The Informed and Uninformed Agent's Price of a Contingent Claim

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Abstract

The existence of an adapted solution to a backward stochastic differential equation which is not adapted to the filtration of the underlying Brownian motion is proved. This result is applied to the pricing of contingent claims. It allows to compare the prices of agents who have different information about the evolution of the market. The problem is considered in both the classical and the Föllmer-Schweizer hedging case.

key-words: backward stochastic differential equations, pricing of contingent claims, anticipative information

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1 Introduction

The existence of an adapted solution to a backward stochastic differential equation has been proved when the filtration under consideration is generated by the underlyng Brownian motion. In the first part of this article we use Girsanov's theorem to derive an existence result for a larger filtration to which the Brownian motion is adapted. This result is applied to the pricing of a contingent claims by using forward-backward stochastic differential equations. This approach has several advantages in comparison to the classical techniques: There is no need to define an optimal hedging strategy and the whole system can be considered under the original measure. The power of this tool will be described here when we derive the price of a claim in both an incomplete and a complete market under different information structures available to the agent. Especially we are interested in the question when these prices are equal. The methods we use are described in [15], where techniques from control theory establish the relation between the price of a claim and the adjoint equation of a trivial control problem.

2 An existence result for a BSDE under additional information

From the results in [18], [19], [20] and [21] and the counterexamples in [1] it is common belief that a BSDE has an ordinary, that is adapted solution only if the underlying filtration is generated by the Brownian motion which is assumed to be augmented to satisfy the usual conditions in the sense of Meyer. In this section we shall establish an existence result for a BSDE where the filtration is greater than the one generated by the Brownian motion.

Let $(F_t)_{t\in[0,T]}$ be a filtration on a given probability space (Ω, F, P) which carries a standard Brownian motion (w_t) adapted to this given filtration. Note that this filtration might be larger than (F_t^w) the one generated by (w_t)

$$F_t^w \subset F_t. \tag{1}$$

Let

$$r : [0,T] \to \Re \tag{2}$$

be a deterministic $B(0,T)$-measurable function and

$$\theta : [0,T] \times \Omega \to \Re \tag{3}$$

be an (F_t)-progressively measurable function such that

$$P(\int_0^T \theta_u^2 du < \infty) = 1 \tag{4}$$

and

$$E(z_T) = 1 \tag{5}$$

for (z_t) the solution of the Doléans-Dade equation

$$\begin{aligned} dz_t &= -z_t \theta_t dw_t \\ z_0 &= 1. \end{aligned} \tag{6}$$

We are given a BSDE

$$dy_t = -\left[y_u r + Z_u^* \theta_u\right] du - Z_u^* dw_u \tag{7}$$

$$y_T = \xi \tag{8}$$

on (Ω, F, F_t, P), where ξ is an F_T-measurable random variable. The problem now consists in finding an (F_t)-adapted pair (y_t, Z_t) which satisfies integrability conditions such that the integral equation

$$y_t = \xi - \int_t^T \left[y_u r + Z_u^* \theta_u\right] du - \int_t^T Z_u^* dw_u. \tag{9}$$

holds P-a.s..

Now consider the auxiliary problem

$$\begin{aligned} d\widetilde{y}_t &= -\widetilde{y}_u r du - \widetilde{Z}_u^* d\widetilde{w}_u \\ &\widetilde{y}_T = \xi \end{aligned} \tag{10}$$

where $(\widetilde{w_t})$is the Girsanov transform of (w_t) with respect to the Girsanov functional associated with θ, i.e.

$$\widetilde{w_t} = w_t + \int_0^T \theta_u du. \tag{11}$$

$(\widetilde{w}_t)$ is a standard F_t-Brownian motion with respect to $\widetilde{P}$ where

$$d\widetilde{P} = z_T dP \tag{12}$$

is a probability measure on (Ω, F). Let $\left(F_t^{\widetilde{w}}\right)$ be the filtration generated by $\widetilde{w}_t$. Clearly,

$$\left(F_t^{\widetilde{w}}\right) \subset (F_t). \tag{13}$$

The crucial assumption which at first sight might appear quite technical is the equality

$$\left(F_T^{\widetilde{w}}\right) = (F_T)\,. \tag{14}$$

In the following section we will show that it is quite natural in an application to finance: (F_T) is the information available to an agent who has anticipative knowledge of the behavior of part of the market. This additional information is given right from the beginning. A second agent has to gather this information as time goes on until in the end the additional information becomes useless:

$$F_T^{\widetilde{w}} = F_T. \tag{15}$$

Under this assumption we may now consider

$$d\widetilde{y}_t = \xi - \int_t^T \widetilde{y}_u r du - \int_t^T \widetilde{Z}_u^* d\widetilde{w}_u \tag{16}$$

on $\left(\Omega, F, F_t^{\widetilde{w}}, \widetilde{P}\right)$ and from the results in [2] and [21] it is straightforward that an $\left(F_t^{\widetilde{w}}\right)$-adapted solution $\left(\widetilde{y}_t, \widetilde{Z}_t\right)$ exists.

Theorem 1 *The unique solution of (16) also solves (9).*

Proof. As ξ is $F_T^{\widetilde{w}} = F_T$-measurable the following equation holds:

$$\begin{aligned} \widetilde{y}_t &= \xi - \int_t^T \widetilde{y}_u r du - \int_t^T \widetilde{Z}_u^* d\widetilde{w}_u \\ &= \xi - \int_t^T \widetilde{y}_u r du - \int_t^T \widetilde{Z}_u^* dw_u - \int_t^T \widetilde{Z}_u^* \theta_u dw_u \\ &= \xi - \int_t^T \left[\widetilde{y}_u r + \widetilde{Z}_u^* \theta_u\right] du - \int_t^T \widetilde{Z}_u^* dw_u \end{aligned} \tag{17}$$

This proves that $\left(\widetilde{y}_t, \widetilde{Z}_t\right)$ solves (9). ■

This result will be applied to the case where an enlargement of filtration can be described by Girsanov's theorem.

3 Setting the hedging problem

We consider - in the usual notation - a (d+1)-dimensional asset consisting of a bond

$$dP_t^0 = r(t) P_t^0 dt \tag{18}$$

and d stocks

$$
\begin{aligned}
dP_t^i &= \mu_i(t,P_t)P_t^i dt + \sigma_{ij}(t,P_t)P_t^j dw_t^j, \\
t &\in [s,T], 1 \le i,j \le d
\end{aligned}
\tag{19}
$$

with initial

$$(1, p_1, ..., p_d). \tag{20}$$

Here (w_t) is a d-dimensional Brownian motion and (F_t) the augmented filtration of (w_t). All processes are assumed to live on (Ω, F, F_t, P) which is assumed to satisfy the usual condition in the sense of Meyer. The coefficients

$$\mu_i : [0,T] \times R^d \to R \tag{21}$$

and

$$\sigma_{ij} : [0,T] \times R^d \to R. \tag{22}$$

are assumed to satisfy conditions such that a pathwise unique solution exists. Finally σ is assumed to be invertible. For details on the conditions, see [8] or [14]. The claim is an F_T-measurable random variable ξ , which e.g. is of the form

$$\xi = g(P_T, \omega) \tag{23}$$

where g is a $B \otimes F_T$-measurable function

$$g : R^d \times \Omega \to R \tag{24}$$

such that ξ is twice integrable. In [15] we showed that the pricing and hedging problem for the above claim and asset may be considered as a trivial control problem. To this end we define the risk premium process θ by

$$\sigma\theta = \mu - r1_d \tag{25}$$

where $\sigma = (\sigma_{ij}), \mu = (\mu_i)$ and 1_d is the d-dimensional vector whose every component is 1.

We assume that θ exists as a d-dimensional bounded, predictable vector process, this ensuring the absence of arbitrage in our model (see [13], [14]). Next consider the following trivial control problem with dynamics

$$dz_{st} = -z_{st}\left[rdt + \theta_t^* dw_t\right] \tag{26}$$

$$z_{ss} = 1 \tag{27}$$

and cost criterion

$$J = E\left[z_{sT}\xi\right]. \tag{28}$$

The formal adjoint of this problem is given by

$$y_t = \xi - \int_t^T \left[y_u r + Z_u^* \theta_u\right] du - \int_t^T Z_u^* dw_u. \tag{29}$$

Following [6], [7], [8], [9] the unique solution (y_t, Z_t) of this forward-backward stochastic differential equation is the hedging price of the claim ξ

$$y_t = \bar{E}\left[\xi \mid F_t\right] \tag{30}$$

where $\bar{E}$ denotes expectation with respect to the Girsanov measure associated with the Girsanov functional (z_{st}). The hedging portfolio for the risk premium process θ is given by

$$\pi_t = \sigma^{-1} Z_t. \tag{31}$$

This may be written in a more familiar way when we consider a generalized pricing system, i.e. a predictable function

$$u : [s, T] \times R^d \times \Omega \to R \tag{32}$$

such that

$$y_t = u(t, P_t). \tag{33}$$

The pricing system satisfies the stochastic partial differential equation

$$u(t,x) = g(x) + \int_t^T \left[\frac{1}{2}\sigma u_{xx}\sigma + (\mu - \sigma\theta)u_x - ru + \sigma k_x - k\theta\right] ds - \int_t^T k(s,x) dw_s. \tag{34}$$

Obviously the hedging portfolio is then given by

$$\pi_t = \sigma^{-1} Z_t = \nabla u(t, P_t) + \sigma^{-1} k(t, P_t). \tag{35}$$

The details of these results can be found in [15].

If in the above spde the coefficients are deterministic then the solution is given by $(u, k) = (u, o)$ and the spde goes over into the form of the classical Black-Scholes formula for the optimal hedging problem. Note however here that for the pricing of the claim in this setting no change of measure and no explicit optimal hedging strategy has to be considered. For the general development of the theory of FBSDE see [6], [7], [18], [19], [20], [21], [22], [25].

4 Available Information

We consider a market with two agents A_0 and A_{in}.The latter has more information about the evolution of the asset than the former. By making use of results by [10], [12] and [23] we will consider the prices under these different information structures (F_t^0) and (F_t^{in}), respectively.

Let $\alpha_1(t), \alpha_2(t)$ be two nonsingular matrix valued deterministic, measurable functions of t with

$$\alpha_1\alpha_1^* + \alpha_2\alpha_2^* = I \tag{36}$$

for all t.

Then

$$w(t) = \alpha_1 w_1(t) + \alpha_2 w_2(t) \tag{37}$$

is again an F_t-Brownian motion if both w_1 and w_2 are independent F_t-Brownian motions. Let in the above model for our market w be decomposed in this way. Let

$$F_t^0 = F_t = F_t^w \text{ and } F_t^{in} = \sigma(w_1(T)) \vee F_t. \tag{38}$$

Both F_t^0 and F_t^{in} are again assumed to satisfy the usual conditions. In this case the results of [10], [12], and [23] allow us to describe the enlargement of filtration by Girsanov's theorem:

Theorem 2 *Let*

$$\triangle_u(x) = \alpha_1(TI - \alpha_1\alpha_1^* t)^{-1}(x - \alpha_1^* w(t)) \tag{39}$$

be the cross variation of the F_t-conditional density of $(w_1(t))$ and (w_t) divided by the density, then

$$B(t) = w(t) - \int_s^t \triangle_u(w_1(T))du \tag{40}$$

is a d-dimensional $\left(\Omega, F_t^{in}, P\right)$-standard Brownian motion.

Remark 1 *Note here that (B_t) is an F_t^{in}-Brownian motion for which in general the natural filtration (F_t^B) will be different from (F_t^{in}). Below we consider an FBSDE with driving Brownian motion (B_t). Following the arguments in the first section we will derive a solution to this FBSDE.*

For the informed agent A_{in} the asset appears as

$$\begin{aligned} dP_t^0 &= r(t)P_t^0 dt \\ dP_t^{in} &= P_t^{in}(\mu_i(t, P_t^{in})P_t^{in} dt + \sigma_{ij}\triangle_t^j(w_1(T))dt + \sigma_{ij}(t, P_t^{in})dB_t^j), \\ t &\in [s,T], 1 \le i,j \le d. \end{aligned}$$

The risk premium processes for the two agents are respectively

$$\theta^0 = \sigma^{-1}(\mu - r1_d) \tag{41}$$

and

$$\theta^{in} = \sigma^{-1}(\mu + \sigma\triangle_t - r1_d). \tag{42}$$

We now formally apply the arguments of section 1 to compute the prices y_t^0 and y_t^{in} for A_0 and A_{in}:
Let z_{st}^0 be a solution of

$$\begin{aligned} dz_{st}^0 &= -z_{st}^0 \left[rdt + \theta^{0*}dw_t\right] \\ z_{ss}^0 &= 1 \end{aligned} \tag{43}$$

and z_{st}^{in} be a solution of

$$\begin{aligned} dz_{st}^{in} &= -z_{st}^{in} \left[rdt + \theta^{in*}dw_t\right] \\ z_{ss}^{in} &= 1, \end{aligned} \tag{44}$$

then the prices for the $F_T^{in} = F_T^0 = F_T$ claim is given by

$$y_t^0 = \xi - \int_t^T \left[y_u^0 r + Z_u^{0*}\theta^0\right] dt - \int_t^T Z_u^{0*} dw_u \tag{45}$$

and

$$y_t^{in} = \xi - \int_t^T \left[y_u^{in} r + Z_u^{in*}\theta^i\right] dt - \int_t^T Z_u^{in*} dB_u \tag{46}$$

on $\left(\Omega, F, F_t^0, P\right)$ and $\left(\Omega, F, F_t^{in}, P\right)$ respectively.

Theorem 3 *The prices (y_t^0) and (y_t^{in}) are equal, if the price (y_t^0) exists. Moreover under the above assumptions it is a standard result that (y_t^0, Z_t^0) exists.*

Proof. Let (y_t^0, Z_t^0) be a solution then (y_t^0) and (Z_t^0) are also F_t^{in}−measurable. Then

$$\begin{aligned} y_t^0 &= \xi - \int_t^T \left[y_u^0 r + Z_u^{0*}\theta^0\right] dt - \int_t^T Z_u^{0*} dw_u \\ &= \xi - \int_t^T \left[y_u^0 r + Z_u^{0*}\triangle_u + Z_u^{0*}\theta^0\right] dt - \int_t^T Z_u^{0*} dw_u \\ &= \xi - \int_t^T \left[y_u^{in} r + Z_u^{in*}\theta^{in}\right] dt - \int_t^T Z_u^{in*} dB_u \\ &= y_t^{in} \end{aligned} \tag{47}$$

so that (y_t^0, Z_t^0) solves for (y_t^i, Z_t^i). ■

Corollary 4 *The optimal hedging strategies are equal for both agents, if* (y_t^0, Z_t^0) *exists.*

Proof. From section 1 the optimal hedging strategies are given by

$$\pi_t^0 = \sigma^{-1} Z_t^0 \text{ and } \pi_t^{in} = \sigma^{-1} Z_t^{in}. \tag{48}$$

As $Z_t^0 = Z_t^{in}$ by the uniqueness of solutions the result follows. Note, however,that the risk premium processes are different. ■

Instead of studying the forward-backward equations we could also look at the corresponding pricing systems:

$$\begin{aligned} u^0(t,x) &= g(x) + \int_t^T \left[\frac{1}{2} \sigma u_{xx}^0 \sigma + (\mu - \sigma\theta^0) u_x^0 - r u^0 + \sigma k_x^0 - k^0 \theta^0 \right] ds \\ &\quad - \int_t^T k^0(s,x) dw_s \end{aligned} \tag{49}$$

and

$$\begin{aligned} u^{in}(t,x) &= g(x) + \int_t^T \left[\frac{1}{2} \sigma u_{xx}^{in} \sigma + (\mu - \sigma\theta^{in}) u_x^{in} - r u^{in} + \sigma k_x^{in} - k^{in} \theta^{in} \right] ds \\ &\quad - \int_t^T k^{in}(s,x) dB_s \end{aligned} \tag{50}$$

in an abbreviated notation.

Again $\left(u_t^0, k^0\right)$ solves the equation for $\left(u_t^{in}, k^{in}\right)$, if it exists, and hence

$$y_t^0 = u^0(t, P_t, \omega) = u^{in}(t, P_t, \omega) = y_t^{in}. \tag{51}$$

Remark 2 *If in the above spde all coefficients are deterministic, then a solution is given by*

$$\left(u_t^0, k^0\right) = \left(u_t^0, 0\right) \text{ and } \left(u_t^{in}, k^{in}\right) = \left(u_t^0, 0\right). \tag{52}$$

Furthermore $y_s^0 = y_s^{in}$ *are deterministic as a consequence of the Markov nature of the processes.*

5 The Föllmer-Schweizer uninformed agent

Here we assume that

$$P_t^p = \left(P_t^1, ..., P_t^m\right)^* \tag{53}$$

are the primary securities which are actually traded, $m < d$. Split up the matrix σ into

$$\begin{aligned} p^1 &= (\sigma_{ij})_{i \leq m, j \leq d} \\ p^2 &= (\sigma_{ij})_{m < i, j \leq d} \\ \sigma &= \begin{pmatrix} p^1 \\ p^2 \end{pmatrix}. \end{aligned} \tag{54}$$

Note that if σ has full rank, then $p^1 p^{1*}$ is invertible. Again let ξ be a contingent claim which is assumed to be square integrable and (F_T^w)-measurable. As there might not be an admissible hedging strategy π^{FS} which finances ξ, i.e. the BSDE

$$\begin{aligned} y_t &= \xi - \int_t^T \left[y_u r + Z_u^{01*} \theta^0 \right] du - \int_t^T Z_u^{01*} dw_u \\ p^{1*} \pi^{FS} &= Z^{01} \end{aligned} \tag{55}$$

might not have a solution. Föllmer and Schweizer introduced a self-financing-in-mean strategy which in terms of FBSDEs is a triple

$$(y_t, Z_t^{01}, O_t) \tag{56}$$

solving

$$y_t = \xi - \int_t^T \left[y_u r + Z_u^{01*} \theta^0 \right] du - \int_t^T Z_u^{01*} dw_u + O_T - O_t, \tag{57}$$

where by definition the *tracking error* $O_T - O_t$ and $\int_t^T Z_u^{01*} dw_u$ are orthogonal martingales. For details the reader is referred to [11] and [7].

The aim of this chapter is to compare the uninformed and informed Föllmer-Schweizer-(FS-)prices and hedging strategies.

We first consider the uninformed agent. His risk premium is given by

$$\theta^0 = \sigma^{-1}(\mu - r 1_d) \tag{58}$$

-in the above notation - if both sets of securities are traded. If only the first m securities are traded the risk premium θ^{01} obviously is the projection of θ^0 onto the range of p^{1*}, so that

$$\theta^{01} = p^{1*}(p^1 p^{1*}) p^1 \theta^0. \tag{59}$$

When we compute the adjoint of the control problem associated with

$$dz_{st}^{01} = -z_{st}^{01} \left[r dt + \theta^{01*} dw_t \right] \tag{60}$$

we find the formal adjoint

$$y_t^{01} = \xi - \int_t^T \left[y_u^{01} r + Z_u^{01*}\theta^{01}\right] dt - \int_t^T Z_u^{01*} dw_u. \tag{61}$$

Decompose Z_t^{01} as

$$Z_t^{01} = Z_t^{011} + Z_t^{012} \tag{62}$$

where

$$\begin{aligned} Z_t^{011} &= proj_{range(p^{1*})} Z_t^{01} \\ Z_t^{012} &= proj_{\ker(p^1)} Z_t^{01}. \end{aligned} \tag{63}$$

and rewrite

$$y_t^{01} = \xi - \int_t^T \left[y_u^{01} r + Z_u^{011*}\theta^{01} + Z_u^{012*}\theta^{01}\right] dt - \int_t^T Z_u^{011*} dw_u - \int_t^T Z_u^{012*} dw_u. \tag{64}$$

As

$$\int_t^T Z_u^{011*} dw_u \text{ and } \int_t^T Z_u^{012*} dw_u \tag{65}$$

are orthogonal y_t^{01} is the F-S-price. The FS-strategy π^{011} is defined by

$$p^{1*}\pi^{011} = Z^{011} \tag{66}$$

or

$$\begin{aligned} p^{1*}\pi^{011} &= p^{1*}(p^1 p^{1*})^{-1} p^1 Z^{01} \\ &= p^{1*}(p^1 p^{1*})^{-1} p^1 (p^{1*}(p^1 p^{1*})^{-1} p^1) Z^{01} + p^{1*}(p^1 p^{1*})^{-1} p^1 Z^{012}. \end{aligned} \tag{67}$$

Note here that $(p^1 p^{2*}) = 0$ implies that

$$p^{1*}\pi^{011} = p^{1*}(p^1 p^{1*})^{-1} p^1 Z^{011} \tag{68}$$

so that this proves the well known fact that the F-S-strategy can be computed by completing the market such that $(p^1 p^{2*}) = 0$ and the optimal strategy is given by

$$\pi^{011} = (p^1 p^{1*})^{-1} p^1 Z^{011} \tag{69}$$

which does not involve p^2. It is also easily seen that then

$$p^{1*}\pi^{011} = p^{1*} proj_{range(p^{1*})} \pi_t^{01}. \tag{70}$$

Next we are going to compare the F-S-price to the price when all securities are traded. To this end we decompose θ^0 into

$$\begin{aligned} \theta^0 &= \theta^{01} + \theta^{02} \\ \theta^{01} &= proj_{range(p^{1*})}\theta^0, \\ \theta^{02} &= proj_{\ker(p^1)}\theta^0. \end{aligned} \tag{71}$$

Then $\int \theta_t^{01*}dw$ and $\int \theta_t^{02*}dw$ are orthogonal and z_{st}^0 may be written as

$$z_{st}^0 = z_{st}^{01} z_{st}^{02} \tag{72}$$

where

$$\begin{aligned} dz_{st}^{01} &= -z_{st}^{01}\left[rdt + \theta^{01*}dw_t\right], \\ dz_{st}^{02} &= -z_{st}^{02}\left[\theta^{02*}dw_t\right]. \end{aligned} \tag{73}$$

With this the price is given by

$$\begin{aligned} y_t^0 &= \xi - \int_t^T \left[y_u^0 r + U_u^{0*}\theta^0\right] du - \int_t^T U_u^{0*} dw_u \\ &= \xi - \int_t^T \left[y_u^0 r + U_u^{0*}(\theta^{01} + \theta^{02})\right] du - \int_t^T U_u^{0*} dw_u \\ &= \xi - \int_t^T \left[y_u^0 r + U_u^{01*}\theta^{01} + U_u^{02*}\theta^{02})\right] du - \int_t^T U_u^{01*} dw_u - \int_t^T U_u^{01*} dw_u \end{aligned} \tag{74}$$

where U^{0i} are the respective projections. Then

$$\begin{aligned} y_t^{02} &= y_t^0 - y_t^{01} \\ &= -\int_t^T \left[(y_u^0 - y_u^{01})r + (U_u^{01*} - Z_u^{011*})\theta^{01} + U_u^{02*}\theta^{02})\right] du \\ &\quad - \int_t^T (U_u^{01*} - Z_u^{011*})dw_u - \int_t^T (U_u^{02*} - Z_u^{012*})dw_u \end{aligned} \tag{75}$$

Theorem 5 *If the tracking error of the FS-strategy*

$$\int_t^T (Z_u^{012*})dw_u \tag{76}$$

is identically zero, then the FS-price and the price for the claim when the complete asset is traded are equal.

Proof. We can rewrite the above equation as

$$y_t^{02} = -\int_t^T \left[y_u^{02} r + V_u^* \theta^{01} + U_u^{02*} \theta^{02})\right] du - \int_t^T V_u^* dw_u - \int_t^T U_u^{02*} dw_u \quad (77)$$

By the uniqueness of solutions, this equation has a solution

$$\left(y^{02}, V, U^{02}\right) = (0,0,0) \,. \quad (78)$$

So in this case also U^{02} is zero, which means that also the optimal hedging strategy against the risk premium θ^{02} is zero. In this sense the secondary securities are not traded even if they are offered, which obviously makes sense. ■

Remark 3 *We are in a very lucky position, when we trade in an almost complete market like the FS-market. In this case, the optimal FS-hedging strategy can be given in closed form as above. This is completely different when we try to apply the above techniques to the following toy problem:*

The two dimensional asset is given as a semimartingale whose martingale part is given by a double martingale in the sense of [3], [4], i.e. the martingale splits into a Brownian martingale w and an (orthogonal) Poisson martingale q. Then

$$dz_{st} = -z_{st}(\theta_1^* dw_t + \theta_2^* dq_t). \quad (79)$$

With this the price is the formal adjoint of a trivial control problem is given by

$$y_t = \xi - \int_t^T \left[y_u r + Z_u^{1*} \theta_1 + Z_u^{2*} \theta_2\right] du - \int_t^T Z_u^{1*} dw_u - \int_t^T Z_u^{2*} dq_u. \quad (80)$$

Under quite strong assumptions this equation is shown in [24] to have a unique solution. When we compare this equation to equations (64) and (68) we are tempted to rewrite (y_t) as

$$y_t = \xi - \int_t^T \left[y_u r + Z_u^{1*} \theta_1\right] du - \int_t^T Z_u^{1*} dw_u - s_t \quad (81)$$

where (s_t) is a special semimartingale with martingale term strongly orthogonal to $\int_t^T Z_u^{1*} dw_u$. But (s_t) depends on the hedging strategy so that the argument following cannot be applied. A possible way out of this difficulty might be to consider a more general definition of what a solution of an FBSDE is: Let a solution of

$$y_t = \xi - \int_t^T \left[y_u r + Z_u^* \theta\right] du - \int_t^T Z_u^* dm_u \quad (82)$$

be defined as a nonanticipative pair (y, Z) which minimizes

$$E\left(y_T - \xi\right)^2 \quad (83)$$

with respect to Z. This leads to a control problem which describes the minimal variance price. Details will be found in the forthcoming paper [16].

6 The F-S- informed agent

The informed agent's price of the asset is seen as

$$\begin{aligned} dP_t^0 &= r(t)P_t^0 dt \quad (84)\\ dP_t^{in} &= P_t^{in}(\mu_i(t,P_t)P_t^{in}dt + \sigma_{ij}\triangle_t^j(w_1(T))dt + \sigma_{ij}(t,P_t)dB_t^j) \end{aligned}$$

in the notation of the preceding sections. By θ_t^{in} we denote the risk premium of the whole asset defined by

$$\sigma\theta^{in} = (\mu + \sigma\triangle_t - r1_d). \quad (85)$$

Again we denote by θ^{i1} the projection of θ^i onto the range of p_1^*, so that

$$\theta^{in1} = p^{1*}(p^1p^{1*})p^1\theta^{in}. \quad (86)$$

By exactly the same arguments as above the price then is given by

$$y_t^{in1} = \xi - \int_t^T \left[y_u^{in1}r + Z_u^{in11*}\theta_u^{i1}\right]du - \int_t^T Z_u^{in11*}dB_u - \int_t^T Z_u^{in12*}dB_u. \quad (87)$$

and the following theorem holds:

Theorem 6 *Let $\left(y^{01}, Z^{011}, Z^{012}\right)$ solve (61), then it also solves (87) and the optimal hedging portfolio is given by*

$$p_1^*\pi^{in11} = Z^{in11} = Z^{011} = p_1^*\pi^{011}. \quad (88)$$

Proof. The result is immediate from the consideration above. ■

7 Conclusion

In the first section we derive an existence result for an adapted solution to a BSDE which is not adapted to the filtration generated by the underlying Brownian motion. This result is used to exemplify the power of the FBSDE-techniques as a tool to derive the prices of a claim in different situations of available information to different agents. Especially we find examples where the prices coincide. Many of these results are known from the literature, but these approaches use the hedging and/or minimal martingale measure approach. The new feature of this contribution is the exclusive use of FBSDE interpreted as the adjoint of a trivial control problem.

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