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DENK, R.

The determinantal method for Hill systems

The Floquet exponents of Hill systems can be described as the zeros of the determinant of an infinite block matrix. Using acceleration of convergence, this approach can be applied numerically. On the other hand, the determinantal method provides the starting values for the calculation of the Floquet exponents as selected eigenvalues of an infinite block matrix.

In many fields of applications the stability of a physical system can be reduced to an equation of the form

$$y''(x) + G(x) \cdot y(x) = 0 \quad (x \in \mathbb{R}) \tag{1}$$

where $G \in L^\infty(\mathbb{R}, \mathbb{R}^{n\times n})$ is a matrix valued function with $G(x+1) = G(x)$ for almost every $x \in \mathbb{R}$. Systems of this type appear, for instance, in the theory of vibrations or in hydrodynamics. In the typical case where the function G depends on some set of parameters, one is not really interested in the solution of (1) but in the values of the parameters for which this equation is stable.

Let $Y : \mathbb{R} \to \mathbb{R}^{2n\times 2n}$ be the fundamental solution of (1), i.e. the solution of the initial value problem

$$Y'(x) = \begin{pmatrix} 0 & I_n \\ -G(x) & 0 \end{pmatrix} \cdot Y(x) \tag{2}$$

with the initial value $Y(0) = I_{2n}$. Then the Floquet exponents (characteristic exponents) can be defined as the complex numbers ν where $\exp(\nu)$ is an eigenvalue of the so-called matrizant $Y(1)$. Thus the question of stability is reduced to the knowledge of the Floquet exponents. For example, asymptotic stability in the sense of Lyapunov is equivalent to the condition that all Floquet exponents have negative real part.

Due to the definition, the Floquet exponents can be computed by integrating the initial value problem (2) numerically (e.g. with Adams methods). On the other hand, they can be described as the zeros of an infinite determinant (cf. [1]). It is the aim of this paper to summarize the determinantal approach and to show that it can be connected with a related eigenvalue description of the Floquet exponents.

We define the infinite block matrix $A(\nu) = (A_{kl}(\nu))_{k,l=-\infty}^{\infty}$ by

$$A_{kl}(\nu) := \frac{(2\pi k - i\nu)^2 \delta_{kl} I_n - \hat{G}(k-l)}{(2\pi k)^2 - \delta_{0,k}}$$

where $\hat{G}(k) \in \mathbb{C}^{n\times n}$ denotes the k-th Fourier coefficient of the periodic function G. Then the limit of the finite section determinants $\lim_{N\to\infty} \Delta_N$ with $\Delta_N := \det \left(A_{kl}(\nu)\right)_{k,l=-N}^{N}$ exists and is called $\det A(\nu)$. The following theorem shows a surprisingly simple connection between the infinite determinant and the fundamental matrix.

T h e o r e m 1. *For all* $\nu \in \mathbb{C}$ *we have* $\exp(\nu n) \cdot \det A(\nu) = \det(Y(1) - \exp(\nu) I_{2n})$.

For a proof of this theorem in the symmetric case where $G(x) = G(-x)$ the reader is referred to [1]. The proof for the general case can be made analogously or using the theory of Hilbert Schmidt operators and their regularized determinants. This theorem has important consequences for calculating the Floquet exponents: First, we see that the zeros of $\det A(\nu)$ are precisely the Floquet exponents of (1). Additionally, we know that $\exp(\nu n) \cdot \det A(\nu)$ is a polynomial in $\exp(\nu)$ with degree $2n$ and leading coefficient 1. Therefore, the Floquet exponents can be computed by evaluating $\det A(\nu) = \lim_N \Delta_N$ for $2n$ different values of ν.

It is no difficulty to compute Δ_N using the LU decomposition with partial pivot search. But the sequence (Δ_N) converges too slowly for numerical applications, for in general we have $\Delta_{N+1} - \Delta_N = \mathrm{O}(N^{-2})$. In the important case where $G(x)$ is a trigonometric polynomial the convergence of the sequence Δ_N can be accelerated, cf. [2], [3], [4] for details. It is possible to achieve a convergence rate of $\mathrm{O}(N^{-6})$ in the general and $\mathrm{O}(N^{-8})$ in the symmetric case. Using this acceleration of convergence, the determinantal method is reasonable also for numerical purposes. But the determinantal approach can also be combined with a related concept which makes use of the eigenvalues of an infinite blockmatrix.

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Theorem 2. *Let the matrix* $B = (B_{kl})_{k,l=-\infty}^{\infty}$ *be given by*

$$B_{kl} := \begin{pmatrix} -2\pi ik I_n & I_n \\ -\hat{G}(0) & -2\pi ik I_n \end{pmatrix} \quad \text{if } k = l\,, \qquad B_{kl} := \begin{pmatrix} 0 & 0 \\ -\hat{G}(k-l) & 0 \end{pmatrix} \quad \text{if } k \neq l\,.$$

Then ν *is a Floquet exponent of (1) if and only if it is an eigenvalue of* B.

In order to understand the expression "eigenvalue" in this theorem, the matrix B should be identified with the unbounded operator in the space $\ell^2(\mathbb{Z},\mathbb{C}^{2n})$ with the dense domain

$$D(B) := \{(c_k) \in \ell^2(\mathbb{Z},\mathbb{C}^{2n}) : (2\pi ikc_k)_k \in \ell^2(\mathbb{Z},\mathbb{C}^{2n})\}$$

which maps every $c \in D(B)$ to the vector $Bc := (\sum_{l=-\infty}^{\infty} B_{kl}c_l)_{k=-\infty}^{\infty}$.

Proof. Let $H(x)$ be the matrix on the right-hand side of equation (2). Obviously the condition for a Floquet exponent is equivalent to the solvability of the differential equation $y'(x) = H(x)y(x)$ with the boundary condition $y(1) = \exp(\nu)y(0)$. By the standard transformation $y(x) \mapsto z(x) := \exp(-\nu x)y(x)$ we see that the solution y corresponds to an eigenvector of the differential operator L with domain $D(L) := \{f \in H^1([0,1],\mathbb{C}^{2n}) : f(0) = f(1)\}$ which maps every $f \in D(L)$ to $Lf := f' - Af$. The Floquet exponents are exactly the eigenvalues of $-L$. Representing $-L$ with respect to the orthonormal base $\{\exp(2\pi ikx)e_j : k \in \mathbb{Z}, j = 1,\ldots,2n\}$ where e_j is the j-th unit vector in $\mathbb{C}^{2n}$ we obtain the infinite matrix B and the result stated in the theorem.

Again we approximate the infinite matrix by finite section matrices. The eigenvalues of the finite sections were computed with the QR algorithm. This approximation turns out to be very precise, but makes no use of the periodicity of the Floquet exponents. Indeed, it is enough to calculate $2n$ eigenvalues of B. But for a computation of selected eigenvalues of a matrix (using inverse vector iteration, for instance) it is important to know good approximations of these eigenvalues which can be provided by the determinantal method. So we suggest a two-step algorithm to compute the Floquet exponents of (1) with high precision:

- Calculate a first estimation for the Floquet exponents using the determinantal method (with convergence acceleration)
- Refine – if necessary – the estimations using the eigenvalue approach and inverse vector iteration.

In the following table, numerical results for one typical example are stated. For this problem n was equal to 2 and the coefficients of the matrix G were given by $2G_{11}(x) = G_{22}(x) = -2\cos 2\pi x$, $G_{12}(x) = 2+3\cos 2\pi x+2\sin 2\pi x$ and $G_{21}(x) = 1 + 3\cos 2\pi x + \sin 2\pi x$.

Table 1. CPU-time for different methods to calculate the Floquet exponents

Method	Relative error				
	10^{-4}	10^{-6}	10^{-8}	10^{-10}	10^{-12}
Numerical integration	1.21	2.03	2.73	3.79	5.11
Determinantal approach	0.04	0.05	0.09	0.18	0.42
QR algorithm	0.28	1.21	1.21	2.20	4.31
Two-step algorithm described above	0.06	0.08	0.08	0.12	0.12

From this and other numerical examples we see that the determinantal approach is much faster than numerical integration. The QR algorithm which computes all eigenvalues of the finite sections of B is not reasonable here. For high precision the dimensions of the finite section determinants become too large so that it is better to combine determinantal and eigenvalue approach.

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Address: Dr. Robert Denk, NWF I - Mathematik, Universität Regensburg, D–93040 Regensburg, Germany.