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Virtual forms

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Abstract

In this article, quadratic forms over a field of characteristic different from two are generalised to so-called *virtual forms* over an arbitrary field. These objects turn out to be useful for the study of the Milnor K-theory of a field. The relation between both areas is established by a sequence of maps corresponding to Delzant's *Stiefel-Whitney classes*.

1 Introduction

Quadratic forms have manifold aspects and therefore they can be generalised in various directions. The aim of this article is to transfer some of the algebraic theory of quadratic forms over fields into a more general context. To this aim *virtual forms* over an arbitrary field are introduced. These objects are are closely related to Milnor K-theory.

To give the idea let us consider a field F of characteristic different from 2. The algebraic theory of quadratic forms over F, as it may be learned from [6] or [10], starts with a few crucial observations. First of all, quadratic forms over F can be diagonalised. Witt's Cancellation Law then gives information on the interplay between orthogonal sum and isometry. This leads to the definition of the Witt ring $W(F)$ and the Witt-Grothendieck ring $\widehat{W}(F)$. Using further Witt's Chain Equivalence Theorem, which describes when exactly two given diagonalisations belong to the same quadratic form, one obtains a description of $W(F)$ and of $\widehat{W}(F)$ by means of generators and relations.

This abstract description of the Witt-Grothendieck ring $\widehat{W}(F)$ will be taken as the guiding principle for the definition of a group $G(F,\ell)$, the *Grothendieck group of ℓ-forms*, where now F is an arbitrary field and $\ell \in \mathbb{N}$. If F is of characteristic different from 2, then $G(F,2)$ coincides with the group $\widehat{W}(F)$. The definition of $G(F,\ell)$ is based on Milnor K-theory. After defining several operations on the elements of $G(F,\ell)$, a descending sequence of subgroups $(G^n(F,\ell))_{n\in\mathbb{N}}$ is constructed, destined to replace the filtration given by the powers of the fundamental ideal in the Witt-Grothendieck ring.

While in general $G(F,\ell)$ is not a commutative group for $\ell \neq 2$, at least the groups $G^n(F,\ell)$ for $n \geq 2$ are contained in the center of $G(F,\ell)$. Moreover, if ℓ is odd, then $G^n(F,\ell)$ vanishes for $n \geq 3$ and any element of $G^1(F,\ell)$ has finite order dividing ℓ. A sequence of maps from $G(F,\ell)$ to the Milnor K-groups modulo ℓ of the field F allows to use ℓ-forms to investigate the Milnor K-theory of F. These maps generalise Delzant's Stiefel-Whitney classes (in the way Milnor defines them in [8]). In certain cases, in particular for odd ℓ, it turns out that the group $G^1(F,\ell)$ can be described as a group extension of the first by the second Milnor K-group modulo ℓ. As an application of some of the new concepts an alternative proof of a result in K-theory due to B. Kahn is given. Finally, a notion of isotropy is introduced.

Part of the results presented here have been announced in [1].

Throughout this article F denotes a field, $F^\times$ its multiplicative group, and ℓ a nonnegative integer.

2 Milnor K-theory

Let us recall the basic definitions from Milnor K-theory (cf. [8]) and fix some notation. Let $n,\ell \in \mathbb{N}$. Let $K_n^{(\ell)}F$ denote the nth K-group modulo ℓ of the Milnor K-theory of F. To be explicite, $K_n^{(\ell)}F$ is the additive abelian group which is generated by so-called *symbols* $\{a_1,\dots,a_n\}$, where $a_1,\dots,a_n \in F^\times$, which are subject to the following relations:

(M1) the natural map $\{\,\} : (F^\times)^n \longrightarrow K_nF$ is $\mathbb{Z}$-multilinear;

(M2) $\{a_1,\dots,a_n\} = 0$ whenever $a_i + a_{i+1} = 1$ in F for some $i < n$;

(M3) $\ell \cdot K_n^{(\ell)}F = 0$.

Obviously (M3) can be ignored if $\ell = 0$. In fact, $K_n^{(0)}F$ is just the 'full' Milnor K-group usually denoted by K_nF, and $K_n^{(\ell)}F$ corresponds to its quotient modulo ℓ. Note that $K_1^{(\ell)}F$ is just the group $F^\times/F^{\times\ell}$ in additive notation if one identifies the element $\{x\}$ of $K_1^{(\ell)}F$ with the class $xF^{\times\ell}$ in $F^\times/F^{\times\ell}$. For $n = 0$ the first two relations have no relevance, so $K_0^{(\ell)}F$ is a cyclic group generated by the empty symbol $\{\}$, and in view of $(M3)$ we identify $K_0^{(\ell)}F = \mathbb{Z}/\ell\mathbb{Z}$. In $K_2^{(\ell)}F$, it follows from (M1) and (M2) that $\{-a,a\} = 0$ and $\{a,b\} = -\{b,a\} = \{b^{-1},a\}$ hold for $a,b \in F^\times$ (cf. [8, §1]).

We denote $K_*^{(\ell)}F = \bigoplus_{i\geq 0} K_i^{(\ell)}F$ and $\widehat{K}_*^{(\ell)}F = \prod_{i\geq 0} K_i^{(\ell)}F$. With

$$\{a_1,\dots,a_r\}\cdot\{a_{r+1},\dots,a_{r+s}\} = \{a_1,\dots,a_r,a_{r+1},\dots,a_{r+s}\}$$

one has a natural multiplication in both $K_*^{(\ell)}F$ and $\widehat{K}_*^{(\ell)}F$. Hence $K_*^{(\ell)}F$ and $\widehat{K}_*^{(\ell)}F$ are endowed with a natural structure as algebras over the ring $K_0^{(\ell)}F = \mathbb{Z}/\,\ell\mathbb{Z}$, and $K_*^{(\ell)}F$ is a graded algebra.

3 Chain equivalence

We consider finite sequences of elements in $F^\times$. We denote by $S_n(F)$ the set of all such sequences of length n and by $[a_1,\ldots,a_n]$ the sequence with entries $a_1,\ldots,a_n \in F^\times$. We use the sign $\diamond$ to denote the concatenation of two sequences: for $\alpha = [a_1,\ldots,a_r] \in S_r(F)$ and $\alpha' = [a_{r+1},\ldots,a_{r+s}] \in S_s(F)$ we set $\alpha \diamond \alpha' = [a_1,\ldots,a_r,a_{r+1},\ldots,a_{r+s}] \in S_{r+s}(F)$.

We introduce now, for any $n \geq 1$, an equivalence relation on $S_n(F)$ which depends on the (fixed) integer $\ell \geq 0$. This relation is introduced first for $n=1$ and $n=2$ and then extended in a canonical way to arbitrary n.

Let $a, a' \in F^\times$. We consider the sequences $[a]$ and $[a']$ in $S_1(F)$ to be equivalent in this relation (depending on ℓ) and write $[a] \overset{\ell}{\sim} [a']$ in case that $\{a\} = \{a'\}$ holds in $K_1^{(\ell)}F$, that is, if $a' = x^\ell a$ for some $x \in F^\times$. Let now $a, b, a', b' \in F^\times$. We write

$$[a,b] \overset{\ell}{\sim} [a',b'] \qquad \text{if} \qquad \left\{ \begin{array}{rcll} \{ab\} & = & \{a'b'\} & \text{in } K_1^{(\ell)}F \quad \text{and} \\ \{a,b\} & = & \{a',b'\} & \text{in } K_2^{(\ell)}F\,. \end{array} \right.$$

We thus have defined an equivalence relation on $S_n(F)$ for $n = 1, 2$.

Let now $n > 2$. We say that the two sequences $\alpha = [a_1,\ldots,a_n]$ and $\beta = [b_1,\ldots,b_n]$ in $S_n(F)$ are *simply ℓ-equivalent* and write $\alpha \overset{\ell}{\approx} \beta$ if there exists a positive integer $k < n$ such that $[a_k,a_{k+1}] \overset{\ell}{\sim} [b_k,b_{k+1}]$ and such that we have $\{a_i\} = \{b_i\}$ in $K_1^{(\ell)}F$ for $1 \leq i < k$ and for $k+1 < i \leq n$. This relation is not transitive in general. In order to obtain an equivalence relation $\overset{\ell}{\sim}$ we take the transitive closure of $\overset{\ell}{\approx}$. We say that $\alpha, \beta \in S_n(F)$ are *chain ℓ-equivalent* and write $\alpha \overset{\ell}{\sim} \beta$ if

$$\alpha \overset{\ell}{\approx} \gamma_1 \overset{\ell}{\approx} \gamma_2 \overset{\ell}{\approx} \ldots \overset{\ell}{\approx} \gamma_r \overset{\ell}{\approx} \beta$$

holds for a suitable choice of $\gamma_1,\ldots,\gamma_r \in S_n(F)$, $r \geq 1$. For $\ell = 0$ we just speak of *simple equivalence* and *chain equivalence* and we write $\sim$ and $\approx$, accordingly; note that simple (resp. chain) equivalence implies simple (resp. chain) ℓ-equivalence modulo any $\ell > 0$.

The definition of chain ℓ-equivalence is motivated by a fundamental result in the algebraic theory of quadratic forms, Witt's Chain Equivalence Theorem (cf. [6, Chap. I, §5]), which can be reformulated in the following way.

3.1 Proposition. *Assume that* $\mathrm{char}(F) \neq 2$*. Two sequences* $[a_1, \dots, a_n]$ *and* $[b_1, \dots, b_n]$ *in* $S_n(F)$ *are chain 2-equivalent if and only if the quadratic forms* $a_1X_1^2 + \dots + a_nX_n^2$ *and* $b_1X_1^2 + \dots + b_nX_n^2$ *over* F *are isometric.*

Proof: For $n = 1$ the statement is obvious. Now we inspect the case $n = 2$.

For any $c, d \in F^\times$, the symbols $\{cd\} \in K_1^{(2)}F$ and $\{c, d\} \in K_2^{(2)}F$ are invariants of the quadratic form $cX^2 + dY^2$ up to isometry. Hence, if the quadratic forms $a_1X^2 + a_2Y^2$ and $b_1X^2 + b_2Y^2$ are isometric, then we have $[a_1, a_2] \overset{2}{\sim} [b_1, b_2]$.

Conversely, assume that $[a_1, a_2] \overset{2}{\sim} [b_1, b_2]$. Since $\{a_1, a_2\} = \{b_1, b_2\}$ holds in $K_2^{(2)}F$, the 3-dimensional quadratic forms $-a_1X^2 - a_2Y^2 + a_1a_2Z^2$ and $-b_1X^2 - b_2Y^2 + b_1b_2Z^2$ are isometric, by [3, Theorem 1.8] and Witt Cancellation (cf. [6, Chap. I, 4.2]). Since in addition we have $b_1b_2 = a_1a_2x^2$ for some $x \in F^\times$, using Witt Cancellation and multiplying by -1 we see that $a_1X^2 + a_2Y^2$ and $b_1X^2 + b_2Y^2$ are isometric.

So far we have shown that 2-equivalence for elements in $S_2(F)$ characterises isometry for the corresponding diagonal quadratic forms in two variables over F. Now, for arbitrary $n \geq 2$, by Witt's Chain Equivalence Theorem (cf. [6, Chap. I, §5]) isometry for quadratic diagonal forms in n variables is entirely determined by isometry for 2-dimensional diagonal subforms. This completes the proof. □

3.2 Remark. Assume that $\ell \geq 2$ and that F contains a primitve ℓth root of unity ζ. In this case, the Merkurjev-Suslin Theorem (cf. [7]) implies that symbols in $K_2^{(\ell)}F$ are in one-to-one correspondence with isomorphism classes of cyclic algebras of degree ℓ over F. More precisely, given $a, b \in F^\times$, the symbol $\{a, b\}$ is identified with the *symbol algebra* $(a, b)_{F,\zeta}$. By definition, $(a, b)_{F,\zeta}$ is the central simple F-algebra of degree ℓ generated by elements u, v which are subject to the relations $u^\ell = a$, $v^\ell = b$, and $vu = \zeta uv$.

Therefore, under the assumptions on ℓ and F, chain ℓ-equivalence can be described in terms of classical algebraic structures not involving K-theory.

Permuting the entries of a sequence does not necessarily yield a sequence which is chain equivalent (modulo ℓ) to the original one. More particularly, for $a, b \in S_2(F)$, one has $[a, b] \overset{\ell}{\sim} [b, a]$ if and only if $2 \cdot \{a, b\} = 0$ in $K_n^{(\ell)}F$.

3.3 Lemma. *For any* $a, b \in F^\times$*, one has* $[a, b] \sim [b^{-1}, ab^2] \sim [a^2b, a^{-1}]$.

Proof: Note first that the product of both entries is the same for any of the three given sequences in $S_2(F)$. It is further easy to check the equalities

$$\{a, b\} = \{b^{-1}, ab^2\} = \{a^2b, a^{-1}\}$$

in K_2F. This yields the statement. □

3.4 Corollary. *If $a = \pm 1$, then $[a, b] \sim [b, a]$ for any $b \in F^\times$.*

Proof: Since $a^{-1} = a$, this is clear from the lemma. □

A generalisation of the last statement will be obtained in (3.9) below.

Let $S_0(F)$ denote the singleton set consisting of the empty sequence $[\,]$. Let further

$$S(F) = \bigcup_{n \in \mathbb{N}} S_n(F)\,.$$

Hence $(S(F), \diamond)$ is the free monoïd generated by $F^\times$.

We are going to consider operations on sequences and how they behave with respect to chain equivalence. We say that a map

$$\Phi : S(F) \longrightarrow S(F)$$

is *compatible with chain ℓ-equivalence* if it preserves lengths of sequences and if, for any $n \in \mathbb{N}$ and any $\alpha, \beta \in S_n(F)$, the relation $\alpha \overset{\ell}{\sim} \beta$ implies that $\Phi(\alpha) \overset{\ell}{\sim} \Phi(\beta)$. We say that Φ as above is *compatible* if it is compatible with chain ℓ-equivalence for every $\ell \in \mathbb{N}$. A compatible map Φ may or may not be an endomorphism of the monoïd $S(F)$, and it does not need to be injective nor surjective.

Examples of compatible maps are given by rising the coefficients of a sequence to some power and by reversing the order of the coefficients.

Let $\alpha = [a_1, \ldots, a_n] \in S_n(F)$. For any $z \in \mathbb{Z}$, we write α^z for the sequence $[a_1^z, \ldots, a_n^z]$. In particular, $\alpha^{-1} = [a_1^{-1}, \ldots, a_n^{-1}]$. Furthermore, we denote by $\widetilde{\alpha}$ the sequence with the entries of α in reversed order, i.e. $\widetilde{\alpha} = [a_n, \ldots, a_1]$.

3.5 Proposition.

(a) *For any $z \in \mathbb{Z}$, the rule $\alpha \longmapsto \alpha^z$ defines a compatible endomorphism of $S(F)$. In particular, $\alpha \longmapsto \alpha^{-1}$ is a compatible involution on $S(F)$.*

(b) *The rule $\alpha \longmapsto \widetilde{\alpha}$ defines a compatible involution on $S(F)$.*

Proof: Only the compatibility (with respect to arbitrary ℓ) requires a proof. In view of the definitions, one needs to verify compatibility only for sequences $\alpha, \beta \in S_n(F)$ where $n \leq 2$, thus to show for those that $\alpha \overset{\ell}{\sim} \beta$ entails $\alpha^z \overset{\ell}{\sim} \beta^z$ for any $z \in \mathbb{Z}$, as well as $\widetilde{\alpha} \overset{\ell}{\sim} \widetilde{\beta}$. For $n = 1$ this is obvious.

Let now $\alpha, \beta \in S_2(F)$ with $\alpha \overset{\ell}{\sim} \beta$. Using that, for any $u, v \in F^\times$ and any $z \in \mathbb{Z}$, one has

$$\{u^z v^z\} = z \cdot \{uv\} \text{ in } K_1^{(\ell)}F \text{ and } \{u^z, v^z\} = z^2 \cdot \{u, v\} \text{ in } K_2^{(\ell)}F\,,$$

one sees that $\alpha^z \overset{\ell}{\sim} \beta^z$, for any $z \in \mathbb{Z}$. Similarily, using that $\{v, u\} = -\{u, v\}$ in $K_2^{(\ell)}F$ for any $u, v \in F^\times$, one sees that $\widetilde{\alpha} \overset{\ell}{\sim} \widetilde{\beta}$. □

Given a sequence $\alpha = [a_1, \ldots, a_n] \in S_n(F)$ and an element $c \in F^\times$, we denote by $c * \alpha$ the sequence $[c^{\varepsilon_1}a_1, \ldots, c^{\varepsilon_n}a_n]$, where $\varepsilon_i = (-1)^i$ for $i = 1, \ldots, n$. We call $c * \alpha$ the *conjugate of α by c*. Note that conjugation is bijective, but it is in general not an endomorphism of the monoïd $S(F)$. However, $(-1) * \alpha = [-a_1, \ldots, -a_n]$, so conjugation by -1 is an automorphism of $S(F)$. As we shall see now, conjugation by an element of $F^\times$ is a compatible operation on $S(F)$.

3.6 Proposition. *Let $\alpha, \beta \in S_n(F)$, $c \in F^\times$, and $\ell \in \mathbb{N}$. Then $\alpha \overset{\ell}{\sim} \beta$ if and only if $c * \alpha \overset{\ell}{\sim} c * \beta$.*

Proof: By the definition of chain ℓ-equivalence, the statement needs to be proven only for $n = 2$. For any $a, b, c \in F^\times$, we have $\{c^{-1}a, cb\} = \{c^{-1}, -ab\} + \{a, b\}$ in $K_2^{(\ell)}F$. Therefore, if $a, b, a', b' \in F^\times$ are such that $[a, b] \overset{\ell}{\sim} [a', b']$, i.e. $\{ab\} = \{a'b'\}$ in $K_1^{(\ell)}F$ and $\{a, b\} = \{a', b'\}$ in $K_2^{(\ell)}F$, then we obtain $\{c^{-1}a, cb\} = \{c^{-1}a', cb'\}$ and $[c^{-1}a, cb] \overset{\ell}{\sim} [c^{-1}a', cb']$. This proves one direction. Replacing c by c^{-1} then immediately yields the converse direction. □

For a sequence $\alpha = [a_1, \ldots, a_n]$ in $S_n(F)$, we denote by α^* the sequence $[-a_n^{-1}, \ldots, -a_1^{-1}]$ in $S_n(F)$. This defines an involution

$$^* : S(F) \longrightarrow S(F)\,,\ \alpha \longmapsto \alpha^*\,.$$

3.7 Corollary. *The involution $^* : S(F) \longrightarrow S(F)$ is compatible.*

Proof: The map * is compatible, because it is the combination of three compatible operations on $S(F)$ (which actually commute with each other), namely $\alpha \mapsto \alpha^{-1}$, $\alpha \mapsto \widetilde{\alpha}$, and $\alpha \mapsto (-1) * \alpha$. □

For $\alpha \in S_n(F)$ and $r \in \mathbb{N}$ we put $\quad r \times \alpha = \alpha \diamond \ldots \diamond \alpha \in S_{rn}(F)$.

3.8 Proposition. *For any $\alpha \in S_n(F)$ one has* $\quad \alpha \diamond \alpha^* \sim n \times [1, -1]\,.$

Proof: It follows from the definition of chain equivalence on $S_2(F)$ that $[a, -a^{-1}] \sim [1, -1]$ holds for any $a \in F^\times$. Since $[1, -1]$ commutes up to chain equivalence with any element of $S(F)$ by (3.4), the result follows. □

After having considered operations on $S(F)$ which turned out to be compatible with chain equivalence, we may now ask which operations stabilise the equivalence classes of certain sequences.

3.9 Proposition. *Let $a_1, \ldots, a_n \in F^\times$ be such that $a_1 \cdots a_n = \pm 1$. Then*

$$[a_1, \ldots, a_n] \sim [a_n^{-1}, a_1^{-1}, \ldots, a_{n-1}^{-1}] .$$

Moreover, if n is odd, then

$$[a_1, \ldots, a_n] \sim [a_1^{-1}, \ldots, a_n^{-1}] \sim [a_n, a_1, \ldots, a_{n-1}] .$$

Proof: Using $n-1$ times Lemma (3.3) we obtain

$$\begin{aligned} [a_1, \ldots, a_n] &\sim [a_1, \ldots, a_n a_{n-1}^2, a_{n-1}^{-1}] \\ &\sim \ldots \\ &\sim [a_n(a_1 \cdots a_{n-1})^2, a_1^{-1}, \ldots, a_{n-1}^{-1}] . \end{aligned}$$

Since by hypothesis $(a_1 \cdots a_{n-1})^2 = a_n^{-2}$, the first part of the statement follows. This also shows the second equivalence of the second part of the statement, which thus holds without condition on n.

Finally, if n is odd, then applying n times the first part yields

$$[a_1, \ldots, a_n] \sim [a_1^{-1}, \ldots, a_n^{-1}] .$$

□

3.10 Remark. Possibly the whole statement of the proposition is true for every n. The second part, which here is under the hypothesis that n is odd, can easily be proven for even n under the assumption that $[a_1, \ldots, a_n]$ is chain equivalent to a sequence having one entry equal to ± 1.

3.11 Proposition. *Let $\alpha = [a_1, \ldots, a_n] \in S_n(F)$ with $(a_1 \cdots a_n) = \pm 1$. Then for any $\beta \in S_m(F)$ one has $\alpha \diamond \beta \sim \beta \diamond \alpha^\varepsilon$, where $\varepsilon = (-1)^m$.*

Proof: We may assume that $m = 1$, so that $\beta = [c]$ for some $c \in F^\times$. Using Lemma (3.3) n times we obtain

$$\begin{aligned} \alpha \diamond \beta = [a_1, \ldots, a_n, c] &\sim [a_1, \ldots, a_{n-1}, c a_n^2, a_n^{-1}] \\ &\sim \ldots \\ &\sim [c(a_1 \cdots a_n)^2, a_1^{-1}, \ldots, a_n^{-1}] = \beta \diamond \alpha^{-1} , \end{aligned}$$

because $(a_1 \cdots a_n)^2 = 1$, by the hypothesis. □

4 Virtual forms

We say that two sequences $\alpha, \alpha' \in S(F)$ are *stably chain ℓ-equivalent* if

$$\alpha \diamond (r \times [1,-1]) \overset{\ell}{\sim} \alpha' \diamond (r \times [1,-1])$$

holds for some $r \in \mathbb{N}$. We write $M(F,\ell)$ for the set of equivalence classes in $S(F)$ modulo this equivalence relation. The elements of this set are called *virtual forms of degree ℓ over F*, or *ℓ-forms* for short. For $\alpha \in S(F)$ we denote by $\langle \alpha \rangle$ the ℓ-form given by α. If $\alpha = [a_1, \ldots, a_n]$ then we may write $\langle a_1, \ldots, a_n \rangle$ instead of $\langle \alpha \rangle$. The class of the empty sequence $[\,]$ is also considered as an element of $M(F,\ell)$, and it is denoted by 0 and called the *trivial form.*

It is not clear whether stable chain ℓ-equivalence is really a coarser equivalence relation than chain ℓ-equivalence.

4.1 Question. *For $\alpha, \alpha' \in S(F)$, does $\langle \alpha \rangle = \langle \alpha' \rangle$ imply that $\alpha \overset{\ell}{\sim} \alpha'$?*

We define a (not necessarily commutative) operation $+$ on $M(F,\ell)$. For $\alpha, \beta \in S(F)$, we set

$$\langle \alpha \rangle + \langle \beta \rangle \;=\; \langle \alpha \diamond \beta \rangle\,.$$

Note that it follows from (3.4) that this operation is well-defined and associative. Therefore $(M(F,\ell),+)$ is a monoïd with neutral element 0. For $n \in \mathbb{N}$ and $\varphi \in M(F,\ell)$, we denote by $n \times \varphi$ the *n-fold sum* $\varphi + \ldots + \varphi$.

4.2 Lemma. *The monoïd $(M(F,\ell),+)$ satisfies the cancellation law.*

Proof: Given $\alpha, \alpha', \beta \in S(F)$ such that $\langle \alpha \rangle + \langle \beta \rangle = \langle \alpha' \rangle + \langle \beta \rangle$, we claim that $\langle \alpha \rangle = \langle \alpha' \rangle$. We may restrict to the case where $\beta \in S_1(F)$, i.e. $\langle \beta \rangle = \langle b \rangle$ for some $b \in F^\times$. Note that $\langle b \rangle + \langle -b^{-1} \rangle = \langle b, -b^{-1} \rangle = \langle 1, -1 \rangle$ by (3.8). Therefore $\langle \alpha \rangle + \langle \beta \rangle = \langle \alpha' \rangle + \langle \beta \rangle$ implies that $\langle \alpha \rangle + \langle 1, -1 \rangle = \langle \alpha' \rangle + \langle 1, -1 \rangle$. This means that $\alpha \diamond [1,-1]$ and $\alpha' \diamond [1,-1]$ are stably chain ℓ-equivalent. The same then holds for α and α' so that $\langle \alpha \rangle = \langle \alpha \rangle$. This shows that cancellation on the right is possible. The proof of cancellation on the left is analogous. $\square$

We write $\mathbb{H}$ for the ℓ-form $\langle 1, -1 \rangle$. Note that $\langle a, -a^{-1} \rangle = \mathbb{H}$ for any $a \in F^\times$ by (3.8). We denote by $G(F,\ell)$ the set of formal differences

$$\varphi \;-\; n \times \mathbb{H}$$

where $\varphi \in M(F,\ell)$ and $n \in \mathbb{N}$. Since by (3.4) the ℓ-form $n \times \mathbb{H}$ lies in the center of $M(F,\ell)$ for any $n \in \mathbb{N}$, the operation $+$ extends naturally to $G(F,\ell)$ and gives it the structure of a monoïd. Since $M(F,\ell)$ satisfies the cancellation law, the same is true for $G(F,\ell)$ and $M(F,\ell)$ can be seen as a submonoïd of $G(F,\ell)$ by identifying an ℓ-form φ with the difference $\varphi \;-\; 0 \times \mathbb{H}$.

4.3 Proposition. *$G(F,\ell)$ is a group.*

Proof: We have to show that, for arbitrary $n \in \mathbb{N}$ and $\alpha \in S(F)$, the difference $\langle\alpha\rangle - n \times \mathbb{H}$ has an inverse in $G(F,\ell)$. In fact, if m is the length of the sequence α, then the inverse is given by $\langle\alpha^*\rangle + (n-m) \times \mathbb{H}$, because $\langle\alpha\rangle + \langle\alpha^*\rangle = m \times \mathbb{H}$ by (3.8). □

4.4 Example. Let us assume that $\mathrm{char}(F) \neq 2$ and consider the case $\ell = 2$. By (3.1) and Witt's Cancellation Theorem, 2-forms over F correspond bijectively to isometry classes of regular quadratic forms over F. In particular, Question (4.1) has a positive answer in this case and $G(F,2)$ is isomorphic to the Witt-Grothendieck group $\widehat{W}(F)$.

We call $G(F,\ell)$ the *Grothendieck group of ℓ-forms* over F. Note that this group is not abelian unless $2 \cdot K_2^{(\ell)}F = 0$ (e.g. for $\ell = 2$).

4.5 Lemma. *For any $a, b \in F^\times$, we have $\langle a,b\rangle = \langle b^{-1}, ab^2\rangle = \langle a^2b, a^{-1}\rangle$.*

Proof: This is immediate from (3.3). □

We next prove a universal property for the group $G(F,\ell)$.

4.6 Lemma. *Let $(G,\circ)$ be a group and $g : F^\times \longrightarrow G$ a map. Assume that the following hold for any $a, b, a', b' \in F^\times$:*

- *if $\{a\} = \{b\}$ in $K_1^{(\ell)}F$, then $g(a) = g(b)$ in G;*
- *if $\{ab\} = \{a'b'\}$ in $K_1^{(\ell)}F$ and $\{a,b\} = \{a',b'\}$ in $K_2^{(\ell)}F$, then $g(a) \circ g(b) = g(a') \circ g(b')$ in G.*

Then there is a unique group homomorphism $\Gamma : G(F,\ell) \longrightarrow G$ such that $\Gamma(\langle a\rangle) = g(a)$ for every $a \in F^\times$.

Proof: We can define a homomorphism of monoïds $S(F) \longrightarrow G$ by the rule $[a_1,\ldots,a_n] \longmapsto g(a_1) \circ \ldots \circ g(a_n)$. In view of the two conditions and since G is a group, it is obvious that the image of a sequence $\alpha \in S(F)$ under this homomorphism depends on α only up to stable chain ℓ-equivalence. This yields a homomorphism $M(F,\ell) \longrightarrow G$ which maps the ℓ-form $\langle a_1,\ldots,a_n\rangle$ to $g(a_1)\circ\ldots\circ g(a_n)$. Since the group $G(F,\ell)$ is generated by $M(F,\ell)$ and since G is a group, it is clear that this map extends to a map $\Gamma : G(F,\ell) \longrightarrow G$ with the desired property. The uniqueness of Γ is obvious. □

4.7 Remark. The lemma shows that $(G(F,\ell),+)$ is equal to the group defined by generators and relations in the following way. $G(F,\ell)$ is generated by elements $\langle a\rangle$ with $a \in F^\times$, and the defining relations are:

$$\text{(V1)} \qquad \langle a\rangle = \langle b\rangle \qquad \text{if} \qquad \{a\} = \{b\} \qquad \text{in } K_1^{(\ell)}F\,;$$

$$\text{(V2)} \qquad \langle a\rangle + \langle b\rangle = \langle a'\rangle + \langle b'\rangle \quad \text{if} \quad \begin{cases} \{ab\} = \{a'b'\} & \text{in } K_1^{(\ell)}F \text{ and} \\ \{a\,,\,b\} = \{a'\,,\,b'\} & \text{in } K_2^{(\ell)}F\,. \end{cases}$$

We will frequently use these rules without particular mention.

Let φ be an ℓ-form over F, say $\varphi = \langle a_1, \ldots, a_r\rangle$. We shall refer to r as the *rank of* φ and denote it by $rk(\varphi)$. We further put $d_1(\varphi) = \{a_1 \cdots a_r\} \in K_1^{(\ell)}F$ and call this the *determinant of* φ. This yields two group homomorphisms

$$\begin{aligned} rk: \ & G(F,\ell) \longrightarrow \mathbb{Z}\,, \\ d_1: \ & G(F,\ell) \longrightarrow K_1^{(\ell)}F\,. \end{aligned}$$

The scrupulous reader may apply (4.6) to check that these homomorphisms are actually well-defined. Note that the group $G(F,\ell)$ is generated by the ℓ-forms of rank one.

For an ℓ-form φ given by $\varphi = \langle\alpha\rangle$ with $\alpha \in S(F)$, we use the notations $\varphi^* = \langle\alpha^*\rangle$, $\varphi^z = \langle\alpha^z\rangle$ for $z \in \mathbb{Z}$, in particular $\varphi^{-1} = \langle\alpha^{-1}\rangle$, and further $c * \varphi = \langle c * \alpha\rangle$ for $c \in F^\times$. By (3.5), (3.6), and (3.7), these operations on ℓ-forms are well-defined.

4.8 Proposition. *For any ℓ-form φ over F, we have:*

(*a*) $\quad \varphi + \varphi^* = rk(\varphi) \times \mathbb{H}\,;$

(*b*) $\quad (c * \varphi)^* = \begin{cases} c * \varphi^* & \textit{if } rk(\varphi) \textit{ is even,} \\ (c^{-1}) * \varphi^* & \textit{if } rk(\varphi) \textit{ is odd,} \end{cases} \quad$ *for any $c \in F^\times$.*

Proof: Part (a) follows from (3.8), and (b) is easily checked. □

We may extend the conjugation operation to elements of $G(F,\ell)$: for $a \in F^\times$, $\varphi \in M(F,\ell)$, and $m \in \mathbb{N}$, we put

$$c * (\varphi - m \times \mathbb{H}) = c * \varphi - m \times \mathbb{H}\,.$$

We denote the inverse of an element $\xi \in G(F,\ell)$ by $-\xi$. (This should not be confused with the element $(-1) * \xi$.) If $\xi, \zeta \in G(F,\ell)$, then $\xi - \zeta$ is meant to be $\xi + (-\zeta)$. Furthermore, with $\xi \in G(F,\ell)$ and $n \in \mathbb{N}$, we write $n \times \xi$ for the *n-fold sum* $\xi + \cdots + \xi$.

4.9 Proposition. *Let φ be an ℓ-form over F with $d_1(\varphi) = 0$ or $d_1(\varphi) = \{-1\}$. Then $\varphi = \varphi^{-1}$. Moreover, if $a_1, \ldots, a_n \in F^\times$ are such that $\varphi = \langle a_1, \ldots, a_n\rangle$, then also $\varphi = \langle a_n, a_1, \ldots, a_{n-1}\rangle$.*

Proof: Let $a_1, \dots, a_n \in F^\times$ be such that $\varphi = \langle a_1, \dots, a_n \rangle$. The hypothesis on $d_1(\varphi)$ says that $a_1 \cdots a_n = \pm c$ for some $c \in F^{\times \ell}$. Since ℓ-forms remain unchanged if one entry of a representing sequence is multiplied by an element of $F^{\times \ell}$, we may assume that $c = 1$, whence $a_1 \cdots a_n = \pm 1$. If n is odd, then the claims now follow directly from (3.9). If n is even, then we apply (3.9) to $\varphi \perp \langle 1 \rangle$ instead and use that $\langle 1 \rangle$ lies in the center of $G(F, \ell)$. □

4.10 Corollary. *The kernel of $d_1 : G(F, \ell) \longrightarrow K_1^{(\ell)} F$ is contained in the center of $G(F, \ell)$.*

Proof: Let φ and ψ be ℓ-forms over F with $d_1(\varphi) = 0$. Then $\varphi + \psi = \psi + \varphi^\varepsilon$ with $\varepsilon = \pm 1$ by (3.11) and $\varphi^\varepsilon = \varphi$ by (4.9). Hence φ and ψ commute. □

4.11 Corollary. *Let φ be an ℓ-form of even rank and trivial determinant over F. Then $c^2 * \varphi = \varphi$ for any $c \in F^\times$.*

Proof: Applying (4.9) to φ and to $c * \varphi$ yields

$$c * \varphi \;=\; (c * \varphi)^{-1} \;=\; c^{-1} * \varphi^{-1} \;=\; c^{-1} * \varphi\,,$$

whence $c^2 * \varphi = \varphi$. □

For any ℓ-form φ over F, we define its *companion form* φ° to be

$$\varphi^\circ = \begin{cases} m \times \mathbb{H} + \langle c \rangle & \text{if } rk(\varphi) = 2m+1, \\ m \times \mathbb{H} + \langle 1, c \rangle & \text{if } rk(\varphi) = 2m+2, \end{cases}$$

where $c \in F^\times$ is chosen such that $d_1(\varphi) = d_1(\varphi^\circ)$.

4.12 Proposition. *Let $\xi \in G(F, \ell)$. To have $\xi = \varphi - \varphi^\circ$ for some ℓ-form φ over F, it is necessary and sufficient that $rk(\xi) = 0$ and $d_1(\xi) = 0$.*

Proof: We write $\xi = \varphi - m \times \mathbb{H}$ with $\varphi \in M(F, \ell)$ and $m \in \mathbb{N}$. If $rk(\xi) = 0$ and $d_1(\xi) = 0$, then $rk(\varphi) = 2m$ and $d_1(\varphi) = d_1(m \times \mathbb{H}) = \{(-1)^m\}$, so that $\varphi^\circ = m \times \mathbb{H}$ and $\xi = \varphi - \varphi^\circ$. Conversely, for any ℓ-form φ over F one has $rk(\varphi - \varphi^\circ) = 0$ and $d_1(\varphi - \varphi^\circ) = d_1(\varphi) - d_1(\varphi^\circ) = 0$. □

Given $x \in \mathbb{R}$, let $[x]$ denote the integral part of x, that is, the unique integer such that $[x] \leq x < [x] + 1$.

4.13 Proposition. *Let φ be an ℓ-form over F and let $m = \left[\frac{1}{2}(rk(\varphi) - 1)\right]$. There exist ℓ-forms $\vartheta_1, \dots, \vartheta_m$ over F each of rank 4 and trivial determinant such that*

$$\varphi - \varphi^\circ \;=\; (\vartheta_1 - 2 \times \mathbb{H}) + \cdots + (\vartheta_m - 2 \times \mathbb{H})\,.$$

Proof: We proceed by induction on m. If $m = 0$, then $\varphi = \varphi^\circ$ and thus the statement holds trivially. Let now $m > 0$. We may write $\varphi = \varphi' + \langle x, y, z\rangle$ for certain $x, y, z \in F^\times$ and an ℓ-form φ'. We put $\vartheta = \langle x, y, z, (xyz)^{-1}\rangle$ and $\psi = \varphi' + \langle -xyz\rangle$. Since $d_1(\vartheta) = 0$, we obtain $\varphi + \mathbb{H} = \varphi' + \vartheta + \langle -xyz\rangle = \psi + \vartheta$ by (4.10). Since $rk(\psi) = rk(\varphi) - 2$ and $d_1(\varphi) = d_1(\psi + \mathbb{H})$, we have $\varphi^\circ = \psi^\circ + \mathbb{H}$. Thus using (4.10) yields

$$\varphi - \varphi^\circ = \psi - \psi^\circ + \vartheta - 2 \times \mathbb{H}\,.$$

Now we apply the induction hypothesis to ψ. □

5 Pfister forms

For $a \in F^\times$, we put $\langle\langle a\rangle\rangle = \langle 1, -a\rangle$ and call this a 1*-fold Pfister form* (*of degree* ℓ). Given $n > 1$ and $a_1, \ldots, a_n \in F^\times$, we define recursively

$$\langle\langle a_1, \ldots, a_n\rangle\rangle = \varphi + (a_1 * \varphi)^* = \varphi + a_1 * \varphi^* \qquad \text{where} \qquad \varphi = \langle\langle a_2, \ldots, a_n\rangle\rangle\,,$$

and we call $\langle\langle a_1, \ldots, a_n\rangle\rangle$ an n*-fold Pfister form*. This is an ℓ-form of rank 2^n. Note that, for $a \in F^\times$, we have $\langle\langle a\rangle\rangle = \langle 1\rangle + (a * \langle 1\rangle)^*$.

5.1 Lemma. *For any $a, b \in F^\times$, we have $\langle\langle a, b\rangle\rangle = \langle 1, -a, -b, (ab)^{-1}\rangle$.*

Proof: We compute $\langle\langle a, b\rangle\rangle = \langle 1, -b\rangle + \langle a^{-1}, -ab\rangle^* = \langle 1, -b, (ab)^{-1}, -a\rangle$ and then use (4.9). □

Let $a, b, c, d \in F^\times$. In order to have that the ℓ-forms $\langle\langle a, b\rangle\rangle$ and $\langle\langle c, d\rangle\rangle$ are equal, it is necessary to have $\{a, b\} = \{c, d\}$ in $K_2^{(\ell)}F$. This will become obvious from (7.4), below. On the other hand, if one has $\{ab\} = \{cd\}$ in $K_1^{(\ell)}F$ and $\{a, b\} = \{c, d\}$ in $K_2^{(\ell)}F$, then it follows easily using (5.1) that $\langle\langle a, b\rangle\rangle = \langle\langle c, d\rangle\rangle$.

5.2 Proposition. *Assume that $a, b, c \in F^\times$ are such that $\{a, b\} = \{b, c\}$ in $K_2^{(\ell)}F$. Then $\langle\langle a, b\rangle\rangle = \langle\langle b, c\rangle\rangle$.*

Proof: Let $a, b, c \in F^\times$ with $\{a, b\} = \{b, c\}$. By (4.9) and (4.5) we have $\langle -a, -b, (ab)^{-1}\rangle = \langle -b, (ab)^{-1}, -a\rangle = \langle -b, -a^{-1}, ab^{-1}\rangle$. With (5.1) we conclude that $\langle\langle a, b\rangle\rangle = \langle\langle b, a^{-1}\rangle\rangle$.

Using now that $\{a, b\} = \{b, c\} = -\{c, b\}$ we obtain

$$\{-a^{-1}, b^{-1}a\} \;=\; \{-a, b\} \;=\; -\,\{-c, b\} \;=\; \{-c, b^{-1}\} \;=\; \{-c, (bc)^{-1}\}\,.$$

Therefore $\langle -a^{-1}, b^{-1}a\rangle = \langle -c, (bc)^{-1}\rangle$ and thus $\langle\langle b, a^{-1}\rangle\rangle = \langle\langle b, c\rangle\rangle$. □

5.3 Corollary. *If $a, b \in F^\times$ are such that $\{a, b\} = 0$, then $\langle\langle a, b\rangle\rangle = 2 \times \mathbb{H}$.*

Proof: This is follows from (5.2), because $\{b, 1\} = \{1, 1\} = 0$. □

5.4 Question. *Assume that $a, b, c, d \in F^\times$ are such that $\{a, b\} = \{c, d\}$ in $K_2^{(\ell)}F$. Does it follow that the ℓ-forms $\langle\langle a, b\rangle\rangle$ and $\langle\langle c, d\rangle\rangle$ coincide?*

In the case where $\ell = 2 \neq \mathrm{char}(F)$, a positive answer to this question is contained in [3, Theorem 1.8.]). We will see in (7.13) that the answer is positive in several other cases, in particular for any odd ℓ.

5.5 Problem. Let $a, b, c, d \in F^\times$ be such that $\{a, b\} = \{c, d\}$ in $K_2^{(\ell)}F$. It is not known whether in this situation one can always find a chain of elements $a_1, \ldots, a_n \in F^\times$ with $a_1 = a$, $a_2 = b$, $a_{n-1} = c$, $a_n = d$ and such that $\{a_{i-1}, a_i\} = \{a_i, a_{i+1}\}$ for $1 < i < n$. For $\ell = 2$ and $\mathrm{char}(F) \neq 2$ this is possible and one can achieve this with $n = 5$ ([6, Chap. V, §4]). If $\ell = 3$ and if F contains a primitive 3rd root of unity (in particular $\mathrm{char}(F) \neq 3$), then an analogous statement where $n = 7$ follows from [9, Corollary 2.2.].

6 Subgroups

We are going to define a sequence of subgroups $(G^n(F, \ell))_{n\in\mathbb{N}}$ of $G(F, \ell)$ generalizing the powers of the fundamental ideal in the Witt-Grothendieck ring of quadratic forms.

We put $G^0(F, \ell) = G(F, \ell)$. For $n > 0$, we define $G^n(F, \ell)$ to be the group generated by the differences $\xi - c * \xi$ where $\xi \in G^{n-1}(F, \ell)$ and $c \in F^\times$. Clearly $G^n(F, \ell)$ is a subgroup of $G^{n-1}(F, \ell)$.

6.1 Proposition. *$G^1(F, \ell)$ is equal to the kernel of $rk : G(F, \ell) \longrightarrow \mathbb{Z}$.*

Proof: It is clear that every element of $G^1(F, \ell)$ has trivial rank. On the other hand, the kernel of rk is generated by the differences $\langle a\rangle - \langle 1\rangle = \langle a\rangle - a * \langle a\rangle$ with $a \in F^\times$, and these belong to $G^1(F, \ell)$. □

6.2 Proposition. *The groups $G^n(F, \ell)$ with $n \geq 2$ lie in the center of $G(F, \ell)$.*

Proof: For $\xi \in G^1(F, \ell)$ and $c \in F^\times$, we have $rk(\xi) = 0$, thus $d_1(c*\xi) = d_1(\xi)$ and $d_1(\xi - c * \xi) = 0$. By (4.10) the kernel of $d_1 : G(F, \ell) \longrightarrow K_1^{(\ell)}F$ is contained in the center of $G(F, \ell)$, so the statement follows. □

6.3 Lemma. *For any $a, b, c \in F^\times$, one has*

$$\langle a, b, c, (abc)^{-1}\rangle = \langle\langle -a, -b\rangle\rangle + \langle\langle -ab, -c\rangle\rangle - \langle\langle -1, -ab\rangle\rangle .$$

Proof: We compute

$$\begin{aligned}\langle a, b, c, (abc)^{-1}\rangle + \langle\langle -1, -ab\rangle\rangle &= \langle 1, a, b, (ab)^{-1}\rangle + \langle 1, ab, c, (abc)^{-1}\rangle \\ &= \langle\langle -a, -b\rangle\rangle + \langle\langle -ab, -c\rangle\rangle\end{aligned}$$

□

6.4 Theorem. *$G^2(F,\ell)$ is equal to the kernel of $d_1 : G^1(F,\ell) \longrightarrow K_1^{(\ell)}F$. Furthermore, $G^2(F,\ell)$ consists of all the differences $\varphi - \varphi^\circ$ with $\varphi \in M(F,\ell)$, and it is generated by the differences $\langle\langle a, b\rangle\rangle - 2 \times \mathbb{H}$ with $a, b \in F^\times$.*

Proof: We denote by H the kernel of $d_1 : G^1(F,\ell) \longrightarrow K_1^{(\ell)}F$. By (4.12), it consists of the differences $\varphi - \varphi^\circ$ with $\varphi \in M(F,\ell)$. By (4.13), H is generated by the differences $\vartheta - 2 \times \mathbb{H}$ where ϑ is an ℓ-form over F of rank 4 and trivial determinant. Note that any such form can be written as $\vartheta = \langle x, y, z, (xyz)^{-1}\rangle$ with $x, y, z \in F^\times$, and then

$$\vartheta - 2\times\mathbb{H} = (\langle\langle -x, -y\rangle\rangle - 2\times\mathbb{H}) + (\langle\langle -xy, -z\rangle\rangle - 2\times\mathbb{H}) - (\langle\langle -1, -xy\rangle\rangle - 2\times\mathbb{H})\,,$$

by (6.3). Therefore H is already generated by the differences $\langle\langle a, b\rangle\rangle - 2 \times \mathbb{H}$ with $a, b \in F^\times$. Since $\langle\langle a, b\rangle\rangle - 2 \times \mathbb{H} = \langle 1, -b\rangle - a * \langle 1, -b\rangle$, it follows that H is contained in $G^2(F,\ell)$. On the other hand, it is clear that $G^2(F,\ell)$ is contained in H, the kernel of $d_1 : G^1(F,\ell) \longrightarrow K_1^{(\ell)}F$. □

6.5 Corollary. *For any $n \geq 1$, the group $G^n(F,\ell)$ is generated by the differences $\pi - 2^{n-1} \times \mathbb{H}$ where π is an n-fold Pfister form.*

Proof: For $n = 1$ this is easy to see and for $n = 2$ the statement is contained in (6.4). We proceed by induction on n. Let $n > 2$. By definition, $G^n(F,\ell)$ is generated by the elements $\xi - c * \xi$ with $\xi \in G^{n-1}(F,\ell)$ and $c \in F^\times$. We want to show that $\xi - c * \xi$ is a sum of elements of the form $\pi - 2^{n-1} \times \mathbb{H}$ where π is an n-fold Pfister form. Applying the induction hypothesis to ξ and using that $G^{n-1}(F,\ell)$ is commutative by (6.2), we restrict to the case where $\xi = \rho - 2^{n-2} \times \mathbb{H}$ for some $(n-1)$-fold Pfister form ρ. We obtain

$$\xi - c * \xi \;=\; \rho - c * \rho \;=\; \rho + c * \rho^* - 2^{n-1} \times \mathbb{H}\,,$$

and since $\rho + c * \rho^*$ is an n-fold Pfister form, this finishes the proof. □

6.6 Question. *Do we have $\bigcap_{n=0}^{\infty} G^n(F,\ell) = 0$?*

In the case where $\ell = 2 \neq \mathrm{char}(F)$, the Arason-Pfister Hauptsatz gives a positive answer to this question (cf. [6, Chap. X, §5]). There are other cases where the answer is positive, in fact by rather simple reasons.

6.7 Theorem. *Assume that ℓ is odd or that $F^\times = F^{\times 2}$. Then $G^3(F,\ell) = 0$.*

Proof: By hypothesis, $F^\times/F^{\times\ell}$ is 2-divisible. Therefore $G^3(F,\ell)$ is generated by elements $\xi - c^2 * \xi$ with $c \in F^\times$ and $\xi \in G^2(F,\ell)$. Now, for $\xi \in G^2(F,\ell)$ there is $\varphi \in M(F,\ell)$ and $m \in \mathbb{N}$ such that $d_1(\varphi) = 0$, $rk(\varphi) = 2m$, and $\xi = \varphi - m \times \mathbb{H}$. Using (4.11), we obtain for any $c \in F^\times$ that

$$\xi - c^2 * \xi \;=\; \varphi - c^2 * \varphi \;=\; 0\,,$$

and this finishes the proof. □

6.8 Proposition. *The commutator subgroup of $G(F,\ell)$ is generated by the elements of the shape $\langle\langle a^2, b\rangle\rangle - \langle\langle 1,1\rangle\rangle$ with $a, b \in F^\times$.*

Proof: Let $\xi, \zeta \in G(F,\ell)$. We want to compute the commutator of ξ and ζ. We choose $a, b \in F^\times$ such that $d_1(\xi) = \{a\}$ and $d_1(\zeta) = \{b\}$ in $K_1^{(\ell)}F$. It follows that $\xi \equiv \langle a\rangle$ and $\zeta \equiv \langle b\rangle$ modulo the center of $G(F,\ell)$. This yields

$$\xi + \zeta - \xi - \zeta \;=\; \langle a\rangle + \langle b\rangle - \langle a\rangle - \langle b\rangle \;=\; \langle a, b, -a^{-1}, -b^{-1}\rangle - \langle\langle 1,1\rangle\rangle\,.$$

Furthermore, we have

$$\langle a, b, -a^{-1}, -b^{-1}\rangle = \langle a, -a, a^{-2}b, -b^{-1}\rangle = \langle 1, -a^2, a^{-2}b, -b^{-1}\rangle = \langle\langle a^2, b\rangle\rangle.$$

Therefore $\xi + \zeta - \xi - \zeta = \langle\langle a^2, b\rangle\rangle - \langle\langle 1,1\rangle\rangle$, and the statement follows. □

6.9 Corollary. *If ℓ is odd, then $G^2(F,\ell)$ is the commutator subgroup of $G(F,\ell)$.*

Proof: If ℓ is odd, then the generators of $G^2(F,\ell)$ given by (6.5) can all be written as $\langle\langle 1,1\rangle\rangle - \langle\langle a^2, b\rangle\rangle$ (for given $c, b \in F^\times$, one may set $a = c^{\frac{\ell+1}{2}}$ in order to have $\langle\langle c, b\rangle\rangle = \langle\langle a^2, b\rangle\rangle$). □

6.10 Theorem. *If $-1 \in F^{\times\ell}$, then $\ell \times \xi = 0$ for any $\xi \in G^1(F,\ell)$.*

Proof: Assuming that $-1 \in F^{\times\ell}$, we have $\langle -x\rangle = \langle x\rangle$ for any $x \in F^\times$.

Let $d \in F^\times$. Induction on i shows that the equalities $\langle d^{i-1}, d\rangle = \langle d^i, 1\rangle$ and $i \times (\langle d\rangle - \langle 1\rangle) = \langle d^i\rangle - \langle 1\rangle$ hold for any $i \geq 1$. In particular, for $\xi = \langle d\rangle - \langle 1\rangle$ we have that $\ell \times \xi = 0$.

Let $a, b \in F^\times$ and $i \geq 1$. Using (5.2) and (4.10), we compute

$$\begin{aligned}
\langle\!\langle a^i, b\rangle\!\rangle + \langle\!\langle a, b\rangle\!\rangle &= \langle\!\langle b, (a^i b)^{-1}\rangle\!\rangle + \langle\!\langle a, b\rangle\!\rangle \\
&= \langle 1, 1, b, a^{-i}b^{-1}, a^i, a, b, (ab)^{-1}\rangle \\
&= \langle 1, 1, b, a^{-i}b^{-1}, a^{i+1}, 1, b, (ab)^{-1}\rangle \\
&= \langle b, a^{-i}b^{-1}, 1, a^{i+1}, b, 1, 1, (ab)^{-1}\rangle \\
&= \langle b, a^{-i}b^{-1}, 1, a^{i+1}, b, (a^{i+1}b)^{-1}, a^{i+1}b, (ab)^{-1}\rangle \\
&= \langle b, a^{-i}b^{-1}\rangle + \langle\!\langle a^{i+1}, b\rangle\!\rangle + \langle a^{i+1}b, (ab)^{-1}\rangle \\
&= \langle\!\langle a^{i+1}, b\rangle\!\rangle + \langle b, a^{-i}b^{-1}, a^{i+1}b, (ab)^{-1}\rangle\,.
\end{aligned}$$

Since $\{a^{-i}b^{-1}, a^{i+1}b\} = \{a^{-i}b^{-1}, a\} = \{a, b\} = \{b^{-1}, ab\}$ in $K_2^{(\ell)}F$, we have further that $\langle a^{-i}b^{-1}, a^{i+1}b\rangle = \langle b^{-1}, ab\rangle$. Therefore we obtain

$$\langle\!\langle a^i, b\rangle\!\rangle + \langle\!\langle a, b\rangle\!\rangle \;=\; \langle\!\langle a^{i+1}b\rangle\!\rangle + \langle b, b^{-1}, ab, (ab)^{-1}\rangle \;=\; \langle\!\langle a^{i+1}, b\rangle\!\rangle + \langle\!\langle 1, 1\rangle\!\rangle.$$

By induction on i we deduce that

$$i \times (\langle\!\langle a, b\rangle\!\rangle - \langle\!\langle 1, 1\rangle\!\rangle) \;=\; \langle\!\langle a^i, b\rangle\!\rangle - \langle\!\langle 1, 1\rangle\!\rangle.$$

Since $\langle\!\langle a^\ell, b\rangle\!\rangle = \langle\!\langle 1, 1\rangle\!\rangle$, we have in particular for $\xi = (\langle\!\langle a, b\rangle\!\rangle - \langle\!\langle 1, 1\rangle\!\rangle)$ that $\ell \times \xi = 0$.

Since $G^2(F, \ell)$ is an abelian group, generated by elements $(\langle\!\langle a, b\rangle\!\rangle - \langle\!\langle 1, 1\rangle\!\rangle)$ with $a, b \in F^\times$, the previous implies that $\ell \times \xi = 0$ holds for any $\xi \in G^2(F, \ell)$.

Let now $\xi \in G^1(F, \ell)$ be arbitrary. With $d \in F^\times$ such that $d_1(\xi) = \{d\}$ in $K_1^{(\ell)}F$ and $\zeta = \xi - \langle d\rangle + \langle 1\rangle$ we have $\zeta \in G^2(F, \ell)$. By what we have shown, $\ell \times \zeta = 0$ and $\ell \times (\langle d\rangle - \langle 1\rangle) = 0$. Since ζ lies in the center of $G(F, \ell)$, it follows that $\ell \times \xi = \ell \times (\zeta + \langle d\rangle - \langle 1\rangle) = 0$. □

Let $R^{(\ell)}F$ denote the subgroup of $F^\times$ consisting of those $x \in F^\times$ satisfying $\{x, y\} = 0$ in $K_2^{(\ell)}F$ for any $y \in F^\times$. We call $R^{(\ell)}F$ the *ℓ-radical of F*. Note that $F^{\times\ell} \subset R^{(\ell)}F$. Let $\bar{R}^{(\ell)}F$ denote the corresponding subgroup of $K_1^{(\ell)}F$, consisting of the symbols $\{r\}$ with $r \in R^{(\ell)}F$. In the case where $\mathrm{char}(F) \neq 2$, the group $R^{(2)}F$ was introduced by Kaplansky in [5] and called 'the radical of F' and denoted by $R(F)$.

6.11 Proposition. *If ℓ is odd, then the center of $G(F, \ell)$ is equal to the preimage of $\bar{R}^{(\ell)}F$ under $d_1 : G(F, \ell) \longrightarrow K_1^{(\ell)}F$.*

Proof: Let $\xi \in G(F, \ell)$. Let $d \in F^\times$ be such that $d_1(\xi) = \{d\}$. Then $\xi \equiv \langle d\rangle$ modulo the center of $G(F, \ell)$. Thus ξ is in the center of $G(F, \ell)$ if and only if the identity $\langle d, e\rangle = \langle e, d\rangle$ holds for every $e \in F^\times$. But for any $e \in F^\times$ we have the equivalences

$$\langle d, e\rangle = \langle e, d\rangle \iff \{d, e\} = \{e, d\} \iff 2 \cdot \{d, e\} = 0 \iff \{d, e\} = 0\,.$$

Therefore ξ belongs to the center of $G(F, \ell)$ if and only if $d \in R^{(\ell)}F$. □

7 Delzant classes

Stiefel-Whitney classes of quadratic forms were defined by Delzant in [2] and later addapted by Milnor in [8] in such way that these maps take their values in K-groups instead of Galois cohomology groups. Here they will be generalised to what shall be named *Delzant classes.* They are disigned as a tool to perform computations in Milnor K-theory.

We denote by $U(F,\ell)$ the multiplicative group of the ring $\widehat{K}_*^{(\ell)}F$ and by $U^1(F,\ell)$ the subgroup consisting of the elements with constant term equal to 1 (in $K_0^{(\ell)}F = \mathbb{Z}/\ell\mathbb{Z}$). By (4.6), there is a unique group homomorphism

$$d : G(F,\ell) \longrightarrow U^1(F,\ell)$$

which maps $\langle x\rangle$ to $1+\{x\}$, for any $x \in F^\times$. We call d the *Delzant homomorphism* (*modulo* ℓ).

For any $n \in \mathbb{N}$ let $d_n : G(F,\ell) \longrightarrow K_n^{(\ell)}F$ be the composition of d with the projection from $\widehat{K}_*^{(\ell)}F$ to the nth component $K_n^{(\ell)}F$. In other terms, for $\xi \in G(F,\ell)$ we write

$$d(\xi) = \sum_{n=0}^{\infty} d_n(\xi)$$

with $d_n(\xi) \in K_n^{(\ell)}F$ ($n \in \mathbb{N}$); we call $d_n(\xi)$ the *n-th Delzant class of ξ.* Note that $d_0 : G(F,\ell) \longrightarrow K_0^{(\ell)}F$ is the constant map 1 and $d_1 : G(F,\ell) \longrightarrow K_1^{(\ell)}F$ is the determinant homomorphism defined earlier.

The interest of these maps in the study of K-groups lies in the fact that $K_n^{(\ell)}F$ is generated by the image of $d_n : G(F,\ell) \longrightarrow K_n^{(\ell)}F$. Indeed $K_n^{(\ell)}F$ ($n \geq 1$) is generated by symbols and any symbol $\{a_1,\dots,a_n\}$ is equal to the nth Delzant class of the ℓ-form $\langle a_1,\dots,a_n\rangle$.

7.1 Proposition. *For any $\varphi \in M(F,\ell)$ and $n > rk(\varphi)$, one has $d_n(\varphi) = 0$.*

Proof: Let φ be an ℓ-form of rank r over F. With $a_1,\dots,a_r \in F^\times$ such that $\varphi = \langle a_1,\dots,a_r\rangle$, we see that $d(\varphi) = (1+\{a_1\})\cdots(1+\{a_r\})$ lies in $\bigoplus_{i=0}^{r} K_i^{(\ell)}F$, whence $d_n(\varphi) = 0$ for $n > r$. □

7.2 Proposition. *For any $\xi_1,\dots,\xi_k \in G(F,\ell)$ one has*

$$d_n(\xi_1+\cdots+\xi_k) = \sum_{\substack{i_1,\dots,i_k\geq 0\\ i_1+\cdots+i_k=n}} \left(d_{i_1}(\xi_1)\cdots d_{i_k}(\xi_k)\right).$$

Proof: This is straightforward from the definitions of d and d_n ($n \geq 1$). □

7.3 Lemma. *Let* $a, b \in F^\times$. *Then* $d\,(\langle\!\langle 1,1\rangle\!\rangle - \langle\!\langle a,b\rangle\!\rangle) \;=\; 1 - \{a,b\}$.

Proof: One easily checks that $d(\langle\!\langle a,b\rangle\!\rangle) = 1 + \{a,b\} + \{-1,-1\}$ and $d(\langle\!\langle 1,1\rangle\!\rangle) = 1 + \{-1,-1\} = (1 - \{a,b\}) \cdot d(\langle\!\langle a,b\rangle\!\rangle)$. The statement now follows because $d : G(F,\ell) \longrightarrow U^1(F,\ell)$ is a group homomorphism. □

7.4 Proposition. *The restricted map* $d_2 : G^2(F,\ell) \longrightarrow K_2^{(\ell)}F$ *is a surjective homomorphism. For any* $a, b \in F^\times$, *it maps* $\langle\!\langle a,b\rangle\!\rangle - \langle\!\langle 1,1\rangle\!\rangle$ *to* $\{a,b\}$.

Proof: For any $\xi, \zeta \in G(F,\ell)$ we obtain from (7.2) that

$$d_2(\xi + \zeta) = d_2(\xi) + d_1(\xi)d_1(\zeta) + d_2(\zeta)\,.$$

The term $d_1(\xi)d_1(\zeta)$ is zero in case one of ξ and ζ belongs to $G^2(F,\ell)$. Therefore $d_2 : G^2(F,\ell) \longrightarrow K_2^{(\ell)}F$ is a homomorphism. It follows from (7.3) that $d_2\,(\langle\!\langle a,b\rangle\!\rangle - \langle\!\langle 1,1\rangle\!\rangle) \;=\; \{a,b\}$ for any $a,b \in F^\times$. Using (6.5), the surjectivity is now obvious. □

7.5 Corollary. *One has* $G^2(F,\ell) = 0$ *if and only if* $K_2^{(\ell)}F = 0$, *and in this case* $G(F,\ell)$ *is commutative and* $G^1(F,\ell)$ *is isomorphic to* $F^\times/F^{\times\ell}$.

Proof: If $G^2(F,\ell) = 0$, then $K_2^{(\ell)}F = 0$ by (7.4). Conversely, assume that $K_2^{(\ell)}F$ vanishes. Then (5.3) and (6.4) together imply that $G^2(F,\ell) = 0$. In this case $d_1 : G^1(F,\ell) \longrightarrow K_1^{(\ell)}F$ is an isomorphism, whence $G^1(F,\ell) \cong K_1^{(\ell)}F \cong F^\times/F^{\times\ell}$, and using (6.8) it follows that $G(F,\ell)$ is commutative. □

The next aim is to show that $G^3(F,\ell)$ is equal to the kernel of the homomorphism $d_2 : G^2(F,\ell) \longrightarrow K_2^{(\ell)}F$.

7.6 Lemma. *Let* $a,b,c \in F^\times$. *Then*

$$\langle\!\langle a,b,c\rangle\!\rangle \;=\; \langle\!\langle a,c\rangle\!\rangle + \langle\!\langle b,c\rangle\!\rangle - \langle\!\langle ab,c\rangle\!\rangle + 2\times\mathbb{H}\,.$$

Proof: Since $\langle\!\langle a,b,c\rangle\!\rangle = \langle\!\langle b,c\rangle\!\rangle + a * \langle\!\langle b,c\rangle\!\rangle^*$, the statement follows from

$$\begin{aligned}
a * \langle\!\langle b,c\rangle\!\rangle^* &= \langle -a^{-1}bc, ac^{-1}, (ab)^{-1}, -a\rangle \\
&\overset{(4.5)}{=} \langle a^{-1}c, -(ab)c^{-1}, (ab)^{-1}, -a\rangle \\
&\overset{(4.9)}{=} \langle -a, a^{-1}c, -(ab)c^{-1}, (ab)^{-1}\rangle \\
&= \langle -a, a^{-1}c\rangle + \langle -c^{-1}, c\rangle + \langle -(ab)c^{-1}, (ab)^{-1}\rangle - \mathbb{H} \\
&= \langle 1, -a, a^{-1}c, -c^{-1}\rangle + \langle c, -(ab)c^{-1}, (ab)^{-1}, -1\rangle - 2\times\mathbb{H} \\
&= \langle\!\langle c^{-1}, a\rangle\!\rangle + \langle\!\langle c^{-1}, ab\rangle\!\rangle^* - 2\times\mathbb{H} \\
&= \langle\!\langle a,c\rangle\!\rangle - \langle\!\langle c^{-1}, ab\rangle\!\rangle + 2\times\mathbb{H} \\
&= \langle\!\langle a,c\rangle\!\rangle - \langle\!\langle ab,c\rangle\!\rangle + 2\times\mathbb{H}\,.
\end{aligned}$$

□

7.7 Corollary. *For any $x, y, z \in F^\times$, one has the following congruences modulo $G^3(F,\ell)$:*

$$\langle\langle xy, z\rangle\rangle + 2\times\mathbb{H} \;\equiv\; \langle\langle x, z\rangle\rangle + \langle\langle y, z\rangle\rangle \quad (1)$$
$$\langle\langle x, yz\rangle\rangle + 2\times\mathbb{H} \;\equiv\; \langle\langle x, y\rangle\rangle + \langle\langle x, z\rangle\rangle \quad (2)$$

Proof: The lemma immediately yields (1). To obtain (2), one uses that $\langle\langle x, yz\rangle\rangle = \langle\langle yz, x^{-1}\rangle\rangle$ by (5.2). □

7.8 Lemma. *For any $a, b, c \in F^\times$ and $n \geq 1$ one has*

$$d_n\left(\langle\langle a,b,c\rangle\rangle - 4\times\mathbb{H}\right) \;=\; \begin{cases} \{-1\}^{n-3}\cdot\{a,b,c\} & \text{if } 4 \mid n\,, \\ 0 & \text{otherwise.} \end{cases}$$

Proof: For any $n \geq 1$, let s_n denote the term on the right hand side in the claimed equality. It is easily checked that

$$\left(1 + \sum_{n=1}^{\infty} s_n\right)\cdot\left(1 - \{ab,c\} + \{-1,a,b,c\}\right) \;=\; 1 - \{ab,c\}\,.$$

Using (7.6) and (7.3) we obtain that

$$\begin{aligned} d\left(\langle\langle a,b,c\rangle\rangle - 4\times\mathbb{H}\right) &= (1-\{a,c\})^{-1}\cdot(1-\{b,c\})^{-1}\cdot(1-\{ab,c\}) \\ &= (1-\{ab,c\}+\{-1,a,b,c\})^{-1}\cdot(1-\{ab,c\})\,. \end{aligned}$$

Therefore $d\left(\langle\langle a,b,c\rangle\rangle - 4\times\mathbb{H}\right) = (1 + \sum_{n=1}^{\infty} s_n)$. □

7.9 Theorem. *$G^3(F,\ell)$ is the kernel of $d_2 : G^2(F,\ell) \longrightarrow K_2^{(\ell)}F$.*

Proof: By (7.4), the restriction of d_2 to $G^2(F,\ell)$ is a homomorphism. It follows from (6.5) and (7.8) that d_2 is trivial on $G^3(F,\ell)$. Hence we obtain a homomorphism $\overline{d}_2 : G^2(F,\ell)/G^3(F,\ell) \longrightarrow K_2^{(\ell)}F$ which maps the difference $\langle\langle a,b\rangle\rangle - \langle\langle 1,1\rangle\rangle$ to the symbol $\{a,b\}$.

Now we consider the pairing $F^\times \times F^\times \longrightarrow G^2(F,\ell)/G^3(F,\ell)$ which associates to a pair (a,b) the class of $\langle\langle a,b\rangle\rangle - \langle\langle 1,1\rangle\rangle$. Since $\langle\langle a,b\rangle\rangle = \langle\langle 1,1\rangle\rangle$ whenever $\{a,b\} = 0$, this induces a homomorphism $K_2^{(\ell)}F \longrightarrow G^2(F,\ell)/G^3(F,\ell)$ which is inverse to $\overline{d}_2$. Thus $\overline{d}_2$ is an isomorphism. □

7.10 Remark. From (7.9) we readily obtain an exact sequence

$$0 \longrightarrow K_2^{(\ell)}F \xrightarrow{(\overline{d}_2)^{-1}} G^1(F,\ell)/G^3(F,\ell) \xrightarrow{\overline{d_1}} K_1^{(\ell)}F \longrightarrow 0\,.$$

This turns out to be the group extension constructed from the 2-cocycle $K_1^{(\ell)}F \times K_1^{(\ell)}F \longrightarrow K_2^{(\ell)}F$ given by the multiplication in $K_*^{(\ell)}F$.

7.11 Corollary. *If* $-1 \in F^{\times \ell}$*, then the kernel of* $d : G^1(F,\ell) \longrightarrow U^1(F,\ell)$ *is equal to* $G^3(F,\ell)$*.*

Proof: The kernel of $d : G^1(F,\ell) \longrightarrow U^1(F,\ell)$ is contained in the kernel of $d_2 : G^2(F,\ell) \longrightarrow K_2^{(\ell)}F$, whence in $G^3(F,\ell)$, by (7.9). Assume now that $-1 \in F^{\times \ell}$, i.e. $\{-1\} = 0$ in $K_1^{(\ell)}F$. Since $\{-1\} = 0$ in $K_1^{(\ell)}F$, (7.8) yields that $d(\pi - 4 \times \mathbb{H}) = 1$ for any 3-fold Pfister form π over F. Using (6.5), this implies that d is trivial on $G^3(F,\ell)$. □

7.12 Corollary. *The homomorphism* $d : G^1(F,\ell) \longrightarrow U^1(F,\ell)$ *is injective in each of the following cases:*

- ℓ *is odd;*
- $F^\times = F^{\times 2}$ *and* $\ell > 0$*;*
- $\mathrm{char}(F) = 2$ *and* F *is perfect.*

Proof: In any of these cases we have $-1 \in F^{\times \ell}$, so that the previous corollary applies. Hence the kernel of $d : G^1(F,\ell) \longrightarrow U^1(F,\ell)$ is equal to $G^3(F,\ell)$. The statement then follows, because $G^3(F,\ell) = 0$ by (6.7). □

7.13 Remark. If $d : G^1(F,\ell) \longrightarrow U^1(F,\ell)$ is injective, then we have immediately a positive answer to (5.4).

7.14 Proposition. *The homomorphism* $d : G^1(F,\ell) \longrightarrow U^1(F,\ell)$ *is surjective if and only if* $K_3^{(\ell)}F = 0$*.*

Proof: Given a symbol σ in $K_3^{(\ell)}F$, it is easy to see that $1+\sigma$ lies in the image of d if and only if $\sigma = 0$. This shows that the condition in the statement is necessary.

Assume now that $K_3^{(\ell)}F = 0$. The elements of $U^1(F,\ell)$ then are of the shape $1 + \alpha + \beta$ with $\alpha \in K_1^{(\ell)}F$ and $\beta \in K_2^{(\ell)}F$. Let such α and β be given. There is an element $\xi \in G^2(F,\ell)$ such that $\beta = d_2(\xi)$. Let $d \in F^\times$ be such that $\alpha = \{d\}$ in $K_1^{(\ell)}F$. For $\zeta = \xi + \langle d \rangle - \langle 1 \rangle \in G^1(F,\ell)$, we obtain

$$d(\zeta) = d(\xi) \cdot d(\langle d \rangle - \langle 1 \rangle) = (1+\beta) \cdot (1+\alpha) = 1 + \alpha + \beta\,,$$

since $\beta \cdot \alpha \in K_3^{(\ell)}F = 0$. Therefore the condition is sufficient. □

7.15 Conjecture. *If* $K_3^{(\ell)}F = 0$*, then* $d : G^1(F,\ell) \longrightarrow U^1(F,\ell)$ *is an isomorphism.*

We finish this section by a result on interdependencies among the different Delzant classes of a form. It generalises [8, Remark 3.4].

7.16 Theorem. *Let n be a positive integer. Let $n_1 > \ldots > n_k$ be the decreasing sequence of 2-powers such that $n = n_1 + \cdots + n_k$. Then for any ℓ-form φ over F one has the equality*

$$d_n(\varphi) \;=\; d_{n_1}(\varphi) \cdots d_{n_k}(\varphi)\,.$$

Proof: We put $d_n^*(\varphi) = d_{n_1}(\varphi) \cdots d_{n_k}(\varphi)$ and want to prove that $d_n^*(\varphi) = d_n(\varphi)$. If $rk(\varphi) = 1$ this is trivial. We proceed by induction on $rk(\varphi)$.

Assume now that $rk(\varphi) > 1$ and write $\varphi = \psi + \langle x\rangle$. For any $m \geq 1$, we have by the induction hypothesis

$$d_m^*(\psi) = d_m(\psi)\,,$$

and we know further from (7.1) and (7.2) that

$$d_m(\varphi) \;=\; d_m(\psi) + d_{m-1}(\psi)\cdot\{x\}\,.$$

We compute

$$\begin{aligned} d_n(\varphi) &= d_n(\psi) + d_{n-1}(\psi)\cdot\{x\} \\ &= d_n^*(\psi) + d_{n-1}^*(\psi)\cdot\{x\} \\ &= d_{n-n_k}^*(\psi)\cdot\big(d_{n_k}(\psi) + d_{n_k-1}^*(\psi)\cdot\{x\}\big) \\ &= d_{n-n_k}^*(\psi)\cdot(d_{n_k}(\psi) + d_{n_k-1}(\psi)\cdot\{x\}) \\ &= d_{n-n_k}^*(\psi)\cdot d_{n_k}(\varphi) \\ &= d_{n_1}(\psi)\cdots d_{n_{k-1}}(\psi)\cdot d_{n_k}(\varphi)\,. \end{aligned}$$

To finish the proof, we shall show for $1 \leq i < k$ that

$$d_{n_i}(\psi)\cdot d_{n_k}(\varphi) = d_{n_i}(\varphi)\cdot d_{n_k}(\varphi)\,.$$

Since n_i is a 2-power greater than n_k, the element $\xi = d_{n_i-1}^*(\psi)\cdot\{x\}$ in $K_*^{(\ell)}F$ is a multiple of $d_{n_k-1}(\psi)\cdot\{x\}$ as well as of $d_{n_k}(\psi)$. Using that $\zeta^2 = \{-1\}^m\cdot\zeta$ holds for any $\zeta \in K_m^{(\ell)}F$ and $m \geq 0$, it follows that

$$d_{n_k-1}^*(\psi)\cdot\{x\}\cdot\xi = \{-1\}^{n_k}\cdot\xi = d_{n_k}(\psi)\cdot\xi\,.$$

Since $d_{n_k-1}^*(\psi) = d_{n_k-1}(\psi)$, we obtain that

$$d_{n_k}(\varphi)\cdot\xi = (d_{n_k}(\psi) + d_{n_k-1}(\psi)\cdot\{x\})\cdot\xi = 0\,.$$

Since further $\xi = d_{n_i-1}(\psi)\cdot\{x\}$, we conclude that

$$d_{n_i}(\varphi)\cdot d_{n_k}(\varphi) \;=\; (d_{n_i}(\psi) + d_{n_i-1}(\psi)\cdot\{x\})\cdot d_{n_k}(\varphi) \;=\; d_{n_i}(\psi)\cdot d_{n_k}(\varphi)\,,$$

which is what we claimed above. □

8 Vanishing of higher K-groups

To illustrate how virtual forms and Delzant classes can be used to do computations in Milnor K-theory, we give an alternative proof of a result due to B. Kahn which, under the hypothesis that $-1 \in F^{\times \ell}$, relates the 'symbol length' of $K_2^{(\ell)}F$ to the vanishing on higher K-groups modulo ℓ.

8.1 Proposition. *Let $\xi \in K_2^{(\ell)}F$ and $m \in \mathbb{N}$. The following are equivalent:*

- *(i) ξ is a sum of m symbols in $K_2^{(\ell)}F$.*
- *(ii) There exists an ℓ-form φ of rank $2m+2$ over F such that $\xi = d_2(\varphi - \varphi^\circ)$.*
- *(iii) There exists an ℓ-form φ of rank $2m+1$ and of determinant $\{(-1)^{m+1}\}$ over F such that $\xi = d_2(\varphi - \varphi^\circ)$.*

Proof: The implication $(iii \Rightarrow ii)$ is obvious.

$(ii \Rightarrow i)$ Assume that $\xi = d_2(\varphi - \varphi^\circ)$, where φ is an ℓ-form of rank $2m+2$ over F. By (4.13), there exist ℓ-forms $\vartheta_1, \ldots, \vartheta_m$ over F, each of rank 4 and of trivial determinant, such that $\varphi - \varphi^\circ = (\vartheta_1 - \langle\langle 1,1\rangle\rangle) + \cdots + (\vartheta_m - \langle\langle 1,1\rangle\rangle)$. For $i = 1, \ldots, m$ we put $\xi_i = d_2(\vartheta_i - \langle\langle 1,1\rangle\rangle)$ and obtain $\xi = \xi_1 + \cdots + \xi_m$. Now, if $\vartheta = \langle x, y, z, (xyz)^{-1}\rangle$, then $d_2(\vartheta - \langle\langle 1,1\rangle\rangle) = \{-xy, -yz\}$. Therefore $\xi_1, \ldots, \xi_m$ are symbols in $K_2^{(\ell)}F$, so ξ is a sum of m symbols.

$(i \Rightarrow iii)$ Assume that ξ is a sum of m symbols in $K_2^{(\ell)}F$. We show by induction on m that there exists an ℓ-form φ of rank $2m+1$ and of determinant $d_1(\varphi) = \{(-1)^{m+1}\}$ such that $\xi = d_2(\varphi - \varphi^\circ)$. If $m = 0$, this is trivial. We assume now that $m > 0$ and write $\xi = \zeta + \{a, b\}$, where $a, b \in F^\times$ and ζ is a sum of $m-1$ symbols in $K_2^{(\ell)}F$. By induction hypothesis, there exists an ℓ-form ψ of rank $2m-1$ and of determinant $\{(-1)^m\}$ such that $\zeta = d_2(\psi - \psi^\circ)$. We write $\psi = \psi' + \langle c\rangle$. Now we consider the ℓ-form $\varphi = \psi' + \langle x, y, z\rangle$, where $x, y, z \in F^\times$ shall be fixed later. Clearly, φ has rank $2m+1$. In order to have $d_1(\varphi) = \{(-1)^{m+1}\}$, we need to have $xyz = -c$. We compute

$$\begin{aligned}
\varphi - \varphi^\circ &= \psi' + \langle x, y, z\rangle - \psi^\circ - \mathbb{H} \\
&= \psi' + \langle x, y, z\rangle + \langle (xyz)^{-1}, -xyz\rangle - \psi^\circ - 2 \times \mathbb{H} \\
&= (\psi' + \langle c\rangle - \psi^\circ) + (\langle x, y, z, (xyz)^{-1}\rangle - \langle\langle 1,1\rangle\rangle \\
&= (\psi - \psi^\circ) + (\langle x, y, z, (xyz)^{-1}\rangle - \langle\langle 1,1\rangle\rangle)
\end{aligned}$$

and obtain then

$$d_2(\varphi - \varphi^\circ) = d_2(\psi - \psi^\circ) + d_2\left(\langle x, y, z, (xyz)^{-1}\rangle - \langle\langle 1,1\rangle\rangle\right) = \zeta + \{-xy, -yz\}\,.$$

Hence, $d_2(\varphi - \varphi^\circ) = \xi$ holds as soon as we have $\{-xy, -yz\} = \{a, b\}$. Together with the condition that $xyz = -c$, this can be achieved by putting $x = -b^{-1}c$, $y = abc^{-1}$ and $z = a^{-1}c$. □

8.2 Theorem (Kahn). *Let $m \in \mathbb{N}$ and assume that $-1 \in F^{\times \ell}$. If every element of $K_2^{(\ell)}F$ is equal to a sum of m symbols, then $K_n^{(\ell)}F = 0$ for any $n \geq 2m + 2$.*

Proof: It obviously suffices to prove the claim for $n = 2m+2$. By hypothesis, we have $\{-1\} = 0$ in $K_1^{(\ell)}F$. As a consequence, for any ℓ-form φ of trivial determinant over F we have the equality $d(\varphi - \varphi^\circ) = d(\varphi)$. We shall use this observation below.

Suppose on the contrary that $K_n^{(\ell)}F \neq 0$. Hence there exist $a_1, \ldots, a_n \in F^\times$ such that $\{a_1, \ldots, a_n\} \neq 0$ in $K_n^{(\ell)}F$. We set $a = (a_1 \cdots a_n)^{-1}$ and $\varphi = \langle a_1, \ldots, a_n, a\rangle$. Using that n is even, an easy computation yields that $d_n(\varphi) = \{a_1, \ldots, a_n\}$.

Suppose now, that the element $d_2(\varphi) \in K_2^{(\ell)}F$ can be written as a sum of m symbols. By (8.1), there exists an ℓ-form ψ over F of rank $2m+1$ and of trivial determinant such that $d_2(\varphi) = d_2(\psi)$. It follows that $\varphi - \varphi^\circ$ and $\psi - \psi^\circ$ represent the same class in $G^2(F, \ell)/G^3(F, \ell)$. With (7.11) we conclude that $d(\varphi) = d(\psi)$. In particular, we have

$$d_n(\psi) \;=\; d_n(\varphi) \;=\; \{a_1, \ldots, a_n\} \;\neq\; 0\,.$$

However, since $n = 2m + 1 = 2$ this is in contradiction to (7.1). □

8.3 Remark. The original statement in [4, Section 3] is in slightly different terms. The proof given there uses divided power operations instead of virtual forms and Delzant classes.

Under the assumption that $-1 \in F^{\times \ell}$, it follows from [4, Appendix A] that the commutative ring $K_{**}^{(\ell)}F = \bigoplus_{i=0}^{\infty} K_{2i}^{(\ell)}F$ is endowed with a *system of divided powers*. That is, there is a unique collection of maps

$$[i]:\; K_{2m}^{(\ell)}F \longrightarrow K_{2im}^{(\ell)}F\,,\; x \longmapsto x^{[i]},$$

for $i, m \geq 0$, with the following properties:

(1) $s^{[0]} = 1$ and $s^{[1]} = s$,

(2) $(st)^{[i]} = s^i t^{[i]}$,

(3) $s^{[i]}t^{[j]} = \binom{i+j}{j}(st)^{[i+j]}$,

(4) $(s+t)^{[k]} = \sum_{i+j=k} s^{[i]}t^{[j]}$,

(5) $(s^{[i]})^{[j]} = \frac{(ij)!}{i!j!^i} s^{[ij]}$,

(6) $s^{[i]} = 0$ if s is a symbol and $i \geq 2$.

It is not just by chance that both strategies apply to prove Kahn's theorem. In fact, using the above rules for divided powers, one can easily prove the following link between divided powers and the Delzant classes of virtual forms: assuming that $-1 \in F^{\times\ell}$, one has for any ℓ-form φ over F and for any $i \geq 1$ the identities

$$d_{2i}(\varphi) = (d_2(\varphi))^{[i]} \quad \text{and} \quad d_{2i+1}(\varphi) = d_1(\varphi) \cdot d_{2i}(\varphi).$$

9 Isotropy and representation

We generalise the concept of isotropy from quadratic forms to virtual forms.

Let φ be an ℓ-form over F. We say that φ is *isotropic*, if $\varphi = \psi + \mathbb{H}$ for some ℓ-form ψ over F, otherwise we say that φ is *anisotropic*. Furthermore, we say that φ is *hyperbolic*, if $\varphi = m \times \mathbb{H}$ for some $m \geq 1$.

Let F'/F be a field extension. Given an ℓ-form $\varphi = \langle a_1, \ldots, a_m \rangle$ over F we denote by $\varphi_{F'}$ the corresponding ℓ-form $\langle a_1, \ldots, a_m \rangle$ over F'. Obviously passing from φ to φ_E defines a homomorphism $G(F, \ell) \longrightarrow G(F', \ell)$.

One may ask whether some well-known theorems about the isotropy behavior of quadratic forms under field extensions can be generalised in some way to virtual forms.

9.1 Question. *Let F'/F be a finite field extension of degree prime to ℓ. Does any anisotropic ℓ-form over F remain anisotropic after extension to F'?*

9.2 Question. *Let ℓ be a prime and assume that F contains an ℓth root of unity. Let $F' = F(\sqrt[\ell]{d})$ for some $d \in F \setminus F^{\times\ell}$ and $\varphi \in M(F, \ell)$. Is it true that $\varphi_{F'}$ is isotropic if and only φ contains a subform β with $rk(\beta) = 2$ and $d_1(\beta) = \{(-d)^i\}$ with $0 \leq i < \ell$?*

In the case where $\ell = 2 \neq \mathrm{char}(F)$, both questions above have a positive answer, by Springer's Theorem (cf. [6, Chap. VII, §2]) and by [6, Chap. VII, §3].

Let $a \in F^\times$ and let φ be an ℓ-form over F. If φ can be written as $\langle a \rangle + \psi$ (resp. as $\psi + \langle a \rangle$) for some ℓ-form ψ, then we say that φ *left-represents* (resp.

right-represents) a; it is easy to see that this is equivalent to saying that the form $\langle -a^{-1}\rangle \perp \varphi$ (resp. $\varphi \perp \langle -a^{-1}\rangle$) is isotropic. Note that an isotropic form (left- and right-)represents any element of $F^\times$.

9.3 Proposition. *Let φ be an ℓ-form over F and $a \in F^\times$.*

(*a*) *If φ has even rank, then φ left-represents a if and only if it right-represents a^{-1}.*

(*b*) *If φ has odd rank, then φ left-represents a if and only if it right-represents a.*

(*c*) *If $d_1(\varphi) = 0$ or $d_1(\varphi) = \{-1\}$, then φ left-represents a if and only if it right-represents a.*

Proof: Parts (a) and (b) follow from (3.3) and (c) is clear from (4.9). □

In view of the proposition, if φ has odd rank or trivial determinant, one may say that φ *represents* some element $a \in F^\times$ if it left-represents a.

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