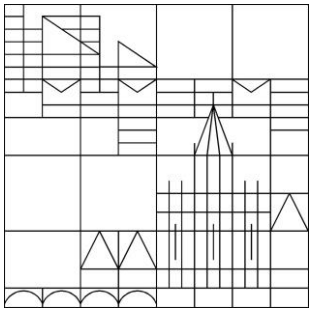

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GLOBAL EXISTENCE VERSUS BLOW-UP FOR MULTI-D HYPERBOLIZED COMPRESSIBLE NAVIER-STOKES EQUATIONS

YUXI HU AND REINHARD RACKE

Abstract. We consider the non-isentropic compressible Navier-Stokes equations in two or three space dimensions for which the heat conduction of Fourier's law is replaced by Cattaneo's law and the classical Newtonian flow is replaced by a revised Maxwell flow. We show that a physical entropy exists for this new model. For two special cases, we show the global well-posedness of solutions with small initial data and the blow-up of solutions in finite time for a class of large initial data. Moreover, for vanishing relaxation parameters, the solutions (if it exists) are shown to converge to solutions of the classical system.

Keywords: compressible Navier-Stokes equations; Cattaneo's law; Maxwell flow; Global solutions; blow-up; relaxation limit

AMS classification code: 35 L 60, 35 B 44

1. Introduction

The classical compressible Navier-Stokes equations in $\mathbb{R}^n \times \mathbb{R}^+$, $n = 2, 3$, are given by

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p = \operatorname{div}(S), \\ \rho \partial_t e + \rho u \cdot \nabla e + p \operatorname{div} u + \operatorname{div} q = S : \nabla u, \end{cases} \tag{1.1}$$

with the constitutive law for a Newtonian fluid,

$$S = \mu \left(\nabla u + \nabla u^T - \frac{2}{n} \operatorname{div} u I_n \right) + \lambda \operatorname{div} u I_n \tag{1.2}$$

and heat conduction given by Fourier's law,

$$q = -\kappa \nabla \theta. \tag{1.3}$$

Here, the functions $\rho, u, e, p, S, q, \theta$ denote the fluid density, velocity, specific internal energy per unit mass, pressure, stress tensor, heat flux and temperature, respectively. I_n denotes the identity matrix in $\mathbb{R}^n$. μ, λ are the shear and the bulk viscosity constant, respectively. κ is the constant heat conduction coefficient.

By combining Newton'saw of viscosity and Hooke's law of elasticity, one obtains the following constitutive equation

$$\tau \dot{S} + S = \mu \left(\nabla u + \nabla u^T - \frac{2}{n} \operatorname{div} u I_n \right) + \lambda \operatorname{div} u I_n, \tag{1.4}$$

where $\dot{S} = \partial_t S + u \cdot \nabla S$, see [38]. The positive parameter τ is the relaxation time describing the time lag in the response of the stress tensor to velocity gradient. A fluid obeying equation (1.4) is called Maxwell flow, see also [26]. We should mention that, even for a simple fluid, the effect of relaxation can not always be neglected, see [28] with the experiments of high-frequency (20GHZ)

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vibration of nanoscale mechanical devices immersed in water-glycerol mixtures. In 2014, Yong [38] proposed a new model in which he divided the stress tensor S into $S_1 + S_2 I_n$ and did relaxation for S_1 and S_2 in the following form

$$\tau_2 \partial_t S_1 + S_1 = \mu(\nabla u + \nabla u^T - \frac{2}{n}\mathrm{div} u I_n), \tag{1.5}$$

$$\tau_3 \partial_t S_2 + S_2 = \lambda \mathrm{div} u. \tag{1.6}$$

Note that there is no quadratic term $u \cdot \nabla S_1$ and $u \cdot \nabla S_2$ in (1.5) resp. (1.6), and and thus does not have the property of Galilean invariance. The constitutive equations (1.5)-(1.6) are called revised Maxwell's law. A similar revised Maxwell model was considered by Chakraborty and Sader [3] for a compressible viscoelastic fluid, where they show that this model provides a general formalism with which to characterize the fluid-structure interaction of nanoscale mechanical devices vibrating in simple liquids.

On the other hand, although Fourier's law (1.3) plays an important role in experimental and applied physics, it has the drawbacks of an inherent infinite propagation speed of signals. In order to overcome this paradox, Cattaneo [2] proposed the following constitutive equation for q,

$$\tau_1 \partial_t q + q + \kappa \nabla \theta = 0, \tag{1.7}$$

which gives rise to heat waves with finite propagation speed. Here $\tau_1 > 0$ is the relaxation time.

In view of the above considerations, we investigate the following system in $\mathbb{R}^n \times \mathbb{R}_+$,

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, \\ \rho \partial_t u + \rho u \cdot \nabla u + \nabla p = \mu \mathrm{div}(\nabla u + \nabla u^T - \frac{2}{n}\mathrm{div} u I_n) + \nabla S_2, \\ \rho \partial_t e + \rho u \cdot \nabla e + p \mathrm{div} u + \mathrm{div} q = \mu(\nabla u + (\nabla u)^T - \frac{2}{n}\mathrm{div} u I_n) : \nabla u + S_2 \mathrm{div} u, \\ \tau_1(\partial_t q + u \cdot \nabla q) + q + \kappa \nabla \theta = 0, \\ \tau_3(\partial_t S_2 + u \cdot \nabla S_2) + S_2 = \lambda \mathrm{div} u, \end{cases} \tag{1.8}$$

where we have taken $\tau_2 = 0$ in (1.5). That is, we do not consider a relaxation in S_1. This case is mathematically not yet accessible, even locally. We consider a Galilean invariant form of (1.6). Besides, we consider the Galilean invariance form of Cattaneo's law $(1.8)_4$ which is proposed by Christov and Jordan [6].

We investigate the Cauchy problem to system (1.8) for the functions

$$(\rho, u, \theta, q, S_2) : \mathbb{R}^n \times [0, +\infty) \to \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}$$

with initial condition

$$(\rho(x,0), u(x,0), \theta(x,0), q(x,0), S_2(x,0)) = (\rho_0, u_0, \theta_0, q_0, S_{20}). \tag{1.9}$$

We are interested in the local and global well-posedness for small data, as well as in a blow-up for large data, both for the cases $\mu > 0$ resp. $\mu = 0$. The latter is not only motivated because it is mathematically accessible with respect to local existence and blow-up, but also with a physical background. In fact, there are recent studies determining the volume viscosity of a variety of gases which were found to be much larger (factor 10^4) than the corresponding shear viscosities, see [32].

In order to close the system (1.8), the equations for the thermodynamic variables p and e should be given. In this paper, we assume that

$$e = C_v \theta + \frac{\tau_1}{\kappa \rho \theta} q^2 + \frac{\tau_3}{2\lambda \rho} S_2^2, \tag{1.10}$$

$$p = R\rho\theta - \frac{\tau_1}{2\kappa\theta} q^2 - \frac{\tau_3}{2\lambda} S_2^2, \tag{1.11}$$

with positive constants C_v, R denoting the heat capacity at constant volume and the gas constant, respectively, such that they satisfy the thermodynamic equation

$$\rho^2 e_\rho = p - \theta p_\theta.$$

The dependence on q^2 term of the internal energy is indicated in paper [7], where they rigorously prove that such constitutive equations are consistent with the second law of thermodynamics if and only if one use the relaxation equation $(1.8)_4$, see also [4, 8, 35]. Since we also consider a relaxation for the stress tensor S_2, it is motivated, naturally, by [7] that the internal energy should also depend on S_2 in a quadratic form. Indeed, under the above constitutive laws, we show in Section 2 that there exists an entropy for the system (1.8), which implies the compatibility with the second law of thermodynamics.

For classical compressible Navier-Stokes equations, the case $\tau_1 = \tau_3 = 0$ in (1.8), there are lots of results concerning the local and global well-posedness of strong and/or weak solutions because of its physical importance and mathematical challenges. In particular, the local existence and uniqueness of smooth solutions was established by Serrin [31] and Nash [27] for initial data far away from vacuum. Later, Matsumura and Nishida [25] got global smooth solutions for small initial data without vacuum. For large data, Xin [37], Cho and Jin [5] showed that smooth solutions must blow up in finite time if the initial data has a vacuum state. See [11, 12, 23, 20, 21, 9] for global existence of weak solutions.

Neglecting the quadratic nonlinear terms $u \cdot \nabla q$ and $u \cdot \nabla S_2$ in $(1.8)_4$ resp. $(1.8)_5$, the cases $\tau_1 > 0, \tau_2 = \tau_3 = 0$ (Cattaneo's law) and $\tau_1 = 0, \tau_2 > 0, \tau_3 > 0$ (revised Maxwell's law) have been studied in $\mathbb{R}^n$, $n \geq 2$, respectively, in our papers [15, 16]. For the *one-dimensional* case, we had considered the relaxation both for q and S with Galilean invariance form. In [17], we showed the global existence of smooth solutions with small initial data and convergence to classical system as relaxation parameter goes to zero. In our paper with Wang [18], a blow-up result for large data was shown, hereby also yielding an interesting example, where the relaxed and the non-relaxed system are close to each other on the linearized level, globally for small data for the nonlinear system, and on any finite time horizon for the nonlinear one, but differ qualitatively for large data.

However, there are few results by considering both relaxations for q and S in the *multi-dimensional* case. The aim of this paper is to solve this problem for two special cases with $\tau_1 > 0, \tau_2 = 0, \tau_3 > 0$ and $\mu > 0$ resp. $\mu = 0$.

Let $G := \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$ denotes the physical state spaces of the unknowns $(\rho, u, \theta, q, S_2)$. Let G_0 and G_1 be any convex compact subset of G such that $G_0 \subset\subset G_1 \subset\subset G$. The following theorem is considered under the assumption $\mu > 0$. We have global existence for small data.

Defining the energy term

$$E(t) := \sup_{0 \leq \tau \leq t} \|(\rho - 1, u, \theta - 1, q, S_2)(\tau, \cdot)\|_{H^3}^2 + \int_0^t \left(\|(\nabla\rho, \nabla\theta)\|_{H^2}^2 + \|(q, S_2)\|_{H^3}^2 + \|\nabla u\|_{H^3}^2 \right) \mathrm{d}t \tag{1.12}$$

and

$$E_0 := E(0),$$

we will prove first a global existence theorem for small data,

Theorem 1.1. *Let $\tau_1 > 0, \tau_2 = 0, \tau_3 > 0$ and $\mu > 0$. Suppose for the initial data*

$$(\rho_0 - 1, u_0, \theta_0 - 1, q_0, S_{20}) \in H^3.$$

Then, there exists a small constant $\delta > 0$ such that if $E_0 < \delta$, then the initial value problem (1.8), (1.9) has a unique solution $(\rho, u, \theta, q, S_2)$ globally in time such that $(\rho - 1, u, \theta - 1, q, S_2) \in C(0, +\infty; H^3)$, $(\nabla\rho, \nabla\theta) \in L^2(0, +\infty; H^2)$, $\nabla u \in L^2(0, +\infty; H^3)$, $(q, S_2) \in L^2(0, +\infty; H^3)$.

For any $t > 0$ we have

$$\|(\rho - 1, u, \theta - 1, q, S_2)\|_{H^3}^2 + \int_0^t \left(\|(\nabla\rho, \nabla\theta)\|_{H^2}^2 + \|\nabla u\|_{H^3}^2 + \|(q, S_2)\|_{H^3}^2 \right) \mathrm{d}t \leq C E_0^2, \tag{1.13}$$

where C is a constant being independent of t and of the initial data. Moreover, the solution decays in the sense

$$\|\nabla(\rho, u, \theta, q, S_2)\|_{L^2} \to 0 \quad \text{as } t \to \infty. \tag{1.14}$$

For $\mu = 0$, the system (1.8) is reduced to a purely hyperbolic one with zero-order damping terms

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \rho \partial_t u + \rho u \cdot \nabla u + \nabla p = \nabla S_2, \\ \rho \partial_t e + \rho u \cdot \nabla e + p \operatorname{div} u + \operatorname{div} q = S_2 \operatorname{div} u, \\ \tau_1(\partial_t q + u \cdot \nabla q) + q + \kappa \nabla \theta = 0, \\ \tau_3(\partial_t S_2 + u \cdot \nabla S_2) + S_2 = \lambda \operatorname{div} u. \end{cases} \tag{1.15}$$

The existence of solutions to the system (1.15) with initial data (1.9), even locally, is not immediately clear, since it is not symmetric nor strictly hyperbolic (nor hyperbolic-parabolic). By carefully calculating the eigenvalues and eigenvectors of the corresponding matrix in the associated first-order system, we show that (1.15) is a *constantly hyperbolic* system, and thus has a local solution. Furthermore, we show in the following theorem that smooth solutions can not exist globally for a class of large initial data.

We define some useful averaged quantities, as in [18]:

$$F(t) := \int_{\mathbb{R}^n} x \cdot \rho(x,t) u(x,t) \mathrm{d}x, \tag{1.16}$$

$$G(t) := \int_{\mathbb{R}^n} (\mathcal{E}(x,t) - \bar{\mathcal{E}}) \mathrm{d}x, \tag{1.17}$$

where $\mathcal{E}(x,t) := \rho(e + \frac{1}{2}u^2)$ is the total energy and $\bar{\mathcal{E}} := \bar{\rho}(\bar{e} + \frac{1}{2}\bar{u}^2) = C_v$.

Theorem 1.2. *Let $(\rho, u, \theta, q, S_2)$ be the local solution to (1.15) on $[0, T_0)$ with initial data (1.9) (given by Theorem 4.1 below). Assume that the initial data $(\rho_0 - 1, u_0, \theta_0 - 1, q_0, S_{20})$ are compactly supported in a ball $B_0(M)$ with radius $M > 0$. Moreover, we assume that*

$$G(0) > 0, \tag{1.18}$$

$$1 < \gamma := 1 + \frac{R}{C_v} < \frac{5}{3} \tag{1.19}$$

Then, there exists u_0 satisfying

$$F(0) > \max\left\{ \frac{128\sigma \max \rho_0}{3(5-3\gamma)}, \frac{8\sqrt{\pi \max \rho_0}}{\sqrt{3(5-3\gamma)}} \right\} M^4, \tag{1.20}$$

such that the length T_0 of the maximal interval of existence of a smooth solution $(\rho, u, \theta, q, S_2)$ of (1.15) is finite, provided the compact support of the initial data is sufficiently large.

Summarizing we present first results for some multi-dimensional hyperbolic compressible Navier-Stokes equations including local well-posedness, global well-posedness for small data and blow-up for large data, using a variety of techniques to overcome the technical difficulties.

The paper is organized as follows. In Section 2 we derive an entropy equation for system (1.8) and present some preliminary lemmas. The global existence result Theorem 1.1 is proved in Section 3. In Section 4 the local existence and the blow-up result for large data, Theorem 1.2, are proved. In Section 5 we give a remark on the singular limit as $\tau \to 0$.

We introduce some notation. $W^{m,p} = W^{m,p}(\mathbb{R}^n)$, $0 \le m \le \infty$, $1 \le p \le \infty$, denotes the usual Sobolev space with norm $\|\cdot\|_{W^{m,p}}$. H^m and L^p stand for $W^{m,2}(\mathbb{R}^n)$ resp. $W^{0,p}(\mathbb{R}^n)$. For $m \times d$-matrices $B = (b_{jk})$, $M = (m_{jk})$, we denote $A : B = \sum_{j=1}^{m} \sum_{k=1}^{d} b_{jk}\, m_{jk}$, and $M^2 = M : M$. For $m \in \mathbb{N}_0$ we denote by $\nabla^m v$ derivatives of v of order m.

2. Entropy equation and some preliminary inequalities

In this part, we first derive an entropy equation for system (1.8). Defining the entropy

$$\eta := C_v \ln\theta - R\ln\rho + \frac{\tau_1}{2\kappa\theta^2\rho}q^2. \tag{2.1}$$

Similar to [17], we have for a local solution

Lemma 2.1.

$$\partial_t(\rho\eta) + \mathrm{div}(\rho u\eta) + \mathrm{div}\left(\frac{q}{\theta}\right) = \frac{q^2}{\kappa\theta^2} + \frac{S_2^2}{\theta\lambda} + \frac{S_1^2}{2\mu\theta}. \tag{2.2}$$

Proof. Dividing equation $(1.8)_3$ by θ, we have

$$\begin{aligned}&\frac{\rho}{\theta}\partial_t(C_v\theta + \frac{\tau_1}{\kappa\theta\rho}q^2 + \frac{\tau_3}{2\lambda\rho}S_2^2) + \frac{\rho}{\theta}u\cdot\nabla(C_v\theta + \frac{\tau_1}{\kappa\theta\rho}q^2 + \frac{\tau_3}{2\lambda\rho}S_2^2) + R\rho\mathrm{div}u\\ &-\frac{\tau_1}{2\kappa\theta^2}q^2\mathrm{div}u - \frac{\tau_3}{2\lambda\theta}S_2^2\mathrm{div}u + \frac{\mathrm{div}q}{\theta} = \frac{S_2\mathrm{div}u}{\theta} + \frac{\mu}{\theta}(\nabla u + \nabla u^T - \frac{2}{n}\mathrm{div}uI_n) : \nabla u.\end{aligned} \tag{2.3}$$

For the term $\frac{\rho}{\theta}\partial_t(\frac{\tau_1}{\kappa\theta\rho}q^2)$, we have

$$\begin{aligned}\frac{\rho}{\theta}\partial_t(\frac{\tau_1}{\kappa\theta\rho}q^2) &= \rho\partial_t(\frac{\tau_1}{\kappa\rho\theta^2}q^2) + \frac{\tau_1\partial_t\theta}{\kappa\theta^3}q^2\\ &= \rho\partial_t(\frac{\tau_1}{\kappa\rho\theta^2}q^2) - \frac{1}{2}\partial_t(\frac{1}{\theta^2})\frac{\tau_1}{\kappa}q^2\\ &= \rho\partial_t(\frac{\tau_1}{\kappa\rho\theta^2}q^2) - \frac{1}{2}\partial_t(\frac{\tau_1}{\kappa\theta^2}q^2) + \frac{\tau_1 q\partial_t q}{\kappa\theta^2}\\ &= \rho\partial_t(\frac{\tau_1}{\kappa\rho\theta^2}q^2) - \frac{1}{2}\partial_t(\frac{\tau_1}{\kappa\theta^2}q^2) - \frac{\tau_1 q}{\kappa\theta^2}u\cdot\nabla q - \frac{q^2}{\kappa\theta^2} - \frac{q\cdot\nabla\theta}{\theta^2}.\end{aligned} \tag{2.4}$$

For the term $\frac{\rho}{\theta}u\cdot\nabla(\frac{\tau_1}{\kappa\theta\rho}q^2)$, we get

$$\begin{aligned}\frac{\rho}{\theta}u\cdot\nabla(\frac{\tau_1}{\kappa\theta\rho}q^2) &= \rho u\cdot\nabla(\frac{\tau_1}{\kappa\rho\theta^2}q^2) + u\frac{\nabla\theta}{\theta^3}\frac{\tau_1}{\kappa}q^2\\ &= \rho u\cdot\nabla(\frac{\tau_1}{\kappa\rho\theta^2}q^2) - u\cdot\nabla(\frac{1}{2}\frac{\tau_1}{\kappa\theta^2}q^2) + \frac{\tau_1 q}{\kappa\theta^2}u\cdot\nabla q.\end{aligned} \tag{2.5}$$

Then we have

$$\begin{aligned}&\frac{\rho}{\theta}\partial_t(\frac{\tau_1}{\kappa\theta\rho}q^2) + \frac{\rho}{\theta}u\cdot\nabla(\frac{\tau_1}{\kappa\theta\rho}q^2) - \frac{\tau_1}{2\kappa\theta^2}q^2\mathrm{div}u\\ &= \left[\rho\partial_t(\frac{\tau_1}{\kappa\rho\theta^2}q^2) + \rho u\cdot\nabla(\frac{\tau_1}{\kappa\rho\theta^2}q^2)\right] - \left[\frac{1}{2}\partial_t(\frac{\tau_1}{\kappa\theta^2}q^2) + \mathrm{div}(u\frac{1}{2}\frac{\tau_1}{\kappa\theta^2}q^2)\right] - \frac{q^2}{\kappa\theta^2} - \frac{q\nabla\theta}{\theta^2}\\ &= \left[\partial_t(\frac{1}{2}\frac{\tau_1}{\kappa\theta^2}q^2) + \mathrm{div}(u\frac{1}{2}\frac{\tau_1}{\kappa\theta^2}q^2)\right] - \frac{q^2}{\kappa\theta^2} - \frac{q\cdot\nabla\theta}{\theta^2}.\end{aligned} \tag{2.6}$$

On the other hand, we have

$$\begin{aligned}
&\frac{\rho}{\theta}\partial_t\left(\frac{\tau_3}{2\lambda\rho}S_2^2\right)+\frac{\rho}{\theta}u\cdot\nabla\left(\frac{\tau_3}{2\lambda\rho}S_2^2\right)-\frac{\tau_3}{2\lambda\theta}S_2^2\mathrm{div}u\\
&=\frac{\tau_3}{\theta\lambda}S_2\partial_tS_2-\frac{\tau_3S_2^2}{2\lambda\rho\theta}\partial_t\rho+\frac{\tau_3}{\theta\lambda}S_2u\cdot\nabla S_2-\frac{\tau_3S_2^2}{2\lambda\rho\theta}u\nabla\rho-\frac{\tau_3}{2\lambda\theta}S_2^2\mathrm{div}u\\
&=\frac{S_2}{\theta\lambda}\left(\tau_3(\partial_tS_2+u\cdot\nabla S_2)\right)-\frac{\tau_3S_2^2}{2\lambda\rho\theta}(\partial_t\rho+u\cdot\nabla\rho)-\frac{\tau_3}{2\lambda\theta}S_2^2\mathrm{div}u\\
&=\frac{S_2}{\theta\lambda}(-S_2+\lambda\mathrm{div}u)+\frac{\tau_3S_2^2}{2\lambda\rho\theta}\rho\mathrm{div}u-\frac{\tau_3}{2\lambda\theta}S_2^2\mathrm{div}u\\
&=-\frac{S_2^2}{\theta\lambda}+\frac{S_2\mathrm{div}u}{\theta}.
\end{aligned}\tag{2.7}$$

Moreover, notice that

$$(\nabla u+\nabla u^T-\frac{2}{n}\mathrm{div}uI_n):\nabla u=\frac{1}{2}|\nabla u+\nabla u^T-\frac{2}{n}\mathrm{div}uI_n|^2=\frac{S_1^2}{2\mu^2},\tag{2.8}$$

since the matrix $\nabla u+\nabla u^T-\frac{2}{n}\mathrm{div}uI_n$ is symmetric and traceless. Therefore, we derive from (2.3), (2.6), (2.7), (2.8) the following equality

$$\begin{aligned}
\partial_t(C_v\rho\ln\theta+\frac{\tau_1}{2\kappa\theta^2}q^2)+\mathrm{div}(C_v\rho u\ln\theta+u\frac{\tau_1}{2\kappa\theta^2}q^2)+R\rho\mathrm{div}u+\mathrm{div}(\frac{q}{\theta})\\
=\frac{q^2}{\kappa\theta^2}+\frac{S_2^2}{\theta\lambda}+\frac{\mu}{2\theta}|\nabla u+\nabla u^T-\frac{2}{n}\mathrm{div}I_n|^2.
\end{aligned}\tag{2.9}$$

Now, we rewrite the equation $(1.8)_1$ as

$$R\rho\partial_t(\ln\rho)+R\rho u\nabla(\ln\rho)+R\rho\mathrm{div}u=0.\tag{2.10}$$

Combining the equations (2.9) and (2.10), we get the desired result. □

Remark 2.1. *In case $\tau_1=\tau_2=\tau_3=0$, the entropy η defined in* (2.1) *is reduced to the quantity $C_v\ln\theta-R\ln\rho$ which is the entropy for classical fluid dynamics. Moreover, the entropy equation* (2.2) *is reduced to the following classical entropy equation*

$$\partial_t(\rho\eta)+\mathrm{div}(\rho u\eta)-\mathrm{div}(\frac{\kappa\nabla\theta}{\theta})=\frac{\kappa|\nabla\theta|^2}{\theta^2}+\frac{1}{\theta}\left(\frac{\mu}{2}|\nabla u+\nabla u^T-\frac{2}{n}\mathrm{div}uI_n|^2+\lambda|\mathrm{div}u|^2\right).\tag{2.11}$$

Combining the equation of conservation of energy

$$\partial_t(\rho e+\frac{1}{2}\rho u^2)+\mathrm{div}(\rho ue+\frac{\rho u}{2}|u|^2)+\mathrm{div}(pu)+\mathrm{div}q=\mathrm{div}(S:u),\tag{2.12}$$

the mass equation $(1.8)_1$ and the entropy equation (2.2), we get

Lemma 2.2.

$$\begin{aligned}
&\partial_t\left[C_v\rho(\theta-\ln\theta-1)+R(\rho\ln\rho-\rho+1)+(1-\frac{1}{2\theta})\frac{\tau_1}{\kappa\theta}q^2+\frac{1}{2}\rho u^2+\frac{\tau_3}{2\lambda}S_2^2\right]\\
&+\mathrm{div}\left[C_v\rho u(\theta-\ln\theta-1)+u(1-\frac{1}{2\theta})\frac{\tau_1}{\kappa\theta}q^2+\frac{\tau_3}{2\lambda}uS_2^2+R\rho u\ln\rho-R\rho u-\frac{q}{\theta}+\frac{1}{2}\rho u|u|^2\right.\\
&\left.+pu+q-\mu u(\nabla u+\nabla u^T-\frac{2}{n}\mathrm{div}uI_n)-S_2u\right]+\frac{q^2}{\kappa\theta^2}+\frac{S_2^2}{\theta\lambda}+\frac{\mu}{2\theta}|\nabla u+\nabla u^T-\frac{2}{n}\mathrm{div}uI_n|^2=0.
\end{aligned}\tag{2.13}$$

Denoting $V := (\rho, u, \theta, q, S_2)$, and writing (2.13) as

$$\partial_t \eta_1(V) + \mathrm{div}[\zeta(V)] + \frac{q^2}{\kappa\theta^2} + \frac{S_2^2}{\theta\lambda} + \frac{S_1^2}{2\mu\theta} = 0,$$

we have a dissipative relation for the convex entropy pair (η_1, ζ) with convex η_1.

This equation will later on imply in particular the lower energy estimates of the solutions which are crucial to get the global existence of solutions for system (1.8). Next, we present some inequalities which is frequently used in the proof of our main results.

Lemma 2.3. *In space dimensions $n = 2, 3$ we have the standard Sobolev imbeddings*
i) $H^2 \hookrightarrow L^\infty$.

ii) $H^1 \hookrightarrow L^p$, for $2 \le p \le 6$.

The following Moser-type inequalities will be used in subsequent sections and can be found as a standard tool for example in [24, 30].

Lemma 2.4. *(i) Let $r, m, n \in \mathbb{N}$, $1 < p \le \infty$, $h \in C^r(\mathbb{R}^m)$, $B := \|h\|_{C^r(\overline{B(0,1)})}$. Then there is a constant $C = C(r, m, n, p) > 0$ such that for all $w = (w_1, \dots, w_m) \in W^{r,p}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ with $\|w\|_{L^\infty} \le 1$ the inequality*

$$\|\nabla^r h(w)\|_{L^p} \le C\, B\, \|\nabla^r w\|_{L^p} \tag{2.14}$$

holds.

(ii) Let $m \in \mathbb{N}$. Then there is a constant $C = C(m, n) > 0$ such that for all $f, g \in W^{m,2} \cap L^\infty$ and $\alpha \in \mathbb{N}_0^n, |\alpha| \le m$, the following inequalities hold:

$$\|\nabla^{|\alpha|}(fg)\|_2 \le C(\|f\|_{L^\infty}\|\nabla^m g\|_2 + \|\nabla^m f\|_2\|g\|_{L^\infty}), \tag{2.15}$$

$$\|\nabla^{|\alpha|}(fg) - f\nabla^{|\alpha|}g\|_2 \le C(\|\nabla f\|_{L^\infty}\|\nabla^{m-1} g\|_2 + \|\nabla^m f\|_2\|g\|_{L^\infty}). \tag{2.16}$$

3. Global existence for small data: proof of Theorem 1.1

We first derive the following equation for θ:

$$\rho e_\theta \partial_t\theta + (\rho u e_\theta - \frac{2q}{\theta})\nabla\theta + \theta p_\theta \mathrm{div} u + \mathrm{div} q = \frac{2}{\kappa\theta}q^2 + \frac{1}{\lambda}S_2^2 + \frac{\mu}{2}|\nabla u + \nabla u^T - \frac{2}{n}\mathrm{div} u I_n|^2. \tag{3.1}$$

Proof of ((3.1). Since

$$\begin{aligned}\rho\partial_t e &= \rho e_\theta \partial_t\theta + \rho e_\rho \partial_t\rho + \rho e_q \partial_t q + \rho e_{S_2}\partial_t S_2,\\ \rho u\cdot\nabla e &= \rho e_\theta u\cdot\nabla\theta + \rho e_\rho u\cdot\nabla\rho + \rho e_q u\cdot\nabla q + \rho e_{S_2} u\cdot\nabla S_2,\end{aligned}$$

we have

$$\begin{aligned}&\rho\partial_t e + \rho u\nabla e\\ &= \rho e_\theta\partial_t\theta + \rho e_\theta u\cdot\nabla\theta + \rho e_\rho(\partial_t\rho + u\cdot\nabla\rho) + \rho e_q(\partial_t q + u\cdot\nabla q) + \rho e_{S_2}(\partial_t S_2 + u\cdot\nabla S_2)\\ &= \rho e_\theta\partial_t\theta + \rho e_\theta u\cdot\nabla\theta - \rho^2 e_\rho \mathrm{div} u + \rho e_q(-\frac{1}{\tau_1}q - \frac{\kappa}{\tau_1}\nabla\theta) + \rho e_{S_2}(-\frac{1}{\tau_3}S_2 + \frac{1}{\tau_3}\lambda\mathrm{div} u)\\ &= \rho e_\theta\partial_t\theta + \rho e_\theta u\cdot\nabla\theta - \rho^2 e_\rho \mathrm{div} u + \rho\frac{2\tau_1}{\kappa\rho\theta}q(-\frac{1}{\tau_1}q - \frac{\kappa}{\tau_1}\nabla\theta) + \rho\frac{\tau_3}{\lambda\rho}S_2(-\frac{1}{\tau_3}S_2 + \frac{1}{\tau_3}\lambda\mathrm{div} u)\\ &= \rho e_\theta\partial_t\theta + (\rho e_\theta u - \frac{2q}{\theta})\cdot\nabla\theta - \rho^2 e_\rho\mathrm{div} u - \frac{2}{\kappa\theta}q^2 - \frac{1}{\lambda}S_2^2 + S_2\mathrm{div} u.\end{aligned}$$

By the valid thermodynamic equation $\rho^2 e_\rho = p - \theta p_\theta$ and (2.8), we derive the desired equation (3.1). □

So, we have

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, \\ \rho \partial_t u + \rho u \cdot \nabla u + \nabla p = \mu(\triangle u + \frac{n-2}{n}\nabla \mathrm{div} u) + \nabla S_2, \\ \rho e_\theta \partial_t \theta + (\rho u e_\theta - \frac{2q}{\theta})\nabla\theta + \theta p_\theta \mathrm{div} u + \mathrm{div} q = \frac{2}{\kappa\theta}q^2 + \frac{1}{\lambda}S_2^2 + \frac{\mu}{2}|\nabla u + (\nabla u)^T - \frac{2}{n}\mathrm{div} u I_n|^2, \\ \tau_1(\partial_t q + u \cdot \nabla q) + q + \kappa \nabla\theta = 0, \\ \tau_3(\partial_t S_2 + u \cdot \nabla S_2) + S_2 = \lambda \mathrm{div} u. \end{cases} \tag{3.2}$$

We first show the *local* existence of solutions in the following theorem.

Theorem 3.1. *Suppose that the initial data* $(\rho_0 - 1, u_0, \theta_0 - 1, q_0, S_{20}) \in H^3$ *and* $(\rho_0, u_0, \theta_0, q_0, S_{20}) \in G_0$. *Then, for each convex open subset* G_1 *satisfying* $G_0 \subset\subset G_1 \subset\subset G$, *there exists* $T = T(\|(\rho_0, \dots, S_{20})\|_{H^3}) > 0$, *such that the system* (3.2) *has an unique classical solution* $(\rho, u, \theta, q, S_2)$ *satisfying*

$$(\rho - 1, \theta - 1, q, S_2) \in C([0,T], H^3) \cap C^1([0,T], H^2)$$
$$u \in C([0,T], H^3) \cap C^1([0,T], H^1)$$

and

$$(\rho, u, \theta, q, S_2) \in G_1, \quad \forall (x,t) \in \mathbb{R}^3 \times [0,T].$$

Proof. First, we write the above system as a hyperbolic-parabolic system:

$$\begin{cases} A_0^1 \partial_t V + \sum\limits_{j=1}^n A_j \partial_{x_j} V = f(V, \nabla u), \\ A_0^2 \partial_t u - \sum\limits_{i=1}^n \sum\limits_{j=1}^n B_{ij} \partial_{x_i x_j} u = g(V, u, \nabla V, \nabla u), \end{cases}$$

where $V = (\rho, \theta, q, S_2)$ and

$$A_0^1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \rho e_\theta & 0 & 0 \\ 0 & 0 & \frac{\tau_1}{\kappa} & 0 \\ 0 & 0 & 0 & \tau_3 \end{pmatrix}, \quad \sum_{j=1}^n A_j \xi_j = \begin{pmatrix} u\cdot\xi & 0 & 0 & 0 \\ 0 & \rho e_\theta u \cdot \xi & \xi^T & 0 \\ 0 & \xi & u\cdot\xi & 0 \\ 0 & 0 & 0 & u\cdot\xi \end{pmatrix},$$

$$f(V, \nabla u) = \begin{pmatrix} \rho \mathrm{div} u \\ -\theta p_\theta \mathrm{div} u + \frac{2}{\kappa\theta}q^2 + \frac{1}{\lambda}S_2^2 + \frac{\mu}{2}|\nabla u + (\nabla u)^T - \frac{2}{n}\mathrm{div} u I_n|^2 \\ -\frac{q}{\kappa} \\ \lambda \mathrm{div} u - S_2 \end{pmatrix},$$

$$A_0^2 = \rho I_n, \quad \sum_{i=1}^n \sum_{j=1}^n B_{ij}\xi_i\xi_j = \mu(I_n + \frac{n-2}{n}\xi\xi^T),$$

$$g(V, u, \nabla V, \nabla u) = -\rho u \cdot \nabla u - p_\rho \nabla\rho - p_\theta \nabla\theta - p_q \nabla q - (p_{S_2} - 1)\nabla S_2,$$

for $\xi \in \mathbb{R}^n$ with $|\xi| = 1$. We observe that A_0^1, A_0^2 are symmetric and positive definite matrices, A_j is a symmetric matrix, B_{ij} are symmetric matrices, and $\sum\limits_{i=1}^n \sum\limits_{j=1}^n B_{ij}\xi_i\xi_j$ is positive definite for all $\xi \in \mathbb{S}^{n-1}$. For the equilibrium state $\bar{V} = (1,1,0,0)$, $\bar{u} = 0$, we have $f(\bar{V}, 0) = 0$ and $g(\bar{V}, \bar{u}, 0, 0) = 0$. Then the local existence theorem follows immediately by using the results of Kawashima, see [22] or the appendix in [30]. □

For the global existence for small data, we do *not* use the results of Kawashima [22] by checking the so-called Kawashima condition. Instead, we use energy estimates and the entropy equation to get the desired result. Since we have quadratic terms in the equation $(3.2)_3$, the Sobolev regularity in Kawashima [22] in $\mathbb{R}^3$ is at least H^4. Here we obtain H^3-estimates; Kawashima's results do not apply to our problem.

Proposition 3.2. *There is a constant $\delta > 0$ such that if $E(t) \le \delta$, $t \in [0,T]$, then for $t \in [0,T]$*

$$E(t) \le C\|(\rho_0 - 1, u_0, \theta_0 - 1, q_0, S_0\|_{H^3}^2 + CE(t)^{\frac{3}{2}}, \tag{3.3}$$

where C is a constant being independent of δ and of the initial data.

The global existence of solutions for small data follows from Proposition 3.2 and Theorem 3.1 immediately by continuing a local solution, if the initial data are chosen to be sufficiently small.

The following series of Lemmas is devoted to prove Proposition 3.2. First, from Lemma 2.2, we derive the following lower energy estimates for a given local solution, cp. [17] for similar arguments.

Lemma 3.3.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\Big(C_v\rho(\theta - \ln\theta - 1) + R(\rho\ln\rho - \rho + 1) + (1 - \frac{1}{2\theta})\frac{\tau_1}{\kappa\theta}q^2 + \frac{1}{2}\rho u^2 + \frac{\tau_3}{2\lambda}S_2^2\Big)\,\mathrm{d}x \\ + \int_{\mathbb{R}^3}\left(\frac{q^2}{\kappa\theta^2} + \frac{S_2^2}{\theta\lambda} + \frac{\mu}{2\theta}|\nabla u + (\nabla u)^T - \frac{2}{n}\mathrm{div}uI_n|^2\right)\mathrm{d}x = 0.\end{aligned}$$

Moreover, there exists $\delta_0 > 0$ such that if $E(t) \le \delta_0$ for $t \in [0,T]$, we have

$$\|(\rho - 1, u, \theta - 1, q, S_2)(\cdot, t)\|_{L^\infty} \le \frac{1}{4} \tag{3.4}$$

and

$$\int_{\mathbb{R}^3}\left((\rho-1)^2 + (\theta-1)^2 + u^2 + q^2 + S_2^2\right)(x,t)\mathrm{d}x + \int_0^t\int_{\mathbb{R}^3}\left(q^2 + S_2^2 + |\nabla u|^2 + |\mathrm{div}u|^2\right)\mathrm{d}x\mathrm{d}t \le CE_0. \tag{3.5}$$

In the sequel we will assume

$$E(t) \le \delta_0, \qquad \text{for } t \in [0,T].$$

Before we concentrate on the higher-order a priori estimates, by using the equations (3.2), we can prove

Lemma 3.4. *There is a small constant $\delta > 0$ such that if $E(t) \le \delta$, $t \in [0,T]$, then for $t \in [0,T]$*

$$\|(\partial_t\rho, \partial_t\theta, \partial_t q, \partial_t S_2)\|_{H^2}^2 + \|\partial_t u\|_{H^1}^2 + \int_0^t \|(\partial_t\rho, \partial_t\theta, \partial_t q, \partial_t S_2, \partial_t u)\|_{H^2}^2\mathrm{d}t \le CE(t). \tag{3.6}$$

Proof. We only give a short proof for the estimates of $\partial_t u$. The other terms can be estimated in a similar way. From equation $(3.2)_2$, we have

$$\begin{aligned}\|\partial_t u\|_{H^1} &\le \|u\cdot\nabla u + \frac{p_\rho}{\rho}\nabla\rho + \frac{p_\theta}{\rho}\nabla\theta + \frac{p_q}{\rho}\nabla q + \frac{p_{S_2}}{\rho}\nabla S_2 + \frac{1}{\rho}\nabla S_2 + \frac{\mu}{\rho}(\triangle u + \frac{n-2}{n}\nabla\mathrm{div}u)\|_{H^1} \\ &\le CE(t)^{\frac{1}{2}} + \|\frac{\mu}{\rho}(\triangle u + \frac{n-2}{n}\nabla\mathrm{div}u)\|_{H^1}.\end{aligned}$$

Since

$$\|\frac{\mu}{\rho}(\triangle u + \frac{n-2}{n}\nabla\mathrm{div}u)\|_{L^2} \le C\|u\|_{H^2} \le CE(t)^{\frac{1}{2}}$$

and

$$\begin{aligned}&\|\nabla\left(\frac{\mu}{\rho}(\triangle u+\frac{n-2}{n}\nabla\mathrm{div}u)\right)\|_{L^2}\\&\le C\left(\|\frac{1}{\rho^2}\nabla\rho\|_{L^3}\|\triangle u+\frac{n-2}{n}\nabla\mathrm{div}u\|_{L^6}+\|\frac{1}{\rho}\|_{L^\infty}\|\triangle u+\frac{n-2}{n}\nabla\mathrm{div}u\|_{H^1}\right)\\&\le C(\|\nabla\rho\|_{H^1}\|u\|_{H^3}+\|u\|_{H^3})\le CE(t)^{\frac12}.\end{aligned}$$

Thus, we have

$$\|\partial_t u\|_{H^1}^2\le CE(t).$$

On the other hand, using similar methods as above, we have

$$\|\partial_t u\|_{H^2}\le C\|(\nabla\rho,\nabla u,\nabla\theta,\nabla q,\nabla S_2,\triangle u)\|_{H^2}$$

which implies by the definition of $E(t)$ that

$$\int_0^t\|\partial_t u\|_{H^2}^2\mathrm{d}t\le CE(t).$$

□

Now, for $1\le|\alpha|\le3$, taking $\nabla\nabla^{|\alpha|-1}$ to the equations $(3.2)_1,(3.2)_3,(3.2)_4,(3.2)_5$ and $\nabla^{|\alpha|-1}$ to the equation $(3.2)_2$, we derive the following system

$$\begin{cases}\partial_t\nabla\nabla^{|\alpha|-1}\rho+u\cdot\nabla(\nabla\nabla^{|\alpha|-1}\rho)+\rho\nabla\mathrm{div}\nabla^{|\alpha|-1}u=-[\nabla\nabla^{|\alpha|-1}(u\cdot\nabla\rho)-u\cdot\nabla(\nabla\nabla^{|\alpha|-1}\rho)]\\\qquad\qquad-[\nabla\nabla^{|\alpha|-1}(\rho\mathrm{div}u)-\rho\nabla\mathrm{div}\nabla^{|\alpha|-1}u]=:f_1,\\\rho\partial_t\nabla^{|\alpha|-1}u-\mu(\triangle\nabla^{|\alpha|-1}u+\frac{n-2}{n}\nabla\mathrm{div}\nabla^{|\alpha|-1}u)+p_\rho\nabla\nabla^{|\alpha|-1}\rho+p_\theta\nabla\nabla^{|\alpha|-1}\theta-\nabla\nabla^{|\alpha|-1}S_2\\+\nabla^{|\alpha|-1}(p_q\nabla q+p_{S_2}\nabla S_2)=-[\nabla^{|\alpha|-1}(p_\rho\nabla\rho)-p_\rho\nabla\nabla^{|\alpha|-1}\rho]-[\nabla^{|\alpha|-1}(p_\theta\nabla\theta)-p_\theta\nabla\nabla^{|\alpha|-1}\theta]\\\qquad\qquad-[\nabla^{|\alpha|-1}(\rho\partial_t u)-\rho\partial_t\nabla^{|\alpha|-1}u]-\nabla^{|\alpha|-1}(\rho u\cdot\nabla u)=:f_2,\\\rho e_\theta\partial_t\nabla\nabla^{|\alpha|-1}\theta+(\rho ue_\theta-\frac{2q}{\theta})\cdot\nabla(\nabla\nabla^{|\alpha|-1}\theta)+\theta p_\theta\nabla\mathrm{div}\nabla^{|\alpha|-1}u+\nabla\mathrm{div}\nabla^{|\alpha|-1}q\\=-(\nabla\nabla^{|\alpha|-1}(\rho e_\theta\partial_t\theta)-\rho e_\theta\partial_t\nabla\nabla^{|\alpha|-1}\theta)-[\nabla\nabla^{|\alpha|-1}(\theta p_\theta\mathrm{div}u)-\theta p_\theta\nabla\mathrm{div}\nabla^{|\alpha|-1}u]\\+\nabla\nabla^{|\alpha|-1}\left(\frac{2}{\kappa\theta}q^2+\frac{1}{\lambda}S_2^2\right)-\left(\nabla\nabla^{|\alpha|-1}((\rho ue_\theta-\frac{2q}{\theta})\nabla\theta)-(\rho ue_\theta-\frac{2q}{\theta})\cdot\nabla(\nabla\nabla^{|\alpha|-1}\theta)\right)\\\qquad\qquad+\frac{\mu}{2}\nabla\nabla^{|\alpha|-1}\left(|(\nabla u+\nabla u)^T-\frac{2}{n}\mathrm{div}uI_n|^2\right)=:f_3,\\\tau_1(\partial_t\nabla\nabla^{|\alpha|-1}q+u\cdot\nabla(\nabla\nabla^{|\alpha|-1}q)+\nabla\nabla^{|\alpha|-1}q+\kappa\nabla^2\nabla^{|\alpha|-1}\theta\\\qquad\qquad=-\tau_1(\nabla\nabla^{|\alpha|-1}(u\cdot\nabla q)-u\cdot\nabla(\nabla\nabla^{|\alpha|-1}q)=:f_4,\\\tau_3(\partial_t\nabla\nabla^{|\alpha|-1}S_2+u\cdot\nabla(\nabla\nabla^{|\alpha|-1}S_2)+\nabla\nabla^{|\alpha|-1}S_2-\lambda\nabla\mathrm{div}\nabla^{|\alpha|-1}u\\\qquad\qquad=-\tau_3(\nabla\nabla^{|\alpha|-1}(u\cdot\nabla S_2)-u\cdot\nabla(\nabla\nabla^{|\alpha|-1}S_2)=:f_5.\end{cases}\tag{3.7}$$

The following lemma gives the estimates of the right-hand side of the above system.

Lemma 3.5. *For any $1\le|\alpha|\le3$, we have*

$$\begin{aligned}\|f_1\|_{L^2}&\le C(E(t))^{\frac12}\|(\nabla\rho,\nabla u)\|_{H^2},\\\|f_2\|_{L^2}&\le C(E(t))^{\frac12}(\|(\nabla\rho,\nabla\theta)\|_{H^1}+\|\nabla u\|_{H^2}+\|\partial_t u\|_{H^1}),\\\|f_3\|_{L^2}&\le C(E(t))^{\frac12}(\|(\nabla\rho,\nabla u,\nabla\theta,\nabla q,\nabla S_2)\|_{H^2}+\|\nabla u\|_{H^3}+\|\partial_t\theta\|_{H^2}),\\\|f_4\|_{L^2}&\le C(E(t))^{\frac12}\|(\nabla q,\nabla u)\|_{H^2},\qquad\|f_5\|_{L^2}\le C(E(t))^{\frac12}\|(\nabla u,\nabla S_2)\|_{H^2}.\end{aligned}$$

Proof. By Sobolev's imbedding theorem and Moser-type inequalities, we have

$$\begin{aligned}\|f_1\|_{L^2} &\le \|\nabla\nabla^{|\alpha|-1}(u\cdot\nabla\rho)-u\cdot\nabla(\nabla\nabla^{|\alpha|-1})\rho\|_{L^2}+\|\nabla\nabla^{|\alpha|-1}(\rho\mathrm{div}u)-\rho\nabla\mathrm{div}\nabla^{|\alpha|-1}u\|_{L^2}\\ &\le C\left(\|\nabla u\|_{L^\infty}\|\nabla^{|\alpha|}\rho\|_{L^2}+\|\nabla\rho\|_{L^\infty}\|\nabla^{|\alpha|}u\|_{L^2}\right)\\ &\le C(E(t))^{\frac12}\|(\nabla^{|\alpha|}\rho,\nabla^{|\alpha|}u)\|_{L^2}\le C(E(t))^{\frac12}\|(\nabla\rho,\nabla u)\|_{H^2}.\end{aligned}$$

Noting that $p_\rho=R\theta, p_\theta=R\rho+\frac{\tau_1}{2\kappa\theta^2}q^2$, we get for $|\alpha|=1$ that

$$\nabla^{|\alpha|-1}(p_\rho\nabla\rho)-p_\rho\nabla\nabla^{|\alpha|-1}\rho=0$$

and for $|\alpha|\ge 2$ that

$$\begin{aligned}\|\nabla^{|\alpha|-1}(p_\rho\nabla\rho)-p_\rho\nabla\nabla^{|\alpha|-1}\rho\|_{L^2}&\le C\|(\nabla\rho,\nabla\theta)\|_{L^\infty}\|(\nabla^{|\alpha|-1}\rho,\nabla^{|\alpha|-1}\theta)\|_{L^2}\\ &\le C(E(t))^{\frac12}\|(\nabla\rho,\nabla\theta)\|_{H^2}.\end{aligned}$$

Similarly, we have for $|\alpha|\ge 1$ that

$$\|\nabla^{|\alpha|-1}(p_\theta\nabla\theta)-p_\theta\nabla\nabla^{|\alpha|-1}\theta\|\le C(E(t))^{\frac12}\|(\nabla\rho,\nabla\theta,\nabla q)\|_{H^2}.$$

Now we estimate the term $\|\nabla^{|\alpha|-1}(\rho\partial_t u)-\rho\partial_t\nabla^{|\alpha|-1}u\|_{L^2}$. For $|\alpha|=1$, this term vanishes. For $|\alpha|=2$, we have as a typical term

$$\|\partial_{x_i}(\rho\partial_t u)-\rho\partial_t\partial_{x_i}u\|_{L^2}=\|(\partial_{x_i}\rho)\partial_t u\|_{L^2}\le(E(t))^{\frac12}\|\partial_t u\|_{L^2}.$$

For $|\alpha|=3$, we have as a typical term

$$\begin{aligned}&\|\partial_{x_i}\partial_{x_j}(\rho\partial_t u)-\rho\partial_t\partial_{x_i}\partial_{x_j}u\|_{L^2}\\ =&\|(\partial_{x_i}\partial_{x_j}\rho)\partial_t u+(\partial_{x_j}\rho)\partial_t\partial_{x_i}u+\partial_{x_i}\rho\partial_{x_j}\partial_t u\|_{L^2}\\ \le&C(\|\nabla^2\rho\|_{L^6}\|\partial_t u\|_{L^3}+\|\nabla\rho\|_{L^\infty}\|\partial_t u\|_{H^1}\\ \le&C\|\nabla\rho\|_{H^2}\|\partial_t u\|_{H^1}\le C(E(t))^{\frac12}\|\partial_t u\|_{H^1}.\end{aligned}$$

So, for $1\le|\alpha|\le 3$, we get

$$\|\nabla^{|\alpha|-1}(\rho\partial_t u)-\rho\partial_t\nabla^{|\alpha|-1}u\|_{L^2}\le C(E(t))^{\frac12}\|\partial_t u\|_{H^1}.$$

On the other hand,

$$\begin{aligned}\|\nabla^{|\alpha|-1}(\rho u\cdot\nabla u)\|_{L^2}&\le C\|\rho u\|_{L^\infty}\|\nabla^{|\alpha|}u\|_{L^2}+C\|\nabla u\|_{L^\infty}\|\nabla^{|\alpha|-1}(\rho u)\|_{L^2}\\ &\le C(E(t))^{\frac12}(\|\nabla u\|_{H^2}+\|\nabla\rho\|_{H^1}).\end{aligned}$$

Combining the above estimates, we get the estimate for f_2. The terms f_3, f_4, f_5 can be estimated in a similar way, we omit the details. This finishes the proof of Lemma 3.5. □

The next lemma gives the higher-order a priori estimates of the solutions.

Lemma 3.6. *For any $0\le t\le T$, we have*

$$\|(\nabla\rho,\nabla u,\nabla\theta,\nabla q,\nabla S_2)\|_{H^2}^2+\int_0^t(\|(\nabla q,\nabla S_2)\|_{H^2}^2+\|\nabla u\|_{H^3}^2)\mathrm{d}t\le C(E_0+E(t)^{\frac32}).\tag{3.8}$$

Proof. Multiply $(3.7)_1,(3.7)_2,(3.7)_3,(3.7)_4,(3.7)_5$ for $|\alpha|\ge 2$ by $\frac{p_\rho}{\rho}\nabla\nabla^{|\alpha|-1}\rho$, $-\triangle\nabla^{|\alpha|-1}u$, $\frac{1}{\theta}\nabla\nabla^{|\alpha|-1}\theta$, $\frac{1}{\kappa\theta}\nabla\nabla^{|\alpha|-1}q$, $\frac{1}{\lambda}\nabla\nabla^{|\alpha|-1}S_2$, respectively, – for $|\alpha|=1$ take the multipliers $\frac{p_\rho}{\rho}\nabla\rho$, $-\triangle u$, $\frac{1}{\theta}\nabla\theta$, $\frac{1}{\kappa\theta}\nabla q$,

$\frac{1}{\lambda}\nabla S_2$, respectively –, and summing up the results, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\left[\frac{p_\rho}{2\rho}|\nabla\nabla^{|\alpha|-1}\rho|^2+\frac{\rho}{2}|\nabla\nabla^{|\alpha|-1}u|^2+\frac{\rho e_\theta}{2\theta}|\nabla\nabla^{|\alpha|-1}\theta|^2+\frac{\tau_1}{2\kappa\theta}|\nabla\nabla^{|\alpha|-1}q|^2+\frac{\tau_3}{2\lambda}|\nabla\nabla^{|\alpha|-1}S_2|^2\right]\mathrm{d}x$$
$$+\int\left(\frac{1}{\kappa\theta}|\nabla\nabla^{|\alpha|-1}q|^2+\frac{1}{\lambda}|\nabla\nabla^{|\alpha|-1}S_2|^2\right)\mathrm{d}x+\mu\int(\triangle\nabla^{|\alpha|-1}u+\frac{n-2}{n}\nabla\mathrm{div}\nabla^{|\alpha|-1}u)\triangle\nabla^{|\alpha|-1}u\mathrm{d}x$$
$$=\sum_{i=1}^{4}G_i+\sum_{i=1}^{4}H_i+\sum_{i=1}^{5}F_i+D+L \tag{3.9}$$

where

$$\int\ldots\mathrm{d}x:=\int_{\mathbb{R}^n}\ldots\mathrm{d}x,$$

and

$$G_1:=\int u\cdot\nabla(\nabla\nabla^{|\alpha|-1}\rho)\cdot\frac{p_\rho}{\rho}\nabla\nabla^{|\alpha|-1}\rho\mathrm{d}x,\quad G_2:=\int(\rho ue_\theta-\frac{2q}{\theta})\cdot\nabla(\nabla\nabla^{|\alpha|-1}\theta)\cdot\frac{1}{\theta}\nabla\nabla^{|\alpha|-1}\theta\mathrm{d}x,$$
$$G_3:=\int\tau_1u\cdot\nabla(\nabla\nabla^{|\alpha|-1}q)\cdot\frac{1}{\kappa\theta}\nabla\nabla^{|\alpha|-1}q\mathrm{d}x,\quad G_4:=\int\tau_3u\cdot\nabla(\nabla\nabla^{|\alpha|-1}S_2)\cdot\frac{1}{\lambda}\nabla\nabla^{|\alpha|-1}S_2\mathrm{d}x,$$

$$H_1:=\int\left((\rho\nabla\mathrm{div}\nabla^{|\alpha|-1}u)\cdot\frac{p_\rho}{\rho}\nabla\nabla^{|\alpha|-1}\rho-p_\rho\nabla\nabla^{|\alpha|-1}\rho\cdot\triangle\nabla^{|\alpha|-1}u\right)\mathrm{d}x,$$
$$H_2:=\int\left(-p_\theta\nabla\nabla^{|\alpha|-1}\theta\cdot\triangle\nabla^{|\alpha|-1}u+\theta p_\theta\nabla\mathrm{div}\nabla^{|\alpha|-1}u\cdot\frac{1}{\theta}\nabla\nabla^{|\alpha|-1}\theta\right)\mathrm{d}x,$$
$$H_3:=\int\left(\frac{1}{\theta}\nabla\mathrm{div}\nabla^{|\alpha|-1}q\cdot\nabla\nabla^{|\alpha|-1}\theta+\kappa\nabla^2\nabla^{|\alpha|-1}\theta\cdot\frac{1}{\kappa\theta}\nabla\nabla^{|\alpha|-1}q\right)\mathrm{d}x,$$
$$H_4=\int\left(\nabla\nabla^{|\alpha|-1}S_2\cdot\triangle\nabla^{|\alpha|-1}u-\lambda\nabla\mathrm{div}\nabla^{|\alpha|-1}u\cdot\frac{1}{\lambda}\nabla\nabla^{|\alpha|-1}S_2\right)\mathrm{d}x,$$

$$F_1:=\int f_1\cdot\nabla\nabla^{|\alpha|-1}\rho\mathrm{d}x,\quad F_2:=-\int f_2\cdot\triangle\nabla^{|\alpha|-1}u\mathrm{d}x,\quad F_3:=\int f_3\cdot\nabla\nabla^{|\alpha|-1}\theta\mathrm{d}x,$$
$$F_4:=\int f_4\cdot\frac{1}{\kappa}\nabla\nabla^{|\alpha|-1}q\mathrm{d}x,\quad F_5:=\int f_5\cdot\nabla\nabla^{|\alpha|-1}S_2\mathrm{d}x,$$

$$D:=\int\frac{1}{2}\left(\partial_t\left(\frac{p_\rho}{\rho}\right)|\nabla\nabla^{|\alpha|-1}\rho|^2+\frac{1}{2}\partial_t\rho|\nabla\nabla^{|\alpha|-1}u|^2\right.$$
$$\left.+\frac{1}{2}\partial_t\left(\frac{\rho e_\theta}{\theta}\right)|\nabla\nabla^{|\alpha|-1}\theta|^2-\nabla\rho\cdot\partial_t\nabla^{|\alpha|-1}u\cdot\nabla\nabla^{|\alpha|-1}u\right)\mathrm{d}x,$$

$$L:=\int-\nabla^{|\alpha|-1}(p_q\nabla q+p_{S_2}\nabla S_2)\cdot\triangle\nabla^{|\alpha|-1}u\mathrm{d}x.$$

Now we estimate the right hand side of the equation (3.9).

$$G_1=\int-\mathrm{div}(\frac{p_\rho}{\rho}u)\cdot\frac{1}{2}|\nabla\nabla^{|\alpha|-1}\rho|^2\mathrm{d}x$$
$$\leq C\|\nabla(\rho,\theta,u)\|_{L^\infty}\int|\nabla\nabla^{|\alpha|-1}\rho|^2\mathrm{d}x\leq C(E(t))^{\frac{1}{2}}\|\nabla\rho\|_{H^2}^2\mathrm{d}x,$$

while G_i, $2\leq i\leq 4$ can be estimated similarly. So, we get

$$\sum_{i=1}^{4}G_i\leq C(E(t))^{\frac{1}{2}}\|(\nabla\rho,\nabla\theta,\nabla q,\nabla S_2)\|_{H^2}^2. \tag{3.10}$$

Integrating by part, we get

$$\begin{aligned}H_1 &= \int p_\rho \nabla\nabla^{|\alpha|-1}\rho \cdot (\nabla \mathrm{div}\nabla^{|\alpha|-1}u - \triangle\nabla^{|\alpha|-1}u)\mathrm{d}x\\ &= \int p_\rho \nabla\nabla^{|\alpha|-1}\rho \cdot \nabla\times\nabla\times(\nabla^{|\alpha|-1}u)\mathrm{d}x\\ &= \int (\nabla p_\rho \times \nabla\nabla^{|\alpha|-1}\rho)\cdot\nabla\times(\nabla^{|\alpha|-1}u)\mathrm{d}x \le CE(t)^{\frac{1}{2}}\|\nabla\rho\|_{H^2}\|\nabla u\|_{H^2}.\end{aligned}$$

Here we used the relation

$$\Delta u = \nabla \mathrm{div} u - \nabla\times\nabla\times u,$$

which also holds in two space dimensions. There, for a scalar field f the rotation is given as the vector $\nabla\times f := (\partial_{x_2}f, -\partial_{x_1}f)^T$, resp. for a vector field F as the scalar $\nabla\times F := \partial_{x_1}F_2 - \partial_{x_2}F_1$. Similarly, we have

$$\begin{aligned}H_2 &= \int p_\theta \nabla\nabla^{|\alpha|-1}\theta(\nabla \mathrm{div}\nabla^{|\alpha|-1}u - \triangle\nabla^{|\alpha|-1}u)\mathrm{d}x\\ &= \int p_\theta \nabla\nabla^{|\alpha|-1}\theta\cdot\nabla\times\nabla\times\nabla^{|\alpha|-1}u\mathrm{d}x \le CE(t)^{\frac{1}{2}}\|\nabla\theta\|_{H^2}\|\nabla u\|_{H^2}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}H_3 &= \int \frac{1}{\theta}\left(\nabla \mathrm{div}\nabla^{|\alpha|-1}q\cdot\nabla\nabla^{|\alpha|-1}\theta + \nabla^2\nabla^{|\alpha|-1}\theta\cdot\nabla\nabla^{|\alpha|-1}q\right)\mathrm{d}x\\ &= \int \frac{1}{\theta}\nabla\nabla^{|\alpha|-1}\theta\cdot(\nabla\mathrm{div}\nabla^{|\alpha|-1}q - \triangle\nabla^{|\alpha|-1}q)\\ &= \int \nabla\left(\frac{1}{\theta}\right)\cdot\nabla\nabla^{|\alpha|-1}\theta\cdot\nabla\times\nabla^{|\alpha|-1}q\mathrm{d}x \le CE(t)^{\frac{1}{2}}\|\nabla\theta\|_{H^2}\|\nabla q\|_{H^2}.\end{aligned}$$

Moreover,

$$\begin{aligned}H_4 &= \int \nabla^{|\alpha|-1}\nabla S_2(\triangle\nabla^{|\alpha|-1}u - \nabla\mathrm{div}\nabla^{|\alpha|-1}u)\mathrm{d}x\\ &= -\int \nabla^{|\alpha|-1}\nabla S_2\cdot\nabla\times\nabla\times\nabla^{|\alpha|-1}u\mathrm{d}x = 0.\end{aligned}$$

So, we derive that

$$\sum_{i=1}^{4} H_i \le C(E(t)^{\frac{1}{2}})\|(\nabla\rho,\nabla u,\nabla\theta,\nabla q)\|_{H^2}^2. \tag{3.11}$$

Using Lemma 3.5, we get

$$\sum_{i=1}^{5} F_i \le CE(t)^{\frac{1}{2}}\left(\|(\nabla\rho,\nabla u,\nabla\theta,\nabla q,\nabla S_2)\|_{H^2}^2 + \|\nabla u\|_{H^3}^2 + \|\partial_t u\|_{H^1}^2 + \|\partial_t\theta\|_{H^1}^2\right). \tag{3.12}$$

For the last terms D and L on the right-hand side of (3.9), we have

$$\begin{aligned}D &\le \|(\rho_t,\theta_t,q_t,(S_2)_t,\nabla\rho)\|_{L^\infty}\int\left(|\nabla\nabla^{|\alpha|-1}\rho|^2 + |\nabla\nabla^{|\alpha|-1}u|^2 + |\nabla\nabla^{|\alpha|-1}\theta|^2 + |\nabla^{|\alpha|-1}u_t|^2\right)\mathrm{d}x\\ &\le CE(t)^{\frac{1}{2}}\left(\|(\nabla\rho,\nabla u,\nabla\theta,\nabla q)\|_{H^2}^2 + \|u_t\|_{H^2}^2\right)\mathrm{d}x.\end{aligned}$$

To estimate L, we first take $|\alpha| = 1$, typically

$$\begin{aligned} L &= \int (p_q \nabla q + p_{S_2} \nabla S_2) \triangle u \mathrm{d}x \\ &\leq C \|(p_q, p_{S_2})\|_{L^\infty} \int (|\nabla q|^2 + |\nabla S_2|^2 + |\triangle u|^2) \mathrm{d}x \\ &\leq C E(t)^{\frac{1}{2}} (\|(\nabla q, \nabla S_2)\|_{H^2}^2 + \|u\|_{H^2}^2), \end{aligned}$$

where we have used the fact that $p_q = -\frac{\tau_1}{\kappa\theta} q, p_{S_2} = -\frac{\tau_3}{\lambda} S_2$, while for $2 \leq |\alpha| \leq 3$,

$$\begin{aligned} L &= -\int \nabla^{|\alpha|-1}(p_q \nabla q + p_{S_2} \nabla S_2) \triangle \nabla^{|\alpha|-1} u \mathrm{d}x \\ &\leq C \|(p_q, \nabla q, p_{S_2}, \nabla S_2)\|_{L^\infty} \int \Big(|\nabla \nabla^{|\alpha|-1} q|^2 + |\nabla^{|\alpha|-1} q|^2 + |\nabla^{|\alpha|-1} \theta|^2 + |\nabla \nabla^{|\alpha|-1} S_2|^2 \\ &\qquad + |\nabla^{|\alpha|-1} S_2|^2 + |\triangle \nabla^{|\alpha|-1} u|^2 \Big) \mathrm{d}x \\ &\leq C E(t)^{\frac{1}{2}} (\|(\nabla q, \nabla \theta, \nabla S_2)\|_{H^2}^2 + \|\nabla u\|_{H^3}^2). \end{aligned}$$

For the last term on the left hand side of (3.9), we observe that

$$\begin{aligned} &\int \nabla \mathrm{div} \nabla^{|\alpha|-1} u \cdot \triangle \nabla^{|\alpha|-1} u \mathrm{d}x \\ =& \int \nabla \mathrm{div} \nabla^{|\alpha|-1} u \cdot (\nabla \mathrm{div} \nabla^{|\alpha|-1} u - \nabla \times \nabla \times \nabla^{|\alpha|-1} u) \mathrm{d}x \\ =& \int \left| \nabla \mathrm{div} \nabla^{|\alpha|-1} u \right|^2 \mathrm{d}x, \end{aligned}$$

which gives

$$\begin{aligned} &\mu \int (\triangle \nabla^{|\alpha|-1} u + \frac{n-2}{n} \nabla \mathrm{div} \nabla^{|\alpha|-1} u) \triangle \nabla^{|\alpha|-1} u \mathrm{d}x \\ &\quad = \mu \int |\triangle \nabla^{|\alpha|-1} u|^2 + \frac{n-2}{n} \left| \nabla \mathrm{div} \nabla^{|\alpha|-1} u \right|^2 \mathrm{d}x \end{aligned} \tag{3.13}$$

Summing $|\alpha|$ from 1 to 3 and integrating the inequality (3.9) over $(0, t)$, using the elliptic inequality $\|u\|_{H^2} \leq C \|\triangle u\|_{L^2}$ as well as Lemma 3.4, we get the desired estimates (3.8), which proves Lemma 3.6. □

Now, we use the ideas from Kawashima [22] , see also [36, 33], to estimate the missing last two terms: $\int_0^t \|\nabla \rho\|_{H^2}^2 \mathrm{d}t$ and $\int_0^t \|\nabla \theta\|_{H^2}^2 \mathrm{d}t$.

We first write the system (3.2) in symmetric hyperbolic-parabolic form. Let $U := (\rho, u, \theta, q, S_2)$ and $\bar{U} = (\bar{\rho}, \bar{u}, \bar{\theta}, \bar{q}, \bar{S}_2) := (1, 0, 1, 0, 0)$, then we have

$$A^0(\bar{U}) U_t + \sum_{j=1}^n A^j(\bar{U}) U_{x_j} + \sum_{j=1}^n \sum_{k=1}^n B^{jk}(\bar{U}) U_{x_j x_k} + L(\bar{U}) U = F(U, \nabla U, \partial_t U), \tag{3.14}$$

where

$$A^0(\bar{U}) = \begin{pmatrix} \bar{p}_\rho & 0 & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 & 0 \\ 0 & 0 & \bar{e}_\theta & 0 & 0 \\ 0 & 0 & 0 & \frac{\tau_1}{\kappa} & 0 \\ 0 & 0 & 0 & 0 & \frac{\tau_3}{\lambda} \end{pmatrix}, \sum_{j=1}^{n} A^j(\bar{U})\xi_j = \begin{pmatrix} 0 & \bar{p}_\rho \xi^T & 0 & 0 & 0 \\ \bar{p}_\rho \xi & 0 & \bar{p}_\theta \xi & 0 & -\xi \\ 0 & \bar{p}_\theta \xi^T & 0 & \xi^T & 0 \\ 0 & 0 & \xi & 0 & 0 \\ 0 & -\xi^T & 0 & 0 & 0 \end{pmatrix},$$

$$\sum_{j=1}^{n}\sum_{k=1}^{n} B^{jk}\xi_j\xi_k = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \mu I_n + \frac{n-2}{n}\mu\xi\xi^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, L(\bar{U}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\kappa} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\lambda} \end{pmatrix},$$

$\xi = (\xi_1, \cdots, \xi_n) \in S^{n-1}$and $F(U, \nabla U, U_t) \equiv (F_1, F_2, F_3, F_4, F_5)$ where

$F_1 := \bar{p}_\rho(-u \cdot \nabla\rho - (\rho - 1)\mathrm{div}u)$

$F_2 := -(\rho - 1)\partial_t u - (p_\rho - \bar{p}_\rho)\nabla\rho - (p_\theta - \bar{p}_\theta)\nabla\theta - p_q\nabla q - p_{S_2}\nabla S_2,$

$F_3 := -(\rho e_\theta - \bar{e}_\theta)\partial_t\theta - (\rho u e_\theta - \frac{2q}{\theta})\nabla\theta - (\theta p_\theta - \bar{p}_\theta)\mathrm{div}u + \frac{2}{\kappa\theta}q^2 + \frac{1}{\lambda}S_2^2 + \frac{\mu}{2}|\nabla u + \nabla u^T - \frac{2}{n}\mathrm{div}u|^2,$

$F_4 := -\frac{\tau_1}{\kappa}u \cdot \nabla q, \quad F_5 := -\frac{\tau_3}{\lambda}u \cdot \nabla S_2$

F is etimated in the following lemma.

Lemma 3.7. *For $1 \le |\alpha| \le 3$, there exists a constant C such that*

$$\begin{aligned} \|\nabla^{|\alpha|-1}F_1\|_{L^2} &\le CE(t)^{\frac{1}{2}}\|(\nabla\rho, \nabla u)\|_{H^2}, \\ \|\nabla^{|\alpha|-1}F_2\|_{L^2} &\le CE(t)^{\frac{1}{2}}\left(\|(\nabla\rho, \nabla\theta, \nabla q, \nabla S_2)\|_{H^2} + \|\partial_t u\|_{H^2}\right), \\ \|\nabla^{|\alpha|-1}F_3\|_{L^2} &\le CE(t)^{\frac{1}{2}}\left(\|(\nabla\theta, \partial_t\theta, \nabla u)\|_{H^2} + \|(q, S_2)\|_{H^2}\right), \\ \|\nabla^{\alpha-1}F_4\|_{L^2} &\le CE(t)^{\frac{1}{2}}\left(\|\nabla q\|_{H^2} + \|\nabla u\|_{H^1}\right), \\ \|\nabla^{|\alpha|-1}F_5\|_{L^2} &\le CE(t)^{\frac{1}{2}}\left(\|\nabla S_2\|_{H^2} + \|\nabla u\|_{H^1}\right). \end{aligned}$$

Proof. For $|\alpha| = 1$, we have

$$\|F_1\|_{L^2} \le \|(u, \rho - 1)\|_{L^\infty}(\|\nabla\rho\|_{L^2} + \|\nabla u\|_{L^2}) \le CE(t)^{\frac{1}{2}}(\|(\nabla\rho, \nabla u)\|_{H^2}).$$

For $2 \le |\alpha| \le 3$, we have

$$\begin{aligned} \|\nabla^{|\alpha|-1}F_1\|_{L^2} &\le C\|(u, \nabla\rho, \rho - 1, \nabla u)\|_{L^\infty}(\|(\nabla^{|\alpha|}\rho, \nabla^{|\alpha|-1}u, \nabla^{|\alpha|}u, \nabla^{|\alpha|-1}\rho)\|_{L^2}) \\ &\le CE(t)^{\frac{1}{2}}\|(\nabla\rho, \nabla u)\|_{H^2}. \end{aligned}$$

The other terms in F_i, $2 \le i \le 5$ can be estimated in a similar way as above except the first term "$-(\rho - 1)u_t$" in F_2 since $\|u_t\|_{L^\infty}$ is not bounded by $E(t)^{\frac{1}{2}}$. In the following, we only estimate this term. Note that for $|\alpha| = 1$,

$$\|(\rho - 1)\partial_t u\|_{L^2} \le \|\rho - 1\|_{L^\infty}\|\partial_t u\|_{L^2} \le CE(t)^{\frac{1}{2}}\|\partial_t u\|_{L^2}.$$

For $|\alpha| = 2$,

$$\begin{aligned} &\|\partial_{x_i}[(\rho - 1)\partial_t u]\|_{L^2} \\ &= \|\partial_{x_i}\partial_t u + (\rho - 1)\partial_t\partial_{x_i}u\| \\ &\le C\|(\nabla\rho, \rho - 1)\|_{L^\infty}\|\partial_t u\|_{H^1} \le CE(t)^{\frac{1}{2}}\|\partial_t u\|_{H^1}. \end{aligned}$$

For $|\alpha|=3$,

$$\begin{aligned}&\|\partial_{x_i}\partial_{x_j}[(\rho-1)\partial_t u]\|_{L^2}\\&=\|\partial_{x_i}\partial_{x_j}\rho\partial_t u+\partial_{x_i}\partial_t\partial_{x_j}u+\partial_{x_j}\rho\partial_t\partial_{x_i}u+(\rho-1)\partial_t\partial_{x_i}\partial_{x_j}u\|\\&\le\|\nabla^2\rho\|_{L^6}\|\partial_t u\|_{L^3}+\|\nabla\rho\|_{L^\infty}\|\nabla\partial_t u\|_{L^2}+\|\rho-1\|_{L^\infty}\|\partial_t u\|_{H^2}\\&\le CE(t)^{\frac12}\|\partial_t u\|_{H^2}.\end{aligned}$$

This finishes the proof of Lemma 3.7. □

Now, we introduce a matrix K^j for $j=1,2,\cdots,n$ as follows,

$$\sum_{j=1}^n K^j\xi_j=\epsilon\begin{pmatrix}0&\bar p_\rho\xi^T&0&0&0\\-\xi&0&0&0&0\\0&0&0&\frac{\kappa N}{\tau_1}\xi^T&0\\0&0&-\frac{N}{\bar e_\theta}\xi&0&\frac{\lambda}{\tau_3}\xi\\0&0&0&-\frac{\kappa}{\tau_1}\xi^T&0\end{pmatrix},\tag{3.15}$$

where $\epsilon>0$ and $N>0$ is yet arbitrary. Then, we calculate

$$\sum_{j=1}^n K^jA^0(\bar U)\xi_j=\epsilon\begin{pmatrix}0&\bar p_\rho\xi^T&0&0&0\\-\bar p_\rho\xi&0&0&0&0\\0&0&0&N\xi^T&0\\0&0&-N\xi&0&\xi\\0&0&0&-\xi^T&0\end{pmatrix}\tag{3.16}$$

which is an anti-symmetric matrix and

$$\begin{aligned}M&:=\sum_{j=1}^n\sum_{i=1}^n\left(\frac12\left(K^iA^j\xi_j\xi_i+(K^iA^j\xi_j\xi_i)^T\right)+B^{ij}\xi_j\xi_i\right)+L\\&=\begin{pmatrix}\epsilon(\bar p_\rho)^2&0&\frac{\epsilon}{2}\bar p_\rho\bar p_\theta&0&-\frac{\epsilon}{2}\bar p_\rho\\0&\mu I_n+(\frac{n-2}{n}\mu-\epsilon\bar p_\rho)\xi\xi^T&0&-\frac{\epsilon}{2}\left(\frac{\bar p_\theta N}{\bar e_\theta}+\frac{\lambda}{\tau_3}\right)\xi\xi^T&0\\\frac{\epsilon}{2}\bar p_\rho\bar p_\theta&0&\epsilon\frac{\kappa N}{\tau_1}&0&-\epsilon\frac{\kappa}{2\tau_1}\\0&-\frac{\epsilon}{2}\left(\frac{\bar p_\theta N}{\bar e_\theta}+\frac{\lambda}{\tau_3}\right)\xi\xi^T&0&\frac{1}{\kappa}I_n-\frac{\epsilon N}{\bar e_\theta}\xi\xi^T&0\\-\frac{\epsilon}{2}\bar p_\rho&0&-\epsilon\frac{\kappa}{2\tau_1}&0&\frac{1}{\lambda}\end{pmatrix}\end{aligned}$$

is symmetric and positive definite for some large N and sufficiently small ϵ. In fact, let $\eta=(\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)^T\in\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}$ be any vector, then

$$\eta^TM\eta=(\eta_1,\eta_3,\eta_5)^TM_1(\eta_1,\eta_3,\eta_5)+(\eta_2,\eta_4)^TM_2(\eta_2,\eta_4),$$

where

$$M_1=\begin{pmatrix}\epsilon(\bar p_\rho)^2&\frac{\epsilon}{2}\bar p_\rho\bar p_\theta&-\frac{\epsilon}{2}\bar p_\rho\\\frac{\epsilon}{2}\bar p_\rho\bar p_\theta&\epsilon\frac{\kappa N}{\tau_1}&-\epsilon\frac{\kappa}{2\tau_1}\\-\frac{\epsilon}{2}\bar p_\rho&-\epsilon\frac{\kappa}{2\tau_1}&\frac{1}{\lambda}\end{pmatrix},M_2=\begin{pmatrix}\mu I_n+(\frac{n-2}{n}\mu-\epsilon\bar p_\rho)\xi\xi^T&-\frac{\epsilon}{2}\left(\frac{\bar p_\theta N}{\bar e_\theta}+\frac{\lambda}{\tau_3}\right)\xi\xi^T\\-\frac{\epsilon}{2}\left(\frac{\bar p_\theta N}{\bar e_\theta}+\frac{\lambda}{\tau_3}\right)\xi\xi^T&\frac{1}{\kappa}I_n-\frac{\epsilon N}{\bar e_\theta}\xi\xi^T\end{pmatrix}$$

So, the positive definiteness of M is equivalent to that of M_1 and M_2. Let d_j denote the j-th principle minor of the matrix M_1, then $d_1=\epsilon(\bar p_\rho)^2>0$ and

$$d_2=\epsilon^2(\bar p_\rho)^2\left(\frac{\kappa N}{\tau_1}-\frac14(\bar p_\theta)^2\right).$$

We can choose N independent of ϵ such that

$$\frac{\kappa N}{\tau_1}>\frac14(\bar p_\theta)^2$$

implying $d_2 > 0$. Now, N is fixed. For small ϵ, we observe that

$$d_3 = \frac{1}{\lambda}(\bar{p}_\rho)^2 \left(\frac{\kappa N}{\tau_1} - \frac{1}{4}(\bar{p}_\theta)^2 \right) \epsilon^2 + O(\epsilon^3), \quad \epsilon \to 0,$$

which gives $d_3 > 0$ by choosing ϵ sufficiently small. Thus, M_1 is a positive definite matrix. Furthermore, we observe that M_2 is a small symmetric perturbation of a positive definite matrix, which implies that M_2 is also positive definite for sufficiently small ϵ. From now on ϵ is fixed.

Applying $K^i \nabla^{|\alpha|-1}$ to the system (3.14), multiplying by $\nabla^{|\alpha|-1} U_{x_i}$ in L^2, and taking the sum over $i = 1, \dots, n$, we obtain

$$\begin{aligned}
\sum_{i=1}^{n} < K^i A^0(\bar{U}) \nabla^{|\alpha|-1} U_t, \nabla^{|\alpha|-1} U_{x_i} > + \sum_{i=1}^{n} \sum_{j=1}^{n} < K^i A^j(\bar{U}) \nabla^{|\alpha|-1} U_{x_j}, \nabla^{|\alpha|-1} U_{x_i} > \\
+ < \sum_{i=1}^{n} K^i \left(\sum_{j=1}^{n} \sum_{k=1}^{n} B^{jk}(\bar{U}) \nabla^{|\alpha|-1} U_{x_j x_k} + L(\bar{U}) \nabla^{|\alpha|-1} U \right), \nabla^{|\alpha|-1} U_{x_i} > \\
= \sum_{i=1}^{n} < K^i \nabla^{|\alpha|-1} F(U, \nabla U, U_t), \nabla^{|\alpha|-1} U_{x_i} > . \qquad (3.17)
\end{aligned}$$

Here $< \cdot, \cdot >$ means the L^2 inner product. Now, we estimate each term of (3.17) as follows.

$$\begin{aligned}
&\sum_{i=1}^{n} < K^i A^0(\bar{U}) \nabla^{|\alpha|-1} U_t, \nabla^{|\alpha|-1} U_{x_i} > \\
=& \sum_{i=1}^{n} \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} < K^i A^0(\bar{U}) \nabla^{|\alpha|-1} U, \nabla^{|\alpha|-1} U_{x_i} >
\end{aligned}$$

where the we use the fact $K^i A^0$ is an anti-symmetric matrix. On the other hand, using the fact that M defined above is a positive definite matrix, we derive that

$$\begin{aligned}
&\sum_{i=1}^{n} \sum_{j=1}^{n} < K^i A^j(\bar{U}) \nabla^{|\alpha|-1} U_{x_j}, \nabla^{|\alpha|-1} U_{x_i} > \\
=& < \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{1}{2} (K^i A^j(\bar{U}) + (K^i A^j(\bar{U}))^T) + B^{ij}(\bar{U}) \right) + L(\bar{U}) \nabla^{|\alpha|-1} U_{x_j}, \nabla^{|\alpha|-1} U_{x_i} > \\
&- < \sum_{i=1}^{n} \sum_{j=1}^{n} \left(B^{ij}(\bar{U}) + L(\bar{U}) \right) \nabla^{|\alpha|-1} U_{x_j}, \nabla^{|\alpha|-1} U_{x_i} > \\
&\geq \beta \|\nabla^{|\alpha|} U\|_{L^2}^2 - C(\|(\nabla^{|\alpha|} u, \nabla^{|\alpha|} q, \nabla^{|\alpha|} S_2)\|_{L^2}^2,
\end{aligned}$$

where $\beta = \beta(M)$ is some positive constant. The positivity of M can be exploited using the Fourier transform. Meanwhile, we have

$$\begin{aligned}
&< \sum_{i=1}^{n} K^i \left(\sum_{j=1}^{n} \sum_{k=1}^{n} B^{jk}(\bar{U}) \nabla^{|\alpha|-1} U_{x_j x_k} + L(\bar{U}) \nabla^{|\alpha|-1} U \right), \nabla^{|\alpha|-1} U_{x_i} > \\
&\leq C(\|\nabla u\|_{H^3}^2 + \|q\|_{H^3}^2 + \|S_2\|_{H^3}^2).
\end{aligned}$$

Integrating the equation (3.17) over $(0, t)$, combing the above estimates, using Lemmas 3.6 and 3.7, we get the following result.

Lemma 3.8. *There exists a constant $C > 0$ such that*

$$\int_0^t \|(\nabla\rho, \nabla\theta)\|_{H^2}^2 \mathrm{d}t \leq C(E_0 + E(t)^{\frac{3}{2}}). \tag{3.18}$$

Combining Lemmas 3.6 and 3.8, we get

Lemma 3.9. *There exists a constant $C > 0$ such that*

$$\|(\nabla\rho, \nabla u, \nabla\theta, \nabla q, \nabla S_2)\|_{H^2} + \int_0^t (\|(\nabla\rho, \nabla\theta, \nabla q, \nabla S_2)\|_{H^2}^2 + \|\nabla u\|_{H^3}^2)\mathrm{d}t \leq C(E_0 + E(t)^{\frac{3}{2}}). \tag{3.19}$$

Thus, the proof of Proposition 3.2 is finished by combining Lemmas 3.3 and 3.9. Moreover, looking at the local existence Theorem 3.1 and the dependence on the norms of the initial data that determine the length of the existence interval, this allows to continue a local solution to a global one. Moreover, from inequality (1.13) and Lemma 3.4, we have

$$\int_0^t \|\nabla(\rho, u, \theta, q, S_2)\|_{L^2} \mathrm{d}t \leq C$$

and

$$\int_0^t \left| \frac{\mathrm{d}}{\mathrm{d}t} \|\nabla(\rho, u, \theta, q, S_2)\|_{L^2} \right| \mathrm{d}t \leq C,$$

which implies the decay estimate (1.14) immediately and thus proves Theorem 1.1. □

4. Blow-up result: Proof of Theorem 1.2

In this part, we consider the local existence and the blow-up of solutions for system (1.15). To this end, we need the following assumption.

Assumption. There exists $\delta > 0$, sufficiently small, such that

$$\min_{x\in\mathbb{R}^n}(\rho_0(x), \theta_0(x)) > 0, \quad \max_{x\in\mathbb{R}^n}(|q_0(x)|, |S_{20}(x)|) \leq \frac{\delta}{2}. \tag{4.1}$$

We remark that for the local solution then will hold

$$\min_{(x,t)}(p_\rho, p_\theta, e_\theta)(t, x) > 0 \ \max_{(x,t)} |p_{S_2}(t, x)| < \frac{1}{2}. \tag{4.2}$$

Replacing the equation $(1.15)_3$ by the equation for the temperature θ given in (3.1), we have

$$\begin{cases} \partial_t\rho + \mathrm{div}(\rho u) = 0, \\ \rho\partial_t u + \rho u\cdot\nabla u + \nabla p = \nabla S_2, \\ \rho e_\theta\partial_t\theta + (\rho u e_\theta - \frac{2q}{\theta})\nabla\theta + \theta p_\theta \mathrm{div}u + \mathrm{div}q = \frac{2}{\kappa\theta}q^2 + \frac{1}{\lambda}S_2^2, \\ \tau_1(\partial_t q + u\cdot\nabla q) + q + \kappa\nabla\theta = 0, \\ \tau_3(\partial_t S_2 + u\cdot\nabla S_2) + S_2 = \lambda\mathrm{div}u. \end{cases} \tag{4.3}$$

Now, we transform the above system into a first-order system for $V := (\rho, u, \theta, q, S_2)^T$. We have

$$\partial_t V + \sum_{j=1}^3 A_j(V)\partial_{x_j} V + B(V)V = F(V), \tag{4.4}$$

where

$$\sum_{j=1}^{3} A_j\xi_j = \begin{pmatrix} u\cdot\xi & \rho\xi^T & 0 & 0 & 0 \\ \frac{p_\rho}{\rho}\xi & (u\cdot\xi)I_n & \frac{p_\theta}{\rho}\xi & \frac{p_q}{\rho}\xi^T & \frac{p_{S_2}-1}{\rho}\xi \\ 0 & \frac{\theta p_\theta}{\rho e_\theta}\xi^T & (u-\frac{2q}{\rho\theta e_\theta})\xi & \xi^T & 0 \\ 0 & 0 & \frac{\kappa}{\tau_1}\xi & (u\cdot\xi)I_n & 0 \\ 0 & -\frac{\lambda}{\tau_3}\xi^T & 0 & 0 & u\cdot\xi \end{pmatrix},$$

$$B(V) = diag\left\{0,0,0,\frac{1}{\tau_1},\frac{1}{\tau_3}\right\}, F(V) = \left(0,0,\frac{2}{\kappa\theta}q^2+\frac{1}{\lambda}S_2^2,0,0\right)^T.$$

Since the $(2n+3)\times(2n+3)$-matrix $\sum_{j=1}^{n} A_j\xi_j$ is *not* symmetric, the system (4.4) is *neither* symmetric-hyperbolic *nor* strictly hyperbolic. So, the local existence does not follow immediately by the classical theory of symmetric-hyperbolic or strictly hyperbolic systems. We shall show that the system (4.4) is a constantly hyperbolic system which also will imply a local existence theorem.

We look only at the three-dimensional case $n = 3$, analogous arguments apply to the case $n = 2$. We first prove that the matrix $\sum_{j=1}^{3} A_j\xi_j$ has nine linearly independent eigenvectors corresponding to five different eigenvalues (one eigenvalue is 5-fold). For $|\xi| = 1$ the characteristic polynomial for $\sum_{j=1}^{3} A_j\xi_j$ is given by (also checked by ©Maple)

$$P(n,\Lambda,\xi) := \det\left(\sum_{j=1}^{3} A_j\xi_j - \Lambda I_9\right) = (u\cdot\xi-\Lambda)^5 g(u\cdot\xi-\Lambda), \tag{4.5}$$

where

$$\begin{aligned} g(z) :=& z^4 - \frac{2q\cdot\xi}{\rho\theta e_\theta}z^3 - \left(\frac{\kappa}{\tau_1}+\frac{\theta p_\theta^2}{\rho^2 e_\theta}+\frac{\lambda(1-p_{S_2})}{\rho\tau_3}+p_\rho\right)z^2 \\ &+ \left(\frac{\kappa\theta p_\theta}{\tau_1\rho^2 e_\theta}p_q\cdot\xi + \left(\frac{\lambda(1-p_{S_2})}{\rho\tau_3}+p_\rho\right)\cdot\frac{2q\cdot\xi}{\rho\theta e_\theta}\right)z + \left(\frac{\lambda(1-p_{S_2})}{\rho\tau_3}+p_\rho\right)\cdot\frac{\kappa}{\tau_1}. \end{aligned} \tag{4.6}$$

We observe the similarity of the characteristic polynomial $P(n,\Lambda,\xi)$ with the corresponding polynomial in one space dimension (n=1), given in [17], actually we have

$$P(n,\Lambda,\xi) = (u\cdot\xi-\Lambda)^{2n-1}g(u\cdot\xi-\Lambda), \qquad \text{for } n = 1,2,3. \tag{4.7}$$

A similar situation – characteristic polynomial in space dimensions $n = 2, 3$ is given by the corresponding polynomial in space dimension $n = 1$ times a power of linear polynomials – is observed in linear thermoelasticity, see [30] or [19].

We first show that there exist five linearly independent eigenvectors corresponding to the eigenvalue $\Lambda = u\cdot\xi$. Let $W = (x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9)^T$ be an eigenvector corresponding to the eigenvalue $\Lambda = u\cdot\xi$. Then we have

$$\begin{aligned} x_2\xi_1 + x_3\xi_2 + x_4\xi_3 &= 0, \\ x_5 &= 0, \\ x_6\xi_1 + x_7\xi_2 + x_8\xi_3 &= 0, \\ p_\rho x_1 + (p_{S_2}-1)x_9 &= 0, \end{aligned}$$

from which we obtain that there exist two linearly independent eigenvectors of the form $W = (0,x_2,x_3,x_4,0,0,0,0,0)^T$ with $(x_2,x_3,x_4)\cdot\xi = 0$, two of the form $W = (0,0,0,0,0,x_6,x_7,x_8,0)^T$ with $(x_6,x_7,x_8)\cdot\xi = 0$, and one of the form $W = (x_1,0,0,0,0,0,0,0,x_9)^T$ with $(x_1,x_9)\cdot(p_\rho,p_{S_2}-1)^T = 0$.

Second, g has four different zeros. Since g is essentially the same as the corresponding one in one space dimension, we only (easily) transfer the considerations from [17] for the reader's convenience.

Note that $g(\pm\infty) = +\infty$, and $g(0) = \left(\frac{\lambda(1-p_{S_2})}{\rho\tau_3} + p_\rho\right) \cdot \frac{\kappa}{\tau_1} > 0$.

Let

$$\mu_\pm := \pm\sqrt{\frac{\lambda(1-p_{S_2})}{\rho\tau_2} + p_\rho},$$

then

$$\mu_- < 0 < \mu_+$$

and

$$g(\mu_\pm) = \mu_- \left(\frac{\theta p_\theta^2}{\rho^2 e_\theta}\mu_+ \mp \frac{\kappa\theta p_\theta}{\tau_1\rho^2 e_\theta} p_q \cdot \xi\right) \equiv \mu_- Q,$$

which implies by assumption (4.1) and (4.2) that

$$Q \geq \min_{(t,x)\in\mathbb{R}^+\times\mathbb{R}^3} \frac{\theta p_\theta^2 \mu_+}{2\rho^2 e_\theta} > 0$$

if

$$|p_q \cdot \xi| \leq \min_{(t,x)\in\mathbb{R}^+\times\mathbb{R}^3} \frac{\tau_1 p_\theta \mu_+}{\kappa},$$

which is satisfied if $|q| < \delta$ for some $\delta > 0$. Therefore, we derive that

$$g(\mu_\pm) < 0.$$

Hence, g has four different real zeros $z_1 < z_2 < 0 < z_3 < z_4$. Thus we have demonstrated that our system is *constantly* hyperbolic.

The constant hyperbolicity implies the local well-posedness [1]. Constantly hyperbolic system are much less investigated than symmetric-hyperbolic or strictly hyperbolic ones, causing in general more difficulties. But we can refer to [1, Thm. 2.3 and Thm. 10.2], and thus have

Theorem 4.1. *Let $s > \frac{n}{2} + 1$ and $(\rho_0, u_0, \theta_0, q_0, S_{20}) : \mathbb{R}^n \to \mathbb{R}^{2n+3}$ be given with*

$$\rho_0 - 1, u_0, \theta_0 - 1, q_0, S_0 \in H^s,$$

$$\min_{x\in\mathbb{R}^n}(\rho_0(x), \theta_0(x)) > 0, \quad \max_{x\in\mathbb{R}^n}(|q_0(x)|, |S_{20}(x)|) \leq \frac{\delta}{2}.$$

Then, there exists a unique local solution $(\rho, u, \theta, q, S_2)$ to system (4.3) in some time interval $[0, T_0)$ with

$$(\rho - 1, u, \theta - 1, q, S_2) \in C^0([0, T_0), H^s) \cap C^1([0, T_0), H^{s-1})$$

and

$$\min_{(x,t)\in\mathbb{R}^n\times[0,T_0)}(\rho(x,t), \theta(x,t)) > 0, \qquad \max_{(x,t)\in\mathbb{R}^n\times[0,T_0)}(|q(x,t)|, |S_2(x,t)|) \leq \delta.$$

The following proposition states the finite propagation speed property which is guaranteed by the hyperbolicity of the system (4.3), see [30, 14].

Proposition 4.2. *Assume the initial data $(\rho_0, u_0, \theta_0, q_0, S_{20})$ satisfy the assumption given in Theorem 4.1 and $(\rho, u, \theta, q, S_2)$ be local solutions to (4.3) on $[0, T_0)$. We further assume that the initial data $(\rho_0 - 1, u_0, \theta_0 - 1, q_0, S_{20})$ are compactly supported in a ball $B_0(M)$ with radius $M > 0$. Then, there exists a constant σ such that*

$$(\rho(\cdot,t), u(\cdot,t), \theta(\cdot,t), q(\cdot,t), S_2(\cdot,t)) = (1, 0, 1, 0, 0) =: (\bar\rho, \bar u, \bar\theta, \bar q, \bar S_2) \tag{4.8}$$

on $D(t) := \{x \in \mathbb{R}^n || x| \geq M + \sigma t\}$, $0 \leq t < T_0$.

In the sequel, without loss of generality, we shall assume that

$$\frac{\bar{\theta}}{2} < \theta < 2\bar{\theta}.$$

We recall the averaged quantities defined in(1.16), (1.17),

$$F(t) = \int_{\mathbb{R}^n} x \cdot \rho(x,t)u(x,t)\mathrm{d}x,$$
$$G(t) = \int_{\mathbb{R}^n} (\mathcal{E}(x,t) - \bar{\mathcal{E}})\mathrm{d}x,$$

with $\mathcal{E}(x,t) = \rho(e + \frac{1}{2}u^2)$ and $\bar{\mathcal{E}} = \bar{\rho}(\bar{e} + \frac{1}{2}\bar{u}^2) = C_v$. We mention that these quantities above exist as finite numbers since the solution $(\rho - 1, u, \theta - 1, q, S)$ is zero on the set $D(t)$ given in Proposition 4.2.

Now, we are ready to prove Theorem 1.2, with an ansatz and with calculations that we have been using starting in [14], essentially going back to ideas of Sideris [34], and that has also been used in the one-dimensional situation [18].

Proof. We will present the case $n = 3$. The case $n = 2$ is obtained with easy modifications. From the equations $(1.15)_{1-3}$, we can get the equation for the total energy E:

$$\partial_t E + \mathrm{div}(uE + up - uS_2 + q) = 0, \tag{4.9}$$

which implies that $G(t)$ is constant and

$$G(t) = G(0) > 0. \tag{4.10}$$

On the other hand, we have

$$\begin{aligned} F'(t) &= \int_{\mathbb{R}^3} \partial_t(\rho u)x\mathrm{d}x \\ &= \int_{\mathbb{R}^3} (-\mathrm{div}(\rho u \otimes u) - \nabla p + \nabla S_2)x\mathrm{d}x \\ &= \int_{\mathbb{R}^3} \rho u^2 \mathrm{d}x + 3\int_{\mathbb{R}^3} (p - \bar{p})\mathrm{d}x - 3\int_{\mathbb{R}^3} S_2 \mathrm{d}x. \end{aligned}$$

By the constitutive equations (1.10) and (1.11), we have

$$\int_{\mathbb{R}^3} (p - \bar{p})\mathrm{d}x = \int_{\mathbb{R}^3} (R\rho\theta - \frac{\tau_1}{2\kappa\theta}q^2 - \frac{\tau_3}{2\lambda}S_2^2 - R\bar{\rho}\bar{\theta})\mathrm{d}x$$

and

$$R\rho\theta = \frac{R}{C_v}\rho e - \frac{\tau_1 R}{C_v\kappa\theta}q^2 - \frac{\tau_3 R}{C_v\lambda}S_2^2.$$

So, using (4.10), we derive that

$$\begin{aligned} \int_{\mathbb{R}^3} (p - \bar{p})\mathrm{d}x &\geq \int_{\mathbb{R}^3} \left((\gamma - 1)(\rho e - \bar{\rho}\bar{e}) - \frac{\tau_1\gamma}{\kappa\theta}q^2 - \frac{\tau_3\gamma}{\lambda}S_2^2\right)\mathrm{d}x \\ &\geq \int_{\mathbb{R}^3} (\gamma - 1)\left((E - \frac{1}{2}\rho u^2) - \bar{E}\right)\mathrm{d}x - \int_{\mathbb{R}^3} \left(\frac{\tau_1\gamma}{\kappa\theta}q^2 + \frac{\tau_3\gamma}{\lambda}S_2^2\right)\mathrm{d}x \\ &\geq -\frac{\gamma - 1}{2}\int_{\mathbb{R}^3} \rho u^2 \mathrm{d}x - \int_{\mathbb{R}^3} \left(\frac{\tau_1\gamma}{\kappa\theta}q^2 + \frac{\tau_3\gamma}{\lambda}S_2^2\right)\mathrm{d}x, \end{aligned}$$

where $\gamma := \frac{R}{C_v} + 1$. Combining the above estimates and using Hölder's inequality, we get

$$F'(t) \geq \frac{5 - 3\gamma}{2}\int_{\mathbb{R}^3} \rho u^2 \mathrm{d}x - 3\int_{\mathbb{R}^3} \frac{\tau_1\gamma}{\kappa\theta}q^2\mathrm{d}x - 3\int_{\mathbb{R}^3} \left(\frac{\tau_3\gamma}{\lambda} + \frac{1}{2}\right)S_2^2\mathrm{d}x - 2\pi(M + \sigma t)^3. \tag{4.11}$$

By the definition of $F(t)$, we conclude

$$\begin{aligned}F^2(t)&=\left(\int_{\mathbb{R}^3}x\cdot\rho(x,t)u(x,t)\mathrm{d}x\right)^2\\&\le\int_{B_t}x^2\rho\mathrm{d}x\cdot\int_{B_t}\rho u^2\mathrm{d}x\\&\le(M+\sigma t)^2\int_{B_t}\rho\mathrm{d}x\cdot\int_{B_t}\rho u^2\mathrm{d}x\\&=(M+\tilde\sigma t)^2\int_{B_0}\rho\mathrm{d}x\cdot\int_{B_t}\rho u^2\mathrm{d}x\\&\le\frac{4\pi}{3}\max\rho_0(M+\tilde\sigma t)^5\int_{\mathbb{R}^3}\rho u^2\mathrm{d}x,\end{aligned}$$

where $B_t:=\{x\in\mathbb{R}^3||x|\le M+\tilde\sigma t\}$ and $\tilde\sigma\ge\sigma$ can be chosen arbitrarily . For simplicity, we still denote $\tilde\sigma$ by σ in the following calculations. Therefore, we get

$$F'(t)\ge\frac{3(5-3\gamma)}{8\pi\max\rho_0(M+\sigma t)^5}F^2-3\int_{\mathbb{R}^3}\left(\frac{\tau_1\gamma}{\kappa\theta}q^2+\frac{2\tau_3\gamma+\lambda}{2\lambda}S_2^2\right)\mathrm{d}x-2\pi(M+\sigma t)^3.\tag{4.12}$$

Let $c_2:=\frac{\sigma}{M}$, $c_3:=\frac{3(5-3\gamma)}{8\pi\max\rho_0M^5}$. Assume for the moment

$$F(t)\ge c_1>0\tag{4.13}$$

and

$$2\pi(M+\sigma t)^3=2\pi M^3(1+c_2t)^3\le\frac{c_3}{2(1+c_2t)^5}F^2,\tag{4.14}$$

where c_1 is to be determined later. Under the above a priori assumptions, we immediately get

$$F'(t)\ge\frac{c_3}{2(1+c_2t)^5}F^2-\frac{6\tau_1\gamma}{\kappa\bar\theta}\int_{\mathbb{R}^3}q^2\mathrm{d}x-\frac{6\tau_3\gamma+3\lambda}{2\lambda}\int_{\mathbb{R}^3}S_2^2\mathrm{d}x.$$

Using the assumption (4.13), the above inequality implies that

$$\frac{F'(t)}{F^2(t)}\ge\frac{c_3}{2(1+c_2t)^5}-\frac{6\tau_1\gamma}{c_1^2\kappa\bar\theta}\int_{\mathbb{R}^3}q^2\mathrm{d}x-\frac{6\tau_3\gamma+3\lambda}{c_1^2 2\lambda}\int_{\mathbb{R}^3}S_2^2\mathrm{d}x.\tag{4.15}$$

Now, we recall the dissipative entropy equation (2.13) given in Lemma 2.2, with $\mu=0$,

$$\begin{aligned}&\partial_t\left[C_v\rho(\theta-\ln\theta-1)+R(\rho\ln\rho-\rho+1)+(1-\frac{1}{2\theta})\frac{\tau_1}{\kappa\theta}q^2+\frac{1}{2}\rho u^2+\frac{\tau_3}{2\lambda}S_2^2\right]\\&+\mathrm{div}\left[\rho uC_v(\theta-\ln\theta-1)+u(1-\frac{1}{2\theta})\frac{\tau_1}{\kappa\theta}q^2+\frac{\tau_3}{2\lambda}uS_2^2+R\rho u\ln\rho-R\rho u-\frac{q}{\theta}\right.\\&\qquad\left.+\frac{1}{2}\rho u|u|^2+pu+q-S_2u\right]+\frac{q^2}{\kappa\theta^2}+\frac{S_2^2}{\theta\lambda}=0.\end{aligned}\tag{4.16}$$

Let

$$W_0=\int_{\mathbb{R}^3}\left(C_v\rho_0(\theta_0-\ln\theta_0-1)+R(\rho_0\ln\rho_0-\rho_0+1)+(1-\frac{1}{2\theta_0})\frac{\tau_1}{\kappa\theta}q_0^2+\frac{\tau_2}{2\lambda}S_{20}^2\right)\mathrm{d}x,$$

then (4.16) implies

$$\int_0^t\int_{\mathbb{R}^3}\frac{q^2}{\kappa\theta^2}\mathrm{d}x\mathrm{d}t+\int_0^t\int_{\mathbb{R}^3}\frac{S_2^2}{\theta\lambda}\mathrm{d}x\mathrm{d}t\le W_0+\frac{\max\rho_0}{2}\|u_0\|_{L^2}^2.$$

Therefore, we have

$$\frac{6\tau_1\gamma}{c_1^2\kappa\bar{\theta}}\int_0^t\int_{\mathbb{R}^3} q^2\mathrm{d}x\mathrm{d}t+\frac{6\tau_3\gamma+3\lambda}{c_1^2 2\lambda}\int_0^t\int_{\mathbb{R}^3} S_2^2\mathrm{d}x\mathrm{d}t\le c_4+c_5\|u_0\|_{L^2}^2, \tag{4.17}$$

where

$$c_4=\frac{3}{c_1^2}\left[\bar{\theta}(8\tau_1\gamma+2\tau_3\gamma+\lambda)W_0\right],\quad c_5=\frac{3}{c_1^2}\left[\bar{\theta}(8\tau_1\gamma+2\tau_3\gamma+\lambda)\frac{\max\rho_0}{2}\right].$$

Integrating (4.15) over $(0,t)$, using the above estimates, we have

$$\frac{1}{F_0}-\frac{1}{F(t)}\ge-\frac{c_3}{8c_2(1+c_2t)^4}+\frac{c_3}{8c_2}-c_4-c_5\|u_0\|_{L^2}^2. \tag{4.18}$$

Now, we assume

$$F_0>\frac{16c_2}{c_3}, \tag{4.19}$$

$$c_4+c_5\|u_0\|_{L^2}^2\le\frac{c_3}{16c_2}. \tag{4.20}$$

Then we have from (4.18) and (4.20)

$$\frac{1}{F_0}\ge\frac{1}{F_0}-\frac{1}{F(t)}\ge-\frac{c_3}{8c_2(1+c_2t)^4}+\frac{c_3}{16c_2} \tag{4.21}$$

which implies that the maximal time of existence T can not arbitrarily large without contradicting (4.19).

Now, we first show that the a priori assumption (4.13) holds. Define

$$c_1:=\frac{4c_2}{c_3},$$

then

$$\frac{1}{F(t)}\le\frac{1}{F_0}+\frac{c_3}{8c_2(1+c_2t)^4}-\frac{c_3}{16c_2}\le\frac{c_3}{8c_2(1+c_2t)^4}$$

which implies

$$F(t)\ge\frac{8c_2}{c_3}(1+c_2t)^4\ge 2c_1. \tag{4.22}$$

This assures the a priori assumption (4.13) being compatible with (4.19) by noting that $F_0\ge 2c_1$.

To show the a priori estimates (4.14) hold, by bootstrap methods, we only need to show

$$2\pi M^3(1+c_2t)^3\le\frac{c_3}{4(1+c_2t)^5}F(t)^2. \tag{4.23}$$

As a first step, we need (4.23) to hold for $t=0$, that is

$$F_0^2\ge\frac{8\pi M^3}{c_3}=\frac{64\pi\max\rho_0}{3(5-3\gamma)}M^8. \tag{4.24}$$

Using (4.22), (4.23) is equivalent to

$$\sigma^2\ge\frac{3(5-3\gamma)}{64\max\rho_0} \tag{4.25}$$

which is satisfied naturally since σ can be chosen arbitrarily large and then be fixed.

Thus, the proof will be finished if there exists u_0 such that (4.19), (4.20) and (4.24) hold and the assumption 1.18 is satisfied. Let (cp. [14, 18])

$$\tilde{v}(r) = \begin{cases} L\cos(\frac{\pi}{2}(r-1)), & r \in [0,1], \\ L, & r \in (1, M-1], \\ \frac{L}{2}\cos(\pi(r-M+1)) + \frac{L}{2}, & r \in (M-1, M], \\ 0, & r \in (M, +\infty), \end{cases} \tag{4.26}$$

where L is a positive constant to be determined later. $\tilde{v}$ is not in $H^3(\mathbb{R}_+)$, but we can think of $\tilde{v}$ being smoothed around the singular points $r = 1, M-1, M$ and put to zero around $r = 0$, yielding a function v, with $\|v\|_{L^2} \leq 2\|\tilde{v}\|_{L^2}$. We choose

$$u_0(x) := v(|x|)\frac{x}{|x|}.$$

Assumption 1.18 can easily be satisfied since it is equivalent to requiring

$$\int_{\mathbb{R}^3} \left(\rho_0 e_0 - \bar{\rho}\bar{e} + \frac{1}{2}u_0^2\right) dx > 0,$$

which is satisfied by choosing $\rho_0\theta_0 > \bar{\rho}\bar{\theta} = 1$. Let $M \geq 5$, then

$$\begin{aligned} F_0 &= \int_{\mathbb{R}^3} x \cdot \rho_0(x)u_0(x)\mathrm{d}x = \int_{\mathbb{R}^3} \rho_0(x)v(|x|)|x|\mathrm{d}x \\ &\geq \min\rho_0 \int_{B_0(M)} v(|x|)|x|\mathrm{d}x \\ &\geq \min\rho_0 \int_0^M v(r)r \cdot 4\pi r^2 \mathrm{d}r \\ &\geq \min\rho_0 \int_2^{M-2} L \cdot 4\pi r^3 \mathrm{d}r \geq \frac{\pi\min\rho_0}{32} LM^4 \end{aligned}$$

We choose L sufficiently large such that

$$\frac{\pi\min\rho_0}{32}L \geq \max\left\{\sqrt{\frac{64\pi\max\rho_0}{3(5-3\gamma)}}, \frac{128\sigma\pi\max\rho_0}{3(5-3\gamma)}\right\}.$$

So, (4.19) and (4.24) hold. On the other hand, since $\|u_0\|_{L^2}^2 \leq 4L^2\frac{4\pi}{3}M^3$, we choose M such that

$$\bar{\theta}(8\tau_1\gamma + 2\tau_2\gamma + \mu)\left(W_0 + \frac{2\pi\max\rho_0 L^2}{3}M^3\right) \leq \frac{16\pi\sigma\max\rho_0}{9(5-3\gamma)}M^4.$$

Therefore, (4.20) holds and the proof of Theorem 1.2 is finished. □

5. Remark on the singular limit

We conclude the paper with a remark and additional result on the *singular limit* $\tau \to 0$, which we can describe for the case $\tau_1 > 0, \tau_2 = 0, \tau_3 > 0$, $\mu > 0$. We assume for simplicity $\tau_1 = \tau_3 =: \tau$. For $\tau > 0$ let $(\rho^\tau, u^\tau, \theta^\tau, q^\tau, S_2^\tau)$ denote the local solution to the system (1.8) defined on $[0, T_\tau)$, where

$$T_\tau = \sup\{T > 0; (\rho^\tau - 1, u^\tau, \theta^\tau - 1, q^\tau, S_2^\tau) \in C([0,T], H^3), (\rho^\tau, u^\tau, \theta^\tau, q^\tau, S_2^\tau) \in G_1\}$$

with initial data $(\rho_0^\tau, u_0^\tau, \theta_0^\tau, q_0^\tau, S_{20}^\tau) \in G_0$. Then we have

Theorem 5.1. *Let (ρ, u, θ) be the smooth solution to the classical compressible Navier-Stokes equations with $(\rho(x,0), u(x,0), \theta(x,0)) = (\rho_0, u_0, \theta_0)$ satisfying $\inf_{(x,t)\in\mathbb{R}^3\times[0,T_*]}(\rho(x,t), \theta(x,t)) > 0$ and*

$$(\rho - 1) \in C([0,T_*], H^6) \cap C^1([0,T_*], H^5),$$
$$(u, \theta - 1) \in C([0,T_*], H^6) \cap C^1([0,T_*], H^4),$$

with finite $T_* > 0$. Moreover, assume that the initial data are well-prepared, i.e.,

$$\|(\rho_0^\tau - \rho_0, u_0^\tau - u_0, \theta_0^\tau - \theta_0, \sqrt{\tau}(q_0^\tau + \kappa\nabla\theta_0), \sqrt{\tau}(S_{20}^\tau - \lambda \mathrm{div} u_0))\|_{H^3} \leq \tau.$$

Then, there exist constants τ_0 and $C > 0$ such that for $\tau \leq \tau_0$,

$$\|(\rho^\tau, u^\tau, \theta^\tau)(\cdot,t) - (\rho, u, \theta)(\cdot,t)\|_{H^3} \leq C\tau, \tag{5.1}$$

and

$$\|(q^\tau + \kappa\nabla\theta, S_2^\tau - \lambda\nabla u)\|_{H^3} \leq C\tau^{\frac{1}{2}}, \tag{5.2}$$

for all $t \in (0, min(T_*, T_\tau))$, and the constant C is independent of τ.

Thus, we have that the solutions for $\tau > 0$ converge to the solution for $\tau = 0$ with a certain order in τ on any finite interval of common existence.

The (long) proof of Theorem 5.1, which we omit here, can be done in the spirit of corresponding considerations in [16], overcoming a higher complexity given here by energy estimates similar to those used in the proof of Theorem 1.1.

Recently, Peng and Zhao [29] studied the 1-d version and obtained in partical a global existence result which is uniform with respect to τ as well as a global convergence result in a weak topology.

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