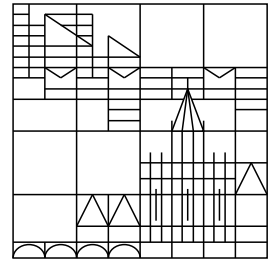


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A general Hilbert-Mumford Criterion

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A GENERAL HILBERT-MUMFORD CRITERION

JÜRGEN HAUSEN

ABSTRACT. We provide a Hilbert-Mumford Criterion for actions of reductive groups G on \mathbb{Q} -factorial complex varieties. The result allows to construct open subsets admitting a good quotient by G from certain maximal open subsets admitting a good quotient by a maximal torus of G . As an application, we indicate how to obtain all invariant open subsets with good quotient for a given G -action on a complete \mathbb{Q} -factorial toric variety.

1. STATEMENT OF THE RESULTS

Let a reductive group G act on a normal complex algebraic variety X . It is a central problem in Geometric Invariant Theory to construct all G -invariant open subsets $V \subset X$ admitting a *good quotient*, i.e. an affine G -invariant morphism $V \rightarrow V//G$ onto a complex algebraic space such that locally $V//G$ is the spectrum of the invariant functions. Let us call these $V \subset X$ for the moment the *good G -sets*.

In principle, it suffices to know all good T -sets $U \subset X$ for some fixed maximal torus $T \subset G$, because these comprise the good G -sets, see [7]. The construction of good T -sets is less hard, and in order to gain good G -sets one studies the following question: *Let $U \subset X$ be a good T -set. When is the intersection $W(U)$ of all translates $g \cdot U$, $g \in G$, a good G -set?*

The classical Hilbert-Mumford Criterion answers this question in the affirmative for sets of T -semistable points of G -linearized ample line bundles. Moreover, A. Białynicki-Birula and J. Świącicka settled the case of good T -sets defined by generalized moment functions [2], and the case $U = X$, see [3]. For $G = \mathrm{SL}_2$, several results can be found in [4], [5], and [12].

Usually, one imposes maximality conditions on the good T -set U , e.g. projectivity or completeness of $U//T$. The most general concept is *T -maximality*: U is not T -saturated in some properly larger good T -set U' , where *T -saturated* means saturated with respect to the quotient map. For complete X and T -maximal $U \subset X$ which are invariant under the normalizer $N(T)$, A. Białynicki-Birula conjectures that $W(U)$ is a good G -set [1, Conj. 12.1].

We shall settle the case of *$(T, 2)$ -maximal* subsets. These are good T -sets $U \subset X$ such that $U//T$ is embeddable into a toric variety, and U is not a T -saturated subset of some properly larger U' having the same properties, compare [14]. We shall assume that X is \mathbb{Q} -factorial, i.e. X is normal and for every Weil divisor some multiple is Cartier. In Section 4, we prove:

Theorem 1.1. *Let a connected reductive group G act on a \mathbb{Q} -factorial complex variety X . Let $T \subset G$ be a maximal torus and $U \subset X$ a $(T, 2)$ -maximal open subset. Then the intersection $W(U)$ of all translates $g \cdot U$, $g \in G$, is open in X , there is a good quotient $W(U) \rightarrow W(U)//G$, and $W(U)$ is T -saturated in U .*

This generalizes results by A. Białyński-Birula and J. Świącicka for $X = \mathbb{P}^n$, see [6, Thm. C], and by J. Świącicka for smooth complete varieties X with $\text{Pic}(X) = \mathbb{Z}$, see [14, Cor. 6.3]. As an application of Theorem 1.1, we obtain:

Corollary 1.2. *Let a connected reductive group G act on a complete \mathbb{Q} -factorial toric variety X , and let $T \subset G$ be a maximal torus. Then we have:*

- (i) *For every T -maximal open subset $U \subset X$ the set $W(U)$ is open and admits a good quotient $W(U) \rightarrow W(U)//G$.*
- (ii) *Every G -invariant open subset $V \subset X$ admitting a good quotient $V \rightarrow V//G$ is a G -saturated subset of some set $W(U)$ as in (i).*

Together with well known fan-theoretical descriptions of the T -maximal open subsets, see e.g. [13], this corollary explicitly solves the quotient problem for actions of connected reductive groups G on \mathbb{Q} -factorial toric varieties. In [1, Problem 12.9] our corollary was conjectured (in fact for arbitrary toric varieties).

2. BACKGROUND ON GOOD QUOTIENTS

We recall basic definitions and facts on good quotients, see also [1, Chap. 7], [3, Sec. 1] and [6, Sec. 2]. Let a reductive group G act morphically on a complex algebraic variety X . The concept of a good quotient is locally, with respect to the étale topology, modelled on the classical invariant theory quotient:

Definition 2.1. An G -invariant morphism $p: X \rightarrow Y$ onto a separated complex algebraic space Y is called a *good quotient* for the G -action on X if Y is covered by étale neighbourhoods $V \rightarrow Y$ such that

- (i) V and its inverse image $U := p^{-1}(V) = X \times_Y V$ are affine varieties,
- (ii) $p^*: \mathcal{O}(V) \rightarrow \mathcal{O}(U)$ defines an isomorphism onto the algebra of G -invariants.

A good quotient $p: X \rightarrow Y$ for the G -action on X is called *geometric*, if its fibres are precisely the G -orbits.

A good quotient $X \rightarrow Y$ for the G -action on X is categorical, i.e. any G -invariant morphism $X \rightarrow Z$ of algebraic spaces factors uniquely through $X \rightarrow Y$. In particular, good quotient spaces are unique up to isomorphism. This justifies the notation $X \rightarrow X//G$ for good and $X \rightarrow X/G$ for geometric quotients.

In the sequel we say that an open subset $U \subset X$ of a G -variety X with good quotient is *G -saturated*, if U is saturated with respect to the quotient map $X \rightarrow X//G$. The following well known properties of good quotients are direct consequences of the corresponding statements in the affine case:

Remark 2.2. Assume that the G -action on X has a good quotient $p: X \rightarrow X//G$.

- (i) If $A \subset X$ is G -invariant and closed, then $p(A)$ is closed in $X//G$, and the restriction $p: A \rightarrow p(A)$ is a good quotient for the action of G on A .
- (ii) If A and A' are disjoint G -invariant closed subsets of X , then $p(A)$ and $p(A')$ are disjoint.
- (iii) If $U \subset X$ is G -saturated and open, then $p(U)$ is open in $X//G$, and the restriction $p: U \rightarrow p(U)$ is a good quotient for the action of G on U .
- (iv) If $A \subset X$ and $U \subset X$ are as in (i) and (iii), then $A \cap U$ is G -saturated in A .

Let X be normal (in particular irreducible) with a good quotient $X \rightarrow X//G$. Then any reductive subgroup $H \subset G$ admits a good quotient $X \rightarrow X//H$, see [7, Cor. 10]. If H is normal in G , then universality of good quotients [1, Thm. 7.1.4] allows to push down the G -action to $X//H$. Moreover, we have:

Proposition 2.3. *Let $H \subset G$ be a reductive normal subgroup such that $X//H$ is an algebraic variety. Then the canonical map $X//H \rightarrow X//G$ is a good quotient for the induced action of G/H on $X//H$. \square*

We turn to the special case of an action of an algebraic torus T on a normal variety X . Good quotients for such torus actions are always affine morphisms of normal algebraic varieties, see [3, Cor. 1.3]. We work with the following maximality concepts for good quotients, compare [14, Def. 4.3]:

Definition 2.4. A T -invariant open subset $U \subset X$ with a good quotient $U \rightarrow U//T$ is called a (T, k) -maximal subset of X if

- (i) the quotient space $U//T$ is an A_k -variety, i.e. any collection $y_1, \dots, y_k \in U//T$ admits a common affine neighbourhood in $U//T$,
- (ii) U does not occur as proper T -saturated subset of some T -invariant open $U' \subset X$ admitting a good quotient $U' \rightarrow U'//T$ with an A_k -variety $U'//T$.

As usual, T -maximal stands for $(T, 1)$ -maximal. The collection of all (T, k) -maximal subsets is always finite, see [14, Thm. 4.4]. The case $k = 2$ can also be characterized via embeddability of the quotient spaces: By [15, Thm. A], a normal variety has the A_2 -property if and only if it embeds into a toric variety.

Proposition 2.5. *Let X be a toric variety, and let the algebraic torus T act on X via a homomorphism $T \rightarrow T_X$ to the big torus $T_X \subset X$. Then the T -maximal subsets of X are precisely the $(T, 2)$ -maximal subsets of X .*

Proof. First observe that every $(T, 2)$ -maximal subset is T -saturated in some T -maximal subset. Hence we only have to show that for any T -maximal $U \subset X$ the quotient space $U//T$ is an A_2 -variety. But this is known: By [13, Cor. 2.4 and 2.5], the set U is T_X -invariant, and $U//T$ inherits the structure of a toric variety from U . In particular, $U//T$ is an A_2 -variety, see [15, p. 709]. \square

3. GLOBALLY DEFINED $(T, 2)$ -MAXIMAL SUBSETS

Let G be a connected reductive group, $T \subset G$ a maximal torus, and X a normal G -variety. In this section, we reduce the construction of $(T, 2)$ -maximal subsets to a purely a toric problem in \mathbb{C}^n . The following notion is central:

Definition 3.1. We say that a $(T, 2)$ -maximal subset $U \subset X$ is *globally defined* in X , if there are T -homogeneous $f_1, \dots, f_r \in \mathcal{O}(X)$ such that each X_{f_i} is an affine open subset of U and any pair $x, x' \in U$ is contained in some X_{f_i} .

Here, as usual, $f \in \mathcal{O}(X)$ is called T -homogeneous, if $f(t \cdot x) = \chi(t)f(x)$ holds with a character $\chi: T \rightarrow \mathbb{C}^*$, and X_f denotes the set of all $x \in X$ with $f(x) \neq 0$. Our reduction is split into two lemmas. The proofs are based on ideas of [11].

Lemma 3.2. *Let X be \mathbb{Q} -factorial, and let $U \subset X$ be $(T, 2)$ -maximal. Then there are an algebraic torus H and a \mathbb{Q} -factorial quasiaffine $(G \times H)$ -variety \widehat{X} such that*

- (i) H acts freely on \widehat{X} with a G -equivariant geometric quotient $q: \widehat{X} \rightarrow X$,
- (ii) $\widehat{U} := q^{-1}(U)$ is a globally defined $(T \times H, 2)$ -maximal subset of \widehat{X} .

Proof. Let $p: U \rightarrow U//T$ be the quotient. By assumption, we can cover $U//T$ by affine open subsets Y_1, \dots, Y_r such that any pair $y, y' \in U//T$ is contained in a common Y_i . Since p is affine, each $p^{-1}(Y_i)$ is affine. Hence each $X \setminus p^{-1}(Y_i)$ is of pure codimension one and, by \mathbb{Q} -factoriality, equals the support $\text{Supp}(D_i)$ of an effective Cartier divisor D_i on X .

The Cartier divisors D_1, \dots, D_r generate a free abelian subgroup Λ of the group of all Cartier divisors of X . Enlarging Λ by adding finitely many generators, we achieve that every $x \in X$ admits an affine neighbourhood $X \setminus \text{Supp}(D)$ for some effective member $D \in \Lambda$. The group Λ gives rise to a graded \mathcal{O}_X -algebra:

$$\mathcal{A} := \bigoplus_{D \in \Lambda} \mathcal{A}_D := \bigoplus_{D \in \Lambda} \mathcal{O}_X(D).$$

After eventually replacing Λ with a subgroup of finite index, we can endow \mathcal{A} with a G -sheaf structure, see [11, Prop. 3.5]: for any $g \in G$ and any open $V \subset X$, we then have a Λ -graded homomorphism $\mathcal{A}(V) \rightarrow \mathcal{A}(g \cdot V)$, these homomorphisms are compatible with restriction of \mathcal{A} and multiplication of G , and the resulting G -representation on $\mathcal{A}(X)$ is rational.

We define the desired data; for details see [10, Sec. 2]. Let $\widehat{X} := \text{Spec}(\mathcal{A})$. The inclusion $\mathcal{O}_X \rightarrow \mathcal{A}$ defines an affine morphism $q: \widehat{X} \rightarrow X$ with $q_*(\mathcal{O}_{\widehat{X}}) = \mathcal{A}$. For the canonical section of an effective $D \in \Lambda$, its zero set in \widehat{X} is just $q^{-1}(\text{Supp}(D))$. In particular, \widehat{X} is covered by affine sets \widehat{X}_f and hence is quasiaffine.

The Λ -grading of \mathcal{A} corresponds to a free action of the torus $H := \text{Spec}(\mathbb{C}[\Lambda])$ on \widehat{X} . This makes $q: \widehat{X} \rightarrow X$ to an H -principal bundle. In particular, q is a geometric quotient for the H -action, and \widehat{X} is \mathbb{Q} -factorial. The G -sheaf structure of \mathcal{A} induces a G -action on \widehat{X} commuting with the H -action and making q equivariant.

We show that $\widehat{U} = q^{-1}(U)$ is $(\widehat{T}, 2)$ -maximal, where we set $\widehat{T} := T \times H$. First note that the restriction $p \circ q: \widehat{U} \rightarrow U//T$ is a good quotient for the \widehat{T} -action. For $(\widehat{T}, 2)$ -maximality, let \widehat{U} be \widehat{T} -saturated in some $(\widehat{T}, 2)$ -maximal $\widehat{U}_1 \subset \widehat{X}$. Then Lemma 2.3 gives a commutative diagram

$$\begin{array}{ccc} \widehat{U}_1 & \xrightarrow{\parallel \widehat{T}} & \widehat{U}_1 // \widehat{T} \\ & \searrow q & \nearrow \parallel T \\ & U_1 & \end{array}$$

where $U_1 := q(\widehat{U}_1)$ is open in X . Since \widehat{U} is \widehat{T} -saturated in \widehat{U}_1 and $\widehat{U}_1 \rightarrow U_1$ is surjective, this diagram shows that U is a T -saturated subset of U_1 . By $(T, 2)$ -maximality of U in X , this implies $U = U_1$ and hence $\widehat{U} = \widehat{U}_1$.

Finally, let $f_i \in \mathcal{O}(\widehat{X})$ be the canonical sections of some large positive multiples of the D_i . The zero set of f_i in \widehat{X} is just $q^{-1}(\text{Supp}(D_i))$. In particular, these zero sets are \widehat{T} -invariant, and hence the f_i are \widehat{T} -homogeneous. By construction, the sets \widehat{X}_{f_i} equal $q^{-1}(p^{-1}(Y_i))$, and thus form an affine cover of \widehat{U} as required in 3.1. \square

Lemma 3.3. *Let X be quasiaffine, and let $U \subset X$ be a globally defined $(T, 2)$ -maximal subset of X . Then there exist a linear G -action on some \mathbb{C}^n and a G -equivariant locally closed embedding $X \rightarrow \mathbb{C}^n$ such that*

- (i) *the maximal torus $T \subset G$ acts on \mathbb{C}^n by means of a homomorphism $T \rightarrow \mathbb{T}^n$ to the big torus $\mathbb{T}^n := (\mathbb{C}^*)^n$,*
- (ii) *there is a \mathbb{T}^n -invariant open $V \subset \mathbb{C}^n$ containing U as a closed subset and admitting a good quotient $V \rightarrow V//T$.*

Proof. Let $f_1, \dots, f_r \in \mathcal{O}(X)$ be as in 3.1, and set $X_i := X_{f_i}$. By [10, Lemma 2.4], we can realize X as a G -invariant open subset of an affine G -variety \overline{X} such that

the f_i extend regularly to \overline{X} and satisfy $\overline{X}_{f_i} = X_i$. Complete the f_i to a system f_1, \dots, f_s of T -homogeneous generators of the algebra $\mathcal{O}(\overline{X})$.

To proceed, we use the standard representation $g.f(x) := f(g^{-1}x)$ of G on $\mathcal{O}(\overline{X})$. Let $M_i \subset \mathcal{O}(\overline{X})$ be the G -module generated by $G.f_i$. Fix a basis f_{i1}, \dots, f_{in_i} of M_i such that all f_{ij} are T -homogeneous and for the first one we have $f_{i1} = f_i$. Denoting by N_i the dual G -module of M_i , we obtain G -equivariant maps

$$\Phi_i: \overline{X} \rightarrow N_i, \quad x \mapsto [h \mapsto h(x)].$$

We identify N_i with \mathbb{C}^{n_i} by associating to a functional of N_i its coordinates z_{i1}, \dots, z_{in_i} with respect to the dual basis $f_{i1}^*, \dots, f_{in_i}^*$. Then the pullback $\Phi_i^*(z_{ij})$ is just the function f_{ij} . Now, consider the direct sum of the G -modules \mathbb{C}^{n_i} ; we write this direct sum as \mathbb{C}^n but still use the coordinates z_{ij} . The maps Φ_i fit together to a G -equivariant closed embedding:

$$\Phi: \overline{X} \rightarrow \mathbb{C}^n, \quad x \mapsto (f_{11}(x), \dots, f_{1n_1}(x), \dots, f_{s1}(x), \dots, f_{sn_s}(x)).$$

In the sequel, we shall regard \overline{X} as a G -invariant closed subset of \mathbb{C}^n . Thus the functions f_{ij} are just the restrictions of the coordinate functions z_{ij} . By construction, the maximal torus T of G acts diagonally on \mathbb{C}^n , that means that T acts by a homomorphism $T \rightarrow \mathbb{T}^n$ to the big torus $\mathbb{T}^n = (\mathbb{C}^*)^n$.

We come to the construction of the desired set $V \subset \mathbb{C}^n$. Let $V_i \subset \mathbb{C}^n$ be the complement of the coordinate hyperplane defined by z_{i1} . Note that $\overline{X} \cap V_i$ equals X_i . In particular, X_i is closed in V_i . Consider the union $V_0 := V_1 \cup \dots \cup V_r$. Then V_0 is invariant under the big torus \mathbb{T}^n . Moreover, we have

$$\overline{X} \cap V_0 = \bigcup_{i=1}^r \overline{X} \cap V_i = \bigcup_{i=1}^r \overline{X}_{f_i} = \bigcup_{i=1}^r X_i = U.$$

Let $V \subset V_0$ be the minimal \mathbb{T}^n -invariant open subset with $U = \overline{X} \cap V$. Then every closed \mathbb{T}^n -orbit of V has nontrivial intersection with U . We show that V admits a good quotient by the action of T . By [11, Prop. 1.2], it suffices to verify that any two points with closed \mathbb{T}^n -orbits in V have a common T -invariant affine open neighbourhood in V .

Let $z, z' \in V$ have closed \mathbb{T}^n -orbits in V . Since these \mathbb{T}^n -orbits meet U , there are $t, t' \in \mathbb{T}^n$ such that $t.z$ and $t'.z'$ lie in U . By the choice of f_1, \dots, f_r , the points $t.z$ and $t'.z'$ even lie in some common X_i . Consider the corresponding V_i and the good quotient $p: V_i \rightarrow V_i//T$. The latter is a toric morphism of affine toric varieties.

Let $Z_i := V_i \setminus V$. Then Z_i is T -invariant and closed in V_i . Moreover, Z_i does not meet the T -invariant closed subset $X_i \subset V_i$. Thus $p(Z_i)$ and $p(X_i)$ are closed in $V_i//T$ and disjoint from each other. In particular, neither $p(t.z)$ nor $p(t'.z')$ lie in $p(Z_i)$. Since Z_i is even \mathbb{T}^n -invariant, also $p(z)$ and $p(z')$ do not lie in $p(Z_i)$.

Consequently, there exists a T -invariant regular function on V_i that vanishes along Z_i but not in the points z and z' . Removing the zero set of this function from V_i yields the desired common T -invariant affine open neighbourhood of the points z and z' in V . This proves existence of a good quotient $V \rightarrow V//T$. \square

4. PROOF OF THE RESULTS

Proof Theorem 1.1. First we reduce to the case of globally defined subsets of quasi-affine varieties. So, assume for the moment that we have Theorem 1.1 in this case. Consider the quasiaffine variety \widehat{X} , the torus H and the geometric quotient $q: \widehat{X} \rightarrow X$ provided by Lemma 3.2.

Then $\widehat{G} := G \times H$ is reductive with maximal torus $\widehat{T} := T \times H$, and $\widehat{U} = q^{-1}(U)$ is a globally defined $(\widehat{T}, 2)$ -maximal subset of \widehat{X} . By assumption, the intersection $W(\widehat{U})$ of all translates $\widehat{g} \cdot \widehat{U}$ is open, admits a good quotient by \widehat{G} , and is \widehat{T} -saturated in \widehat{U} . Since each $\widehat{g} \cdot \widehat{U}$ is H -invariant and $q: \widehat{X} \rightarrow X$ is G -equivariant, we obtain

$$q(W(\widehat{U})) = q\left(\bigcap_{\widehat{g} \in \widehat{G}} \widehat{g} \cdot \widehat{U}\right) = \bigcap_{g \in G} g \cdot q(\widehat{U}) = \bigcap_{g \in G} g \cdot U = W(U).$$

In particular, $W(U)$ is open in X . Moreover, restricting q gives a geometric quotient $W(\widehat{U}) \rightarrow W(U)$ for the H -action. Lemma 2.3 tells us that the induced map from $W(U)$ onto $W(\widehat{U})//\widehat{G}$ is a good quotient for the G -action on $W(U)$. Similarly, we infer T -saturatedness of $W(U)$ in U from the commutative diagram

$$\begin{array}{ccc} \widehat{U} & \xrightarrow{\parallel \widehat{T}} & \widehat{U} // \widehat{T} \\ & \searrow q & \nearrow \\ & U & \parallel T \end{array}$$

We are left with proving 1.1 for quasiaffine X and globally defined $(T, 2)$ -maximal $U \subset X$. By Lemma 3.3, we may view X as a G -invariant locally closed subset of a G -module \mathbb{C}^n , where T acts via a homomorphism $T \rightarrow \mathbb{T}^n$ and U is closed in some \mathbb{T}^n -invariant open $V \subset \mathbb{C}^n$ with good quotient $V \rightarrow V//T$. We regard \mathbb{C}^n as the G -invariant open subset of \mathbb{P}^n obtained by removing the zero set of the homogeneous coordinate z_0 .

Let $V' \subset \mathbb{P}^n$ be a T -maximal open subset containing V as a T -saturated subset. Let \overline{X} be the closure of X in \mathbb{P}^n , and set $X' := \overline{X} \cap V'$. Then X' is closed in V' , and we have $U = X' \cap V$. Using 2.2 (i), (iii) and (iv), we subsume the situation in a commutative cube

$$\begin{array}{ccccc} & & U & \longrightarrow & V \\ & \swarrow & \downarrow & & \downarrow \\ X' & \longrightarrow & V' & & \\ \downarrow & & \downarrow & & \downarrow \\ & & U//T & \longrightarrow & V//T \\ \downarrow & \swarrow & \downarrow & & \downarrow \\ X'//T & \longrightarrow & V'//T & & \end{array}$$

where the downwards arrows are good quotients by the respective actions of T , the right arrows are closed inclusions, the upper diagonal arrows are T -saturated inclusions and the lower diagonal arrows are open inclusions.

According to [6, Thm. C], the intersection $W(V')$ of all translates $g \cdot V'$ is open in \mathbb{P}^n and admits a good quotient by the action of G . Recall from [6, Lemma 8.4] that $W(V')$ is T -saturated in V' . We transfer the desired properties step by step from $W(V')$ to $W(U)$. First note that by G -invariance of \overline{X} we have

$$W(X') = \bigcap_{g \in G} g \cdot X' = \bigcap_{g \in G} g \cdot (\overline{X} \cap V') = \overline{X} \cap W(V') = X' \cap W(V').$$

Thus $W(X')$ is open in X' , and by 2.2 (iv) it is T -saturated in X' . In particular, the T -action on $W(X')$ has a good quotient. Moreover, $W(X')$ is G -invariant and closed in $W(V')$. Thus 2.2 (i) ensures the existence of a good quotient

$$u: W(X') \rightarrow W(X')//G.$$

Consider $B := X' \setminus X$. Since X is open in \overline{X} and B equals $(\overline{X} \setminus X) \cap X'$, the set B is closed in X' . The intersection $W(B)$ of the translates $g \cdot B$, where $g \in G$, is G -invariant and closed in $W(X')$. We claim that it suffices to verify

$$(1) \quad W(U) = W(X') \setminus u^{-1}(u(W(B))).$$

Indeed, suppose we have (1). Then $W(U)$ is open in X' , hence in U , and thus in X . Property 2.2 (iii) provides a good quotient $W(U) \rightarrow W(U)//G$. Moreover, $W(U)$ is T -saturated in $W(X')$, because it is G -saturated and we have the induced map from $W(X')//T$ onto $W(X')//G$. Since $W(X')$ and U are T -saturated in X' , we obtain that $W(U)$ is T -saturated in U .

We verify (1). Let $v: X' \rightarrow X'//T$ be the quotient map. As a subvariety, $X'//T$ inherits the A_2 -property from $V'//T$, which in turn has it by 2.5. Thus, since U is $(T, 2)$ -maximal in X , it is necessarily the maximal T -saturated subset of X' which is contained in $X \cap X'$. In terms of $B = X' \setminus X$ this means:

$$(2) \quad U = X' \setminus v^{-1}(v(B)).$$

We check the inclusion “ \subset ” of (1). Let $x \in u^{-1}(u(W(B)))$. Then, by 2.2 (ii), the closure of $G \cdot x$ meets $W(B)$. The classical Hilbert-Mumford Lemma [8, Thm. 4.2] says that for some maximal torus $T' \subset G$ the closure of $T' \cdot x$ meets $W(B)$. Let $g \in G$ with $T = gT'g^{-1}$. Then the closure of $T \cdot g \cdot x$ meets $W(B)$. Hence $g \cdot x$ lies in $v^{-1}(v(B))$. By (2), the point x cannot belong to $W(U)$.

We turn to the inclusion “ \supset ” of (1). For this, consider the set $A := (X \cap X') \setminus U$. Then X' is the disjoint union of U , A and B . Consequently, we have

$$W(U) = \bigcap_{g \in G} g \cdot (X' \setminus (A \cup B)) = W(X') \setminus \bigcup_{g \in G} g \cdot A \cup g \cdot B.$$

So we have to show that u maps a given $x \in W(X') \cap g \cdot (A \cup B)$ to $u(W(B))$. Since $g^{-1} \cdot x \notin U$ holds, we infer from (2) that $g^{-1} \cdot x$ lies in $v^{-1}(v(B))$. According to 2.2 (ii), the closure of $T \cdot g^{-1} \cdot x$ in X' meets B . Since $W(X')$ is T -saturated in X' , this implies that the closure of $T \cdot g^{-1} \cdot x$ meets $W(X') \cap B$. But we have

$$W(X') \cap B = W(X') \setminus X = \bigcap_{g \in G} g \cdot (X' \setminus X) = W(B).$$

Hence we obtained that the closure of the orbit $G \cdot x$ intersects $W(B)$. This in turn shows that the image $u(x)$ lies in $u(W(B))$. \square

Proof Corollary 1.2. Recall from [9, Sec. 4] that the automorphism group of X is a linear algebraic group having the big torus $T_X \subset X$ as a maximal torus. Thus, by conjugating T_X we achieve that $T \subset G$ acts on X via a homomorphism $T \rightarrow T_X$. Proposition 2.5 then ensures that each T -maximal subset of X is as well $(T, 2)$ -maximal, and statement (i) follows from Theorem 1.1.

For statement (ii), let $V \subset X$ be open and G -invariant with good quotient $V \rightarrow V//G$. Then [7, Cor. 10] provides a good quotient $V \rightarrow V//T$. Let $U \subset X$ be a T -maximal subset containing V as T -saturated subset. Then we have $V \subset W(U)$. Again by 2.5, the set U is $(T, 2)$ -maximal. Thus Theorem 1.1 says that $W(U)$ is open, has a good quotient $u: W(U) \rightarrow W(U)//G$, and is T -saturated in U .

For G -saturatedness of V in $W(U)$ we have to show that any $x \in u^{-1}(u(V))$ with closed G -orbit in $W(U)$ belongs to V . For this note that V is T -saturated in $W(U)$, because both sets are so in U . Now, let $y \in V$ with $u(y) = u(x)$. Then x lies in the closure of $G \cdot y$. Thus [8, Thm. 4.2] provides a $g \in G$ such that the closure

of $T \cdot g \cdot y$ meets $G \cdot x$. Since $g \cdot y$ lies in V and V is T -saturated in $W(U)$, we obtain $G \cdot x \subset V$, and hence $x \in V$. \square

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