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On a queer binomial sum

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Abstract. In this paper the binomial sum

$$S_n(r) = \sum_{m=0}^{n} \binom{n}{m} \frac{(-1)^m}{m^r + 1} \quad (r > 0,\ n \in \mathbb{N})$$

is investigated. It turns out that the behaviour of this sum for $n \to \infty$ depends on the parameter r and changes dramatically at the values $r = 1$ and $r = 2$. In particular, for $r \geq 2$ we obtain an oscillatory behaviour while for $r < 2$ the sequence $S_n(r)$ is monotonous at least for large values of n.

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1. Introduction. For $n \in \mathbb{N}$ and real $r > 0$ put

$$S_n(r) := \sum_{m=0}^{n} \binom{n}{m} \frac{(-1)^m}{m^r + 1} .$$

It is easy to show that $S_n(1) = \frac{1}{n+1}$ (see e.g. [1], Formula 0.155). In [2] the fact that $0 < S_n(r) \leq \frac{1}{2}$ for $0 < r < 1$ was needed. The second author of the present paper furnished the proof: a formula for $S_n(r)$ has been established from which not only $S_n(r) > 0$ can be read off but $S_n(r) \downarrow 0$ for $n \to \infty$ as well.

We resume this subject here and determine the behaviour of $S_n(r)$ for $n \to \infty$ for every fixed $r > 0$. It is queer indeed. At the values $r = 1$ and $r = 2$ it takes unexpectedly a dramatic turn.

To formulate our results we have to introduce the following notation. For $r > 0$, $r \notin \mathbb{N}$, and complex z not belonging to the negative real axis we define $z^r := \exp(r \log z)$ where log denotes the principal branch. The function $z^r + 1$ has no zeros for $0 < r < 1$. For $r \geq 1$ it has the simple zeros $w_k(r) = \exp(\frac{i(2k+1)\pi}{r})$ $\quad (-\frac{r+1}{2} < k \leq \frac{r-1}{2})$. They occur in conjugate pairs. We single out $w(r) := w_0(r) = e^{i\pi/r}$ with maximal real part $\cos\frac{\pi}{r}$.

The main results are contained in the following theorem.

Theorem 1. *Set* $A_n(r) = 0$ *for* $0 < r < 1$ *and*

$$A_n(r) := \frac{1}{r} \sum_k \frac{n!}{\prod_{l=1}^{n}(l - w_k(r))} \qquad \text{for } r \geq 1 .$$

For $r \in \mathbb{N}$ *set* $B_n(r) = 0$. *For* $r > 0,\ r \notin \mathbb{N}$, *put*

$$\begin{aligned} B_n^*(r) &:= \int_0^\infty \frac{u^r}{|1+u^r e^{ir\pi}|^2}\, du \int_0^\infty e^{-ut}(1-e^{-t})^n\, dt\,, \\ B_n(r) &:= \frac{\sin r\pi}{\pi}\ B_n^*(r)\,. \end{aligned}$$

a) For every $r > 0$ *and* $n \in \mathbb{N}$ *we have*

$$(1) \qquad S_n(r) = A_n(r) + B_n(r)\,.$$

b) For every $r > 0,\ r \notin \mathbb{N}$, *the asymptotics*

$$(2) \qquad B_n^*(r) = \Gamma(r)(\log n)^{-r}\Big(1 + O\Big((\log n)^{-\min\{1,r\}}\Big)\Big)$$

hold for $n \to \infty$ *with an* O*-constant depending only on* r.

If we combine the results of Theorem 1 with

$$(3) \qquad \frac{n!}{\prod_{l=1}^n (l-w)} = \Gamma(1-w)\, n^w + O\Big(n^{\operatorname{Re} w - 1}\Big) \qquad \text{for } w \in \mathbb{C}\backslash\mathbb{N}$$

we are able to explain the behaviour of $S_n(r)$ for $n \to \infty$, $r > 0$ fixed. (Cf. also Fig. 1 for the behaviour of $S_n(r)$ for $0 < r \le 4$ and $1 \le n \le 100$.) In the following, the symbol $\downarrow$ denotes the monotonous decrease of a sequence.

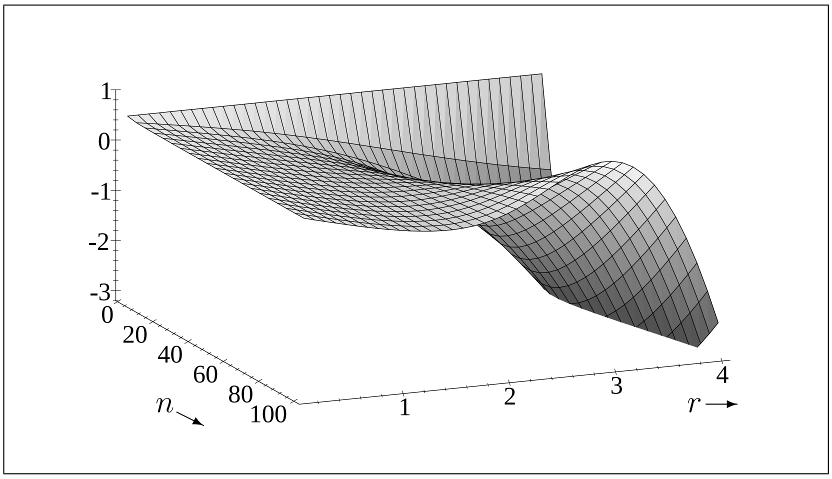


Figure 1: The function $S_n(r)$ for $0 < r \le 4$ and $1 \le n \le 100$.

Theorem 2. *The sum $S_n(r)$ has the following behaviour for fixed $r > 0$ and $n \to \infty$:*

$\underline{0 < r < 1}$*: Here we have $S_n(r) = B_n(r)$, $S_n(r)\downarrow 0$ for $n \to \infty$ and $S_n(r) \sim \Gamma(r)(\log n)^{-r}$.*

$\underline{1 < r < 2}$*: In this case $S_n(r) = B_n(r) + O(n^{\cos\frac{\pi}{r}})$, $-S_n(r)\downarrow 0$ for $n \to \infty$ $\quad (n \geq n_0(r))$ and $-S_n(r) \sim \Gamma(r)(\log n)^{-r}$.*

$\underline{r \geq 2}$*: Put $a(r) := \sin\frac{\pi}{r}$, $b(r) := \arg\Gamma(1 - w(r))$, and $c(r) := \frac{2}{r}|\Gamma(1 - w(r))|$, where $w(r) = e^{i\pi/r}$. Then*

$$s_n(r) := \frac{S_n(r)}{n^{\cos\frac{\pi}{r}}} = c(r)\cos\left[a(r)\log n + b(r)\right] + O(n^{-\delta(r)})$$

with some $\delta(r) > 0$ which could be specified.

Now we use the fact that $\{\cos(a\log n + b) : n \in \mathbb{N}\}$ is dense on $[-1, 1]$ for $a \neq 0$ and $b \in \mathbb{R}$. To see this, we first remark that for every $a \neq 0$ one of the numbers $\frac{a}{2\pi}\log 2$ or $\frac{a}{2\pi}\log 3$ is irrational. Let $\frac{a}{2\pi}\log q \notin \mathbb{Q}$ with $q = 2$ or $q = 3$. Then, by Kronecker's approximation theorem (see, for instance, [3], p. 150), for every $x \in \mathbb{R}$ and $\varepsilon > 0$ there exist $m \in \mathbb{N}_0$ and $h \in \mathbb{Z}$ such that

$$\left|\left(\frac{a}{2\pi}\log q\right)m + \frac{b - x}{2\pi} - h\right| \leq \frac{\varepsilon}{2\pi}\,.$$

Let $y \in [-1, 1]$ and $x \in \mathbb{R}$ with $\cos x = y$. Then for $n = q^m$ we obtain

$$|\cos(a\log n + b) - y| \leq |(a\log n + b) - (x + 2\pi h)| \leq \varepsilon$$

which shows the denseness result stated above. From this and from Theorem 2 we infer that for $r \geq 2$ the set $\{s_n(r) : n \in \mathbb{N}\}$ is dense on $[-c(r), c(r)]$ and that

$$\liminf_{n\to\infty} s_n(r) = -c(r)\,, \quad \limsup_{n\to\infty} s_n(r) = c(r)\,.$$

We mention that $\lim\limits_{r\to\infty} c(r) = \frac{2}{\pi}$. In particular, for $r = 2$ we get

$$(4) \qquad S_n(2) = \operatorname{Re}\frac{n!}{\prod_{l=1}^n (l - i)} = c(2)\cos\left[\log n + \arg\Gamma(1 - i)\right] + O\Big(\frac{1}{n}\Big)\,.$$

$S_n(2)$ is dense on the interval $[-c(2), c(2)]$, and we obtain

$$\liminf_{n\to\infty} S_n(2) = -c(2)\,, \quad \limsup_{n\to\infty} S_n(2) = c(2)\,,$$
$$c(2) = \left(\frac{2\pi}{e^{\pi} - e^{-\pi}}\right)^{1/2}.$$

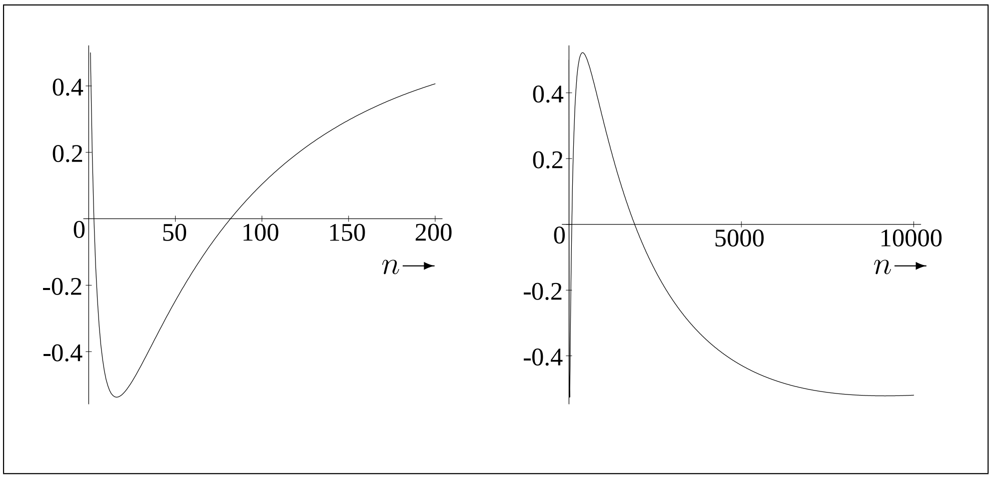


Figure 2: The function $S_n(2)$ for $1 \leq n \leq 200$ (left) and $1 \leq n \leq 10000$ (right).

In Fig. 2 one can see the oscillatory behaviour of $S_n(2)$ described mainly by the term $\cos[\log n]$ in (4).

We proof (1) for $r \in \mathbb{N}$ in Section 2 and for $r > 0,\ r \notin \mathbb{N}$, in Section 3. In Section 4 we prove (2).

The following identity will be used all the time: for $\operatorname{Re}\alpha < 0,\ \operatorname{Re}\beta \leq 0$, and $m \in \mathbb{N}_0$ one has

$$(5) \quad I(\alpha, \beta, m) := \int_0^\infty e^{\alpha t}(1 - e^{\beta t})^m \, dt = -\frac{m!\, \beta^m}{\prod_{l=0}^m (\alpha + l\beta)} \,.$$

This identity can be derived from [1], Formula 3.312, or proved directly by induction with respect to m.

2. The case $r \in \mathbb{N}$. We have

$$\begin{aligned} S_n(1) &= \sum_{m=0}^n \binom{n}{m} (-1)^m \int_0^1 u^m \, du = \int_0^1 (1-u)^n \, du = \frac{1}{n+1} \,, \\ S_n(2) &= \sum_{m=0}^n \binom{n}{m} (-1)^m \operatorname{Re} \int_0^1 u^{im} \, du = \operatorname{Re} I(n) \text{ with} \\ I(n) &:= \int_0^1 (1 - u^i)^n \, du = \int_0^\infty e^{-v}(1 - e^{-iv})^n \, dv = I(-1, -i, n) \\ &= -\frac{n!(-i)^n}{\prod_{l=0}^n (-1 - li)} = \frac{n!}{\prod_{l=1}^n (l - i)} \,, \end{aligned}$$

where we used (5). Now let $r \geq 3$. In this case we use the decomposition into partial fractions and get

$$\frac{1}{z^r+1} = -\frac{1}{r}\sum_k \frac{w_k}{z-w_k} = -\frac{1}{r}\sum_k w_k \int_0^\infty \exp\left[(w_k-z)t\right] dt$$

for $\operatorname{Re} z > \cos\frac{\pi}{r}$. In particular we obtain

$$\frac{1}{m^r+1} = -\frac{1}{r}\sum_k w_k \int_0^\infty \exp\left[(w_k-m)t\right] dt \qquad \text{for } m \in \mathbb{N}\,.$$

From this we infer

$$S_n(r) = 1 - \frac{1}{r}\sum_k w_k \int_0^\infty e^{w_k t}((1-e^{-t})^n - 1)\, dt\,.$$

Integrating by parts we find

$$\text{(6)} \qquad \int_0^\infty e^{w_k t}((1-e^{-t})^n - 1)\, dt = \frac{1}{w_k} - \frac{n}{w_k} I(w_k-1,-1,n-1)\,.$$

Therefore

$$S_n(r) = \frac{1}{r}\sum_k nI(w_k-1,-1,n-1) = A_n(r) \qquad \text{by (5)}\,.$$

3. The case $r \notin \mathbb{N}$. Denote by $\mathbb{C}^-$ the complex plane cut along the negative real axis. For $0 < r < 1$ the function $\frac{1}{z^r+1}$ is holomorphic on $\mathbb{C}^-$. For $r > 1$ it is meromorphic there with simple poles at w_k $(-\frac{r+1}{2} < k \le \frac{r-1}{2})$.

Let $\operatorname{Re} z > 1$. Then $|z| > 1$. Therefore

$$\frac{1}{z^r+1} = \sum_{j=1}^\infty (-1)^{j-1} z^{-rj} = \sum_{j=1}^\infty \frac{(-1)^{j-1}}{\Gamma(rj)} \int_0^\infty t^{rj-1} e^{-zt}\, dt$$

from which we infer

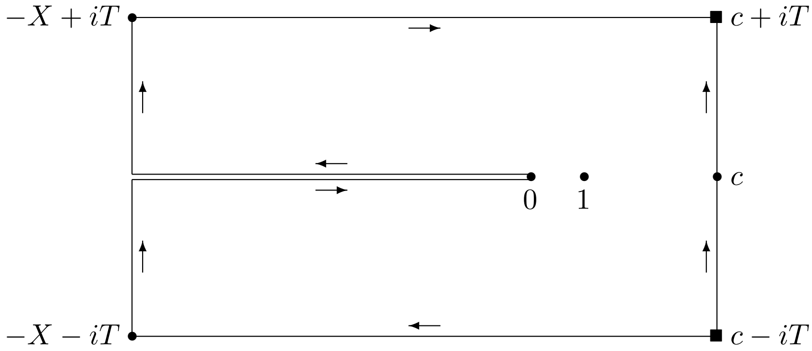
$$\text{(7)} \qquad \frac{1}{z^r+1} = \int_0^\infty f(r,t) e^{-zt}\, dt \quad (\operatorname{Re} z > 1)$$

$$\text{with } f(r,t) := \sum_{j=1}^\infty \frac{(-1)^{j-1}}{\Gamma(rj)} t^{rj-1} \quad (t > 0)\,.$$

By the inversion formula of Laplace ([4], (1.4.2)) we get

$$f(r,t) = \lim_{T\to\infty} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{e^{tz}}{z^r+1}\, dz \quad (c > 1,\, t > 0)\,.$$

Let $X \ge 2$ be given. We deform the interval $[c-iT, c+iT]$ into the following contour:



We apply the theorem of residues and take first the limit $X \to \infty$ and then the limit $T \to \infty$. This leads to the representation

(8) $\quad f(r,t) = g(r,t) + h(r,t)$

where

$$\begin{aligned} g(r,t) &:= \sum_k \operatorname{Res}_{w_k} \frac{e^{tz}}{z^r+1} \qquad (r>1)\,, \\ h(r,t) &:= \frac{1}{2\pi i} \int_{\leftrightarrows \bullet 0} \frac{e^{tz}}{z^r+1}\, dz\,. \end{aligned}$$

We find

$$\begin{aligned} (9) \qquad g(r,t) &= -\frac{1}{r} \sum_k w_k e^{w_k t} \qquad (r>1)\,, \\ (10) \quad h(r,t) &= \frac{\sin r\pi}{\pi}\, h^*(r,t) \qquad \text{where} \\ (11) \quad h^*(r,t) &:= \int_0^\infty \frac{u^r}{|1+u^r e^{ir\pi}|^2} e^{-tu}\, du\,. \end{aligned}$$

We have

$$(12) \quad \int_0^\infty h^*(r,t)\, dt = \int_0^\infty \frac{u^{r-1}}{|1+u^r e^{ir\pi}|^2}\, du < \infty$$

since the integrand on the right-hand side is $\sim u^{-r-1}$ for $u \to \infty$. Now observe that

$$\begin{aligned} (13) \quad \int_0^\infty h^*(r,t)(1-e^{-t})^n\, dt &= B_n^*(r)\,, \\ (14) \quad \int_0^\infty h(r,t)(1-e^{-t})^n\, dt &= B_n(r)\,. \end{aligned}$$

From (7) and (8) we infer

$$(15)\quad \frac{1}{z^r+1} = G(r,z) + H(r,z) \quad (\operatorname{Re} z > 1)$$

where

$$(16)\quad G(r,z) \;:=\; \int_0^\infty g(r,t)e^{-zt}\,dt \quad (\operatorname{Re} z > \cos\frac{\pi}{r}\,,\; r>1)\,,$$

$$(17)\quad H(r,z) \;:=\; \int_0^\infty h(r,t)e^{-zt}\,dt \quad (\operatorname{Re} z \geq 0\,,\; r>0)\,.$$

Let $r > 1$. From (9) we deduce for $\operatorname{Re} z > \cos\frac{\pi}{r}$

$$(18)\quad G(r,z) \;=\; -\frac{1}{r}\sum_k w_k \int_0^\infty \exp[(w_k - z)t]\,dt\,,$$

$$(19)\quad G(r,z) \;=\; -\frac{1}{r}\sum_k \frac{w_k}{z-w_k}\,.$$

From (12) we see that

(20) $H(r,z)$ is continuous for $\operatorname{Re} z \geq 0$ and holomorphic on $\operatorname{Re} z > 0$.

From this we derive by analytic continuation and continuity the following facts:

$$(21)\quad \frac{1}{z^r+1} = H(r,z) \quad (\operatorname{Re} z \geq 0) \text{ for } 0<r<1\,,$$

(22) for $1<r<2$ the equation (15) with $G(r,z)$ given by (18) holds true for $\operatorname{Re} z \geq 0$ (note that $\cos\frac{\pi}{r} < 0$ for $1<r<2$),

(23) for $r>2$ the equation (15) with $G(r,z)$ given by (18) holds true for $\operatorname{Re} z > \cos\frac{\pi}{r}$; in particular it holds true for $\operatorname{Re} z \geq 1$,

(24) for $r>1$ the equation (15) with $G(r,z)$ given by (19) holds true for real $z \geq 0$.

In particular,

(25) $1 = \frac{1}{r}\sum_k 1 + H(r,0)$.

<u>$0<r<1$</u>: From (21) and (14) we obtain

$$S_n(r) = \sum_{m=0}^{n} \binom{n}{m} (-1)^m H(r,m) = B_n(r)\,.$$

<u>$1<r<2$</u>: There are two zeros w, $\overline{w}$ where $w = w(r)$. From (22) we get

$$\frac{1}{m^r+1} = -\frac{2}{r}\operatorname{Re} w \int_0^\infty \exp[(w-m)t]\,dt + H(r,m) \qquad \text{for } m \in \mathbb{N}_0$$

which implies

$$S_n(r) = -\frac{2}{r}\operatorname{Re} wI(w,-1,n) + \sum_{m=0}^{n}\binom{n}{m}(-1)^m H(r,m)\,.$$

By (5) and (14) this is equal to $A_n(r) + B_n(r)$.

$\underline{r > 2}$: From (23) we get

$$\frac{1}{m^r+1} = -\frac{1}{r}\sum_k w_k \int_0^\infty \exp[(w_k - m)t]\,dt + H(r,m) \quad \text{ for } m \in \mathbb{N}\,.$$

From this together with (14) we infer

$$S_n(r) = 1 - \frac{1}{r}\sum_k w_k \int_0^\infty e^{w_k t}[(1-e^{-t})^n - 1]\,dt + B_n(r) - H(r,0)\,.$$

We apply (6), (25) and (5) and obtain $S_n(r) = A_n(r) + B_n(r)$.

4. Evaluation of $B_n^*(r)$, conclusion.

Let $r > 0$, $r \notin \mathbb{N}$. By definition we have

$$B_n^*(r) = \int_0^\infty f_r(u) g_n(u)\,du\,,$$

where

$$f_r(u) := \frac{u^r}{|1 + u^r e^{ir\pi}|^2}$$

and

$$g_n(u) = I(-u,-1,n) = \frac{n!}{\prod_{l=0}^n (u+l)}$$

by (5).

Put $u_0(r) := (1/2)^{1/r}$ and split up

$$B_n^*(r) = \Big(\int_0^{u_0(r)} + \int_{u_0(r)}^\infty\Big)\ldots\,du = x + y\,.$$

Estimation of y. Write $g_n(u) = u^{-1}h_n(u)$. Then

$$y \le h_n(u_0(r)) \int_0^\infty f_r(u) u^{-1}\,du\,.$$

Since

$$h_n(u_0(r)) = u_0(r)\, n^{-u_0(r)}\Big(n^{u_0(r)} g_n(u_0(r))\,\Big)$$

and $n^{u_0(r)} g_n(u_0(r)) \to \Gamma(u_0(r))$ we get $y \ll n^{u_0(r)}$.

Estimation of x. For $0 \le u \le u_0(r)$ one has

$$\begin{aligned} \frac{1}{|1+u^r e^{ir\pi}|^2} &= 1 + O(u^r)\,, \\ g_n(u) &= n^{-u}\Gamma(u)\Big(1+O\Big(\frac{1}{n}\Big)\Big)\,, \\ \Gamma(u) &= \frac{1}{u}\,\Big(1+O(u)\Big) \end{aligned}$$

with O-constants depending on r at most.

We conclude that

$$f_r(u) g_n(u) = u^{r-1}\exp(-u\log n)\Big(1+O\Big(u^r+u+\frac{1}{n}\Big)\,\Big)$$

for $0 \le u \le u_0(r)$. From this it follows that

$$x = \Gamma(r)(\log n)^{-r} + O\Big((\log n)^{-2r} + (\log n)^{-r-1}\Big)\,.$$

It remains to show that $T_n(r) := -S_n(r)\downarrow 0$ $(n\to\infty,\ n \ge n_0(r))$ in the case $1<r<2$. We have

$$T_n(r) - T_{n+1}(r) = (A_{n+1}(r) - A_n(r)) + \Big(-\frac{\sin r\pi}{\pi}\Big)\,(B_n^*(r) - B_{n+1}^*(r))\,.$$

Since

$$A_n(r) = \frac{2}{r}\,\mathrm{Re}\,[\Gamma(1-w(r))n^{w(r)}] + O\Big(n^{\,\mathrm{Re}\,w(r)-1}\Big)$$

we get $A_{n+1}(r) - A_n(r) \ll n^{\cos\frac{\pi}{r}-1}$. From the representation

$$B_n^*(r) - B_{n+1}^*(r) = \int_0^\infty \frac{u^r}{|1+u^r e^{i\pi r}|^2}\, I(-(u+1),-1,n)\, du$$

and

$$\begin{aligned} I(-(u+1),-1,n) &= \frac{1}{u+1}\prod_{l=1}^{n}\Big(1+\frac{u+1}{l}\Big)^{-1} \\ &\ge \frac{1}{u+1}\exp[-(u+1)L(n)]\,, \quad L(n) := \log n + 1\,, \\ &\ge \frac{1}{en}\,\frac{1}{u+1}\exp[-uL(n)] \end{aligned}$$

we obtain

$$\begin{aligned} B_n^*(r) - B_{n+1}^*(r) &\geq \frac{1}{8en}\int_0^1 u^r \exp(-uL(n))\,du \\ &= \frac{1}{8en}\Big(\frac{\Gamma(r+1)}{L(n)^{r+1}} - \int_0^\infty v^r e^{-v}\,dv\Big) \\ &\geq \frac{1}{8en}\Big(\frac{\Gamma(r+1)}{L(n)^{r+1}} - \frac{L(n)^r}{n}\Big)\,. \end{aligned}$$

We conclude that

$$T_n(r) - T_{n+1}(r) \geq \Big(-\frac{\sin r\pi}{\pi}\Big)\,\frac{\Gamma(r+1)}{16e}\,\frac{1}{nL(n)^{r+1}} + O\Big(n^{\cos\frac{\pi}{r}-1}\Big)\,.$$

Since $\cos\frac{\pi}{r} < 0$ the expression on the right-hand side is positive for $n \geq n_0(r)$.

5. Final remarks. We can see from the proofs above that the queer behaviour of $S_n(r)$ is determined mainly by the zeros of the function z^r+1 in $\mathbb{C}^-$. These zeros which appear in the term $A_n(r)$ are the reason for the quite complicated structure of $S_n(r)$. If we replace the term m^r+1 in the definition of $S_n(r)$ by m^r and consider

$$\widetilde{S}_n(r) := \sum_{m=1}^{n}\binom{n}{m}\frac{(-1)^n}{m^r}\,,$$

the situation is much simpler. Indeed, we can write

$$\begin{aligned} \widetilde{S}_n(r) &= \sum_{m=1}^{n}\binom{n}{m}(-1)^m\frac{1}{\Gamma(r)}\int_0^\infty t^{r-1}e^{-mt}\,dt \\ &= -\frac{1}{\Gamma(r)}\int_0^\infty t^{r-1}[1-(1-e^{-t})^n]\,dt \\ &= -\frac{1}{\Gamma(r)}\sum_{m=0}^{n-1} J_m(r) \end{aligned}$$

with $J_m(r) := \int_0^\infty t^{r-1}(1-e^{-t})^m e^{-t}\,dt$. It is not difficult to see that

$$J_m(r) = \frac{[\log(m+1)]^{r-1}}{m+1}\,\Big[1 + O\Big(\frac{1}{\sqrt{\log(m+1)}}\Big)\Big]\,,$$

and consequently

$$\lim_{n\to\infty}\frac{\widetilde{S}_n(r)}{(\log n)^r} = -\,\frac{1}{\Gamma(r)}\,.$$

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