Universität Konstanz

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Konstanzer Schriften in Mathematik und Informatik
Nr. 108, Januar 2000

ISSN 1430–3558

Categorical Quotients of Simplicial Toric Varieties

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Abstract

We prove a criterion for the existence of a categorical quotient for the action of a subtorus on a simplicial non–degenerate toric variety in the category of $\mathbb{Q}$–factorial algebraic varieties.

Introduction

In this note we give a criterion for the existence of categorical quotients for subtorus actions on toric varieties. Recall that a categorical quotient for a regular action of an algebraic group G on an algebraic variety X is a G–invariant regular map $p: X \to Y$ that is universal with respect to G–invariant regular maps from X to varieties Z. In general, the problem of existence of such quotients is quite delicate.

In order to obtain existence statements it is often useful, to treat the problem in a modified category. For proper actions the category of algebraic spaces is reasonable (see e.g. [Ko] and [Ke;Mo]). Another approach is the category of dense constructible subsets, proposed by A. Białynicki–Birula in [BB]. In both cases the category of algebraic varieties is enlarged.

Sometimes it is also suitable to consider smaller categories: in [AC,Ha;1] it is proved by an algorithmic construction that every subtorus action on a toric variety admits a *toric quotient*, i.e., a categorical quotient in the category of toric varieties. This result is used in [AC,Ha;2] to solve the problem of existence of categorical quotients for subtorus actions for the category of quasi–projective varieties.

In the present article we consider the analogue in the category of $\mathbb{Q}$–factorial varieties: Let X be a simplicial non–degenerate toric variety and let H be a closed subgroup

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of the big torus of X. Let $p: X \to Y$ be the toric quotient constructed in [AC,Ha;1]. Then we obtain (see Corollary 3.3):

Theorem. *If p is surjective and Y is simplicial, then p is a categorical quotient in the category of $\mathbb{Q}$–factorial algebraic varieties.*

If we drop surjectivity of p in the above statement, then we obtain existence of categorical quotients in the category of $\mathbb{Q}$–factorial dense constructible subsets (see Corollary 3.2). Both results are obtained as corollaries of the decomposition theorem 3.1 according to which every H–invariant regular map from a toric variety to a $\mathbb{Q}$–factorial variety can be split into an H–invariant toric part and a non–toric part.

In the proof of the decomposition theorem, toric *prevarieties* play a central role: In order to decompose a given H–invariant regular map $f: X \to Z$ we use a modification of Włodarczyk's embedding theorem to realize Z as a closed subvariety of a certain toric prevariety Z' and consider the situation in the framework of toric prevarieties. The technical heart of the proof then consists of representing the toric prevarieties in question as quotients of quasi–affine toric varieties (see Section 1) and a lifting result for f (see Section 2).

1 Toric Prevarieties as Quotients

Recall that a toric prevariety is by definition a normal irreducible prevariety X together with an effective regular action of an algebraic torus T that has an open orbit. We refer to T as the big torus of X and, after fixing a base point x_0 in the open T–orbit we identify T with $T \cdot x_0$ via the orbit map.

A toric morphism of toric prevarieties X and X' with respective big tori $T \subset X$ and $T' \subset X'$ is a regular map $f: X \to X'$ that maps T homomorphically to T' and is equivariant with respect to the actions of T and T', i.e. satisfies to $f(t{\cdot}x) = f(t){\cdot}f(x)$ on $T \times X$. The kernel of a toric morphism $f: X \to X'$ is defined to be $\ker(f) := \ker(f|_T)$. For details on toric prevarieties we refer to [AC,Ha;3].

In this section we state some facts on representing toric varieties and prevarieties as quotients of quasi–affine toric varieties. We begin by recalling a well–known result on toric varieties due to D. Cox (see [Co]):

1.1 Proposition. *For every non–degenerate toric variety X there is an open toric subvariety $\mathrm{C}(X)$ of some $\mathbb{C}^r$ such that $\mathbb{C}^r \setminus \mathrm{C}(X)$ is of dimension at most $r-2$, and a toric morphism $p: \mathrm{C}(X) \to X$ that is a good quotient with respect to an algebraic subgroup of $(\mathbb{C}^*)^r$.*

Here a toric (pre–)variety is called *non–degenerate* if it does not contain a proper torus factor. A second important representation of certain toric varieties as quotient spaces was given by T. Kajiwara (see[Ka]). We need a non–separated version of this construction. We use the following notion of quotient:

Let X be a complex algebraic prevariety and assume that a reductive group G acts on X by means of a regular map $G \times X \to X$. A G–invariant regular map $p\colon X \to Y$ onto a prevariety Y is called a *good prequotient* for the action of G on X if it is affine and $\mathcal{O}_Y$ is the sheaf $(p_* \mathcal{O}_X)^G$ of invariants. A good prequotient is called *geometric* if it separates orbits.

Let X be a toric prevariety with big torus T. Then X is covered by finitely many maximal open affine toric subvarieties X_i. The variety X is called of *affine intersection*, if the intersection of any two X_i and X_j is again affine. Moreover, X is said to have enough Cartier divisors, if every complement $X \setminus X_i$ is the support of some effective T–stable Cartier divisor on X. For example, this is true for every simplicial toric prevariety.

In the sequel assume that X is non–degenerate, of affine intersection and moreover has enough Cartier divisors. We briefly sketch the construction of a quasi–affine toric variety $\mathrm{K}(X)$ and a quotient representation $q\colon \mathrm{K}(X) \to X$. Thereby we fix some notation used later. For details and proofs we refer to [AC,Ha;4].

The toric prevariety X arises from an affine system of fans $\mathcal{S} = (\Delta_{ij})_{i,j \in I}$ in a lattice N. In our situation the most important facts are that each Δ_{ij} is the fan of faces of a (strictly convex polyhedral) cone σ_{ij} in N, the Δ_{ij} are subfans of $\Delta_{ii} \cap \Delta_{jj}$, and X is obtained as the glueing of the affine toric varieties $X_i := X_{\Delta_{ii}}$ along the open toric subvarieties $X_{\Delta_{ij}}$.

The idea of the construction of $\mathrm{K}(X)$ is to make all T–invariant Cartier divisors of X principal. In order to describe the group $\mathrm{CDiv}^T(X)$ of T–invariant Cartier divisors, let $M := \mathrm{Hom}(N, \mathbb{Z})$ and consider the kernel $\widetilde{M}$ of the following lattice homomorphism

$$\bigoplus_{i \in I} M/(M \cap \sigma_{ii}^{\perp}) \ \to \ \bigoplus_{i,j \in I} M/(M \cap \sigma_{ij}^{\perp}), \qquad (\overline{u_i})_{i \in I} \mapsto (\overline{u_i - u_j})_{i,j \in I},$$

where $\sigma_{ij}^{\perp}$ is the space of linear forms in $M \otimes_{\mathbb{Z}} \mathbb{R}$ vanishing along σ_{ij}. Then the elements of $\widetilde{M}$ are in canonical one–to–one correspondence to the T–invariant Cartier divisors of X. Note that if $\mathcal{S}$ arises from a fan Δ, then $\widetilde{M}$ is just the lattice of support functions on Δ. The effective T–invariant Cartier divisors correspond to the set

$$\widetilde{M}_{\geq 0} := \{(\overline{u_i})_{i \in I};\ u_i \in \sigma_{ii}^{\vee} \text{ for all } i \in I\}.$$

of non–negative "support functions" in $\widetilde{M}$. Here $\sigma_{ii}^{\vee}$ as usual denotes the dual cone of σ_{ii}. Let $\widetilde{N} := \mathrm{Hom}(\widetilde{M}, \mathbb{Z})$. Then the canonical map

$$Q^* \colon M \to \widetilde{M}, \qquad u \mapsto (\overline{u}, \ldots, \overline{u})$$

defines a lattice homomorphism $Q\colon \widetilde{N} \to N$. Let $\widetilde{\sigma} \subset \widetilde{N} \otimes_{\mathbb{Z}} \mathbb{R}$ denote the dual of the cone generated by $\widetilde{M}_{\geq 0}$. The cone $\widetilde{\sigma}$ turns out to be strictly convex. Moreover, for $i \in I$ let $\widetilde{\sigma}_i$ be the dual cone of the cone generated by

$$\{(\overline{u_j})_{j \in I};\ u_i \in \sigma_{ii}^{\vee}\} \subset \widetilde{M}.$$

These cones $\widetilde{\sigma}_i$ are the maximal cones of a subfan $\widetilde{\Delta}$ of the fan of faces of $\widetilde{\sigma}$. Define toric varieties

$$\mathrm{K}(X) := X_{\widetilde{\Delta}}, \qquad \overline{\mathrm{K}(X)} := X_{\widetilde{\sigma}}.$$

1.2 Proposition. *In the above notations we have:*

i) $\mathcal{O}(\mathrm{K}(X)) = \mathcal{O}(\overline{\mathrm{K}(X)}) = \mathbb{C}[\widetilde{M}_{\geq 0}]$.

ii) *Q defines a toric morphism $q\colon \mathrm{K}(X) \to X$, which is a geometric prequotient for the action of $H := \ker(q)$ on $\mathrm{K}(X)$.*

iii) *If the T–invariant Cartier divisor D on X is defined by $\widetilde{u} \in \widetilde{M}$, then $q^*(D) = \mathrm{div}(\chi^{\widetilde{u}})$ holds on $\mathrm{K}(X)$.*

iv) *If $X_i \subset X$ is a T–stable affine chart and $\widetilde{u} \in \widetilde{M}$ defines a T–stable effective Cartier divisor having $X \setminus X_i$ as support, then $q^{-1}(X_i) = \overline{\mathrm{K}(X)} \setminus |\mathrm{div}(\chi^{\widetilde{u}})|$.* □

For a degenerate toric prevariety X' of affine intersection with enough Cartier divisors we choose a decomposition of the form $X' = X \times T''$, where T'' is a torus and X a non–degenerate toric prevariety. We set $\mathrm{K}(X') := \mathrm{K}(X) \times T''$ and $q' := q \times \mathrm{id}_{T''}\colon \mathrm{K}(X') \to X'$. Then q' is also a geometric prequotient for the action of $\ker(q')$ on $\mathrm{K}(X')$.

2 Lifting of Morphisms

Let X be a non–degenerate toric variety, X' a toric prevariety of affine intersection with enough Cartier divisors, and let T, T' denote the respective big tori. Consider the quotients $p\colon \mathrm{C}(X) \to X$ and $q\colon \mathrm{K}(X') \to X'$ we discussed in the previous section. Let $\widetilde{T}'$ denote the big torus of $\mathrm{K}(X')$. We prove the following lifting result for arbitrary regular maps from X to X' that are not necessarily toric morphisms:

2.1 Theorem. *Let $f\colon X \to X'$ be a regular map such that $f(X)$ intersects the open orbit of X'. Then there exists a regular map $F\colon \mathrm{C}(X) \to \mathrm{K}(X')$ such that the diagram*

$$\begin{array}{ccc} \mathrm{C}(X) & \xrightarrow{F} & \mathrm{K}(X') \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{f} & X' \end{array}$$

is commutative. If f is constant on the orbits of an algebraic subgroup H of T, then there is a regular homomorphism from $H' := p^{-1}(H)$ to $\widetilde{T}'$ such that F is equivariant with respect to the induced actions of H' on $\mathrm{C}(X)$ and $\mathrm{K}(X')$.

In order to prepare the proof of this theorem we consider the following special case: Assume that $X = \mathrm{C}(X)$, in other words that X is an open toric subvariety of some $\mathbb{C}^r$ such that $\mathbb{C}^r \setminus X$ is of dimension at most 2. Moreover, assume that X' is non–degenerate.

Since X' is a toric prevariety it arises from an affine system of fans $\mathcal{S} = (\Delta_{ij})_{i,j \in I}$ in a lattice N. In the notation of the preceeding section, choose a basis $\widetilde{u}_1, \ldots, \widetilde{u}_m$ of $\widetilde{M}$ and denote the T'–invariant Cartier divisor on X' corresponding to $\widetilde{u}_k$ by D_k.

Since $f(X)$ intersects the open T'–orbit, we have well–defined pullbacks $f^*(D_k)$ in $\mathrm{CDiv}(X)$ for $k = 1, \ldots, m$. Since $\mathbb{C}^r \setminus X$ is of codimension at least two we can extend $f^*(D_k)$ in a unique way to $\mathbb{C}^r$. Choose rational functions $h_k \in \mathbb{C}(x_1, \ldots, x_r)$ with

$$f^*(D_k) = \mathrm{div}(h_k).$$

2.2 Lemma. *If f is H–invariant for some algebraic subgroup H of $(\mathbb{C}^*)^r$, then each h_k is homogeneous with respect to a character $\eta_k \colon H \to \mathbb{C}^*$.*

Proof. Since the divisor $f^*(D_k)$ is locally defined by H–invariant regular functions, it is invariant under the induced action of H on $\mathrm{WDiv}(X)$. Hence, for any $g \in H$, we have

$$\mathrm{div}(g \cdot h_k) = g \cdot \mathrm{div}(h_k) = \mathrm{div}(h_k).$$

Thus $g \cdot h_k = \alpha_k(g) h_k$ holds for some $\alpha_k(g) \in \mathbb{C}^*$. The function $\alpha_k \colon H \to \mathbb{C}^*$ is in fact a character, and h_k is homogeneous with respect to $\eta_k := \alpha_k^{-1}$. □

Now consider the characters $\chi_k := \chi^{\widetilde{u}_k}$ of the big torus $\widetilde{T}'$ of $\mathrm{K}(X')$. We obtain for every $t \in (\mathbb{C}^*)^m$ an algebra homomorphism

$$F_t^* \colon \mathcal{O}(\widetilde{T}') \to \mathbb{C}(x_1, \ldots, x_r), \qquad F_t^*(\chi_k) := t_k \cdot h_k \, .$$

2.3 Lemma. *The algebra homomorphism $F_t^* \colon \mathcal{O}(\widetilde{T}') \to \mathbb{C}(x_1, \ldots, x_r)$ has the following properties:*

i) *Let $\widetilde{u} \in \widetilde{M}$ and let D be the corresponding T'–invariant Cartier divisor on X'. Then we have $f^*(D) = \mathrm{div}(F_t^*(\chi^{\widetilde{u}}))$ on X.*

ii) $F_t^*(\mathcal{O}(\overline{\mathrm{K}(X')})) \subset \mathbb{C}[x_1, \ldots, x_r]$.

Proof. i) A given $\widetilde{u} \in \widetilde{M}$ can be written in the form $\widetilde{u} = \sum_{k=1}^m n_k \widetilde{u}_k$ for some $n_k \in \mathbb{Z}$. Then by definition

$$F_t^*(\chi^{\widetilde{u}}) = F_t^*\Big(\prod_{k=1}^m \chi_k^{n_k}\Big) = \prod_{k=1}^m t_k^{n_k} \cdot h_k^{n_k}.$$

This implies

$$\mathrm{div}(F_t^*(\chi^{\widetilde{u}})) = \sum_{k=1}^m n_k \, \mathrm{div}(h_k) = f^*\Big(\sum_{k=1}^m n_k D_k\Big) = f^*(D).$$

For ii), it suffices to show that for every $\widetilde{u} \in \widetilde{M}_{\ge 0}$ the corresponding character $\chi^{\widetilde{u}}$ is mapped by F_t^* to a regular function on X. So let $\widetilde{u} \in \widetilde{M}_{\ge 0}$. Then the associated Cartier divisor D is effective. By i) also $\operatorname{div}(F_t^*(\chi^{\widetilde{u}}))$ is effective. Hence $F_t^*(\chi^{\widetilde{u}})$ is regular on X. □

By the second assertion of the above lemma, F_t^* defines a regular map from $\mathbb{C}^r$ to $\overline{\mathrm{K}(X')}$. Thus, by restricting, we obtain a regular map $F_t \colon X \to \overline{\mathrm{K}(X')}$.

2.4 Lemma. *The regular map $F_t \colon X \to \overline{\mathrm{K}(X')}$ has the following properties:*

i) *For every T'–stable chart X_i' of X' we have $F_t(f^{-1}(X_i')) \subset q^{-1}(X_i')$; in particular $F_t(X) \subset \mathrm{K}(X')$ holds.*

ii) *If f is H–invariant for some algebraic subgroup H of $(\mathbb{C}^*)^r$, then there exists a homomorphism $\varphi \colon H \to \widetilde{T}'$ such that F_t is equivariant with respect to the induced actions of H on $\mathrm{C}(X)$ and $\mathrm{K}(X')$.*

Proof. For i), choose $\widetilde{u} \in \widetilde{M}_{\ge 0}$ such that the corresponding effective T'–invariant Cartier divisor D on X' has support $X' \setminus X_i'$. By Lemma 2.3 i) and Proposition 1.2 iv) we have

$$f^{-1}(X_i') = X \setminus |f^*(D)| = X \setminus |\operatorname{div}(F_t^*(\chi^{\widetilde{u}}))| = F_t^{-1}(\overline{\mathrm{K}(X')} \setminus |\operatorname{div}(\chi^{\widetilde{u}})|) = F_t^{-1}(q^{-1}(X_i')),$$

which yields the claim. To prove ii), let $\eta_k \colon H \to \mathbb{C}^*$ be as in Lemma 2.2. Then $\varphi^*(\chi_k) := \eta_k$ defines the desired homomorphism. □

Proof of Theorem 2.1. Of course, we may assume that $X = \mathrm{C}(X)$. If $X' = X'' \times T''$ holds for some torus T'' and a toric prevariety X'', then we have decompositions, $\mathrm{K}(X') = \mathrm{K}(X'') \times T''$ and $q = q'' \times id_{T''}$. Therefore any lifting $F' \colon X \to \mathrm{K}(X'')$ of $f' := \mathrm{pr}_1 \circ f$ yields a lifting $F = (F', \mathrm{pr}_2 \circ f) \colon X \to \mathrm{K}(X')$ of f. Hence it suffices to treat the case that X' is non–degenerate.

In this situation we have the maps F_t defined above. If f is invariant with respect to some $H \subset T$ then, by Lemma 2.4 ii), the F_t are H–equivaraint. Moreover, Lemma 2.4 i) shows that we can restrict the regular maps f and $q \circ F_t$ to obtain well–defined regular maps from $f^{-1}(X_i')$ to X_i' for every $i \in I$.

Because X_i' is separated, it suffices to prove that for a suitable choice of $t \in (\mathbb{C}^*)^m$ these regular maps coincide on the dense open subset $f^{-1}(T')$ of $f^{-1}(X_i')$. To this end it suffices to find a $t \in (\mathbb{C}^*)^m$ such that the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{C}(X) & \xleftarrow{F_t^*} & \mathbb{C}(\mathrm{K}(X')) \\ & {\scriptstyle f^*} \nwarrow \quad \nearrow {\scriptstyle q^*} & \\ & \mathcal{O}(T') & \end{array} .$$

Now consider $u \in M := \operatorname{Hom}(N, \mathbb{Z})$. The image of u under the natural lattice homomorphism $Q^* \colon M \to \widetilde{M}$ is the constant family $\widetilde{u} = (\overline{u}, \ldots, \overline{u})$. This implies that

$q^*(\chi^u) = \chi^{\widetilde{u}}$. The constant family $\widetilde{u}$ corresponds to the principal divisor $D := \mathrm{div}(\chi^u)$ on X'. By Lemma 2.3 i) we obtain

$$\mathrm{div}(F_t^*(q^*(\chi^u))) = \mathrm{div}(F_t^*(\chi^{\widetilde{u}})) = f^*(D) = \mathrm{div}(f^*(\chi^u)).$$

Since $\mathbb{C}^r \setminus X$ is of codimension at least two in $\mathbb{C}^r$, this consideration shows that $f^*(\chi^u)$ can only differ from $F_t^*(q^*(\chi^u))$ by some constant.

Now, let $u_1, \ldots, u_l$ be a $\mathbb{Z}$–basis of M. Write the corresponding principal divisors in the form $\mathrm{div}(\chi^{u_j}) = \sum_{k=1}^m n_{jk} D_k$, where n_{jk} are suitable integers. By Proposition 1.2 ii) one has $q^*(D) = \mathrm{div}(\chi_k)$ on $\mathrm{K}(X')$ and hence we obtain

$$q^*(\chi^{u_j}) = \prod_{k=1}^m \chi_k^{n_{jk}}.$$

Consequently, we can compute the pullback of χ^{u_j} with respect to $F_t \circ q$ as follows:

$$F_t^*(q^*(\chi^{u_j})) = F_t^*\Big(\prod_{k=1}^m \chi_k^{n_{jk}}\Big) = \prod_{k=1}^m (t_k h_k)^{n_{jk}} = \Big(\prod_{k=1}^m (t_k)^{n_{jk}}\Big)\Big(\prod_{k=1}^m (h_k)^{n_{jk}}\Big).$$

We have already seen that $f^*(\chi^{u_j})$ only differs from this expression by some constant. In other words, for every $j = 1, \ldots, l$ there is a constant $c_j \in \mathbb{C}^*$ with

$$f^*(\chi^{u_j}) = c_j \prod_{k=1}^m (h_k)^{n_{jk}}.$$

Thus, in order to make the above diagram commutative, we have to find a solution $t = (t_1, \ldots, t_m) \in (\mathbb{C}^*)^m$ of the system of equations

$$c_j = \prod_{k=1}^m (t_k)^{n_{jk}}, \quad j = 1, \ldots, l.$$

To this end note that coordinate functions χ^{u_j} of T' and χ_k of $\widetilde{T}'$ respectively provide identifications $T' \cong (\mathbb{C}^*)^l$ and $\widetilde{T}' \cong (\mathbb{C}^*)^m$. Under these identifications the homomorphism $\widetilde{T}' \to T'$ obtained by restricting q to the big tori, corresponds to the homomorphism

$$(\mathbb{C}^*)^m \to (\mathbb{C}^*)^l, \qquad (t_1, \ldots, t_m) \mapsto \Big(\prod_{k=1}^m t_k^{n_{1k}}, \ldots, \prod_{k=1}^m t_k^{n_{lk}}\Big).$$

Since the group homomorphism $q|_{\widetilde{T}'} \colon \widetilde{T}' \to T'$ is surjective, the desired solution t of our system of equations does exist. □

3 Decomposition of Morphisms

As before let X denote a non–degenerate toric variety. Let T be the big torus of X and let $H \subset T$ be a closed subgroup. We prove the following decomposition result for regular maps:

3.1 Theorem. *Let $f: X \to Z$ be an H–invariant regular map to a $\mathbb{Q}$–factorial algebraic variety Z. Then there exists a dominant H–invariant toric morphism $g: X \to X'$, an open subset $U \subset X'$ with $g(X) \subset U$ and a regular map $h: U \to Z$ such that $f = h \circ g$.*

We apply this result to investigate quotients. We formulate our statement in the category of so–called *dc–subsets*, introduced by A. Białynicki–Birula: The objects of this category are pairs (U, X), where U is a dense constructible subset of an algebraic variety X. A morphism from a pair (U, X) to (U', X') is given by a rational map $\varphi: X \to X'$ such that U is contained in the domain of definition of φ and $\varphi(U) \subset U'$ holds.

A dc–subset (U, X) is called $\mathbb{Q}$–factorial, if X is $\mathbb{Q}$–factorial. Now suppose that an algebraic group G acts regularly on a $\mathbb{Q}$–factorial variety X, and let U be a G–stable dense constructible subset of X. A categorical quotient for the action of G on (U, X) in the category of $\mathbb{Q}$–factorial dc–subsets is defined to be a G–invariant morphism p from (U, X) to a $\mathbb{Q}$–factorial dc–subset such that every G–invariant morphism from (U, X) factors uniquely through p.

We turn back to the action of the algebraic subgroup H of the big torus on X. A *toric quotient* for the action of H on X is a toric morphism $p: X \to X /_{\mathrm{tor}} H$ that is universal with respect to H–invariant toric morphisms. Such a toric quotient does always exist:

According to [AC,Ha;1] there is a toric quotient $p_0: X \to X /_{\mathrm{tor}} H^0$ for the action of the identity component H^0 of H. The quotient space $X /_{\mathrm{tor}} H^0$ is again a toric variety with an big torus T_0 that is a quotient of T/H^0. Note that the action of a finite subgroup of the big torus on a toric variety always admits a geometric quotient that is automatically a toric quotient. So we can divide the space $X /_{\mathrm{tor}} H^0$ by the induced action of the finite group H/H^0 to obtain a toric quotient $p: X \to X /_{\mathrm{tor}} H$ for the action of H on X.

Now, assume that X is simplicial. As an immediate consequence of the above decomposition theorem we obtain the following existence results for categorical quotients:

3.2 Corollary. *If the toric quotient variety $X /_{\mathrm{tor}} H$ is simplicial, then the map $X \to p(X)$ induced by the toric quotient p of X by H is a categorical quotient in the category of $\mathbb{Q}$–factorial dc–subsets.* □

3.3 Corollary. *If moreover the toric quotient p is surjective, then it is a categorical quotient in the category of $\mathbb{Q}$–factorial algebraic varieties.* □

Proof of Theorem 3.1. First w.l.o.g. we can make some assumptions on the situation. Consider Cox's construction for X to obtain a toric morphism $p: \mathrm{C}(X) \to X$, where $\mathrm{C}(X)$ is an open toric subvariety of some $\mathbb{C}^n$ with at least 2–codimensional boundary and p is a good (and hence also toric) quotient for the action of $H_1 := \ker(p)$ on $\mathrm{C}(X)$. Set $H' := p^{-1}(H)$.

Suppose that we can decompose the H'–invariant regular map $f_1 := f \circ p$ into $f_1 = h_1 \circ g_1$ where g_1 is an H'–invariant toric morphism, and where h_1 is a regular map that is defined on an open neighbourhood U_1 of the image of g_1. Since $H_1 \subset H'$ and p is a toric quotient for the action of H_1, there is a dominant toric morphism g with $g_1 = g \circ p$. Moreover g is H–invariant and, since p is surjective, the image of g is contained in U_1. So $f = h_1 \circ g$ is the desired decomposition. Therefore we can assume $X = \mathrm{C}(X)$ in this proof.

As a further simplification we note that it suffices to give a proof for the case that H is connected: Suppose that X', g, h and U satisfy the assertion for the identity component H^0 of H. Then g induces an action of $\Gamma := H/H^0$ on X'. Let $r\colon X' \to X''$ be the geometric quotient for this action.

By appropriate shrinking, we can achieve that U is Γ–invariant. Since r is geometric, this means that $r^{-1}(r(U)) = U$ holds, $r(U)$ is open in X'' and the restriction $r\colon U \to r(U)$ is again a geometric quotient for the action of Γ. The regular map h is necessarily Γ–invariant, and therefore factors through r. So one has $h = h' \circ r$ for some regular map $h'\colon r(U) \to Z$. It follows that $f = h' \circ (r \circ g)$ is the desired decomposition. Consequently we may assume in our proof that H is connected.

Finally we consider the map f. In order to obtain the desired factorization of f, one can view f as map from X to the closure Y of $f(X)$ in Z. We claim that we can realize Y as a closed subvariety of a simplicial toric prevariety Y' of affine intersection such that the subset $f(X)$ of Y intersects the open orbit of Y'. Let us verify this claim.

Since Z is $\mathbb{Q}$–factorial, we can apply the modification of Włodarczyk's embedding theorem given in [Ha] to obtain a closed embedding $\imath\colon Z \to Z'$ of Z into a simplicial toric prevariety Z' of affine intersection. Hence we also have an embedding $\imath\colon Y \to Z'$ into the simplicial toric prevariety Z' of affine intersection. Let $Y' \subset Z'$ the minimal orbit closure of the big torus of Z' such that $\imath(Y) \subset Y'$. Then Y' is again a simplicial toric prevariety of affine intersection and $\imath(f(X))$ intersects the open orbit of Y'. So our claim is verified.

After these preparations we can begin with the proof of the theorem. Since Y' is simplicial, it has enough Cartier divisors. In particular, as explained in Section 1, we can present Y' as a geometric prequotient $q\colon \mathrm{K}(Y') \to Y'$ of an open toric subvariety $\mathrm{K}(Y')$ of some affine toric variety $\overline{\mathrm{K}(Y')}$.

Let $F\colon X \to \mathrm{K}(Y')$ denote the lifting of $f\colon X \to Y'$ with respect to q as it was constructed in Theorem 2.1. Recall that there is a homomorphism from H to the big torus $\widetilde{T}'$ of $\overline{\mathrm{K}(Y')}$ such that F is equivariant with respect to the H–actions on X and $\mathrm{K}(Y')$. After choosing a toric embedding $\overline{\mathrm{K}(Y')} \to \mathbb{C}^s$, we may assume that $\overline{\mathrm{K}(Y')}$ is a closed $\widetilde{T}'$–invariant subvariety of some $\mathbb{C}^s$.

Since we assumed that X is open in $\mathbb{C}^n$ with at least 2–codimensional boundary, the components of F are in fact polynomials. By writing these components as linear combinations of monomials, we obtain a decomposition of F in the form $F = S \circ g'$, with a toric morphism $g'\colon X \to \mathbb{C}^r$ and a linear map $S\colon \mathbb{C}^r \to \mathbb{C}^s$. Note that S is H–equivariant with respect to the actions of H induced on $\mathbb{C}^r$ by g and on $\mathbb{C}^s$ by F.

Let W denote the normalization of the closure of $g'(X)$ in $\mathbb{C}^r$. Then W is an affine toric variety with big torus $g'(T)$. We can lift the toric morphism g' to a dominant toric morphism $\widehat{g}\colon X \to W$, and pull back S to a regular map $\widehat{S}\colon W \to \mathbb{C}^s$. Both, $\widehat{g}$ and $\widehat{S}$, are again equivariant for the induced H–action on W. So far we are in the following situation:

$$\begin{array}{ccc}
W & \xrightarrow{\widehat{S}} & \mathbb{C}^s \\
{\scriptstyle \widehat{g}}\uparrow & & \cup \\
X & \xrightarrow{F} & \mathrm{K}(Y') \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle q} \\
Y & \subset & Y'
\end{array} \quad .$$

Since $\widehat{g}(X)$ is dense in W, we have $\widehat{S}(W) \subset \overline{\mathrm{K}(Y')}$. Now consider the subset $V := \widehat{S}^{-1}(\mathrm{K}(Y'))$ of W. Then V is open in W and H–stable. Since the above diagram is commutative, we have $\widehat{g}(X) \subset V$. By taking closures in $\mathrm{K}(Y')$ we obtain

$$\overline{\widehat{S}(V)} = \overline{\widehat{S}(\widehat{g}(X))} = \overline{F(X)} \subset q^{-1}(Y).$$

In particular it follows that $\widehat{S}(V) \subset q^{-1}(Y)$. We claim that the restriction $\widehat{S}\colon V \to q^{-1}(Y)$ is an affine map. To check this, note first, that $\widehat{S}\colon W \to \mathbb{C}^s$ is affine. Thus also $\widehat{S}\colon W \to \overline{\mathrm{K}(Y')}$ and $\widehat{S}\colon V \to \mathrm{K}(Y')$ are affine. Since $q^{-1}(Y)$ is closed in $\mathrm{K}(Y')$ our claim follows.

Moreover, restricting of q yields an affine H–invariant regular map $q\colon q^{-1}(Y) \to Y$. So $q \circ \widehat{S}\colon V \to Y$ is an affine H–invariant regular map. With [Ra], Lemma 4.1 (see also [Ne], Prop. 3.12), we can conclude that V admits a good quotient $o\colon V \to V/\!/H$ for the action of H. So we obtain the following commutative diagram of regular maps:

$$\begin{array}{ccc}
V & \xrightarrow{\widehat{S}} & q^{-1}(Y) \\
{\scriptstyle o}\downarrow & & \downarrow{\scriptstyle q} \\
V/\!/H & \xrightarrow{h} & Y
\end{array} \quad .$$

By a result of J. Święcicka (see [Sw]), there is an open toric subvariety V_1 of W containing V that has a good toric quotient $o_1\colon V_1 \to V_1/\!/H$ such that the map $V/\!/H \to V_1/\!/H$ induced by $V \subset V_1$ is an open inclusion.

As the final diagram shows, the data for the desired decomposition of f are the toric variety $X' := V_1/\!/H$, the open subset $U := V/\!/H$, the toric morphism $g := o_1 \circ \widehat{g}$ and the regular map h:

$$\begin{array}{ccccc} X & \xrightarrow{\widehat{g}} & V & \subset & V_1 \\ {\scriptstyle f}\downarrow & {\scriptstyle g}\searrow & {\scriptstyle o}\downarrow & & \downarrow{\scriptstyle o_1} \\ Y & \xleftarrow{h} & V/\!/H & \subset & V_1/\!/H \quad . \end{array} \qquad \square$$

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