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1 Introduction

A compact complex space X is called a *Moishezon space* if each irreducible component X_i of X has $\dim_{\mathbb{C}} X_i$ algebraically independent global meromorphic functions. A point x of a Moishezon space X is *schematic* if there exist meromorphic functions on X that provide local coordinates at x.

In general the set of non schematic points of a normal Moishezon space X is analytic of codimension at least 2, and X is algebraic if and only if every $x \in X$ is schematic. So it is natural to look for criteria for a point $x \in X$ to be schematic.

In the present article we consider normal Moishezon spaces X that are equivariant compactifications of a reductive complex Lie group G in the following sense: G acts algebraically on X (see section 2) and there exists a point $x_0 \in X$ with open dense orbit $G \cdot x_0$ and trivial isotropy group G_{x_0}.

D. Luna proved (see [Lu], Proposition): If a smooth Moishezon space X is an equivariant compactification of G then every fixed point of G is schematic. The purpose of this note is to generalize Luna's result to a wider class of points. As a corollary of our result (see section 4) we obtain:

Suppose that the normal Moishezon space X is an equivariant compactification of the reductive Lie group G and let $p \in X$. If the isotropy group G_p of p is reductive then p is schematic.

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2 Algebraic Group Actions on Moishezon Spaces

The purpose of this section is to define the concept of an algebraic action of an algebraic group on a Moishezon space. For the convenience of the reader we first recall some basic properties of Moishezon spaces in an elementary setting. A more advanced treatment of these facts is given in [Kn].

Let X be a Moishezon space of complex dimension n and let x_0 be a schematic point of X. Then, by definition, there are meromorphic functions $f_1, \ldots, f_n$ on X that define local coordinates near x_0. Consider the meromorphic map

$$F : X \to \mathbb{P}_n, \quad x \mapsto [1, f_1(x), \ldots, f_n(x)]$$

(for an introduction to meromorphic maps see for example [Do], section 2, or [Ue], section I.2). As in [Do], proof of Proposition 5, we see that the closure of the graph $\Gamma(F) \subset X \times \mathbb{P}_n$ is a projective variety. Let

$$\mathrm{pr}_X : \overline{\Gamma(F)} \to X, \quad (x, y) \mapsto x.$$

Denote by $U \subset X$ the maximal (open) set such that $x_0 \in U$ and the restriction of pr_X to $\mathrm{pr}_X^{-1}(U)$ is an isomorphism onto its image. Then we obtain (see [Do], Proposition 2):

2.1 Remark. *Every $x \in U$ is schematic and $X \setminus U$ is an analytic set in X. If X is normal then $X \setminus U$ is of codimension at least two.* □

By Chow's Theorem we obtain that $\mathrm{pr}_X^{-1}(U)$ is a Zariski open subset of the projective variety $\overline{\Gamma(F)}$. In other words U is biholomorphically equivalent to an algebraic variety. Moreover, if x' is a second schematic point with local coordinates $f_1', \ldots, f_n'$ then for the associated algebraic chart pr_X' we obtain that the transistion map $\mathrm{pr}_X^{-1} \circ \mathrm{pr}_X'$ is algebraic. So we conclude

2.2 Proposition. *The set X_{sch} of schematic points of X is an algebraic variety. Moreover $A := X \setminus X_{\mathrm{sch}}$ is an analytic set in X and if X is normal then A is of codimension at least two.* □

Now suppose we have a holomorphic action $G \times X \to X$ of an algebraic group G on X. Then the set of schematic points of X is invariant under this action. We call the action of G on X *algebraic*, if the restricted action $G \times X_{\mathrm{sch}} \to X_{\mathrm{sch}}$ is algebraic.

3 Meromorphic Continuation of K-finite Functions

Let X be a reduced complex space with countable topology and let G be a reductive complex Lie group acting holomorphically on X. We fix a maximal compact subgroup K of G (consequently G is isomorphic to $K^{\mathbb{C}}$, the complexification of K).

For every K-invariant open set $U \subset X$ one defines a representation of K on the complex vector space $\mathcal{O}(U)$ of holomorphic functions on U by setting $(k{\cdot}f)(x) := f(k^{-1}{\cdot}x)$. Similarly one defines a representation of K on $\mathcal{M}(U)$, the space of meromorphic functions on U.

A holomorphic function f defined on an open K-invariant set $U \subset X$ is called *K-finite* if the set $\{k{\cdot}f;\ k \in K\}$ is contained in a finitely generated vector subspace of $\mathcal{O}(U)$. For the proof of our main result we need the following (well-known, compare [Ri], sections 2 and 3) fact:

3.1 Lemma. *Let x be a point of a K-invariant Stein open set $U \subset X$. Then there exist K-finite functions $f_1, \dots, f_r \in \mathcal{O}(U)$ that provide local coordinates at x.*

Proof. Since U is Stein, we can choose functions $h_1, \dots, h_r \in \mathcal{O}(U)$ that provide local coordinates at x. A result of Harish-Chandra (see [Bo], Chap. 3) implies that the set of the K-finite functions of $\mathcal{O}(U)$ is dense in $\mathcal{O}(U)$ with respect to the topology of compact convergence. By replacing each h_i by a sufficently good K-finite approximation $f_i \in \mathcal{O}(U)$ of h_i we obtain the desired result. □

The rest of this section is devoted to give a condition on that locally defined K-finite functions can be extended to global meromorphic functions (see Lemma 3).

An open K-invariant set $U \subset X$ is called *K-irreducible*, if there exists an irreducible component U' of U such that $U = K{\cdot}U'$. In particular the induced action of K on the set of the irreducible components of U is transitive.

For the algebra $\mathcal{M}(U)^K$ of K-invariant meromorphic functions of a K-irreducible open set U we note:

3.2 Lemma. *Suppose that X is almost homogeneous with respect to the action of G (i.e., there exists an orbit $W := G{\cdot}x_0$ that is open and dense in X). Then $\mathcal{M}(U)^K \cong \mathbb{C}$ for every K-irreducible open set $U \subset X$.*

Proof. Let $f \in \mathcal{M}(U)^K$. Since U is K-irreducible, it suffices to show that f is constant on a non empty open subset U_1 of U. We choose a non empty open $U_1 \subset W \cap U$ such that f is holomorphic on U_1 and U_1 is connected.

Now we fix a point $x \in U_1$ and consider the isomorphism $\varphi : G/G_x \to W$ induced by the orbit map $g \mapsto g \cdot x$. By assumption $f \circ \varphi$ is constant on the (non empty) subset $KG_x \cap \varphi^{-1}(U_1)$ of G/G_x.

Since $\varphi^{-1}(U_1)$ is connected, we can apply the identity principle 1.3 of [He] to obtain that $f \circ \varphi$ is constant on all $\varphi^{-1}(U_1)$. Clearly this implies that f is constant on U_1. □

3.3 Lemma. *Suppose that*

i) *there exists an open dense orbit $W := G{\cdot}x_0$ such that the isotropy group of x_0 is trivial (in particular W is in a natural way an affine algebraic variety),*

ii) *every rational function on W can be extended to a meromorphic function on X.*

Then every K-finite function f defined on an open K-irreducible set $U \subset X$ can be extended to a meromorphic function on X.

Proof. First we consider the complex vector space $V \subset \mathcal{O}(U)$ spanned by $K \cdot f$, where K acts on $\mathcal{O}(U)$ as above. Then V is of finite dimension and the action of K on V is continuous linear.

Next let us consider the complex vector space $\mathbb{C}[W]$ of regular functions of the affine variety W. By definition of the affine structure of W (property i)), the action of G on W is algebraic. Thus setting $g{\cdot}h(x) := h(g^{-1}{\cdot}x)$ for $h \in \mathbb{C}[W]$ yields a rational representation of G on $\mathbb{C}[W]$. By property ii) we have $\mathbb{C}[W] \subset \mathcal{M}(X) \subset \mathcal{M}(U)$.

Denote by $R \subset \mathcal{M}(U)$ the $\mathbb{C}$-algebra generated by V and $\mathbb{C}[W]$. Then R is finitely generated and $K{\cdot}R = R$. Moreover the natural K-action on R can be uniquely extended to an action of G on R which is linear and rational. Hence R defines an affine G-variety $Y := \operatorname{spec}(R)$.

The inclusion $\mathbb{C}[W] \subset R$ gives rise to a G-morphism $\varphi : Y \to W$. We claim that φ is an isomorphism:

Since G acts freely on W, it follows that the restriction of φ to any G-orbit in Y is an isomorphism. In particular all G-orbits of Y have the same dimension. Hence every such orbit is closed in Y. Now, by Lemma 2, every G-invariant function lying in $R \subset \mathcal{M}(U)$ is constant. But this implies that there exists precisely one closed orbit in Y. So we obtain our claim.

In particular this shows that $\mathbb{C}[W] = R$, so one has $f \in \mathbb{C}[W] \subset \mathcal{M}(X)$. □

4 Main Results

4.1 Proposition. *Let the normal Moishezon space X be an equivariant compactification of the reductive Lie group $G \cong K^{\mathbb{C}}$. Suppose that $p \in X$ has an open Stein K-invariant neighbourhood $U \subset X$. Then p is schematic.*

Proof. First we check that the requirements of Lemma 3 are given. For i) it is sufficient to show that the complement of the open orbit W is an analytic set in X. But this is well known (compare for example [Po], Proposition 1).

Property ii) follows from the facts that X is a normal Moishezon space and G acts algebraically: W is an affine Zariski open subset of the algebraic variety $X_{\rm sch}$ of schematic points of X. So we get an isomorphism of the algebras of rational functions $\mathbb{C}(W)$ and $\mathbb{C}(X_{\rm sch})$. Since $X \setminus X_{\rm sch}$ is analytic of codimension ≥ 2 and X is normal, one obtains an embedding $\mathbb{C}(X_{\rm sch}) \to \mathcal{M}(X)$.

So in our situation all the assumptions made on X and G in Lemma 3 are valid. According to Lemma 1 there exist K-finite functions $f_1, \dots, f_r \in \mathcal{O}(U)$ that provide local coordinates at p. By appropriate shrinking we can achieve that $K{\cdot}p$ meets every connected component of U. Since X is normal, U then has to be K-irreducible. So, by Lemma 3, each f_i can be extended to an element of $\mathcal{M}(X)$. □

4.2 Corollary. *Suppose that the normal Moishezon space X is an equivariant compactification of the reductive Lie group G and let $p \in X$. If the isotropy group G_p of p is reductive then p is schematic.*

Proof. Let K be a maximal compact subgroup of G. Since the isotropy group of p is reductive, it follows that there exists a $g \in G$ such that $y := g{\cdot}p$ is a totally real K-point (i.e., the inclusion $K_y \subset G_y$ induces an isomorphism $K_y^{\mathbb{C}} \to G_y$, see [He;Lo], Section 2.2).

Clearly it suffices to show that y is schematic. But this follows from Proposition 4.1 and the fact that $K \cdot y$ has open Stein K-invariant neighbourhoods in X (compare [He;Lo], Section 2.3). □

It should be mentioned, that our criterion for a point to be schematic is not a necessary condition: In the following we construct a smooth algebraic equivariant compactification X of $SL(2, \mathbb{C})$ such that there exist points in X that have no Stein $SU(2)$-invariant neighbourhood.

4.3 Example. Let B denote the Borel subgroup of $SL(2, \mathbb{C})$ consisting of all upper triangular matrices of determinant one. We consider the (algebraic) action of B on $\mathbb{P}_2$ given by

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \cdot [z_0, z_1, z_2] := [az_0 + bz_1, a^{-1}z_1, z_2].$$

Then $p_0 := [0, 0, 1]$ is a fixed point of this action and for $p_1 := [1, 1, 1]$ the orbit map $B \to \mathbb{P}_2$, $h \mapsto h{\cdot}p_1$ is an open embedding. We set

$$X := SL(2, \mathbb{C}) \times_B \mathbb{P}_2 := \big(SL(2, \mathbb{C}) \times \mathbb{P}_2\big)/B,$$

where B acts on $SL(2, \mathbb{C}) \times \mathbb{P}_2$ by $h{\cdot}(g, p) := (gh^{-1}, h{\cdot}p)$. Endowed with the $SL(2, \mathbb{C})$-action defined by

$$g_1 \cdot [g, p] := [g_1 g, p],$$

X is a smooth algebraic equivariant compactification of $SL(2,\mathbb{C})$ (in fact the orbit map $g \mapsto g\cdot[e,p_1]$ is an open embedding with dense image). The orbit

$$A := SL(2,\mathbb{C})\cdot[e,p_0] \cong SL(2,\mathbb{C})/B$$

is a compact curve in X. Now every element g of $SL(2,\mathbb{C})$ can be written as a product $g = uh$ with $u \in SU(2)$ and $h \in B$. Hence it follows that

$$SU(2)\cdot[e,p_0] = A.$$

Since A is a compact curve in X, it cannot be contained in any open Stein neighbourhood of $[e,p_0]$. In particular the point $[e,p_0]$ does not possess a $SU(2)$-invariant open Stein neighbourhood in X.

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