be the graded $\mathcal{O}_X$-algebra associated to D, and let $U:=X^{ss}(D,T)$. Since we assumed $X=G{\cdot}U$, Proposition 4.5 tells us that the divisor D is in fact ample. Moreover, replacing D with a multiple, we may even assume that D is very ample, and that there are T-invariant $f_1,\ldots,f_r\in\mathcal{A}_D(X)$ such that the sets $X\setminus Z(f_i)$ are affine and cover U.

Choose any G-invariant vector subspace $M\subset\mathcal{A}_D(X)$ of finite dimension such that $f_1,\ldots,f_r\in M$ holds, and the corresponding morphism $\imath\colon X\rightarrow\mathbb{P}(N)$ is a locally closed embedding, where N is the dual G-module of M. Then $\imath$ is G-equivariant, and $\mathcal{A}_D$ equals as a G-sheaf the pullback of $\mathcal{O}(1)$. Moreover, by construction, the f_i extend to T-invariant sections of $\mathcal{O}(1)$. Together this gives the assertion. □

Proof of Theorem 7.4. First note that U is as well a qp-maximal N-set. Thus we can choose a G-linearized Weil divisor D on X as in Proposition 7.3 (i). By Proposition 4.5, D is an ample Cartier divisor on $X_0:=G{\cdot}U$. By Proposition 4.8, the G-linearization is strong over X_0. In particular, on X_0 the G-sheaf $\mathcal{A}_D$ is the sheaf of sections of a G-linearized line bundle.

Now choose a locally closed G-equivariant embedding $X_0\subset\mathbb{P}_n$ as in Lemma 7.7, and let $\overline{X}_0$ denote the closure of X_0 in $\mathbb{P}_n$. Since $U/\!\!/T$ is complete, we obtain

$$U=\overline{X}_0\cap\mathbb{P}_n^{ss}(\mathcal{O}(1),T).$$

Moreover, from Lemma 7.6 we infer that the translates $g\cdot U$, where $g\in G$, cover $\overline{X}_0$. But this means that we have $X_0=\overline{X}_0$. In particular X_0 is projective, $X=X_0$ holds and D is ample. □

It is unclear, how far one can go with statements like Theorem 7.4 in the case of a non normal variety X. However, our methods yield the following (as before, G is a connected semisimple group, $T\subset G$ is a maximal torus, and $N\subset G$ denotes the normalizer of G):

Corollary 7.8. *Let X be an arbitrary G-variety, and let $U\subset X$ an N-invariant open subset $U\subset X$ admitting a good quotient $U\rightarrow U/\!\!/T$. Suppose that $U/\!\!/T$ is projective and that there is a normal point $x\in U$ with a closed T-orbit. Then X is complete with a projective normalization, and we have $X=G{\cdot}U$.*

Proof. Consider the G-equivariant normalization $\nu\colon X'\rightarrow X$. Let $U'\subset X'$ be the inverse image of $U\subset X$, and let $\mu\colon Y'\rightarrow Y$ be the normalization of the quotient space $Y := U /\!\!/ T$. Then the universal property of the normalization gives a commutative diagram

$$\begin{array}{ccc} U' & \xrightarrow{\nu} & U \\ \downarrow & & \downarrow \\ Y' & \xrightarrow[\mu]{} & Y \end{array}$$

We claim that $U' \to Y'$ is a good quotient for the T-action on U'. First note that $U' \to Y'$ is affine, because the remaining three morphisms of the diagram are so. This implies existence of a good quotient $U' \to Y''$ for the action of T on U'. Consider the induced morphism $Y'' \to Y'$. This morphism is also affine. Indeed, for any affine open $V' \subset Y'$ its inverse image $W \subset U'$ is affine and saturated with respect to $U' \to Y''$. Thus, the inverse image V'' of V' under $Y'' \to Y'$ is affine.

Let $V \subset Y$ be the intersection of the set of normal points of Y and the open kernel of the image of the set of normal points of U. Since by assumption there is a normal point with closed T-orbit in U, the set V is nonempty. Now, over V the maps $U' \to U$ and $Y' \to Y$ are isomorphisms. Consequently, $Y'' \to Y'$ is birational. By Richardson's Lemma, see [24, p. 106], the map $Y'' \to Y'$ is an isomorphism. This verifies the claim.

Being the normalization of a projective variety, Y' is again projective, see e.g. [22, Thm. III.8.4]. Since we already proved the Theorem in the normal case, we can conclude that the normalization X' of X is projective. In particular, we infer that X is complete. Finally, $X = G \cdot U$ follows from G-equivariance of $X' \to X$ and $X' = G \cdot U'$. □

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