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BOOK REVIEW

# Dan A. Lee: “Geometric Relativity”

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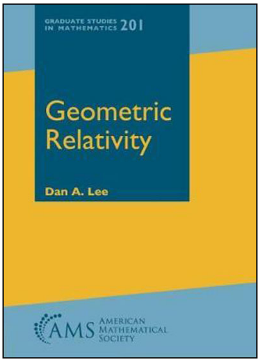


The book under review is an invitation to general relativity. It addresses readers with a good mathematical background in differential geometry, more precisely Riemannian geometry and the geometry of submanifolds, in (linear elliptic) partial differential equations and in algebraic topology. It does not require previous knowledge in physics. Although the topics of this book belong to general relativity, they are formulated and studied using the mathematical language of the three areas mentioned above. A recurrent topic are theorems related to the positivity of mass.

The book consists of two parts. In the first part, the author considers questions that can be formulated in Riemannian manifolds, while, in the second part, he addresses questions that make more use of Lorentzian manifolds. Each part starts with an introduction and considers some classical problems as well as recent developments.

The part “Riemannian geometry” starts with a review of Riemannian geometry with a strong focus on scalar curvature including the result that compact 3-manifolds admit a positive scalar curvature metric if and only if they are a connected sum of spherical space forms and copies of $S^2 \times S^1$.

In Chap. 2, the author considers hypersurfaces and the first and second variation of volume. Consequences are the vanishing of the mean curvature for volume minimizers, but also a theorem of R. Schoen and S.-T. Yau [7] that states that in an oriented 3-manifold with positive scalar curvature, every stable, two-sided closed minimal surface must be a topological sphere. Here and in the following, we will tacitly assume connectedness when necessary. The chapter also includes results involving more topology, gives a quick glimpse on currents in geometric measure theory and

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contains discussions about the situation in higher dimensions and includes rigidity theorems.

Chapter 3 discusses the Riemannian positive mass theorem, whose first version is by R. Schoen and S.-T. Yau [6]. It states that the ADM mass

$$m_{\mathrm{ADM}}(M,g) = \lim_{\rho\to\infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_\rho} \sum_{i,j=1}^{n} (g_{ij,i} - g_{ii,j}) \frac{x^j}{|x|} \, \overline{d\mu}_{S_\rho},$$

named after R. Arnowitt, S. Deser and C. Misner, of each end of a complete asymptotically flat manifold with nonnegative scalar curvature is nonnegative and equality only occurs in Euclidean space. In particular, there exist no such manifolds, where the scalar curvature is nonnegative, does not vanish identically and is zero outside of a compact set. Here, $\omega_n$ refers to the volume of the unit ball in $\mathbb{R}^n$, $S_\rho$ to a Euclidean coordinate sphere in an asymptotically flat coordinate chart corresponding to that end and a bar refers to a quantity with respect to the Euclidean metric. Around that result, the author introduces Schwarzschild space, asymptotic flatness, weighted spaces, the ADM mass and a motivation for that definition of mass. Then he discusses the Riemannian positive mass theorem in several special cases, in particular the conformal situation. For the proof of the positive mass theorem, he uses a simplification of J. Lohkamp [5] to reduce the problem to a related rigidity question. The chapter ends with a sketch of an alternative proof using Ricci flow.

For readers not familiar with the definition of the ADM mass, wondering why this strange expression should have something to do with mass, we extract some thoughts from Section "3.1.3 Motivation for mass" in condensed form skipping technical details and setting physical constants equal to one: In Newtonian gravity, the gravitational acceleration due to a point mass $m$ at $x_0$ on a test particle at $x$ is $-\frac{m}{|x-x_0|^2}\frac{x-x_0}{|x-x_0|}$. This equals $-\nabla V(x)$ for the Newtonian potential $V(x) = -\frac{m}{|x-x_0|}$. If $\rho\colon \mathbb{R}^3 \to \mathbb{R}$ denotes the mass density, the total mass is given by $m = \int_{\mathbb{R}^3} \rho(y)\,dy$, and the gravitational potential is obtained by superposition or integration as

$$V(x) = -\int_{\mathbb{R}^3} \frac{\rho(y)}{|x-y|}\,dy.$$

Comparing this expression with the representation formula for harmonic functions yields $\Delta V = 4\pi\rho$. We now use the divergence theorem and obtain

$$\begin{aligned} m &= \int_{\mathbb{R}^3} \rho(x)\,dx \\ &= \int_{\mathbb{R}^3} \frac{1}{4\pi} \Delta V(x)\,dx \\ &= \lim_{r\to\infty} \int_{B_r(0)} \frac{1}{4\pi} \operatorname{div}(\nabla V(x))\,dx \\ &= \lim_{r\to\infty} \int_{S_r} \frac{1}{4\pi} \left\langle \nabla V(x), \frac{x}{|x|} \right\rangle d\mu_{S_r}. \end{aligned}$$

Hence, we see that in Newtonian gravity, the total mass $m$ of a system can be written as a limit of integrals over spheres as in the definition of the ADM mass.

In general relativity, scalar curvature is proportional to mass density: $R_g = 16\pi\rho$. Hence, if general relativity were a linear theory, the total mass $m$ should be

$$\frac{1}{16\pi}\int_{\mathbb{R}^3} R_g(x)\, d\mu_g.$$

Assume now, that $(M, g)$ is close to Euclidean space $(\mathbb{R}^3, \bar{g})$. The first variation formula for the scalar curvature of the metric $h$ in direction $g$ is given by

$$DR|_h(g) = -\Delta_h(\mathrm{tr}_h\, g) + \mathrm{div}_h(\mathrm{div}_h\, g) - \langle \mathrm{Ric}_h, g\rangle_h.$$

Applying this with the Euclidean metric $h = \bar{g}$ of zero scalar and Ricci curvature, we obtain $R|_g \approx 0 + DR|_{\bar{g}}(g - \bar{g})$. In the following calculation, all differential operators and traces are with respect to the Euclidean metric. We integrate our approximation and use the divergence theorem to obtain

$$\begin{aligned} m &\approx \frac{1}{16\pi}\int_{\mathbb{R}^3} DR|_{\bar{g}}(g-\bar{g})\, dx \\ &= \lim_{r\to\infty} \frac{1}{16\pi}\int_{B_r(0)} -\Delta(\mathrm{tr}\, g) + \mathrm{div}(\mathrm{div}\, g) + 0\, dx \\ &= \lim_{r\to\infty} \frac{1}{16\pi}\int_{S_r} (-d(\mathrm{tr}\, g) + \mathrm{div}\, g)\frac{x}{|x|}\, \overline{d\mu}_{S_r} \\ &= \lim_{r\to\infty} \frac{1}{16\pi}\int_{S_r} \sum_{i,j=1}^{3} (-g_{ii,j} + g_{ij,i})\frac{x^j}{|x|}\, \overline{d\mu}_{S_r}. \end{aligned}$$

This is precisely the expression in the definition of the ADM mass.

Chapter 4 focuses on the Riemannian Penrose inequality conjecture. Assume that $(M^n, g)$ is a complete one-ended asymptotically flat manifold of nonnegative scalar curvature with boundary $\partial M$ which is an apparent horizon. Being an apparent horizon means that $\partial M \subset M$ is a minimal hypersurface and there is no other minimal hypersurface in $M$ enclosing $\partial M$. Then the Riemannian Penrose inequality conjectures that

$$m_{\mathrm{ADM}}(M, g) \geq \frac{1}{2}\left(\frac{|\partial M|}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}}$$

and equality occurs if and only if $M$ is isometric to half of the Schwarzschild space of mass $m_{\mathrm{ADM}}(M, g)$. As $|\partial M| \geq 0$, it generalizes the positive mass theorem. This chapter addresses two approaches to that conjecture:

G. Huisken and T. Ilmanen [4] use inverse mean curvature flow to prove it for $n = 3$, a certain decay rate in the asymptotic flatness condition and $|\partial M|$ replaced by the area of any connected component of $\partial M$. The idea of the proof is to consider surfaces $(\Sigma_t)_{t\geq 0}$ that solve inverse mean curvature flow and flow from a connected component of the apparent horizon to infinity. The author presents several calculations that lead to the result if a classical solution exists, e.g. that the Hawking mass

$$m_{\text{Haw}} = \sqrt{\frac{|\Sigma_t|}{16\pi}}\left(1 - \frac{1}{16\pi}\int_{\Sigma_t} H^2\, d\mu_t\right)$$

is weakly increasing. He also gives a fair amount of details of a weak level set flow that has to be studied in general and allows for changes in the topology of the evolving surfaces.

H. Bray [1] proves the conjecture for $n < 8$. He evolves the metric $(g_t)_{t\ge 0}$ with a conformal harmonic factor varying from 0 at some evolving inner boundary $\partial\Omega_t$ to $-1$ at infinity, where $\Omega_t$ is obtained as a strictly minimizing hull. This flow also allows for changes in the topology of $\partial\Omega_t$. The proof uses that $\partial\Omega_t \subset (M, g_t)$ has constant volume, monotonicities, e.g. for the ADM mass, and the convergence of metrics to a Schwarzschild metric.

Chapter 5 reviews spin geometry and then focuses on the spinor proof of the positive mass theorem in arbitrary dimensions for spin manifolds by E. Witten [9]. It uses the Schrödinger-Lichnerowicz formula, the divergence theorem, the interpretation of a boundary term as ADM mass and the solvability of the Dirac equation in weighted spaces.

Chapter 6 considers the Bartnik quasi-local mass. Given a compact Riemannian manifold $(\Omega, g)$ with nonempty boundary, this mass is defined as the infimum of the ADM masses of all extensions of $(\Omega, g)$ to complete asymptotically flat manifolds with nonnegative scalar curvature. The author discusses, whether there exist minimizers that realize this infimum.

The second part is concerned with asymptotically flat initial data sets $(M^n, g, k)$, where $(M^n, g)$ is a Riemannian manifold and $k$ should be thought of as a second fundamental form that describes how the Riemannian manifold $M$ sits in an $n+1$-dimensional Lorentzian manifold. It starts with an introduction to general relativity in Chap. 7 for readers familiar with Riemannian geometry. Then, it introduces Lorentzian metrics, causality, the Einstein field equations, the Schwarzschild spacetime of some mass $m$, initial data sets, the dominant energy condition $\mu \ge |J|_g$, where

$$\mu = \frac{1}{2}\left[R_g + (\operatorname{tr}_g k)^2 - |k|_g^2\right] \quad \text{and} \quad J = (\operatorname{div}_g k)^\sharp - \nabla(\operatorname{tr}_g k),$$

embeddings with these data into Lorentzian manifolds, asymptotic flatness and the ADM energy momentum $(E, P)$ with

$$E = \lim_{\rho\to\infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_\rho} \sum_{i,j=1}^{n} (g_{ij,i} - g_{ii,j})\overline{\nu}^j\, \overline{d\mu}_{S_\rho}$$

and

$$P_i = \lim_{\rho\to\infty} \frac{1}{(n-1)\omega_{n-1}} \int_{S_\rho} \sum_{j=1}^{n} (k_{ij} - (\operatorname{tr}_g k) g_{ij})\overline{\nu}^j\, \overline{d\mu}_{S_\rho},$$

the spacetime positive mass conjecture, black holes, marginally outer trapped surfaces and the Penrose incompleteness theorem and the spacetime Penrose inequality conjecture.

Chapter 8 is devoted to the spacetime positive mass theorem established successively under different assumptions by R. Schoen and S.-T. Yau [8], E. Witten [9], M. Eichmair [2] and M. Eichmair, L.-H. Huang, D.A. Lee and R. Schoen [3]. It states that an asymptotically flat initial data set $(M^n, g, k)$, $n \geq 3$, satisfying the dominant energy condition fulfills $E \geq |P|$ in each end if $n < 8$ or $M$ is spin. The idea of proof is to use marginally outer trapped surfaces and to see that a counterexample in dimension $n$, $3 < n < 8$, also yields a counterexample in dimension $n-1$. If $M$ is spin, it is possible to proceed similarly as in the proof of the Riemannian positive mass theorem. The corresponding rigidity statement is essentially that $E = 0$ implies that $(M, g, k)$, $3 \leq n < 8$, sits inside Minkowski space.

In the final chapter on density theorems for the constraint equations, the author establishes a result employed in the proof of the spacetime positive mass theorem. If the initial data set $(M, g, k)$ satisfies the dominant energy condition, then there exist initial data sets that approximate $(g, k)$, have harmonic asymptotics in each end and fulfill the strict dominant energy condition $\mu > |J|_g$.

In the appendix, the author collects a large amount of background material concerning second-order linear elliptic operators, more precisely, a strong maximum principle, a priori estimates on compact manifolds, the Laplacian as in isomorphism, eigenfunctions and harmonic expansions. Then similar results are discussed in weighted spaces on asymptotically flat manifolds.

The book ends with an extensive bibliography and an index.

The book contains many results. Therefore, it is quite natural that not all of them can be proved in detail. Technical aspects are sometimes described in words and not discussed rigorously in full detail. Those descriptions help to grasp the main ideas. Further explanations, e.g. concerning the next result or the next step in a proof, are also very helpful for the reader. More explicitly, it is helpful for acquiring an intuition on the subject that the author addresses questions like "Why is a certain term natural?" or "Why is it not possible to relax a certain assumption?". This combination of rigorous mathematical presentation and additional explanations makes the book very readable. Once you have understood the main ideas, you might want to study full details in the original publications before starting research about a topic of the book. Because technical details are left out, I consider this book more an invitation to learn about geometric topics in general relativity than a self-contained textbook. On the other hand, this style is nice for people (researchers or graduate students) with some background in nearby areas that want to see an overview of classical topics and related recent developments in geometric relativity. They will also find a few open problems and many – sometimes advanced – exercises.

The readability of the book is good, but some typos made it to the final version. Names appear in different formats and it is often not straight-forward to find a reference in the bibliography, because it is ordered according to the names of the authors and not by the labels of the references.

To summarize, the book gives a good overview of results concerning the mass of asymptotically flat manifolds. It discusses central results like the Riemannian and spacetime positive mass theorem and the Riemannian Penrose inequality. Those results and many other results are proved. Technically complicated parts in the proofs of the main results are sometimes replaced by descriptions of the ideas in the background that make the book very readable. Therefore, "Geometric Relativity" is a pleasant invitation to the subject that gives a good impression of the subject and orientation for further reading in original publications.

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## References

1. Bray, H.L.: Proof of the Riemannian Penrose inequality using the positive mass theorem. J. Differ. Geom. **59**(2), 177–267 (2001)
2. Eichmair, M.: The Jang equation reduction of the spacetime positive energy theorem in dimensions less than eight. Commun. Math. Phys. **319**(3), 575–593 (2013)
3. Eichmair, M., Huang, L.-H., Lee, D.A., Schoen, R.: The spacetime positive mass theorem in dimensions less than eight. J. Eur. Math. Soc. **18**(1), 83–121 (2016)
4. Huisken, G., Ilmanen, T.: The inverse mean curvature flow and the Riemannian Penrose inequality. J. Differ. Geom. **59**(3), 353–437 (2001)
5. Lohkamp, J.: Scalar curvature and hammocks. Math. Ann. **313**(3), 385–407 (1999)
6. Schoen, R., Yau, S.-T.: On the proof of the positive mass conjecture in general relativity. Commun. Math. Phys. **65**(1), 45–76 (1979)
7. Schoen, R., Yau, S.-T.: On the structure of manifolds with positive scalar curvature. Manuscr. Math. **28**(1–3), 159–183 (1979)
8. Schoen, R., Yau, S.-T.: Proof of the positive mass theorem. II. Commun. Math. Phys. **79**(2), 231–260 (1981)
9. Witten, E.: A new proof of the positive energy theorem. Commun. Math. Phys. **80**(3), 381–402 (1981)