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Abstract. Let X be a normal algebraic variety endowed with a regular action of a connected linear algebraic group G. We provide a simple proof for the fact that the union $G{\cdot}U$ of all translates of a given quasiprojective open subset $U \subset X$ is again quasiprojective.

Statement and proof of the result

Let X be a normal algebraic variety, that is a normal integral scheme of finite type over some algebraically closed field, and suppose that a connected linear algebraic group G acts by means of a regular map $G \times X \to X$ on X. The purpose of this little note is to provide a reference (and a simple proof) for the following fact:

Proposition 1. *For every quasiprojective open subset $U \subset X$, the set $G{\cdot}U$ is again quasiprojective. In particular, every maximal quasiprojective open subset of X is G-invariant.*

If X admits a normal completion for which the factor group of Weil divisors modulo $\mathbb{Q}$-Cartier divisors is of finite rank, then [5, Theorem A] asserts that X has only finitely many maximal open quasiprojective subvarieties. In particular, for such X, Proposition 1 holds even with an arbitrary connected algebraic group G, see [5, Theorem D].

In [3, Lemma 8] a weaker version of Theorem 1 is stated: Every point of a normal G-variety admits a G-invariant quasiprojective open neighbourhood. The proofs presented in [1, p. 62] and [3] work only for divisorial X, that is, for any $x \in X$ there must be an effective Cartier divisor D such that $X \setminus \mathrm{Supp}(D)$ is an affine neighbourhood of x. However, [4, Theorem 3.8] fills this gap.

Of course, connectedness of the group G is essential for Theorem 1: Hironaka constructed an example of a smooth complete threedimensional variety X with an action of $G := \mathbb{Z}/2\mathbb{Z}$ such that the orbit space X/G is not an algebraic variety, compare [2, p. 14]. In particular, the maximal quasiprojective subsets of X cannot be invariant. Two immediate consequences of Proposition 1 are:

Proposition 2. i) *X is quasiprojective if and only if there is an open quasiprojective $U \subset X$ that meets all closed G-orbits.*

ii) *Suppose X has only finitely many G-orbits. Then X is quasiprojective if and only if there is an open affine $U \subset X$ that meets all closed G-orbits.* □

Proof of Proposition 1. Let $U \subset X$ be an open quasiprojective subset. Clearly we may assume that $X = G{\cdot}U$ holds. Moreover, we may assume that U is a maximal quasiprojective open subset of X. Then the complement $X \setminus U$ is of codimension at least two in X, compare [5, Lemma 3].

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Let D_U be an ample Cartier divisor on U. Then, by closing components, D_U extends uniquely to a Weil divisor D on X. Replacing D with a suitable multiple, we may assume that there are sections $s_1, \ldots, s_r$ in $\mathcal{O}_D(U)$ such that the sets U_{s_i} are affine and cover U.

The restriction D' of D to the regular locus $X' \subset X$ is Cartier. In particular, we can view $\mathcal{O}_{D'}$ as the sheaf of sections of a line bundle. After replacing D again with a suitable multiple, we can choose a G-linearization of this bundle and endow $\mathcal{O}_{D'}$ with the structure of a G-sheaf, see e.g. [1].

This G-sheaf structure on $\mathcal{O}_{D'}$ extends canonically to $\mathcal{O}_D$: For an open set $V \subset X$ let $V' := V \cap X'$. Given a section $s \in \mathcal{O}_D(V)$, we define its translates $g \cdot s$ as follows: translate the restriction $s' \in \mathcal{O}_D(V')$ to a section $g \cdot s' \in \mathcal{O}_D(g \cdot V')$ and then extend $g \cdot s'$ to the desired section $g \cdot s \in \mathcal{O}_D(g \cdot V)$.

Using the G-sheaf structure on $\mathcal{O}_D$, we see that locally $\mathcal{O}_D$ is generated by a single section. That means that the Weil divisor D is in fact a Cartier divisor.

We show now that D is ample: By our assumption on U, the codimension of $X \setminus U$ in X is at least two. So each of the above sections $s_i \in \mathcal{O}_D(U)$ extends in a unique way to a section $\overline{s}_i \in \mathcal{O}_D(X)$. Consider the complements

$$A := X \setminus X_{\overline{s}_i}, \qquad B := X \setminus U_{s_i}.$$

Both, A and B, are of pure codimension one in X. Moreover, since $U_{s_i} \subset X_{\overline{s}_i}$ holds, A is a union of certain irreducible components of B. Since $B \setminus A$ is contained in $X \setminus U$ and the latter set has codimension at least two in X, we obtain $A = B$. This implies $X_{\overline{s}_i} = U_{s_i}$. In particular, every $X_{\overline{s}_i}$ is affine.

Since U is covered by the $X_{\overline{s}_i}$ and $X = G{\cdot}U$ holds, we obtain that X is covered by the affine open sets $g_j \cdot X_{s_i}$ with certain $g_1, \ldots, g_n \in G$. But we have $g_j \cdot X_{\overline{s}_i} = X_{g_j \overline{s}_i}$ and $g_j \cdot \overline{s}_i \in \mathcal{O}_D(X)$. Thus D is in fact an ample Cartier divisor on X. Consequently X is quasiprojective. □

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