Bachelor Thesis

PROJECTIVE DUALS OF SMOOTH CUBIC CURVES

Universität Konstanz

Fachbereich Mathematik und Statistik

Submitted by Julian Weigert

Supervised by Prof. Dr. Mateusz Michałek

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Contents

1. Abstract

In this thesis we will discuss the notion of projective dual varieties and in particular the case of a projective curve in $\mathbb{P}^2$. Starting with a projective variety X, the dual variety will encode the tangent spaces of X at every point $x \in X$ and hence can be used as a tool to examine tangency properties of X. One example is 3.3.4, where we are interested in lines that are tangent to a curve at two different points. We can use the duality to turn this into a problem of finding double points of the dual variety.

In chapter 2 we will introduce general definitions and show that irreducibility of a variety can be transfered to the dual variety..In the second part of this chapter we discuss how to actually compute the dual variety in the case of a curve in $\mathbb{P}^2$ and use this to see first examples. The definitions and statements that we use in this section mainly come from chapter 1 of [GKZ94].

The third chapter focuses on the special case of a curve X in $\mathbb{P}^2$. The aim of this section is to relate the two key invariants of dimension and degree of X to the dimension and degree of the dual variety $X^\vee$. We will first prove that the dual of a curve is a curve again and then introduce the notion of general elements in a topological space to prepare the main result of this section. We conclude by showing a special case of the Plücker relations, namely a result that compares the degree of a curve to the degree of its dual curve. More precisely we will prove that in the general case, the dual of a degree $d > 1$ curve must have degree $d(d-1)$.

The main result of chapter 3 gives us a rational map $\mathbb{P}(S^3\mathbb{C}^3) \dashrightarrow \mathbb{P}(S^6\mathbb{C}^3)$, sending the defining equation of a smooth cubic curve in $\mathbb{P}^2$ to the degree 6 polynomial that defines the dual curve. We will see that this is a rational map, given by 28 polynomials $F_0, \ldots, F_{27}$. The goal of this last section is to find the degree of these 28 polynomials, which we will be doing in two steps. First we find a theoretical upper bound, by looking at some properties of the resultant of a collection of polynomials, then we will calculate lower bounds by substituting linearly to reduce the number of variables in $F_0, \ldots, F_{27}$ and therefore reduce the computational effort. Finally we will get the lower bound to match our theoretical upper bound and hence fix the degree of $F_0, \ldots, F_{27}$ to be 16. Most of the ideas that we use in this chapter resulted from a conversation that I had with Prof. Dr. Bernd Sturmfels and Prof. Dr. Mateusz Michałek at the University of Konstanz.

2. Projective Dual Varieties

2.1. Basic definitions and first properties

Definition 2.1.1. Let V be an $n+1$-dimensional $\mathbb{C}$- vector space, where $n \in \mathbb{N}$. By $\mathbb{P}(V)$ we denote the *projectivization of* V, namely the set of all 1-dimensional vector subspaces of V. For $\mathbb{P}(\mathbb{C}^{n+1})$, where $n \in \mathbb{N}$, we will just write $\mathbb{P}^n$. If $W \subseteq V$ is a vector subspace, then 1-dimensional subspaces of W are also 1-dimensional in V, hence $\mathbb{P}(W) \subseteq \mathbb{P}(V)$ naturally. We say in this case that $\mathbb{P}(W)$ is a projective subspace of $\mathbb{P}(V)$. If the dimension of W is two, we will as usual call $\mathbb{P}(W)$ a line in $\mathbb{P}(V)$ and if W is of codimension one we will call $\mathbb{P}(W)$ a hyperplane in $\mathbb{P}(V)$.

Remark 2.1.2. When working with projective spaces, we will often view them as topological spaces. This remark describes how to put homogeneous coordinates on a projective space and how to equip a projective space $\mathbb{P}(V)$ with the Zariski-topology.
Since a 1-dimensional vector subspace of V is specified by one nonzero vector, $\mathbb{P}(V)$ is just the set of equivalence classes $[v]$, for $v \in V \setminus \{0\}$, with the relation $v_1 \sim v_2$ if and only if v_1, v_2 define the same 1-dimensional vector subspace, namely if and only if $v_1 = \lambda v_2$ for some $\lambda \in \mathbb{C} \setminus \{0\}$. By fixing a base of V, we can talk about coordinates of a point in V. For a fixed, nonzero point $a = (a_0, \dots, a_n) \in V$, we write $(a_0 : \dots : a_n) \in P(V)$ for the line that passes through a, especially we have $(a_0 : \dots : a_n) = (\lambda a_0 : \dots : \lambda a_n)$ for any nonzero $\lambda \in \mathbb{C}$.
Zariski-Topology makes $\mathbb{P}(V)$ into a topological space: A subset of $\mathbb{P}(V)$ is closed, if and only if it is the set of common zeroes of some homogeneous polynomials in $\mathbb{C}[x_0, \dots, x_n]$. Notice that, while in general we cannot evaluate a polynomial at a point $(a_0 : \dots : a_n) \in \mathbb{P}(V)$,[1] it makes sense to ask if a homogeneous polynomial f vanishes at $a = (a_0 : \dots : a_n) \in \mathbb{P}^n$. Since $\lambda^{\deg(f)} \cdot f(a) = f(\lambda \cdot a), \lambda \in \mathbb{C}$, the evaluation will only change by a multiple of $\lambda \neq 0$, so if we look at zeroes this has no effect on the result. For homogeneous polynomials $f_1, \dots, f_m$, we also write $\mathcal{V}_{\mathbb{P}}(f_1, \dots, f_m)$ for the set of common zeroes of these polynomials in $\mathbb{P}(V)$ and call it a projective variety. With induced topology from $\mathbb{P}(V)$, this becomes a topological space again, in particular a variety has a dimension, namely the supremum of integers n, such that there exists a chain $Z_0 \subsetneq \dots \subsetneq Z_n$ of closed irreducible subsets of the variety.
We also assign an affine variety, called the affine cone, to every projective variety, simply by taking the zeros of the same polynomials as a subset of $\mathbb{A}^{n+1}$. By definition $\mathbb{A}^{n+1}$ is the topological space $\mathbb{C}^{n+1}$ with Zariski-Topology, i.e. the closed sets are common zero sets of (not necessarily homogeneous) polynomials in $\mathbb{C}[x_0, \dots, x_n]$. To distinguish (projective) varieties in $\mathbb{P}^n$ from (affine) varieties in $\mathbb{A}^{n+1}$, we will denote the former $\mathcal{V}_{\mathbb{P}}(f_1, \dots, f_m)$, for homogeneous $f_1, \dots, f_m \in \mathbb{C}[x_0, \dots, x_n]$ and the latter $\mathcal{V}_{\mathbb{A}}(f_1, \dots, f_m)$ for arbitrary $f_1, \dots, f_m \in \mathbb{C}[x_0, \dots, x_n]$.

Definition 2.1.3 (projective dual space)**.** To a finite dimensional $\mathbb{C}$-vector space V we can assign the dual V^*, which is the vector space of all $\mathbb{C}$-linear functions from V to $\mathbb{C}$. We

[1] the evaluation of a polynomial at p can change when we scale a point p by a nonzero scalar, but $p = \lambda p \in \mathbb{P}(V)$, for $\lambda \in \mathbb{C} \setminus \{0\}$

define the *dual projective space* of V to be $\mathbb{P}(V)^* := \mathbb{P}(V^*)$, namely the projectivization of the dual space.

Remark 2.1.4. The two projective spaces $\mathbb{P} = \mathbb{P}(V)$ and $\mathbb{P}^* = \mathbb{P}(V)^*$ satisfy the following dual relation:

(i) A point $p \in \mathbb{P}^*$, is by definition a line in V^*, hence a vector space of the form $\{a \cdot v : a \in \mathbb{C}\}$, where $v : V \to \mathbb{C}$ is an element of V^*. The kernel of v is a subspace of V of codimension 1 and it uniquely determines v up to scaling, hence we may identify p with the hyperplane $\mathbb{P}(\ker(v)) \subseteq \mathbb{P}(V)$.

(ii) Conversely, a point $p \in \mathbb{P}$, is by definition a dimension 1 subspace l_p of V, hence the set of all $v \in V^*$, such that $\ker(v) \supseteq l_p$ is a codimension one subspace $W \subseteq V^*$. So $\mathbb{P}(W) \subseteq \mathbb{P}^*$ is a hyperplane. The other way round we can recover p from $\mathbb{P}(W)$ in the following way: Since $W \subseteq V^*$ has codimension 1, the set of all points $x \in V$ such that $v(x) = 0$ for every $v \in W$, is a dimension 1 subspace of V, namely a point in $\mathbb{P}$. By the way we constructed W, this point is p. Hence points in $\mathbb{P}$ uniquely correspond to hyperplanes in $\mathbb{P}^*$.

Point (i) tells us that we may also view $\mathbb{P}^*$ as the set of all hyperplanes of $\mathbb{P}$. The condition $\ker(v) \supseteq l_p$ in (ii) exactly means that p lies on the hyperplane defined by $\ker(v)$. Hence the hyperplane of $\mathbb{P}^*$ that corresponds to the point $p \in \mathbb{P}$ (via (ii)) is just given by the set of all hyperplanes of $\mathbb{P}$, passing through p.

Now that we have a dual object for projective spaces, the next step will be to define a dual object for varieties. More precisely, to a variety $X = \mathcal{V}_{\mathbb{P}}(f_1, \ldots, f_m) \subseteq \mathbb{P}^n$ we want to assign a dual variety, which will be a closed subset of $(\mathbb{P}^n)^*$. To do this, we shortly recall the definitions of smooth points and tangent spaces of a projective variety.
A point $(a_0 : \ldots : a_n) \in X = \mathcal{V}_{\mathbb{P}}(f_1, \ldots, f_m)$ is a smooth point of X, if and only if the point $(a_0, \ldots, a_n) \in \mathbb{C}^{n+1}$ is a smooth point of the affine cone $\hat{X}$ of X, i.e if and only if the rank of the Jacobian matrix at $(a_0, \ldots, a_n)$ is greater than the codimension of $\hat{X}$:

$$\operatorname{rank} \begin{pmatrix} \frac{\partial f_1}{\partial x_0} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_m}{\partial x_0} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} (a_0, \ldots, a_n) \geq \operatorname{codim}(\hat{X}, \mathbb{C}^{n+1}). \tag{1}$$

Notice that being a smooth point is an open condition: The set of all smooth points of X is exactly the complement of the variety generated by $f_1, \ldots, f_m$, together with all $k \times k$-minors of the Jacobian matrix, where $k = \operatorname{codim}(\hat{X}, \mathbb{C}^{n+1})$, hence an open subset of $X \subseteq \mathbb{P}^n$.
The tangent space T_aX of the variety $X = \mathcal{V}_{\mathbb{P}}(f_1, \ldots, f_m)$ at the point $a = (a_0, \ldots, a_n) \in X$ is the projective subspace of $\mathbb{P}^n$ defined by the linear equations

$$\sum_{i=0}^{n} \frac{\partial f_j}{\partial x_i}(a) \cdot x_i = 0, \qquad j = 1, \ldots, m.$$

As further described in A.1.2, (ii), one can identify $\mathcal{V}_{\mathbb{P}}(f_1, \ldots, f_m) \subseteq \mathbb{P}^n$ with the scheme $X = (\operatorname{Proj}(\mathbb{C}[x_0, \ldots, x_n]/\langle f_1, \ldots, f_m\rangle), \mathcal{O})$, then the usual definitions of smooth points and tangent spaces go like this: The tangent space on X at the point $x \in X$ is the dual vector space of the $(\mathcal{O}_x/m_x)$- vector space m_x/m_x^2, where m_x is the maximal ideal of the local ring $\mathcal{O}_x$. A point is then smooth if and only if the dimension of its tangent space is the (Krull)-dimension of $\mathcal{O}_x$, i.e. if and only if $\mathcal{O}_x$ is a regular ring. How these (more general) definitions relate to the definitions we gave in the projective setting, can be checked in [Sha94, chapter II, sections 1.3 and 1.4].

Definition 2.1.5 (projectively dual variety)**.** Let $\mathbb{P} = \mathbb{P}^n$ be a projective space and $X \subseteq \mathbb{P}$ an irreducible projective variety, i.e. a Zariski-closed, irreducible subset of $\mathbb{P}$. We say that a hyperplane $H \subseteq \mathbb{P}$ is tangent to X at the smooth point $x \in X$, if $x \in H$ and the tangent space to H at x (i.e. H itself) contains the tangent space to X at x. Since hyperplanes of $\mathbb{P}$ give points of $\mathbb{P}^*$, we can define the *projectively dual variety of* X, denoted by $X^\vee$, to be the (Zariski-)closure of the set of all hyperplanes of $\mathbb{P}$ that are tangent to X at some smooth point $x \in X$. Denoting by X_{sm} the set of all smooth points of X this means

$$X^\vee := \overline{\{H \in \mathbb{P}^* | \exists x \in X_{sm} : T_x X \subseteq H\}}.$$

The name "dual variety" suggests that for a variety $X \subseteq \mathbb{P}$ we have $(X^\vee)^\vee = X$. This result is part of the Biduality Theorem, which can be found in [GKZ94, chapter 1, Theorem 1.1].

Example 2.1.6. In this example we show that being tangent to a (fixed) smooth point of a projective variety $X = \mathcal{V}_\mathbb{P}(f_1, \dots, f_m) \subseteq \mathbb{P}^n$ is a polynomial condition.
A point $b = (b_1 : \dots : b_{n+1}) \in (\mathbb{P}^n)^*$ corresponds to the hyperplane $\mathbb{P}(\ker(u)) \subseteq \mathbb{P}^n$, where $u : \mathbb{C}^{n+1} \to \mathbb{C}, (y_1, \dots, y_{n+1}) \mapsto \sum_{i=0}^{n+1} b_i y_i$ (see 2.1.4). By definition this is tangent to X at a point $a = (a_1 : \dots : a_{n+1})$, if and only if $\ker(u)$ contains the tangent space $T_a\mathcal{V}_\mathbb{P}(f_1, \dots, f_m)$, which is if and only if $(b_1, \dots, b_{n+1}) \in \operatorname{span}_\mathbb{C}(v_1, \dots, v_m)$, where we define $v_i := (\left(\frac{\partial}{\partial x_0} f_i\right)(a), \dots, \left(\frac{\partial}{\partial x_n} f_i\right)(a)) \in \mathbb{C}^{n+1}$ (notice that this condition does not change when we scale b with a nonzero element of $\mathbb{C}$). Let $v_1, \dots, v_{\tilde{m}}$ be a maximal linearly independent subset of the $v_1, \dots, v_m$, hence a base of $\operatorname{span}_\mathbb{C}(v_1, \dots, v_m)$, then we can add vectors $v_{\tilde{m}+1}, \dots, v_{n+1}$, such that $v_1, \dots, v_{n+1}$ form a basis of $\mathbb{C}^{n+1}$. Clearly now $b \in \operatorname{span}_\mathbb{C}(v_1, \dots, v_m)$ if and only if the coefficients of b in the new base $v_1, \dots, v_n$ are zero in front of any v_i where $i \geq \tilde{m} + 1$. This is even a linear condition on the (original) entries of b: Let S be the matrix of the upper base change from the standard base to $v_1, \dots, v_{n+1}$. When we describe b in the basis $v_1, \dots v_{n+1}$, the coefficient in front of v_i is exactly the dot product of the i-th row of S and b (considered as a column vector), i.e. $\sum_{j=1}^{n+1} S_{ij} b_j$. [2] Clearly this is a homogeneous polynomial of degree one in the b_j's.

We can also ask, what happens, when we change a. Clearly the matrix of base change S depends on a, but it is not straightforward to see that S_{ij} have to be (homogeneous) polynomials in $a_1, \dots, a_{n+1}$, since the last rows of S come from the vectors $v_{\tilde{m}+1}, \dots, v_{n+1}$ that we added. However, we can see this using schemes.

[2]here S_{ij} is the entry in row i and column j of the matrix S

In the product $\mathbb{P}^{\tilde{m}} \times (\mathbb{P}^n \times (\mathbb{P}^n)^*)$ we can consider the set of all triples (c, a, b), such that

$$b_1 = \sum_{i=1}^{\tilde{m}} c_i \left(\frac{\partial}{\partial x_1} f_i \right) (a)$$
$$\vdots$$
$$b_{n+1} = \sum_{i=1}^{\tilde{m}} c_i \left(\frac{\partial}{\partial x_{n+1}} f_i \right) (a).$$

These conditions are polynomials $F_1, \ldots, F_{n+1}$ in the c_i, b_i, a_i and F_j homogeneous in b_i's and in c_i's of degree 1 and in a_i's of degree $\deg(f_j) - 1$. But then these polynomials define a closed subscheme of the fibered product $\mathbb{P}^{\tilde{m}} \times_{\mathbb{C}} (\mathbb{P}^n \times_{\mathbb{C}} (\mathbb{P}^n)^*)$ that is locally given, on an affine open $c_i = 1 \times_{\mathbb{C}} (a_j = 1 \times_{\mathbb{C}} b_k = 1)$ by

$$\mathrm{Spec}(\mathbb{C}[c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_{n+1}, a_1, \ldots, a_{j-1}, c_{j+1}, \ldots, a_{n+1}, b_1, \ldots, b_{k-1}, c_{k+1}, \ldots, b_{n+1}]/I),$$

where $I := \mathrm{rad}(F_1, \ldots, F_{n+1})_{c_i = a_j = b_k = 1}$.
The fibered product $\mathbb{P}^{\tilde{m}} \times_{\mathbb{C}} (\mathbb{P}^n \times_{\mathbb{C}} (\mathbb{P}^n)^*)$ comes with two projections, the second one going to $\mathbb{P}^n \times_{\mathbb{C}} (\mathbb{P}^n)^*$. This must be a proper morphism, since it is a base change of the projective (hence proper) morphism $\mathbb{P}^{\tilde{m}} \to \mathrm{Spec}(\mathbb{C})$.[3] But proper morphisms by definition are closed, so the image of the second projection is a closed subset of $\mathbb{P}^n \times_{\mathbb{C}} (\mathbb{P}^n)^*$. However on closed points, the image of this projection contains exactly those pairs (a, b), such that there exists a $c \in \mathbb{P}^n$, where (c, a, b) fulfills the equations $F_1, \ldots, F_{n+1}$, hence exactly those pairs (a, b), where $b \in \mathrm{span}_{\mathbb{C}}(v_1, \ldots, v_m)$ in the notation above. This means that a pair (a, b), where $b \in (\mathbb{P}^n)^*$ and a is a smooth point of X is in the image of the projection $\mathbb{P}^n \times_{\mathbb{C}} (\mathbb{P}^n \times_{\mathbb{C}} (\mathbb{P}^n)^*) \to \mathbb{P}^n \times_{\mathbb{C}} (\mathbb{P}^n)^*$, if and only if it corresponds to a hyperplane tangent to X at a. Notice that we do not claim that, if a pair (a, b) is in this image, then a needs to be a smooth point; only when a is a smooth point, being in the image means that b is tangent to X at a.
Now the image of the projection $\mathbb{P}^n \times_{\mathbb{C}} (\mathbb{P}^n \times_{\mathbb{C}} (\mathbb{P}^n)^*) \to \mathbb{P}^n \times_{\mathbb{C}} (\mathbb{P}^n)^*$ needs to be closed, because the projection is a closed map. However, such a closed subset must also be closed, when restricted to any open affine, but closed subsets of the spectrum of a polynomial ring are given by polynomial equations.[4] Homogenizing those equations in the a_i's and in the b_i's gives us the desired polynomials that vanish on a pair (a, b), where a is a smooth point of X if and only if b is a hyperplane tangent to X at a.
To compute these equations on an affine flag, one would need to eliminate $c_1, \ldots, c_{n+1}$ from the ideal I.

Our next goal is to compute the dual variety of a given irreducible $X \subseteq \mathbb{P}$. We will follow the proof of [GKZ94, chapter 1, Prop. 1.3], as it gives us a first idea about how to do this.

Proposition 2.1.7. For an irreducible variety $V \subseteq \mathbb{P} = \mathbb{P}^n$, its dual variety is irreducible.

[3] see [Har77], chapter II, 4.8 and 4.9

[4] by [Har77], chapter 2, exercise 3.11 b, any closed subscheme of $\mathrm{Spec}(A), A$ a ring, is of the form $\mathrm{Spec}(A/I)$ for an ideal $I \subseteq A$. In our case the polynomial equation will just be any set of generators of I.

By following the definition of dual variety, the first step is to consider the smooth points of V, but this is an open subset of V (see (1)) and therefore it is not a variety in the sense of 2.1.2. We will therefore pass to schemes, where we have a more general notion of a variety, namely a reduced, separated scheme of finite type over an algebraically closed field. We will call such a scheme an abstract variety (see also A.1.1). Notice that for us, an abstract variety does not need to be irreducible, in contrast to the definition given in [Har77] for instance.
The main idea of this proof will be to consider all pairs (x, H), where x is a smooth point of V and H is a hyperplane tangent to V at x. By definition we can then obtain the dual variety by projecting to the second coordinate. This idea will also be very helpful in sections 3.1 and 3.3, when we discuss dimension and degree of dual curves, therefore we will now provide a formal way of making the set of all such pairs (x, H) an abstract subvariety.

Lemma 2.1.8. Let $V = \mathcal{V}_{\mathbb{P}}(f_1, \dots, f_m)$ be an irreducible projective variety and let $X = \mathrm{Proj}(\mathbb{C}[x_0, \dots, x_n]/\mathrm{rad}(f_1, \dots, f_m))$ be the associated abstract variety in the sense of A.1.2, (ii). Consider the fibered product of schemes $X \times_{\mathbb{C}} (\mathbb{P}^n)^*$, where we identify $(\mathbb{P}^n)^* := \mathrm{Proj}(\mathbb{C}[u_0, \dots, u_n])$.

(i) The closed points of $X \times_{\mathbb{C}} (\mathbb{P}^n)^*$ are exactly all tuples (a, b), where $a \in V$ and $b \in (\mathbb{P}^n)^*$. Furthermore, if π_1, π_2 are the scheme-theoretic projections from the fibered product $X \times_{\mathbb{C}} (\mathbb{P}^n)^*$ to X and to $(\mathbb{P}^n)^*$, then π_1 sends a closed point (a, b) to the closed point of X that corresponds to $a \in V$ and π_2 sends the same point to the closed point $b \in (\mathbb{P}^n)^*$.

(ii) There exists a closed subscheme W_0 of an open subscheme of $X \times_{\mathbb{C}} (\mathbb{P}^n)^*$, such that W_0 contains exactly those closed points (a, b) of $X \times_{\mathbb{C}} (\mathbb{P}^n)^*$ where a is a smooth point of V and b corresponds to a hyperplane in $\mathbb{P}^n$ that is tangent to V at a. In particular W_0 is an abstract variety.

Proof. (i) For any closed point $x \in X \times_{\mathbb{C}} (\mathbb{P}^n)^*$, we can find an affine open $U = X \cap \{x_i = 1\} \subseteq X$ and an affine open $A = \{u_j = 1\} \subseteq (\mathbb{P}^n)^*$, such that

$$\begin{aligned} x &\in U \times_{\mathbb{C}} A \\ &= \mathrm{Spec}(\mathbb{C}[x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n]/I_{x_i=1}) \times_{\mathbb{C}} \mathrm{Spec}(\mathbb{C}[u_0, \dots, u_{j-1}, u_{j+1}, \dots, u_n]) \\ &= \mathrm{Spec}(\mathbb{C}[x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n]/I_{x_i=1} \otimes_{\mathbb{C}} \mathbb{C}[u_0, \dots, u_{j-1}, u_{j+1}, \dots, u_n]) \\ &= \mathrm{Spec}(\mathbb{C}[x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n, u_0, \dots, u_{j-1}, u_{j+1}, \dots, u_n]/I^e_{x_i=1}). \end{aligned}$$

Here $I := \langle f_1, \dots, f_m \rangle \subseteq \mathbb{C}[x_0, \dots, x_n]$ and I^e denotes the ideal in $\mathbb{C}[x_0, \dots, x_n, u_0, \dots, u_{j-1}, u_{j+1}, \dots, u_n]$, generated by the same polynomials $f_1, \dots, f_m$. Since x is closed in $X \times_{\mathbb{C}} (\mathbb{P}^n)^*$, it is closed in $U \times_{\mathbb{C}} V$ too and hence it corresponds to a maximal ideal of $\mathbb{C}[x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n, u_0, \dots, u_{j-1}, u_{j+1}, \dots, u_n]$ that contains $I^e_{x_i=1}$. As x varies, such ideals are of the form $\langle x_0 - a_0, \dots, x_{i-1} - a_{i-1}, x_{i+1} - a_{i+1}, \dots, x_n - a_n, u_0 - b_0, \dots, u_{j-1} - b_{j-1}, u_{j+1} - a_{j+1}, \dots, u_n - b_n \rangle, a_l, b_k \in \mathbb{C}$,[5] i.e. they give tupels $(a, b) = (a_0 : \dots : a_n : b_0 : \dots : b_n)$ (where we set $a_i = b_j = 1$),

[5] by Hilberts Nullstellensatz, see e.g. [Har77], chapter I, example 1.4.4

where $a \in V$ and $b \in (\mathbb{P}^n)^*$. The other way round, a tupel $(a, b) \in V \times (\mathbb{P}^n)^*$ gives a maximal ideal in every open affine $U \times_{\mathbb{C}} A$ that contains (a, b), hence it is a closed point of $X \times_{\mathbb{C}} (\mathbb{P}^n)^*$. We conclude that closed points of the abstract variety $X \times_{\mathbb{C}} (\mathbb{P}^n)^*$,[6] exactly correspond to pairs of points (a, b) in the cartesian product $V \times (\mathbb{P}^n)^*$.
On the open affine $U \times_{\mathbb{C}} A$ from above, π_1 just sends a prime ideal

$$p \in \mathrm{Spec}(\mathbb{C}[x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, u_0, \ldots, u_{j-1}, u_{j+1}, \ldots, u_n]),$$

that contains $I_{x_i=1}$, to $p \cap \mathbb{C}[x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n] \in U$, meaning that π_1 on closed points corresponds exactly to the (set-theoretic) first projection from the cartesian product $V \times (\mathbb{P}^n)^*$ to V. The same holds of course for π_2.

(ii) As described in (1), the set V_{sm} of smooth points of V is an open subset of $\mathbb{P}^n$ and a subset of V, hence an open subset of V. Via the homeomorphism between points of V and closed points of X, we get an open subset of the set of closed points of the scheme X. As we considered this with induced topology from X, there exists an open subset of X, which restricts to exactly those closed points of X that correspond to points in $V_{sm} \subseteq V$. Fix such an open subset of X, this gives an open subscheme $X_{sm} \subseteq X$. In particular X_{sm} is an abstract variety by A.1.4. By construction of the fibered product $X_{sm} \times_{\mathbb{C}} (\mathbb{P}^n)^*$ is an open subscheme of $X \times_{\mathbb{C}} (\mathbb{P}^n)^*$. From 2.1.6 we know that there exist polynomials $F_1, \ldots, F_k \in \mathbb{C}[x_0, \ldots, x_n, u_0, \ldots, u_n]$, homogeneous in the x_i's and homogeneous in the u_i's, such that a pair $(a, b) \in V_{sm} \times (\mathbb{P}^n)^*$, is a common zero of $F_1, \ldots, F_k$ if and only if b corresponds to a hyperplane in $\mathbb{P}^n$ that is tangent to V at a. Consider the set of pairs

$$V_0 := \{(a, b) \in \mathbb{P}^n \times (\mathbb{P}^n)^* : f_1(a) = \ldots = f_m(a) = F_1(a, b) = \ldots = F_k(a, b) = 0\}.$$

Clearly V_0 is a closed subset of $X \times (\mathbb{P}^n)^*$: Indeed on any open affine $U \times_{\mathbb{C}} A$ as above, it is just the closed subscheme defined by the polynomials obtained by dehomogenizing $f_1, \ldots, f_m, F_1, \ldots, F_k$ with respect to the variables x_i, u_j, and since such affine opens cover $X \times_{\mathbb{C}} (\mathbb{P}^n)^*$, V_0 is also closed there. Hence V_0 can be made into a closed subscheme with reduced scheme structure ([Har77], chapter II, example 3.2.6).
Define

$$W_0 := (X_{sm} \times_{\mathbb{C}} (\mathbb{P}^n)^*) \cap V_0,$$

then clearly the closed points of W_0 are exactly pairs (a, b) where a is a smooth point of V and b is a hyperplane tangent to V at a. Also notice that $X_{sm} \times_{\mathbb{C}} (\mathbb{P}^n)^*$ is an open subscheme of the abstract variety $X \times_{\mathbb{C}} (\mathbb{P}^n)^*$ and hence it is an abstract variety itself (A.1.4). W_0 is a closed, reduced subscheme of $(X_{sm} \times_{\mathbb{C}} (\mathbb{P}^n)^*)$ and therefore it is an abstract variety too (A.1.4).

□

[6] see A.1.5, X is irreducible since V is, by A.1.2

Now the key statement that we need to proof 2.1.7, is that W_0 is irreducible. For this, we will use the following result about abstract varieties.

Proposition 2.1.9. Let $f : X \to Y$ be a closed morphism of two abstract varieties (i.e. a morphism of schemes that are also abstract varieties) and assume that Y is irreducible. As f is continuous, for any closed point $y \in Y$, the fiber $f^{-1}(y)$ has to be a closed subset of X, we assume that this closed subset is nonempty, irreducible of some fixed dimension r, which does not depend on the choice of y. Then X has to be irreducible too.

Proof. Assume $X = \bigcup_{i=1}^k Z_i$ for irreducible, closed subsets Z_i of X and k as small as possible. Notice that X, as an abstract variety, is of finite type over $\mathbb{C}$, hence noetherian as a scheme and in particular as a topological space, so the union can be assumed to be finite. We want to prove $k = 1$. As a map of topological spaces, f is also surjective onto closed points, as the fiber of a closed point is nonempty. However as f is closed, the image of f is a closed subset of Y, so since closed points are dense,[7] f must really be surjective onto all points of Y as a map of topological spaces. In particular $\bigcup_{i=1}^k f(Z_i) = f(\bigcup_{i=1}^k Z_i) = f(X) = Y$ describes Y as a union of closed subsets. By irreducibility of Y, we can w.l.o.g assume that $f(Z_1) = Y$.
For any closed point $y \in Y$, by assumption, the fiber $f^{-1}(y)$ is irreducible. However, we clearly have $f^{-1}(y) = \bigcup_{i=1}^k (f^{-1}(y) \cap Z_i)$, which is a union of closed sets, so we must have $f^{-1}(y) \subseteq Z_i$ for some $i =\in \{1, \ldots, k\}$ for every closed point $y \in Y$.
Next we need a result that relates the dimension of a general fiber of a dominant map of irreducible varieties, namely 3.1.1, (ii). Applying it to the restriction of f to the irreducible component Z_1 (which is surjective on topological spaces, hence dominant) allows us to find a nonempty open subset $U \subseteq Y$, such that for any $y \in Y$, we have

$$\dim(f^{-1}(y) \cap Z_1) \geq \dim(Z_1) - \dim(Y), \tag{2}$$

and when $y \in U$, we even get equality. By minimality of the number of irreducible components k, we know that $V := X \setminus \left(\bigcup_{i=2}^k Z_i\right) \subseteq Z_1$ is a nonempty open subset of W_0. A closed point in $v \in V$ has the property that the fiber of $f(v)$ is completely contained in Z_1, because we know already that it is contained in some irreducible component and it can't be any Z_i for $i > 1$, as they do not contain v. To conclude we have three special properties of fibers:

(i) If $y \in U$, then $\dim(f^{-1}(y) \cap Z_1) = \dim(Z_1) - \dim(Y)$,

(ii) If $y \in Y$ is a closed point, then $\dim(f^{-1}(y)) = r$ by assumption,

(iii) if $y = f(v)$ for some $v \in V$, then $f^{-1}(y) \cap Z_1 = f^{-1}(y)$.

Notice that we can finish the proof if we have an element $y_0 \in Y$ that has all three properties at the same time: The three properties give us $\dim(Z_1) - \dim(Y) = \dim(f^{-1}(y_0) \cap Z_1) = \dim(f^{-1}(y_0)) = r$, but then for every closed point $y \in Y$, we get

$$\dim(f^{-1}(y) \cap Z_1) \geq \dim(Z_1) - \dim(Y) = r = \dim(f^{-1}(y)),$$

[7] see A.1.3

by (2). As Z_1 is irreducible, this means $f^{-1}(y) \subseteq Z_1$, meaning Z_1 contains all fibers of closed points. But then in particular Z_1 contains every closed point of X, because f as a closed map must map closed points to closed points. As closed points of X are dense in X by A.1.3 and as Z_1 is closed we get $X = Z_1$ and hence $k = 1$ by minimality.
To finish we need to find a point $y_0 \in Y$ that has all three properties (i),(ii),(iii). First we can make V an open subscheme of X, by A.1.4 V is an abstract variety. But then closed points of V are dense in V as we prove in A.1.3 and the closed points of V are exactly the closed points of Z_1 that belong to V. We trivially have $Z_1 = V^c \cup \overline{V}$, where V^c denotes the complement of V in Z_1. As V is a nonempty open subset of Z_1 and as Z_1 is irreducible, V needs to be dense in Z_1. Now take any closed subset C of Z_1 that contains every closed point of Z_1 that belongs to V. As those points are dense in V, we have $C \cap V = V$ and therefore C is a closed subset of Z_1 that contains the dense subset V, so $C = Z_1$. This means that, even when we take only the closed points of Z_1 that belong to V, we still get a dense subset of Z_1. Mapping those points under the surjective map f gives a dense subset $D \subseteq Y$ and as f is closed, D only contains closed points of Y. This means the points in D already have property (ii) and (iii) and it suffices to show that $D \cap U \neq \emptyset$. However this is obvious, because D is dense and U is open, so if $D \cap U = \emptyset$, we had $D \subseteq U^c$ and hence $Y = \overline{D} \subseteq U^c \subseteq Y$ as U^c is closed. This is a contradiction to U being nonempty. □

Proof of 2.1.7. Consider the abstract varieties X, X_{sm}, W_0 from 2.1.8. We have projections p_1, p_2 from W_0 to X_{sm} and to $(\mathbb{P}^n)^*$, namely the compositions of the closed immersion $W_0 \to (X_{sm} \times_{\mathbb{C}} (\mathbb{P}^n)^*)$, the open immersion $(X_{sm} \times_{\mathbb{C}} (\mathbb{P}^n)^*) \to X \times_{\mathbb{C}} (\mathbb{P}^n)^*$ and the projections π_1, π_2 to X and to $(\mathbb{P}^n)^*$. By definition, the dual of our variety $\mathcal{V}_{\mathbb{P}}(f_1, \ldots, f_m)$ is exactly the closure of (closed points in) the image of the second projection $W_0 \to (\mathbb{P}^n)^*$. We will denote by $X^\vee$ the subscheme of $(\mathbb{P}^n)^*$, which is given by the closure of the image of the second projection, with reduced induced subscheme structure. Notice that, by construction, the closed points of $X^\vee$ are homeomorphic to the dual variety of V in the way we defined the dual in 2.1.5.
Now by assumption V is irreducible, so by A.1.2 X is an irreducible scheme too. X_{sm} is an open subscheme of X, so it must be irreducible aswell:
Assume that $Z_1 \cap Z_2 = X_{sm}$ for two closed subsets Z_1, Z_2 of X_{sm}. Since topology on X_{sm} is induced from topology on X, the closed sets Z_1, Z_2 must be restrictions of closed sets in X to X_{sm}, i.e. we find $Z_1', Z_2' \subseteq X$, such that $Z_1 = X_{sm} \cap Z_1'$ and $Z_2 = X_{sm} \cap Z_2'$. But now $Z_1' \cup Z_2'$ is a closed subset of X that contains the open subset X_{sm}. We claim that X_{sm} is either dense in X or empty.[8] Indeed, when $X_{sm} \neq \emptyset$, the complement X_{sm}^c of X_{sm} in X and the closure $\overline{X_{sm}}$ of X_{sm} in X are two closed subsets of X, with $X_{sm}^c \cup \overline{X_{sm}} = X$, so by irreducibility of X and as $X_{sm}^c \neq X$ we must have $\overline{X_{sm}} = X$, meaning X_{sm} is dense in X. In this case we need to have $Z_1' \cup Z_2' = X$, as it is a closed subset, which contains a dense subset of X and again by irreducibility of X we have $Z_1' = X$ or $Z_2' = X$. But then we also have $Z_1' \cap X_{sm} = X \cap X_{sm} = X_{sm}$ or $Z_2' \cap X_{sm} = X \cap X_{sm} = X_{sm}$, making X_{sm} irreducible. If on the other hand $X_{sm} = \emptyset$, it is trivially irreducible.
Next we claim that W_0 is irreducible. Consider the first projection $p_1 : W_0 \to X_{sm}$ and take any closed point $x \in X_{sm}$. Such a point corresponds to a smooth point of the variety

[8] actually X_{sm} can not be empty as every reduced variety has at least some smooth points

V and p_1 maps a closed point of W_0 to x if and only if it corresponds to a pair (x, u), where $u \in (\mathbb{P}^n)^*$ represents a hyperplane of $\mathbb{P}^n$ that is tangent to V at x. However, being tangent to V at x means to contain T_xV, which by smoothness of V at the point x and [Sha94, II.1.4 Theorem 3] is a projective subspace of $\mathbb{P}^n$ of dimension equal to $r := \dim(\mathcal{V}_{\mathbb{P}}(f_1, \dots, f_m))$. This means that, the set of all hyperplanes which are tangent to $\mathcal{V}_{\mathbb{P}}(f_1, \dots, f_m)$ at x, are all codimension one subspaces of $\mathbb{P}^n$ that contain T_xX, hence all codimension one subspaces of $\mathbb{C}^{n+1}$ that contain a fixed vector subspace V of dimension $r + 1$ (T_xX is the projectivization of V). This is exactly the set of all hyperplanes of the vector space $\mathbb{C}^{n+1}/V$, i.e. the dual of the projectivization of this space $\mathbb{P}(\mathbb{C}^{n+1}/V)^*$. Hence the fiber $p_1^{-1}(x)$ forms a projective subspace of $(\mathbb{P}^n)^*$ of dimension $n - r - 1$ and this dimension is independent of x. In particular, the fiber of a closed point is irreducible. Furthermore, the projection p_1 is exactly the composition of a closed immersion i and a base change φ of the projective (hence proper) morphism $\pi : (\mathbb{P}^n)^* \to \mathrm{Spec}(\mathbb{C})$ (see [Har77, chapter II, 4.9]) that makes $(\mathbb{P}^n)^*$ a projective abstract variety.

$$\begin{array}{ccccc} & & W_0 & & \\ & \swarrow^{p_1} & \downarrow i & & \\ X_{sm} & \xleftarrow[\varphi]{} & X_{sm} \times_{\mathbb{C}} (\mathbb{P}^n)^* & \longrightarrow & (\mathbb{P}^n)^* \\ & \searrow & & \swarrow_{\pi} & \\ & & \mathrm{Spec}(\mathbb{C}) & & \end{array} \tag{3}$$

As closed immersions are proper and as being proper is stable under base change (see [Har77, chapter II, 4.8]), the projection p_1 is a proper and therefore closed morphism. But now $p_1 : W_0 \to X_{sm}$ satisfies all the assumptions from 2.1.9, so W_0 needs to be irreducible. We can use the map $p_2 : W_0 \to (\mathbb{P}^n)^*$, to see that $X^\vee$ is irreducible:
Once more assume $X^\vee = Z_1 \cap Z_2$ for two closed subsets $Z_1, Z_2 \subseteq X^\vee$. By construction, as a topological space, $X^\vee$ is exactly the closure of the image of the second projection $p_2 : W_0 \to (\mathbb{P}^n)^*$, which as a morphism of schemes must be continous on the level of topological spaces. In particular $p_2^{-1}(Z_1)$ and $p_2^{-1}(Z_2)$ must be closed subsets of W_0. But now $W_0 = p_2^{-1}(X^\vee) = p_2^{-1}(Z_1) \cup p_2^{-1}(Z_2)$, so by irreducibility of W_0, we can w.l.o.g assume $W_0 = p_2^{-1}(Z_1)$. Applying p_2 gives us $p_2(W_0) = p_2(p_2^{-1}(Z_1)) \subseteq Z_1$, so as Z_1 is closed in $X^\vee$ and hence also in $(\mathbb{P}^n)^*$, Z_1 must contain the closure of $p_2(W_0)$, which is exactly $X^\vee$.
Now as $X^\vee$ is irreducible, also the (dense) subset of closed points is irreducible, but this is exactly homeomorphic to the dual variety of V in the sense of 2.1.5, hence finishing the proof. □

In the diagramm (3), in general the morphism $X_{sm} \to \mathrm{Spec}(\mathbb{C})$ is not a projective, but only a quasi-projective morphism, since it is the composition of the open immersion $X_{sm} \to X$ and the projective morphism $X \to \mathbb{P}^n \to \mathrm{Spec}(\mathbb{C})$. However, when X is a smooth variety, the open immersion is actually identity, meaning $X_{sm} \to \mathrm{Spec}(\mathbb{C})$ is projective. This allows the following simplification in the definition of the dual variety.

Corollary 2.1.10. When X is a smooth variety, the set of all points in $(\mathbb{P}^n)^*$ that correspond to hyperplanes tangent to X at some point is closed. In particular we don't need to take the closure in the definition of $X^\vee$ in that case.

Proof. For smooth X, the second projection $p_2 : W_0 \to (\mathbb{P}^n)^*$ is the base change of the projective, hence proper morphism $X = X_{sm} \to \mathrm{Spec}(\mathbb{C})$ and therefore proper itself by [Har77, chapter II, 4.8 and 4.9]. In particular p_2 is a closed morphism, so the image $p_2(W_0)$ is closed in $(\mathbb{P}^n)^*$. □

2.2. Computing $X^\vee$

Just like in the proof of 2.1.7, we can compute the dual variety of a smooth irreducible $X \subseteq \mathbb{P}$ by constructing the set $W_0 \subseteq \mathbb{P} \times \mathbb{P}^*$ of pairs (x, H), where $x \in X$ and H is a hyperplane tangent to X at x, and then projecting it to the second component. We will now carry this out in the case of a smooth curve in $\mathbb{P}^2$.

Example 2.2.1. Working over $\mathbb{P}^2$, let $f(x, y, z) \in \mathbb{C}[x, y, z]$ be a smooth, irreducible, homogeneous polynomial and let $p_0 := (x_0 : y_0 : z_0) \in \mathcal{V}_\mathbb{P}(f)$ be a zero of f. The tangent space of $X = \mathcal{V}_\mathbb{P}(f)$ at the point p_0 is given by

$$T_{p_0}X = \{(a : b : c) \in \mathbb{P}^2 : af_x(p_0) + bf_y(p_0) + cf_z(p_0) = 0\}, \tag{4}$$

where f_x, f_y, f_z denote the three partial derivatives of f with respect to the given variable. $T_{p_0}X$ is a linear subspace of $\mathbb{P}^2$ of codimension one, hence $T_{p_0}X$ itself is the only hyperplane tangent to X at p_0. The set $W_0 \subseteq \mathbb{P}^2 \times (\mathbb{P}^2)^*$ therefore is just the set of all (p, T_pX), where $p \in X$.
Next we introduce homogeneous coordinates on $\mathbb{P}^2$ and $(\mathbb{P}^2)^*$ by fixing the standard base e_1, e_2, e_3 on $\mathbb{C}^3$ and its dual base e_1^*, e_2^*, e_3^* on $(\mathbb{C}^3)^*$. To express W_0 in terms of these homogeneous coordinates, fix again $p_0 \in \mathcal{V}_\mathbb{P}(f)$ like above. 2.1.4 exactly tells us how to view $T_{p_0}X$ as a point of $(\mathbb{P}^2)^*$, namely $(u : v : w) \in (\mathbb{P}^2)^*$ corresponds to $T_{p_0}X$, if and only if the kernel of the $\mathbb{C}$-linear map $a \cdot e_1^* + b \cdot e_2^* + c \cdot e_3^*$ is $T_{p_0}X$. However from (4) it is immediately clear that the point $(f_x(p_0) : f_y(p_0) : f_z(p_0))$ corresponds to $T_{p_0}X$, so any point $(u : v : w)$ has this property, if and only if it is a nonzero scalar multiple of $(f_x(p_0) : f_y(p_0) : f_z(p_0))$.
We conclude: A point $((x : y : z), (u : v : w)) \in \mathbb{P}^2 \times (\mathbb{P}^2)^*$ belongs to W_0 if and only if

- $f(x, y, z) = 0$, i.e. $(x : y : z) \in X$ and
- there exists some $\lambda \in \mathbb{C} \setminus \{0\}$, such that $u = \lambda f_x(x, y, z), v = \lambda f_y(x, y, z), w = \lambda f_z(x, y, z)$. Or equivalent: $uf_y(x, y, z) - vf_x(x, y, z) = uf_z(x, y, z) - wf_x(x, y, z) = vf_z(x, y, z) - wf_y(x, y, z) = 0$

Now we can compute the dual easily on any affine flag. For example fix $x = 1, u = 1$, then W_0 intersects this affine flag in the affine variety

$$\mathcal{V}_\mathbb{A}(f, f_y - vf_x, f_z - wf_x, vf_z - wf_y) \subseteq \mathbb{A}^2 \times \mathbb{A}^2 = \mathbb{A}^4. \tag{5}$$

We project this to the last two coordinates by eliminating the variables y, z from the defining ideal, to get the defining ideal of $X^\vee \cap \{u = 1\}$. Notice also,that the generator $vf_z - wf_y$ can be omitted because $vf_z - wf_y = v(f_z - wf_x) - w(f_y - vf_x)$

Example 2.2.2. Consider the smooth, cubic curve $f = x^3 - xy^2 - x^2z + z^3 \in \mathbb{C}[x, y, z]$. On the affine flag $\{x = 1\}$ we get the classical elliptic curve defined by $h = z^3 - y^2 - z + 1$ (see Figure 1). The variety in (5) is defined by the ideal

$$I = \langle z^3 - y^2 - z + 1, -2y - v(-y^2 - 2z + 3), 3z^2 - 1 - w(-y^2 - 2z + 3)\rangle.$$

Eliminating the variables y, z, we get a principal ideal generated by

$$\begin{aligned} h^\vee &:= 23v^6 - 18v^4w^2 - v^2w^4 - 4w^6 - 24v^4w - 36v^2w^3 - 4w^5 \\ &- 54v^4 - 30v^2w^2 + 4w^3 + 27v^2 \in \mathbb{C}[v, w]. \end{aligned}$$

We get the dual curve by homogenizing in the third variable u, so the dual variety of $\mathcal{V}_\mathbb{P}(f)$ is $\mathcal{V}_\mathbb{P}(f^\vee)$, where

$$\begin{aligned} f^\vee &:= 23v^6 - 18v^4w^2 - v^2w^4 - 4w^6 - 24v^4wu - 36v^2w^3u - 4w^5u \\ &- 54v^4u^2 - 30v^2w^2u^2 + 4w^3u^3 + 27v^2u^4 \in \mathbb{C}[u, v, w]. \end{aligned} \tag{6}$$

Notice that, in general, homogenizing and dehomogenizing a polynomial are not inverse operations: The polynomial $x^2 - xy$ is homogeneous of degree two, but when we substitute $x = 1$ we get $1 - y$, which by homogenizing with respect to x becomes the homogeneous degree one polynomial $x - y$. Still in our case we get the dual curve from homogenizing, because the dual curve must have degree 6 (see chapter 3). Clearly there is only one homogeneous degree 6 polynomial in u, v, w that restricts to $h^\vee$ on the affine flag $\{u = 1\}$, namely the homogenization.

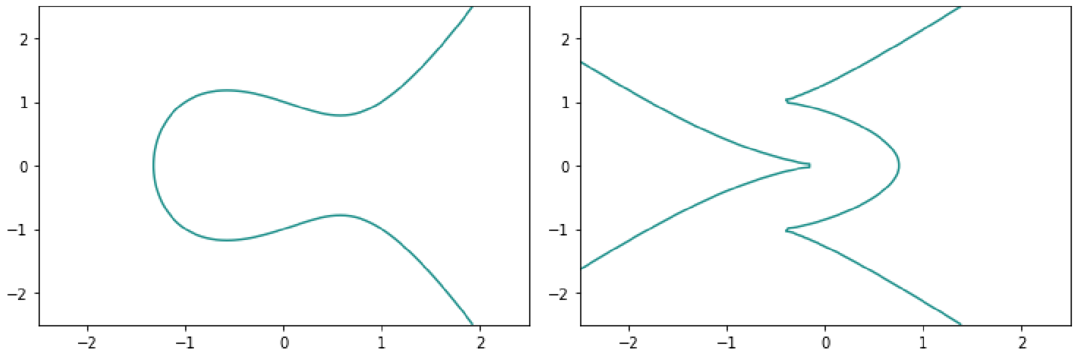

Figure 1: The curve $y^2 = z^3 - z + 1$ (left) and its dual $h^\vee$ (right)in $\mathbb{R}^2$.

3. Dual curves in $\mathbb{P}^2$

3.1. Dimension of the dual curve

In this section we will focus on 1-dimensional, irreducible varieties, i.e. curves in $\mathbb{P}^2$. First notice that in this case, the dual variety is a curve again:
We can obtain the dual variety like in 2.1.8, by considering the set $W_0 \subseteq X_{sm} \times (\mathbb{P}^2)^*$ of all pairs (x, H), where x is a smooth point of X and a hyperplane H that is tangent to X at the point x, then projecting it to $(\mathbb{P}^2)^*$. However, for every smooth point $x \in X$, the tangent space to X at x, by [Sha94, II.1.4, Theorem 3], is of the same dimension as X and therefore a hyperplane of $\mathbb{P}^2$, i.e. a point of $(\mathbb{P}^2)^*$ already. This means that all fibers of closed points of X_{sm} of the first projection $W_0 \to \mathbb{P}^2$ are of dimension 0. The following theorem helps us to relate the dimension of a fiber to dimension of domain and codomain of a dominant map.

Lemma 3.1.1. Let $\varphi : W \to V$ be a dominant morphism of irreducible (abstract) varieties. Then

(i) $\dim(W) \geq \dim(V)$;

(ii) For $P \in \varphi(W)$ we have

$$\dim(\varphi^{-1}(P)) \geq \dim(W) - \dim(V),$$

and equality holds on a nonempty open subset of V.

Proof. See [Mil17, Theorem 9.9]. □

Applying (ii) to the dominant map $\pi_1 : W_0 \to X_{sm} \subseteq X$ we conclude that W_0 is of dimension $\dim(X) + \dim(\pi_2^{-1}(P)) = 1 + 0 = 1$ for a closed point $P \in X_{sm}$ that also lays in the nonempty open subset V of X, where equality holds in (ii) (such a point exists, as closed points are dense by A.1.3). We also get that the image under the second projection is of dimension at most 1 by (i), so as by 2.1.7, $X^\vee$ is irreducible, it is either a curve or a point. The case when $X^\vee$ is just one point exactly means that the tangent space at every point of the curve is the same, which can only appear, when we consider degree 1 curves, i.e. lines in $\mathbb{P}^2$.

Remark 3.1.2. The fact, that for a smooth point p of a curve $C = \mathcal{V}_\mathbb{P}(f), f \in S^d\mathbb{C}^3$, we find exactly one hyperplane tangent to C at p also allows us to build a (regular) map from smooth points C_{sm} of C to $C^\vee$. This map exactly sends a point $p \in C$ to the point in $(\mathbb{P}^2)^*$ that corresponds to the tangent space of C at p:

$$C_{sm} \to C^\vee, p \mapsto \left(\left(\frac{\partial}{\partial x} f\right)(p) : \left(\frac{\partial}{\partial y} f\right)(p) : \left(\frac{\partial}{\partial z} f\right)(p)\right)$$

Definition 3.1.3. For an irreducible, homogeneous polynomial $f \in \mathbb{C}[x, y, z]$ of degree greater than 1, we denote by $f^\vee$ the polynomial, which defines the dual curve of $\mathcal{V}_\mathbb{P}(f)$.

Notice that $f^\vee$ is only defined up to multiplication with a nonzero scalar, as also $\lambda \cdot f^\vee, \lambda \in \mathbb{C} \setminus \{0\}$ defines the curve dual to $\mathcal{V}_\mathbb{P}(f)$. This is mostly not a problem, as we are more interested in properties of the dual curve, than in properties of the defining polynomial. Formally to make $f^\vee$ unique, one could choose a monomial order and require $f^\vee$ to be normalized in this order.
Also note that in the upper considerations, the reason for the 1-dimensionality of $X^\vee$ was that X was of codimension one, hence if we work over $\mathbb{P}^n$ where $n > 2$, the dual of a curve will in general not be a curve again.

Example 3.1.4. When we compute the dual curve using a computer, it will be much easier to work over the field $\mathbb{Q}$ instead of $\mathbb{C}$. In this case we can also see that the dual of a curve is one dimensional by using the resultant. In 2.2.1 we have already seen that the key to computing $\mathcal{V}_\mathbb{P}(f)^\vee$ is to eliminate two variables from the ideal

$$I = \langle f, f_y - vf_x, f_z - wf_x, vf_z - wf_y \rangle = \langle f, f_y - vf_x, f_z - wf_x \rangle.$$

Recall that in the mentioned example we already picked coordinates on x, y, z on $\mathbb{P}^2$ and u, v, w on $(\mathbb{P}^2)^*$ and chose affine flags $x = 1$ and $u = 1$, so this ideal is in $\mathbb{Q}[y, z, v, w]$ and we want to eliminate y, z. To eliminate two variables from an ideal with three generators, we can use the resultant: By [MS21, Theorem 4.11], the elimination ideal $I \cap \mathbb{Q}[v, w]$ is exactly the principal ideal generated by the resultant

$$\operatorname{Res}(f, f_y - vf_x, f_z - wf_x) \in \mathbb{Q}[v, w].$$

Hence over $\mathbb{Q}$, finding $f^\vee$ is the same as computing the resultant.

3.2. Properties of general elements

Next we want to relate the degree of a homogeneous polynomial $f \in \mathbb{C}[x, y, z]$ to the degree of its dual $f^\vee$. Here we will not do it for a completely arbitrary polynomial f, but only for most polynomials. To make this more precise we introduce general elements of a topological space.

Definition 3.2.1. Let X be a topological space. We say that a property $\mathcal{P}$ holds for a general element of X, if the set of points $x \in X$, for which $\mathcal{P}$ holds, contains a dense open subset of X.

As the next lemma shows, the notion of general elements works best for irreducible topological spaces like $\mathbb{A}^n$ and $\mathbb{P}^n, n \in \mathbb{N}$.

Lemma 3.2.2. Let X be an irreducible topological space, then:

(i) If the property $\mathcal{P}$ holds on a nonempty open subset of X, then $\mathcal{P}$ holds for a general element (this means we don't need to check for density of the open subset).

(ii) If the set of all elements of X that do not have a property $\mathcal{P}$ is not dense in X, then $\mathcal{P}$ holds for a general element of X.

(iii) If $\mathcal{P}_1, \ldots, \mathcal{P}_n$ are properties,that hold for a general element of X, then also $\mathcal{P}_1 \wedge \ldots \wedge \mathcal{P}_1$ holds for a general element of X.

Proof. (i) In irreducible topological spaces, nonempty open subsets are dense. Indeed, if $U \subseteq X$ is open and nonempty, then U^c is a proper closed subset of X and clearly $\overline{U} \cup U^c = X$, so by irreducibility U is dense.

(ii) Let $C \subsetneq X$ be the closure of the set of all elements that don't have $\mathcal{P}$. By assumption, this is a closed subset that contains every point which does not have the property $\mathcal{P}$ Then C^c is a nonempty open subset of X and all elements of C^c have the property $\mathcal{P}$, hence we conclude by (i).

(iii) For $i = 1, \ldots, n$, let U_i be a dense open subset of X, where $\mathcal{P}_i$ holds. Then especially $U_i \neq \emptyset$ and hence U_i^c are proper closed subsets of X. Clearly $\bigcap_{i=1}^n U_i$ is an open subset of X, where all of the properties $\mathcal{P}_1, \ldots, \mathcal{P}_n$ hold simultaneously, so by (i) it suffices to show that this intersection is nonempty. For contradiction assume it was, then

$$\bigcup_{i=1}^n U_i^c = \left(\bigcap_{i=1}^n U_i \right)^c = \emptyset^c = X.$$

However X is irreducible and U_i^c are all closed, proper subsets of X, so this is a contradiction.

□

One straight forward example of a property that holds for a general element $(a_1, \ldots, a_n) \in \mathbb{A}^n$ (over $\mathbb{C}$) is having a nonzero first entry a_1: This is clearly an open condition, so we conclude by point (ii) of 3.2.2. Indeed the same argument holds for any entry of $(a_1, \ldots, a_n)$, so by 3.2.2, part (iii), we can even say that a general element of A^n will have no zero entries at all.

To give some more advanced examples, we will consider the projectivization of the d-th symmetric power of $\mathbb{C}^3$, denoted $\mathbb{P}(S^d\mathbb{C}^3)$, with Zariski-toplogy. This is just $\mathbb{P}_{\mathbb{C}}^{n-1}$ for $n = \binom{2+d}{d}$, the number of degree d monomials in 3 variables, so we can think of the elements of this space as of coefficients of a homogeneous degree d polynomial over $\mathbb{C}$ up to scaling. Geometrically, the elements of $S^d\mathbb{C}^3$ give curves in $\mathbb{P}^2$ by taking their varieties. Since scaling a polynomial results in the same variety, it makes sense to consider the projectivization. A general element here has the property of being a complete, smooth and irreducible polynomial (complete meaning that every single degree d monomial in three variables appears with a nonzero coefficient in f). Notice that all these properties do not change, if we multyply a homogeneous polynomial with a nonzero constant, hence it makes sense to assign these properties to an element of $\mathbb{P}(S^d\mathbb{C}^3)$. Since we will use this fact later on, we shall provide a formal proof now.

Lemma 3.2.3. Consider $\mathbb{P}(S^d\mathbb{C}^3)$ with Zariski-topology, then a general element is a complete, smooth and irreducible polynomial, where any first partial derivative is smooth again.

Proof. First notice that in the homogeneous case in just three variables, smooth polynomials are irreducible:
Take $f \in \mathbb{P}(S^d\mathbb{C}^3)$ that is reducible. Then we find $f = f_1 \cdot f_2$ for two non-constant polynomials f_1, f_2. First we show that f_1, f_2 are homogeneous again. Indeed assume they were not, pick any degree-compatible monomial order. Let m_i be the smallest monomial in f_i and M_i the largest monomial in f_i, $(i = 1, 2)$. By assumption $\deg(M_1M_2) > \deg(m_1m_2)$, however both of these monomials need to appear in $f = f_1f_2$ with nonzero coefficient, contradicting the fact that f is homogeneous. This means that f_1, f_2 give us two one-dimensional (possibly reducible) varieties in $\mathbb{P}^2$ and their union is exactly f. Since in $\mathbb{P}^2$ any two curves intersect (see for instance [Sha94, chapter I, section 6.2 corollary 1 of Proposition 1]), especially any two one-dimensional varieties intersect (these are just finite unions of curves), so we find a point $(x_0 : y_0 : z_0) \in \mathbb{P}^2$ where f_1 and f_2 vanish simultaneously. At this point the partial derivative of f with respect to the variable x is exactly

$$\left(\frac{\partial}{\partial x} f_1\right)(x_0, y_0, z_0) \cdot \underbrace{f_2(x_0, y_0, z_0)}_{=0} + \left(\frac{\partial}{\partial x} f_2\right)(x_0, y_0, z_0) \cdot \underbrace{f_1(x_0, y_0, z_0)}_{=0} = 0.$$

With the same argument the other two partial derivatives vanish, meaning (x_0, y_0, z_0) is a singular point of f.
Hence it suffices to show that a general polynomial is smooth. To see this, we consider the set H of all pairs $(p, f) \in \mathbb{P}^2 \times \mathbb{P}(S^d\mathbb{C}^3)$, where p is a singular point of f. Notice that a point (p, f) is in H if and only if

$$\left(\frac{\partial}{\partial x} f\right)(p) = \left(\frac{\partial}{\partial y} f\right)(p) = \left(\frac{\partial}{\partial z} f\right)(p) = 0.$$

These three conditions are polynomials F_1, F_2, F_3, homogeneous both in the entries of p and in the coefficients of f, so they define a closed subscheme of the fibered product $\mathbb{P}^2 \times_{\mathbb{C}} \mathbb{P}(S^d\mathbb{C}^3)$, which on an affine flag is given by the polynomials obtained from dehomogenizing F_1, F_2, F_3 in the entries of p and in the coefficients of f. As a closed subscheme of the abstract variety $\mathbb{P}^2 \times_{\mathbb{C}} \mathbb{P}(S^d\mathbb{C}^3)$, we can view H as an abstract variety itself (see A.1.4 and A.1.5)and the closed points of the scheme H will exactly correspond to pairs (p, f) where p is a singular point of f.
The second projection of H to $\mathbb{P}(S^d\mathbb{C}^3)$, on the level of closed points, maps exactly onto the set of non-smooth polynomials, so we want this to be not dense, then we can use part (ii) of 3.2.2. Clearly the first projection to $\mathbb{P}^2$ is surjective onto closed points, especially dominant, so we can use 3.1.1 on the map $\pi_1 : H \to \mathbb{P}^2$. To get the dimension of a general fiber, fix $p = (p_0 : p_1 : p_2) \in \mathbb{P}^2$. At least one of the entries p_0, p_1, p_2 is nonzero, without loss of generality we assume that this is p_0, then by scaling with the nonzero number p_0^{-1}, we can achieve $p = (1, p_1', p_2')$. This describes the point p in coordinates $(x : y : z)$, however we can change the coordinates to new ones, which we call $(x' : y' : z')$, by setting

$$x' = x, \qquad y' = y - p_1x, \qquad z' = z - p_2x. \tag{7}$$

In these coordinates, the point p is represented by $(1 : p_1 - p_1 \cdot 1 : p_2 - p_2 \cdot 1) = (1 : 0 : 0)$. Also if $f \in \mathbb{P}(S^d\mathbb{C}^3)$ was a homogeneous polynomial in the old coordinates $(x : y : z)$, then we get a homogeneous degree d polynomial $f'(x', y', z') := f(x', y' + p_1 x', z' + p_2 x')$ in the new coordinates. This also behaves well with smooth points: A point given in the old coordinates $(x : y : z)$ is a smooth point of f if and only if the corresponding point in the new coordinates (x', y', z') is a smooth point of f'. This is due to the fact that we substitute linearly, so we have

$$\begin{aligned}
\frac{\partial}{\partial x'} f'(x', y', z') &= \left(\frac{\partial}{\partial x} f\right)(x', y' + p_1 x', z' + p_2 x') + \left(\frac{\partial}{\partial y} f\right)(x', y' + p_1 x', z' + p_2 x') \cdot p_1 \\
&\quad + \left(\frac{\partial}{\partial z} f\right)(x', y' + p_1 x', z' + p_2 x') \cdot p_2 \\
\frac{\partial}{\partial y'} f'(x', y', z') &= \left(\frac{\partial}{\partial y} f\right)(x', y' + p_1 x', z' + p_2 x') \\
\frac{\partial}{\partial z'} f'(x', y', z') &= \left(\frac{\partial}{\partial z} f\right)(x', y' + p_1 x', z' + p_2 x').
\end{aligned}$$

We can immediately see that these three partial derivatives vanish at a point $(x' : y' : z')$ in new coordinates, if and only if the three partial derivatives of f vanish at the corresponding point $(x : y : z) = (x' : y' + p_1 x' : z' + p_2 x')$.
$f \in \mathbb{P}(S^d\mathbb{C}^3)$ is singular in p if and only if

(i) $f(p) = 0$, by the choice of p this exactly means that the coefficient in front of the monomial x^d is zero

(ii) $\left(\frac{\partial}{\partial y} f\right)(p) = \left(\frac{\partial}{\partial z} f\right)(p) = 0$, again by choice of p this means that the coefficients in front of the monomials yx^{d-1} and zx^{d-1} are zero.

(iii) $\left(\frac{\partial}{\partial x} f\right)(p) = 0$, but this condidtion also means that the coefficient of x^d is zero, so it is the same condition as (i).

Hence the fiber $\pi_1^{-1}(\{p\})$ is the set of all polynomials in $S^d\mathbb{C}^3$, where 3 fixed coefficients vanish, this is of dimension $\dim(\mathbb{P}(S^d\mathbb{C}^3)) - 3 = \dim(\mathbb{P}^{\binom{d+2}{d}-1}) - 3 = \binom{2+d}{d} - 4$. $\mathbb{P}^2$ is a variety of dimension two (any affine flag is a dense open subscheme isomorphic to $\mathbb{A}^2$, which has dimension 2 by [Har77, chapter I, 1.9]), so by 3.1.1 (ii), the dimension of H is exactly $\dim(\pi_1^{-1}(\{p\})) + \dim(\mathbb{P}^2) = \binom{d+2}{d} - 4 + 2 = \binom{d+2}{d} - 2$. This by point (i) of 3.1.1 implies that $\dim(\overline{\pi_2(H)}) \leq \dim(H) = \binom{d+2}{d} - 2 < \binom{d+2}{d} - 1 = \dim(\mathbb{P}(S^d\mathbb{C}^3))$. Especially the image of the second projection of H is not dense in $\mathbb{P}(S^d\mathbb{C}^3)$.
Clearly the set of elements of $\mathbb{P}(S^d\mathbb{C}^3)$ that are complete is open and nonempty, as the condition of any fixed coefficient being zero is a closed condition.
For the property about the first partial derivative, notice that by the upper part we find a nonempty open subset U of $\mathbb{P}(S^{d-1}\mathbb{C}^3)$, where all polynomials in U are smooth. Now

take any complete $f \in \mathbb{P}(S^d\mathbb{C}^3)$, then we can write

$$f = \sum_{i+j+k=d} a_{i,j,k} x^i y^j z^k,$$

for some $a_{i,j,k} \in \mathbb{C} \setminus \{0\}$ (which are only fixed upto a nonzero constant). Consider the first partial derivative with respect to z (of course the others work completely analog). This gives the polynomial

$$\frac{\partial}{\partial z} f = \sum_{i+j+k=d-1} (k+1) a_{i,j,k+1} x^i y^j z^k \in \mathbb{P}(S^{d-1}\mathbb{C}^3).$$

We claim that

$$U' := \{(a_{i,j,k})_{i+j+k=d} : ((k+1) \cdot a_{i,j,k+1})_{i+j+k=d-1} \in U \subseteq \mathbb{P}(S^{d-1}\mathbb{C}^3)\} \subseteq \mathbb{P}(S^d\mathbb{C}^3),$$

is open and nonempty (here we identify elements of $\mathbb{P}(S^d\mathbb{C}^3)$ with the tuple of coefficients of the polynomial). It is clear that this is nonempty, as U is nonempty: If $(b_{i,j,k})_{i+j+k=d-1} \in U$, just define

$$a_{i,j,k} := \begin{cases} \frac{b_{i,j,k-1}}{k}, & \text{if } k \neq 0 \\ 0, & \text{else} \end{cases},$$

for all i, j, k with $i + j + k = d$, then clearly $(a_{i,j,k})_{i+j+k=d} \in U'$. As U was open in $\mathbb{P}(S^{d-1}\mathbb{C}^3)$, its complement is Zariski-closed, which means we find homogeneous polynomials $g_1, \ldots, g_n$ in the coefficients $(a_{i,j,k})_{i+j+k=d-1}$ of a homgeneous degree $d-1$ polynomial, that vanish on the coefficients of any polynomial in $\mathbb{P}(S^{d-1}\mathbb{C}^3) \setminus U$. The $g_1, \ldots, g_n$ give rise to polynomials $g'_1, \ldots, g'_n$ in the coefficients of a homogeneous degree d polynomial like this:

$$\begin{aligned} g'_1((a_{i,j,k})_{i+j+k=d}) &:= g_1((k \cdot a_{i,j,k})_{\substack{i+j+k=d \\ k\neq 0}}) \\ &\vdots \\ g'_n((a_{i,j,k})_{i+j+k=d}) &:= g_n((k \cdot a_{i,j,k})_{\substack{i+j+k=d \\ k\neq 0}}). \end{aligned}$$

(Notice that in the new polynomials $g'_1, \ldots, g'_n$, the only variables that appear are the ones that represent a coefficient of a homogeneous degree d polynomial in front of a monomial, which contains the variable z. These are only $\binom{1+d}{d-1}$ many of the total $\binom{2+d}{d}$ variables.) Now clearly

$$\begin{aligned} (a_{i,j,k})_{i+j+k=d} \in \mathcal{V}_{\mathbb{A}}(g'_1, \ldots, g'_n) &\Leftrightarrow g'_1((a_{i,j,k})_{i+j+k=d}) = \ldots = g'_n((a_{i,j,k})_{i+j+k=d}) = 0 \\ &\Leftrightarrow g_1((k \cdot a_{i,j,k})_{\substack{i+j+k=d \\ k\neq 0}}) = \ldots = g_n((k \cdot a_{i,j,k})_{\substack{i+j+k=d \\ k\neq 0}}) = 0 \\ &\Leftrightarrow (k \cdot a_{i,j,k})_{\substack{i+j+k=d \\ k\neq 0}} \notin U \\ &\Leftrightarrow (a_{i,j,k})_{i+j+k=d} \notin U'. \end{aligned}$$

Hence $U' \subseteq \mathbb{P}(S^d\mathbb{C}^3)$ is a nonempty open subset. also by definition of U', we have that if $(a_{i,j,k})_{i+j+k=d} \in U'$, then the first partial derivative with respect to z of the polynomial $f \in \mathbb{P}(S^d\mathbb{C}^3)$, with coefficients $a_{i,j,k}$ in front of the monomial $x^iy^jz^k$ is in $U \subseteq S^{d-1}\mathbb{C}^3$ and hence smooth. This shows that the property "having a smooth first partial derivative with respect to z" is true for a general element of $\mathbb{P}(S^d\mathbb{C}^3)$.
Now we have proven all of the properties

- being smooth and irreducible
- being complete
- having a smooth first partial derivative with respect to any (fixed) variable

are true for a general element individually, we can conclude by 3.2.2, part (iii) that a general element of $\mathbb{P}(S^d\mathbb{C}^3)$ has all three properties at once. □

In the previous proof, we were relying on the fact that we worked with homogeneous polynomials, which represent projective curves in $\mathbb{P}^2$. For instance, in the first part we showed that smooth curves in $\mathbb{P}^2$ are irreducible, which is not true for curves in $\mathbb{A}^2$. The key difference here is that in $\mathbb{P}^2$ any two curves intersect, while in $\mathbb{A}^2$ one can think of two parallel lines, to see that the same result fails here. However, if we have a homogeneous polynomial $f \in S^d\mathbb{C}^3$ (i.e. a curve in $\mathbb{P}^2$) that has all the nice properties from 3.2.3, then we can cut it with an affine piece $\mathbb{A}^2 \subseteq \mathbb{P}^2$, to get an affine curve, while maintaining all of the properties.

Lemma 3.2.4. Let $f \in S^d\mathbb{C}^3$ be a complete, smooth and irreducible polynomial, such that the first partial derivative with respect to any variable is smooth again. Write f in the variables x, y, z, then the polynomial $f(1, y, z) \in \mathbb{C}[y, z]_{\leq d}$ is also complete (meaning that every monomial of degree $\leq d$ in y, z appears with nonzero coefficient in f), smooth and irreducible and the two first partial derivatives $\frac{\partial}{\partial y} f(1, y, z)$, $\frac{\partial}{\partial z} f(1, y, z)$ are smooth again.

Proof.

- Completeness: Let $y^jz^k, j + k =: d' \leq d$ be any monomial of degree at most $d' \leq d$, then the coefficient in $f(1, y, z)$ in front of this monomial is exactly the coefficient in front of $x^{d-d'}y^jz^k$, which is nonzero as f is complete.
- Irreducibility: Assume we find $g_1, g_2 \in \mathbb{C}[y, z]\backslash\mathbb{C}$, such that $g_1g_2 = f(1, y, z)$. Notice that by completeness of f, we can recover f from $f(1, y, z)$ by homogenizing it with respect to z. However the operation of homogenizing a (non-constant) polynomial is multiplicative, hence we can homogenize g_1, g_2 to get two non-constant factors of f. This contradicts the irreducibility of f.
- Smoothness: Assume $f(1, y, z)$ has a singular point $p \in \mathbb{A}^2$, then by a change of coordinates like in (7) in the last proof, we may assume $p = (0, 0)$. The fact that p is a zero of $f(1, y, z)$ means $f(1, 0, 0) = 0$ and hence the coefficient $a_{d,0,0}$ in front of the monomial x^d in f is zero (notice that we might lose completeness when we change coordinates, so this is not a contradiction yet). As p is a singular point, both $\left(\frac{\partial}{\partial y} f\right)(1, 0, 0) = 0$ and $\left(\frac{\partial}{\partial z} f\right)(1, 0, 0) = 0$. However also the partial derivative of f

with respect to x needs to vanish at $(1:0:0)$, as $\left(\frac{\partial}{\partial x} f\right)(1,0,0) = d \cdot a_{d,0,0} = 0$. This contradicts the smoothness of f (which is maintained under a change of coordinates as proven above).

- Smoothness of first derivatives: Just apply the last point to $\frac{\partial}{\partial y} f \in S^{d-1}\mathbb{C}^3$ and notice that $\left(\frac{\partial}{\partial y} f\right)(1,y,z) = \left(\frac{\partial}{\partial y} f(1,y,z)\right)$. The same of course holds for $\frac{\partial}{\partial z} f$.

□

3.3. Degree of $f^\vee$

Working over $\mathbb{P}^2$, we will now describe the degree of the dual of a general curve. The main result will be the following Theorem.

Theorem 3.3.1. Let $d \geq 2$. For a general homogeneous polynomial $f \in \mathbb{P}(S^d\mathbb{C}^3)$, the dual curve $f^\vee$ has degree $d(d-1)$.

Remark 3.3.2. We will prove this theorem as stated above for general polynomials, but there is also a version of this that works for any polynomial. This result is known as Plücker-relations and also factors in different sorts of singularities of a curve. Especially in the form as we stated it above, one could replace the "general homogeneous polynomial" with any smooth homogeneous polynomial. The interested reader can find the extended version in [GH78, chapter 2, section 4].
In section 2.1, we stated the Biduality Theorem, which predicates that taking the dual twice results in the original curve. At first glance this seems to contradict our theorem, as for $d > 2$ we claim that the degree of the dual curve is larger than the degree of the original curve. This contradiction is resolved by the fact that the dual of a smooth curve does not need to be smooth again. In fact we have seen such an example already in2.2.2, where we computed the dual of the smooth curve $f = x^3 - xy^2 - x^2z + z^3$. The result was the degree 6 polynomial $f^\vee$ in (6), which is not smooth. To see this, consider the ideal

$$I = \langle f^\vee, \frac{\partial}{\partial u} f^\vee, \frac{\partial}{\partial v} f^\vee, \frac{\partial}{\partial w} f^\vee \rangle \subseteq \mathbb{C}[u,v,w].$$

If $f^\vee$ was smooth, the projective variety defined by the generators of I would need to be empty, but then the affine variety defined by I would only contain one point, namely $(0,0,0)$. Using Hilbert's Nullstellensatz, the radical of I would need to be equal to the maximal ideal $\langle u,v,w \rangle$ that defines the variety $\{(0,0,0)\}$. Using a software like Macaulay2, one can easily check that this is not the case, hence $f^\vee$ needs to have singular points apart from $(0,0,0)$ in $\mathbb{A}^3$, but such points give singular points in $\mathbb{P}^2$.

The following example shows two sources of singularities for plane curves.

Example 3.3.3. (i) If the original curve has a tangent of higher intersection multiplicity than two, the dual curve will have a cusp. The easiest example is the curve $h := y - z^3$ which intersects its tangent line at $(0,0)$ with multiplicity 3. Homogenizing this in y we get $f := x^2y - z^3 \in \mathbb{C}[x,y,z]$, which is not a smooth curve. However, the only singularity of f in $\mathbb{P}^2$ is the point $(0:1:0)$, which does not belong to the affine flag $\{x = 1\}$. Therefore we can still compute the dual on this affine flag as we described in 2.2.1, while this procedure would of course fail if we chose $\{y = 1\}$. So choosing $\{x = 1\}$ and $\{v = 1\}$ for the coordinates on $\mathbb{P}^2$ and $(\mathbb{P}^2)^*$ we get the dual $h^\vee := 4w^3 + 27u^2$ on $\{v = 1\}$. The point $(u, w) = (0, 0)$ is the singularity of $h^\vee$ that corresponds to the tangent at $(0, 0)$ on h: Indeed the singularity $(0, 0)$ on the affine flag $\{v = 1\}$ gives the point $(0 : 1 : 0) \in (\mathbb{P}^2)^*$. This point represents the hyperplane $\mathbb{P}(\ker(s))$, where $s : \mathbb{C}^3 \to \mathbb{C}, (a, b, c) \mapsto 0 \cdot a + 1 \cdot b + 0 \cdot c$, i.e. the hyperplane $\mathbb{P}^1 \subseteq \mathbb{P}^2$ given by all points with second coordinate $y = 0$. This intersects the chart $\{x = 1\}$ in the line $\{(y, z) \in \mathbb{A}^2 : y = 0\}$, which is precisely the tangent to h at $(0, 0)$.

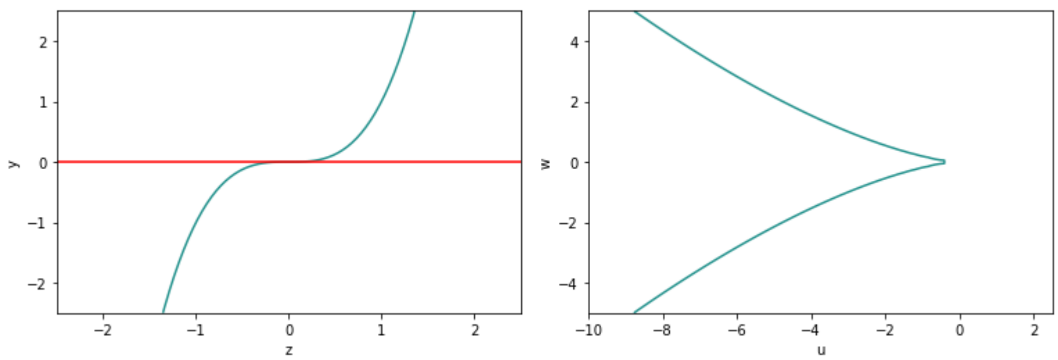

Figure 2: The curve $y = z^3$ with tangent line (red) at $(0, 0)$ (left) and its dual $4w^3 + 27u^2$ with a cusp at $(0, 0)$ (right) in $\mathbb{R}^2$.

(ii) The original curve might have two distinct points with the same tangent. This results in a self-intersection of the dual curve. Let us consider the curve $h = y - z^4 + 2z^2 \in \mathbb{C}[y, z]$, where the points $(-1, -1)$ and $(-1, 1)$ have the same tangent space. We homogenize in x to get $f = x^3y - z^4 + 2x^2z^2$. Again, the only singular point of f in $\mathbb{P}^2$ is $(0 : 1 : 0)$, so we choose $\{x = 1\}$ and $\{u = 1\}$ to compute the dual and get

$$h^\vee := 32v^2w^2 - 27w^4 + 256v^3 - 288vw^2 - 512v^2 + 256v \in \mathbb{C}[v, w].$$

We can see in Figure 3 that the point $(v, w) = (1, 0)$ is a double point of $h^\vee$. Again this point corresponds to $(1 : 0 : 1) \in (\mathbb{P}^2)^*$, which describes the hyperplane of $\mathbb{P}^2$ that consists of all points $(a : b : c)$, where $a + c = 0$. This hyperplane intersects $\{x = 1\}$ in the set $\{(y, z) \in \mathbb{A}^2 : z = -1\}$, which is exactly the tangent line to the points $(-1, 1)$ and $(-1, -1)$ of h. So the double point of $h^\vee$ indeed comes from the double tangent of h.

Before starting with the proof of 3.3.1, we will need a few helpful results. The first Lemma is just to exclude the special case of a line, which is tangent to the curve at two distinct points (see point (ii) of the last example).

Lemma 3.3.4. Let $f \in S^d\mathbb{C}^3$ be an irreducible polynomial, then for a general point $p \in \mathbb{P}^2$ there is no hyperplane $H \subseteq \mathbb{P}^2$ that contains p and is tangent to the curve defined by f in $\mathbb{P}^2$ at two different smooth points.

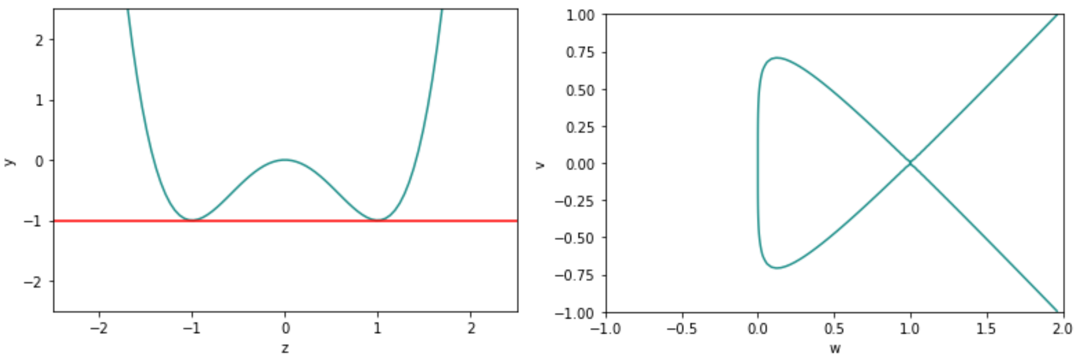

Figure 3: The curve $y = z^4 + 2z^2$ with the same tangent line (red) at $(-1,-1)$ and $(1,-1)$ (left) and its dual $h^\vee$ with self-intersection point at $(1,0)$ (right) in $\mathbb{R}^2$.

Proof. Let $C = \mathcal{V}_\mathbb{P}(f) \subseteq \mathbb{P}^2$ and consider two different smooth points $p_1, p_2 \in X$, such that there exists a hyperplane $H \subseteq \mathbb{P}^2$ that is tangent to X at p_1 and at p_2. As described in 3.1, this already means that $T_{p_1}C = H = T_{p_2}C$, as all three spaces are irreducible of codimension one and H contains the two tangent spaces. H gives a point $s \in C^\vee \subseteq (\mathbb{P}^2)^*$, we claim that s is not a smooth point of the curve $C^\vee$. Assume for contradiction $C^\vee$ was smooth in s, then the tangent space $T_sC^\vee$ to $C^\vee$ at s would be a hyperplane in $(\mathbb{P}^2)^*$ and therefore a point in $\mathbb{P}^2$. By [oT20, Lemma 1.7.15, point (iii)], this point needs to equal to p_1 and equal to p_2, however we assumed $p_1 \neq p_2$, so we have a contradiction. Therefore s is a singular point of $C^\vee$.
Now $C^\vee$ as a curve in $(\mathbb{P}^2)^*$ is of dimension one and irreducible by 2.1.7. From [CLO18, chapter 9, §6, Theorem 8, (iii)], we know that the singular points of $C^\vee$ form a strict subvariety, called the singular locus of $C^\vee$. However a strict closed subset of an irreducible topological space of dimension one has to be zero dimensional. A zero dimensional variety is the union of all its irreducible components, which need to be points, so by noetherianity we conclude that $C^\vee$ can only have finitely many singular points.
Going back to the original problem, we now know that there can only be finitely many hyperplanes in $\mathbb{P}^2$ which are tangent to C at two different points. But a collection S of finitely many lines in $\mathbb{P}^2$ is clearly a variety: A single line is the variety defined by a homogeneous degree one polynomial, so S is the variety of a product of finitely many homogeneous degree one polynomials. In particular S is a strict closed subset of $\mathbb{P}^2$, so S^c is a nonempty open subset that contains every point, which is not contained in any double-tangent of C. By 3.2.2, this means that a general point $p \in \mathbb{P}^2$ is not contained in a hyperplane tangent to XC at two different points. □

The following notion of transversal intersection will be helpful for the proof of 3.3.1. This definition formalizes the idea of two curves, which intersect with multiplicity one only, meaning they intersect, while not having the same tangent space at the intersection point.

Definition 3.3.5. Let $V_1, V_2 \subseteq \mathbb{A}^n_k$ be two smooth varieties. A point $x \in V_1 \cap V_2$ is a *point of transversal intersection* of V_1 and V_2, if the tangent spaces T_xV_1 and T_xV_2 span the ambient space $\mathbb{A}^n$ (as a $\mathbb{C}$-vector space). We say V_1 and V_2 *intersect transversally*, if every intersection point is a point of transversal intersection.

Example 3.3.6. In $\mathbb{A}^2$, first consider the two curves given by$f := y - x^2$ and $g := y + x^2$ (see Figure 4, left). Clearly they only intersect in $(x, y) = (0, 0)$, but we claim that this is not a point of transversal intersection. Indeed the tangent space to $\mathcal{V}_{\mathbb{A}}(f)$ at $(0, 0)$ is

$$\begin{aligned}\{(a,b) : \left(\frac{\partial}{\partial x} f\right)(0,0) \cdot a + \left(\frac{\partial}{\partial y} f\right)(0,0) \cdot b = 0\} &= \{(a,b) : 0 \cdot a + 1 \cdot b = 0\} \\ &= \{(a,b) : b = 0\},\end{aligned}$$

but the same computation shows that this is also the line tangent to $\mathcal{V}_{\mathbb{A}}(g)$ at $(0, 0)$. Now the two tangent spaces are the same one-dimensional subspace of $\mathbb{A}^2$, hence they can not span the two-dimensional $\mathbb{C}$ vector space $\mathbb{A}^2$.
Now consider a third curve given by $h = y-(x-4)^2$ (see Figure 4, right) . $\mathcal{V}_{\mathbb{A}}(h)$ intersects $\mathcal{V}_{\mathbb{A}}(f)$ in the point $(x, y) = (2, 4)$, but this time the intersection is transversal. To see this we compute the two tangent spaces at $(2, 4)$. For f we get

$$\begin{aligned}\{(a,b) : \left(\frac{\partial}{\partial x} f\right)(2,4) \cdot a + \left(\frac{\partial}{\partial y} f\right)(2,4) \cdot b = 0\} &= \{(a,b) : -4 \cdot a + 1 \cdot b = 0\} \\ &= \{(a,b) : b = 4 \cdot a\} \\ &= \operatorname{span}_{\mathbb{C}}((1,4)),\end{aligned}$$

while for h we have

$$\begin{aligned}\{(a,b) : \left(\frac{\partial}{\partial x} h\right)(2,4) \cdot a + \left(\frac{\partial}{\partial y} h\right)(2,4) \cdot b = 0\} &= \{(a,b) : 4 \cdot a + 1 \cdot b = 0\} \\ &= \{(a,b) : b = -4 \cdot a\} \\ &= \operatorname{span}_{\mathbb{C}}((1,-4)).\end{aligned}$$

Clearly the vectors $(1, 4)$ and $(1, -4)$ span $\mathbb{A}^2$ as a $\mathbb{C}$-vector space, so the intersection is transversal.

The next lemma connects the geometric notion of transversal intersection of two smooth curves, to an algebraic property of the ideal that defines their intersection. For this we recall the concept of a minimal primary decomposition of an ideal I of a noetherian ring R. A primary decomposition of any ideal I in any ring R is a representation

$$I = \bigcap_{i=1}^{k} q_k,$$

where $k \in \mathbb{N}$ and q_i for $i = 1, \ldots, k$ are primary ideals, i.e. ideals that have the property, that if $ab \in q_i$ for $a, b \in R$ and $a \notin I$, then $b^n \in q_i$ for some large enough $n \in \mathbb{N}$. We say that a primary representation is minimal if and only if

(i) for $i \neq j$ we have $\operatorname{rad}(q_i) \neq \operatorname{rad}(q_j)$,

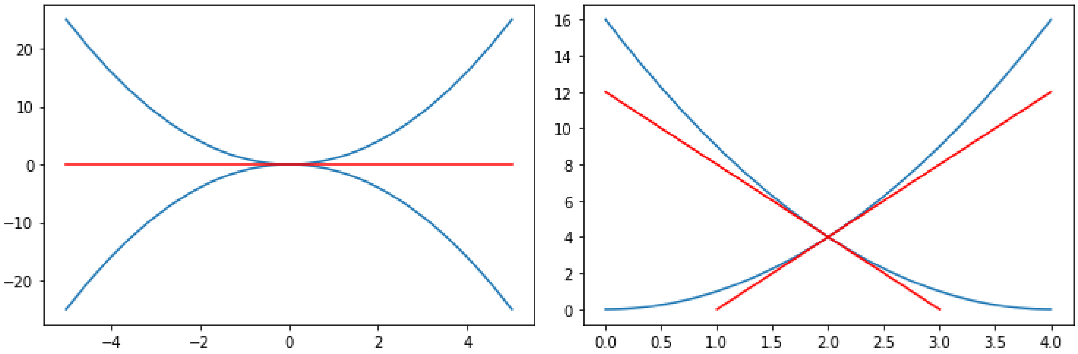

Figure 4: Left: The curves f, g that intersect non-transversally in $(0,0)$ and their tangent spaces (red). Right: The curves f, h that intersect transversally in $(2,4)$ and their respective tangent lines at that point (red). Notice that the red lines are of course not the tangent spaces we computed above, as they are not even vector spaces

(ii) for $i = 1, \ldots, k$ we have $\bigcup_{j \neq i} q_j \nsubseteq q_i$, meaning none of the q_i is redundant.

Working over a noetherian ring, every ideal has a (not necessarily unique) minimal primary decomposition, a proof can be found in [MS21, chapter 3.2.]. Geometrically, if the ideal $I = \langle f_1, \ldots, f_m$ describes a variety $V := \mathcal{V}_{\mathbb{A}}(I) = \mathcal{V}_{\mathbb{A}}(f_1, \ldots, f_m)$, the prime ideals $\mathrm{rad}(q_i), i = 1, \ldots, k$ describe the irreducible components of V.
Apart from primary decomposition, we will use the notion of dimension and degree of an ideal. Formal definitions of these invariants, as well as the results that we use from this topic can be checked in A.2.

Lemma 3.3.7. Let $f, g \in \mathbb{C}[y, z] \setminus \mathbb{C}$ be smooth polynomials, such that $\mathcal{V}_{\mathbb{A}}(f)$ and $\mathcal{V}_{\mathbb{A}}(g)$ intersect transversally. Furthermore let $\bigcap_{i=1}^n q_i = \langle f, g\rangle$ be a minimal primary decomposition of the ideal generated by f and g, then we have $\deg(q_i) = 1$ for every $i = 1, \ldots, n$.

Proof. Fix some index i, then the primary ideal q_i is contained in a maximal ideal $m \subseteq \mathbb{C}[y, z]$. Such a maximal ideal is of the form $m = \langle y - a, z - b\rangle$ for some $a, b \in \mathbb{C}$, but by a change of coordinates we can assume $a = b = 0$ (geometrically fixing m corresponds to looking at one specific intersection point and changing the coordinates means shifting both curves, such that this point is $(0,0)$). We claim $m = \mathrm{rad}(q_i)$. Indeed, as $q_i \subseteq m$ and m is maximal, hence also radical, we need to have $\mathrm{rad}(q_i) \subseteq m$. However $\mathrm{rad}(q_i)$ is the radical of a primary ideal, hence prime and $f \in \langle f, g\rangle \subseteq q_i$, so we find an irreducible factor f_i of f, which belongs to q_i. As $\mathbb{C}[y, z]$ is factorial, $\langle f_i\rangle$ is a prime ideal and we therefore have the following chain of prime ideals:

$$\langle 0\rangle \subsetneq \langle f_i\rangle \subseteq \mathrm{rad}(q_i) \subseteq m. \tag{8}$$

The Krull-dimension of $\mathbb{C}[y, z]$ is two (see A.2.6) , so at least one of these inclusions needs to be equality. For contradiction, assume the first inclusion is equality, then $\langle f, g\rangle \subseteq q_i \subseteq \mathrm{rad}(q_i) = \langle f_i\rangle$ means that f and g share a common factor (namely f_i). Lets say $f = \tilde{f} f_i$

and $g = \tilde{g} f_i$, then if p is a zero of f_i (notice that $f_i \notin \mathbb{C}$ as it is contained in a prime ideal), the tangent space to the curve of f at p is

$$\begin{aligned} T_p \mathcal{V}_{\mathbb{A}}(f) &= \left\{ (y,z) \in \mathbb{A}^2 : \left(\frac{\partial}{\partial y} f \right)(p) \cdot y + \left(\frac{\partial}{\partial z} f \right)(p) \cdot z = 0 \right\} \\ &= \left\{ (y,z) \in \mathbb{A}^2 : \tilde{f}(p) \left(\frac{\partial}{\partial y} f_i \right)(p) \cdot y + \tilde{f}(p) \left(\frac{\partial}{\partial z} f_i \right)(p) \cdot z = 0 \right\} \\ &= \left\{ (y,z) \in \mathbb{A}^2 : \left(\frac{\partial}{\partial y} f_i \right)(p) \cdot y + \left(\frac{\partial}{\partial z} f_i \right)(p) \cdot z = 0 \right\}. \end{aligned}$$

For the last equality we use that $\tilde{f}(p) \neq 0$, as else p was a singular point of f, but f is smooth. Completely analog we get

$$T_p \mathcal{V}_{\mathbb{A}}(g) = \left\{ (y,z) \in \mathbb{A}^2 : \left(\frac{\partial}{\partial y} f_i \right)(p) \cdot y + \left(\frac{\partial}{\partial z} f_i \right)(p) \cdot z = 0 \right\}$$

as well, so the tangent spaces of g and f at p are the same one-dimensional subspace and therefore can not span $\mathbb{A}^2$. By definition, this makes p a point of non-transversal intersection of the curves defined by f and g, which contradicts our main assumption. We conclude that the first inclusion in (8) has to be strict and therefore the last one needs to be equality, meaning $m = \mathrm{rad}(q_i)$ as claimed.
Now it suffices to show that q_i is radical, because then we can conclude $\langle y, z \rangle = m = \mathrm{rad}(q_i) = q_i$ and therefore $\deg(q_i) = \dim_{\mathbb{C}}(\mathbb{C}[y,z]/\langle y,z \rangle) = \dim_{\mathbb{C}}(\mathbb{C}) = 1$. Here we use that the ideal $\langle y, z \rangle$ is of dimension zero, as its variety is just one point (A.2.4) and hence the degree of this ideal is defined as the dimension of the $\mathbb{C}$-vector space $\mathbb{C}[y,z]/\langle y,z \rangle$.
As $\mathrm{rad}(q_i) = m = \langle y, z \rangle$, we need to show that $y, z \in q_i$, while we know that $y^{m_1}, z^{m_2} \in q_i$ for some large enough natural numbers m_1, m_2. By symmetry it will be enough to show $y \in q_i$.
We claim that for any natural number k, we find a polynomial $h \in \mathbb{C}[y,z]$ with the properties that

(i) every monomial that appears with a nonzero coefficient in h is of degree at least k and

(ii) $y = h \mod q_i$, i.e. that $y - h \in q_i \subseteq \mathbb{C}[y,z]$.

Notice that the claim immediately finishes the proof, since if we use it on $k = m_1 + m_2$, by point (i), the polynomial h can only contain monomials that are divisable by y^{m_1} or by z^{m_2}, which are in q_i. But then $h \in q_i$ and hence by (ii) also $y \in q_i$ as we wanted.
Now to proof the claim we will once more use transversal intersection. Since $\langle f, g \rangle = \bigcap_{j=1}^n q_j \subseteq q_i \subseteq m = \langle y, z \rangle$, we can write

$$\begin{aligned} f &= ay + bz + f^+, \qquad && a, b \in \mathbb{C} \\ g &= a'y + b'z + g^+, \qquad && a', b' \in \mathbb{C}, \end{aligned}$$

where f^+, g^+ are polynomials in $\mathbb{C}[y,z]$ that only use monomials of degree greater than or equal to two. In particular $(0,0)$ is a point where both f and g vanish, so as the varieties of f and g intersect transversally, the respective tangent spaces at $(0,0)$ need to span the whole plane $\mathbb{A}^2$. The tangent space to f at $(0,0)$ is simply the one-dimensional $\mathbb{C}$-vector space

$$\begin{aligned} T_{(0,0)}\mathcal{V}_{\mathbb{A}}(f) &= \left\{(y,z) \in \mathbb{A}^2 : \left(\frac{\partial}{\partial y} f\right)(0,0)\cdot y + \left(\frac{\partial}{\partial z} f\right)(0,0)\cdot z = 0\right\} \\ &= \{(y,z) \in \mathbb{A}^2 : a\cdot y + b\cdot z = 0\} \\ &= \mathrm{Span}_{\mathbb{C}}((b,a)), \end{aligned}$$

where for the last equality we use that f is smooth in $(0,0)$, hence a and b can not both be zero. Completely analog, for g we get the one-dimensional tangent space $T_{(0,0)}\mathcal{V}_{\mathbb{A}}(g) = \mathrm{Span}_{\mathbb{C}}((b',a'))$. These two $\mathbb{C}$-vector spaces span $\mathbb{A}^2$ if and only if the two vectors (b,a) and (b',a') are linearly independent. Therefore we need to have $a\cdot b' \neq a'\cdot b$. This allows us to define two polynomials in $\langle f,g\rangle$ as follows:

$$\begin{aligned} h_1 &:= \frac{b'f - bg}{b'a - a'b} = \frac{(b'a - a'b)y + b'f^+ - bg^+}{b'a - a'b} = y - h_1^+ \\ h_2 &:= \frac{a'f - ag}{ba' - ab'} = \frac{(ba' - ab')z + a'f^+ - ag^+}{ba' - ab'} = z - h_2^+. \end{aligned}$$

Here we define

$$h_1^+ := \frac{bg^+ - b'f^+}{b'a - a'b}, \qquad h_2^+ := \frac{ag^+ - a'f^+}{ba' - ab'},$$

which of course are both polynomials in $\mathbb{C}[y,z]$ that only use monomials of degree two or higher (as f^+, g^+ had this property).
From the definitions of h_1, h_2 it is immediately clear that $h_1, h_2 \in q_i$, as they are even $\mathbb{C}$-linear combinations of f, g. This means that modulo q_i, we have $y = h_1^+$ and $z = h_2^+$. Now the claim follows by induction on k: For $k = 1$ take $h = y$, if $k > 1$ let $\tilde{h}$ be a polynomial that satisfies (i) and (ii) for $k-1$. For any monomial $y^{k_1}z^{k_2}$, appearing in $\tilde{h}$ with nonzero coefficient and $k_1+k_2 = k-1$, we replace it with $(h_1^+)^{k_1}(h_2^+)^{k_2}$. Modulo q_i this doesn't change the polynomial $\tilde{h}$, as $y = h_1^+, z = h_2^+$ in $\mathbb{C}[y,z]/q_i$, but in $(h_1^+)^{k_1}(h_2^+)^{k_2}$, all appearing monomials are of degree greater than or equal to $2\cdot k_1 + 2\cdot k_2 = 2(k-1) \geq k+1$, because h_1^+, h_2^+ only contain degree two or higher monomials. Hence if we do this process with all (finitely many) degree $k-1$ monomials in $\tilde{h}$, we obtain a monomial $h \in \mathbb{C}[y,z]$ with properties (i) and (ii) for k, which proves the claim. □

Now the proof of 3.3.1 will go in two steps: First we will consider a homogeneous polynomial that has all the properties from 3.2.3 and find a nice condition for when the dual has the desired degree, then we will show that this is condition is met on a nonempty open subset of $\mathbb{P}(S^d\mathbb{C}^3)$, which allows us to conclude by 3.2.2.

Proof of Theorem 3.3.1. Let $f \in S^d\mathbb{C}^3$ be a complete, smooth and irreducible polynomial, where all the first partial derivatives are smooth again, and let $f^\vee$ be its dual. The degree

of $f^\vee$ as a polynomial is the same as the degree of $\mathcal{V}_{\mathbb{P}}(f^\vee)$ as a projective variety,[9] however the latter is also the number of intersections of $\mathcal{V}_{\mathbb{P}}(f^\vee)$ with a general line in $(\mathbb{P}^2)^*$ (see A.2.5). Now a line l in $(\mathbb{P}^2)^*$, by 2.1.4, corresponds to a point $p \in \mathbb{P}^2$, such that l is the collection of all lines in $\mathbb{P}^2$ that pass through p. Any intersection point of l with the dual curve of $\mathcal{V}_{\mathbb{P}}(f)$, therefore corresponds to a line in $\mathbb{P}^2$ that passes through p and is tangent to $\mathcal{V}_{\mathbb{P}}(f)$ at some point. Hence to get the degree of $f^\vee$, we need to fix a general point $p \in \mathbb{P}^2$ and count the number of lines through it that are tangent to our curve at some point. Clearly every such line gives a tangency point of the curve $\mathcal{V}_{\mathbb{P}}(f)$ (meaning a point where the tangent space contains p), however two different tangency points might give the same tangent line through p. In 3.3.4 we exactly proved, that for a general point p, this case can't appear, therefore it is enough to count the number of points on $\mathcal{V}_{\mathbb{P}}(f)$, for which the tangent space contains p.
Fix homogeneous coordinates x, y, z on $\mathbb{P}^2$, such that the point p is given as $p = (0:0:1)$. For a point $q \in \mathcal{V}_{\mathbb{P}}(f)$, the tangent space of the curve at this point is

$$T_q(\mathcal{V}_{\mathbb{P}}(f)) = \left\{(a:b:c) \in \mathbb{P}^2 : a\left(\frac{\partial}{\partial x}f\right)(q) + b\left(\frac{\partial}{\partial y}f\right)(q) + c\left(\frac{\partial}{\partial z}f\right)(q) = 0\right\}.$$

Clearly $p = (0:0:1)$ is on this line if and only if $\left(\frac{\partial}{\partial z}f\right)(q) = 0$. Therefore the number of points of the curve, where the tangent contains p is exactly the number of common zeroes of f and $\frac{\partial}{\partial z}f$, i.e. the set-theoretic cardinality of the variety $V := \mathcal{V}_{\mathbb{P}}(f, \frac{\partial}{\partial z}f)$. For now we fix the affine flag $\{x = 1\}$ and count the number of points in $V \cap \{x = 1\}$. Slightly abusing notation, we will replace f with $f(1, y, z) \in \mathbb{C}[y, z]$, then $V \cap \{x = 1\} = \mathcal{V}_{\mathbb{A}}(f, \frac{\partial}{\partial z}f)$ is an affine variety, which we will call V again. Notice that by 3.2.4, the new f will still be a complete, smooth and irreducible polynomial with smooth first partial derivatives.
Next we will compute the dimension and degree of V using the (affine) Hilbert-function (we are only interested in the degree, the dimension will be zero, as the degree of $f^\vee$, has to be a finite number, hence V needs to be a finite collection of points).

Proposition 3.3.8. *V is of dimension zero and of degree $d(d-1)$.*

Proof of Proposition 3.3.8. First consider the following sequence of morphisms of rings

$$0 \to \mathbb{C}[y,z] \xrightarrow{\cdot f} \mathbb{C}[y,z] \to \mathbb{C}[y,z]/\langle f\rangle \to 0,$$

where the first map $\mathbb{C}[y,z] \to \mathbb{C}[y,z]$ is just multiplication by f and the second map is the natural projection $\mathbb{C}[y,z] \to \mathbb{C}[y,z], g \mapsto g + \langle f\rangle$. First notice that this sequence behaves well with degree in the sense that for any $n \in \mathbb{N}$, we can restrict the first map to $\mathbb{C}[y,z]_{\leq n}$ and the second map to $\mathbb{C}[y,z]_{\leq n+d}$ and receive a well-defined sequence of $\mathbb{C}$-vector spaces

$$0 \to \mathbb{C}[y,z]_{\leq n} \xrightarrow{\varphi} \mathbb{C}[y,z]_{\leq n+d} \xrightarrow{\pi} \mathbb{C}[y,z]_{\leq n+d}/\langle f\rangle_{\leq n+d} \to 0.$$

This sequence is exact because

[9] The degree of $f^\vee$ is also the degree of the ideal $\langle f^\vee\rangle$, as we define it in A.2. This follows for example from the computations we make in 3.3.8.

- φ is injective as f is of degree $d \geq 2$, hence not the zero polynomial and $\mathbb{C}[y,z]$ is integral,
- $\operatorname{im}(\varphi) = \langle f \rangle_{\leq n+d} = \ker(\pi)$,
- π is surjective as any $g + \langle f \rangle_{\leq n+d} \in \mathbb{C}[y,z]_{\leq n+d}/\langle f \rangle_{n+d}$ is the image under π of $g \in \mathbb{C}[y,z]_{\leq n+d}$.

This tells us how the dimensions of the three $\mathbb{C}$-vector spaces in the sequence above relate, namely

$$\dim_{\mathbb{C}}(\mathbb{C}[y,z]_{\leq n+d}) - \dim_{\mathbb{C}}(\mathbb{C}[y,z]_{\leq n}) = \dim_{\mathbb{C}}(\mathbb{C}[y,z]_{\leq n+d}/\langle f \rangle_{\leq n+d}). \tag{9}$$

The two dimensions on the left side of the equation are easy to find by a combinatorial argument: The set of all monomials in $\mathbb{C}[y,z]$ of degree at most n is clearly a base of $\mathbb{C}[y,z]_{\leq n}$ as a $\mathbb{C}$-vector space, hence $\dim_{\mathbb{C}}(\mathbb{C}[y,z]_{\leq n})$ is exactly the number of different monomials in two variables of degree at most n. We can get this number by summing up all the numbers of degree (exactly) i monomials in two variables, where i ranges from zero to n. However a degree i monomial in two variables y, z is uniquely determined by the power, to which y appears in the monomial (if y appears to power $0 \leq j \leq n$, then z needs to appear to power $i - j$), hence there are exactly $i+1$ different degree i monomials in two variables. We conclude

$$\dim_{\mathbb{C}}(\mathbb{C}[y,z]_{\leq n}) = \sum_{i=0}^{n}(i+1) = n+1+\sum_{i=0}^{n} i = n+1+\frac{n(n+1)}{2} = \frac{n^2+3n}{2},$$

and by applying the same reasoning for $n+d$ instead of n we get

$$\dim_{\mathbb{C}}(\mathbb{C}[y,z]_{\leq n+d}) = \frac{(n+d)^2+3(n+d)}{2}.$$

From (9) we conclude

$$\dim_{\mathbb{C}}(\mathbb{C}[y,z]_{\leq n+d}/\langle f \rangle_{\leq n+d}) = \frac{(n+d)^2+3(n+d)}{2} - \frac{n^2+3n}{2} = \frac{2nd+d^2+3d}{2},$$

and hence the affine Hilbert function of $\langle f \rangle$ sends $n+d \mapsto \frac{2nd+d^2+3d}{2}$ for every $n \in \mathbb{N}$. By shifting with d, we get

$$\dim_{\mathbb{C}}(\mathbb{C}[y,z]_{\leq n}/\langle f \rangle_{\leq n}) = \frac{2d(n-d)+d^2+3d}{2} = dn + \frac{3d-d^2}{2} \qquad \text{for } n > d. \tag{10}$$

From this we could easily get the Hilbert polynomial and hence dimension and degree of $\langle f \rangle$, but we really need it for $\langle f \frac{\partial}{\partial z} f \rangle$, so we apply the same trick once more. Fix $n \in \mathbb{N}$ and consider the following sequence of $\mathbb{C}$-vector spaces:

$$0 \to \mathbb{C}[y,z]_{\leq n}/\langle f \rangle_{\leq n} \xrightarrow{\varphi'} \mathbb{C}[y,z]_{\leq n+d-1}/\langle f \rangle_{\leq n+d-1} \xrightarrow{\pi'} \mathbb{C}[y,z]_{\leq n+d-1}/\langle f, \frac{\partial}{\partial z} f \rangle_{\leq n+d-1} \to 0.$$

Here the first map φ' sends

$$\mathbb{C}[y,z]_{\leq n}/\langle f\rangle_{\leq n} \ni g+\langle f\rangle_{\leq n} \mapsto \left(\frac{\partial}{\partial z}f\right)\cdot g+\langle f\rangle_{\leq n+d-1} \in \mathbb{C}[y,z]_{\leq n+d-1}/\langle f\rangle_{\leq n+d-1},$$

while the second map π' is again just natural projection. Notice that $\deg(\frac{\partial}{\partial z}f)=\deg(f)-1=d-1$, because f is complete, so φ' is well-defined. Again this sequence is exact:

- φ' is injective. Indeed assume $\varphi'(g+\langle f\rangle_{\leq n})=0$, then $g\cdot\left(\frac{\partial}{\partial z}f\right)\in\langle f\rangle_{n+d-1}$. However f is irreducible, hence prime in the factorial ring $\mathbb{C}[y,z]$, and the degree $d-1$ polynomial $\frac{\partial}{\partial z}f$ can not be divided by the degree d polynomial f, hence $g\in\langle f\rangle_{\leq n}$.

- $\operatorname{im}(\varphi')=\langle\frac{\partial}{\partial z}f\rangle_{\leq n+d-1}\ker(\pi')$.

- π' is clearly surjective, as any $g+\langle f,\frac{\partial}{\partial z}f\rangle_{\leq n+d-1}\in\mathbb{C}[y,z]_{\leq n+d-1}/\langle f,\frac{\partial}{\partial z}f\rangle_{\leq n+d-1}$ is the image under π' of $g+\langle f\rangle_{\leq n+d-1}\in\mathbb{C}[y,z]_{\leq n+d-1}/\langle f\rangle_{\leq n+d-1}$.

Again this gives us the following equation for the dimensions of these $\mathbb{C}$-vector spaces:

$$\begin{aligned}\dim_{\mathbb{C}}(\mathbb{C}[y,z]_{\leq n+d-1}/\langle f,\frac{\partial}{\partial z}f\rangle_{\leq n+d-1}) &= \dim_{\mathbb{C}}(\mathbb{C}[y,z]_{\leq n+d-1}/\langle f\rangle_{\leq n+d-1})\\ &\quad-\dim_{\mathbb{C}}(\mathbb{C}[y,z]_{\leq n}/\langle f\rangle_{\leq n}).\end{aligned}$$

However, the dimensions on the right side were already computed in (10), so we immediately get

$$\begin{aligned}\dim_{\mathbb{C}}(\mathbb{C}[y,z]_{\leq n+d-1}/\langle f,\frac{\partial}{\partial z}f\rangle_{\leq n+d-1}) &= d(n+d-1)+\frac{3d-d^2}{2}-(dn+\frac{3d-d^2}{2})\\ &= d(d-1),\end{aligned}$$

as long as $n>d$. This means that for high enough values of n, the Hilbert-function of $\langle f,\frac{\partial}{\partial z}f\rangle$ is given as

$$\dim_{\mathbb{C}}(\mathbb{C}[y,z]_{\leq n}/\langle f,\frac{\partial}{\partial z}f\rangle_{\leq n})-\dim_{\mathbb{C}}(\mathbb{C}[y,z]_{\leq n-1}/\langle f,\frac{\partial}{\partial z}f\rangle_{\leq n-1})=d(d-1)-(d(d-1))=0,$$

and therefore the Hilbert-Polynomial is the zero polynomial. In this case, by definition, the ideal $\langle f,\frac{\partial}{\partial z}f\rangle$ has dimension 0 and degree $\dim_{\mathbb{C}}(\mathbb{C}[y,z]/\langle f\frac{\partial}{\partial z}f\rangle)$. We claim the degree is $d(d-1)$. We already know that for large enough n, $\dim_{\mathbb{C}}(\mathbb{C}[y,z]_{\leq n}/\langle f,\frac{\partial}{\partial z}f\rangle_{\leq n})=d(d-1)$, which is independent of n. Now pick any base $g_1+\langle f,\frac{\partial}{\partial z}f\rangle_{\leq n},\dots g_{d(d-1)}+\langle f,\frac{\partial}{\partial z}f\rangle_{\leq n}$ of this $\mathbb{C}$-vector space, then as the $g_1,\dots,g_{d(d-1)}$ are of degree at most n, their classes are still linearly independent in every single $\mathbb{C}[y,z]_{\leq m}/\langle f,\frac{\partial}{\partial z}f\rangle_{\leq m}$ for $m>n$, aswell as in $\mathbb{C}[y,z]/\langle f,\frac{\partial}{\partial z}f\rangle$. But then as the former $\mathbb{C}$-vector space is of dimension $d(d-1)$ it already is a base of every single $\mathbb{C}[y,z]_{\leq m}/\langle f,\frac{\partial}{\partial z}f\rangle_{\leq m}$ for $m>n$. Now $g_1,\dots,g_{d(d-1)}$ also needs to generate $\mathbb{C}[y,z]/\langle f,\frac{\partial}{\partial z}f\rangle$. Indeed take $g+\langle f\frac{\partial}{\partial z}f\rangle\in\mathbb{C}[y,z]/\langle f,\frac{\partial}{\partial z}f\rangle$ and let $m=\deg(g)$, then $g + \langle f \frac{\partial}{\partial z} f\rangle_{\leq m} \in \mathbb{C}[y,z]_{\leq m}/\langle f, \frac{\partial}{\partial z} f\rangle_{\leq m}$ and hence we find

$$g = \sum_{i=1}^{d(d-1)} \lambda_i g_1 \qquad \mod \langle f, \frac{\partial}{\partial z} f\rangle_{\leq m},$$

for some $\lambda_i \in \mathbb{C}$. In particular, the last equation holds modulo $\langle f, \frac{\partial}{\partial z} f\rangle$, so $g_1 + \langle f, \frac{\partial}{\partial z} f\rangle, \ldots$ $g_{d(d-1)} + \langle f, \frac{\partial}{\partial z} f\rangle$ is a base of $\mathbb{C}[y,z]/\langle f \frac{\partial}{\partial z} f\rangle$, which therefore needs to be of dimension $d(d-1)$. $\square_{3.3.8}$

So now we know that V is zero dimensional and of degree $d(d-1)$. This means that V is a finite collection of points and we get $d(d-1)$, when we count those points with multiplicity. However, we need the actual number of distinct point of V, so our next goal will be to get a condition on f that makes V have no points of multiplicity greater than one. To make this formal, we use primary decomposition.

Let $I := \langle f \frac{\partial}{\partial z} f\rangle$, then as I is a zero-dimensional ideal, the ring $\mathbb{C}[y,z]/I$ is of Krull-dimension 0. We claim that this ring can be written as a direct sum of zero-dimensional, local rings. To see this, first notice that $\mathbb{C}[y,z]$ is noetherian, hence the ideal I has a minimal primary decomposition, meaning we can write $I = q_1 \cap \ldots \cap q_n$, where $q_i's$ are primary ideals of $\mathbb{C}[y,z]$ with pairwise different radicals $p_i := \mathrm{rad}(q_i)$ and $q_i \nsubseteq \bigcap_{j\neq i} q_j$.

Claim: $q_i's$ are pairwise coprime.

Consider q_i, q_j for $i \neq j$, then as taking the radical commutes with intersection, we have

$$I \subseteq \mathrm{rad}(I) = \bigcap_{k=1}^{n} \mathrm{rad}(q_k) = \bigcap_{k=1}^{n} p_k \subsetneq p_i,$$

and completely analog$I \subsetneq p_j$. Since the radical of a primary ideal is clearly a prime, this allows us to view p_i, p_j as prime ideals not only in $\mathbb{C}[y,z]$, but also in $\mathbb{C}[y,z]/I$. However $\mathbb{C}[y,z]/I$ was of Krull-dimension zero, so we can't have any prime ideal in $\mathbb{C}[y,z]$ strictly containing p_i or p_j, as else we found a chain of primes of length one in $\mathbb{C}[y,z]$. This means p_i, p_j are maximal ideals of $\mathbb{C}[y,z]$ and not equal, so they are coprime. Now two ideals are coprime if and only if their union contains the neutral element 1, which is if and only if their radicals have this property, so also q_i, q_j are coprime, proving the claim.

Using Chinese Remainder Theorem we get

$$\mathbb{C}[y,z]/I = \mathbb{C}[y,z]/\bigcap_{i=1}^{n} q_i \cong \mathbb{C}[y,z]/q_1 \times \ldots \times \mathbb{C}[y,z]/q_n. \tag{11}$$

Notice that each $\mathbb{C}[y,z]/q_i$ is a local ring, since by [MS21, Lemma 3.12], p_i is the unique smallest prime in $\mathbb{C}[y,z]$ that contains q_i. Now any prime ideal of $\mathbb{C}[y,z]/q_i$ is a prime ideal of $\mathbb{C}[y,z]$ that contains q_i, but then by minimality of p_i it also contains p_i, which is a maximal ideal in $\mathbb{C}[y,z]$. Hence the only maximal ideal in $\mathbb{C}[y,z]/q_i$ is the ideal corresponding to p_i. In particular $\mathbb{C}[y.z]/q_i$ is a zero-dimensional, local ring.

Furthermore V contains exactly one point for every p_i, as

$$V = \mathcal{V}_{\mathbb{A}}(I) = \mathcal{V}_{\mathbb{A}}(\bigcap_{i=1}^{n} q_i) = \bigcup_{i=1}^{n} \mathcal{V}_{\mathbb{A}}(q_i) = \bigcup_{i=1}^{n} \mathcal{V}_{\mathbb{A}}(\mathrm{rad}(q_i)) = \bigcup_{i=1}^{n} \mathcal{V}_{\mathbb{A}}(p_i).$$

We already know that $p_i's$ are even maximal ideal of $\mathbb{C}[y,z]$, hence $\mathcal{V}_{\mathbb{A}}(p_i)$ is just one point for every p_i. Since $p_i \neq p_j$, also $\mathcal{V}_{\mathbb{A}}(p_i) \neq \mathcal{V}_{\mathbb{A}}(p_j)$ for $i \neq j$, so we conclude that the number of points in V (now counted without multiplicity) is exactly n, i.e. the number of primary ideals appearing in a minimal primary decomposition. However, as $\dim(I) = \dim(q_i) = 0, i = 1, \dots, n$, we have

$$\begin{aligned} d(d-1) &= \deg(I) \\ &= \dim_{\mathbb{C}}(\mathbb{C}[y,z]/I) \\ &\overset{(11)}{=} \dim_{\mathbb{C}}(\mathbb{C}[y,z]/q_1 \times \ldots \times \mathbb{C}[y,z]/q_n) \\ &= \sum_{i=1}^{n} \dim(\mathbb{C}[y,z]/q_i) \\ &= \sum_{i=1}^{n} \deg(q_i). \end{aligned}$$

This tells us that, under the assumption that each q_i is of degree one, V has exactly $d(d-1)$ points.
In 3.3.7 we already showed that this assumption is met, when $\mathcal{V}_{\mathbb{A}}(f)$ and $\mathcal{V}_{\mathbb{A}}(\frac{\partial}{\partial z} f)$ intersect transversally. Consider any point $s \in V$, then we have $f(s) = \left(\frac{\partial}{\partial z} f\right)(s) = 0$. Since f is smooth, we must have $\left(\frac{\partial}{\partial y} f\right)(s) \neq 0$, so the tangent space of the curve defined by f at s is

$$\begin{aligned} T_s(\mathcal{V}_{\mathbb{A}}(f)) &= \{(a,b) \in \mathbb{A}^2 : \left(\frac{\partial}{\partial y} f\right)(s) \cdot a + \left(\frac{\partial}{\partial z} f\right)(s) \cdot b = 0\} \\ &= \{(a,b) \in \mathbb{A}^2 : \left(\frac{\partial}{\partial y} f\right)(s) \cdot a = 0\} \\ &= \{(0,b) \in \mathbb{A}^2\}. \end{aligned}$$

If $\left(\frac{\partial^2}{\partial z^2} f\right)(s) = 0$, by smoothness of the first derivative of f, also $\left(\frac{\partial^2}{\partial y \partial z} f\right)(s) \neq 0$, so the tangent space of $\frac{\partial}{\partial z} f$ at s is the same one-dimensional subspace of $\mathbb{A}^2$ as $T_s(\mathcal{V}_{\mathbb{A}}(f))$, meaning s is not a point of transversal intersection. On the other hand, if $\left(\frac{\partial^2}{\partial z^2} f\right)(s) \neq 0$, clearly the tangent spaces of f and $\frac{\partial}{\partial z} f$ at s are two different, one-dimensional subspaces of $\mathbb{A}^2$ and hence span $\mathbb{A}^2$ as a $\mathbb{C}$-vector space. This means that $\mathcal{V}_{\mathbb{A}}(f)$ and $\mathcal{V}_{\mathbb{A}}(\frac{\partial}{\partial z} f)$ intersect transversally, if and only if there is no point $s \in \mathbb{A}^2$, such that

$$f(s) = \left(\frac{\partial}{\partial z} f\right)(s) = \left(\frac{\partial^2}{\partial z^2} f\right)(s) = 0. \tag{12}$$

This is finally a nice condition on the polynomial f, under which V contains exactly

$d(d-1)$ points.
We will now go back to homogeneous polynomials in $\mathbb{P}(S^d\mathbb{C}^3)$ and proof that, for a general element f of $\mathbb{P}(S^d\mathbb{C}^3)$,

(i) there exists no point $s \in \mathbb{A}^2$, such that (12) holds for $f(1, y, z)$.

(ii) there exists no point $(0 : y : z) \in \mathbb{P}^2$, such that $f(0, y, z) = \left(\frac{\partial}{\partial z} f\right)(0, y, z) = 0$.

First notice that with these two statements we can finish the proof. Indeed, from 3.2.3 together with 3.2.2 (iii), we can conclude that a general element f of $\mathbb{P}(S^d\mathbb{C}^3)$ is smooth, irreducible, has smooth first derivatives and (i), (ii) hold. Above we proved, that the first three properties, together with (i) imply that $V = \mathcal{V}_{\mathbb{P}}(f, \frac{\partial}{\partial z} f)$ contains exactly $d(d-1)$ points on the affine flag $\{x = 1\}$ and (ii) exactly tells us that there are no points in$V \setminus \{x = 1\}$. However in the start of the proof, we showed that the number of points in V is also the degree of $f^\vee$, hence this must be $d(d-1)$. So on the open dense subset, where all of the properties smooth, irreducible, smooth first derivative, (i) and (ii) hold simultaneously, also all elements have the property that their dual has degree $d(d-1)$. This exactly proves the Theorem.
Now to show (i) and (ii), we will use a similar trick as in 3.1.
For (i): We first define a subset$X$ in the product $\mathbb{P}(S^d\mathbb{C}^3) \times \mathbb{A}^2$ as

$$\{(f, (s_1, s_2)) \in \mathbb{P}(S^d\mathbb{C}^3) \times \mathbb{A}^2 : f(1, s_1, s_2) = \left(\frac{\partial}{\partial z} f\right)(1, s_1, s_2) = \left(\frac{\partial^2}{\partial z^2} f\right)(1, s_1, s_2) = 0\}.$$

First notice that the condition for a point (f, s) being in X is a polynomial condition, homogeneous in the entries of f. Just like we did in 2.1.8, we can make X a closed subscheme of the fibered product $\mathbb{P}(S^d\mathbb{C}^3) \times \mathbb{A}^2$,[10] such that the closed points of this scheme correspond to points of X and the restrictions π_1, π_2 of the scheme-theoretic projections from $\mathbb{P}(S^d\mathbb{C}^3) \times \mathbb{A}^2$ to $\mathbb{P}(S^d\mathbb{C}^3)$ and $\mathbb{A}^2$ just send a closed point (f, s) to the closed point f (resp. s) of $\mathbb{P}(S^d\mathbb{C}^3)$ (resp $\mathbb{A}^2$).
We want to compute the fiber of any closed point $s \in \mathbb{A}^2$ under the projection $\pi_2 : X \to \mathbb{A}^2$. Fix any $s \in \mathbb{A}^2$, then by a homogeneous change of coordinates, we might assume $s = (0, 0)$ (we provided all the details of the coordinate change in 3.2.3 ,(7)). Now for any $f \in \mathbb{P}(S^d\mathbb{C}^3)$

- $f(1, s_1, s_2) = 0$ means that the coefficient in front of the monomial x^d must be zero,
- $\left(\frac{\partial}{\partial z} f\right)(1, s_1, s_2) = 0$ means that the coefficient in front of the monomial zx^{d-1} must be zero and
- $\left(\frac{\partial^2}{\partial z^2} f\right)(1, s_1, s_2) = 0$ means that the coefficient in front of the monomial z^2x^{d-2} must be zero.

hence the fiber $\pi_2^{-1}(s)$ is exactly the set of all elements in $\mathbb{P}(S^d\mathbb{C}^3) = \mathbb{P}^{n-1}, n = \binom{2+d}{d}$, where three fixed entries are zero, i.e. a projective subspace $\mathbb{P}^{n-1-3} = \mathbb{P}^{n-4} \subseteq \mathbb{P}^{n-1}$.
The scheme X is noetherian (as it is of finite type over $\mathbb{C}$), so we can cover X by finitely

[10]This makes X an abstract variety by A.1.5,A.1.4

many irreducible components $Z_1, \dots, Z_k$ and clearly the dimension of X is the maximum of $\dim(Z_1), \dots, \dim(Z_k)$.[11] Let us assume Z_1 is an irreducible component of maximal dimension. We restrict the map π_2 to Z_1 and get a map $\pi_{2|_{Z_1}} : Z_1 \to \overline{\mathrm{im}(\pi_{2|_{Z_1}})} \subseteq \mathbb{A}^2$. Notice that now also the codomain of this map needs to be irreducible, since Z_1 is irreducible (we proved a very similar statement in the proof of 2.1.7). Also this map is clearly dominant, so we can use 3.1.1, (ii), for any closed point $s \in \overline{\mathrm{im}(\pi_{2|_{Z_1}})} \subseteq \mathbb{A}^2$, to get

$$\begin{aligned} n - 4 &= \dim(\mathbb{P}^{n-4}) \\ &= \dim(\pi_2^{-1}(s)) \\ &\geq \dim(\pi_{2|_{Z_1}}^{-1}(s)) \\ &\geq \dim(Z_1) - \dim(\overline{\mathrm{im}(\pi_{2|_{Z_1}})}) \\ &\geq \dim(X) - \dim(\mathbb{A}^2) \\ &= \dim(X) - 2. \end{aligned}$$

So $\dim(X) \leq n - 2$. Now for an arbitrary irreducible component $Z_i, i = 1, \dots, k$ of X, the restriction of the first projection $\pi_{1|_{Z_i}} : Z_i \to \overline{\mathrm{im}(\pi_{1|_{Z_i}})} \subseteq \mathbb{P}(S^d\mathbb{C}^3)$ is a dominant map of irreducible varieties, so by the first part of 3.1.1, we need to have

$$\dim(\overline{\mathrm{im}(\pi_{1|_{Z_i}})}) \leq \dim(Z_i) \leq \dim(X) \leq n - 2.$$

But now

$$\dim(\overline{\mathrm{im}(\pi_1)}) = \dim\left(\overline{\bigcup_{i=1}^k \mathrm{im}(\pi_{1|_{Z_i}})}\right) = \dim\left(\bigcup_{i=1}^k \overline{\mathrm{im}(\pi_{1|_{Z_i}})}\right) = \max_{i=1,\dots,k}(\dim(\overline{\mathrm{im}(\pi_{1|_{Z_i}})})) \leq n - 2.$$

However, the dimension of $\mathbb{P}(S^d\mathbb{C}^3)$ is $n - 1$, so the image of the first projection $X \to \mathbb{P}(S^d\mathbb{C}^3)$ is not dense in $\mathbb{P}(S^d\mathbb{C}^3)$. Now notice that the image of the first projection (on closed points) is exactly the set of homogeneous degree d polynomials, such that (i) does not hold. By 3.2.2, part (ii) we can conclude that a general polynomial in $\mathbb{P}(S^d\mathbb{C}^3)$ has the property (i) as claimed.

For (ii): With the goal of applying the same reasoning once more, we consider the set X of all pairs $(f, (s_1 : s_2)) \in \mathbb{P}(S^d\mathbb{C}^3) \times \mathbb{P}^1$, such that $f(0s_1, s_2) = \left(\frac{\partial}{\partial z} f\right)(0, s_1, s_2) = 0$. Again, this set is given by polynomial equations, both homogeneous in the entries of f and in (s_1, s_2), so we can consider X as an abstract variety in the fibered product $\mathbb{P}(S^d\mathbb{C}^3) \times_{\mathbb{C}} \mathbb{P}^1$. The closed points of X correspond again to pairs (f, s) that have the property described above and the scheme theoretic restricions π_1, π_2 of the projections from $\mathbb{P}(S^d\mathbb{C}^3) \times_{\mathbb{C}} \mathbb{P}^1$ to $\mathbb{P}(S^d\mathbb{C}^3)$ and $\mathbb{P}^1$ send a closed point (f, s) to the closed point $f \in \mathbb{P}(S^d\mathbb{C}^3)$ and $s \in \mathbb{P}^1$ respectively.

To compute the dimension of a fiber of a closed point $s = (s_1, s_2) \in \mathbb{P}^1$, where $s_1 \neq 0$,[12] under the second projection, we first choose new homogeneous coordinates $(x : y : z)$ of

[11] Remember that the dimension of X is just the length of the longest chain of closed irreducible subsets in X, such a maximal chain needs to end in one of the Z_i, which then has the same dimension as X

[12] this just makes sure that the condition $\left(\frac{\partial}{\partial z} f\right)(0, s_1, s_2)$ does not change, when we change the coordinates.

$\mathbb{P}^2$, such that the point $(0 : s_1 : s_2) \in \mathbb{P}^2$ has the form $(0 : 1 : 0)$ in the new coordinates. Now for any $f \in \mathbb{P}(S^d\mathbb{C}^3)$

- $f(0,1,0) = 0$ means that the coefficient in front of the monomial y^d must be zero,
- $\left(\frac{\partial}{\partial z} f\right)(0,1,0) = 0$ means that the coefficient in front of the monomial zy^{d-1} must be zero.

In other words, the fiber of s under π_2 contains exactly those polynomials in $\mathbb{P}(S^d\mathbb{C}^3)$ that have two fixed coefficients zero. Therefore $\pi_2^{-1}(s)$ is a projective subspace of $\mathbb{P}(S^d\mathbb{C}^3) = \mathbb{P}^{n-1}$ of dimension $n - 1 - 2 = n - 3$.
Again let us write X as the union of its irreducible components $Z_1, \ldots, Z_k$ and assume that $\dim(X) = \dim(Z_1)$. We apply 3.1.1,(ii) to the dominant map $\pi_{2|_{Z_1}} : Z_1 \to \overline{\mathrm{im}(\pi_{2|_{Z_1}})}$ of irreducible varieties and to the point $s = (s_1 : s_2) \neq (0 : 1)$ to get

$$\begin{aligned} n - 3 &= \dim(\pi_2^{-1}(s)) \\ &\geq \dim(\pi_{2|_{Z_1}}^{-1}(s)) \\ &\geq \dim(Z_1) - \dim(\overline{\mathrm{im}(\pi_{2|_{Z_1}})}) \\ &\geq \dim(X) - \dim(\mathbb{P}^1) \\ &= \dim(X) - 1. \end{aligned}$$

This implies $\dim(X) \leq n-2$. Now we can proceed completely analog to the proof of point (i) above, to see that the first projection $\pi_1 : X \to \mathbb{P}(S^d\mathbb{C}^3)$ can not have a dense image. However the (closed) points in the image of π_1 are exactly those polynoials $f \in \mathbb{P}(S^d\mathbb{C}^3)$ that do not have property (ii), so by 3.2.2,(ii), a general polynomial in $\mathbb{P}(S^d\mathbb{C}^3)$ must have the property as claimed. □

This proof used several nice properties, that a general polynomial has, such as most importantly smoothness and smoothness of partial derivatives. When we do not have these properties, the result will in general fail, but from the first part of the proof, we can at least derive an inequality.

Corollary 3.3.9. Let $f \in S^d\mathbb{C}^3$ be any irreducible polynomial, then the dual $f^\vee$ has degree at most $d(d-1)$.

Proof. Just like in the proof of 3.3.1, the degree of $f^\vee$ is the number of points in $V := \mathcal{V}_\mathbb{P}(f, \frac{\partial}{\partial z} f)$. With the same methods as above, we can compute the Hilbert function of the ideal $I := \langle f, \frac{\partial}{\partial z} f\rangle \subseteq \mathbb{C}[x, y, z]$ to be the polynomial $\mathrm{HP}_I(n) = d(d-1)$. This is a degree zero polynomial in n and therefore I by definition is an ideal of dimension one and degree $d(d-1)$. By A.2.4, this means, that the affine cone of V is a variety of dimension one and degree $d(d-1)$, but then as a projective variety, V has dimension zero and degree $d(d-1)$. Hence we get $d(d-1)$ points in V, when we count them with multiplicity, so counting every point only once, we get a number less then or equal to $d(d-1)$. □

4. The map $\cdot^\vee : \mathbb{P}(S^3\mathbb{C}^3) \dashrightarrow \mathbb{P}(S^6\mathbb{C}^3)$

4.1. Theoretical approach

We will now focus on the case of a smooth cubic curve in $\mathbb{P}^2$, meaning a projective variety in $\mathbb{P}^2$ of dimension one and degree 3. Such a curve is given as $\mathcal{V}_\mathbb{P}(f)$, where $f \in S^3\mathbb{C}^3$ is a homogeneous degree 3 polynomial in 3 variables (x, y, z). In chapter 3 we exactly proofed that the dual variety will also be of dimension one (3.1) and for a general $f \in \mathbb{P}(S^3\mathbb{C}^3)$, the dual will have degree $3(3-1) = 6$. This gives us a rational map, taking the $\binom{3+2}{3} = 10$ coefficients of a homogeneous degree 3 polynomial in 3 variables to the $\binom{6+2}{6} = 28$ coefficients of the degree 6 polynomial that defines the dual curve:

$$\begin{aligned} \cdot^\vee : \mathbb{P}(S^3\mathbb{C}^3) &\dashrightarrow \mathbb{P}(S^6\mathbb{C}^3) \\ f &\mapsto f^\vee. \end{aligned}$$

Notice that $\cdot^\vee$ is not globally defined, we only know there exists a nonempty open subset U of $\mathbb{P}(S^3\mathbb{C}^3)$, where the dual will actually have degree 6. We proved in 3.3.1 that such U exists, using the more general Plücker relations (see [GH78, chapter 2 section 4]) we can even take U to be the set of all smooth (and hence irreducible, see the proof of 3.2.3) polynomials in $\mathbb{P}(S^3\mathbb{C}^3)$.
When we identify a polynomial with the tupel of its coefficients in some fixed order, we can write $\cdot^\vee$ as

$$\begin{aligned} \mathbb{P}^9 \supseteq U &\to \mathbb{P}^{27} \\ a = (a_0, \dots, a_9) &\mapsto (F_0(a), \dots, F_{27}(a)), \end{aligned}$$

where for now $F_1, \dots, F_{27}$ are any maps $U \to \mathbb{P}^1$. The goal of this section is to show that all the F_i are actually degree 16 polynomials in $a_0, \dots, a_9$.

Example 4.1.1. The open subset U is a proper subset, meaning there exist polynomials in $\mathbb{P}(S^3\mathbb{C}^3)$, such that the dual does not have degree 6. We already saw one example in 3.3.3, where we started with the non-smooth curve defined by $f = x^2 \cdot y - z^3$. We found that the dual curve needs to be given by the equation $4w^3 + 27u^2 = 0$ on the affine flag $\{v = 1\}$. But then the polynomial $f^\vee \in \mathbb{C}[u, v, w]$ needs to be a polynomial, where dehomogenizing in v gives $4w^3 + 27u^2$. We know that the degree can be at most 6 (3.3.9), but still there are several possibilities, namely

$$\begin{aligned} &4w^3v^3 + 27u^2v^4, \\ &4w^3v^2 + 27u^2v^3, \\ &4w^3v + 27u^2v^2, \\ &4w^3 + 27u^2v. \end{aligned}$$

The correct equation is $f^\vee = 4w^3 + 27u^2v$, since this is the only irreducible polynomial in

this list and we already know that the dual is irreducible by 2.1.7.[13] So in this case the dual has degree $3 < 6$.
Another example for polynomials not in U are all reducible polynomials, as for those we did not define the dual curve.

Algorithm 4.1.2. From 2.2.1 we know an algorithm for computing $f^\vee$ for a given (smooth, irreducible) f:

1. Compute the three first order partial differentials $f_x = \frac{\partial}{\partial x} f, f_y = \frac{\partial}{\partial y} f, f_z = \frac{\partial}{\partial z} f$.

2. Dehomogenize f and the three partial differentials with respect to x (or any other variable).

3. Define the ideal $I = \langle f, f_y - v f_x, f_z - w f_x \rangle \subseteq \mathbb{C}[y, z, v, w]$ and eliminate the variables y, z from it.

4. Homogenize the generator of the resulting Ideal with respect to a new variable u.

We know from chapter 3 that the result is a degree 6 polynomial and the coefficients will exactly be the maps $F_0, \ldots, F_{27}$, evaluated at the coefficients of f. This means that to compute $F_0, \ldots, F_{27}$, we only need to do the upper algorithm with arbitrary coefficients $a_0, \ldots, a_9$ of f. In other words, we can get $F_0, \ldots, F_{27}$ by eliminating y, z from the ideal

$$I = \langle f, f_y - v f_x, f_z - w f_x \rangle \subseteq \mathbb{C}[y, z, v, w, a_0, \ldots, a_9],$$

where

$$f = a_0 x^3 + a_1 x y^2 + a_2 x y z + a_3 x z^2 + a_4 x^2 y + a_5 x^2 z + a_6 y^3 + a_7 y^2 z + a_8 y z^2 + a_9 z^3,$$

and f_x, f_y, f_z are the three partial derivatives that now also depend on $a_0, \ldots, a_9$. This results in a principal ideal, where the coefficients of the generator are exactly our maps $F_0, \ldots, F_{27}$ in $(a_0, \ldots, a_9)$ (up to multiplying by a nonzero constant). In particular this shows that $F_1, \ldots, F_{27} \in \mathbb{C}[a_0, \ldots, a_9]$ are polynomials.
We can use the theory of resultants to get an upper bound on the degree of the $F_1, \ldots, F_{27}$ in $a_0, \ldots, a_9$. For this we use the following theorem.

Theorem 4.1.3. Consider $m + 1$ polynomials $g_1, \ldots, g_{m+1}$ in m variables $z_1, \ldots, z_n$ with variable coefficients $x_1, \ldots, x_n$ and let $d_i = \deg_{z_1, \ldots, z_n}(g_i), i = 1, \ldots, m + 1$. Let $I = \langle g_1, \ldots, g_{m+1} \rangle \subseteq \mathbb{Q}[x_1, \ldots, x_n, z_1, \ldots, z_m]$ be their ideal, then the elimination ideal $I \cap \mathbb{Q}[x_1, \ldots, x_n]$ is principal, we call the generator $\operatorname{Res}(g_1, \ldots, g_{m+1})$[14].
The generator $\operatorname{Res}(g_1, \ldots, g_{m+1})$ is a polynomial in the coefficients $x_1, \ldots, x_n$ of $g_1, \ldots, g_{m+1}$ and has degree $d_1 \cdots d_{i-1} d_{i+1} \cdots d_{m+1}, i = 1, \ldots, m + 1$, in the coefficients of g_i.

Proof. [MS21, Theorem 4.11.]. □

[13] Notice that a polynomial f is irreducible in $\mathbb{C}[u, v, w]$ if and only if $\mathcal{V}_{\mathbb{P}}(f)$ is an irreducible variety.
[14] Notice that this is of course only defined up to multiplying with a nonzero constant

When we compute the dual of a homogeneous degree 3 polynomial f we have an ideal with three generators

$$\begin{aligned} g_1 &:= f, \\ g_2 &:= f_y - v f_x, \\ g_3 &:= f_z - w f_x, \end{aligned}$$

and need to eliminate two variables y, z, so indeed $f^\vee = \text{Res}(g_1, g_2, g_3)$. But now the second part of the theorem allows us to get the degree of $F_1, \dots, F_{27}$ in the coefficients of g_1, g_2, g_3.
First let $a_0, \dots, a_9$ be the 10 coefficients of $g_1 = f$, let $b_0, \dots, b_5$ and $c_0, \dots, c_5$ be the coefficients of g_2 and g_3 and, then $\text{Res}(g_1, g_2, g_3)$ is a polynomial in the variables $a_0, \dots, a_9, b_0, \dots, b_5, c_0, \dots, c_5$. In y, z, the degrees of our starting polynomials are

$$\deg(g_1) = 3, \ \deg(g_2) = 2, \ \deg(g_3) = 2,$$

so by the last theorem we conlude, that $\text{Res}(g_1, g_2, g_3)$ has

- degree $2 \cdot 2 = 4$ in the coefficients $a_0, \dots, a_9$,
- degree $3 \cdot 2 = 6$ in the coefficients $b_0, \dots, b_5$,
- degree $3 \cdot 2 = 6$ in the coefficients $c_0, \dots, c_5$.

The total degree in all of the coefficients $a_0, \dots, a_9, b_0, \dots, b_5, c_0, \dots, c_6$ can therefore be at most $4 + 6 + 6 = 16$. However, the coefficients $b_i, c_i, i = 0, \dots, 5$ are really polynomials in $a_0, \dots, a_9, v, w$ that are of degree one (hence linear) in $a_0, \dots, a_9$. Now clearly we can not increase the degree of a polynomial strictly, just by substituting the variables with linear forms. This means that in the coefficients $a_0, \dots, a_9$ of f, the resultant $\text{Res}(g_1, g_2, g_3)$ has degree at most 16. As the polynomials $F_0, \dots, F_{27}$ appear as coefficients in $\text{Res}(g_1, g_2, g_3)$ in front of monomials in v, w, we can in particular conclude that $\deg(F_i) \leq 16$ for $i = 0, \dots, 27$.

Remark 4.1.4. From the theorem we can only get an upper bound, as the theorem gives us degree 16 only in the coefficients $a_0, \dots, a_9, b_0, \dots, b_5, c_0, \dots, c_6$. When we substitute linear polynomials in $a_0, \dots, a_9$ for the variables $b_0, \dots b_5, c_0, \dots, c_5$, the degree might strictly decrease. As an easy example consider the polynomial $f(x, y) = 5x - y \in \mathbb{C}[x, y]$, that has degree one in x, y, but when we substitute the linear form $y = 5x - 3$ we get $f(x, 5x - 3) = 3$, which is of degree zero.

4.2. Computing the degree of $\cdot^\vee$

To actually get the degree of $F_0, \dots, F_{27}$ we needed to actually carry out algorithm 4.1.2 with completely variable coefficients $a_0, \dots, a_9$ of the polynomial f. The problem hereby is that step 3 of the algorithm requires us to eliminate two variables from an ideal in a polynomial ring over 14 variables. For fixed coefficients, we just had to eliminate 2 out of 4 variables, which was easily possible with a computer, but the large number of extra variables increases the computational expenses drastically. To better understand this, we shortly recall the necessary computational effort, when eliminating variables from an ideal.

Theorem 4.2.1. Working over $\mathbb{Q}[x_1, \ldots, x_m, x_{m+1}, \ldots, x_n]$, let $\mathcal{G}$ be a lexicographic Gröbner basis of an ideal I. Then $\mathcal{G}' := \mathcal{G} \cap \mathbb{Q}[x_1, \ldots, x_m]$ is a Gröbner basis of the elimination ideal $I \cap \mathbb{Q}[x_1, \ldots, x_m]$.

Proof. [MS21, Theorem 4.5]. □

This theorem exactly tells us how to compute the elimination ideal, namely by computing a lexicographic Gröbner basis, where the variable that we want to eliminate, are the large variables in this order.

Remark 4.2.2. When looking at the proof of 4.2.1, one can see that we can replace the lexicographic order by any weighted order, where the variables that we want to eliminate are weighted much more than the other variables. This weight difference needs to be large enough to ensure that every polynomial in $\mathcal{G}$, that contains a variable that we want to eliminate, must also use such a variable in its initial monomial. We can do this by choosing the weights higher than the degree of any polynomial that might appear during the Gröbner basis computation.

We can compute a Gröbner basis using Buchberger's algorithm, but the complexity of this algorithm increases rapidly, when increasing the number of variables.[15] This makes it impossible for us to compute the dual of a polynomial with completely variable coefficients on a computer, by using algorithm 4.1.2.
So since we can not use 10 new variables for the coefficients, let us use only one new variable c and fix all the other coefficients of f to be some arbitrary numbers. The degree of the resulting polynomial in c can not be greater than the total degree of the $F_i, i = 0, \ldots, 27$ in all coefficients $a_0, \ldots, a_9$, since we just substitute some numbers for a few of the coefficients, while keeping only one of them variable. This gives us at least some lower bounds on the desired total degree of the $F_0, \ldots, F_{27}$.

Computation 4.2.3. Starting with $f = x^3 + y^3 + z^3 + cxyz \in \mathbb{C}[x, y, z, c]$ we can get a first lower bound of 7. Indeed the dual curve is given by

$$\begin{aligned} f^\vee = {} & u^2v^2w^2c^7 + 4\,u^3v^3c^6 + 4\,u^3w^3c^6 + 4\,v^3w^3c^6 + 18\,u^4v\,w\,c^5 + 18\,u\,v^4w\,c^5 \\ & + 18\,u\,v\,w^4c^5 + 135\,u^2v^2w^2c^4 - 27\,u^6c^3 + 162\,u^3v^3c^3 - 27\,v^6c^3 + 162\,u^3w^3c^3 \\ & + 162\,v^3w^3c^3 - 27\,w^6c^3 + 486\,u^4v\,w\,c^2 + 486\,u\,v^4w\,c^2 + 486\,u\,v\,w^4c^2 \\ & + 2916\,u^2v^2w^2c - 729\,u^6 + 1458\,u^3v^3 - 729\,v^6 + 1458\,u^3w^3 + 1458\,v^3w^3 - 729\,w^6 \end{aligned}$$

Here we can also observe one more problem, when carrying out the calculations on a computer. While our starting polynomial had only four terms and all coefficients were ones (or zeroes), the coefficients of the dual curve are much larger numbers. This becomes

[15] For a detailed discussion of the complexity of Buchberger's algorithm see [DMY95, Theorem 11]

even more apparent, when we start with a polynomial with more nonzero coefficients, like for example

$$f = 2x^3 + 6xy^2 + xyz + 4x^2y + 8x^2z + 5y^3 + xz^2 + 6y^2z + 7yz^2 + cz^3.$$

All coefficients here are numbers form one to ten, but the coefficients of the dual polynomial have up to 16 digits. Handling these large numbers increases the computational effort, but in the end we are not really interested in these big coefficients. The dual curve is only defined up to multiplying with a nonzero constant, so we do not need to know the actual coefficients of $f^\vee$, we only need to know which coefficients are zero and which are nonzero.

A simple way to get around this problem is by changing the field over which we do the computations to be a finite field. By replacing $\mathbb{Q}$ with the finite field $F := \mathbb{Z}/101/\mathbb{Z}$, all coefficients that will appear at some point of the calculation will be numbers from 0 to 100 and therefore much cheaper to store and process. The only disadvantage is that some coefficients, that were nonzero before might now become zero as they were divisible by 101. On the other hand coefficients that were zero over $\mathbb{Q}$ or of course zero over F. This means that, working over F, we might lose some monomials, but we can not get any new terms. In particular it can only decrease the total degree of $F_0, \dots, F_{27}$, when we compute them over F instead of computing them over $\mathbb{Q}$.

Another problem that we have, when only replacing one coefficient a_0 of f with a new variable c, is that we will only be able to retrieve information about the degree of $F_0, \dots, F_{27}$ in a_0, but we want to get the total degree in all of the $a_0, \dots, a_9$. We can approach this problem by replacing all of the coefficients $a_0, \dots, a_9$ with linear terms in the new variable c. As we already described in 4.1.4, this can cause terms of f (or any polynomial appearing in the process of computing $f^\vee$) to cancel out and therefore the degree of $f^\vee$ in c can be lower (but not higher), than the total degree in $a_0, \dots, a_9$ would be.

Computation 4.2.4. In macaulay2 we generate 10 random linear terms in the new variable c as coefficients of our starting polynomial f. More precisely we set $a_i = \text{random}(101) \cdot c + \text{random}(101), i = 0, \dots, 9$, where the macaulay2 command "random(101)" outputs a random number from 0 to 100. We get the following polynomial:

$$\begin{aligned} f = {} & 35\,x^3c + 33\,x^2y\,c + 39\,x\,y^2c - 33\,y^3c + 45\,x^2z\,c - 45\,x\,y\,z\,c - y^2z\,c - 50\,x\,z^2c + 43\,y\,z^2c \\ & - 16\,z^3c - 37\,x^3 - 38\,x^2y - 14\,x\,y^2 - 11\,y^3 + 43\,x^2z + 22\,x\,y\,z + 20\,y^2z + 24\,x\,z^2 + 47\,y\,z^2 \\ & + 8\,z^3 \in F[x, y, z, c]. \end{aligned}$$

Using algorithm 4.1.2, while working over $F = \mathbb{Z}/101\mathbb{Z}$, we get a large polynomial in $F[u, v, w, c]$, which can be found in A.3. Indeed we can not only see that the result $f^\vee$ has degree 16 in c, but we can even find a monomial of the form $x^iy^jz^kc^{16}$ for any $i, j, k \in \{0, \dots, 6\}, i+j+k = 6$. As the polynomials $F_0, \dots, F_{27}$ are exactly the coefficients of $f^\vee$ when we consider it as a polynomial in the variables u, v, w, this means that the linear substitution of a_i's makes the polynomials $F_0, \dots, F_{27} \in F[a_0, \dots, a_9]$ into degree (exactly) 16 polynomials in the new variable c over F. As we discussed, this makes 16 a lower bound on the total degree of $F_0, \dots, F_{27}$ in $a_0, \dots, a_9$. However, 16 was also our theoretical upper bound, so this proves that all 28 polynomials $F_0, \dots, F_{27} \in \mathbb{C}[a_0, \dots, a_9]$

have total degree 16.

A. Appendix

A.1. Abstract Varieties

Abstract varieties generalize the definition of a variety as a zero set of some polynomials to a scheme theoretic setting, with the advantage of obtaining a much large class. Our source for any scheme related definition is [Har77, chapter II], except the definition of an abstract variety below, which for us will not be irreducible. We will now show how to assign an abstract variety to zero set of (homogeneous) polynomials and show that open subschemes and reduced closed subschemes of abstract varieties are abstract varieties itself.

Definition A.1.1. An abstract variety is a reduced, separated scheme of finite type over an algebraically closed field k.

Lemma A.1.2. (i) Let $f_1, \dots, f_m \in k[x_1, \dots, x_n]$ be polynomials, then $\mathcal{V}_{\mathbb{A}}(f_1, \dots, f_m)$ is homeomorphic to the set of closed points of the scheme $\operatorname{Spec}(k[x_1, \dots, x_n]/I)$, where $I = \operatorname{rad}(f_1, \dots, f_m)$ and this scheme is an abstract variety. This abstract variety is irreducible if and only if $\mathcal{V}_{\mathbb{A}}(f_1, \dots, f_m)$ is irreducible.

(ii) Let $f_1, \dots, f_m \in k[x_0, \dots, x_n]$ be homogeneous polynomials, then $\mathcal{V}_{\mathbb{P}}(f_1, \dots, f_m)$ is homeomorphic to the set of closed points of the scheme $\operatorname{Proj}(k[x_0, \dots, x_n]/I)$, where $I = \operatorname{rad}(f_1, \dots, f_m)$ and this scheme is an abstract variety. This abstract variety is irreducible if and only if $\mathcal{V}_{\mathbb{P}}(f_1, \dots, f_m)$ is irreducible.

Proof. The homeomorphism is described in [Har77, chapter II, 2.6.]. Notice that while in Hartshorne, varieties are irreducible, this is not used in the proof. The fact that these schemes are abstract varieties is proven in [Har77, chapter II, example 3.2.1 and proposition 4.10], with the only difference that we do not have irreducibility, but only reducedness (I is radical), so our schemes will not be Integral, but only reduced in general. □

Lemma A.1.3. Let X be an abstract variety, then the closed points of X form a dense subset.

Proof. Let $Z \subseteq X$ be any closed subset of X that contains every closed point of X. For contradiction assume $Z \neq X$, then Z^c is a nonempty open subset of X. As a scheme, X can be covered by spectra of rings, so we find some ring A, such that $Z^c \cap \operatorname{Spec}(A) \neq \emptyset$. In particular $Z^c \cap \operatorname{Spec}(A)$ is a nonempty open subset of $\operatorname{Spec}(A)$. The latter, as a topological space has a base of topology given by sets of the form $D(a) := \{p \in \operatorname{Spec}(A) : a \notin p\}, a \in A \setminus \{0\}$, so in particular we find an element $a \in A \setminus \{0\}$, such that $D(a)$ is contained in Z^c, hence disjoint from Z. However, all maximal ideals of A are closed points in $\operatorname{Spec}(A)$ and therefore also in X, so they are in Z and therefore not in $D(a)$. But then they have to contain a, which means

$$a \in \bigcap_{\substack{m \text{ maximal ideal} \\ \text{of } A}} m = \operatorname{nil}(A) = \{0\},$$

where we use that A is reduced, since X, as an abstract variety, is a reduced scheme. This contradicts the choice of a, hence $Z = X$. Therefore the closure of the set of all closed points in X is X as claimed. □

Lemma A.1.4. (i) An open subscheme of an abstract variety is an abstract variety itself.

(ii) A reduced, closed subscheme of an abstract variety is an abstract variety itself.

Proof. (i) Let X be an abstract variety over the algebraically closed field k, then we have a separated morphism $f : X \to \mathrm{Spec}(k)$ of finite type. Let $U \subset X$ be any open subscheme of X and let $i : U \hookrightarrow X$ be the corresponding open immersion. Clearly U is reduced, since any stalk of U at an element $x \in U$ is isomorphic to the stalk of x in the scheme X and hence reduced. The composition $f \circ i : U \to \mathrm{Spec}(k)$ makes U into a scheme over k. By [Har77, chapter II, 4.6] open immersions are separated and the composition of separated morphisms is separated, so $f \circ i$ makes U separated over k.
Since X is of finite type over k, we can cover X by finitely many open affines $\mathrm{Spec}(A_i), i = 1, \dots, n$, where A_i is a finitely generated k-algebra and hence a noetherian ring. Now the (finitely many) sets $\mathrm{Spec}(A_i) \cap U, i = 1, \dots, n$ cover U, so it is enough, to show that each $\mathrm{Spec}(A_i) \cap U$ can be covered by finitely many spectra of finitely generated k-algebras. As the set $\mathrm{Spec}(A_i) \cap U$ is open in $\mathrm{Spec}(A_i)$ it can be covered by affine open subschemes of the form $\mathrm{Spec}((A_i)_f)$, where $f \in A_i$ (here $(A_i)_f$ means the localization of A_i with respect to the multiplicative set $\{1, f, f^2, \dots\}$). Clearly if A_i is a finitely generated k-algebra, then so is $(A_i)_f$, as we can just add $\frac{1}{f}$ to the generators. Since $\mathrm{Spec}(A_i) \cap U$ is a subspace of the noetherian space $\mathrm{Spec}(A_i)$, it is quasi-compact, so we find a finite subcover.

(ii) Now let $j : Z \to X$ be a closed immersion that makes Z into a reduced closed subscheme of X, then by [Har77, chapter II, 4.8] closed immersions are separated and the composition of separated morphisms is separated, so $f \circ j : Z \to \mathrm{Spec}(k)$ makes Z a reduced, separated scheme over k. By [Har77, chapter II, exercise 3.13 (a) and (c)], closed immersions, aswell as compositions of morphisms of finite type, are of finite type again, so $f \circ j$ also makes Z of finite type over k.

□

Lemma A.1.5. The fibered product of irreducible abstract varieties X, Y over the same algebraically closed field k is an irreducible abstract variety aswell.

Proof. The fibered product is again separated over k by [Har77, chapter II, 4.6.], it is of finite type over k by [Har77, chapter II, exercise 3.13 (e)]. Furthermore irreducible abstract varieties are irreducible and reduced, hence integral (see [Har77, chapter II, 3.1]), so as X, Y are of finite type over k, we can cover them by finitely many open affines $\mathrm{Spec}(A_i), i = 1, \dots, n$ (for X) and $\mathrm{Spec}(B_j), j = 1, \dots, k$ (for Y), where A_i, B_j are finitely generated k algebras that are also integral domains. Now the fibered product $X \times_k Y$ by construction is covered by all $\mathrm{Spec}(A_i \otimes_k B_j), i = 1, \dots, n, j = 1, \dots, k$ and is therefore integral, hence reduced and irreducible itself (here we use the fact from commutative algebra that the tensor product of two finitely generated k-algebras that are also integral domains, is again an integral domain). □

A.2. Dimension and Degree of Ideals

Definition A.2.1. Let $I \subseteq \mathbb{C}[x_1, \dots, x_n]$ be an ideal. We define the *affine Hilbert function* to be $\mathrm{HF} : \mathbb{N} \to \mathbb{N}, q \mapsto \dim_{\mathbb{C}}(\mathbb{C}[x_1, \dots, x_n]_{\leq q}/I_{\leq q})$. The *Hilbert function* of I is then defined to be $h_I : \mathbb{N} \to \mathbb{N}, q \mapsto \mathrm{HF}_I(q) - \mathrm{HF}_I(q-1)$. There exists a polynomial called *Hilbert polynomial* HP_I, such that $h_I(q)$ and $\mathrm{HP}_I(q)$ agree for q large enough (see [MS21, theorem 1.25]).
Writing the Hilbert polynomial in the form

$$\mathrm{HP}_I(q) = \frac{g}{(d-1)!} q^{d-1} + \text{ lower order terms,}$$

we define d to be the dimension of I and g to be its degree, if HP_I is not the zero polynomial. Otherwise (if $\mathrm{HP}_I = 0$) we set the dimension of I to be zero and the degree to be the finite number $\dim(\mathbb{C}[x_1, \dots, x_n]/I)$.

Definition A.2.2. The *Krull-dimension* of a ring R is defined as the supremum over all n, such that there exists a chain of prime ideals

$$p_0 \subseteq p_1 \subseteq \cdots \subseteq p_n$$

in R.

Definition A.2.3. For an affine variety $V = \mathcal{V}_{\mathbb{A}}(f_1, \dots, f_m)$ we define the degree of V to be the degree of the ideal $I = \langle f_1, \dots, f_m \rangle \subseteq \mathbb{C}[x_1, \dots, x_n]$. For a projective variety, we define the degree as the degree of its affine cone.

Now we have dimension for varieties (just dimension as a topological space), rings and ideals. To a variety $V = \mathcal{V}_{\mathbb{A}}(f_1, \dots, f_m) \subseteq \mathbb{A}^n$ we can assign an ideal $I = \langle f_1, \dots, f_m \rangle \subseteq \mathbb{C}[x_1, \dots, x_n]$ and a coordinate ring $\mathbb{C}[x_1, \dots, x_n]/I$. Here all three definitions of dimension are compatible.

Lemma A.2.4. The dimension of a variety is the same as the dimension of its ideal and the same as the dimension of its coordinate ring.

Proof. We use the notation from above. Clearly the dimension of the affine variety V and the Krull-dimension of its coordinate ring are equal, since chains of prime ideals that contain I give chains of irredcible subvarieties of V and the other way round. The fact that the dimension of V as a variety is the dimension of the ideal I is known as "Dimension Theorem" and can be found in [CLO18, chapter 9, §3, Theorem 8]. □

Also notice that trivially the dimension of a variety agrees with the dimension of its associated scheme via A.1.2.

Lemma A.2.5. The degree of a variety V (it can be projective or affine) is the number of intersection points of V with a general codimension r subspace of $\mathbb{A}^n$ in the affine case and of $\mathbb{P}^n$ in the projective case. Here $r = \dim(V)$ must be nonzero.

Proof. The proof is sketched in [MS21, chapter 2.1.]. □

Lemma A.2.6. The Krull-dimension of a polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$ is n.

Proof. [Eis95, chapter 10, corollary 1.3 (a)]. □

Notice that by A.2.4, this also shows that $\mathbb{A}^n$ (and hence $\mathbb{P}^n$ by going to an affine flag) are n-dimensional topological spaces.

A.3. Full result of computation 4.2.4

$$-48 u^6c^{16} - 44 u^5 v c^{16} - 33 u^4v^2c^{16} + 10 u^3v^3c^{16} - 16 u^2v^4c^{16} - 30 u v^5c^{16} + v^6c^{16} + 39 u^5 w c^{16} +$$
$$35 u^4 v w c^{16} + 26 u^3v^2w c^{16} + 20 u^2v^3w c^{16} + 23 u v^4 w c^{16} - 22 v^5 w c^{16} + 42 u^4w^2c^{16} + 11 u^3 v w^2 c^{16} +$$
$$18 u^2v^2w^2c^{16} + 5 u v^3w^2c^{16} - 35 v^4w^2c^{16} + 46 u^3w^3c^{16} - 39 u^2 v w^3c^{16} + 14 u v^2w^3c^{16} - 28 v^3w^3c^{16} -$$
$$8 u^2w^4c^{16} + 18 u v w^4c^{16} - 21 v^2w^4c^{16} + u w^5c^{16} - 11 v w^5 c^{16} + 26 w^6c^{16} + 17 u^6c^{15} + 4 u^5 v c^{15} +$$
$$12 u^4v^2c^{15} + 48 u^3v^3c^{15} + u v^5c^{15} + 27 v^6c^{15} - 33 u^5 w c^{15} + 19 u^4 v w c^{15} + 35 u^3v^2 w c^{15} -$$
$$2 u^2v^3w c^{15} - 50 u v^4 w c^{15} - 25 v^5 w c^{15} + 40 u^4w^2c^{15} - 19 u^3 v w^2c^{15} + 50 u^2v^2w^2c^{15} - 25 u v^3w^2c^{15} -$$
$$2 v^4w^2c^{15} - 12 u^3w^3c^{15} - 28 u^2 v w^3c^{15} - 42 u v^2w^3c^{15} + 42 v^3w^3c^{15} + 43 u^2w^4c^{15} - 32 u v w^4c^{15} -$$
$$37 v^2w^4c^{15} + 17 u w^5c^{15} + 7 v w^5c^{15} - 15 w^6c^{15} + 39 u^6c^{14} + 14 u^5 v c^{14} + 33 u^4v^2c^{14} + 39 u^3v^3c^{14} +$$
$$14 u^2v^4c^{14} + 47 u v^5c^{14} - 36 v^6c^{14} + 4 u^5 w c^{14} + 32 u^4 v w c^{14} + 24 u^3v^2 w c^{14} - 35 u^2v^3 w c^{14} -$$
$$7 u v^4 w c^{14} + 31 v^5 w c^{14} - 47 u^4w^2c^{14} - 17 u^3 v w^2c^{14} + 39 u^2v^2w^2c^{14} + 5 u v^3w^2c^{14} - 14 v^4w^2c^{14} -$$
$$36 u^3w^3c^{14} + 7 u^2 v w^3c^{14} - 29 u v^2w^3c^{14} - v^3w^3c^{14} - 12 u^2w^4c^{14} - 38 u v w^4c^{14} + 11 v^2w^4c^{14} -$$
$$44 u w^5c^{14} - 19 v w^5c^{14} + 46 w^6c^{14} + 30 u^6c^{13} - 33 u^5 v c^{13} - 5 u^4v^2c^{13} - 23 u^3v^3c^{13} - 15 u^2v^4c^{13} -$$
$$48 u v^5c^{13} - 12 v^6c^{13} + 9 u^5 w c^{13} - 50 u^4 v w c^{13} - 49 u^3v^2 w c^{13} + 40 u^2v^3 w c^{13} + 2 u v^4 w c^{13} -$$
$$21 v^5 w c^{13} + 49 u^4w^2c^{13} + 11 u^3 v w^2c^{13} + 46 u^2v^2w^2c^{13} + 7 u v^3w^2c^{13} + 7 v^4w^2c^{13} + 19 u^3w^3c^{13} -$$
$$4 u^2 v w^3c^{13} + 48 u v^2w^3c^{13} + 35 u^2w^4c^{13} + 20 u v w^4c^{13} - 50 v^2w^4c^{13} + 7 u w^5c^{13} + 7 v w^5c^{13} -$$
$$35 w^6c^{13} - 35 u^6c^{12} + 35 u^5 v c^{12} + 2 u^4v^2c^{12} - 11 u^3v^3c^{12} - 46 u^2v^4c^{12} + 16 u v^5c^{12} - 9 v^6c^{12} +$$
$$46 u^5 w c^{12} + 29 u^4 v w c^{12} - 46 u^3v^2 w c^{12} - 19 u^2v^3 w c^{12} + 50 u v^4 w c^{12} - 11 v^5 w c^{12} + 21 u^4w^2c^{12} -$$
$$20 u^3 v w^2c^{12} + 33 u^2v^2w^2c^{12} + 44 u v^3w^2c^{12} - 44 v^4w^2c^{12} + 31 u^3w^3c^{12} + 45 u^2 v w^3c^{12} + 6 u v^2w^3c^{12} +$$
$$19 v^3w^3c^{12} + 33 u^2w^4c^{12} - 43 u v w^4c^{12} - 28 v^2w^4c^{12} - 35 u w^5c^{12} - 27 v w^5c^{12} + 2 w^6c^{12} -$$
$$8 u^6c^{11} - 47 u^5 v c^{11} + 14 u^4v^2c^{11} - 39 u^3v^3c^{11} + u^2v^4c^{11} + 26 u v^5c^{11} - 39 v^6c^{11} - 22 u^5 w c^{11} -$$
$$47 u^4 v w c^{11} - 31 u^3v^2 w c^{11} - 19 u^2v^3 w c^{11} + 23 u v^4 w c^{11} - 40 v^5 w c^{11} - 22 u^4w^2c^{11} - 34 u^3 v w^2c^{11} +$$
$$38 u^2v^2w^2c^{11} + 5 u v^3w^2c^{11} - 3 v^4w^2c^{11} + 26 u^3w^3c^{11} - 15 u^2 v w^3c^{11} - 10 u v^2w^3c^{11} - 38 v^3w^3c^{11} -$$
$$6 u^2w^4c^{11} - 27 u v w^4c^{11} + 35 v^2w^4c^{11} + 22 u w^5c^{11} - 11 v w^5c^{11} - 8 w^6c^{11} - 3 u^6c^{10} + 46 u^5 v c^{10} -$$
$$47 u^4v^2c^{10} - 42 u^3v^3c^{10} - 24 u^2v^4c^{10} - 18 u v^5c^{10} + 21 v^6c^{10} - 22 u^5 w c^{10} - 41 u^4 v w c^{10} +$$
$$44 u^3v^2 w c^{10} - 16 u^2v^3 w c^{10} - 14 u v^4 w c^{10} - 31 v^5 w c^{10} + 32 u^4w^2c^{10} - 42 u^3 v w^2c^{10} + 4 u^2v^2w^2c^{10} -$$
$$2 u v^3w^2c^{10} - 11 v^4w^2c^{10} - 28 u^3w^3c^{10} + 23 u^2 v w^3c^{10} + 4 u v^2w^3c^{10} + 36 v^3w^3c^{10} + 36 u^2w^4c^{10} -$$
$$11 u v w^4c^{10} + 50 v^2w^4c^{10} + 50 u w^5c^{10} + 43 v w^5c^{10} - 21 w^6c^{10} - u^6c^9 + 32 u^5 v c^9 - 24 u^4v^2c^9 +$$
$$31 u^3v^3c^9 + 7 u^2v^4c^9 + 37 u v^5c^9 + 37 v^6c^9 - 44 u^5 w c^9 - 25 u^4 v w c^9 - 24 u^3v^2 w c^9 + 29 u^2v^3 w c^9 -$$
$$45 u v^4 w c^9 - 40 v^5 w c^9 + 23 u^4w^2c^9 - 30 u^3 v w^2c^9 + 46 u^2v^2w^2c^9 - 17 u v^3w^2c^9 + 23 v^4w^2c^9 -$$
$$6 u^3w^3c^9 - 50 u^2 v w^3c^9 - 18 u v^2w^3c^9 - 31 v^3w^3c^9 + 35 u^2w^4c^9 - 38 u v w^4c^9 + 37 v^2w^4c^9 +$$
$$39 u w^5c^9 - 3 v w^5c^9 - 39 w^6c^9 - 28 u^6c^8 - 4 u^5 v c^8 + 29 u^4v^2c^8 - 41 u^3v^3c^8 - 3 u^2v^4c^8 + 14 u v^5c^8 -$$
$$42 v^6c^8 + 25 u^5 w c^8 - 41 u^4 v w c^8 - 25 u^3v^2 w c^8 + 42 u^2v^3 w c^8 + 15 u v^4 w c^8 - 18 v^5 w c^8 -$$
$$25 u^4w^2c^8 - 32 u^3 v w^2c^8 - 12 u^2v^2w^2c^8 - 42 u v^3w^2c^8 + 33 v^4w^2c^8 + 10 u^3w^3c^8 + 18 u^2 v w^3c^8 +$$
$$39 u v^2w^3c^8 + 17 v^3w^3c^8 - 33 u^2w^4c^8 - 20 u v w^4c^8 - 49 v^2w^4c^8 + 43 u w^5c^8 - 21 v w^5c^8 + w^6c^8 -$$
$$46 u^6c^7 + 24 u^5 v c^7 - 4 u^4v^2c^7 - 46 u^3v^3c^7 - 45 u^2v^4c^7 + 42 u v^5c^7 + 8 v^6c^7 - 26 u^5 w c^7 + u^4 v w c^7 -$$
$$28 u^3v^2 w c^7 - 14 u^2v^3 w c^7 - 3 u v^4 w c^7 - 46 u^4w^2c^7 - 39 u^3 v w^2c^7 - 5 u^2v^2w^2c^7 - 18 u v^3w^2c^7 +$$
$$3 v^4w^2c^7 + 15 u^3w^3c^7 + 2 u^2 v w^3c^7 + 10 u v^2w^3c^7 + 26 v^3w^3c^7 - 19 u^2w^4c^7 - 2 u v w^4c^7 +$$

$19\,v^2w^4c^7 + 39\,u\,w^5c^7 - 17\,v\,w^5c^7 + 34\,w^6c^7 - 2\,u^6c^6 + 21\,u^5v\,c^6 - 41\,u^4v^2c^6 + 25\,u^3v^3c^6 - 34\,u^2v^4c^6 + 20\,u\,v^5c^6 - 15\,v^6c^6 - 46\,u^5w\,c^6 - 34\,u^4v\,w\,c^6 - 14\,u^3v^2w\,c^6 - 25\,u^2v^3w\,c^6 + 16\,u\,v^4w\,c^6 + 32\,v^5w\,c^6 + 25\,u^4w^2c^6 + 15\,u^3v\,w^2c^6 - 18\,u^2v^2w^2c^6 - 9\,u\,v^3w^2c^6 + 14\,v^4w^2c^6 + 25\,u^3w^3c^6 + 45\,u^2v\,w^3c^6 - 37\,u\,v^2w^3c^6 + 25\,v^3w^3c^6 + 46\,u^2w^4c^6 + 20\,u\,v\,w^4c^6 - 30\,v^2w^4c^6 - 30\,u\,w^5c^6 + 18\,v\,w^5c^6 - 13\,w^6c^6 + 33\,u^6c^5 + 28\,u^5v\,c^5 - 39\,u^4v^2c^5 + 31\,u^3v^3c^5 - 45\,u^2v^4c^5 - 18\,u\,v^5c^5 - 27\,v^6c^5 - 2\,u^5w\,c^5 - 33\,u^4v\,w\,c^5 - 11\,u^3v^2w\,c^5 - 19\,u^2v^3w\,c^5 + 47\,u\,v^4w\,c^5 + 49\,v^5w\,c^5 + 7\,u^4w^2c^5 + 30\,u^3v\,w^2c^5 + 37\,u^2v^2w^2c^5 + 7\,u\,v^3w^2c^5 + 18\,v^4w^2c^5 - 33\,u^3w^3c^5 - 33\,u^2v\,w^3c^5 + 46\,u\,v^2w^3c^5 - 9\,v^3w^3c^5 - 33\,u^2w^4c^5 + 42\,u\,v\,w^4c^5 - 39\,v^2w^4c^5 - 21\,u\,w^5c^5 + 22\,v\,w^5c^5 - 11\,w^6c^5 - 23\,u^6c^4 - 48\,u^5v\,c^4 - 40\,u^4v^2c^4 - 7\,u^3v^3c^4 - 33\,u^2v^4c^4 + 40\,u\,v^5c^4 - 30\,v^6c^4 - 6\,u^5w\,c^4 + 4\,u^4v\,w\,c^4 + 26\,u^3v^2w\,c^4 - 25\,u^2v^3w\,c^4 + 38\,u\,v^4w\,c^4 + 24\,v^5w\,c^4 - 45\,u^4w^2c^4 + 5\,u^3v\,w^2c^4 - 14\,u^2v^2w^2c^4 - 34\,u\,v^3w^2c^4 + 8\,v^4w^2c^4 - 49\,u^3w^3c^4 + 10\,u^2v\,w^3c^4 - u\,v^2w^3c^4 - 18\,v^3w^3c^4 - 29\,u^2w^4c^4 + 24\,u\,v\,w^4c^4 + 15\,v^2w^4c^4 + 38\,u\,w^5c^4 - 49\,v\,w^5c^4 - 15\,w^6c^4 + 20\,u^6c^3 - 7\,u^5v\,c^3 + 10\,u^4v^2c^3 - 35\,u^3v^3c^3 - 42\,u^2v^4c^3 + 37\,u\,v^5c^3 - 37\,v^6c^3 + 32\,u^5w\,c^3 + 16\,u^3v^2w\,c^3 - 22\,u^2v^3w\,c^3 - 11\,u\,v^4w\,c^3 + 15\,v^5w\,c^3 + u^4w^2c^3 - 7\,u^3v\,w^2c^3 - 22\,u^2v^2w^2c^3 + 25\,u\,v^3w^2c^3 - 23\,v^4w^2c^3 - 26\,u^3w^3c^3 - 26\,u^2v\,w^3c^3 - 45\,u\,v^2w^3c^3 - 9\,v^3w^3c^3 + 48\,u^2w^4c^3 + 45\,u\,v\,w^4c^3 - 4\,v^2w^4c^3 + 26\,u\,w^5c^3 - 3\,v\,w^5c^3 - 27\,w^6c^3 + 49\,u^6c^2 + 15\,u^5v\,c^2 + 47\,u^4v^2c^2 + 31\,u^3v^3c^2 + 11\,u^2v^4c^2 - 24\,u\,v^5c^2 + 29\,v^6c^2 + 39\,u^5w\,c^2 + 32\,u^4v\,w\,c^2 + 15\,u^3v^2w\,c^2 - 5\,u^2v^3w\,c^2 - 26\,u\,v^4w\,c^2 + 50\,v^5w\,c^2 - 43\,u^4w^2c^2 + 37\,u^3v\,w^2c^2 - 49\,u^2v^2w^2c^2 - 14\,u\,v^3w^2c^2 - 29\,v^4w^2c^2 - 28\,u^3w^3c^2 + 21\,u^2v\,w^3c^2 + 47\,u\,v^2w^3c^2 - 39\,v^3w^3c^2 - 28\,u^2w^4c^2 + 45\,u\,v\,w^4c^2 - 13\,v^2w^4c^2 + 22\,u\,w^5c^2 - 27\,v\,w^5c^2 + 30\,w^6c^2 - 6\,u^6c - 27\,u^5v\,c + 17\,u^4v^2c - 50\,u^3v^3c - 24\,u\,v^5c + 46\,v^6c + 42\,u^5w\,c - 24\,u^4v\,w\,c - 10\,u^3v^2w\,c + 43\,u^2v^3w\,c + 22\,u\,v^4w\,c + 23\,v^5w\,c + 45\,u^4w^2c + 6\,u^3v\,w^2c - 38\,u^2v^2w^2c - 3\,u\,v^3w^2c - 24\,v^4w^2c - 11\,u^2v\,w^3c + 7\,u\,v^2w^3c + 22\,v^3w^3c - u^2w^4c + 11\,u\,v\,w^4c - 22\,v^2w^4c + 48\,u\,w^5c - 40\,v\,w^5c + 42\,w^6c + 31\,u^6 - 45\,u^5v + 25\,u^4v^2 + 24\,u^3v^3 + 17\,u^2v^4 + v^6 - 14\,u^5w + 16\,u^4v\,w + 46\,u^3v^2w + 36\,u^2v^3w + 12\,u\,v^4w + 50\,v^5w - 34\,u^4w^2 - 3\,u^3v\,w^2 - 34\,u^2v^2w^2 - 18\,u\,v^3w^2 + 47\,v^4w^2 + 7\,u^3w^3 + 36\,u^2v\,w^3 + 45\,u\,v^2w^3 - 23\,v^3w^3 - 40\,u^2w^4 + 38\,u\,v\,w^4 - 4\,v^2w^4 + 47\,u\,w^5 - 44\,v\,w^5 - 19\,w^6$

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Declaration of Authorship

I declare that the submitted thesis

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is my own unaided work. Direct ir indirect sources are acknowledged as references. This thesis was not previously presented to another examination board and has not been published before.

Konstanz, July 18, 2022

Julian Weigert