Dimensions of Faces of Gram Spectrahedra

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Introduction

Given a real homogeneous polynomial \( f \in \mathbb{R}[x_1, \ldots, x_n]_{2d} \) that is a sum of squares, i.e. that can be written in the form \( f = \sum_{i=1}^{r} p_i^2 \) for some \( p_1, \ldots, p_r \in \mathbb{R}[x_1, \ldots, x_n]_d \), there are in general many non-equivalent ways to write \( f \) as a sum of squares. The Gram spectrahedron \( \text{Gram}(f) \) parametrizes those representations up to equivalence. It carries in a natural way the structure of a spectrahedron, which is the intersection of an affine linear space with the cone of positive semi-definite matrices. As a closed, convex set its boundary is the union of its faces. The facial structure of spectrahedra has been studied by Ramana and Goldman [RG95] who showed that every face of a spectrahedron is given by all points of the spectrahedron whose matrices contain a given subspace in their kernel. In [Sch] Scheiderer formulated those results in a coordinate-free way, i.e. instead of symmetric matrices, he talks about symmetric tensors. The advantage of this point of view, especially for Gram spectrahedra, is that more structure is readily available and is not hidden by the choice of a basis and coordinates. We write \( S_2^d \mathbb{R}[x_1, \ldots, x_n] \) for the second symmetric power of the space \( \mathbb{R}[x_1, \ldots, x_n]_d \) which we understand as the set of all symmetric tensors in \( \mathbb{R}[x_1, \ldots, x_n]_d \otimes \mathbb{R}[x_1, \ldots, x_n]_d \).

We define the Gram spectrahedron of \( f \) as the set
\[
\text{Gram}(f) = \{ \vartheta \in S_2^d \mathbb{R}[x_1, \ldots, x_n]_d : \vartheta \geq 0, \mu(\vartheta) = f \}
\]
consisting of all symmetric, positive-semidefinite tensors \( \vartheta \) that are mapped to \( f \) via the Gram map \( \mu: S_2^d \mathbb{R}[x_1, \ldots, x_n]_d \to \mathbb{R}[x_1, \ldots, x_n]_{2d}, \vartheta = \sum_{i=1}^{r} p_i \otimes q_i \mapsto \sum_{i=1}^{r} p_i q_i \). Extreme points of Gram spectrahedra play an important role in finding certificates for non-negativity in optimization. They have been studied for example in [PSV11], [Sch17], and [CPSV17] where the authors were especially interested in extreme points of smallest rank. We do not study extreme points of Gram spectrahedra but their facial structure as a whole. For ternary quartics (\( n = 3, d = 2 \)) this leads to a complete understanding of their Gram spectrahedra in terms of their facial structure.

Let \( f \in \Sigma_{n,2d} \). As was shown in [Sch] we may associate to any face \( F \subseteq \text{Gram}(f) \) a subspace \( U \subseteq \mathbb{R}[x]_{2d} \) where the dimension of \( U \) equals the rank of any relative interior point of \( F \) and such that the dimension of the face \( F \) is given by \( \binom{\dim U + 1}{2} - \dim U^2 \) where \( U^2 \subseteq \mathbb{R}[x]_{2d} \) is the vector space spanned by all products \( p q \) with \( p, q \in U \). Especially, the dimension of \( F \) only depends on \( \dim U^2 \) if the dimension of \( U \) is fixed.

The thesis consists of two main parts which are chapters 3 and 4. In Chapter 3 we investigate maximal possible dimensions of faces. If \( U \subseteq \mathbb{R}[x]_{2d} \) is a generic subspace of dimension \( r \), then the face associated to \( U \) on any Gram spectrahedron has minimal possible dimension (Proposition 2.3.22) compared to all other faces of rank \( r \). We are therefore interested in upper bounds for dimensions of faces if we fix their rank. This is equivalent to finding upper bounds for the codimension of \( U^2 \) if we fix the dimension of \( U \). In [BC18, Proposition 2.2] Boij and Conca observed that the maximal codimension of \( U^2 \) can always be achieved by a strongly stable subspace \( U \), i.e. a monomial subspace with an additional combinatorial property (see Definition 3.1.1). This allows us to compute the maximal codimension of \( U^2 \).
for a fixed dimension of $U$ with any computer algebra software if the number of variables and the degree are small (Remark 3.1.17). However, all strongly stable subspaces satisfy $\mathcal{V}(U) \neq \emptyset$, hence if $F \subseteq \text{Gram}(f)$ is a face on some Gram spectrahedron corresponding to $U$, then the form is singular at these points. More precisely, $f$ has a non-trivial real zero.

We therefore study subspaces $U$ such that $\mathcal{V}(U) = \emptyset$ afterward. Led by the fact that in the cases $\text{codim} U \in \{1, 2\}$ we can find a bound for $\text{codim} U^2$ that does not depend on $n$ or $d$ if $\mathcal{V}(U) = \emptyset$ (Proposition 3.4.3, Theorem 3.4.8), we find bounds for $\text{codim} U^2$ for larger codimensions of $U$ that are independent of $n$ and $d$. Let $U \subseteq \mathbb{C}[x_1, \ldots, x_n]_d$ be a subspace of codimension $k$. The main result is Theorem 3.8.12 where we show that if the codimension $k$ of $U$ is at most $d - 1$ and $\mathcal{V}(U) = \emptyset$ then

$$\text{codim} U^2 \leq 2k^2 + \binom{k + 2}{3}.$$ 

As desired, this gives a bound that is independent of $n$ and $d$. For Gram spectrahedron of non-singular forms this gives an upper bound on the dimension of faces of large rank. We note that this does not imply that the dimension of the faces is bounded by an integer independent of $n$ or $d$.

In Chapter 4 we are mostly concerned with the case $n = 3$, $d = 2$ of ternary quartics. Sum of squares representations of ternary quartics have already been studied by Hilbert [Hil88] in 1888, who showed that every real psd ternary quartic can also be written as a sum of squares, and every such quartic can be written as a sum of three squares. Later, it was shown in [PRSS04] that every smooth psd quartic admits exactly eight such representations as a sum of three squares (up to orthogonal equivalence).

In [PSV11] the authors relate the sum of squares representation of length 3 to the 28 bitangent lines of the quartic curve and their combinatorial structure. They also observe that the eight corresponding Gram tensors of rank 3 split into two groups of four such that in every group all line segments between two of them are contained in the boundary of Gram$(f)$. The stronger statement that additionally for any two Gram tensors in different groups the line segment is not part of the boundary is also claimed for generic forms but only partially proven. We complete the proof of this statement for all smooth quartics (Theorem 4.3.16). The graph having these eight Gram tensors as vertices is called the Steiner graph of the form $f$.

This mostly finishes our study of the rank 3 Gram tensors and we turn to tensors of rank 4 and 5. These were not understood at all, it was even unknown whether there exist any faces of positive dimension other than the faces containing the line segments represented in the Steiner graph ([PSV11, p.18]). These rank 4 and rank 5 tensors form the whole boundary of Gram$(f)$ together with the eight rank 3 tensors for any smooth form $f$. We show that faces of rank 4 may either be extreme points or have dimension 1. These 1-dimensional faces however, may only appear on Gram spectrahedra of smooth forms if the automorphism group of the curve $\{f = 0\}$ has even order (Corollary 4.4.8). Faces of rank 5 are very different in this aspect. Firstly, faces of rank 5 can only be of dimension 0 or 2 if the quartic is smooth (Theorem 4.1.2). Moreover, the Gram spectrahedron of every smooth psd quartic has faces of rank 5 both of dimension 0 and 2 (Corollary 4.4.20, Theorem 4.4.23). As the Gram spectrahedron of every smooth psd quartic has extreme points of rank 4 (Proposition 4.4.12) this also shows that for every smooth psd quartic, the Gram spectrahedron has points of all ranks in the Pataki interval which contains the numbers 3, 4, 5. This is the smallest combination of dimension and size of the matrices.
where the Pataki interval has length three. For any combination of rank $r$ and dimension $s$ we also determine the dimension of the set of tensors on the Gram spectrahedron having rank $r$ and lying in the interior of a face of dimension $s$. These results are summarized in Remark 4.4.25.

Afterward, we study the structure of the faces and show that except for 0- and 1-dimensional faces, no other face can be polyhedral (Theorem 4.5.9), which is contrasting the case of binary forms where lots of polyhedral faces exist in general (May21).

We finish with a study of the normal cones of Gram spectrahedra in Section 4.6. This is needed in the proof of Theorem 4.4.23 as well as to be able to study the dual convex body of Gram spectrahedra (Section 4.7).

We briefly comment on the methods used in the thesis. These include two very different types of arguments. To understand dimensions of faces, we make use of the fact that these can be calculated purely algebraically. Let $f \in \Sigma_{n,2d}$ and $F \subseteq \text{Gram}(f)$ a face. Let $\vartheta \in F$ be a relative interior point with image $U \subseteq \mathbb{R}[x_1,\ldots,x_n]_d$. The dimension of $F$ is now given by $(\dim U + 1) - \dim U^2$. It is therefore enough to study possible dimensions of squares of subspaces. To do this, we use two main techniques. On the one hand, we study the spaces $(U : l)_{d-1}$ in Section 3.5 where $l \in \mathbb{C}[x_1,\ldots,x_n]$ is a generic linear form and $(U : l)$ denotes the quotient ideal. On the other hand, we consider $U \cap \mathbb{C}[x_1,\ldots,x_{n-1}]_d$ in Section 3.7 which is a subspace in one less variable. Both can be studied using theorems about Hilbert functions, for example by Macaulay, Green and Gotzmann (see Section 3.3).

In Chapter 4 we follow the classical approach of [PSV11] and [PRSS04], connecting the sos-representations of a smooth form $f$ with the 28 bitangent lines of the curve $\{f = 0\}$. A large amount of results on the combinatorial structure of the bitangents and their connection to determinantal representations is also contained in the book [Dol12] by Dolgachev which we regularly use as a reference. This is especially necessary to show additional properties of the Steiner graph (Proposition 4.4.14).

As to the organization of the thesis. In Chapter 1 we introduce our notation and show some preparatory results used throughout. In Chapter 2 we introduce Gram spectrahedra and review results about their facial structure as proven in [RG95] and [Sch]. The first main chapter, Chapter 3 contains the study of upper bounds of dimension of faces of Gram spectrahedra for non-singular forms in any number of variables, whereas in Chapter 4 we focus mostly on the case $n = 3$, $d = 2$ of ternary quartics.

Lastly, we will make a few remarks on still open problems. Firstly, in the case of ternary quartics, we have a necessary condition on the form $f$ for its Gram spectrahedron to have a 1-dimension face (Corollary 4.4.8). It would be great to have a full characterization of this behavior. This would mean finding a condition on the form $f$ or the bitangents which ensures that two of the tensors in Corollary 4.4.8 (ii) are real and psd.

In the general case (arbitrary $n$ and $d$), there are two main points we want to mention. Firstly, this is the following (Conjecture 3.7.11)

**Conjecture.** Let $k \leq d - 1$, $n - 1$, and $n \geq 3$. Let $W \subseteq A_d$ be a subspace of dimension $k$ and suppose that $W$ contains no $d$-th power of a linear form. Then for a generic linear form $l \in \mathbb{C}[x_1,\ldots,x_n]_1$ it holds that either

(i) $W$ contains no $d$-th power of a linear form, or

(ii) $n = k + 1$ and $W = L_1^{d-1}\mathbb{C}[L_2,\ldots,L_{k+1}]_1$ for linearly independent linear forms $L_1,\ldots,L_{k+1}$. 


By Proposition 3.7.8 the conjecture holds for $n \geq 3k + 1$ and we can also easily show it for $k = 1$. We did check it for all monomial subspaces in small cases $(n, d \leq 5)$ where it holds true.

The second point concerns the bound for $\text{codim } U^2 (V(U) = \emptyset)$ found in Theorem 3.8.12. It would be great to have some small examples where we know the bound exactly and can compare it to our bound. However, already in the case $n = 2, d = 5$ and $\text{codim } U = 2$, we do not know a tight bound. One way to construct a different bound is shown in Proposition 3.2.3. However, this bound is usually not tight for subspaces of small codimension. Hence, this does not allow us a good comparison to the general bound which only holds for small codimensions.

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1 Preliminaries and notations

1.1 Algebraic geometry

Let $\mathbb{N}$ be the set of positive integers. Let $n \in \mathbb{N}$. We denote by $\mathbb{P}^n$ the $n$-dimensional projective space over the complex numbers $\mathbb{C}$, that is the set of all 1-dimensional subspaces of $\mathbb{C}^{n+1}$. We write $P = (p_0 : \cdots : p_n) \in \mathbb{P}^n$ for a point in projective space. Moreover for any finite-dimensional $K$-vector space $V$ ($K \in \{ \mathbb{R}, \mathbb{C} \}$), we write $\mathbb{P}V$ for the projectivization of $V$, that is $\mathbb{P}V = \{ [v] \in V : v \in V \setminus \{ 0 \} \}$ where $[v]$ denotes the 1-dimensional subspace spanned by $v$ inside $V$.

A projective variety $X \subseteq \mathbb{P}^n$ is the zero set of finitely many homogeneous polynomials (forms) in $\mathbb{C}[x_0, \ldots, x_n]$. I.e. there exist forms $f_1, \ldots, f_r \in \mathbb{C}[x_0, \ldots, x_n]$ such that

$$X = \mathcal{V}(f_1, \ldots, f_r) = \{ \xi \in \mathbb{P}^n : f_1(\xi) = \cdots = f_r(\xi) = 0 \}.$$ 

Let $I = I(X)$ be the homogeneous ideal generated by all forms in $\mathbb{C}[x_0, \ldots, x_n]$ vanishing identically on $X$. Then we denote the coordinate ring of $X$ by $\mathbb{C}[X] := \mathbb{C}[x_0, \ldots, x_n]/I$.

We equip the projective space with the Zariski topology, i.e. the closed sets are given by varieties.

An analogous an affine variety $X \subseteq \mathbb{A}^n = \mathbb{C}^n$ is the zero set of finitely many polynomials (not necessarily homogeneous) in $\mathbb{C}[x_1, \ldots, x_n]$.

If $X$ can be defined by real forms, we also consider its real coordinate ring $\mathbb{R}[X] = \mathbb{R}[x_0, \ldots, x_n]/I$ where $I \subseteq \mathbb{R}[x_0, \ldots, x_n]$ is the ideal generated by all forms in $\mathbb{R}[x_0, \ldots, x_n]$ that vanish identically on $X$.

By this definition a variety does not have to be irreducible, however, it is always reduced, i.e. its coordinate ring is reduced.

Let $X \subseteq \mathbb{P}^n$ be a projective variety such that $I(X)$ is generated by the homogeneous polynomials $f_1, \ldots, f_r \in \mathbb{C}[x_0, \ldots, x_n]$. Let $P = (p_0: \ldots : p_n) \in X$. We define the tangent space of $X$ at $P$ to be the projective linear space spanned by the forms

$$x_0 \frac{\partial f_1}{\partial x_0}(P) + \cdots + x_n \frac{\partial f_1}{\partial x_n}(P)$$

for $i = 1, \ldots, r$. The point $P$ is called non-singular or smooth if the dimension of the tangent space equals the dimension of $X$. The variety $X$ is called non-singular or smooth if every point of $X$ is non-singular. By [Har92 Lecture 14] the tangent space does not depend on the choice of the generators. A form $f \in \mathbb{C}[x_0, \ldots, x_n]$ is called smooth or non-singular if $f$ is irreducible and $\mathcal{V}(f)$ is non-singular.

We say that some property holds generically if it holds outside a Zariski-closed subset which is not the whole space. The ambient space should always be clear from the context.

Let $d \in \mathbb{N}$ and $U \subseteq \mathbb{C}[x_0, \ldots, x_n]_d$ be a subspace where $\mathbb{C}[x_0, \ldots, x_n]_d$ denotes the vector space in $\mathbb{C}[x_0, \ldots, x_n]$ consisting of all polynomials of degree $d$.

We say that $U$ is base-point-free if $\mathcal{V}(U) = \emptyset$. If $X \subseteq \mathbb{P}^n$ is a variety, and $U \subseteq \mathbb{C}[X]_d$ is a subspace, then we say that $U$ is base-point-free if $\mathcal{V}(U) \cap X = \emptyset$. 

1.2 Convexity and sums of squares

We introduce some facts about convex sets and connections between sums of squares and convexity.

Let $V$ be a finite-dimensional $\mathbb{R}$-vector space and let $C \subseteq V$ be a subset. The set $C$ is called convex if for any two points $x, y \in C$ and every $\lambda \in [0, 1]$ the point $\lambda x + (1 - \lambda)y$ is also contained in $C$. Whenever $C$ is convex, it especially follows that for any $r \in \mathbb{N}$ and any points $x_1, \ldots, x_r \in C$ and $\lambda_1, \ldots, \lambda_r \geq 0$ with $\sum_{i=1}^{r} \lambda_i = 1$, the point $\sum_{i=1}^{r} \lambda_i x_i$ is again contained in $C$. For any subset $C \subseteq V$, its convex hull is defined as the smallest convex set containing it. The convex hull is given by

$$\text{conv}(C) = \left\{ \sum_{i=1}^{n} \lambda_i v_i : n \in \mathbb{N}, v_i \in C, \lambda_i \geq 0, \forall i = 1, \ldots, n, \sum_{i=1}^{n} \lambda_i = 1 \right\}.$$

Let $C$ be a convex set and let $F \subseteq C$ be a convex subset. The set $F$ is called a face of $C$ if for any two points $x, y \in C$ such that there exists $\lambda \in (0, 1)$ with $\lambda x + (1 - \lambda)y \in F$, the points $x$ and $y$ are already contained in $F$. Assume that the interior of $C$, denoted by $\text{int}(C)$, is non-empty. Then every point on the boundary of $C$, which we denote by $\partial C$, is contained in a non-trivial face, that is, a face which is neither empty nor all of $C$. On the other hand, every non-trivial face is contained in the boundary of $C$. Since every compact convex set $C$ is the convex hull of its boundary points, i.e. every point is $C$ can be written as $\lambda x + (1 - \lambda)y$ for some $x, y \in \partial C$ and some $\lambda \in [0, 1]$, understanding the facial structure describes the full set $C$.

The intersection of two faces is once again a face. Hence for every subset $M \subseteq C$ there is a smallest face $F$ of $C$ that contains $M$, called the supporting face of $M$.

A special kind of convex set is the convex cone. A convex set $C$ is called a convex cone if for every point $x \in C$ and every $\lambda \geq 0$ the point $\lambda x$ is also contained in $C$, i.e. non-negative multiples of every point are again in $C$.

A subset $S \subseteq \mathbb{R}^n$ is called semi-algebraic if it is a finite boolean combination of sets of the form

$$S = \{ x \in \mathbb{R}^n : p(x) > 0 \}$$

for some $p \in \mathbb{R}[x_1, \ldots, x_n]$.

Two of the most famous examples of convex cones connected to sums of squares are the cone of non-negative forms and the cone of sums of squares $(n \geq 1, d \geq 1)$

$$P_{n,d} := \{ f \in \mathbb{R}[x_1, \ldots, x_n][2d] : \forall \xi \in \mathbb{R}^n : f(\xi) \geq 0 \},$$

$$\Sigma_{n,d} := \{ f \in \mathbb{R}[x_1, \ldots, x_n][2d] : \exists r \in \mathbb{N}, f_1, \ldots, f_r \in \mathbb{R}[x_1, \ldots, x_n][d] : f = \sum_{i=1}^{r} f_i^2 \}.$$

It is easy to see that $\Sigma_{2d}$ is contained in $P_{n,2d}$. However, the other inclusion is in general not true as Hilbert showed in 1888. In fact equality only holds in some special cases, those are $(n, d) \in \{(3, 2), (2, d), (1, d), (n, 2) : n \geq 2, d \geq 1 \}$.

Another important example of a convex cone is the cone of positive semidefinite matrices of size $n$

$$\mathbb{S}_+^n := \{ A \in \mathbb{S}^n : A \succeq 0 \}$$

where $\mathbb{S}^n$ denotes the set of all real symmetric matrices of size $n$.

We will see that positive semidefinite matrices are closely connected to sum of squares representations of forms.
1.3 Commutative algebra

Let $K$ be the field of the real or the complex numbers and denote by $A$ the polynomial ring over $K$ in $n$ variables $K[x_1, \ldots, x_n]$. We always consider $A$ with its standard $\mathbb{Z}$-grading.

For any homogeneous ideal $I \subseteq A$, we define the Hilbert function of $I$ to be the Hilbert function of the quotient $A/I$. That is the function $t \mapsto \dim_K (A/I)_t$ with $t \in \mathbb{N}$ where $(A/I)_t$ denotes the degree $t$ component of the graded ring $A/I$. We denote the Hilbert function by $h_t(I) := \dim (A/I)_t$. Furthermore, we usually identify this function with its sequence of values, i.e. the sequence $(h_t)_{t \geq 0}$ with $h_t = \dim (A/I)_t$.

Let $I \subseteq A$ be a homogeneous ideal and let $R := A/I$. An element $y \in R$ is called $R$-regular if $yz = 0$ implies $z = 0$ for all $z \in R$. A sequence $y_1, \ldots, y_r$ in $R$ is called regular if $(y_1, \ldots, y_r) \neq R$ and for every $i = 1, \ldots, r$ the element $y_i \in R/(y_1, \ldots, y_{i-1})$ is $R/(y_1, \ldots, y_{i-1})$-regular. Here $(y_1, \ldots, y_r)$ denotes the ideal generated by $y_1, \ldots, y_r$ in $R$.

Let $p \subseteq R$ be a homogeneous prime ideal. The depth of the local ring $R_p$ is defined as the length of a maximal $R_p$-regular sequence in $pR_p$. It is denoted by depth$(R_p)$. The local ring $R_p$ is called a Cohen-Macaulay ring if

$$\text{depth}(R_p) = \dim(R_p) (= \text{Krull dimension of } R_p).$$

The ring $R$ is called Cohen-Macaulay if $R$ localized at the irrelevant ideal $\bigoplus_{i > 0} R_i$ is Cohen-Macaulay as a local ring in the sense above.

The next proposition is an easy fact about regular sequences.

**Proposition 1.3.1.** Let $d \geq 1$ and let $R = A/I$ be as above. If $p_1, \ldots, p_r$ is a regular sequence in $R$ and $\deg p_i = d$ for every $i = 1, \ldots, r$, then the kernel of the map

$$\psi: R_d^r \to R_{2d}, \quad (q_1, \ldots, q_r) \mapsto \sum_{i=1}^r p_i q_i$$

is spanned by the $\binom{r}{2}$ elements

$$(0, \ldots, 0, p_i, 0, \ldots, 0, -p_j, 0, \ldots, 0) \quad (1 \leq j < i \leq r)$$

where $p_i$ is the $j$-th entry and $-p_j$ is the $i$-th entry.

**Proof.** For every $i = 1, \ldots, r$ the map

$$\psi_i: (R/(p_1, \ldots, p_{i-1}))_d \to (R/(p_1, \ldots, p_{i-1}))_{2d}$$

is injective by definition of a regular sequence.

Let $s = \dim R_d$, then we show that $\dim (p_1, \ldots, p_i)_{2d} = i \cdot s - \binom{i}{2}$ by induction on $i$. For $i = 1$ we have $\dim (p_1 R_d) = s$. For $i > 1$ we have

$$\dim (p_1, \ldots, p_i)_{2d} = \dim (p_1, \ldots, p_{i-1})_{2d} + \dim p_i R_d - \dim (p_i R_d \cap (p_1, \ldots, p_{i-1})_{2d})$$

$$= (i - 1) \cdot s - \binom{i-1}{2} + s - \dim (p_i R_d \cap (p_1, \ldots, p_{i-1})_{2d}).$$

Let $q \in p_i R_d \cap (p_1, \ldots, p_{i-1})_{2d}$. Then we can write $\overline{q} \in p_i \overline{\mathbb{Q}} \in (R/(p_1, \ldots, p_{i-1}))_d \subseteq (R/(p_1, \ldots, p_{i-1}))_{2d}$. Especially $\overline{q} \in \text{im}(\psi_i)$. Since $q \in (p_1, \ldots, p_{i-1})_{2d}$ and $\psi_i$ is injective, it follows that $\overline{p} \in (p_1, \ldots, p_{i-1})_d$ and hence $\dim (p_i R_d \cap (p_1, \ldots, p_{i-1})_{2d}) = i - 1$. 


Combined with the calculations above we get \( \dim \langle p_1, \ldots, p_i \rangle_{2d} = i \cdot s - \binom{i - 1}{2} - (i - 1) = i \cdot s - \binom{i}{2} \).

For the map \( \psi \) we have \( \dim \ker \psi = i \cdot s - \binom{i}{2} \) by the argument above and therefore \( \dim \ker \psi = \binom{i}{2} \). The \( \binom{i}{2} \) elements of the form \( (0, \ldots, 0, p_i, 0, \ldots, 0, -p_j, 0, \ldots, 0) \) are contained in the kernel and are linearly independent, thus they span the kernel. \( \Box \)

Next, we introduce Gorenstein rings. Let \( R \neq \{0\} \) be as above and assume that \( R \) has Krull dimension 0, then it is also local with maximal ideal \( \mathfrak{m} = \langle R_1 \rangle = \langle x_1, \ldots, x_n \rangle \).

**Definition 1.3.2.** The socle of \( R \) is defined as

\[ \text{Soc}(R) := \{0: \mathfrak{m} \} = \{ p \in R: p\mathfrak{m} = 0 \} = \{ p \in R: pR_1 = 0 \} \]

**Definition 1.3.3.** The ring \( R \) is called Gorenstein if \( \dim \text{Soc}(R) = 1 \) as a \( K \)-vector space. The degree of the element spanning \( \text{Soc}(R) \) is called the socle degree of \( R \).

**Remark 1.3.4.** Since \( R \) is 0-dimensional, there exists \( d \geq 0 \) such that \( R_d \neq 0 \) and \( R_s = 0 \) for every \( s > d \). Hence \( R_d \) is always contained in the socle of \( R \). Being Gorenstein is therefore equivalent to \( \dim R_d = 1 \) and \( R_d = \text{Soc}(R) \).

Let \( R = A/I \) be as above but not necessarily of dimension 0. For any integer \( s \geq 0 \) we denote by \( R_s := \text{Hom}_K(R, K) \) the dual vector space.

**Definition 1.3.5.** Let \( \alpha \in R^*_s \) be a linear functional. The homogeneous ideal \( I(\alpha) \) in \( R \) generated by the set

\[ \{ p \in R_k : k > s \text{ or } \alpha(pq) = 0 \text{ for all } q \in R_{s-k} \} \]

is called the Gorenstein ideal with socle \( \alpha \).

**Remark 1.3.6.** Let \( \alpha \in R^*_s \). Then the ring \( S = R/I(\alpha) \) is Gorenstein: indeed, by definition of \( I(\alpha) \) the quotient \( S \) is a local, 0-dimensional ring. Let \( p \in R_k \) such that \( \mathfrak{p} \in \text{Soc}(S) \), then \( pR_1 \in I(\alpha)_{k+1} \). If \( k < s \), then \( \alpha(pR_1R_{s-k-1}) = 0 \) and therefore \( p \in I(\alpha)_k \), hence \( \mathfrak{p} = 0 \) in \( S \). Since \( \alpha \) is a linear form, the kernel of the map \( R_s \to K \), \( p \to \alpha(p) \) has codimension 1 in \( R_s \) and therefore \( \dim S_s = 1 \). Hence the socle is exactly given by \( S_s \) and has dimension 1. This means that \( S \) is Gorenstein.

**Remark 1.3.7.** If \( R \) is a Gorenstein ring with socle degree \( 2d \) for some \( d \geq 1 \), then the Hilbert function of \( R \) is symmetric around \( d \), i.e. \( \dim R_i = \dim R_{2d-i} \) for every \( 0 \leq i \leq 2d \). This immediately follows from the fact that the multiplication map \( R_i \times R_{2d-i} \to R_{2d} \cong K \) is a non-degenerate bilinear form for every \( 0 \leq i \leq 2d \) by the definition of Gorenstein ring.

Lastly, we also talk about orthogonal complements of subspaces. We always do this wrt the apolarity pairing.

**Definition 1.3.8.** For \( i = 1, \ldots, n \) define the differential operator \( \partial_i := \frac{\partial}{\partial x_i} \) and \( \partial := (\partial_1, \ldots, \partial_n) \), \( \partial^\alpha = \partial^{a_1} \cdots \partial^{a_n} \) for \( \alpha \in \mathbb{Z}_+^n \). For \( f = \sum_{\alpha} c_\alpha x^\alpha \in A \) define \( f(\partial) := \sum_{\alpha} c_\alpha \partial^\alpha \). For every \( m \geq 0 \) we then have the following bilinear form on \( A_m \):

\[ \langle f, g \rangle := \frac{1}{m!} f(\partial)(g) = \frac{1}{m!} g(\partial)(f), \quad \forall f, g \in A_m \]

This bilinear form is called the apolarity pairing. It is a perfect pairing and in the real case a scalar product. Furthermore for every \( u \in \mathbb{C}^n \) and \( l := \sum_{i=1}^n u_i x_i \in A_1 \) and \( f \in A_m \) we have \( \langle l^n, f \rangle = f(u) \). (See for example [IK99a, Lemma 1.15])
2 Gram spectrahedra

2.1 An introduction to Gram spectrahedra

We always assume $n \in \mathbb{N}$ and $n \geq 2$. Let $f \in \mathbb{R}[x]_{2d}$ such that $f(x) \geq 0$ for every $x \in \mathbb{R}^n$. Such a form $f$ is called positive semi-definite or psd for short. One way to certify this property is to write $f$ as a sum of squares (sos), that is we find $r \in \mathbb{N}$ and $f_1, \ldots, f_r \in \mathbb{R}[x]_{d}$ such that $f = f_1^2 + \cdots + f_r^2$. Now, $f$ obviously takes only non-negative values on $\mathbb{R}^n$. However, as already mentioned, Hilbert showed that not every $f$ which is psd has a sum of squares representation. But even if such a representation does exist, how can we find one? Moreover, in general there are many inequivalent ways to do so and finding a short sum of squares representation is certainly preferable in optimization. To solve these problems we describe the Gram matrix method originally due to Choi, Lam, and Reznick [CLR95].

Let $f \in \Sigma_{n,2d}$ and assume we have a representation $f = \sum_{i=1}^{n} f_i^2$ for some linearly independent $f_1, \ldots, f_r \in \mathbb{R}[x]_{d}$. The length of the representation is then defined as $r$. Let $X$ be the row vector containing the monomial basis of $\mathbb{R}[x]_{d}$ ordered wrt the lexicographic order but such that $x_1 x_2 \cdots x_n$, i.e. for $\alpha \neq \beta \in \mathbb{Z}_+^n$ we have $x^\alpha > x^\beta$ if and only if $\alpha_i \geq \beta_i$ for $i = j, \ldots, n$ and $\alpha_j > \beta_j$ for some $j \in \{1, \ldots, n\}$. Considering this ordering of the variables simplifies notation later on.

Let $N := \dim \mathbb{R}[x]_{d}$, and let $H$ be the $r \times N$ matrix containing the coordinates of $f_i$ wrt $X$ in its $i$-th row. Then we have

$$f = (f_1, \ldots, f_r)(f_1, \ldots, f_r)^T = (XH^T)(XH^T)^T = X(H^TH)X^T.$$ 

The matrix $H^T H$ is symmetric, psd, and has rank $r$ by construction. The matrix $H^T H$ is called the Gram matrix corresponding to the representation $f = \sum_{i=1}^{r} f_i^2$.

On the other hand, every psd matrix $G \in \mathbb{S}^n_+$ with $XGX^T = f$ can be written as $G = H^T H$ for some matrix $H$ of size $r \times N$ where $r$ is the rank of $G$. Therefore, the matrix $G$ gives rise to a sos representation of $f$ via $f = XH^T HX^T = (XH^T)(XH^T)^T$. Any matrix $G$ with this property is called a positive semidefinite Gram matrix of $f$ and the map $\mu : \mathbb{S}^N \to \mathbb{R}[x]_{2d}, G \mapsto XGX^T$ is called the Gram map.

This correspondence however is not one-to-one as the orthogonal group $O(r)$ acts on length $r$ representations of $f$ via right multiplication, i.e. let $U \in O(r)$ and let $p_1, \ldots, p_r \in \mathbb{R}[x]_{d}$ such that $(f_1, \ldots, f_r)U = (p_1, \ldots, p_r)$, then $f = \sum_{i=1}^{r} p_i^2$. Indeed, we have

$$f = (f_1, \ldots, f_r)(f_1, \ldots, f_r)^T = (f_1, \ldots, f_r)(UU^T)(f_1, \ldots, f_r)^T = \left((f_1, \ldots, f_r)U\right)\left((f_1, \ldots, f_r)U\right)^T = (p_1, \ldots, p_r)(p_1, \ldots, p_r)^T.$$ 

Moreover, the corresponding Gram matrices are the same: with $H$ as above, the Gram matrix corresponding to the sos representation $f = \sum_{i=1}^{r} f_i^2$ is given by $H^T H$ and the
Gram matrix corresponding to \( f = \sum_{i=1}^{r} p_i^2 \) is given by
\[
\begin{align*}
f &= (p_1, \ldots, p_r)(p_1, \ldots, p_r)^T = \left( (f_1, \ldots, f_r) \right) \left( (f_1, \ldots, f_r) \right)^T \\
&= \left( XH^T \right) \left( XH^T \right)^T = X(H^T UU^T H)X^T = X(H^T)X^T.
\end{align*}
\]

This however is the only reason, Gram matrices and sos representations of \( f \) are not in one-to-one correspondence.

**Proposition 2.1.1** ([CLR95, Proposition 2.10.]). Let \( f \in \Sigma_{n,2d} \) such that \( f = \sum_{i=1}^{r} p_i^2 = \sum_{i=1}^{r} q_i^2 \) with \( p_1, \ldots, p_r, q_1, \ldots, q_r \in \mathbb{R}[x] \) and such that the sets \( p_1, \ldots, p_r \) and \( q_1, \ldots, q_r \) are linearly independent. Then the Gram matrices corresponding to the two sos representations are the same if and only if there exists an orthogonal matrix \( U \in O(r) \) such that \( (p_1, \ldots, p_r)U = (q_1, \ldots, q_r) \).

Therefore, up to orthogonal equivalence psd Gram matrices of \( f \) correspond exactly to sos representations of \( f \).

By a change of coordinates, we always mean a linear change of coordinates on the variables over \( \mathbb{C} \) if not explicitly stated otherwise.

**Example 2.1.2.** Let \( d = 1 \) and \( f \in \Sigma_{n,2} \). After a real change of coordinates the quadratic form \( f \) can be written as \( f = \sum_{i=1}^{r} x_i^2 \) for some \( r \in \mathbb{N} \). Since \( f \) is psd, all the signs have to be +. As is well-known for quadratic forms, this representation is unique up to the action of the orthogonal group. Hence, the Gram spectrahedron of \( f \) is just a single point.

We will see more interesting examples after showing some results about the faces of Gram spectrahedra.

Next, we want to understand the structure of the set of all Gram matrices of \( f \). We denote by
\[
\text{Gram}(f) = \{ G \in S^N : XGX^T = f, G \succeq 0 \}
\]
the **Gram spectrahedron** of \( f \), which is the set consisting of all psd Gram matrices of \( f \). It is indeed a spectrahedron, i.e. the intersection of an affine-linear subspace with the cone of psd matrices, as it is the preimage of \( f \) under the Gram map intersected with the cone of psd matrices.

This immediately shows that \( \text{Gram}(f) \) is a closed, convex, and semi-algebraic subset of \( S^N \).

By definition, \( \text{Gram}(f) \) consists of all psd matrices in the affine-linear subspace \( A + \ker(\mu) \) where \( A \) is some Gram matrix of \( f \). Especially, picking a basis \( A_1, \ldots, A_s \) of \( \ker(\mu) \) we can write
\[
\text{Gram}(f) = \left\{ G \in S^N : \exists x_1, \ldots, x_r \in \mathbb{R} : G = A + \sum_{i=1}^{s} x_i A_i \succeq 0 \right\}
\]
or reading this inside the affine-linear subspace \( A + \ker(\mu) \) wrt the basis above, \( \text{Gram}(f) \) is linearly isomorphic to
\[
\left\{ (x_1, \ldots, x_s) \in \mathbb{R}^s : A + \sum_{i=1}^{s} x_i A_i \succeq 0 \right\}.
\]

The boundary of \( \text{Gram}(f) \) can now be described as follows.
Proposition 2.1.3 ([RC95] Corollary 5). Let $A_0,\ldots,A_n \in S^m$ and let $S = \{(x_1,\ldots,x_n) \in \mathbb{R}^n: A_0 + \sum_{i=1}^n x_i A_i \succeq 0\}$ be the spectrahedron defined by $A_0,\ldots,A_n$. Assume there exists a point $y = (y_1,\ldots,y_n) \in \mathbb{R}^n$ where the matrix $A(y) := A_0 + \sum_{i=1}^n y_i A_i$ is positive definite ($A(y) > 0$). Then the boundary $\partial S$ of $S$ is given by all points $x \in S$ such that $\det(A(x)) = 0$ and the relative interior of $S$ is given by all $x \in \mathbb{R}^n$ such that $A(x) > 0$.

Moreover, the Zariski-closure of $\partial S$ in $\mathbb{R}^n$ is given by the zero-set of a polynomial $D \in \mathbb{R}[x_1,\ldots,x_n]$ such that $D$ divides $\det(A(x))$.

Definition 2.1.4. For a spectrahedron $S \subseteq \mathbb{R}^n$, the Zariski-closure of $\partial S \subseteq \mathbb{R}^n$ is called the algebraic boundary of $S$, and is denoted by $\partial_a S$.

By Proposition 2.1.3 the algebraic boundary of a spectrahedron is a hypersurface.

Lemma 2.1.5. Let $S = \{(x_1,\ldots,x_n) \in \mathbb{R}^n: A_0 + \sum_{i=1}^n x_i A_i \succeq 0\}$ be a spectrahedron with $A_0,\ldots,A_n \in S^m$ such that there exists a point $y = (y_1,\ldots,y_n) \in \mathbb{R}^n$ where the matrix $A(y) := A_0 + \sum_{i=1}^n y_i A_i$ is positive definite and such that $\det(A_0 + \sum_{i=1}^n x_i A_i) \in \mathbb{C}[x_1,\ldots,x_n]$ is irreducible. Let $z \in \partial S$, then $\text{rk } A(z) \leq m - 2$ if and only if $z$ is a singular point of the algebraic boundary of $S$.

Proof. For any two real symmetric matrices $A, B$ of size $m$, we write $\langle A,B \rangle := \text{tr}(A B)$. This defines a symmetric, non-degenerate bilinear form on the space of real symmetric matrices of size $m$. Moreover if $A$ and $B$ are psd, then $\text{tr}(AB) = 0$ is equivalent to $AB = 0$.

If the algebraic boundary of $S$ is defined by the polynomial $D \in \mathbb{R}[x_1,\ldots,x_n]$, then $D$ divides $\det(A(x))$. If $z \in \mathbb{R}^n$ is such that $D(z) = 0$ and all partial derivatives of $D$ vanish at $z$, then the same holds for $\det(A(x))$ and all of its partial derivatives at $z$. Therefore the assertion is equivalent to showing that $\text{rk } A(z) \leq m - 2$ if and only if all partial derivatives of $\det(A(x))$ vanish at $z$.

By Jacobi’s formula we have

$$\frac{\partial}{\partial x_i} \det(A(x)) = \langle A(x)^{\text{adj}}, A_i \rangle \quad \forall \, i = 1,\ldots,n,$$

and

$$A(x)A(x)^{\text{adj}} = A(x)^{\text{adj}}A(x) = \det(A(x)) I_m$$

where $I_m$ denotes the identity matrix of size $m$ and $A(x)^{\text{adj}}$ denotes the adjugate matrix of $A(x)$, i.e. the matrix whose entries are the $m - 1$ minors of $A(x)$ up to a factor $\pm 1$.

Let $z \in \partial S$ and assume that $\text{rk } A(z) \leq m - 2$. Then all $m - 1$ minors vanish, hence $A(z)^{\text{adj}}$ is the zero-matrix and all partial derivatives of $\det(A(x))$ vanish at $z$ by the formula above.

Let $z \in \partial S$ such that all partial derivatives of $\det(A(x))$ vanish at $z$. Then $\langle A(z)^{\text{adj}}, A_i \rangle = 0$ for $i = 1,\ldots,n$. By the second formula above we then have

$$\langle A, A(z)^{\text{adj}} \rangle = \langle A, A(z)^{\text{adj}} \rangle + \sum_{i=1}^n z_i \langle A_i, A(z)^{\text{adj}} \rangle = \langle A(z), A(z)^{\text{adj}} \rangle = \text{tr}(A(z) A(z)^{\text{adj}}) = 0.$$

Therefore we especially see $\langle A(z)^{\text{adj}}, A(y) \rangle = 0$. Since $A(z)$ is psd, the same holds for $A(z)^{\text{adj}}$.

Indeed, if the eigenvalues of $A(z)$ are given by $\lambda_1,\ldots,\lambda_m$, the eigenvalues of $A(z)^{\text{adj}}$ are given by $\prod_{i \neq j} \lambda_i$ for $j = 1,\ldots,m$. Hence all eigenvalues of $A(z)^{\text{adj}}$ are greater or equal to zero.

Since $y \in S$, we also know that $A(y)$ is psd, hence $A(z)^{\text{adj}} A(y) = 0$. From this and the fact that $A(y)$ is invertible, we see that $A(z)^{\text{adj}} = 0$. By definition of $A(z)^{\text{adj}}$ this means that all $m - 1$ minors of $A(z)$ vanish and therefore $\text{rk } A(z) \leq m - 2$. \qed
A similar result also holds for hyperbolic forms and their hyperbolicity cones and this lemma is a special case. However, we prefer not to introduce hyperbolic forms and keep the proof self-contained.

**Corollary 2.1.6.** Let \( f \in \operatorname{int} \Sigma_{n, 2d} \), then Gram matrices of corank \( 1 \) are dense in the boundary of \( \operatorname{Gram}(f) \).

**Proof.** The algebraic boundary of \( \operatorname{Gram}(f) \) is a hypersurface and hence the union of irreducible hypersurfaces. Let \( \vartheta \in \partial \operatorname{Gram}(f) \) be generic. Then \( \vartheta \) lies on exactly one irreducible component and is a smooth point of this component. Hence, by Lemma 2.1.5 the corank of \( \vartheta \) is equal to 1. Any such non-singular point that also lies on \( \partial \operatorname{Gram}(f) \) has corank 1, hence corank 1 points are dense in \( \partial \operatorname{Gram}(f) \). \( \square \)

Next, we take a closer look at the boundary by studying faces more closely. Moreover, instead of matrices we work coordinate-free. This is also the point of view we will take thereafter.

### 2.2 Facial structure of spectrahedra

We explain the facial structure of spectrahedra and particularly Gram spectrahedra. All results about spectrahedra are originally due to Ramana and Goldman [RG95], and the coordinate-free approach we are going to use has been introduced by Scheiderer [Sch].

Let \( R \) be an \( \mathbb{R} \)-algebra and let \( V \subseteq R \) be a finite-dimensional \( \mathbb{R} \)-vector space. We denote by \( S_2 V \) the second symmetric power of \( V \) and identify it with the space of all symmetric \( r \times r \) matrices and write \( \varphi : V^r \to \operatorname{S}_2 V \) for the set of all psd tensors in \( \operatorname{S}_2 V \). Additionally, if \( b_0(\lambda, \lambda) > 0 \) for every \( \lambda \in V^\ast \) we say that \( \vartheta \) is positive definite and write \( \vartheta > 0 \).

After choosing a basis \( p_1, \ldots, p_r \) of \( V \), every tensor \( \vartheta \in \operatorname{S}_2 V \) can be written as \( \vartheta = \sum_{i,j=1}^r a_{ij} p_i \otimes p_j \) with \( a_{ij} = a_{ji} \in \mathbb{R} \). This identifies \( \vartheta \) with the symmetric matrix \( (a_{ij})_{i,j} \in \mathbb{S}^r \). Hence, \( \operatorname{S}_2 V \) is identified with the space of all symmetric \( r \times r \) matrices and \( \operatorname{S}_2 V \) with the cone of all psd \( r \times r \) matrices. Especially, \( \operatorname{S}_2 V \) is a full-dimensional, convex cone inside \( \mathbb{S}_2 V \).

Furthermore, since every real symmetric matrix can be diagonalized, we can write \( \vartheta = \sum_{i=1}^r a_i q_i \otimes q_i \) for some \( q_1, \ldots, q_r \in V \) and \( a_1, \ldots, a_r \in \mathbb{R} \). Moreover, \( \vartheta \) is psd if and only if all \( a_i \) are non-negative.

**Lemma 2.2.1.** Let \( U \subseteq V \) be a linear subspace and \( \vartheta, \gamma \in \operatorname{S}_2 V \). The following hold:

(i) \( \operatorname{im} \vartheta \subseteq U \iff \vartheta \in \operatorname{S}_2 U \),

(ii) if \( \vartheta, \gamma \) are psd, then \( \operatorname{im} (\vartheta + \gamma) = \operatorname{im} \vartheta + \operatorname{im} \gamma \),

(iii) if \( \vartheta \) is psd and \( \operatorname{im} \gamma \subseteq \operatorname{im} \vartheta \), then there exists \( \varepsilon > 0 \) such that \( \vartheta - \varepsilon \gamma \) is psd.
Proof. (i): $\Leftarrow$ is clear. $\Rightarrow$: Let $\text{im} \vartheta \subseteq U$ and write $\vartheta = \sum_{i=1}^{r} q_i$ where $p_1, \ldots, p_r \in V$ are linearly independent and $q_1, \ldots, q_r \in V$ are linearly independent. Let $\lambda_1, \ldots, \lambda_r \in V^*$ such that $\lambda_i(p_j) = \delta_{ij}$ for all $i, j \in \{1, \ldots, r\}$, i.e. $(\lambda_i)_{1 \leq i \leq r}$ is the dual basis of $(p_i)_{1 \leq i \leq r}$. Then $q_i = \phi_\vartheta(\lambda_i) \in \text{im} \vartheta = U$. Choosing a dual basis of $q$ analogously shows that $p_1, \ldots, p_r \in U$, hence $\vartheta \in S_2 U$.

(ii) and (iii) are both well-known for matrices and the proofs are analogous.

Now we come to the facial structure of spectrahedra. Let $L \subseteq S_2 V$ be an affine-linear subspace and let $S := L \cap S_2^+ V$ be the spectrahedron defined by $L$. For every non-empty face $F \subseteq S$ we denote by

\[ U(F) := \sum_{\vartheta \in F} \text{im(}\vartheta) \]

the corresponding subspace. And on the other hand for every subspace $U \subseteq V$, we denote by

\[ F(U) := \{ \vartheta \in S : \text{im} \vartheta \subseteq U \} \]

the corresponding face on $S$.

**Definition 2.2.2.** Let $U \subseteq V$ be a subspace. The subspace $U$ is called $S$-facial, if there exists $\vartheta \in S$ such that $\text{im} \vartheta = U$.

In this case, $\vartheta$ is a relative interior point of $F(U)$ and $F(U)$ is the supporting face of $\vartheta$. The next proposition was first shown in [RG95] and in this formulation in [Sch].

**Proposition 2.2.3.** There is an inclusion-preserving bijection between non-empty faces $F$ of $S$ and $S$-facial subspaces $U \subseteq V$ given by

\[ F \mapsto U(F) \]

\[ F(U) \mapsto U. \]

**Proof.** Let $F$ be a non-empty face of $S$ and let $U = U(F)$. We claim that $U$ is $S$-facial and $F(U) = F$. We first show the second part. Let $\vartheta \in F$, then by definition we have $\text{im} \vartheta \subseteq U$, hence $\vartheta \in F(U)$. On the other hand since dim $U$ is finite, there exist $\vartheta_1, \ldots, \vartheta_r \in F$ such that $U = \sum_{i=1}^{r} \text{im} \vartheta_i$. By Lemma 2.2.1 (ii) we then see that $\vartheta := \frac{1}{r} \sum_{i=1}^{r} \vartheta_i$ has $U$ as its image and lies in $F$. Thus we need to show that $F(\text{im} \vartheta) \subseteq F$. Let $\gamma \in F(\text{im} \vartheta)$, then $\text{im} \gamma \subseteq \text{im} \vartheta$. By Lemma 2.2.1 (iii), there exists $0 < \varepsilon < 1$ such that $\vartheta - \varepsilon \gamma \geq 0$. Then

\[ \delta := \frac{1}{1-\varepsilon} \vartheta - \frac{\varepsilon}{1-\varepsilon} \gamma \in S \]

and thus

\[ \vartheta = (1-\varepsilon) \delta + \varepsilon \gamma \in F, \]

where $\delta, \gamma$ are in $S$. Hence by definition $\gamma \in F$. Since $U = \text{im} \vartheta$ we also see that $U$ is $S$-facial.

Conversely, if $U \subseteq V$ is a $S$-facial subspace, then $U(F(U)) = U$: Since $U$ is $S$-facial there exists $\vartheta \in S$ such that $\text{im} \vartheta = U$. Then $\vartheta \in F(U)$ and hence $U(F(U)) \supseteq U$. The other inclusion is clear by definition.

The fact that the bijection is inclusion-preserving is clear from the definition of $F$ and $U$. \qed
Corollary 2.2.4. If \( F \subseteq S \) is a face and \( U = \mathcal{U}(F) \) is the corresponding subspace, then the relative interior of \( F \) is given by all tensors \( \vartheta \) that satisfy \( \text{im} \vartheta = U \).

Remark 2.2.5. Since for any face the image of every relative interior point is the same, we see in particular that the ranks are the same. It is therefore well-defined to call this rank (of any relative interior point) the rank of the face.

To determine the dimensions of faces later on, we will need the following proposition.

Proposition 2.2.6. If \( F \subseteq S \) is a non-empty face with corresponding subspace \( U \), then 
\[ \dim F = \dim (L \cap S_2 U). \]

Proof. Let \( \text{aff}(F) \) denote the affine hull of \( F \). Since \( F = L \cap S_2^+ U \), it follows that \( \text{aff}(F) \subseteq L \cap S_2 U \). Conversely let \( \vartheta \in \text{relint}(F) \), then \( \text{im} \vartheta = U \). Pick any \( \gamma \in L \cap S_2 U \), then there exists \( 0 < \varepsilon < 1 \) such that \( \vartheta + \varepsilon \gamma, \vartheta - \varepsilon \gamma \succeq 0 \). Particularly 
\[ \vartheta_1 := \frac{1}{1 + \varepsilon} (\vartheta + \varepsilon \gamma), \vartheta_2 := \frac{1}{1 - \varepsilon} (\vartheta - \varepsilon \gamma) \in S. \]
Choosing \( \lambda = \frac{\varepsilon + 1}{2} \in (0, 1) \) we get \( \lambda \vartheta_1 + (1 - \lambda) \vartheta_2 = \vartheta \in F \), and therefore \( \vartheta_1, \vartheta_2 \in F \). Now choose \( \lambda = \frac{1 + \varepsilon}{2 \varepsilon} \), then \( \lambda \vartheta_1 + (1 - \lambda) \vartheta_2 = \gamma \), thus \( \gamma \in \text{aff}(F) \).

The last proposition enables us to calculate the dimension algebraically although faces are semi-algebraic in nature.

A special kind of face is the extreme point. These are by definition faces of dimension 0. The next result is originally due to Pataki in [Pat00], this formulation can be found in [CPSV17, Proposition 3.1.]. It determines bounds for the ranks of extreme points of spectrahedra.

Proposition 2.2.7. Let \( \dim V = n \), let \( L \subseteq S_2 V \) be an affine-linear subspace with \( \dim L = m \), and let \( S = L \cap S_2^+ V \).

(i) For every extreme point \( \vartheta \) of \( S \), the rank \( \text{rk} \vartheta = r \) satisfies
\[ m + \left( \frac{r + 1}{2} \right) \leq \left( \frac{n + 1}{2} \right). \]

(ii) When \( L \) is chosen generically among all affine-linear subspaces of dimension \( m \), every \( \vartheta \in S \) satisfies
\[ m \geq \left( \frac{n - \text{rk} \vartheta + 1}{2} \right). \]

This defines the so-called Pataki interval, in which all ranks of extreme points of generic spectrahedra have to lie.

2.3 Facial structure of Gram spectrahedra

We continue our coordinate-free approach but specialize to Gram spectrahedra.

Let \( R \) be an \( \mathbb{R} \)-algebra and \( V \subseteq R \) a finite-dimensional \( \mathbb{R} \)-vector space. The multiplication map \( R \otimes R \to R, f \otimes g \mapsto fg \) induces a multiplication map \( \mu \colon S_2 R \to R \) on the symmetric tensors. This is exactly the Gram map, written in a coordinate-free way.

Let \( \mu_{\mathbb{C}} \colon S_2 (\mathbb{C} \otimes R) \to \mathbb{C} \otimes R \) be the multiplication map extended to \( \mathbb{C} \) by tensoring with \( \mathbb{C} \) over \( \mathbb{R} \).
Definition 2.3.1. Let \( f \in \mathbb{C} \otimes R \), then any \( \vartheta \in \mu^{-1}(f) \subseteq S_2(\mathbb{C} \otimes R) \) is called a Gram tensor of \( f \). If \( f \in R \), the convex set

\[
\text{Gram}_V(f) := \mu^{-1}(f) \cap S^n_+ V,
\]

of all such psd Gram tensors in \( S^n_+ V \) is called the Gram spectrahedron of \( f \) wrt \( V \).

Remark 2.3.2. We also call complex tensors Gram tensors and only restrict to real tensors whenever necessary, which is certainly the case if we need to speak about them being psd.

Remark 2.3.3. Since \( S^n_+ V \) identifies canonically with the set of all symmetric psd matrices of size \( \text{dim}(V) \), and \( \text{Gram}_V(f) \) is the intersection of \( S^n_+ V \) with an affine-linear subspace, it follows that \( \text{Gram}_V(f) \) is a spectrahedron.

Remark 2.3.4. From now on, we write \( \mu \) to denote the Gram/multiplication map \( S_2 B \to B^2 \) for any \( \mathbb{C} \)- or \( \mathbb{R} \)-algebra \( B \) or any subspace \( B \) of a \( \mathbb{C} \)- or \( \mathbb{R} \)-algebra. Furthermore, if \( B \) is a \( \mathbb{C} \)-algebra, we denote by \( \text{im}_C(\vartheta) \) the image of \( \vartheta \in S_2 B \). If \( B \) is an \( \mathbb{R} \)-algebra, we write \( \text{im}(\vartheta) \) for the image of the linear map \( \varphi \mu \) as earlier and \( \text{im}_C(\vartheta) := \mathbb{C} \otimes \text{im}(\vartheta) \).

This is mostly used in Chapter IV where we consider Gram spectrahedra of ternary quartics, since we find nice generators of \( \text{im}_C(\vartheta) \) but not of \( \text{im}(\vartheta) \).

Remark 2.3.5. For \( R = \mathbb{R}[x_1, \ldots, x_n] \), any homogeneous polynomial \( f \) of degree 2d and \( V = R_d \), the set \( \text{Gram}_V(f) \) is the full Gram spectrahedron of \( f \).

Example 2.3.6. Consider the Fermat quartic \( f = x^4 + y^4 + z^4 \in \mathbb{R}[x, y, z]_4 \), then we have the rank 3 sos decomposition \( f = (x^2)^2 + (y^2)^2 + (z^2)^2 \). The corresponding Gram tensor can for example be written as \( \vartheta = x^2 \otimes x^2 + y^2 \otimes y^2 + z^2 \otimes z^2 \), since we have \( \mu(\vartheta) = (x^2)^2 + (y^2)^2 + (z^2)^2 = f \) and \( \vartheta \) is psd.

As with Gram matrices, any diagonalized Gram tensor of a form \( f \) immediately gives rise to a sos representation of \( f \).

From now on, we write \( V^2 := \mu(S_2 V) \) for the subspace generated by all products \( fg \) with \( f, g \in V \). This is indeed the same notation we also use for the cartesian product \( V \times V \). However, on the occasions where we also use the cartesian product, it should be immediately clear from the context what the square means. Furthermore, we write \( \Sigma V^2 \) for the set of all sums of squares in \( V \), i.e. the set of all \( f \in V^2 \) such that there exist \( p_1, \ldots, p_r \in V \) with \( f = \sum_{i=1}^r p_i^2 \).

If we do not restrict to polynomial rings, Gram spectrahedra do not have to be compact, but can in general be unbounded. This does not happen in the following case.

Lemma 2.3.7. If \( \sum_{i=1}^r p_i^2 = 0 \) implies \( p_1 = \cdots = p_r = 0 \) for all \( p_1, \ldots, p_r \in V \), then \( \text{Gram}_V(f) \) is bounded for all \( f \in V^2 \).

Proof. Assume that \( \text{Gram}_V(f) \) is unbounded. Then there exist \( \vartheta \in \Sigma V^2 \) and \( \vartheta \in \text{Gram}_V(f) \) such that \( \vartheta + \lambda \gamma \in \text{Gram}_V(f) \) for all \( \lambda \geq 0 \). It follows that \( \gamma \geq 0 \) and \( f = \mu(\vartheta) + \lambda \mu(\gamma) \), hence \( \mu(\gamma) = 0 \). Since \( \gamma \) is psd, we can write \( \gamma = \sum_{i=1}^r p_i \otimes p_i \), and get \( 0 = \mu(\gamma) = \sum_{i=1}^r p_i^2 \).

Example 2.3.8. Consider the ring \( R = \mathbb{R}[x, y, z]/(x^2 + y^2 + z^2) \) and its subspace \( V = R_1 \). Let \( f \in \mathbb{R}[x, y, z]_2 \) be the form \( x^2 + y^2 + z^2 \). In order to be positive definite for every \( \lambda > 0 \) it is also a sum of squares. More precisely, we have \( f + \lambda (x^2 + y^2 + z^2) = (\lambda + 1)x^2 + \lambda y^2 + \lambda z^2 \).
and this sos representation is unique (up to orthogonal equivalence). Therefore, the Gram spectrahedron \( \text{Gram}_V(f) \) contains the tensor \( (\lambda + 1)\pi \otimes \pi + \lambda \gamma \otimes \gamma + \lambda \pi \otimes \pi \) for every \( \lambda > 0 \). Especially, the Gram spectrahedron is unbounded. In fact, this is the whole Gram spectrahedron which is a half ray, and the boundary point is given by the Gram tensor \( \pi \otimes \pi \).

Now we look at the facial structure of Gram spectrahedra and use our findings of the previous section to find formulas for the dimensions of faces of Gram spectrahedra.

**Proposition 2.3.9.** Let \( f \in \Sigma V^2 \) and let \( F \subseteq \text{Gram}_V(f) \) be a face, then \( F = \text{Gram}_V(f) \) for some subspace \( U \subseteq V \). On the other hand, for every Gram \( V \)-facial subspace \( U \) of \( V \) the set \( \text{Gram}_V(f) \) is a face of \( \text{Gram}_V(f) \).

**Proof.** By Lemma 2.2.1(iii) we have
\[
\text{Gram}_V(f) = \mu^{-1}(f) \cap S_2^+ U = \{ \vartheta \in \text{Gram}_V(f) : \text{im} \vartheta \subseteq U \} = F(U).
\]
Hence everything follows from Proposition 2.2.3.

For simplicity, we say that a subspace is \( f \)-facial rather than Gram \( V \)-facial if \( V \) is understood.

**Proposition 2.3.10.** Let \( U \subseteq V \) be \( f \)-facial, then \( \dim F(U) = \binom{\dim U + 1}{2} - \dim U^2 \).

**Proof.** By Proposition 2.2.6 we know that \( \dim F(U) \) is the same as the dimension of \( L := \mu^{-1}(f) \cap S_2 U \) as an affine-linear subspace. Since \( S_2 U \xrightarrow{\mu} U^2 \) is surjective, we have \( \dim L = \dim \ker(\mu) = \dim S_2 U - \dim U^2 \). Furthermore \( \dim S_2 U = \binom{\dim U + 1}{2} \).

**Corollary 2.3.11.** Let \( U \subseteq V \) be \( f \)-facial and let \( p_1, \ldots, p_r \) be a basis of \( U \). The face \( \text{Gram}_V(f) = F(U) \) is an extreme point of \( \text{Gram}_V(f) \) if and only if the products \( p_ip_j \) with \( 1 \leq i \leq j \leq r \) are linearly independent.

**Proof.** The dimension of \( F(U) \) is given by the dimension of the kernel of \( S_2 U \xrightarrow{\mu} U^2 \). Hence, \( F(U) \) is an extreme point if and only if the map is injective or equivalently the products \( p_ip_j \) (\( 1 \leq i \leq j \leq r \)) are linearly independent.

**Remark 2.3.12.** Let \( f \in \Sigma V^2 \) and \( U \subseteq V \) a subspace. Then \( U \) is \( f \)-facial if and only if \( f \in \text{int} \Sigma U^2 \). If \( U \) is \( f \)-facial, there exists a tensor \( \vartheta \in \text{Gram}_V(f) \) such that \( \text{im} \vartheta = U \). Therefore, \( \vartheta \) has the maximal possible rank of tensors in \( S_2 U \) and therefore \( f = \mu(\vartheta) \in \text{int} \Sigma U^2 \). Conversely, if \( f \in \text{int} \Sigma U^2 \), there is a Gram tensor of \( f \) of rank \( \dim U \) in \( S_2 U \) whose image is equal to \( U \). We used the following well-known lemma to conclude that being an interior point of \( \Sigma U^2 \) is equivalent to having a Gram tensor in \( S_2 U \) of maximal rank.

**Proposition 2.3.13.** Let \( U \subseteq V \) be a subspace and let \( f \in \Sigma V^2 \). Then \( U \) is \( f \)-facial if and only if \( f \in \text{int} \Sigma U^2 \).

**Proof.** By the next lemma applied to the Gram map \( S_2 U \xrightarrow{\mu} U^2 \) and the set of all psd tensors in \( S_2 U \) an element \( g \in V^2 \) is contained in \( \text{int} \Sigma U^2 \) if and only if there exists a Gram tensor of \( g \) in \( S_2 U \) of rank \( \dim U \).

If \( U \) is \( f \)-facial, there exists a tensor \( \vartheta \in \text{Gram}_V(f) \) such that \( \text{im} \vartheta = U \). Therefore \( f = \mu(\vartheta) \in \text{int} \Sigma U^2 \).

On the other hand, if \( f \in \text{int} \Sigma U^2 \) there exists a Gram tensor \( \vartheta \) of \( f \) in \( S_2 U \) of rank \( \dim U \), hence \( \text{im} \vartheta = U \) and therefore \( U \) is \( f \)-facial.

\[ \square \]
We give the proof of [CPSV17] Lemma 1.5.

Lemma 2.3.14. Let $m, n \in \mathbb{N}$ and let $\phi: \mathbb{R}^m \rightarrow \mathbb{R}^n$ be a surjective linear map. If $C \subseteq \mathbb{R}^m$ is a convex set with $\text{int}(C) \neq \emptyset$, then $\phi(\text{int}(C)) = \text{int}(\phi(C))$.

Proof. Since $\phi$ is linear, the map is also open, especially $\phi(\text{int}(C)) \subseteq \text{int}(\phi(C))$.

To show the other inclusion, let $x \in \mathbb{R}^n$ be a point such that $\phi^{-1}(x) \cap \text{int}(C) = \emptyset$. Using separation theorems for convex sets [Web94, Corollary 2.4.11.], there exists a linear functional $\lambda \in (\mathbb{R}^m)^\ast$ with $\lambda(\phi^{-1}(x)) = 0$ and $\lambda(\text{int}(C)) > 0$.

By assumption the interior of $C$ is non-empty, therefore $\lambda(C) \geq 0$. Moreover, since $\phi$ is linear, the fiber of a point $w \in \mathbb{R}^n$ is a translation of the fiber of $x$, i.e. $\phi^{-1}(w) = y + \phi^{-1}(x)$ for some $y \in \mathbb{R}^m$. Since $\lambda(\phi^{-1}(x)) = 0$, it follows that $\lambda$ is constant on the fiber of any element.

We thus have linear functional on $\mathbb{R}^m$ given by

$$\lambda': \mathbb{R}^n \rightarrow \mathbb{R}, \quad y \mapsto \lambda'(y) := \lambda(\phi^{-1}(y)).$$

It satisfies $\lambda'(x) = 0$ and $\lambda'(\phi(C)) = \lambda(C) \geq 0$. Therefore $x \notin \text{int}(\phi(C))$. \]

Notation 2.3.15. Whenever we work with the polynomial ring $R = \mathbb{R}[x_1, \ldots, x_n]$, a polynomial $f \in R_{2d}$ and $V = R_d$, we also write $\text{Gram}(f)$ instead of $\text{Gram}_V(f)$.

Example 2.3.16. Consider once again the Fermat quartic $f = x^4 + y^4 + z^4$. Since $f$ is positive definite, we see that $f \in \text{int} \Sigma_{3,4}$. Then as $\mathbb{R}[x]_2$ is $f$-facial, the dimension of the Gram spectrahedron of $f$ is given by

$$\dim \text{Gram}(f) = \left(\dim \mathbb{R}[x]_2 + 1\right) / 2 - \dim \mathbb{R}[x]_4 = 21 - 15 = 6.$$

Moreover the Gram tensor $\vartheta = x^2 \otimes x^2 + y^2 \otimes y^2 + z^2 \otimes z^2$ is an extreme point of $\text{Gram}(f)$. Indeed, the six forms $x^4, x^2y^2, x^2z^2, y^4, y^2z^2, z^4$ are linearly independent and therefore $\vartheta$ is an extreme point of $\text{Gram}(f)$ by Corollary 2.3.11.

Remark 2.3.17. If we are working with a fixed number of variables and a fixed degree, the dimension of a face only depends on the dimension of $U$ and the dimension of $U^2$. This means that we can determine dimensions purely algebraically. It turns out that it is more convenient to talk about the codimensions of $U$ and $U^2$ most of the time. In these cases this should always be understood as the codimension of $U$ as a subspace of $R_d$. Analogously reading $U^2$ as a subspace of $R_{2d}$.

Remark 2.3.18. For calculations, we do fix a basis and are going to work with Gram matrices. As with the definition of Gram tensor, we will not require Gram matrices to be real.

Remark 2.3.19. Let $X = (M_1, \ldots, M_N)$ be the monomial basis of $R_d$ ($N = \dim R_d$) and let $f \in \Sigma_{n,2d}$ be a sum of squares. Let $F \subseteq \text{Gram}(f)$ be a face and let $\vartheta \in F$ be a relative interior point with corresponding subspace $U$. Write $\vartheta = \sum_{i=1}^r p_i \otimes q_i$ and denote by $c(p)$ the coordinate vector of $p \in R_d$ wrt $X$, i.e. $c(p)X^T = p$. Then $\vartheta$ gives rise to the Gram matrix

$$G = (c(q_1)^T, \ldots, c(q_r)^T) \begin{pmatrix} c(p_1) \\ \vdots \\ c(p_r) \end{pmatrix}.$$
By definition $XGX^T = \sum_{i=1}^r p_i q_i = f$ and the Gram map is given by

$$S^N \to R_{2d}, \ G \mapsto XGX^T.$$ 

With this notation, a generating system of $U$ is also given by the entries of the vector $GX^T$. The dimension of $U$, or equivalently the rank of $F$, is given by the length of the sos representation above or equivalently the rank of $G$.

Now, we revisit Pataki’s interval and show that the bound holding for generic spectrahedra, also holds for generic Gram spectrahedra.

**Proposition 2.3.20.** Let $f \in \mathbb{R}$ and $n = \dim V$. Then every extreme point $\vartheta$ of $\text{Gram}_V(f)$ satisfies

$$\left(\frac{\text{rk} \vartheta + 1}{2}\right) \leq \dim V^2.$$ 

If $f$ is chosen generically, then every $\vartheta \in \text{Gram}_V(f)$ satisfies

$$\left(\frac{n + 1}{2}\right) \geq \left(\frac{n - \text{rk} \vartheta + 1}{2}\right) + \dim V^2.$$ 

**Proof.** The first inequality also follows from Proposition 2.2.7, however, we will give a different proof.

Write $\text{Gram}(f) := \text{Gram}_V(f)$ and let $\vartheta \in \text{Gram}(f)$ be an extreme point of rank $r$. Furthermore, write $m := \dim V^2$ and let $U = \text{im} \vartheta$. By Proposition 2.3.10 the dimension of the supporting face $F(U)$ of $\vartheta$ is given by the dimension of the kernel of $S_2 U \xrightarrow{\mu} U^2 \subseteq V^2$. If $S_2 U = \left(\frac{r + 1}{2}\right) > \dim V^2 = m$, the map cannot be injective and thus $\vartheta$ cannot be an extreme point.

For the other inequality consider the sum of squares map

$$\phi: V^r \to V^2, \ (p_1, \ldots, p_r) \mapsto \sum_{i=1}^r p_i^2$$

and its differential at the point $p = (p_1, \ldots, p_r)$

$$d\phi_r(p): V^r \to V^2, \ (q_1, \ldots, q_r) \mapsto 2 \sum_{i=1}^r p_i q_i.$$ 

Since all the trivial relations $(0, \ldots, 0, p_i, 0, \ldots, 0, -p_j, 0, \ldots, 0), i \leq j$ are contained in the kernel of $d\phi_r(p)$, we have $\dim \text{im}(d\phi_r(p)) \leq r \dim V - \binom{r}{2} = rn - \binom{r}{2}$.

If the inequality we claim was not true, and we had $\left(\frac{n + 1}{2}\right) < \left(\frac{n - \text{rk} \vartheta + 1}{2}\right) + \dim V^2$, then one easily checks that

$$\dim \text{im} (d\phi_r(p)) \leq rn - \binom{r}{2} \leq \left(\frac{n + 1}{2}\right) - \left(\frac{n - \text{rk} \vartheta + 1}{2}\right) < \dim V^2.$$ 

This means that the differential is not surjective for any $p \in V^r$ and therefore by Sard’s Theorem the image of $\phi$ is nowhere dense. Thus, if $f$ is chosen generically, $\text{Gram}(f)$ contains no tensor of rank $r$. 

$\square$
Example 2.3.21. Let \( n = 3, d = 2 \), and let \( f \in \Sigma_{3,4} \) be a generic ternary quartic. Then the ranks \( r \) of extreme points of the Gram spectrahedron \( \text{Gram}(f) \) \((R = \mathbb{R}[x], V = \mathbb{R}[x]_2)\), satisfy the inequalities

\[
\left(\frac{r + 1}{2}\right) \leq \dim \mathbb{R}[x]_4 = 15 \quad \text{and} \quad \left(\frac{7 - r}{2}\right) \geq 15
\]

which is equivalent to \( r \leq 5 \) and \( r \geq 3 \). Hence, the Pataki interval is given by \( 3 \leq r \leq 5 \).

In this case, a generic form \( f \) does have a Gram tensor of every rank in the Pataki interval as we will see in Chapter \([\text{IV}]\). This however is not always the case.

Let \( r \in \mathbb{N} \), and let \( \text{Grass}(r, A_d) \subseteq \mathbb{P}^N \) denote the Grassmannian of \( r \)-dimensional subspaces of \( A_d \) embedded via the Plücker embedding. We say that a subspace \( U \subseteq A_d \) of dimension \( r \) is generic if \( U \in \text{Grass}(r, A_d) \) is generic in the sense of Section \([1.1]\).

**Proposition 2.3.22.** Let \( U \subseteq \mathbb{R}[x]_d \) be a generic subspace of codimension \( k \). Then codim \( U^2 \) is minimal. I.e. for every subspace \( V \subseteq \mathbb{R}[x]_d \) of codimension \( k \) we have codim \( V^2 \geq \) codim \( U^2 \).

**Proof.** Let \( p_1, \ldots, p_r \in \mathbb{R}[x]_d \) such that \( U = \text{span}(p_1, \ldots, p_r) \). Then \( U^2 \) is spanned by all products \( p_ip_j \) \((1 \leq i, j \leq r)\). The span of these products then has maximal dimension whenever \( p_1, \ldots, p_r \) are chosen generically.

**Remark 2.3.23.** Let \( U \subseteq \mathbb{R}[x]_d \) be a generic subspace of dimension \( r \). Then every \( f \in \text{int} \Sigma U^2 \) has a face corresponding to \( U \) which therefore is of dimension \((r+1)/2 - \dim U^2\). By Proposition \([2.3.22]\) this is the smallest possible face of rank \( r \) on any Gram spectrahedron of a form in \( \Sigma_{n,2d} \).

It is not clear in general what the value of \( \dim U^2 \) is for a generic subspace \( U \subseteq \mathbb{R}[x]_d \) of dimension \( r \) and fixed \( n \) and \( d \). One might expect that the dimension of \( U^2 \subseteq A_{2d} \) is always the largest possible, i.e. either the map \( \mu : \mathcal{S}_2U \to A_{2d} \) is injective or surjective. This however is not true in general as the next example shows.

**Proposition 2.3.24 (BC18 Proposition 2.8.)**. Let \( n = 4, d = 2 \) and \( U \subseteq \mathbb{R}[x]_2 \) be a subspace of codimension 2. Then \( \dim U^2 \leq 34 < 35 = \min(\dim \mathbb{R}[x]_4, \dim \mathcal{S}_2U) \).

**Proof.** The dimension of \( U^2 \) is maximal for generic \( U \). Hence, we can assume that \( U \) is generic. Let \( W = U^\perp \). Since \( W \) is generic it contains a quadratic form \( q \) of rank 4. After a change of coordinates (over \( \mathbb{C} \)) which does not change the dimension of \( U^2 \), we can assume that \( q = x_1^2 + x_2^2 + x_3^2 + x_4^2 \) and that \( W = \text{span}(q, a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2) \) for some \( a_1, \ldots, a_4 \in \mathbb{C} \). By construction \( U \) contains all polynomials apolar to both of these forms. Since all monomials \( x_ix_j \) with \( i \neq j \) are apolar to \( W \) they are contained in \( U \). Thus, there are two quadratic relations, namely \((x_1x_4)(x_2x_3) = (x_1x_2)(x_3x_4)\) and \((x_1x_3)(x_2x_4) = (x_1x_2)(x_3x_4)\). And thus the kernel of the map \( \mathcal{S}_2U \to U^2 \) has dimension at least two. Since \( \dim \mathcal{S}_2U = 36 \) we see that \( \dim U^2 \leq 34 \).

**Remark 2.3.25.** We change perspective compared to Remark \([2.3.23]\) and instead consider a generic form \( f \in \Sigma_{n,2d} \). As we show in Chapter \([\text{IV}]\) a generic form \( f \in \Sigma_{3,4} \) has a face of rank 5 and dimension 2. However, a generic subspace \( U \subseteq \mathbb{R}[x_1, x_2, x_3]_2 \) of dimension 5 satisfies \( U^2 = \mathbb{R}[x_1, x_2, x_3]_4 \), and for any \( f \in \text{int} \Sigma U^2 \) corresponds to an extreme point of rank 5.
This shows that if we want to understand Gram spectrahedra of generic forms, it is not enough to consider generic subspaces, but instead we need to study all subspaces of a certain dimension.
3 Bounds for dimensions of faces

Most of the results contained in this chapter are also contained in the same or a very similar form in the preprint [Vil20]. For the rest of this chapter, let \( n, d \in \mathbb{N} \), \( n, d \geq 2 \). We denote by \( A := \mathbb{C}[x_1, \ldots, x_n] \) the polynomial ring over \( \mathbb{C} \). We also write \( A(n) \) to emphasize the number of variables but usually omit the \( n \) in the notation. For the polynomial ring over the reals we write \( \mathbb{R}[x] := \mathbb{R}[x_1, \ldots, x_n], \mathbf{x} = (x_1, \ldots, x_n) \). As we have seen in Chapter 2, the crucial number to understand is the codimension of \( U^2 \) for a subspace \( U \subseteq A_d \). Hence the following definition.

**Definition 3.0.1.** For \( n \geq 2 \) and \( d, k \geq 1 \) let

\[
m(n, d, k) = \max \{ \text{codim} \ U^2 : U \subseteq A_d \text{ subspace, codim} \ U = k \}.
\]

We say that a subspace \( U \subseteq A_d \) of codimension \( k \) realizes \( m(n, d, k) \) if \( \text{codim} \ U^2 = m(n, d, k) \).

Assume we know \( m(n, d, k) \) for some values of \( n, d \), and \( k \). Let \( f \in \Sigma_{n, 2d} \) and let \( F \subseteq \text{Gram}(f) \) be a face of corank \( k \) (equivalently of rank \( \dim A_d - k \)) with corresponding subspace \( U \). This means that \( U \subseteq \mathbb{R}[x]_d \) is a subspace of codimension \( k \). Since the codimension of \( U^2 \) is invariant under base-field extension we can also consider the subspace \( V := U \otimes_{\mathbb{R}} \mathbb{C} \subseteq A_d \) of codimension \( k \). For this subspace, we know \( \text{codim} \ V^2 \leq m(n, d, k) \). Since \( \dim F = (\dim A_d - k + 1) - \dim A_{2d} + \text{codim} U^2 \), this translates into an upper bound for the dimension of the face \( F \). Therefore, in this whole chapter we determine bounds for \( \text{codim} U^2 \), which means that we are only considering algebraic questions throughout.

All subspaces \( U \) we find in the first section however will have a real base-point. Therefore, any \( f \in \Sigma U^2 \) lies on the boundary of the psd cone. This leads us to consider base-point-free subspaces afterward to find better upper bounds whenever our forms are non-singular. Contrary to the base-point case, in the base-point-free case, upper bounds for \( \text{codim} U^2 \) do not depend on \( n \), as long as the degree is large enough. This can be seen as a generalization of the fact that base-point-free subspaces of codimension 1 satisfy \( \text{codim} U^2 \in \{0, 1\} \) for \( d \geq 3 \) (Proposition 3.4.3).

If \( \succeq \) is a monomial ordering and \( I \subseteq A \) is a homogeneous ideal, let \( \text{in}_{\succeq}(I) \) be the initial ideal of \( I \), i.e. the ideal in \( A \) generated by all initial terms of forms in \( I \). Whenever we talk about the initial ideal of \( \langle U \rangle \), we also write \( \text{in}_{\succeq}(U) \) instead of \( \text{in}_{\succeq}(\langle U \rangle) \), where \( \langle U \rangle \) is the ideal generated by \( U \) in \( A \).

Most claims do hold for every monomial ordering or at least any elimination ordering, however for simplicity, if not explicitly stated otherwise, we always work with the lexicographic-ordering (lex-ordering) such that \( x_1 < \cdots < x_n \). I.e. for \( \alpha \neq \beta \in \mathbb{Z}_+^n \) we have \( x^\alpha > x^\beta \) if and only if \( \alpha_i \geq \beta_i \) for \( i = j, \ldots, n \) and \( \alpha_j > \beta_j \) for some \( j \in \{1, \ldots, n\} \). In our applications ordering the variables in this way, is usually preferable. We therefore also write \( \text{in}(I) \) instead of \( \text{in}_{\succeq}(I) \) as the ordering is understood.
3.1 STRONGLY STABLE SUBSPACES

In this first section, we find upper bounds for codim $U^2$. The subspaces realizing the bounds are monomial subspaces, that is, subspaces that have a basis consisting of monomials. We introduce strongly stable subspaces and show some combinatorial results (e.g. Lemma 3.1.7 Lemma 3.1.8 Lemma 3.1.9) for later use, especially in Section 3.4. Strongly stable subspaces and generic initial ideals form one of our main tools for the rest of the chapter.

Definition 3.1.1. Let $U \subseteq A_d$ be a monomial subspace. $U$ is called strongly stable if for every $1 \leq i \leq n$ the following holds: For every monomial $M \subseteq U$ such that $x_i | M$ and every $i < j \leq n$, the monomial $x_j \frac{M}{x_i}$ is contained in $U$.

For every monomial ordering $\succeq$ where $x_1 < \cdots < x_n$ and every monomial $M$ such that $x_i | M$ we have $x_j \frac{M}{x_i} \succeq M$ for every $j > i$.

Remark 3.1.2. Another way of thinking about strongly stable subspaces is via their complements. If $U \subseteq A_d$ is strongly stable and $W = U^\perp$, then for every $1 \leq i \leq n$ the following holds: For every monomial $M \subseteq W$ such that $x_i | M$ and every $1 \leq j < i$, the monomial $x_j \frac{M}{x_i}$ is contained in $W$, i.e. the inequality sign is reversed.

Example 3.1.3. Let $U \subseteq A_d$ be a strongly stable subspace, then the subspace $U^2 \subseteq A_{2d}$ is also strongly stable. Indeed, since $U$ is generated by monomials, the same holds for $U^2$, and every monomial in $U^2$ is the product of two monomials in $U$. Let $M \subseteq U^2$ be a monomial and let $i \in \{1, \ldots, n\}$ such that $x_i | M$ and let $j \in \{i+1, \ldots, n\}$. We write $M = ST$ for two monomials $S, T \subseteq U$. Since $x_i | M$, we can assume wlog that $x_i | S$. Since $U$ is strongly stable, the monomial $x_j \frac{S}{x_i}$ is contained in $U$. Therefore, $x_j \frac{M}{x_i} = \left(x_j \frac{S}{x_i}\right) T \subseteq U^2$ and thus $U^2$ is strongly stable.

Example 3.1.4. (i) We consider the subspace $U = \text{span}(x_2^2, x_2x_3, x_3^2) \subseteq A(3)_2$ of codimension 3. This subspace is strongly stable: We have to check that the monomials we receive after dividing the first two monomials by $x_2$ and then multiplying with $x_3$, are still contained in $U$. This is certainly true.

Another way to check this is to look at $U^\perp = \text{span}(x_1^2, x_1x_2, x_1x_3)$. Here we need to check that if a monomial $M$ is contained in $U^\perp$ and $x_i$ divides $M$, then also $x_j \frac{M}{x_i} \subseteq U^\perp$ for all $j < i$. For example, dividing $x_1x_3$ by $x_3$ and then multiplying with $x_1$ or $x_2$, we get the monomials $x_1^2$ and $x_1x_2$ which are also contained in $U^\perp$.

This second point of view seems more convenient in our setting later on since the subspaces we are dealing with have low codimensions.

(ii) Let $k \in \mathbb{N}$ and $n \geq k$. Let $U \subseteq A_d$ be a strongly stable subspace of codimension $k$ and assume that $x_1^{-1}x_k$ is not contained in $U$, i.e. is contained in $U^\perp$. Hence, also the monomials $x_i x_1^{-1}x_k$ are contained in $U^\perp$ for every $1 \leq i \leq k$. Especially, these are exactly $k$ monomials, and therefore $U^\perp = \text{span}(x_1^d, x_1^{d-1}x_2, \ldots, x_1^{d-1}x_k)$.

This shows that a very limited number of conditions on $U$ might already determine the space. Especially, knowing that a monomial that contains $x_i$ for some large $i$, is not contained in $U$ determines a large number of monomials that are also not contained in $U$. 
Example 3.1.5. The second point of view in Example 3.1.4 can also be visualized in the case \( n = 3 \). We consider the monomials arranged in the following way.

\[
\begin{align*}
x_1^d & \\
x_1^{d-1}x_2 & x_1^{d-1}x_3 \\
x_1^{d-2}x_2^2 & x_1^{d-2}x_2x_3 & x_1^{d-2}x_3^2 \\
x_1^{d-3}x_2^3 & x_1^{d-3}x_2^2x_3 & x_1^{d-3}x_2x_3^2 & x_1^{d-3}x_3^3 \\
& \vdots & \vdots & \vdots & \vdots 
\end{align*}
\]

In the \( i \)-th row, every monomial contains \( x_1 \) to the power \( d - i + 1 \) and in the \( j \)-th column, every monomial contains \( x_3 \) to the power \( j - 1 \).

Let \( U \subseteq A(3)_d \) be a strongly stable subspace and let \( M \) be any monomial not contained in \( U \). Let \((i, j)\) be the position of \( M \) in the diagram above, then every monomial which is to the top left of \( M \) is also not contained in \( U \), i.e. every monomial whose position \((k, l)\) satisfies \( k \leq i \) and \( l \leq j \).

Example 3.1.6. Let \( n = 3 \) and \( d = 4 \). We want to determine all strongly stable subspaces \( U \) of codimension 4 using the diagram above. In this case, we get

\[
\begin{array}{cccc}
x_1^4 & \\
x_1^3x_2 & x_1^3x_3 & \\
x_1^2x_2^2 & x_1^2x_2x_3 & x_1^2x_3^2 & \\
x_1x_2^3 & x_1x_2^2x_3 & x_1x_2x_3^2 & x_1x_3^3 & \\
x_2^4 & x_2^3x_3 & x_2^2x_3^2 & x_2x_3^3 & x_3^4 & \\
\end{array}
\]

We take blocks at the top-left to build \( U^\perp \) which has dimension 4. There are only two such blocks, both are marked in the diagram.

Equivalently, we can build \( U \) using a bottom-right block. Both possible subspaces are given by the complements of the two marked blocks. However, as we are mostly interested in small codimensions, it is preferable to consider \( U^\perp \) instead of \( U \).

Lemma 3.1.7. Let \( U \subseteq A_d \) be a strongly stable subspace of codimension \( 1 \leq k \leq d \). Then

(i) every monomial in \( U^\perp \) is divisible by \( x_1^s \) with \( s := d - k + 1 \), and

(ii) the subspace \( V := x_1U \oplus \mathbb{C}[x_2, \ldots, x_n]_{d+1} \) is also strongly stable.

Proof. (i) Assume there exists a monomial \( M \in U^\perp \) that is not divisible by \( x_1^s \). Since \( M \) has degree \( d \), there exists \( \alpha \in \mathbb{Z}^n_+ \), \(|\alpha| = d\) such that \( M = x_1^\alpha \) and by assumption \( \alpha_1 < s \). We now construct \( k + 1 \) monomials that are not contained in \( U \). Since \( \sum_{i=2}^n \alpha_i > d - s = k - 1 \), we find \( k - 1 \) monomials \( T_1, \ldots, T_{k-1} \in \mathbb{C}[x_2, \ldots, x_n] \) such that \( \deg T_i = i \) and \( T_i|M \) for \( i = 1, \ldots, k-1 \). Define \( T_0 := 1 \).
Then, for \( i = 0, \ldots, k - 1 \) the monomial \( M_i := x_1^{d-i} T_i \) is not contained in \( U \). Indeed, if there exists \( j \in \{ 0, \ldots, k - 1 \} \) such that \( M_j \in U \), then the monomial \( \frac{M}{x_1^{d-j-\alpha_1}} \) has degree \( d-j-\alpha_1 \) and is not divisible by \( x_1 \). Using the definition of a strongly stable subspace and not by \( x_1 \) times we see that

\[
M = \left( \frac{M}{x_1^{d-j-\alpha_1}} \right) \in U,
\]

which is a contradiction. Hence, the \( k+1 \) monomials \( M_0, \ldots, M_{k-1}, M \) are not contained in \( U \). However, these monomials are all different since by construction \( M_i \) is divisible by \( x_1^{d-i} \) and not by \( x_1^{d-i+1} \) for \( i = 0, \ldots, k - 1 \) and \( M \) is not divisible by \( x_1^s \) with \( s = d-k+1 \). This is impossible since \( U \) has codimension \( k \).

(ii) Let \( M \in V \) be a monomial. If \( x_1 \) does not divide \( M \), then \( M \in \mathbb{C}[x_2, \ldots, x_n]_{d+1} \) and for every \( i \in \{ 1, \ldots, n \} \) such that \( x_i | M \) and every \( i < j \leq n \) the monomial \( x_j T_i \) is again contained in \( \mathbb{C}[x_2, \ldots, x_n]_{d+1} \subseteq V \).

If \( x_1 \) divides \( M \), we can write \( M = x_1 M' \) for some monomial \( M' \in U \). Let \( i \in \{ 1, \ldots, n \} \) such that \( x_i | M \) and let \( i < j \leq n \). Either \( x_i | M' \), then \( x_j M' \in U \), hence

\[
x_j \frac{M}{x_i} = x_1 \left( x_j \frac{M'}{x_i} \right) \in V
\]
or \( i = 1 \) and \( x_1 \) does not divide \( M' \). In this case \( x_j \frac{M}{x_i} = x_j M' \in \mathbb{C}[x_2, \ldots, x_n]_{d+1} \subseteq V \). \( \square \)

**Lemma 3.1.8.** Let \( U \subseteq A_d \) be a strongly stable subspace of codimension \( k \leq n \), then every monomial in \( U^\perp \) is contained in \( A(k)_d = \mathbb{C}[x_1, \ldots, x_k]_d \).

**Proof.** The proof works just as the proof of Lemma 3.1.7(i). Assume there exists a monomial \( M \in U^\perp \) that is not contained in \( A(k)_d \). Then there exists \( j \in \{ k+1, \ldots, n \} \) such that \( M = x_j M' \) for some monomial \( M' \in A_{d-1} \). Since \( U \) is strongly stable, the monomials \( M_i := x_j M' \ (i = 1, \ldots, j-1) \) are also contained in \( U^\perp \). But this amounts to a total of \( (j-1) + 1 = j > k \) monomials that are contained in \( U^\perp \) which is impossible, since \( \text{codim} U = k \). \( \square \)

The main reason why strongly stable subspaces are particularly useful are the propositions.

**Proposition 3.1.9** ([Ei95, Theorem 15.18]). Let \( I \subseteq A \) be a homogeneous ideal. There exists a Zariski-open subset \( V \subseteq \text{GL}_n(\mathbb{C}) \) such that for every \( G_1, G_2 \in V \) the initial ideals satisfy \( \text{in}(G_1 I) = \text{in}(G_2 I) \) where \( G_i I := \{ p(G_i^{-1} x) : p \in I \} \) is the ideal \( I \) is mapped to by the coordinate change \( G_i \), for \( i = 1, 2 \).

**Definition 3.1.10.** Let \( I \subseteq A \) be a homogeneous ideal and \( G \in V \) as in Proposition 3.1.9 then

\[
\text{gin}(I) := \text{in}(G I)
\]

is called the *generic initial ideal* of \( I \).

Generic initial ideals have already been used by Hartshorne in 1966 ([Har66]) to show the connectedness of Hilbert schemes, and later on to get hold of invariants of projective varieties. The first systematic study of generic initial ideals in characteristic 0 was done by Galligo in [Gal74].
Proposition 3.1.11 ([Eis95, Theorem 15.20, 15.23]). Let $I \subseteq A$ be a homogeneous ideal, then for every $s \geq 0$ the vector space $\text{gin}(I)_s$ is strongly stable.

The main idea is the following easy observation.

Lemma 3.1.12. Let $I \subseteq A$ be a homogeneous ideal then $\text{in}(I^2) \subseteq \text{in}(I^2)$.

Proof. Let $\text{in}(I) = \langle m_1, \ldots, m_s \rangle$ for some monomials $m_1, \ldots, m_s \in A$. Then

$$\text{in}(I^2) = \langle m_im_j : 1 \leq i, j \leq s \rangle.$$ Let $i, j \in \{1, \ldots, s\}$ then there exist $p, q \in I$ such that $\text{in}(p) = m_i$ and $\text{in}(q) = m_j$. Therefore $m_im_j = \text{in}(pq) \in \text{in}(I^2)$. \hfill $\square$

Remark 3.1.13. In general we have $\text{in}(U^2) \nsubseteq \text{in}(U^2)$. Consider $U = \text{span}(x_1^4 + x_2^2) \subseteq A_2$ with $n \geq 3$. Then $U$ is spanned by all monomials except for $x_1^2$ and $x_2^2$ and by the binomial $x_1^2 - x_2^2$. The initial ideal $\text{in}(U)_2$ is spanned by all monomials except for $x_1^2$, i.e. $\text{in}(U)_2 = \text{span}(x_1^2)$. Now one easily checks that $\text{codim} \text{in}(U^2)_4 = \text{codim} U^2 = 2$ and $\text{codim} \text{in}(U^2)_2 = n$.

Proposition 3.1.14 ([BC18, Proposition 2.2]). For all positive integers $n \geq 2$, $d \geq 2$ and $1 \leq k \leq \text{dim} A_d$, there exists a strongly stable subspace $U \subseteq A_d$ of codimension $k$ such that

$$\text{m}(n, d, k) = \text{codim} U^2.$$ Proof. For every subspace $U \subseteq A_d$ we have $\text{in}(U^2) \subseteq \text{in}(U^2)$. Moreover, the Hilbert function of $\text{in}(U^2)$ is the same as the Hilbert function of $\langle U^2 \rangle$. Therefore $\text{dim}(\text{in}(U^2))_{2d} \leq \text{dim} U^2$ and hence the minimal value of $\text{dim} U^2$ is attained by a monomial subspace $U$. Applying a generic change of coordinates we can assume that $\text{in}(U)$ is the generic initial ideal of $U$. Hence, it is strongly stable by Proposition 3.1.11. \hfill $\square$

We now calculate the value $m(n, d, k)$ for some small examples.

Proposition 3.1.15. Let $n, d \geq 2$, and $k \in \mathbb{N}$, then the following hold.

(i) $\text{m}(n, d, 1) = n$,

(ii) $\text{m}(n, d, 2) = 2n$,

(iii) $\text{m}(2, d, k) = 2k$.

Proof. We know that $m(n, d, k)$ is realized by a strongly stable subspace $U$. For $k = 1$, the only strongly stable subspace is $U = \text{span}(x_1^d)$. Then $U^2$ is the space of all forms in $A_{2d}$ vanishing doubly at the point $(1 : 0 : \cdots : 0)$ which has dimension $n$.

For $k = 2$, there is also only one strongly stable subspace, namely $U = \text{span}(x_1^d, x_1^d - x_2)$. We check which monomials are not contained in $U^2$ and see

$$\langle U^2 \rangle = \text{span}(x_1^{2d-1}x_i, x_1^{2d-2}x_2x_j, x_1^{2d-3}x_2, i = 1, \ldots, n, j = 2, \ldots, n)$$

hence $\text{codim} U^2 = 2n$.

If $n = 2$, then for every $1 \leq k \leq d$ the only strongly stable subspace of codimension $k$ is given by

$$U = \text{span}(x_1^d, x_1^{d-1}x_2, \ldots, x_1^{d-k+1}x_2^{k-1})$$

Then $U^2$ is spanned by all monomials that are divisible by $x_2^{2k}$, hence $\text{codim} U^2 = 2k$. \hfill $\square$
Remark 3.1.16. In fact, any strongly stable subspaces $U \subseteq A_d$ of codimension $k \leq d$ is the space of all forms of degree $d$ vanishing at some $k$ points (counted with multiplicity). Or equivalently, the Hilbert function $t \mapsto \dim (A/(U))_t$ is equal to $k$ for any $t \geq d$. However, it is not clear in general which configuration of $k$ points realizes $m(n,d,k)$.

Remark 3.1.17. For small $n,d,k$ this is a list of $m(n,d,k)$ for $n = 2, 3, 4, 5, 6$. This has been calculated using SAGE [The19] by first finding all strongly stable subspaces of some fixed codimension and then finding the maximum of all $\text{codim } U$. 

<table>
<thead>
<tr>
<th>$d$</th>
<th>$n = 3$</th>
<th>$n = 4$</th>
<th>$n = 5$</th>
<th>$n = 6$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2 3 4 5 6 7 8 9</td>
<td>2 3 4 5 6 7 8 9</td>
<td>2 3 4 5 6 7 8 9</td>
<td>2 3 4 5 6 7 8 9</td>
</tr>
<tr>
<td>$\text{codim } U =$</td>
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<tr>
<td>1</td>
<td>3 3 3 3 3 3 3 3</td>
<td>4 4 4 4 4 4 4 4</td>
<td>3 3 3 3 3 3 3 3</td>
<td>4 4 4 4 4 4 4 4</td>
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<tr>
<td>2</td>
<td>6 6 6 6 6 6 6 6</td>
<td>8 8 8 8 8 8 8 8</td>
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<td>3</td>
<td>10 10 10 10 10 10 10 10</td>
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<td>13 13 13 13 13 13 13 13</td>
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<tr>
<td>4</td>
<td>12 13 13 13 13 13 13 13</td>
<td>20 20 20 20 20 20 20 20</td>
<td>12 13 13 13 13 13 13 13</td>
<td>20 20 20 20 20 20 20 20</td>
</tr>
<tr>
<td>8</td>
<td>25 27 27 28 28 29 27 27</td>
<td>32 39 40 41 41 43 40 40</td>
<td>25 27 27 28 28 29 27 27</td>
<td>32 39 40 41 41 43 40 40</td>
</tr>
<tr>
<td>9</td>
<td>27 30 31 31 32 33 31 31</td>
<td>34 45 45 45 47 47 49 45</td>
<td>27 30 31 31 32 33 31 31</td>
<td>34 45 45 45 47 47 49 45</td>
</tr>
</tbody>
</table>

As shown in Proposition 3.1.15 we have $m(n,d,1) = n$ and $m(n,d,2) = 2n$ in every case shown in the table.

Moreover, we see that in these examples $m(n,d,k) = m(n,k,k)$ for any $d \geq k$, i.e. values are constant on the right side of the diagonal. This is always true as we show later (see Corollary 3.5.5).
Remark 3.1.18. We now discuss the asymptotic behavior of $m(n, d, k)$. As we have mentioned, for fixed $n$ and $k$, the number $m(n, d, k)$ stabilizes for large $d$. To be more precise, we have $m(n, d, k) = m(n, k, k)$ for every $d \geq k$.

Determining exactly the growth of $m(n, d, k)$ for increasing $n$ or $k$ seems to be a rather difficult and probably mostly combinatorial problem. However, it is clear that increasing $n$ or $k$ while fixing the other value and the degree, results in larger values for $m(n, d, k)$. More precisely, we do get a lower bound for the growth using the Alexander-Hirschowitz Theorem.

Let $X \subseteq \mathbb{P}^{n-1}$ be a set of $k$ general points, in the sense of the Alexander-Hirschowitz Theorem. Assume that $d$ is large enough, then the subspace $U$ of forms in $A_d$ vanishing on $X$ has codimension $k$. By the Alexander-Hirschowitz Theorem, the space $V$ of all forms of degree $2d$ vanishing to order at least 2 at every point of $X$ has codimension $kn$. Since $U^2 \subseteq V$, it follows that $\text{codim } U^2 \geq kn$, especially $m(n, d, k) \geq kn$. And therefore also $m(n, d', k) \geq kn$ for every $d' \geq k$.

Remark 3.1.19. Let $U \subseteq A_d$ be a strongly stable subspace such that $U \neq A_d$. If $x^d_i \in U$, we see from the definition that $U = A_d$. Therefore, $x^d_i \in U^+$ which reveals a base-point of $U$. This shows that if $f \in U^2$, then $f$ is singular at the point $(1 : 0 : \cdots : 0)$.

Next, we generalize the statement that generic initial ideals realize $m(n, d, k)$ to quotients of the polynomial ring.

Proposition 3.1.20. Let $k \in \mathbb{N}$, $1 \leq k \leq \dim A_d$, $I \subseteq A$ a homogeneous ideal and $R = A/I$. There exists a strongly stable subspace $V \subseteq A_d$ of codimension $k$ such that $\text{gin}(I)_d \subseteq V$ and for all $U \subseteq R_d$ of codimension $k$ we have

$$\text{codim}_{R_d} U^2 \leq \text{codim}_{A_d} (V^2 + \text{gin}(I)_d).$$

Proof. Let $U \subseteq R_d$ be any subspace of codimension $k$. Let $U' \subseteq A_d$ be the biggest subspace such that $U' R/I = U$ where $\pi R/I$ denotes the residue modulo $I$. Now apply a generic change of coordinates $\phi$ to $A$. This puts $U'$ in generic coordinates, as well as $I$, and we have $V' := \phi U' = U'' \oplus (\phi I)_d$ for some subspace $U'' \subseteq A_d$ with $\dim U'' = \dim U$. Since $(\phi I)_d \subseteq V'$ and $\phi$ is generic we have $\text{gin}(I)_d \subseteq \text{in}(V') =: V$ and $V$ is strongly stable. Especially, we have

$$\dim U = \dim V^{A/\phi I} = \dim V^{A/\text{gin}(I)}.$$

Let $W := \text{gin}(I)_d$, then on the one hand

$$\dim (\text{in}(V')^2 + W)/W \leq \dim (\text{in}(V')^2 + W)/W \leq \dim (V^2 + W)/W$$

where the last inequality follows from Lemma [3.1.21]. And on the other hand

$$\dim (V^2 + W)/W = \dim ((U')^2 + I_{2d})/I_{2d} = \dim U^2,$$

hence

$$\dim (V^2 + W)/W = \dim (\text{in}(V')^2 + W)/W \leq \dim U^2.$$

Lemma 3.1.21. For any subspace $U \subseteq A_d$ we have $\dim (\text{in}(U) + W)/W \leq \dim (U + W)/W$ whenever $W$ is a monomial subspace of $A_d$. \qed
3.1 STRONGLY STABLE SUBSPACES

Proof. Let $p_1, \ldots, p_r \in A_d$ be a basis of $U$ such that $\text{in}(p_1), \ldots, \text{in}(p_r)$ form a basis of $\text{in}(U)_d$ and $p_i$ does not contain the monomials $\text{in}(p_j)$ for any $j \neq i$, i.e. if we write the basis into the rows of a matrix wrt the monomial basis in the correct order, this matrix is in reduced row echelon form, after scaling the $p_i$.

This matrix thus has the form $(I_r|*)$ and doing the same with $\text{in}(U)_d$ we get the matrix $(I_r|0)$ by choice of the basis. Since $W$ is a monomial vector space, the dimension of the two quotients is given by the ranks of the two matrices after removing the columns corresponding to the monomials in $W$. Since the rank of the first matrix is always greater or equal to the rank of the second matrix after removing columns in this fashion, the claim follows. □

Remark 3.1.22. Proposition 3.1.20 is a generalization of Proposition 3.1.14 since we may choose $I = \{0\}$.

However, contrary to Proposition 3.1.14 this bound may not be tight, i.e. there might not exist any subspace $U \subseteq R_d$ such that equality holds. This is because on the right side of the inequality we do not consider our subspace in $R = A/I$ but inside $A/\text{gin}(I)$.

The bound is however tight if the ideal $I$ satisfies $I = \text{gin}(I)$. In this case, we have

$$\text{codim}_{R_{2d}} U^2 \leq \text{codim}_{A_{2d}} (V^2 + \text{gin}(I)_{2d}) = \text{codim}_{R_{2d}} (V + \text{gin}(I))^2,$$

and the image $V + \text{gin}(I)$ of the strongly stable subspace $V$ in $R$ realizes the bound. This is for example the case in Proposition 3.1.14 where $I = \{0\}$.

Example 3.1.23. Let $f \in A_4$ be a form of degree $s$ and let $I = \langle f \rangle$ be the ideal generated by $f$. Then $\text{gin}(I) = \langle x_n^n \rangle$. We consider subspaces of $A/I$ of codimension 1. Let $k = 1$ and $d \geq 2$, then there is one strongly stable subspace $V$ of codimension 1 in $A_d$, namely the one apolar to $x_1^d$. Then

$$V^2 = \text{span}(x_1^{2d-1}x_i; 1 \leq i \leq n) .$$

If $s > 2d$ there is nothing to show since $A_{2d} = (A/I)_{2d}$. If $2d \geq s \geq 2$, then $\text{gin}(I)_{2d} = x_n^s A_{2d-s}$ and $\text{gin}(I)_{2d} \subseteq V^2$, hence $\text{codim}(V^2 + \text{gin}(I)_{2d}) = n$. If $s = 1$, then $x_1^{2d-1}x_n \in \text{gin}(I)_{2d} \setminus V^2$ and we get $\text{codim}(V^2 + \text{gin}(I)_{2d}) = n - 1$.

Example 3.1.24 (ternary quartics). Consider the case $n = 3, d = 2$ and let $f \in \text{int}(\Sigma_{3,4})$ be a non-singular quartic. The Gram spectrahedron of $f$ has dimension 6 which is also the rank of any interior point of Gram$(f)$. The boundary of Gram$(f)$ therefore consists of points of rank at most 5. We apply Proposition 3.1.14 and Remark 3.1.17 to determine the possible dimensions of faces.

We first look at faces of rank 5. A face $F$ of rank 5 corresponds to a subspace $U \subseteq A_2$ of dimension 5 or equivalently of codimension 1. Remark 3.1.17 therefore shows that $\text{codim} U^2$ is at most 3. Using Proposition 2.3.10 we can calculate the dimension of $F$ as

$$\dim F = \left(\frac{5 + 1}{2}\right)^2 - \dim A_4 + \text{codim} U^2 \leq 3.$$

Hence, the largest possible face of rank 5 has dimension 3. However, the subspaces we know of that realize this bound have a real base-point, and therefore the form whose Gram spectrahedron we are looking at has a real zero and does not lie in the interior of the sos cone. As we show in Theorem 3.1.2 there is no form in the interior of the sos cone, whose Gram spectrahedron has a face of dimension 3.
Next, we consider points of rank 4. Again a face \( F \) of rank 4 has a corresponding subspace \( U \) of dimension 4, or equivalently of codimension 2. By the table in Remark 3.1.17\(^\text{[3.1.17]} \) this means \( \text{codim} U^2 \leq 6 \), which translates to the bound \( \text{dim} F \leq 1 \).

Lastly, we look at points of rank 3. Let \( F \) be a face of rank 3 and \( U \) its corresponding subspace. By the table in Remark 3.1.17\(^\text{[3.1.17]} \) we get \( \text{codim} U^2 \leq 10 \) and

\[
\text{dim} F = \binom{4}{2} - \text{dim} A_4 + \text{codim} U^2 \leq 1.
\]

However, \( U \) has to be spanned by a regular sequence since \( f \) is non-singular. If \( U \) is spanned by \( p_1, p_2, p_3 \), then \( U^2 = p_1 U + p_2 U + p_3 U \) and since \( p_1, p_2, p_3 \) form a regular sequence, it follows that \( \text{dim} U^2 = 3 \cdot \text{dim} U - 3 = 6 \) (see Proposition 3.2.3\(^\text{[3.2.3]} \)). Especially, \( \text{codim} U^2 = 9 \) and \( \text{dim} F = 0 \). This shows that although the bounds we determined are tight, they might not be if we restrict ourselves to non-singular forms.

This is the idea in Section 3.2 to find bounds for non-singular forms.

For later reference, we now consider base-point-free monomial subspaces \( U \) and find bounds for \( \text{codim} U^2 \).

**Lemma 3.1.25.** Let \( d \geq 2 \) and let \( U \subseteq A_d \) be a base-point-free, monomial subspace of codimension 1. Then the following hold:

(i) If \( d = 2 \) then \( \text{codim} U^2 = 2 \),

(ii) if \( d \geq 3 \) then \( \text{codim} U^2 \in \{0,1\} \).

**Proof.** Let \( U^1 = \text{span}(M) \) for some monomial \( M \in A_d \). Up to permutation of the variables there are only two monomials in \( A_{2d} \) that have only one decomposition into a product of monomials of degree \( d \), those are \( x_1^{2d} \) and \( x_1^{2d-1} x_2 \).

Let \( T \in A_{2d} \) be any monomial that is not \( x_1^{2d} \) or \( x_1^{2d-1} x_2 \) (after permutation of the variables). Then there are two decompositions into a product of two monomials of degree \( d \). Especially, one of the decompositions does not use the monomial \( M \), hence \( T \in U^2 \).

The decompositions of the two monomials above are \( x_1^{2d} = (x_1^d)(x_1^d) \) and \( x_1^{2d-1} x_2 = x_1^d(x_1^{d-1} x_2) \). Therefore, both are not contained in \( U^2 \) if and only if \( M = x_1^d \), and only \( x_1^{2d-1} x_2 \) is not contained in \( U^2 \) if and only if \( M = x_1^{d-1} x_2 \). In the first case, \( U \) has a base-point, in the second case the only monomial not contained in \( U^2 \) is \( x_1^{2d-1} x_2 \) if \( d \geq 3 \) and thus \( \text{codim} U^2 = 1 \).

If \( d = 2 \) and \( U \) is base-point-free, \( U^1 \) is spanned by \( x_1 x_2 \) (after permutation of the variables). We easily check that \( U^2 \) contains every monomial of degree 4 except for \( x_1^3 x_2 \) and \( x_1 x_2^3 \). Hence \( \text{codim} U^2 = 2 \).

**Lemma 3.1.26.** Let \( U \subseteq A_d \) be a base-point-free, monomial subspace of codimension 2. Then the following hold:

(i) \( \text{codim} U^2 \leq 6 \) if \( d = 2 \),

(ii) \( \text{codim} U^2 \leq 4 \) if \( d \in \{3,4\} \),

(iii) \( \text{codim} U^2 \leq 2 \) if \( d \geq 5 \).

Moreover the bound is tight for \( d \leq 4 \).
3.2 A FIRST UPPER BOUND FOR BASE-POINT-FREE SUBSPACES

Proof. For \( n = 2 \) this follows from Proposition 3.1.14. Moreover for \( d = 3, 4 \), the subspaces \( U_i = \text{span}(x_1^d, x_2^d) \), \( U_2 = \text{span}(x_1^d, x_2^d, x_1^2x_2^2) \) satisfy \( \text{codim}(U_i)^2 = 4 \) \( (i = 1, 2) \).

From now on we may assume \( n \geq 3 \). For \( d = 2 \) let \( W = U \perp \). Then there are five possibilities up to permutation of the variables for \( W \), namely the ones in the following table:

<table>
<thead>
<tr>
<th>( W )</th>
<th>( (U^2) \perp )</th>
<th>( \text{codim } U )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_1^2, x_2^2 )</td>
<td>( x_1^2x_1, x_2^2x_1 ) ( \forall i )</td>
<td>( 2n )</td>
</tr>
<tr>
<td>( x_1^2, x_1x_2 )</td>
<td>( x_1^2x_1, x_1^2x_2 ) ( \forall i )</td>
<td>( n + 1 )</td>
</tr>
<tr>
<td>( x_1^3, x_2x_3 )</td>
<td>( x_1^3x_1, x_2^3x_2, x_3^3x_2 ) ( \forall i )</td>
<td>( n + 2 )</td>
</tr>
<tr>
<td>( x_1x_2, x_1x_3 )</td>
<td>( x_1^3x_1, x_2^3x_2, x_3^3x_2 ) ( x_1x_2^2x_3, x_1x_2x_2^2 )</td>
<td>( 6 )</td>
</tr>
<tr>
<td>( x_1x_2, x_3x_4 )</td>
<td>( x_1^3x_1, x_2^3x_2, x_3^2x_4, x_4^2x_3 )</td>
<td>( 4 )</td>
</tr>
</tbody>
</table>

In the first three cases, \( U \) has a base-point. The other two satisfy \( \text{codim } U^2 \leq 6 \).

Let \( d \geq 3 \). Up to permutation of the variables, there are five monomials of degree \( 2d \) that have two or fewer representations as a product of two monomials of degree \( d \), namely

\[
\begin{align*}
 x_1^{2d} &= x_1^d x_1^d \\
 x_1^{2d-1} x_2 &= x_1^d (x_1^{d-1} x_2) \\
 x_1^{2d-2} x_2 x_3 &= x_1^d (x_1^{d-2} x_2 x_3) = (x_1^{d-1} x_2)(x_1^{d-1} x_3) \\
 x_1^{2d-2} x_2 &= x_1^d (x_1^{d-2} x_2) = (x_1^{d-1} x_2)^2 \\
 x_1^{2d-3} x_2 x_3 &= x_1^d (x_1^{d-3} x_2 x_3) = (x_1^{d-1} x_2)(x_1^{d-2} x_2) 
\end{align*}
\]

Choosing a subspace \( U \) is equivalent to excluding two monomials from \( A_d \). Since only these five monomials can therefore miss from \( U^2 \) we immediately see that for \( d = 3, 4 \) we have \( \text{codim } U^2 \leq 4 \) as \( x_1^d \) has to be contained in \( U \).

If \( d > 5 \) we easily check that for any two monomials of degree \( d \) we exclude from \( A_d \) at most two of the five monomials above are not contained in \( U^2 \), since we cannot exclude \( x_1^d \) because \( U \) is base-point-free. Thus we get \( \text{codim } U^2 \leq 2 \). \( \square \)

3.2 A first upper bound for base-point-free subspaces

All upper bounds in the last section were realized by subspaces with a real base-point. If a subspace \( U \) has a base-point (real or complex), then any form \( f \in U^2 \) is singular at the base-point. To understand Gram spectrahedra of non-singular forms in the interior of the sos cone we now turn to base-point-free subspaces.

The idea in this section is that whenever \( U \) is base-point-free, it contains a regular sequence, which then gives a bound for \( \text{codim } U^2 \).

Lemma 3.2.1. Let \( X \subseteq \mathbb{P}^n \) be an irreducible, non-degenerate (i.e. not contained in any hyperplane) variety of dimension \( r \) such that \( \mathbb{C}[X] \) is Cohen-Macaulay and let \( U \subseteq \mathbb{C}[X]_d \) be a base-point-free subspace. Then there exist elements \( p_0, \ldots, p_r \in U \) that form a regular sequence in \( \mathbb{C}[X] \).
Proof. Let $s$ be the dimension of $U$ as a $\mathbb{C}$-vector space and consider the morphism given by $U$, that is $\phi : X \to \mathbb{P}^{s-1}$, $P \mapsto (f_1(P) : \ldots : f_s(P))$ where $f_1, \ldots, f_s$ form a basis of $U$. Let $Y$ be the image of $\phi$. Since $X$ is an irreducible closed projective variety, the same holds for $Y$. Furthermore $\mathbb{C}[Y] \cong \mathbb{C} \oplus \bigoplus_{i \geq 1} U^i$. Especially, the degree 1 component has dimension $s$, and hence $Y$ is non-degenerate.

Since $U$ is base-point-free, there exists $i \in \mathbb{N}$ such that $UC[X]_i = \mathbb{C}[X]_{d+i}$. Therefore, the ringhomomorphism $\mathbb{C}[Y] \to \mathbb{C}[X]$ is finite and $\dim Y = \dim X = r$.

Since $Y$ has dimension $r$, the intersection of $Y$ with $r+1$ general hyperplanes $H_0, \ldots, H_r \in \mathbb{C}[y_1, \ldots, y_s] = \mathbb{C}[\mathbb{P}^{s-1}]_1$ is empty. Hence so is the intersection of $Y$ with the elements $P_1, \ldots, P_r \in \mathbb{C}[Y]_1 \cong U$.

Since the map $X \to Y$ is surjective, we find elements $h_0, \ldots, h_r \in U$ with no common zero on $X$. Since $\mathbb{C}[X]$ is Cohen-Macaulay, these form a regular sequence in $\mathbb{C}[X]$. \hfill $\blacksquare$

Corollary 3.2.2. Let $U \subseteq A_d$ be a base-point-free subspace, then $U$ contains a regular sequence in $A$.

This results in the following upper bound for $\text{codim } U^2$ or equivalently a lower bound for $\dim U^2$.

Proposition 3.2.3. Let $U \subseteq A_d$ be a base-point-free subspace of dimension $r$. Then

$$\dim U^2 \geq nr - \binom{n}{2}.$$

Proof. Since $U$ is base-point-free it follows that $\dim U \geq n$ and $U$ contains a regular sequence $p_1, \ldots, p_n$ by Lemma 3.2.1. Consider the map

$$U^n \to A_{2d}, \quad (q_1, \ldots, q_n) \mapsto \sum_{i=1}^n p_i q_i.$$

Since $p_1, \ldots, p_n$ is a regular sequence, the syzygies are the obvious ones, namely the kernel is spanned by the vectors $(0, \ldots, 0, p_j, 0, \ldots, 0, -p_i, 0, \ldots, 0)$ with $i < j$ and $p_j$ at position $i$ in the vector and $-p_i$ at position $j$ by Proposition 1.3.1. There are exactly $\binom{n}{2}$ of those vectors, hence the image $\text{span}(p_1, \ldots, p_n)U$ has dimension $n \cdot \dim U - \binom{n}{2}$ and the image is contained in $U^2$. \hfill $\blacksquare$

Example 3.2.4. The easiest example where this bound is tight is $U = \text{span}(x_1^d, \ldots, x_n^d)$. Since $U$ is spanned by a regular sequence the condition $\text{span}(p_1, \ldots, p_n)U = U^2$ is certainly true.

In general, one should expect this bound to be good whenever the dimension of the subspace is close to $n$ and rather bad whenever $\text{codim } U$ is small.

However, the bound can also be tight or almost tight even for larger subspaces. To show this we first need a simple lemma.

Lemma 3.2.5. Let $U \subseteq A_d$ be a subspace with monomial basis $p_1, \ldots, p_n (n \leq r)$ where $p_1, \ldots, p_n$ form a regular sequence in $A$. Let $V := \text{span}(p_1^s, \ldots, p_n^s)$ for some integer $s \geq 1$, then $\dim V^2 = \dim U^2$ and $\dim \text{span}(p_1, \ldots, p_n)U = \dim \text{span}(p_1^s, \ldots, p_n^s)V$. 

3 BOUNDS FOR DIMENSIONS OF FACES
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Proof. Since $p_1, \ldots, p_r$ are monomials all linear relations between the products $p_ip_j$, $(1 \leq i \leq j \leq r)$ have the form $p_ip_j = p_kp_l$ for some $i, j, k, l \in \{1, \ldots, r\}$. Let $p_ip_j = p_kp_l$ for some $i, j, k, l \in \{1, \ldots, r\}$, then $\tilde{p}_i^k \tilde{p}_j^l = \tilde{p}_k^i \tilde{p}_l^j$ hence gives a linear relation between the products $\tilde{p}_i^k \tilde{p}_j^l$ $(1 \leq i \leq k \leq r)$.

On the other hand, since the $\tilde{p}_i^k$ are also monomials every relations $\tilde{p}_i^k \tilde{p}_j^l = \tilde{p}_k^i \tilde{p}_l^j$ also gives rise to a linear relation of the elements $p_ip_j$ $(1 \leq i \leq j \leq r)$. Therefore, the kernels of the maps $S_2U \rightarrow U^2$ and $S_2V \rightarrow V^2$ have the same dimension and the same holds for the kernels of the maps $U^n \rightarrow U^2$, $(q_1, \ldots, q_n) \mapsto \sum_{i=1}^n p_i q_i$ and $V^n \rightarrow V^2$, $(q_1, \ldots, q_n) \mapsto \sum_{i=1}^n p_i q_i$. Since $\dim U = \dim V$, the result follows.

Proposition 3.2.6. In any of the following cases, there exists a base-point-free subspace $U \subseteq A_d$ of dimension $r$ such that the bound in Proposition 3.2.3 is tight.

(i) $r = n$,

(ii) $d$ is even, $n \geq 3$ and $r = n + 3$,

(iii) there exists $s \in \mathbb{N}$ such that $s|d$, and $r = n + s - 1$.

In the next two cases, there exists a base-point-free subspace $U \subseteq A_d$ of dimension $r$ such that the bound is 1 off, i.e. $\dim U^2 = nr - \binom{n}{2} + 1$.

(iv) $d$ is even, $n \geq 4$ and $r = n + 6$,

(v) $3|d$, $n \geq 3$ and $r = n + 7$.

Proof. The case $r = n$ is Example 3.2.4. For the next case we look at the subspace

$$U = \text{span}(x_1^2, \ldots, x_n^2) \oplus \text{span}(x_1x_2, x_1x_3, x_2x_3) = \text{span}(x_1^2, \ldots, x_n^2) \oplus (A(3))_2.$$

Since $\text{span}(x_1^2, x_2^2, x_3^2)A(3)_2 = A(3)_4$, it follows that $\text{span}(x_1^2, \ldots, x_n^2)U = U^2$: let $M = ST \in U^2$ be any monomial with $S, T \in U$. If either $S$ or $T$ is contained in $\text{span}(x_1^2, \ldots, x_n^2)$, then $M = ST \in \text{span}(x_1^2, \ldots, x_n^2)U$. If both $S$ and $T$ are contained in $\text{span}(x_1x_2, x_1x_3, x_2x_3)$, then there exists $S' \in \text{span}(x_1^2, x_2^2, x_3^2)$ and $T' \in A(3)_2$ such that $M = S'T'$, and thus $M = ST' \in \text{span}(x_1^2, \ldots, x_n^2)U$.

Using Lemma 3.2.5, it follows that for every $l \geq 1$ the subspace

$$V = \text{span}(x_1^{2l}, \ldots, x_n^{2l}) \oplus \text{span}((x_1x_2)^l, (x_1x_3)^l, (x_2x_3)^l)$$

also satisfies $\text{span}(x_1^{2l}, \ldots, x_n^{2l})V = V^2$.

The third case works exactly the same. We consider

$$V = \text{span}(x_3^2, \ldots, x_n^2) \oplus A(2)_s$$

and use the fact that $\text{span}(x_1^2, x_3^2)A(2)_2 = A(2)_4$. Again this is a monomial subspace and with the same argument as above the result follows.

For the last two cases, we notice that $\text{span}(x_1^2, \ldots, x_n^2)A(4)_2$ has codimension 1 in $A(4)_4$ and the same holds for $\text{span}(x_1^2, x_2^2, x_3^2)A(3)_3$ in $A(3)_6$. Hence, building the subspaces in the same way as above yields the results.

Remark 3.2.7. One can also prove the above statements using the fact that the second Veronese of $\mathbb{P}^2$ and the $s$-th Veronese of $\mathbb{P}^1$ ($s \in \mathbb{N}$) are both varieties of minimal degree and the second (resp. third) Veronese of $\mathbb{P}^3$ (resp. $\mathbb{P}^2$) is an arithmetically Cohen-Macaulay variety of almost minimal degree.
Remark 3.2.8. The third case is especially interesting since it shows that if the degree is large enough, there exist subspaces of any dimension in any number of variables such that the bound is tight.

For the rest of this chapter, we want to look at the case where this bound is usually not useful, namely the case where the codimension of $U$ is small.

3.3 Some commutative algebra

For the following sections, we need some knowledge about the Hilbert functions of ideals generated by subspaces. We introduce theorems of Macaulay and Gotzmann concerning Hilbert functions and Green’s Hyperplane Restriction Theorem for later reference.

Definition 3.3.1. Let $a, d \in \mathbb{N}$, then $a$ can be uniquely written in the form

$$a = \binom{k(d)}{d} + \binom{k(d-1)}{d-1} + \cdots + \binom{k(1)}{1},$$

where $k(d) > k(d-1) > \cdots > k(1) \geq 0$, called the $d$-th Macaulay representation of $a$ (see [BH98, Lemma 4.2.6]). For any integers $s, t \in \mathbb{Z}$ define

$$a_{(d)}^s := \binom{k(d) + s}{d + t} + \binom{k(d-1) + s}{d - 1 + t} + \cdots + \binom{k(1) + s}{1 + t}.$$  

Furthermore for $a < b$ we define $\binom{a}{b} = 0$.

Theorem 3.3.2 (Macaulay’s Theorem, [IK99b, Corollary C.7], [BH98, Theorem 4.2.10]). Let $I \subseteq A$ be a homogeneous ideal and let $H = (h_i)_{i \geq 0}$ be the Hilbert function of $I$. Then

(i) $h_{i+1} \leq (h_i)_{(d)}^1$ for every $i \geq 0$, and

(ii) if there exists $j \in \mathbb{N}$ such that $j \geq h_j$, then $h_i \geq h_{i+1}$ for every $i \geq j$.

In fact, Macaulay showed in 1927 [Mac27] that whenever we have a sequence $H = (h_i)_{i \geq 0}$ which satisfies property (i) in Theorem 3.3.2 there exists $n \geq 2$ and a homogeneous ideal $I \subseteq A$ such that the Hilbert function of $I$ is exactly $H$. This ideal $I$ can even be chosen to be monomial.

Theorem 3.3.3 (Gotzmann’s Persistence Theorem, [IK99b, Corollary C.17], [AMS18, Theorem 2.6]). Let $d \geq 0$ be an integer and let $I$ be a homogeneous ideal that is generated in degrees at most $d$ ($I = \langle I_{\leq d} \rangle$). Denote by $H = (h_i)_{i \geq 0}$ the Hilbert function of $I$. If $h_{d+1} = (h_{d})_{(d)}^1$, then $h_{d+i+1} = (h_d)_{(d)}^i$ for all $i \geq 1$.

By Macaulay’s Theorem $h_{d+1} \leq (h_d)_{(d)}^1$. Therefore, Gotzmann’s Theorem determines the complete Hilbert function whenever we have maximal growth from some degree $d$ to the next degree $d + 1$. Namely, the growth is maximal for all following degrees as well. For ideals generated by subspaces, this has the following meaning.

Corollary 3.3.4. Let $U \subseteq A_d$ be a subspace of codimension $k \leq d$ and let $H = (h_i)$, $h_i := h_{(U)}(i)$ be the Hilbert function of $U$. Then

(i) $h_{d+1} = \text{codim } A_1 U \leq k$ and
(ii) if \( h_{d+1} = k \), then \( h_{d+1} = \text{codim} A_i U = k \) for all \( i \geq 1 \).

In case (ii) \( \mathcal{V}(U) \neq \emptyset \) is finite.

**Proof.** (i): We have codim \( U = h_d = k \leq d \) and therefore codim \( A_i U = h_{d+1} \leq h_d = k \) by Theorem \ref{thm:3.3.2} (ii).

(ii): We first note that the \( d \)-th Macaulay representation of \( h_d \) is given by

\[
h_d = \binom{d}{d} + \cdots + \binom{d-k+1}{d-k+1} = \sum_{i=0}^{k-1} \binom{d-i}{d-i}
\]

and therefore \( (h_d)(d) \leq h_d = k \). Hence, the assumption \( h_{d+1} = k = (h_d)(d) \) allows us to use Theorem \ref{thm:3.3.3} from which we get

\[
h_{d+1} = (h_d)(d) = \left( \frac{d+i}{d+i} \right) + \cdots + \left( \frac{d-h_d+1+i}{d-h_d+1+i} \right) = k
\]

for every \( i \geq 1 \).

In (ii) the Hilbert polynomial is the constant polynomial \( k \), hence \( \mathcal{V}(U) \) is non-empty and finite. \( \square \)

**Corollary 3.3.5.** Let \( U \subseteq A_d \) be a base-point-free subspace with codim \( U = k \leq d \). Then \( h_{(U)}(2d-1) \leq 1 \). If \( k < d \) then \( h_{(U)}(2d-1) = 0 \).

**Proof.** The Hilbert function of \( \langle U \rangle \) has to be smaller than \( (\ldots, k, k-1, k-2, \ldots, 1, 0) \) (dimension dropping by at least 1 in every degree): indeed, if we had equality in any two consecutive degrees \( s \) and \( s+1 \) with \( s \geq d \) such that \( h_s \neq 0 \), it follows from Corollary \ref{cor:3.3.4} (ii) that \( \mathcal{V}(U) \neq \emptyset \). Therefore, we get the inequality on the degree \( 2d-1 \) component of \( A/(U) \). \( \square \)

For the degree \( 2d \) component, there is a stronger result due to Blekherman using Cayley-Bacharach duality.

**Theorem 3.3.6 ([Ble15, Theorem 2.5.]).** Let \( n \geq 3 \), \( d \geq 3 \) and let \( U \subseteq A_d \) be a base-point-free subspace. If \( \text{codim} U < 3d-2 \), then \( U A_d = A_{2d-2} \). If \( n \geq 4 \), \( d = 2 \) and \( \text{codim} U < 5 \), then \( U A_2 = A_4 \).

**Remark 3.3.7.** If we would use the same argument as in Corollary \ref{cor:3.3.5}, we only get \( h_{(U)}(2d) = 0 \) if \( \text{codim} U \leq d \), instead of whenever \( \text{codim} U < 3d-2 \) (\( < 5 \) if \( d = 2 \)).

This also shows that the bound \( \text{codim} U \leq d \) in Corollary \ref{cor:3.3.5} is far off from being necessary to obtain \( h_{(U)}(2d-1) = 0 \) in general.

However, Theorem \ref{thm:3.3.6} only tells us something about \( U A_d \) and not about \( U A_{d-1} \) and the proof does not easily generalize to other degrees but is very specific to the degree \( 2d \) component \( U A_d \).

**Definition 3.3.8.** Let \( I \subseteq A \) be a homogeneous ideal and \( p \in A_s \) for some \( s \geq 1 \). We define the ideal quotient

\[
(I : p) := \bigoplus_{l \geq 0} (I : p)_l
\]

where

\[
(I : p)_l := \{ q \in A_l : pq \in I \} \subseteq A_l
\]

for every \( l \geq 0 \). If \( U \subseteq A_d \) is a subspace, we write \( (U : p) := (\langle U \rangle : p)_{d-s} \subseteq A_{d-s} \).
3.3.9. We consider the following setup. Let $I \subseteq A$ be a homogeneous ideal and $l \in A_1$ a linear form. We have the graded exact sequence

$$0 \to A/(I : l)(-1) \to A/I \to A/(I, l) \to 0.$$ 

Let $h_i = \dim(A/I)_i$ and $c_i = \dim(A/(I, l))_i$.

In this situation, we have the following theorem due to Green.

**Theorem 3.3.10** (Green’s Hyperplane Restriction Theorem, [Gre89, Theorem 1]). For any $d \geq 0$ and a generic linear form $l \in A_1$ we have

$$c_d \leq \frac{(h_d)_d}{d}.$$ 

This can either be seen as a lower bound for $\dim(I, l)_d$ or equivalently as an upper bound for $\dim(I : l)_{d-1}$ which tells us how many elements in $I$ are divisible by $l$.

Notation-wise this means that if $h_d = \binom{k(d)}{d} + \binom{k(d-1)}{d-1} + \cdots + \binom{k(1)}{1}$, then

$$c_d \leq \binom{k(d)}{d} + \binom{k(d-1)}{d-1} + \cdots + \binom{k(1)}{1}.$$ 

**Example 3.3.11.** Let $U \subseteq A_d$ be a subspace of codimension 1 and let $l \in A_1$ be a generic linear form. This means $h_d = A_d/U = 1 = \binom{d}{d}$. Therefore, Theorem 3.3.10 shows

$$c_d \leq \binom{d-1}{d} = 0.$$ 

On the one hand, this means $(U, l)_d = A_d$, and on the other hand

$$\text{codim}(U : l)_{d-1} = \text{codim} U - \dim(I, l)_d = 1.$$ 

I.e. the subspace $(U : l)_{d-1}$ also has codimension 1.

### 3.4 Subspaces of codimension 1 and 2

We start by determining bounds for $\text{codim} U^2$ in the cases $\text{codim} U = 1, 2$. We show that there is a uniform bound for $\text{codim} U^2$ not depending on $n$ or $d$. This is also our main motivation for the next sections where we generalize this result to higher codimensions. Furthermore, we show Theorem 3.4.3 which is our main tool in the next sections to reduce the number of variables.

**Lemma 3.4.1.** Let $U \subseteq A_d$ be a base-point-free subspace and $W := U^\perp$. If $\mathcal{V}(W)$ is not contained in any linear variety of codimension 2, then there exists a change of coordinates such that $\text{in}(U)_d$ is base-point-free.

In the case $n = 2$, this should be understood as $\mathcal{V}(W) \neq \emptyset$, i.e. $\dim \mathcal{V}(W) \in \{0, 1\}$.

**Proof.** By assumption there exist linearly independent linear forms $l_1, \ldots, l_{n-1} \in A_1$ such that $l_1^d, \ldots, l_{n-1}^d \in U$. After a change of coordinates, we can assume that $x_1^d, \ldots, x_n^d \in U$. Since $x_1 < x_2 < \cdots < x_n$, it holds that $x_i^d \geq x^a$ for any $a = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n = \{(a_1, \ldots, a_n) \in \mathbb{Z}^n : a_i \geq 0, \forall i = 1, \ldots, n\}$, $|a| := \sum_{i=1}^n \alpha_i = d$. Since $U$ is base-point-free, there exists a form in $U$ such that $x_i^d$ occurs in it. Hence, $\text{in}(U)_d$ contains $x_1^d, \ldots, x_n^d$ which shows that the subspace $\text{in}(U)_d$ is base-point-free. 

\qed
With this lemma in place, we look at subspaces of codimension 1. Firstly, we consider the simple case where our subspace does have a base-point.

**Lemma 3.4.2.** If \( U \subseteq A_d \) is a subspace of codimension 1 and \( U \) has a base-point, then \( \text{codim} U^2 = n \).

**Proof.** We can apply a change of coordinates such that \( U^⊥ = \text{span}(x_i^d) \). Then \( U \) is the subspace spanned by all monomials except \( x_i^d \). Now we see that for every \( 1 \leq i \leq n \) the monomial \( x_i^{d-1}x_i \) is not contained in \( U^2 \) and thus \( \text{codim} U^2 = n \). \( \square \)

**Proposition 3.4.3.** Let \( d \geq 2 \) and \( U \subseteq A_d \) be a base-point-free subspace of codimension 1. Then the following hold:

(i) If \( d \geq 3 \), then \( \text{codim} U^2 \leq 1 \),

(ii) If \( d = 2 \), then \( \text{codim} U^2 \leq 2 \).

**Proof.** Write \( W := U^⊥ = \text{span}(q) \) for some \( q \in A_d \). No hypersurface is contained in a linear variety of codimension 2, hence by Lemma 3.4.1 we can apply a change of coordinates and assume that the subspace \( \text{in}(U)_d \) is base-point-free. Then \( \dim U^2 = \dim \text{in}(U^2)_{2d} \geq \dim (\text{in}(U^2)_{2d}) \) by Lemma 3.1.12. From Lemma 3.1.25 which is the monomial case, we get

\[
\text{codim} (\text{in}(U^2)_{2d}) \leq \begin{cases} 
1 & \text{if } d \geq 3 \\
2 & \text{if } d = 2 
\end{cases}.
\]

\( \square \)

**Remark 3.4.4.** (i) We use a different method in Section 4.1 to show that in the case \( d = 2 \) we even have \( \text{codim} U^2 \in (0, 2) \), i.e. the case \( \text{codim} U^2 = 1 \) does not occur.

(ii) For Gram spectrahedra this means the following. If \( d \geq 3 \), then for any \( f \in \text{int}(\Sigma_{n,2d}) \) and any face \( F \subseteq \text{Gram}(f) \) of rank \( \dim A_d - 1 \), we have \( \dim F = \binom{n+1}{d} - \dim A_{2d} + \varepsilon \) with \( \varepsilon \in \{0, 1\} \). This stays true as long as \( f \notin \partial P_{n,2d} \) (see Corollary 4.1.10). We note that in this case, we do not have to assume \( f \) to be non-singular as the subspace corresponding to \( F \) has codimension 1 and is real, hence cannot have a pair of complex conjugate base-points.

Now we turn to the codimension 2 case. We find a bound for \( \text{codim} U^2 \) by reducing either to monomial subspaces or subspaces of binary forms.

First, we show how to reduce the number of variables. The idea of the proof is the following: if \( U \subseteq A[x_{n+1}]_d \subseteq \mathbb{C}[x_1, \ldots, x_{n+1}]_d \) is a subspace of the form \( U = x_{n+1}A[x_{n+1}]_{d-1} \oplus U' \) with \( U' \subseteq A_d \), then \( U^2 = x_{n+1}^2A[x_{n+1}]_{2d-2} \oplus x_{n+1}A_{d-1}U' \oplus (U')^2 \). This shows

\[
\text{codim} U^2 = \text{codim}(U')^2 + \text{codim} A_{d-1}U'.
\]

If \( U \) does not have this nice form, we have to argue slightly more carefully using the same idea.

**Theorem 3.4.5.** Let \( U \subseteq A_d \) be a subspace of codimension \( k \). If there exists \( 2 \leq m \leq n \) (\( R := A(m) \)) such that \( U' := U \cap R_d \) satisfies \( \text{codim}_{A_d} U' = k \), then

\[
\text{codim}_{A_d} U^2 \leq (n - m) \text{codim}_{R_{2d}} U' R_{2d-1} + \text{codim}_{R_{2d}} (U')^2.
\]
Proof. Let $m = \langle x_{m+1}, \ldots, x_n \rangle \subseteq A_d$, then $m_d = \sum_{i=m+1}^n x_i A_{d-1}$. We write

$$U = U' \oplus V \oplus W$$

with $U' \subseteq R_d$, $V \subseteq m_d$ and $W = \text{span}(p_i + q_i : i = 1, \ldots, s)$ where $p_i \in R_d$ and $0 \neq q_i \in m_d$ for $i = 1, \ldots, s$. By assumption $\text{codim}_{R_d} U' = k$ which means $U + R_d = A_d$ and thus

$$V \oplus \text{span}(q_1, \ldots, q_s) = m_d. \quad (3.1)$$

Calculating $U^2$ we get

$$U^2 = (U')^2 + (V + W)^2 + U'(V + W).$$

Since we are working with the lex-ordering (and $x_1 < \cdots < x_n$), any monomial of degree $d$ containing any $x_i$, $i \geq m + 1$ is bigger than any monomial in $R_d$.

Firstly, fix any monomial $x^\alpha$ such that $\alpha \in \mathbb{Z}_+^n, |\alpha| = 2d$ and $\sum_{j \geq m+1} \alpha_j \geq 2$, then there exist $\beta, \gamma \in \mathbb{Z}_+^n, |\beta| = |\gamma| = d$ and $x_i, x_j, i, j \geq m + 1$ such that $x_i |^{2d}, x_j |^{2d}$ and $x^\alpha = x^\beta x^\gamma$. Then we have $x^\beta + p\beta, x^\gamma + p\gamma \in V + W$ for some $p\beta, p\gamma \in R_d$. Hence

$$x^\alpha = \text{in}((x^\beta + p\beta)(x^\gamma + p\gamma)) \in \text{in}((V + W)^2)_{2d} \subseteq \text{in}(U^2)_{2d}.$$

Secondly, we have

$$\text{in}(U'(V + W)) \supseteq \text{in}(U'm_d) \quad \text{(by equation (3.1))}$$

$$= \text{in} \left( \bigoplus_{i=m+1}^n x_i (U'A_{d-1}) \right)$$

$$\supseteq \text{in} \left( \bigoplus_{i=m+1}^n x_i (U'R_{d-1}) \right).$$

This shows that for every $i = m + 1, \ldots, n$ we have

$$m^2 A_{2d-2}, \text{in}((U')^2)_{2d}, \text{in}(x_i U'R_{d-1})_{2d} \subseteq \text{in}(U^2)_{2d}.$$ 

Counting dimensions, we get

$$\text{codim}_{A_{2d}} \text{in}(U^2)_{2d} \leq (n - m) \text{codim}_{R_{2d-1}} U'R_{d-1} + \text{codim}_{R_{2d}} (U')^2.$$ \hfill \Box

Remark 3.4.6. The bound is sharp whenever $U = \langle x_{m+1}, \ldots, x_n \rangle A_{d-1} \oplus U'$ as can be seen from the comment above Theorem 3.4.5.

Corollary 3.4.7. If the subspaces $U, U'$ in Theorem 3.4.5 are base-point-free and $k \leq d - 1$, then

$$\text{codim}^2 \leq \text{codim}(U')^2.$$ 

Proof. By Corollary 3.3.5 the degree $2d - 1$ component of $R/\langle U' \rangle$ has dimension 0. Therefore, the result follows from Theorem 3.4.5. \hfill \Box
Theorem 3.4.8. Let $U \subseteq A_d$ be a base-point-free subspace of codimension 2. Then the following hold:

(i) If $d = 2$, then $\text{codim} U^2 \leq 6$,

(ii) if $d \geq 3$, then $\text{codim} U^2 \leq 4$.

For $d \leq 4$ the bounds are tight.

Proof. Let $W = U^\perp$. If $\mathcal{V}(W) \neq \mathcal{V}(l, l')$ for any two linear forms $l, l' \in A_1$, the claim follows from Lemma 3.4.1 and Lemma 3.1.26 with the same arguments as in the codimension 1 case (Proposition 3.4.3) as we can reduce to base-point-free monomial subspaces.

Otherwise we can assume after a change of coordinates that $\mathcal{V}(W) = \mathcal{V}(x_1, x_2)$ and thus $x_3^d, \ldots, x_n^d \in U$, $x_1^d, x_2^d \not\in U$. Hence, we can write

$$U = \text{span}(x^\alpha + \nu_\alpha x_1^d + \lambda_\alpha x_2^d; \alpha \in \mathbb{Z}_+^n, |\alpha| = d, \exists i \geq 3: x_i|\alpha^\lambda \oplus U'$$

where $U' \subseteq \mathbb{C}[x_1, x_2]_d$ is a subspace of codimension 2. We distinguish two cases. Either (a) for all $\alpha$ we have $\nu_\alpha = \lambda_\alpha = 0$, or (b) there exists $\alpha$ such that $(\nu_\alpha, \lambda_\alpha) \neq (0, 0)$.

(a): Here $U$ has the form

$$U = \text{span}(x_3, \ldots, x_n)A_{d-1} \oplus U'.$$

If $d = 2$, this case cannot occur since $\dim U' = 1$ and thus $U$ has a base-point. Hence, we can assume that $d \geq 3$. Since $U$ is base-point-free it follows that $U'$ is base-point-free as a subspace of $\mathbb{C}[x_1, x_2]_d$. Then $\text{codim} U^2 \leq \text{codim}(U')^2 \leq 4$ by Corollary 3.4.7 and Proposition 3.1.15.

(b): Fix $\alpha \in \mathbb{Z}_+^n, |\alpha| = d$ such that $(\nu_\alpha, \lambda_\alpha) \neq (0, 0)$. Consider the subspace

$$V := U' \oplus \text{span}(\nu_\alpha x_1^d + \lambda_\alpha x_2^d) \subseteq \mathbb{C}[x_1, x_2]_d.$$

This subspace has codimension 1 and thus $V^\perp = \text{span}(h)$ for some $h \in \mathbb{C}[x_1, x_2]_d$. Especially, there exists $l \in \mathbb{C}[x_1, x_2]_l$ such that $l^d \in V$, namely the one evaluating $h$ in one of its zeroes. Therefore, there exist $a \in \mathbb{C}$ and $b \in \mathbb{Z}_+^n, |b| = d$, $x^b \not\in \mathbb{C}[x_1, x_2]$ such that $ax^b + l^d \in U$. Write $x^b = x_1^\beta x_2^\gamma M$ with $M \in \mathbb{C}[x_1, \ldots, x_n]$. Let $\phi$ be a change of coordinates on $\mathbb{C}[x_1, x_2]$ that maps $l$ to $x_2$. Then $aMg + x_2^d \in \phi(U)$ with $g \in \mathbb{C}[x_1, x_2]$ the image of $x_1^\alpha x_2^\beta$ under $\phi$. Now take any monomial ordering such that $x_1 > x_2 > \cdots > x_n$ and such that $x_2^d$ is greater than any monomial in $Mg$, for example, the ordering given in Remark 3.4.9. Write this ordering in $\phi(U)_d$ contains $x_1^d, \ldots, x_n^d$ and is therefore base-point-free: the monomials $x_3^d, \ldots, x_n^d$ are contained in $U$ and therefore in $\phi(U)$ by assumption. The monomial $x_1^d$ is the initial monomial of $aMg + x_2^d$ and $x_1^d$ appears in some form in $\phi(U)$ since it is base-point-free and by the choice of the monomial ordering $x_2^d$ is the initial monomial of that form. Now we finish as earlier, $\text{codim} U^2 = \text{codim} \phi(U)^2 \leq \text{codim}(\text{in}(\phi(U))_d)^2$ and using Lemma 3.1.26 we get the bounds we wanted.

The bounds are tight for $d \leq 4$ by Lemma 3.1.26.

Remark 3.4.9. We want to define a monomial ordering such that $x_1 > x_2 > \cdots > x_n$ and such that $x_2^d$ is greater than any monomial of degree $d$ that is divisible by $x_i$ for any $i \in \{3, \ldots, n\}$. □
We consider a block ordering $\succeq_b$ on the sets $\{x_1, x_2\}$ and $\{x_3, \ldots, x_n\}$ and on each set the graded-lexicographic-ordering (grlex). Let $\alpha, \beta \in \mathbb{Z}_n^+$, then the grlex ordering is defined as

$$x^\alpha \succeq_{\text{grlex}} x^\beta : |\alpha| > |\beta| \text{ or } |\alpha| = |\beta| \text{ and } x^\alpha \succeq_{\text{lex}} x^\beta$$

where $\succeq_{\text{lex}}$ is the usual lex-ordering and the variables are ordered as $x_1 > x_2 > \cdots > x_n$.

Then the block-ordering is defined as follows. Let $\alpha, \beta \in \mathbb{Z}_n^+$, then

$$x^\alpha \succeq_b x^\beta : x_1^{\alpha_1} x_2^{\alpha_2} \succeq_{\text{grlex}} x_1^{\beta_1} x_2^{\beta_2} \text{ or } x_1^{\alpha_1} x_2^{\alpha_2} = x_1^{\beta_1} x_2^{\beta_2} \text{ and } \frac{x^\alpha}{x_1^{\alpha_1} x_2^{\alpha_2}} \succeq_{\text{grlex}} \frac{x^\beta}{x_1^{\beta_1} x_2^{\beta_2}}.$$ 

Now let $x^\alpha \in A_d$ be any monomial such that $\alpha_i > 0$ for some $i \in \{3, \ldots, n\}$. Then $\alpha_1 + \alpha_2 < d$ and therefore $x^\alpha \not\succeq_b x^\beta$.

**Remark 3.4.10.** The idea to choose a monomial ordering such that the degree $d$ component of the initial ideal is base-point-free is unlikely to work as easily for higher codimensions.

The lex-plus-powers conjecture (or EGH conjecture) due to Eisenbud, Green, and Harris [EGH93] predicts that for any homogeneous ideal $I \subseteq A$ containing a regular sequence $p_1, \ldots, p_n$ with $d_i := \deg p_i$, there is also a monomial ideal containing $x_i^{a_i}$ for $i = 1, \ldots, n$ with the same Hilbert function as $I$. The conjecture has only been proven in some special cases, see for example [CM08].

This is certainly not exactly what we are after. On the one hand, we are only interested in the case where all $p_i$ are of the same degree and the ideal is generated in that degree. On the other hand, it is not enough that the monomial ideal has the same Hilbert function, since the Hilbert function of $I$ does not determine the Hilbert function of $I^2$.

This however shows that we should not expect a reduction to monomial ideals to easily work in more general cases.

**Remark 3.4.11.** Since what we showed is an upper bound for the codimension, we would also like to see which values are attainable. It turns out that for $d = 4$ and $n \geq 3$ every possible value $0, \ldots, 4$ for $	ext{codim } U^2$ is attained by some base-point-free subspace $U$.

A generic subspace $U \subseteq A(3)_4$ of codimension 2 satisfies $	ext{codim } U^2 = 0$. The orthogonal complements of $\text{span}(x_1^1 x_2, x_1^2 x_2^2)$ and $\text{span}(x_1^1 x_2, x_1^2 x_3)$ satisfy $	ext{codim } U^2$ equal to 1 and 2, and the values 3 and 4 are attained by subspaces in two variables. These are $\text{span}(x_1^1, x_2^3, x_2 x_3)$ and $\text{span}(x_1^1, x_2^3, x_2 x_3)$.

However, for $d \geq 5$ not all values can be attained. One can show $	ext{codim } U^2 < 4$, but we also do not know an example such that $	ext{codim } U^2 = 3$. The values 0, 1, 2 are attained by a generic subspace $U \subseteq A(3)_5$, and the subspaces $\text{span}(x_1^1 x_2, x_1^2 x_3) \subseteq A(3)_5$, and $\text{span}(x_1^1 x_2, x_1^2 x_3) \subseteq A(3)_5$.

We have seen that in the codimension 1 case the codimension of $U^2$ can be bounded by 1 if $d \geq 3$ (resp. 2 if $d = 2$) and in the case codim $U = 2$, the codimension can be bounded by 4 if $d \geq 3$ (resp. 6 if $d = 2$). We would like to generalize this to higher codimensions. It seems that the correct way to do this is to show that there is a bound for codim $U^2$ that is not dependent on $n$ or $d$, as long as $d$ is large enough.

We have already seen how we can try to reduce the number of variables and therefore make our bounds independent of $n$ using Theorem 3.4.52.

In the next section, we show how to find bounds that are independent of the degree $d$. 


3.5 Reducing the degree

In this section, we show that for \( d \geq k \) the function \( d \mapsto m(n, d, k) \) is constant for every fixed \( n, k \). By definition, this is equivalent to showing that for certain subspaces \( U \subseteq A_d \) of codimension \( k \) there exists a subspace \( V \subseteq A_{d-1} \) of codimension \( k \) such that \( \text{codim} U^2 = \text{codim} V^2 \) whenever \( d > k \).

For the next proofs, let us recall that for any subspace \( U \subseteq A_d \) we have the exact sequence in \ref{3.3.9}:

\[
0 \to A_{d-1}/(U : l) \to A_d/U \to A_d/(U, l)_d \to 0.
\]

Especially, if \( (U, l)_d = A_d \) it follows that \( \text{codim}(U : l) = \text{codim} U \).

**Lemma 3.5.1.** Let \( U \subseteq A_d \) be a subspace of codimension \( k \) and \( k \leq d \). Then for a generic linear form \( l \in A_1 \) we have \( (U, l)_d = A_d \) and \( \text{codim}(U : l) = \text{codim} U \).

**Proof.** With the notation from \ref{3.3.9} and with \( I = \langle U \rangle \) we have

\[
h_I(d) = h_d = k = \sum_{i=0}^{k-1} \frac{(d-i)}{d-i}
\]

since \( k \leq d \). Hence, by Green’s Theorem

\[
\dim A_d/(U, l)_d = c_d \leq (h_d)_{(d)_0}^{-1} = \sum_{i=0}^{k-1} \frac{(d-i-1)}{d-i} = 0
\]

which means \( A_d/(U, l)_d = 0 \), and therefore the first claim follows. The second one is immediate from the exact sequence above.

**Theorem 3.5.2.** Let \( U \subseteq A_d \) be a subspace of codimension \( k \leq d \) and let \( l \in A_1 \) be a generic linear form. With \( V := (U, l)_d = A_{d-1} \) the following inequality holds

\[
\text{codim} U^2 \leq \text{codim} UV.
\]

If furthermore \( k \leq d - 1 \), then

\[
\text{codim} U^2 \leq \text{codim} V^2.
\]

**Proof.** Since \( l \) is generic and \( k \leq d \) it follows from Lemma \ref{3.5.1} that \( \text{codim} V = \text{codim} U \) and \( (U, l)_d = A_d \). Furthermore, we have

\[
A_{2d} = (U, l)_d^2 \subseteq (U^2, l)_2d,
\]

hence \( \text{codim}(U^2 : l) = \text{codim} U^2 \) by the exact sequence in \ref{3.3.9}. Since \( UV \subseteq (U^2 : l) \) we have

\[
\text{codim} U^2 = \text{codim} (U^2 : l) \leq \text{codim} UV.
\]

Now we do the same for \( UV \) if \( k \leq d - 1 \). If we show that \( (V, l)_{d-1} = A_{d-1} \), then

\[
A_{2d-1} = (U, l)_{d-1}(V, l)_{d-1} \subseteq (UV, l)_{2d-1}.
\]

Thus \( \text{codim}(UV : l) = \text{codim} UV \) and \( V^2 \subseteq (UV : l) \) which means \( \text{codim} UV \leq \text{codim} V^2 \).
It is left to show that \( (V, l)_{d-1} = A_{d-1} \). This is equivalent to showing that \( \text{codim}(V : l) = \text{codim} V \). Since \( ((U : l) : l) = (U : l') \) this again is equivalent to showing that \( \text{codim}(U : l') = \text{codim} V = \text{codim} U \) or \( (U, l')_d = A_d \). Since \( l \in A_1 \) is generic, we can also apply a generic change of coordinates to \( U \), hence assume that \( \text{in}(U) = \text{gin}(U) \) and \( l = x_1 \). Then

\[
\dim (U, x_1^2)_d = \dim \text{in}(U, x_1^2) \geq \dim (\text{in}(U), x_1^2)_d.
\]

Here the first equality follows from the fact that any ideal and its initial ideal have the same Hilbert function, the second one is immediate since \( \text{in}(U, x_1^2) \supseteq \text{in}(U, x_1^2) \).

It is therefore enough to show that \( (\text{gin}(U), x_1^2)_d = A_d \). Since \( k \leq d-1 \) every monomial of degree \( d \) not contained in \( \text{gin}(U)_d \) is divisible by \( x_1^2 \) by Lemma 3.1.7(i). But this means exactly that \( \text{gin}(U)_d + x_1^2A_{d-2} = A_d \).

\[\hfill \square\]

\begin{remark}
(i) The reason we pass to initial ideals in the second part of the proof is that we need to show \( (UV, l)_{2d-1} = A_{2d-1} \). As we have seen \( (U, l)_d = A_d \) and if we take another generic linear form \( l' \) we also have \( (V, l')_{d-1} = A_{d-1} \). However, since \( V = (U : l) \) we do not know that \( l \) behaves generically for \( V \).

(ii) It is not true in general that \( (U : l) \) is base-point-free if \( U \) is. Let \( n = 3 \) and let \( U = \text{span}(x^2y, x^2z, xy^2) \subseteq \mathbb{C}[x, y, z]_3 \). Then \( U \) contains \( A_1 \text{span}(yz, z^2) \), and thus for a generic linear form \( l \in A_1 \) we have \( (U : l) = \text{span}(yz, z^2) \oplus \text{span}(p) \) for some \( p \in A_2 \). Hence, the space \( (U : l) \) has a base-point, namely \( V(z, p) \).

One can show however that \( (U : l) \) is base-point-free whenever the degree is large enough.
\end{remark}

Now we show the reversed inequality from Theorem 3.5.2 in the case of strongly stable subspaces.

\begin{proposition}
Let \( U \subseteq A_d \) be a strongly stable subspace of codimension \( k \leq d-1 \) and let \( V := (U : x_1) \). Then \( \text{codim} V^2 \leq \text{codim} U^2 \)
\end{proposition}

\[\begin{proof}
Let \( M \in A_{d-2} \setminus V^2 \). We show that if \( x_1^2M \notin U^2 \), then \( x_1^2((V^2)^\perp) \subseteq (U^2)^\perp \), and the claimed inequality follows.

Assume \( x_1^2M \in U^2 \). Then either

(i) there exist monomials \( S, T \in A_{d-1} \) such that \( x_1^2M = (x_1S)(x_1T) \) and \( x_1S, x_1T \in U \), or

(ii) there exist \( S \in A_{d-2} \) and \( T \in A_d \) such that \( x_1^2 = (x_1S)T \) and \( x_1^2S, T \in U \).

In both cases \( M = ST \). In case (i) we have \( S, T \in V \) since \( V = (U : x_1) \) and hence \( M = ST \in V^2 \), a contradiction.

In case (ii) we see \( x_1S \in V \). If \( x_1T \), then \( \frac{T}{x_1} \in V \) and again we have \( M = (x_1S)\frac{T}{x_1} \in V^2 \). Thus we can assume that \( x_1 \) does not divide \( T \). Since \( k \leq d-1 \) every monomial of degree \( d-1 \) not contained in \( V \) is divisible by \( x_1 \) by Lemma 3.1.7(i). Hence, for every \( i \in \{2, \ldots, n\} \) such that \( x_1T \), the monomial \( \frac{T}{x_i} \) is contained in \( V \). Fix any such \( i \in \{2, \ldots, n\} \). Since \( V \) is strongly stable and \( x_1S \in V \), the monomial \( x_iS \) is also contained in \( V \). Combined this gives

\[ M = ST = (x_1S)\frac{T}{x_1} \in V^2, \]

which is again a contradiction.
\end{proof}\[\hfill \square\]
Combining the two inequalities of Theorem 3.5.2 and Proposition 3.5.4 we get the following result.

**Corollary 3.5.5.** If \( k \leq d \) then

\[
m(n, d, k) = m(n, k, k).
\]

**Proof.** If \( d = k \) there is nothing to show, we can thus assume that \( k \leq d - 1 \). Let \( V \subseteq A_d \) be a subspace of codimension \( k \) such that \( \text{codim} U^2 = m(n, d, k) \) and \( k < d \). By Theorem 3.5.2 we have \( \text{codim} U^2 \leq \text{codim} V^2 \) with \( V = (U : l) \) for a generic linear form \( l \in A_1 \). By definition \( \text{codim} V^2 \leq m(n, d - 1, k) \), thus \( m(n, d, k) \leq m(n, d - 1, k) \). On the other hand, let \( V \subseteq A_{d-1} \) be a strongly stable subspace of codimension \( k \) such that \( \text{codim} V^2 = m(n, d - 1, k) \). Let \( U := x_1 V \oplus \mathbb{C}[x_2, \ldots, x_n]_d \) then \( V = (U : x_1) \) and \( U \) is strongly stable by Lemma 3.1.7(ii). By Proposition 3.5.4 it follows that \( m(n, d - 1, k) = \text{codim} V^2 \leq \text{codim} U^2 \leq m(n, d, k) \). Combined this gives \( m(n, d, k) = m(n, d - 1, k) \) and we are done by induction. \( \square \)

**Remark 3.5.6.** The proofs also show that if \( V \subseteq A_d \) is a strongly stable subspace of codimension \( k \) such that \( \text{codim} V^2 = m(n, d, k) \) and \( k \leq d \), the subspace \( U := x_1 V \oplus \mathbb{C}[x_2, \ldots, x_n]_d \) satisfies \( \text{codim} U^2 = m(n, d + 1, k) \).

**Example 3.5.7.** We now look at some examples for \( n = 3 \). Using SAGE one easily checks that \( m(3, 5, 5) = 16 \) and \( m(3, 9, 9) = 31 \). It then follows from Corollary 3.5.5 that \( m(3, 5, 5) = m(3, 5, 5) \) and \( m(3, 9, 9) = m(3, 9, 9) \) for \( d \geq 5 \) (resp. \( d \geq 9 \)).

For \( k = 5 \), let

\[
U = \text{span}(x_1^5, x_1^3 x_2, x_1^3 x_3, x_1^5 x_2^2, x_1^5 x_2 x_3)^1 \subseteq A_5,
\]

then \( \text{codim} U^2 = m(3, 5, 5) \). From the last remark we then know that one strongly stable subspace realizing \( m(3, 5, 5) \) for \( d \geq 5 \) is given by

\[
U = (x_1^{d-5} \text{span}(x_1^5, x_1^3 x_2, x_1^3 x_3, x_1^5 x_2^2, x_1^5 x_2 x_3))^1 \subseteq A_d.
\]

For \( k = 9 \) we cannot take the nine smallest monomials (wrt the lex-ordering) to realize the maximum. In this case, we take

\[
U = \text{span}(x_1^9, x_1^7 x_2, x_1^7 x_3, x_1^7 x_2^2, x_1^7 x_2 x_3, x_1^7 x_3^2, x_1^7 x_2^2 x_3, x_1^7 x_3^2 x_2 x_3, x_1^7 x_3^2 x_2 x_3^1)^1 \subseteq A_9
\]

and as above for \( d \geq 9 \) we multiply by \( x_1^{d-9} \) to get subspaces realizing \( m(3, 9, 9) \) for \( d \geq 9 \).

This shows however that it is not clear in general which subspace realizes the maximum, even in the case \( n = 3 \). We can for example not take lex-segment ideals which realize the bound in Macaulay’s Theorem 3.3.2 (i), these are ideals such that every degree is spanned by (the correct number of) the greatest monomials wrt the lex-ordering.

### 3.6 Lifting subspaces

In Theorem 3.4.5 we showed how to reduce the number of variables, now we also want to increase that number while preserving \( \text{codim} U^2 \).

**Definition 3.6.1.** Let \( U \subseteq A_d \) be a subspace of codimension \( k \). Define

\[
U^{(1)} := x_{n+1} A(n+1)_{d-1} \oplus U \subseteq A(n+1)_d
\]

and for any \( l \geq 2 \)

\[
U^{(l)} := (U^{(l-1)})^{(1)} \subseteq A(n + l)_d
\]

\( U^{(0)} := U \).
For any \( l \geq 1 \) the subspace \( U^{(l)} \) also has codimension \( k \) in \( A(n+l)d \). And in fact, we know the whole Hilbert function of \( U^{(l)} \).

**Proposition 3.6.2.** Let \( U \subseteq A_d \) be a subspace of codimension \( k \). Let \( H = (h_i)_{i \geq 0} \) be the Hilbert function of \( \langle U \rangle \). Then for every \( l \geq 0 \) the following hold:

(i) The Hilbert function \( K = (k_i)_{i \geq 0} \) of the ideal generated by \( U^{(l)} \) in \( A(n+l) \) satisfies

\[
\begin{align*}
&k_i = \dim A(n+l)_i \text{ for } 0 \leq i \leq d-1 \text{ and } \\
&k_i = h_i \text{ for } i \geq d.
\end{align*}
\]

Furthermore, we have

(ii) \( \text{codim}_{A(n+l)2d}(U^{(l)})^2 = \text{codim}_{A2d} U^2 + l \cdot h_{2d-1} \).

**Proof.** It is enough to show this for \( l = 1 \) since the rest follows by induction. Write \( A' = A[y] \) with a new indeterminate \( y \), then

\[
V := U^{(1)} = yA'_{d-1} \oplus U \subseteq A'_d.
\]

For any \( s \geq 0 \), we have

\[
VA' = yA'_{d-1}A' + UA' = \left( \bigoplus_{i=1}^{d+s} y^i A_{d+s-i} \right) + A_d U + yA_{d-1} U + \cdots + y^s U
\]

\[
= \bigoplus_{i=1}^{d+s} y^i A_{d+s-i} \oplus UA
\]

which shows (i) since \( A'_{d+s} = \bigoplus_{i=0}^{d+s} y^i A_{d+s-i} \).

For (ii) we calculate \( V^2 \) and with the same argument as above we get

\[
V^2 = y^2 A'_{2d-2} + yA'_{d-1}U + U^2 = \bigoplus_{i=2}^{2d} y^i A_{2d-i} \oplus y(A_{d-1} U) \oplus U^2
\]

and

\[
\text{codim}_{A_{2d}} V^2 = \text{codim}_{A_{2d}} U^2 + h_{2d-1}.
\]

This enables us to determine the Hilbert function of codimension 2 subspaces of \( A_d \) as an easy application.

For generic \( U \) the Hilbert function of \( \langle U \rangle \) is as small as possible. In the codimension 2 case this means that the Hilbert function is \((1, n, 2)\) generically. We show that this holds whenever \( U \) is base-point-free.

**Proposition 3.6.3.** Let \( U \subseteq A_2 \) be a base-point-free subspace of codimension 2. Then the Hilbert function of \( \langle U \rangle \) is \((1, n, 2)\).

**Proof.** By Theorem 3.3.2, the Hilbert function is smaller or equal to \((1, n, 2, 2, \ldots)\). So assume \( h_{\langle U \rangle}(3) > 0 \). Then by Proposition 3.6.2, the subspace \( U^{(l)} \subseteq A(n+l)2d \) has codimension 2 and for \( l \geq 7 \) we have \( \text{codim}_{A(n+l)4}(U^{(l)})^2 \geq 7 \) which is not possible by Theorem 3.4.8.
Corollary 3.6.4. Let \( U \subseteq A_d \) be a base-point-free subspace of codimension \( k \in \{1, 2\} \) and \( \text{codim} U^2 = s \). Then for every \( N \geq n \), there exists a base-point-free subspace \( V \subseteq A(N)_d \) of codimension \( k \) such that \( \text{codim} V^2 = s \).

Proof. By Proposition 3.6.3 and Corollary 3.4.7, the degree \( 2d - 1 \) component of \( A/\langle U \rangle \) has dimension \( 0 \). Hence

\[
\text{codim}_{A(n)_2d} U^2 = \text{codim}_{A(n+l)_2d} (U^{(l)})^2.
\]

by Proposition 3.6.2 (ii).

\( \square \)

3.7 Arbitrary codimension

We show bounds for \( \text{codim} U^2 \) for base-point-free subspaces \( U \) of any codimension that are independent of \( n \) and \( d \), if \( d \) is large enough. The most important step is to also consider the orthogonal complement alongside our starting space. This is made precise in Lemma 3.7.2.

The main idea is the following: if \( U \subseteq A_d \) is a base-point-free subspace of codimension \( k \), we consider \( U' := U \cap A(m)_d \) for some \( 2 \leq m \leq n \). To use Theorem 3.4.5 we need to make sure that \( \text{codim} U = \text{codim} U' \) and to get bounds that are independent of \( n \) we want \( U' \) to be base-point-free as well (and \( k \leq d - 1 \)), we then use Corollary 3.4.7 to conclude \( \text{codim} U^2 \leq \text{codim} (U')^2 \). To get a bound that is independent of \( d \), we use the results of Section 3.5.

We still always assume that \( n, d \geq 2 \), and \( k \in \mathbb{N} \).

Remark 3.7.1 (The dual problem). Let \( U \subseteq A_d \) be a base-point-free subspace of codimension \( k \). Instead of asking if \( U' := U \cap A(m)_d \) satisfies

(i) \( \text{codim}_{A(m)_d} U' = k \) and

(ii) \( \mathcal{V}(U') = \emptyset \) with \( \mathcal{V}(U') \subseteq \mathbb{P}^{m-1} \),

as in Theorem 3.4.5 and Corollary 3.4.7, we can also look at the dual problem:

Let \( W = U^\perp \). Does \( W' := W(x_1, \ldots, x_m, 0, \ldots, 0) \) have the same dimension as \( W \) and does \( W' \) not contain the \( d \)-th power of a linear form.

The next lemma is an easy statement from linear algebra.

Lemma 3.7.2. Let \( U \subseteq A_d \) be a subspace and \( W := U^\perp \). Let \( l_1, \ldots, l_s \in A_1 \) be linearly independent linear forms and \( V := \mathbb{C}[l_1, \ldots, l_s]_d \subseteq A_d \). Then

\[
(U \cap V)^\perp \cong (W + V^\perp)/V^\perp,
\]

and \( V^\perp = \text{span}(\lambda_1, \ldots, \lambda_{n-s})_{A_{d-1}} \) where \( \text{span}(l_1, \ldots, l_s)^\perp = \text{span}(\lambda_1, \ldots, \lambda_{n-s}) \subseteq A_1 \).

Write \( \overline{W} \) for \( (W + V^\perp)/V^\perp \), then we especially have \( \text{codim} U \cap V = \dim \overline{W} \) and \( U \cap V \) is base-point-free if and only if \( \overline{W} \) contains no \( d \)-th power of a linear form.

Now we want to work on condition (i) in Remark 3.7.1 to ensure that \( \dim W = \dim \overline{W} \).

Since \( W \) will play the role of \( U^\perp \), \( k \) will usually denote the dimension of \( W \), and not the codimension.

Proposition 3.7.3. Let \( k < n \) and let \( W \subseteq A_d \) be a subspace of dimension \( k \). Let \( l \in A_1 \) be generic. Then \( \dim \overline{W} = \dim W \) where \( \overline{W} = W + \langle l \rangle_A/\langle l \rangle_d \).
Proof. Using the notation of \textsection{3.3.9} with \( I = \langle W \rangle \), we have

\[ h_I(d) = h_d = \dim A_d - k = \sum_{i=0}^{d-2} \left( \binom{n-2+d-i}{d-i} \right) + \left( \frac{n-k}{1} \right) . \]

Green’s Theorem \textsection{3.3.10} then shows

\[ \dim A_d / \langle W, l \rangle_d = c_d \leq \sum_{i=0}^{d-2} \left( \binom{n-2+d-i-1}{d-i} \right) + \left( \frac{n-k-1}{1} \right) \]

As can be easily verified we have \( C = \dim A(n-1)_d - (n-1) \) and therefore

\[ c_d \leq \dim A(n-1)_d - (n-1) + \left( \frac{n-k-1}{1} \right) = \dim A(n-1)_d - k. \]

Equivalently \( \dim W \geq k \). Since \( \dim W \leq \dim W = k \), it follows that \( \dim W = k \). \( \square \)

\textbf{Remark 3.7.4.} The condition of Proposition \textsection{3.7.3} on the dimension, namely \( k < n \) is necessary and tight in the following sense. Let \( 0 \neq F \in A_{d-1} \) and let \( W = FA_1 \). Then \( \dim W = n \) and \( \dim W = n - 1 < \dim W \) for a generic linear form \( l \in A_1 \).

In fact, it follows from [AMS18, Theorem 3.2] that every subspace \( W \) of dimension \( n \) such that \( \dim W < \dim W \) for a generic linear form \( l \in A_1 \), has the form \( FA_1 \) for some \( F \in A_{d-1} \).

\textbf{Corollary 3.7.5.} Let \( U \subseteq A_d \) be a subspace of codimension \( k \) with \( k \leq n \). Then

\[ \codim(U \cap \mathbb{C}[l_1, \ldots, l_k]_d) = k \]

for generic linear forms \( l_1, \ldots, l_k \in A_1 \).

Especially, after applying a generic change of coordinates to \( U \), we have

\[ \codim(U \cap A(k)_d) = k. \]

\textbf{Proof.} Let \( W = U^\perp \). If \( k = n \) there is nothing to show, hence assume \( k < n \). By Lemma \textsection{3.7.2} it is enough to consider the dimension of \( \dim W \subseteq A_d / \langle l_{k+1}, \ldots, l_n \rangle_d \) for any basis \( l_{k+1}, \ldots, l_n \) of the orthogonal complement of \( \text{span}(l_1, \ldots, l_k) \). Since \( k < n \), it follows from Proposition \textsection{3.7.3} that \( \dim W = \dim W \).

\textbf{Corollary 3.7.6.} Let \( W \subseteq A_d \) be a subspace of dimension \( k < n \) and let \( l \in A_1 \) a generic linear form. Then \( \dim (W : l) = 0 \).

\textbf{Proof.} This follows from Proposition \textsection{3.7.3} and the exact sequence

\[ 0 \rightarrow A_{d-1} / (W : l) \rightarrow A_d / W \rightarrow A_d / \langle W, l \rangle_d \rightarrow 0. \]

in \textsection{3.3.9} the space in the middle has dimension \( \dim A_d - k \) and the space on the right has dimension \( \dim A(n-1) - k \) by Proposition \textsection{3.7.3} Hence, the one on the left-hand side has dimension \( A_d - k - \dim A(n-1) + k = \dim A_{d-1} \) and thus \( \dim (W : l) = 0 \). \( \square \)
This shows that condition (i) in Remark 3.7.1 is satisfied whenever \( k < n \) and we go down by one variable. And it is in general not satisfied if \( n \leq k \) since we can take \( W = FA_1 \) for some \( 0 \neq F \in A_{d-1} \).

Now we want to look at condition (ii) in Remark 3.7.1. By Lemma 3.7.2 asking whether \( U' := U \cap A(n - 1) \) is base-point-free is the same as asking if the orthogonal complement contains no \( d \)-th power of a linear form. Thus assume that \( U \) is base-point-free and \( W \) contains no \( d \)-th powers.

Is it true in general though that \( W \subseteq (A/\langle \ell \rangle)_d \) contains no \( d \)-th powers for generic \( \ell \in A_1 \) whenever \( \dim W = \dim W' \)? Sadly this is not the case as the next example shows.

**Example 3.7.7.** Let \( W := x_n^{d-1}A(n - 1)_1 \subseteq A_d \) and let \( \ell \in A_1 \) be a generic linear form. After scaling \( W \), we can write \( W' = x_n + \ell' \) for some \( \ell' \in A(n - 1)_1 \), hence \( W' \subseteq (A/\langle \ell \rangle)_d \) is isomorphic to \( (\ell')^{d-1}A(n - 1)_1 \). Then \( (\ell')^d \in W \) and \( \dim W = \dim W' \).

However, it is true whenever the number of variables is large as the next theorem shows.

**Proposition 3.7.8.** Let \( W \subseteq A_d \) be a subspace of dimension \( k \) and let \( n \geq 3k + 1 \). If \( W \) contains no \( d \)-th power of a linear form, then the same holds for \( \overline{W} = W + \langle \ell \rangle_d/\langle \ell \rangle_d \subseteq (A/\langle \ell \rangle)_d \) where \( \ell \in A_1 \) is a generic linear form.

**Proof.** Assume this is wrong and let \( \ell \in A_1 \) be \( W \)-generic. Since \( W \) contains a \( d \)-th power, there exist \( L \in A_1 \) and \( g \in A_{d-1} \) such that \( L^d + lg \in W \).

Let \( s \leq k + 1 \) be the largest integer such that there exist linearly independent linear forms \( l_1, \ldots, l_s \) that are \( W \)-generic and such that the \( 2s \) linear forms \( l_1, \ldots, l_s, L_1, \ldots, L_s \) are linearly independent with \( L_1^d + l_ig_i \in W \) (\( L_1, \ldots, L_s \in A_1, g_1, \ldots, g_s \in A_{d-1} \)).

If \( s = k + 1 \), then after a change of coordinates the elements

\[
x_1^d + x_{k+1}^d + x_{k+2}p_{k+1}
\]

are contained in \( W \) and are linearly dependent since \( \dim W = k \) (with \( p_i \) the image of \( g_i \) under this change of coordinates for \( i = 1, \ldots, k + 1 \)). Let \( \lambda_1, \ldots, \lambda_{k+1} \in \mathbb{C} \) not all zero such that

\[
\sum_{i=1}^{k+1} \lambda_i x_i^d + \sum_{i=1}^{k+1} \lambda_i x_{k+1+i}p_i = 0.
\]

For \( 1 \leq i \leq k+1 \) the element \( x_i^d \) cannot appear in the second sum, hence both sums are zero. But looking at the first sum, it immediately follows that all \( \lambda_i \) are zero, a contradiction.

If \( s < k + 1 \), let \( h_1, \ldots, h_{k+1} \) be generic linear forms with \( H_i^d + h_ip_i \in W \) (\( H_1, \ldots, H_k \in A_1, p_1, \ldots, p_k \in A_{d-1} \)). Since \( h_1, \ldots, h_{k+1} \) are generic, we may assume that for every \( j \in \{2, \ldots, k+1\} \) the linear form \( h_j \) is \( \text{span}(h_1p_1, \ldots, h_{j-1}p_{j-1}) \)-generic. By assumption we have

\[
H_i \in \text{span}(l_1, \ldots, l_s, L_1, \ldots, L_s, h_i) \quad \text{for all} \quad i = 1, \ldots, k+1.
\]

By changing \( p_i \) we can further assume that \( H_i \in \text{span}(l_1, \ldots, l_s, L_1, \ldots, L_s) \) for every \( i = 1, \ldots, k+1 \): indeed, for every \( i = 1, \ldots, k + 1 \) we can write \( H_i = H_i' + c \cdot h_i \) for some \( H_i' \in \text{span}(l_1, \ldots, l_s, L_1, \ldots, L_s) \) and some \( c \in \mathbb{C} \), then

\[
H_i^d + h_ip_i = (H_i' + c \cdot h_i)^d + h_ip_i = (H_i')^d + h_ip_i.
\]
for some $p'_i \in A_{d-1}$. Since $\dim W = k$, these $k+1$ forms are linearly dependent and there exist $\lambda_1, \ldots, \lambda_{k+1} \in \mathbb{C}$ not all zero such that

$$
\sum_{i=1}^{k+1} \lambda_i H_i^d + \sum_{i=1}^{k+1} \lambda_i p_i = 0.
$$

Since $h_1, \ldots, h_{k+1}$ are generic linear forms and $n \geq 3k + 1$, the linear forms $l_1, \ldots, l_s$, $L_1, \ldots, L_s$, $h_1, \ldots, h_{k+1}$ are linearly independent. Again after a change of coordinates mapping $l_1, \ldots, l_s, L_1, \ldots, L_s$ to $x_1, \ldots, x_{2s}$ and $h_1, \ldots, h_{k+1}$ to $x_{2s+1}, \ldots, x_{2s+k+1}$, the first sum is contained in $\mathbb{C}[x_1, \ldots, x_{2s}]$ and the second sum contains no monomial in the first $2s$ variables. It follows that both sums separately have to be zero. Especially,

$$
\sum_{i=1}^{k+1} \lambda_i h_i p_i = 0.
$$

By assumption not all $\lambda_i$ are zero. If only one was non-zero, say $\lambda_j \neq 0$ for some $j \in \{1, \ldots, k+1\}$, then it follows that $h_j p_j = 0$, hence $H_j^d \in W$ which is a contradiction. Therefore, we can assume that at least two of the $\lambda_i$ are non-zero. Let $j := \max\{i \in \{1, \ldots, k+1\} : \lambda_i \neq 0\}$, then

$$
h_j p_j = \sum_{i=1}^{j-1} \lambda_i h_i p_i
$$

and therefore $h_j p_j \in \text{span}(h_1 p_1, \ldots, h_{j-1} p_{j-1}) =: V \neq \{0\}$. By our choice of $h_1, \ldots, h_{k+1}$, the linear form $h_j$ is $V$-generic. We have $\dim V \leq j - 1 \leq k < n$ which means that $\dim(V : l) = 0$ for a $V$-generic linear form $l$ by Corollary 3.7.6. Especially, this holds for $h_j$, but $h_j p_j \in V$, a contradiction.

**Theorem 3.7.9.** Let $k \leq d - 1$. Then for every $n \geq 2$ and every base-point-free subspace $U \subseteq A_d$ of codimension $k$ we have

$$
\text{codim} U^2 \leq m(3k, k, k).
$$

Especially, the number $m(3k, k, k)$ is independent of $n$ and $d$.

**Proof.** If $n < 3k$ and $U \subseteq A_d$ is a base-point-free subspace of codimension $k$, then by Proposition 3.6.2 we have $\text{codim} U^2 = \text{codim} \left(U^{(3k-n+1)}\right)^2$ and $\text{codim} U = \text{codim} U^{(3k-n+1)}$ with $U^{(3k-n+1)} \subseteq A(3k+1)_d$. It is therefore enough to only consider the case $n > 3k$.

We apply a generic change of coordinates to $U$. By Corollary 3.7.5 it follows that $V := U \cap A(3k)_d$ has codimension $k$ in $A(3k)_d$ and by Proposition 3.7.8 the subspace $V$ is still base-point-free.

Using Corollary 3.4.7 we get

$$
\text{codim} U^2 \leq \text{codim} V^2 \leq m(3k, d, k).
$$

By Corollary 3.5.5 we finally have $m(3k, d, k) = m(3k, k, k)$ which concludes the proof. 

**Remark 3.7.10.** Assuming that $d$ is large enough is essential. Consider the following example: Let $R = A(4)$, $m = \langle x_5, \ldots, x_n \rangle \subseteq A(n), n \geq 5$ and

$$
U = \text{span}(x_3^3, x_2^3, x_1^3, x_3^2 x_2 + x_2^2 x_4) \subseteq A(4)_4.
$$
3.8 DETERMINING THE UPPER BOUND

This subspace has codimension 15 in $A(4)_3$. We check with SAGE [The19] that the Hilbert function of $\langle U \rangle$ is given by $(1, 4, 10, 15, 7, 1)$. Define the subspace

$$V := U^{(n-4)} = \bigoplus_{i=1}^{3} m^i R_{3-i} \oplus U \subseteq A(n)_3.$$  

This subspace also has codimension 15 in $A(n)$ by Proposition 3.6.2. Again by Proposition 3.6.2 we know that $\text{codim } V^2 = \text{codim } U^2 + 7 \cdot (n - 4)$.

This shows that we cannot have a uniform bound for this combination of codimension and degree not depending on $n$.

It seems likely that one cannot only reduce to $3k$ variables in Proposition 3.7.8 but actually to $k + 1$ variables. This is at least the only counterexample we know of (for $k \leq n - 1$).

**Conjecture 3.7.11.** Let $k \leq d - 1$, $n - 1$, and $n \geq 3$. Let $W \subseteq A_d$ be a subspace of dimension $k$ and suppose that $W$ contains no $d$-th power of a linear form. Then for a generic linear form $l \in A_1$ it holds that either

(i) $W$ contains no $d$-th power of a linear form, or

(ii) $n = k + 1$ and $W = L^d_1 C[L_2, \ldots, L_{k+1}]$ for some basis $L_1, \ldots, L_{k+1}$ of $A(k+1)_1$.

This would allow us to show $\text{codim } U^2 \leq m(k, k, k)$ in Theorem 3.7.9 with an additional argument.

**Remark 3.7.12.** (i) The conjecture is certainly false if $n = k$ is allowed: for $n = k$ let $W = x_1^{d-1} C[x_2, \ldots, x_n] \oplus \text{span}(p)$ for some generic $p \in A_d$. Since $p$ is generic, $W$ contains no $d$-th powers, but $W \cong x_1^{d-1} C[x_1, \ldots, x_{n-1}] \oplus \text{span}(p)$ does contain one where $W \subseteq A_d / \langle l \rangle_1$ for a generic linear form $l \in A_1$.

(ii) The conjecture is true for $n \geq 3k + 1$ by Proposition 3.7.8. For $k = 1$ it also follows from a simple geometric observation: if $W = \text{span}(p)$ and $p$ is not a power of a linear form, then $\mathcal{V}(p)$ is non-degenerate. Hence, the same holds for a generic hyperplane section which therefore cannot be defined by the power of a linear form.

**Remark 3.7.13.** By definition of $m(n, d, k)$ we are considering complex subspaces of $A_d$. However, when studying faces of Gram spectrahedra we are only interested in real subspaces. As we have seen $m(n, d, k)$ is realized by a monomial subspace, hence we can consider it as a real subspace of $\mathbb{R} [x]_d$. So the bound is also tight in the real case.

In general, we do not know all attainable values for $\text{codim } U^2$ while fixing $n$, $d$, and $k$. Even in the case of codimension 2 subspaces, we do not know whether some values are possible (see Remark 3.4.11).

3.8 Determining the upper bound

In this section, we determine the value $m(3k, k, k)$. Moreover, we find the strongly stable subspace that realizes $m(k, k, k)$, and show that it is unique. We rely on Proposition 3.8.8 and Proposition 3.8.11 to do most of the combinatorial work for the main results, which are Theorem 3.8.3 and Theorem 3.8.12.

We always assume $k \in \mathbb{N}$, $k \geq 2$. 

Lemma 3.8.1. For any $k \geq 2$ we have $m(3k, k, k) \leq 2k^2 + m(k, k, k)$.

Proof. Let $U \subseteq A(3k)_k$ be some strongly stable subspace of codimension $k$ that realizes $m(3k, k, k)$. By Lemma 3.1.8, every monomial not contained in $U$ is contained in $A(k)_k$. Especially, the codimension of $V := U \cap A(k)_k$ is $k$. Using Theorem 3.4.3 we see
\[
\text{codim} U^2 \leq \text{codim} V^2 + (3k - k) \dim(A(k)/(V))_{2k-1} \\
\leq m(k, k, k) + 2k \dim(A(k)/(V))_{2k-1}.
\]
Since $U \subseteq A(k)_k$ has codimension $k$, we see from Corollary 3.3.4 (i) that
\[
\dim(A(k)/(V))_{2k-1} \leq k
\]
which finishes the proof. \hfill \Box

In fact, it follows from Remark 3.1.16 and Remark 3.4.6 that we have equality in the last lemma.

To determine $m(3k, k, k)$ it now suffices to determine a bound for $m(k, k, k)$. We show that the only strongly stable subspace realizing the bound $m(k, k, k)$ for $k \geq 2$ is $U = \text{span}(x_1^k, x_1^{k-1}x_2, \ldots, x_1^{k-1}x_k)^\perp$ and $m(k, k, k) = \binom{k+2}{3}$.

First, we show that the subspace $U$ above does realize the bound $m(k, k, k)$.

Lemma 3.8.2. Let $U \subseteq A(k)_k$ be the subspace of codimension $k$ spanned by all monomials of degree $k$ except for $x_1^d, x_1^{d-1}x_2, \ldots, x_1^{d-1}x_k$. Then $\text{codim} U^2 = \binom{k+2}{3}$.

Proof. We easily see that the only monomials not contained in $U^2$ are
\begin{itemize}
  \item $x_1^{2k-1}x_i$ for $i = 1, \ldots, k$,
  \item $x_1^{2k-2}x_ix_j$ for $i, j = 2, \ldots, k$,
  \item $x_1^{2k-3}x_ix_jx_l$ for $i, j, l = 2, \ldots, k$
\end{itemize}
or equivalently $x_1^{2k-3}x_ix_jx_l$ for $i, j, l = 1, \ldots, n$. These are a total of $\frac{1}{6}(k^3 + 3k^2 + 2k) = \binom{k+2}{3}$ monomials. \hfill \Box

Theorem 3.8.3. For any $k \geq 2$ we have $m(k, k, k) = \binom{k+2}{3}$.

Proof. We prove this by induction on $k$. For $k = 2$ the claim follows from Proposition 3.1.15 (iii). Assume the claim holds for some $k \geq 2$. Let $U \subseteq A(k+1)_{k+1}$ be a strongly stable subspace realizing the bound $m(k+1, k+1, k+1)$. We write $d = k + 1$ for the degree. If $x_1^{d-1}x_{k+1}$ is not contained in $U$, then $U = \text{span}(x_1^d, x_1^{d-1}x_2, \ldots, x_1^{d-1}x_{k+1})^\perp$ by Lemma 3.8.7.

By Lemma 3.8.2 we then have $\text{codim} U^2 = \binom{k+3}{3} = \binom{k+1}{2}$. We show that this is the only strongly stable subspace realizing the upper bound. For the sake of contradiction assume that $x_1^{d-1}x_{k+1}$ is contained in $U$, then no monomial in $U^\perp$ is divisible by $x_{k+1}$. Indeed, if there was a monomial $M \in U^\perp$ divisible by $x_{k+1}$, we can write $M = Tx_{k+1}$ with $T \in A(k+1)_{d-1}$, then $M = Tx_1^{d-1}x_{k+1} \in U$ as $x_1^{d-1}x_{k+1} \in U$ and $U$ is strongly stable. Therefore, we can write
\[
U = x_{k+1}A(k+1)_{d-1} \oplus V
\]
where $V \subseteq A(k)_d$ is a strongly stable subspace of codimension $k + 1$. By Theorem 3.4.5 we now have

$$m(k + 1, d, k + 1) = \operatorname{codim} U^2 \leq \operatorname{codim} V^2 + k + 1 \leq m(k, d, k + 1) + k + 1.$$ 

In Proposition 3.8.11 we show that $m(k, d + 1) < m(k, k + 1) + \binom{k + 1}{2}$, then we have

$$m(k + 1, k + 1, k + 1) = \operatorname{codim} U^2 < m(k, k + 1) + \binom{k + 1}{2} + k + 1 \leq \binom{k + 3}{3}.$$ 

However, the subspace $W := \operatorname{span}(x_1^d, x_1^{d-1}x_2, \ldots, x_1^{d-1}x_{k+1})$ satisfies $\operatorname{codim} W^2 = \binom{k + 3}{3}$. This yields the following contradiction

$$\binom{k + 3}{3} = \operatorname{codim} W^2 \leq m(k + 1, k + 1, k + 1) < \binom{k + 3}{3}.$$ 

It is left to prove Proposition 3.8.11 which we used in the last proof. For this we first need some more preparation.

**Definition 3.8.4.** For any monomial $M \in A_d$ denote by $p(M)$ the smallest integer $j > 1$ such that $x_j | M$. If $M = x_1^d$, we define $p(M) := 1$. By $M^-$ we denote the reduction of $M$, which is the monomial $x_1 \frac{M}{x_{p(M)}}$. On the other hand, we write $M^+$ for the set of all monomials $T \in A_d$ such that $M$ is the reduction of $T$.

**Example 3.8.5.** Consider the monomial $M = x_1^2x_2x_3x_4 \in A(4)_4$, then $p(M) = 3$ and $M^- = x_1^2x_4$. The set $M^+$ consists of the monomials $x_1x_2x_3x_4$ and $x_1x_2^2x_4$. The monomial $x_1x_3x_2^2$ is not contained in $M^+$ since its reduction is $x_1^2x_2^2$.

If $M = x_1^d$, then by definition $M^- = x_1^d$ and $M^+ = \{x_1^{d-1}x_i : 1 \leq i \leq n\}$.

**Lemma 3.8.6.** Let $M \in A_d$ be a monomial divisible by $x_1$ with $p(M) > 1$. Then $|M^+| = p(M) - 1$.

*Proof.* $M^+$ consists of the elements $x_j \frac{M}{x_1}$ for $j = 2, \ldots, p(M)$; The variables $x_2, \ldots, x_{p(M)-1}$ do not appear in $M$ by definition of $p(M)$. Hence, for $2 \leq j \leq p(M)$ we see $p\left( x_j \frac{M}{x_1} \right) = j$, and therefore

$$\left( x_j \frac{M}{x_1} \right)^- = \frac{x_1}{x_j} \left( x_j \frac{M}{x_1} \right) = M.$$ 

**Lemma 3.8.7.** Let $U \subseteq A_d$ be a strongly stable subspace of codimension $k \leq n$. If there exists a monomial $M \in U^\perp$ such that $p(M) = k$, then $U = \operatorname{span}(x_1^d, x_1^{d-1}x_2, \ldots, x_1^{d-1}x_k)^\perp$. 
**Proof.** Since \( M \not\subseteq U \) and \( U \) is strongly stable, the monomials \( x_i \frac{M}{x_i} \) are also not contained in \( U \) for \( i = 1, \ldots, k - 1 \). Hence, \( U^\perp \) is spanned by the monomials \( x_i \frac{M}{x_i} \) (\( i = 1, \ldots, k - 1 \)) and \( M \). Especially, if \( M = x_1^{d-1} x_k \), then \( U \) has the asserted form.

Assume \( M \neq x_1^{d-1} x_k \), hence \( M = x_a^2 x_T \) for some \( a \in \mathbb{N} \) and some monomial \( 1 \neq T \in \mathbb{C}[x_1, \ldots, x_n] \) of degree \( d - a - 1 \). Just as above, the monomials \( x_1^a x_T \) are not contained in \( U \) for \( i = 1, \ldots, k \). However, since \( T \neq 1 \), there exists \( k < i \leq n \) such that \( x_i | T \), hence \( x_1^a x_T x_i \) is also not contained in \( U \). This is a contradiction since \( k + 1 \) different monomials are not contained in \( U \). \( \square \)

**Proposition 3.8.8.** Let \( U \subseteq A_d \) be a strongly stable subspace of codimension \( k \) with \( 2 \leq k \leq n \). Let \( S \) be the set of all monomials of degree \( d \) not contained in \( U \). Then

\[
\left| \bigcup_{M \in S} M^+ \right| \leq \binom{k}{2} + n
\]

with equality if and only if \( U = \text{span}(x_1^1, x_1^{d-1} x_2, \ldots, x_1^{d-1} x_k)^\perp \). Furthermore, every monomial in \( S \) is contained in \( \bigcup_{M \in S} M^+ \).

**Remark 3.8.9.** We give a brief idea why Proposition 3.8.8 should hold. Assume we know that equality in Proposition 3.8.8 holds whenever \( U = \text{span}(x_1^1, x_1^{d-1} x_2, \ldots, x_1^{d-1} x_k)^\perp \). We swap one monomial in \( U^\perp \) with some other monomial in \( A_d \). For \( U \) to stay strongly stable, we need to remove \( x_1^{d-1} x_k \) and add \( x_1^{d-2} x_2 \). By Lemma 3.8.6 we know that \( |(x_1^{d-2} x_2)| = 1 \) and \( |(x_1^{d-1} x_k)| = k - 1 \), hence the inequality is now strict.

Continuing doing this, we have to remove \( x_1^{d-1} x_j \) for some large \( j \) and add a monomial \( M \) in fewer variables, especially \( p(M) < p(x_1^{d-1} x_j) = j \) and by Lemma 3.8.6 the set \( \bigcup_{M \in S} M^+ \) contains even fewer elements now.

**Proof of Proposition 3.8.8.** Let \( T \in S \) such that \( p(T) = \max\{p(M): M \in S \} \). Since \( U \) is strongly stable, the elements \( x_j \frac{T}{x_j} \frac{x_T}{x_j} \) for \( j = 2, \ldots, p(T) - 1 \) are also not contained in \( U \) and therefore lie in \( S \). Moreover, they satisfy \( p(x_j \frac{T}{x_j} \frac{x_T}{x_j}) = j \) for all \( j = 2, \ldots, p(T) - 1 \). Thus we have \( p(T) - 1 \) elements in \( S \) for which we know the value \( p(\cdot) \), namely

\[
T, x_j \frac{T}{x_j} \frac{x_T}{x_j} \quad (j = 2, \ldots, p(T) - 1).
\]

Next, we look at the element \( x_1^1 \in S \). We have \( (x_1^1)^+ = \{x_1^1, \ldots, x_1^{d-1} x_n\} \), and therefore \( |(x_1^1)^+| = n \).

In total, we identified \( p(T) \) elements in \( S \) for which we know the value \( p(\cdot) \) and all other \( k - p(T) \) elements \( M \) in \( S \) satisfy \( p(M) \leq p(T) \) by the choice of \( T \). We write

\[
S' = S \setminus \{x_1^1, x_2 \frac{T}{x_2} \frac{x_T}{x_2}, \ldots, x_{p(T)-1} \frac{T}{x_{p(T)-1}} \frac{x_T}{x_{p(T)-1}}, T\}
\]

for the set where we removed all monomials from \( S \) of which we determined the value \( p(\cdot) \).
3.8 DETERMINING THE UPPER BOUND

We calculate

\[ \left| \bigcup_{M \in S} M^+ \right| \leq \sum_{i=2}^{p(T)} \left| \left( x_i, \frac{T}{x_{p(T)}} \right)^+ \right| + \left| (x_1^d)^+ \right| + \left| \bigcup_{M \in S'} M^+ \right| \]

(Lemma 3.8.6) \[= \sum_{i=2}^{p(T)} (i - 1) + n + \frac{|S| - p(T) - (p(T) - 1)}{=|S|} \]

\[= \sum_{i=1}^{p(T) - 1} i + (k - p(T))(p(T) - 1) + n \]

\[= \sum_{i=1}^{p(T) - 1} i + \sum_{j=p(T)}^{k-1} (p(T) - 1) + n \]

\[\leq \sum_{i=1}^{k-1} i + n \]

\[= \left( \frac{k}{2} \right) + n. \]

Next, we show that equality holds if and only if \( U = \text{span}(x_1^d, x_1^{d-1} x_2, \ldots, x_1^{d-1} x_k)^\perp. \)

First, we note that \( p(T) = k \) implies \( S = \{x_1^d, x_1^{d-1} x_2, \ldots, x_1^{d-1} x_k\} \) by Lemma 3.8.7.

Assume \( U \) is not of this form, then \( p(T) < k \) and therefore the sum \( \sum_{j=p(T)}^{k-1} (p(T) - 1) \) is non-zero. Moreover, we have a strict inequality

\[\sum_{j=p(T)}^{k-1} (p(T) - 1) < \sum_{j=p(T)}^{k-1} j\]

which means the second to last inequality above is also strict in this case.

Therefore, equality can only hold if \( U = \text{span}(x_1^d, x_1^{d-1} x_2, \ldots, x_1^{d-1} x_k)^\perp. \) Assume we are in this case, then \( p(T) = k. \) First, we note that this means that \( S' = \emptyset, \) hence we get

\[ \left| \bigcup_{M \in S} M^+ \right| \leq \sum_{i=2}^{p(T)} \left| \left( x_i, \frac{T}{x_{p(T)}} \right)^+ \right| + \left| (x_1^d)^+ \right| \]

(Lemma 3.8.6) \[= \sum_{i=2}^{p(T)} (i - 1) + n \]

\[= \sum_{i=1}^{k-1} i + n \]

\[= \left( \frac{k}{2} \right) + n. \]

We thus need to show that the inequality is an equality or equivalently that the sets \( M^+ \) with \( M \in S \) are disjoint.

Let \( x_1^{d-1} x_i \in S, i \geq 2, \) then \( (x_1^{d-1} x_i)^+ = \{x_1^{d-l} x_l x_i : 2 \leq l \leq i\} \). We see that if \( (x_1^{d-1} x_i)^+ \) and \( (x_1^{d-1} x_j)^+ \), \( i, j \geq 2 \) contain a common element, it has the form \( x_1^{d-2} x_l x_j \) with \( i \leq j \)
and \( j \leq i \), which means \( i = j \). For \( x_i^d \) we have \((x_i^d)^+ = \{x_i^d, \ldots, x_i^{d-1}x_n\}\) and every element has degree at least \( d - 1 \) in \( x_i \), hence it is not contained in \((x_i^{d-1}x_i)^+\) for any \( i \geq 2 \). This finishes the first part.

The last claim we need to prove is \( S \subseteq \bigcup_{M \in S} M^+ \). Let \( T \in S \). Since \( U \) is strongly stable, the element \( x_1 \frac{T}{x_{p(T)}} = T^- \) is also contained in \( S \). But by definition this means

\[
T \in \left( x_1 \frac{T}{x_{p(T)}} \right)^+ \subseteq \bigcup_{M \in S} M^+
\]

which finishes the proof.

**Lemma 3.8.10.** Let \( U \subseteq A_d \) be a strongly stable subspace of codimension \( k + 1 \), then there exists a strongly stable subspace \( V \subseteq A_d \) of codimension \( k \) such that \( U \subseteq V \).

**Proof.** If \( U = \{0\} \), take \( V = \text{span}(x_i^d) \). Now assume \( U \neq \{0\} \). Let \( M \) be any monomial not contained in \( U \), and let \( I := \{i \in \mathbb{N} : x_i|M| \setminus \{x_i\} \} \). Then either \( x_{i+1} \frac{M}{x_i} \in U \) for every \( i \in I \) or there exists \( j \in I \) such that \( x_{j+1} \frac{M}{x_j} \notin U \). If all of them are contained in \( U \), we may add \( M \) to \( U \) and the subspace \( V := U \oplus \text{span}(M) \) is still strongly stable. Indeed, let \( i \in I \) and \( i < l \leq n \), then

\[
x_{i+1} \frac{M}{x_i} = x_i \frac{x_{i+1}}{x_i} \left( \frac{x_{i+1}}{x_i} \frac{M}{x_i} \right) \in U.
\]

Any other monomial in \( V \) except for \( M \) is already contained in \( U \) and \( U \) is strongly stable.

If \( x_{i+1} \frac{M}{x_i} \) is not contained in \( U \) for some \( j \in I \), then for any monomial ordering \( \geq \) we have \( x_{j+1} \frac{M}{x_j} \geq M \). Now we continue with \( x_{j+1} \frac{M}{x_j} \) instead of \( M \). After finitely many steps we find \( M \) such that \( x_{i+1} \frac{M}{x_i} \in U \) for every \( i \in I \), since we reach \( x_{n-1}x_n^{d-1} \) and this monomial is only divisible by \( x_{n-1} \) and \( x_n \) and \( x_n \frac{x_n-x_n^{d-1}}{x_{n-1}} = x_n \in U \neq \{0\} \).

**Proposition 3.8.11.** For every \( k \geq 2 \) we have

\[
m(k, k + 1, k + 1) < m(k, k, k) + \binom{k+1}{2}
\]

**Proof.** It is enough to show \( m(k, k+1, k+1) < m(k, k+1, k) + \binom{k+1}{2} \) since by Corollary 3.5.5 we have \( m(k, k+1, k) = m(k, k, k) \). Again we denote the degree by \( d := k+1 \). Let \( U \subseteq A(k)_d \) be a strongly stable subspace of codimension \( k + 1 \) realizing \( m(k, d, k+1) \) and let \( V \subseteq A(k)_d \) be any strongly stable subspace of codimension \( k \) containing \( U \). Such a subspace \( V \) always exists by Lemma 3.8.10.

Now we compare \( U^2 \) and \( V^2 \). We can write \( V = U \oplus \text{span}(M) \) for some monomial \( M \in A(k)_d \) and hence \( V^2 = U^2 + \text{span}(M)V \).

Let \( MT \in \text{span}(M)V \) for some monomial \( T \in V \). We claim that

\[
MT = \left( x_{p(T)} \frac{M}{x_1} \right) T^-
\]

is contained in \( U^2 \) except for at most \( \binom{k+1}{2} - 1 \) choices of \( T \). We note that the monomial \( x_{p(T)} \frac{M}{x_1} \) is always contained in \( U \) since \( U \) is strongly stable. After we show this, we are done as follows:

\[
dim V^2 \leq \dim U^2 + \binom{k+1}{2} - 1
\]
or equivalently
\[ \text{codim } V^2 \geq \text{codim } U^2 - \binom{k+1}{2} + 1, \]
hence
\[ m(k, k+1, k+1) = \text{codim } U^2 \leq \text{codim } V^2 + \binom{k+1}{2} - 1 \]
\[ \leq m(k, k+1, k) + \binom{k+1}{2} - 1, \]
which we wanted to show.

Whenever \( T^- \) is not contained in \( U \) this means that \( T \) is contained in \( M^+ \) for some monomial \( M \in A(k+1) \setminus U \). The subspace \( U \) has codimension \( k+1 \) and is contained in \( A(k+1) \). Therefore, it cannot be the orthogonal complement of \( \text{span}(x_1^d, x_1^{d-1}x_2, \ldots, x_1^{d-1}x_{k+1}) \) in \( A(k+1) \) as the variable \( x_{k+1} \) appears.

Let \( S \) be the set of all monomials in \( A(k) \) that are not contained in \( U \). It follows from Proposition 3.8.8 that \( | \bigcup_{M \in S} M^+ | < \binom{k+1}{2} + k \). This means that there are at most \( \binom{k+1}{2} + k - 1 \) monomials \( M \in A(k) \) such that \( M^- \notin U \).

Since \( U \subseteq V \) and \( V \) is strongly stable, all monomials not contained in \( V \) are also not contained \( U \), hence are in \( S \) and therefore also in \( \bigcup_{M \in S} M^+ \) by Proposition 3.8.8.

We can now finish the proof: Let \( T \in V \) such that \( T^- \notin U \). This means
\[ T \in \left( \bigcup_{M \in S} M^+ \right) \cap V = \left( \bigcup_{M \in S} M^+ \right) \setminus V^\perp \]
and we just showed that the set on the right-hand side has cardinality at most
\[ \left( \binom{k+1}{2} + k - 1 \right) - k = \binom{k+1}{2} - 1. \]

\[ \square \]

**Theorem 3.8.12.** Let \( k \leq d - 1 \). Then for every \( n \geq 2 \) and every base-point-free subspace \( U \subseteq A(n) \) of codimension \( k \) we have
\[ \text{codim } U^2 \leq 2k^2 + \binom{k+2}{3} = \frac{1}{6}(k^3 + 15k^2 + 2k). \]

*Proof.* From Theorem 3.7.9 we know \( \text{codim } U^2 \leq m(3k, k, k) \). From Lemma 3.8.1 we see \( m(3k, k, k) \leq 2k^2 + m(k, k, k) \) and Theorem 3.8.3 shows \( m(k, k, k) = \binom{k+2}{3} \). \( \square \)

**Theorem 3.8.13.** Let \( k \geq 1 \) and let \( n, d \geq k \). Then \( m(n, d, k) = \binom{k+2}{3} + (n - k)k \).
Furthermore the only strongly stable subspace \( U \subseteq A_d \) realizing \( m(n, d, k) \) is given by
\[ U = \text{span}(x_1^d, x_1^{d-1}x_2, \ldots, x_1^{d-1}x_{k+1}) \perp \subseteq A_d. \]
Proof. For $k = 1$, this follows from Lemma 3.4.2 and the fact that there only exists a single strongly stable subspace of codimension 1. For $k \geq 2$ we calculate

\[
m(n, d, k) \leq m(k, d, k) + (n-k)k
\]

and equality holds if and only if $f$.

For the first equality, we also use Remark 3.4.6 and the fact that the Hilbert function of the ideal generated by the subspace $U = \text{span}(x_1^d, x_1^{d-1}x_2, \ldots, x_1^{d-1}x_k)^\perp \subseteq A(k)_k$ is $k$ for any degree at least $k$: from Corollary 3.3.4 we see that the Hilbert function is at most $k$ in those degrees, but we immediately check that $x_1^d, x_1^{d-1}x_2, \ldots, x_1^{d-1}x_k$ are not contained in $U A_{d-k}$.

Let $U \subseteq A_d$ be a strongly stable subspace of codimension $k$ realizing $m(n, d, k)$. By Lemma 3.1.8 every monomial not contained in $U$ is contained in $A(k)_d$, hence codim $V = k$ where $V := U \cap A(k)_d$. Now we have

\[
m(k, k, k) + (n-k)k \leq \text{codim } U^2 \leq m(k, d, k) + (n-k)k \leq m(k, k, k) + (n-k)k,
\]

and therefore both inequalities are equalities and we have codim $V^2 = m(k, k, k)$. We note that $V \subseteq A(k)_d$, and $d$ might still be larger than $k$. By the last part of the proof of Theorem 3.5.2 we have $m(k, k, k) = \text{codim } V^2 \leq \text{codim}(V : x_1^{d-k})^2 \leq m(k, k, k)$. Especially, $(V : x_1^{d-k}) \subseteq A(k)_k$ is a subspace of codimension $k$ realizing $m(k, k, k)$ and thus it is equal to $\text{span}(x_1^k, \ldots, x_1^{k-1}x_k)^\perp \subseteq A(k)_k$. This shows $V^\perp = x_1^d \cdot \text{span}(x_1^k, \ldots, x_1^{k-1}x_k) \subseteq A(k)_d$ and thus also $U = \text{span}(x_1^d, \ldots, x_1^{d-1}x_k)^\perp \subseteq A_d$.

\[\square\]

3.9 Application to Gram spectrahedra

After showing results about squares of subspaces in the last sections, we now want to interpret their meaning for Gram spectrahedra.

Firstly, we can use our results about subspaces of codimension 1 to show that the largest (non-trivial) faces of Gram spectrahedra only appear for forms on the boundary of the psd cone.

Proposition 3.9.1. Let $f \in \Sigma_{n,2d}$, and $F \subseteq \text{Gram}(f)$ a face. Let $\varnothing$ be a relative interior point of $F$ and $U = \text{im } \varnothing$. If $U \neq \mathbb{R}[x]_d$, then

\[
\dim F \leq \left(\frac{\dim \mathbb{R}[x]_d}{2}\right) - \dim \mathbb{R}[x]_{2d} + n
\]

and equality holds if and only if $f \in \partial P_{n,2d}$ with exactly one real zero and $F = \text{Gram}(f)$ or if $n = 2$ and codim $U = 1$. 

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Proof. By Proposition 3.1.14 there exists a strongly stable subspace \( V \subseteq \mathbb{R}[x]_d \) such that \( \text{codim } U^2 \leq \text{codim } V^2 \) and \( \dim U = \dim V \). Since \( V \) is strongly stable, \( V \) has \((1 : 0 : \cdots : 0)\) as a base-point.

Let \( L := \text{span}(x_1^d, x_1^{d-2}x_2) \perp \). If \( \text{codim } U \geq 2 \), then \( V \subseteq L \) since \( V \) is strongly stable. Let \( W \) be any linear complement of \( V \) in \( L \), then \( L = V \oplus W \). Hence
\[
L^2 = V^2 + VW + W^2
\]
and it follows that
\[
\dim(L^2) \leq \dim(V^2) + \dim(VW) + \dim(W^2) \\
\leq \dim(V^2) + \dim(V) \dim(W) + \left( \frac{\dim(W) + 1}{2} \right).
\]

We calculate the dimension of the face \( F \) as follows
\[
\dim F = \left( \frac{\dim(U) + 1}{2} \right) - \dim(U^2) \\
\leq \left( \frac{\dim(U) + 1}{2} \right) - \dim(V^2) \\
\leq \left( \frac{\dim(U) + 1}{2} \right) + \dim(V) \dim(W) + \left( \frac{\dim(W) + 1}{2} \right) - \dim(L^2) \\
= \left( \frac{\dim(L) + 1}{2} \right) - \dim(L^2)
\]
(\text{Lemma 3.4.2})
\[
= \left( \frac{\dim \mathbb{R}[x]_d - 1}{2} \right) - \dim \mathbb{R}[x]_{2d} + 2n \\
< \left( \frac{\dim \mathbb{R}[x]_d}{2} \right) - \dim \mathbb{R}[x]_{2d} + n
\]
where the last inequality is easy to check.

If \( \text{codim } U = 1 \), then \( V = \text{span}(x_1^d)^\perp \) and we get
\[
\dim F = \left( \frac{\dim(U) + 1}{2} \right) - \dim(U^2) \quad \tag{3.2}
\leq \left( \frac{\dim(U) + 1}{2} \right) - \dim(V^2) \quad \tag{3.3}
= \left( \frac{\dim \mathbb{R}[x]_d}{2} \right) - \dim \mathbb{R}[x]_{2d} + n. \quad \tag{3.4}
\]

We thus showed that equality can only hold if \( \text{codim } U = 1 \). Assume that \( f \in \text{int } \Sigma_{n,2d} \) and \( F \subseteq \text{Gram}(F) \) is a face of corank 1 with corresponding subspace \( U \). By Proposition 3.4.3 it follows that \( \text{codim } U^2 \leq 2 \), hence the inequality in (3.4) is strict whenever \( n \geq 3 \). By Corollary 4.1.10 no form in \( \partial \Sigma_{n,2d} \setminus \partial P_{n,2d} \) has a face of corank 1. Hence, equality can only hold if \( f \in \partial P_{n,2d} \) and \( \text{Gram}(f) \) has a face of corank 1.

But since no sos form in \( \partial P_{n,2d} \) has a Gram tensor of rank \( \dim \mathbb{R}[x]_d \), it follows that this face of corank 1 is the whole Gram spectrahedron. Especially, the form can only have one real zero.

In this case, the equality \( \dim F = \left( \frac{\dim \mathbb{R}[x]_d}{2} \right) - \dim \mathbb{R}[x]_{2d} + n \) does hold by Lemma 3.4.2.
Example 3.9.2. Equality in the last proposition holds for example in the following situation. Let $U \subseteq \mathbb{R}[x_d]$ be any subspace of codimension 1 such that $V(U)$ contains exactly one real point, for example $U = \text{span}(x_1^2)$, and let $p_1, \ldots, p_r$ be a basis of $U$. Then the Gram spectrahedron of $f := \sum_{i=1}^r p_i \otimes p_i$ has dimension $\left( \dim \mathbb{R}[x_d] - k \right) - \dim \mathbb{R}[x_d] + n$ by Lemma 3.4.2.

We now continue discussing the bound $m(n, d, k)$ and what we showed in the last sections. First, we get the following from Theorem 3.8.12.

Theorem 3.9.3. Let $f \in \Sigma_{n, 2d}$ be non-singular. If $F \subseteq \text{Gram}(f)$ is a face of corank $k$ with $1 \leq k \leq d - 1$, then

$$\dim F \leq \left( \frac{\dim \mathbb{R}[x_d] - k + 1}{2} \right) - \dim \mathbb{R}[x_d] + 2k^2 + \left( k + \frac{2}{3} \right).$$

Next, we compare this bound to the bound we get from $m(n, d, k)$. I.e. we compare Gram spectrahedra of singular forms to Gram spectrahedra of non-singular forms. We always assume $1 \leq k \leq d - 1$.

Remark 3.9.4 (singular form). Let $U \subseteq \mathbb{R}[x_d]$ be a strongly stable subspace of codimension $k$ realizing $m(n, d, k)$ and let $f \in \text{int} \Sigma U^2$. Since $U$ has a real base-point, the form $f$ lies on the boundary of the psd cone $P_{n, 2d}$. The Gram spectrahedron $\text{Gram}(f)$ has a face corresponding to the subspace $U$ since $f \in \text{int} \Sigma U^2$ (Proposition 2.3.13). Especially, this face $F$ has dimension

$$\dim F = \left( \frac{\dim \mathbb{R}[x_d] - k + 1}{2} \right) - \dim \mathbb{R}[x_d] + m(n, d, k) \quad \text{ (Remark 1.1.19)} \geq \left( \frac{\dim \mathbb{R}[x_d] - k + 1}{2} \right) - \dim \mathbb{R}[x_d] + kn.$$

Remark 3.9.5 (non-singular form). Let $f \in \Sigma_{n, 2d}$ be a non-singular form and let $F \subseteq \text{Gram}(f)$ be a face with corresponding subspace $U$ of codimension $k$. Using Theorem 3.9.3 we have an upper bound for the dimension of $F$,

$$\dim F = \left( \frac{\dim \mathbb{R}[x_d] - k + 1}{2} \right) - \dim \mathbb{R}[x_d] + \text{codim } U^2 \leq \left( \frac{\dim \mathbb{R}[x_d] - k + 1}{2} \right) - \dim \mathbb{R}[x_d] + 2k^2 + \left( \frac{k + 2}{3} \right).$$

We note that although the upper bound for the codimension of $U^2$ is independent of $n$, the dimension of $F$ certainly is not. This is not surprising as the dimension of the kernel of the map $\mathcal{S}_2 U \overset{\partial}{\to} \mathbb{R}[x_d]$ has to depend on $n$ if the codimension of $U$ is small.

Remark 3.9.6. We see that for large enough $n$, the dimensional differences between faces of Gram spectrahedra of singular and non-singular forms are arbitrarily large. Moreover for $n$ large and any $k < d$, every face of corank $k$ of the Gram spectrahedron of the singular form has a higher dimension than any face of corank $k$ of the Gram spectrahedron of the non-singular form. Indeed, the codimension of $U^2$ for any base-point-free subspace is bounded by $2k^2 + \left( \frac{k + 2}{3} \right)$, whereas $\text{codim } U^2 \geq kn$ if $U$ has a base-point.

Next, we comment on the sum of squares representations corresponding to faces of small corank.
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Remark 3.9.7. Let $n \geq 3$, $d \geq 3$ and let $r > \dim \Sym^{2d} - (3d - 2)$. Then every form in $\Sym^{2d}$ has a Gram tensor of rank $\leq r$. This can be seen as follows: Consider the sum of squares map given by
\[
\phi: \Sym^{r} \to \Sym^{2d}, \quad p \mapsto \sum_{i=1}^{r} p_{i}^{2}.
\]
Its differential at the point $q = (q_{1}, \ldots, q_{r}) \in \Sym^{r}$ is given by
\[
d\phi(q): \Sym^{r} \to \Sym^{2d}, \quad p \mapsto 2 \sum_{i=1}^{r} p_{i} q_{i}.
\]
Let $U = \text{span}(q_{1}, \ldots, q_{r})$, then \(\text{codim} U < 3d - 2\). Moreover, $U$ is base-point-free if $q_{1}, \ldots, q_{r}$ have no common zero in $\mathbb{P}^{n-1}$. Therefore, by Theorem 3.3.6 the differential is surjective at $q$ as long as $q_{1}, \ldots, q_{r}$ do not have a common zero. Now we may finish the proof as in [Sch17, 4.6.], which is also explicitly stated in [BSSV19, Theorem 2.2.] in terms of the Pythagoras number.

Remark 3.9.8. (i) We note that understanding powers of subspaces is also interesting in itself since the coordinate ring of the image of the rational map
\[
\mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{r}, \quad x \mapsto (f_{1}(x) : \cdots : f_{r}(x))
\]
is isomorphic to $\bigoplus_{i=0}^{\infty} U^{i}$ if $f_{1}, \ldots, f_{r} \in A_{d}$ and $U = \text{span}(f_{1}, \ldots, f_{r})$. Here $U^{i}$ denotes the subspace spanned by all products $f_{j_{1}} \cdots f_{j_{i}}$ with $1 \leq j_{1} \leq \cdots \leq j_{i} \leq n$ (cf. [BC18]).

From this point of view we only considered the degree 2 component of this ring and saw that the dimension of $U^{2}$ is minimal if and only if the map is not a morphism (if $r \geq \dim A_{d} - d + 1$ and $n$ is large enough).

(ii) Similarly, understanding the Hilbert functions of base-point-free subspaces is also an interesting topic. Here we give Blekherman’s paper [Ble15] as a reference where also several applications to optimization are given. He studies the conditions under which a base-point-free subspace $U \subseteq A_{d}$ satisfies $UA_{d} \neq A_{2d}$ (see Theorem 3.3.6).

(iii) We mention again the EGH conjecture (see Remark 3.4.10) where one is interested in understanding Hilbert functions of ideals containing regular sequences using monomial ideals. Possible Hilbert function of homogeneous ideals $I \subseteq A$ have been completely characterized by Macaulay. However, adding the condition in Remark 3.4.10 that the monomial ideal contains a regular sequence in the same degrees, the problem becomes much harder.

This is very similar to our situation, as finding bounds for \(\text{codim} U^{2}\) is rather easy if we allow subspaces to have base-points, but becomes more difficult if we require $U$ to be base-point-free.

Next, we study some semi-algebraic properties of Gram spectrahedra. For this, we show Proposition 3.9.11 which is a fiber theorem for semi-algebraic maps and a generalization of [Sch Corollary 5.4.].

Proposition 3.9.9. Let $S \subseteq \mathbb{R}^{n}$, $T \subseteq \mathbb{R}^{m}$ be semi-algebraic sets. Let $T$ be convex and $\phi: S \to T$ be a continuous, semi-algebraic map (i.e. its graph is a semi-algebraic subset of $\mathbb{R}^{n+m}$) such that the image is dense in $T$. Then there exists an open, dense, semi-algebraic subset $U \subseteq T$ such that for every $x \in U$, the dimension of the fiber is given by
\[
\dim \phi^{-1}(x) = \dim S - \dim T.
\]
Proof. By [BCR98, Theorem 9.3.2 (Semi-algebraic triviality)] there exist semi-algebraic subsets $T_1, \ldots, T_r$ of $T$ such that $T = \bigcup_{l=1}^r T_l$ and homeomorphisms $\vartheta_l : T_l \times F_l \to \phi^{-1}(T_l)$ with semi-algebraic sets $F_1, \ldots, F_r$.

From this we see that for every $l \in \{1, \ldots, r\}$ we have

$$\dim \phi^{-1}(T_l) = \dim T_l + \dim F_l.$$

Let $U \subseteq T$ be the union of all the sets $\text{int}(T_l)$ ($l = 1, \ldots, r$). Let $T_l$ be such that $\text{int}(T_l) \neq \emptyset$. We show that $\dim F_l = \dim S - \dim T$. Since $\text{int}(T_l) \neq \emptyset$, there exists a ball $B_\varepsilon$ of radius $\varepsilon > 0$ inside $T_l$. Since $f$ is continuous there also exists a ball with radius $\delta > 0$ such that $\phi^{-1}(B_\delta) \subseteq \phi^{-1}(T_l)$ which shows that the interior of $\phi^{-1}(T_l)$ is non-empty and thus $\dim \phi^{-1}(T_l) = \dim S$. Therefore, we have

$$\dim F_l = \dim \phi^{-1}(T_l) - \dim T_l = \dim S - \dim T.$$

This shows that $\dim \phi^{-1}(x) = \dim S - \dim T$ holds for every $x \in U$.

Since $T$ is semi-algebraic and convex, it is regular, that is $\text{int}(T) = T$. Let $x \in T$, and let $B$ be any open ball around $x$. Then $B$ contains an interior point of $T$ since $T$ is regular, and therefore also a ball $B'$ contained in the interior of $T$. Since the interior of $B'$ is non-empty, $B'$ cannot be contained in $T \setminus U$. It follows that there exists a point in $B' \cap U$ and hence $x$ is contained in the closure of $U$.

\[\square\]

Remark 3.9.10. The proof of Proposition 3.9.9 also shows that if $T$ is not convex, the result still holds in a slightly weaker way. That is, the subset $U$ is no longer dense in $T$, however, it still is a open, semi-algebraic subset with non-empty interior.

For any degree $d \geq 1$ and any integers $r, s \geq 0$ let $W_{r,s} \subseteq \mathcal{S}_d^+ \mathbb{R}[x]_{2d}$ be the semi-algebraic set consisting of all psd tensors $\vartheta \in \mathcal{S}_d \mathbb{R}[x]_{2d}$ of rank $r$ such that the map $\mathcal{S}_2 \text{im}(\vartheta) \to \mathbb{R}[x]_{2d}$ has a kernel of dimension $s$. For any $f \in \Sigma_{n,2d}$ we denote by $S_f(r,s)$ the fiber of $f$ under the map $W_{r,s} \to \Sigma_{n,2d}$. Hence, $S_f(r,s)$ is the set of all psd Gram tensors of $f$ of rank $r$ that lie in the relative interior of a face of dimension $s$.

Proposition 3.9.11. Let $r, s \geq 0$. If the image of the map $W_{r,s} \to \Sigma_{n,2d}$ is dense in $\Sigma_{n,2d}$, there exists an open, dense, semi-algebraic subset $U \subseteq \Sigma_{n,2d}$ such that for every $f \in U$ the dimension of the fiber is given by

$$\dim S_f(r,s) = \dim W_{r,s} - \mathbb{R}[x]_{2d}.$$

Especially, if $W_{r,s}$ is dense in the set of all rank $r$ psd tensors in $\mathcal{S}_2 \mathbb{R}[x]_{2d}$, we have

$$\dim S_f(r,s) = r \dim \mathbb{R}[x]_{2d} - \binom{r}{2} - \dim \mathbb{R}[x]_{2d}.$$

Proof. We use Proposition 3.9.9 on the Gram map $\mu$. The cone $\Sigma_{n,2d}$ is full-dimensional inside $\mathbb{R}[x]_{2d}$, hence the dimensions are equal. Let $W_r$ be the set of all psd tensors of rank $r$ inside $\mathcal{S}_2 \mathbb{R}[x]_{2d}$. The dimension of $W_r$ is given by $r \dim \mathbb{R}[x]_{2d} - \binom{r}{2}$: this is well-known for matrices, where it can be seen as follows. Every rank $r$ psd matrix can be factorized as $H^T H$ for a $r \times \dim \mathbb{R}[x]_{2d}$ real matrix $H$ and $G^T G$ (where $G$ is another $r \times \dim \mathbb{R}[x]_{2d}$ real matrix) holds if and only if there is an orthogonal matrix $O \in \text{O}(r)$ such that $OH = G$. As the dimension of $\text{O}(r)$ is given by $\binom{r}{2}$, the claim follows. If $W_{r,s} \subseteq W_r$ is dense, the two sets have the same dimension.

\[\square\]
Next, we show that the algebraic boundary of Gram spectrahedra of generic forms is a ruled hypersurface in the following sense. Let \( s = \max(0, (\dim \mathbb{R}[x])_2 - \dim \mathbb{R}[x]_{2d}) \) and \( r = \dim \mathbb{R}[x]_d \). Let \( f \in \Sigma_{n, 2d} \) and consider the set \( X \) of all subspaces \( U \) of dimension \( r \) such that \( U \) is \( f \)-facial, i.e. there is a point \( \vartheta \in \text{Gram}(f) \) with \( \text{im}(\vartheta) = U \), and such that \( \dim \ker(\mathcal{S}_U \to U^2) = s \), or equivalently \( \vartheta \) lies in the relative interior of a face of rank \( r \) and dimension \( s \). For every \( U \in X \) the affine hull of the corresponding face on \( \text{Gram}(f) \) is an affine subspace of dimension \( s \). We claim that the union of all these subspaces is Zariski-dense in the algebraic boundary of \( \text{Gram}(f) \).

We note that \( s \) is the minimal possible dimension of a face of rank \( r \). It holds that \( s = (\dim \mathbb{R}[x]_2) - \dim \mathbb{R}[x]_{2d} \) if \( n \geq 3 \), \( d \geq 2 \) or \( n = 2 \) and \( d \geq 4 \) and \( s = 0 \) if \( n = 2 \) and \( d = 2, 3 \).

**Lemma 3.9.12.** There exists a subspace \( U \subseteq \mathbb{R}[x]_d \) of codimension 1 such that

- \( U^2 = \mathbb{R}[x]_{2d} \) if \( n \geq 3 \), \( d \geq 2 \) or \( n = 2 \) and \( d \geq 4 \),
- \( \dim U^2 = 3 \) if \( n = 2 \) and \( d = 2 \), and
- \( \dim U^2 = 6 \) if \( n = 2 \) and \( d = 3 \).

Moreover, in the cases \( n = 2 \) and \( d = 2, 3 \) this is the maximal possible dimension for \( U^2 \) for any subspace of codimension 1.

**Proof.** From the proof of Lemma 3.1.25 we know that in the case \( n \geq 3 \), \( d \geq 3 \) we may choose \( U = \text{span}(x_1^{d-2}, x_2, x_3) \subseteq A_d \). In the case \( n \geq 3 \) and \( d = 2 \) we may choose \( U = \text{span}(x_1^2 + x_2^2, x_3^2) \subseteq A_2 \) by Proposition 4.1.5. If \( n = 2 \) and \( d \geq 4 \) consider \( U = \text{span}(x_1^{d-2}, x_2^2) \subseteq A(2)_d \).

Then with the same argument as in Lemma 3.1.25 we see that \( U^2 = \mathbb{R}[x]_{2d} \).

Let \( n = 2 \), \( d = 2 \) and let \( U = \text{span}(x_1^2, x_2^2) \). Then \( \dim U^2 = 3 \). For \( n = 2 \) and \( d = 3 \) we choose \( U = \text{span}(x_1^2, x_2^2, x_1 x_2) \) and calculate \( \dim U^2 = 6 \).

This shows that in these two cases the Gram map \( \mathcal{S}_U \colon \mathbb{R}[x]_d \to \mathbb{R}[x]_{2d} \) is injective as the dimension of the space on the left is 3 and 6 respectively. Hence, this is the maximal dimension for \( U^2 \) for any subspace \( U \) of codimension 1.

**Proposition 3.9.13.** Let \( f \in \Sigma_{n, 2d} \) be generic. Let \( r = \dim \mathbb{R}[x]_d - 1 \) and let \( s = \max(0, (\dim \mathbb{R}[x]_2) - \dim \mathbb{R}[x]_{2d}) \). Then \( S_f(r, s) \) is dense in \( \partial \text{Gram}(f) \) (in the euclidean topology).

Especially, the Zariski-closure of \( S_f(r, s) \) is equal to the algebraic boundary \( \partial^0 \text{Gram}(f) \).

**Proof.** We already know that rank \( r \) points are dense in the boundary by Corollary 2.1.6. Moreover, we showed in Proposition 2.3.22 that a generic subspace \( U \subseteq \mathbb{R}[x]_d \) of codimension 1 maximizes \( \dim U^2 \). This maximal value is given in Lemma 3.9.12. If \( f \) has a face corresponding to such a subspace, the dimension of this face is given by \( s \). We claim that for generic \( f \), points in the relative interior of such faces are dense in the boundary of \( \text{Gram}(f) \).

For this, we show that rank \( r \) points that are in the relative interior of faces of dimension \( s + i \) for some \( i \geq 1 \) do not form a semi-algebraic set of the same dimension as the boundary.

We use a slight variation of the proof of Proposition 3.9.11. With the same notation as in Proposition 3.9.11 let \( i \geq 1 \) and consider the map \( W_{r, s+i} \to \Sigma_{n, 2d} \). If the image is nowhere dense, then a generic \( f \in \Sigma_{n, 2d} \) does not have a face of dimension \( s + i \) and rank \( r \). Hence, assume that the image \( \mu(W_{r, s+i}) \) is a full-dimensional semi-algebraic set. Then there exists a
(convex) ball $B$ in the image. Now consider the map $\mu^{-1}(B) \xrightarrow{\mu} B$. Using Proposition 3.9.9 we see that for a generic $f \in B$ the fiber has dimension

$$\dim(\mu^{-1}(B)) - \dim B \leq \dim W_{r,s+1} - \dim \mathbb{R}[x]_{2d}.$$ 

The dimension of the boundary of $\text{Gram}(f)$ is given by

$$\dim \text{Gram}(f) - 1 = \binom{r+2}{2} - \dim \mathbb{R}[x]_{2d} - 1.$$ 

We now show that the dimension of the boundary is strictly greater than the dimension of the fiber. From this we see that the union of all rank $r$ points that are not in the relative interior of a face of dimension $s$ has dimension strictly smaller than the dimension of the boundary, hence is nowhere dense. Especially, $S_f(r,s)$ is dense since rank $r$ points are dense.

Let $W \subseteq S_2 \mathbb{R}[x]_d$ be the set of all rank $r$ tensors. It is well-known that the Zariski-closure of $W$ is irreducible. The set of all rank $r$ tensors such that $\dim(\text{im } \vartheta)^2$ is maximal is Zariski-open in $W$: indeed, we may assume $\vartheta$ is given as $\vartheta = \sum_{i=1}^r p_i \otimes p_i$. If $\text{span}(p_1, \ldots, p_r)$ is generic, $\dim(\text{im } \vartheta)^2$ is maximal. Hence, the sets $W_{r,s+1}$ are contained in a hypersurface (inside $W$) and satisfy $\dim W_{r,s+1} < \dim W = r(r+1) - \binom{r}{2}$. Especially

$$\dim \partial \text{Gram}(f) = \binom{r+2}{2} - \dim \mathbb{R}[x]_{2d} - 1$$ 

$$= r(r+1) - \binom{r}{2} - \dim \mathbb{R}[x]_{2d}$$ 

$$= \dim W - \dim \mathbb{R}[x]_{2d}$$ 

$$> \dim W_{r,s+1} - \dim \mathbb{R}[x]_{2d}$$ 

$$\geq \dim(\mu^{-1}(B)) - \dim B$$

which finishes the proof. 

This finishes our discussion about general Gram spectrahedra. We now continue by studying Gram spectrahedra of ternary quartics.
4 Ternary quartics

Sum of squares representations of ternary quartics have already been investigated by Hilbert \cite{Hil88} in 1888, who showed that every real psd ternary quartic can also be written as a sum of squares, and every such quartic can be written as a sum of three squares (over $\mathbb{R}$). Later it was shown in \cite{PRSS04} that every smooth psd quartic admits exactly eight such representations as a sum of three squares (up to orthogonal equivalence). Ternary quartics, or geometrically, plane curves of degree four have already been investigated in the 19th century by many mathematicians, including Salmon \cite{Sal79}, Plücker \cite{Pl34}, Cayley, and Hesse. A modern treatise of the topic can be found in Dolgachev’s book \cite{Dol12} which we regularly use as a reference. For a less technical point of view, we also mention \cite{PSV11} for which some of the following results should be seen as a follow-up.

Let $f \in \Sigma_3, 4$ be a generic ternary quartic. We determine the dimension of the semi-algebraic set consisting of all extreme points of fixed rank. As well as the dimension of the set of all rank 5 psd Gram tensors that lie in the interior of a face of dimension 2. Except for faces of rank 4 and dimension 1, the ones above are the only possible combinations for the dimension and rank of any face, if the quartic is smooth. We take a separate look at faces of rank 4 and dimension 1 to find conditions on the quartic such that its Gram spectrahedron contains such a face.

4.1 Faces of corank 1 for quartic forms

We first turn our attention to quartic forms in any number of variables $n \geq 2$ and improve the bound $\text{codim } U^2 \leq 2$ for subspaces $U \subseteq A_2$ of codimension 1. We show that such a subspace never satisfies $\text{codim } U^2 = 1$.

Remark 4.1.1. If $n = 2$, then $\dim A_2 = 3$ and for any subspace $U$ of codimension 1 we have $\text{codim } U^2 = 2$.

If $U$ has a base-point, it follows from Lemma 3.4.2 that $\text{codim } U^2 = 2$. If $U$ is base-point-free, then any basis of $U$ is a regular sequence in $A$ and thus $\dim U^2 = 3$ or equivalently $\text{codim } U^2 = 2$.

For the rest of this section, we assume $n \geq 3$. The aim is to show the following theorem.

Theorem 4.1.2. Let $f \in \Sigma_{n, A}$, and let $F \subseteq \text{Gram}(f)$ be a face with corresponding subspace $U \subseteq \mathbb{R}[z]_2$ such that $\text{codim } U = 1$. Let $c = (\dim \mathbb{R}[z]_2) - \dim \mathbb{R}[z]_4$. Then either

(i) $f$ has a real zero, $F = \text{Gram}(f)$, and $\dim F = c + n$, or

(ii) $f \in \text{int } \Sigma_{n, A}$, $F \neq \text{Gram}(f)$, and $\dim F \in \{c, c + 2\}$.

Furthermore, both cases occur for every $n \geq 3$. Especially, such $f$ never satisfies $f \in \partial \Sigma_{n, A} \setminus \partial P_{n, A}$. 

Proposition 4.1.4. Let $U \subseteq A_2$ be a base-point-free subspace of codimension 1. Let $U^\perp = \text{span}(q)$ with $q \in A_2$ and $r := n - \text{rk}(q)$ where $\text{rk}(q)$ is the rank of $q$ as a quadratic form. After a change of coordinates

$$U = \text{span}(x_1, \ldots, x_r) \text{span}(x_1, \ldots, x_n) + \text{span}(x_ix_j : i, j \in \{r + 1, \ldots, n\}, i \neq j)$$

$$+ \text{span}(x_{r+1}^2 + x_{r+2}^2, \ldots, x_{r+1}^2 + x_n^2),$$

i.e. $U$ is spanned by all monomials of degree 2 in $A_2$ except for $x_{r+1}^2 + \ldots, x_{r+1}^2$, and the binomials $x_{r+1}^2 + x_{r+2}^2, \ldots, x_{r+1}^2 + x_n^2$.

Proof. After a change of coordinates $q = \sum_{i=r+1}^n x_i^2$. If $r + 1 = n$, then $q = x_n^2$ and thus $U$ has a base-point. By definition of $U^\perp = \text{span}(q)$, all monomials except for $x_{r+1}^2, \ldots, x_n^2$ are contained in $U$. Let $r + 1 \leq s \leq n$ then

$$\langle q, x_s^2 \rangle = 1,$$

and thus

$$\langle q, x_s^2 - x_{r+1}^2 \rangle = 0.$$

This means that for every $r + 2 \leq s \leq n$ the binomial $x_s^2 - x_{r+1}^2$ is also contained in $U$. Taking the change of coordinates $x_{r+1} \mapsto \sqrt{-1} x_{r+1}$ shows that $U$ has the required form. □

Proposition 4.1.5. Let $U \subseteq A_2$ be a base-point-free subspace of codimension 1 and let $U^\perp = \text{span}(q)$ for some $q \in A_2$. Then the following statements hold:

(i) If $\text{rk}(q) = 2$, then $\text{codim} U^2 = 2$,

(ii) if $\text{rk}(q) \geq 3$, then $U^2 = A_4$.

Proof. The case $\text{rk}(q) = 1$ is not possible since $U$ would have a base-point.

By Proposition 4.1.4 we can apply a change of coordinates such that $U$ has the form

$$U = \text{span}(x_1, \ldots, x_r) \text{span}(x_1, \ldots, x_n) + \text{span}(x_ix_j : i, j \in \{r + 1, \ldots, n\}, i \neq j)$$

$$+ \text{span}(x_{r+1}^2 + x_{r+2}^2, \ldots, x_{r+1}^2 + x_n^2)$$

where $r = n - \text{rk}(q)$. Thus $U$ is generated by all monomials in $A_2$ except for $x_{r+1}^2, \ldots, x_n^2$, and all the binomials $x_{r+1}^2 + x_{r+2}^2, \ldots, x_{r+1}^2 + x_n^2$.

(i): Since the quadratic form $q$ has rank 2, it is reducible. We write $q = l_1l_2$ for some $l_1, l_2 \in A_1$. Since $\text{rk}(q) = 2$, the two linear forms are linearly independent and after a change of coordinates we may assume that $q = x_1x_2$. This means $U$ is spanned by all monomials in $A_2$ except for $x_1x_2$.

We show that $U^2$ is spanned by all monomials of $A_4$ except for $x_1^2x_2$ and $x_1x_2^2$. Let $M \in A_4$ be any monomial. If $M$ is not divisible by $x_1x_2$, then there exists a decomposition $M = M_1M_2$ for two monomials $M_1, M_2 \in A_2$, and both are contained in $U$ as they are different from $x_1x_2$. If $M$ is divisible by any variable $x_i$ with $3 \leq i \leq n$ and by $x_1x_2$, we may write $M = x_1x_2x_ix_j$ for some $1 \leq j \leq n$. Two decompositions of $M$ are now given by $M = (x_1x_i)(x_2x_j) = (x_1x_j)(x_2x_i)$. If $j \neq 1$, the first decomposition shows that $M \in U^2$, if
4.1 FACES OF CORANK 1 FOR QUARTIC FORMS

\[ j = 1, \text{the second decomposition does. Hence, we may now assume} \ M \in \mathbb{C}[x_1, x_2]_4. \text{ Since}\ U \cap \mathbb{C}[x_1, x_2]_2 = \text{span}(x_1^2, x_2^2), \text{we immediately see that} \ M \text{ is contained in} U^2 \text{ if and only if} M \in \{x_1^4, x_2^4, x_1^2 x_2^2\}. \text{ Together this shows that} \ \text{codim} U^2 = 2 \text{ and the only monomials of degree 4 missing in} U^2 \text{ are} x_1^4 x_2 \text{ and} x_1 x_2^4.

(ii): \ Let \ M \in A_4 \text{ be a monomial such that each variable occurs to the power at most 2. Then it is a product of two monomials} x, x_j \ \text{and} \ x_k x_l \ \text{with} \ i \neq j \ \text{and} \ k \neq l. \text{ Hence,} \ M \in U^2. \\text{ Moreover,} \ x_1^4 x_j = x_1^2(x_1 x_j) \in U^2 \text{ for any} \ 1 \leq i \leq r \text{ and} \ 1 \leq j \leq n. \text{ We are therefore left to check the monomials} x_1^i x_j \text{ and} x_1^j x_i \text{ with} \ r+1 \leq i \leq n \text{ and} \ 1 \leq j \leq n.

Let \ u, v, w \text{ be three distinct variables such that} u^2 + v^2, u^2 + w^2 \in U. \\text{ Since monomials in which each variable appears to the power at most 2 are all contained in} U^2, \text{ we have}

\[
0 \equiv (2u^2 + v^2 + w^2) \equiv 4u^4 + v^4 + w^4 \mod U^2 \tag{4.1}
\]
\[
0 \equiv (u^2 + v^2) \equiv u^4 + v^4 \mod U^2 \tag{4.2}
\]
\[
0 \equiv (u^2 + v^2) \equiv u^4 + w^4 \mod U^2. \tag{4.3}
\]

Subtracting equation (4.2) and (4.3) from equation (4.1), we see \( u^4 \in U^2 \text{ and therefore also} v^4, w^4 \in U^2 \text{ by equation (4.2) and (4.3)}. \text{ Lastly, we have}

\[
(u^2 + w^2)uv \equiv u^3 v \mod U^2
\]
\[
(u^2 + v^2)uw \equiv v^3 w \mod U^2
\]
\[
(u^2 + v^2)uw \equiv u^3 w \mod U^2
\]
\[
(u^2 + u^2)uw \equiv u^3 w \mod U^2.
\]

\text{ Hence} u^4, v^4, w^4, u^3 v, u^3 w, v^3 w, w^3 \in U^2. \text{ By the above observation all fourth powers are contained in} U^2 \text{ and it is also easy to check that} x_1^2 x_j \in U^2 \text{ for all} \ r+1 \leq i \leq r, j \leq n \text{ using} \ r = n - \text{rk}(q) \leq n - 3.

\text{ It is left to check} x_1^2 x_j \text{ for} \ r+1 \leq i \leq n \text{ and} \ 1 \leq j \leq r. \text{ Since} \ r+1 \leq i \text{ there exists} \ r+1 \leq k \leq n \text{ with} \ k \neq i \text{ such that} x_1^2 + x_k^2 \in U. \\text{ Then} x_1^2 x_j \equiv x_1^2 x_j \mod U^2.

\text{ Therefore,} \ U^2 = A_4 \text{ as required.} \]

\textbf{Lemma 4.1.6. }\text{Let} \ U \subseteq A_4 \text{ be a base-point-free subspace of codimension} 1, \text{ and let} \ U^\perp = \text{span}(q) \text{ for some} \ q \in A_2. \text{ Then} \ r := n - \text{rk}(q) \equiv \dim \text{Soc}(A/\langle U \rangle) - 1.

\textbf{Proof. }\text{By Proposition 4.1.4 we can assume}

\[
U = \text{span}(x_1, \ldots, x_r) \text{span}(x_1, \ldots, x_n) + \text{span}(x_i x_j; i, j \in \{r+1, \ldots, n\}, i \neq j) + \text{span}(x_{r+1}^2 + x_{r+2}^2, \ldots, x_n^2).
\]

\text{Since} \ x_1, \ldots, x_r \text{ are contained in the socle of} A/\langle U \rangle \text{, and as the Hilbert function of} (U) \text{ is given by} (1, n, 1), \text{ there is also an element of degree 2 contained in the socle. In fact, this element is} \ q. \text{ Hence} \ r \leq \dim \text{Soc}(A/\langle U \rangle) - 1.

\text{ Since the Hilbert function of} (U) \text{ is} (1, n, 1), \text{ the degree 2 component of the socle can only have dimension 1. Let} \ l \in A_1 \text{ such that} \ l \in \text{Soc}(A/\langle U \rangle). \text{ Since} x_1, \ldots, x_r \text{ are contained in the}
socle we can assume that \( l = \sum_{i=r+1}^{n} a_i x_i \) for some \( a_{r+1}, \ldots, a_n \in \mathbb{C} \). Let \( s \in \{ r+1, \ldots, n \} \), then by assumption \( x_s l \in U \). We have

\[
a_s x_s^2 = x_s l - \sum_{i=r+1, i \neq s}^{n} a_i x_i x_s \in U.
\]

If \( a_s \) was non-zero for any \( s \in \{ r+1, \ldots, n \} \), then \( x_s^2 \in U \), hence \( U = A_2 \) by the description of \( U \) above. Therefore, \( l \in \text{span}(x_1, \ldots, x_r) \) and thus the socle has dimension \( r + 1 \). \( \square \)

**Corollary 4.1.7.** Let \( n = 3 \) and \( U \subseteq A_2 \) be a subspace of codimension 1. Then

\[
U^2 = A_4 \iff A/\langle U \rangle \text{ is Gorenstein.}
\]

**Proof.** Let \( U^\perp = \text{span}(q) \) for some \( q \in A_2 \). Both sides show that \( U \) has to be base-point-free. By Proposition 4.1.5 we have \( U^2 = A_4 \) if and only if \( 3 - \text{rk}(q) \leq n - 3 = 0 \), which is equivalent to \( \text{rk}(q) = 3 \). It follows from Lemma 4.1.6 that \( \text{rk}(q) = 3 \) is equivalent to \( \dim \text{Soc}(A/\langle U \rangle) = 1 \) which is equivalent to \( A/\langle U \rangle \) being Gorenstein. \( \square \)

**Remark 4.1.8.** If \( n = 3 \) and \( U \subseteq A_2 \) is a subspace of codimension 1, the fact that \( \text{codim} U^2 \neq 1 \) can also be checked with a computer algebra program. We have the Gram map \( S_2 U \xrightarrow{\partial} A_4 \) which is linear and therefore can be defined by a matrix after choosing bases for the two spaces.

After a change of coordinates we may assume that \( U \) does not contain the monomial \( x_3^3 \) and therefore a basis of \( U \) is given by

\[
x_1^2 + \lambda_1 x_2^2, x_1 x_2 + \lambda_2 x_3^2, x_1 x_3 + \lambda_3 x_2^2, x_2^2 + \lambda_4 x_3^2, x_2 x_3 + \lambda_5 x_3^2
\]

for some \( \lambda_1, \ldots, \lambda_5 \in \mathbb{C} \).

The matrix defining the Gram map has size \( 15 \times 15 \) and the entries are polynomials in \( \lambda_1, \ldots, \lambda_5 \) of degree at most 2.

We can then check with a computer that whenever the determinant vanishes, so do all the \( 14 \times 14 \) minors when we use indeterminates \( \lambda_1, \ldots, \lambda_5 \). This means that \( \text{codim} U^2 \neq 1 \).

For \( n = 4 \) this matrix already has size \( 35 \times 45 \) and thus we would need to check that if all \( 35 \times 35 \) minors vanish, the same holds for the \( 34 \times 34 \) minors. This is no longer easily possible.

For the next proposition, we use the following notation. For \( \alpha \in \mathbb{R}[x]_d \) define the bilinear form

\[
b_\alpha : \mathbb{R}[x]_{2d} \times \mathbb{R}[x]_{2d} \to \mathbb{R}, \quad (p, q) \mapsto \alpha(pq).
\]

**Proposition 4.1.9.** Let \( f \in \partial \Sigma^\bullet, \partial \in \text{int} \text{Gram}(f) \), and \( U := \text{im}(\partial) \). If \( \text{codim} U \leq 3d - 3 \) (\( d \geq 3 \)) or \( \text{codim} U \leq 5 \) (\( d = 2 \)) then \( f \in \partial \Sigma^\bullet, \partial \) and thus it has a real zero.

**Proof.** Let \( \Sigma := \Sigma^\bullet_{0,2d} \). Consider the face \( \Sigma^\bullet(U) := \{ \alpha \in \mathbb{R}[x]_{2d}; U \subseteq \ker b_\alpha \} \) of the dual cone \( \Sigma^\bullet \) and let \( \alpha \in \Sigma^\bullet(U) \) be an extreme ray. Consider the Gorenstein ideal with socle \( \alpha \) as in Definition 3.3.3

\[
I := I(\alpha) = \bigoplus_{s \geq 0} \{ p \in \mathbb{R}[x]_s ; s > 2d \text{ or } \alpha(pq) = 0 \ (\forall q \in \mathbb{R}[x]_{2d-s}) \}.
\]

By definition, we have \( U \subseteq I_d \) and thus \( \text{codim} I_d \leq 3d - 3 \) (resp. \( \leq 5 \)). Assume \( f \) has no real zero, then \( \alpha \) cannot be a point evaluation at a real point since \( \alpha(f) = 0 \). Since \( \alpha \) is
4.2 Introduction to ternary quartics

For the rest of this section, we fix \( n = 3 \). We keep the notation \( A = \mathbb{C}[x, y, z] \), \( \mathcal{T} = (x, y, z) \), \( \mathbb{R}[z] = \mathbb{R}[x, y, z] \) and write \( \Sigma := \Sigma_{3,4} \).

Since we will do some calculations in this section, we also sometimes fix a basis of \( \mathbb{R}[z]^3 \) and write Gram tensors as Gram matrices wrt to this basis. Whenever we do so, we use the monomial basis ordered as follows:

\[
\sum_{\alpha \in \mathbb{Z}_+^3, |\alpha| = 4} c_\alpha z^\alpha
\]

and \( c_\alpha \in \mathbb{R} \) for \( \alpha \in \mathbb{Z}_+^3 \). Denote by \( X \) the row vector \( (x^2, y^2, z^2, xy, xz, yz) \) containing the six monomials of degree 2. Then by definition \( X G X^T = f \) for every Gram matrix \( G \) of \( f \).

For a positive definite ternary quartic, the dimension of its Gram spectrahedron is 6 and wrt \( X \) is given by all matrices of the form above that are psd.

Next, we give an introduction to bitangents and determinantal representations of ternary quartics. A modern and more abstract point of view can be found in [Dol12]. We mostly use a similar notation to [PSV11].
Definition 4.2.1. Let \( f \in \mathbb{R}[x]_{4} \) be smooth and let \( L \subseteq \mathbb{P}^{2} \) be a line. Then \( L \) is called a bitangent of \( f \), if all intersection points of \( \mathcal{V}(f) \) and \( L \) have even multiplicity.

We refer to [Ful89] for an introduction to intersection numbers of plane curves.

An important tool for the main proof of Section 4.3 is Noether’s \( AF + BG \) Theorem, a special version of Lasker’s Theorem. A proof as well as several historical notes can be found in [EGH96] or [Ful89].

Theorem 4.2.2 (Noether’s \( AF + BG \) Theorem, [Ful89, section 5.5.], [EGH96, Theorem 8]). Let \( f, g \in \mathbb{C}[x] \) be forms of degrees \( d_{1}, d_{2} \) such that \( \mathcal{V}(f, g) \) is finite, and let \( h \in \mathbb{C}[x]_{d} \) with \( d \geq d_{1}, d_{2} \). If for every point \( P \in \mathcal{V}(f, h) \), the intersection multiplicity of \( f \) and \( h \) in \( P \) is at least the intersection multiplicity of \( f \) and \( g \) in \( P \), then there exist \( a \in \mathbb{C}[x]_{d-d_{1}} \) and \( b \in \mathbb{C}[x]_{d-d_{2}} \) such that \( h = af + bg \).

Remark 4.2.3. Let \( L = \mathcal{V}(l) \) for some linear form \( l \in A_{1} \). By Theorem 4.2.2 \( L \) is a bitangent of \( f \) if and only if there exist \( q \in A_{2} \) and \( h \in A_{3} \) such that \( f = q^{2} + hl \).

Most of the time, we work with such an equation rather than considering bitangents as subsets of \( \mathbb{P}^{2} \).

Notation 4.2.4. Let \( L = \mathcal{V}(l), l \in A_{1} \), be a bitangent of \( f \). For convenience, we also call the linear form \( l \) a bitangent of \( f \), as well as its projective point \([l] \in \mathbb{P}A_{1} \).

We then say that two bitangents \( l_{1}, l_{2} \in A_{1} \) are different if \( \mathcal{V}(l_{1}), \mathcal{V}(l_{2}) \subseteq \mathbb{P}^{2} \) are different lines.

Especially, when counting bitangents, this should be understood as counting lines \( L \subseteq \mathbb{P}^{2} \), and not counting forms \( l \in A_{1} \), as every non-zero scalar multiple of such a linear form is also a bitangent.

In 1834, Plücker showed that every smooth ternary quartic has exactly 28 different bitangents. As was more recently shown in [Leh05], the 28 bitangents uniquely determine the smooth quartic \( f \in A_{4} \) up to a non-zero scalar.

Theorem 4.2.5 ([Plü34]). If \( f \in \mathbb{R}[x]_{4} \) is smooth, then \( f \) has exactly 28 different bitangents.

Using the 28 bitangents of a smooth form \( f \in \Sigma \), it was shown in [PSVT11] that one can calculate all linear symmetric determinantal representations of \( f \), as well as all eight sos representations of \( f \) as a sum of three squares.

Definition 4.2.6. Let \( f \in \mathbb{R}[x]_{4} \). Then a linear symmetric determinantal representation of \( f \) is a matrix \( M = Ax + By + Cz \) such that \( f = \det(M) \), and \( A, B, C \) are symmetric \( 4 \times 4 \) matrices over \( \mathbb{C} \).

Let \( f \in \mathbb{R}[x]_{4} \), then two linear symmetric determinantal representations of \( f \) are called equivalent if they are conjugate to each other under the action of \( \text{GL}_{4}(\mathbb{C}) \).

Theorem 4.2.7 ([Hes55]). Every smooth quartic \( f \in \mathbb{R}[x]_{4} \) has exactly 36 inequivalent linear symmetric determinantal representations.

Definition 4.2.8. Let \( f \in \mathbb{R}[x]_{4} \) be a smooth quartic and fix one linear symmetric determinantal representation \( M = Ax + By + Cz \) of \( f \). The Cayley octad of \( M \) is defined as the eight solutions \( O_{1}, \ldots, O_{8} \in \mathbb{P}^{3} \) of the system \( uAu^T = uBu^T = uCu^T = 0 \) with \( u = (u_{0} : \cdots : u_{3}) \in \mathbb{P}^{3} \).
Intersecting three quadratic hypersurfaces in $\mathbb{P}^3$, we expect there to be exactly eight different intersection points $O_1, \ldots, O_8$. Using the fact that $f$ is smooth this is indeed true as proven in [Dol12 Proposition 6.3.3].

This enables us to enumerate the bitangents of $f$ in the following way.

**Proposition 4.2.9** ([PSV11 Proposition 3.3.]). Let $f \in \mathbb{R}[x]_4$ be a smooth quartic. Let $M$ be a linear symmetric determinantal representation of $f$ with Cayley octad $O_1, \ldots, O_8$. Then the 28 bitangents of $f$ are given by $L_{ij}$ with $L_{ij} = V(O_iMO_j^T)$, $1 \leq i < j \leq 8$.

**Remark 4.2.10.** (i) Since $M$ is symmetric, we have $O_iMO_j^T = O_jMO_i^T$ for every $i, j \in \{1, \ldots, 8\}$, $i \neq j$. Thus, for every $i, j \in \{1, \ldots, 8\}$, $i \neq j$, the line $L_{ij} = V(O_iMO_j^T) = L_{ji}$ is a bitangent of $f$ and we do not have to pay attention to the order of the indices.

(ii) With a fixed Cayley octad, we write $b_{ij} \in A_1$ ($1 \leq i < j \leq 8, i \neq j$) for the bitangents of $f$ where $L_{ij} = V(b_{ij})$ and set $b_{ij} := b_{ji}$ for all $1 \leq j < i \leq 8, i \neq j$. This is mostly for convenience to avoid unnecessary scaling later on.

(iii) This enumeration of the bitangents is dependent on the choice of a linear symmetric determinantal representation of $f$ and on the order of the Cayley octad.

(iv) By [Vin93 §8] every smooth real quartic has a real linear symmetric determinantal representation. The Cayley octad corresponding to this linear symmetric determinantal representation contains only real points or points appear in complex conjugate pairs.

**Remark 4.2.11.** Recall that we defined Gram tensors not only as the real tensors that are mapped to a form via the Gram map, but also the complex ones.

By Gram tensor we always mean a possibly complex one, and where necessary we say real Gram tensor to specify.

Next, we follow the algorithm in [PSV11 §5] to construct all 63 Gram tensors of $f$ and to determine combinatorially from a Cayley octad which are real.

**Definition 4.2.12.** Let $f \in \mathbb{R}[x]_4$. Then a quadratic symmetric determinantal representation (QDR) of $f$ is a matrix $Q = \left(\begin{array}{cc} q_0 & q_1 \\ q_1 & q_2 \end{array} \right)$ with $q_0, q_1, q_2 \in A_2$ such that $f = \det(Q)$.

**Proposition 4.2.13** ([Sal79 section 254], [PSV11 Proposition 5.7.]). Let $f \in \mathbb{R}[x]_4$ be a smooth quartic and let $Q$ be a QDR of $f$. Then the variety $\{\lambda Q\lambda^T : \lambda \in \mathbb{P}^1\} \subseteq \mathbb{P}A_2$ contains exactly six products of two bitangents of $f$. I.e. there exist twelve bitangents $l_1, l_1', \ldots, l_6, l_6' \in A_1$ of $f$ such that $[l_i, l'_i] \in \lambda Q\lambda^T$.

4.2.14. Let $f \in \mathbb{R}[x]_4$ be a smooth quartic, and let $Q = \left(\begin{array}{cc} q_0 & q_1 \\ q_1 & q_2 \end{array} \right)$ be a QDR of $f$.

This means $f = q_0q_2 - q_1^2$ which gives rise to a sos representation of $f$ over $\mathbb{C}$ as follows

$$f = \left(\frac{1}{2}q_0 + \frac{1}{2}q_2\right)^2 + \left(\frac{1}{2}q_0 - \frac{1}{2}q_2\right)^2 + (iq_1)^2.$$

We call the Gram tensor $\vartheta$ corresponding to this sos representation of $f$ the Gram tensor corresponding to $Q$. Then $\text{im}_\mathbb{C} \vartheta = \text{span}(q_0, q_1, q_2)$. $\vartheta$ is well-defined: let $f = p_0p_2 + p_1^2$ ($p_0, p_1, p_2 \in A_2$) be any other representation of $f$ such that $\text{span}(p_0, p_1, p_2) = \text{span}(q_0, q_1, q_2)$. Then the Gram tensor corresponding to this equation has the same image as $\vartheta$. As the map $S_2 \text{im}_\mathbb{C}(\vartheta) \xrightarrow{\cong} A_2$ is injective, we see that there is only one Gram tensor with this image, hence the two are equal.
Corresponding to every quadratic symmetric determinantal representation of a form \( f \), we cannot only associate a Gram tensor of \( f \) as above, but also a Steiner complex. These are also in one-to-one correspondence with quadratic symmetric determinantal representations of \( f \) but are of a more combinatorial nature.

**Theorem 4.2.15.** Let \( f \in \mathbb{R}[x,y,z] \) be a smooth quartic, and let \( l_1, l_2, \ldots, l_6, l_7 \in A_1 \) be twelve different bitangents of \( f \). Write \( S = \{(l_i, l'_i) : i = 1, \ldots, 6\} \), then the following are equivalent:

(i) The six quadrics \([l_i,l'_i],[l_i,l'_i] \) are on the hypersurface \( \lambda Q \lambda' \), \( \lambda \in \mathbb{P}^1 \), for a quadratic symmetric determinantal representation \( Q \) of \( f \).

(ii) For ever \( i \neq j \), there exist \( q \in A_2 \) and \( \lambda \in \mathbb{C} \) such that \( f = \lambda l_i l'_i l'_j + q^2 \), i.e. the intersection points of \( V(l_i l'_i l'_j) \) and \( V(f) \) lie on a conic \( V(q) \).

(iii) Let \( M \) be a linear symmetric determinantal representation of \( f \) and \( O_1, \ldots, O_8 \) the corresponding Cayley octad. Then

\[
S = \{(b_{ik}, b_{jk}) : \{i,j \} = I, k \in I^c \} \quad \text{for some } I \subseteq \{1, \ldots, 8\}, |I| = 2 \text{ or } \\
S = \{(b_{ij}, b_{kl}) : \{i,j,k,l \} = I \text{ or } \{i,j,k,l \} = I^c \} \quad \text{for some } I \subseteq \{1, \ldots, 8\}, |I| = 4.
\]

\((I^c = \{1, \ldots, 8\} \setminus I \) denotes the complement of \( I \) in \( \{1, \ldots, 8\} \)) Any sextuple \( S \) satisfying the equivalent conditions (i)-(iii), is called a Steiner complex of \( f \).

From this, we can also calculate the number of Steiner complexes. There are \( \binom{8}{3} = 28 \) Steiner complexes of the first type and \( \frac{1}{2} \cdot \binom{8}{3} = 35 \) of the second type. In the second case we get the same Steiner complex if we choose \( I \) or \( I^c \).

We say that two quadratic symmetric determinantal representation are equivalent if the images of the corresponding Gram tensors are the same. Up to this equivalence there are exactly 63 quadratic symmetric determinantal representation of a smooth quartic \( f \) and these are in one-to-one correspondence with the 63 Steiner complex.

After fixing a linear symmetric determinantal representation and a Cayley octad of \( f \), the rank 3 Gram tensors of \( f \) can be enumerated in the same way the Steiner complexes can, i.e. by subsets of \( \{1, \ldots, 8\} \) as above.

For ease of notation, we write \( I = ijkI \) instead of \( I = \{i,j,k,l \} \subseteq \{1, \ldots, 8\}, |I| = 4 \) and \( I = ijI \) instead of \( I = \{i,j \} \subseteq \{1, \ldots, 8\}, |I| = 2 \).

There was already some computational evidence found by Powers and Reznick [PR00] that a smooth psd ternary quartic has exactly 15 real Gram tensors of rank 3. Later, in [PRSS04] this was proven. In [PSV11] the authors found a way to identify these representations using the combinatorial properties of Steiner complexes.

Let \( f \in \Sigma \) be a smooth quartic, and let \( M \) be a real linear symmetric determinantal representation of \( f \). Since \( f \) is smooth, the curve defined by \( f \) contains no real points, i.e. \( \mathcal{V}(f)(\mathbb{R}) = \emptyset \). Using table 1 in [PSV11], this means that there is no real point in the Cayley octad, hence it consists of four conjugate pairs. After reordering the Cayley octad, we may assume that \( O_i = O_{i+1} \) for \( i = \{1,3,5,7\} \) where \( \overline{\sigma} \) denotes complex conjugation. Then the real Gram tensors of rank 3 of \( f \) are given as follows.

**Theorem 4.2.16** ([PSV11] Theorem 6.2. & 6.3.). The eight real psd Gram tensors of rank 3 of \( f \) correspond to the following eight Steiner complexes:

\[
1357, \quad 1368, \quad 1458, \quad 1467, \\
1358, \quad 1367, \quad 1457, \quad 1468.
\]
The other seven real Gram tensors of rank 3 correspond to the Steiner complexes given by
1234, 1256, 1278,
12, 34, 56, 78.

4.3 The Steiner graph

Definition 4.3.1. Let $f \in \Sigma$ be smooth. Consider the graph whose vertices correspond to the eight real psd rank 3 Gram tensors of $f$ (or equivalently their Steiner complexes). We draw an edge between two vertices if the line segment connecting the two corresponding Gram tensors on $\text{Gram}(f)$ is contained in the boundary of $\text{Gram}(f)$. This graph is called the Steiner graph of $f$.

We show that the Steiner graph of a smooth psd quartic is the union of two $K_4$, where $K_4$ denotes the complete graph on four vertices. This graph has already been considered in [PSV11] where it is shown that it contains the disjoint union of two $K_4$. The fact that there are no additional edges in the Steiner graphs seems to be claimed as well in the paper, but without proof. In this section, we show that this is indeed the case. For this, we need some more facts about possible arrangements of bitangents which we show first.

A similar graph was constructed and studied for binary forms in [Sch].

Remark 4.3.2. We show that for a smooth quartic $f \in \Sigma$ the Steiner graph of $f$ has the form in Fig. 4.1. It is the disjoint union of two complete graphs on four vertices and the vertices in them are given by the four Steiner complexes in a row in Theorem 4.2.16. The proof that the graph below is contained in the Steiner graph is given in Lemma 4.3.7 which is taken from [PSV11] Lemma 6.4.

![Figure 4.1: The Steiner graph of a smooth quartic.](image)

We fix a smooth quartic $f \in \Sigma$, a linear symmetric determinantal representation of $f$, and a Cayley octad.

Proposition 4.3.3. Let $S_1, S_2$ be two different Steiner complexes. Then one of the following holds:

(i) There exist four different bitangents $l_1, \ldots, l_4$ such that $\{l_1, l_2\}, \{l_3, l_4\} \in S_1$ and $\{l_1, l_3\}, \{l_2, l_4\} \in S_2$, or

(ii) there exist six bitangents $l_1, \ldots, l_6$ such that $\{l_i, l_i'\} \in S_1$ and $\{l_i, l_i''\} \in S_2$ for every $i = 1, \ldots, 6$ where $l_i, l_i', l_i''$, $i = 1, \ldots, 6$ are 18 different bitangents.
Proof. This follows from [Dol12 Lemma 5.4.8., Proposition 6.1.6].

**Definition 4.3.4.** In the situation of Proposition 4.3.3, the two Steiner complexes are called **syzygetic** if they satisfy (i) and **azygetic** if they satisfy (ii).

With a fixed linear symmetric determinantal representation and a Cayley octad, being syzygetic or azygetic for a pair of Steiner complexes translates into the following statements about subsets of \{1, \ldots, 8\} representing the Steiner complexes.

**Lemma 4.3.5.** Let \( I, J \subseteq \{1, \ldots, 8\} \) be subsets such that \( |I|, |J| = 4 \) and \( I \neq J, J^c \). Let \( S_1, S_2 \) be the corresponding Steiner complexes. Then \( S_1 \) and \( S_2 \) are syzygetic if and only if \( |I \cap J| = 2 \).

**Proof.** If \( S_1 \) and \( S_2 \) are syzygetic, there exist four different bitangents \( l_1, l_2, l_3, l_4 \) such that

\[
\{l_1, l_2\}, \{l_3, l_4\} \in S_1 \quad \text{and} \quad \{l_1, l_3\}, \{l_2, l_4\} \in S_2.
\]

Let \( I = i_1i_2i_3i_4 \) and \( l_1 = b_{i_1i_2}, l_2 = b_{i_3i_4} \) with \( \{1, \ldots, 8\} = \{i_1, \ldots, i_8\} \). As \( \{l_1, l_3\} \) is contained in \( S_2 \), the set \( J \) contains \( i_1, i_2 \) (after possibly swapping \( J \) for its complement \( J^c \)). Since \( \{l_2, l_4\} \in S_2 \), either \( i_3, i_4 \in J \) or \( i_3, i_4 \in J^c \). In the first case \( J = i_1i_2i_3i_4 \) and \( I = J \). In the second case \( i_1, i_2 \in J \) and \( i_3, i_4 \in J^c \). Especially \( J = i_1i_2i_3i_6 \), since \( i_3, i_4 \notin J \). Therefore \( |I \cap J| = 2 \).

If \( I \) and \( J \) intersect in a size of two, we may assume that \( I = i_1i_2i_3i_4 \) and \( J = i_1i_2i_5i_6 \) where \( \{1, \ldots, 8\} = \{i_1, \ldots, i_8\} \). Then we may take as bitangents \( l_1 = b_{i_1i_2}, l_2 = b_{i_3i_4}, l_3 = b_{i_5i_6}, l_4 = b_{i_7i_8} \) and get \( \{l_1, l_2\}, \{l_3, l_4\} \in S_1 \) and \( \{l_1, l_3\}, \{l_2, l_4\} \in S_2 \) which shows that the two Steiner complexes are syzygetic.

**Example 4.3.6.** We consider the Steiner complexes 1358 and 1457 in the Steiner graph. These form a syzygetic pair. Indeed, by Lemma 4.3.5 we only have to check that the sets 1358 and 1457 contain exactly two common elements. This is true and the elements are 1 and 5.

Moreover, we see from the proof of Lemma 4.3.5 that the four bitangents in Proposition 4.3.3 (i) are given by \( b_{15}, b_{38}, b_{47}, b_{26} \). The pairs \( \{b_{15}, b_{38}\}, \{b_{47}, b_{26}\} \) are contained in 1358 and \( \{b_{15}, b_{47}\}, \{b_{38}, b_{26}\} \) are contained in 1457. It is now natural to consider the Steiner complex containing \( \{b_{15}, b_{26}\}, \{b_{47}, b_{38}\} \) which is the third possibility to form two pairs from the four bitangents. This Steiner complex is therefore 1256. We note that it does correspond to a real rank 3 tensors which is not psd by Theorem 4.2.16.

We can now give the proof that the Steiner graph of \( f \) contains all the edges shown in Remark 4.3.2.

**Lemma 4.3.7.** ([PSV11 Lemma 6.4.1]). Let \( S_1, S_2 \) be two Steiner complexes corresponding to real psd Gram tensors, that are in the same line in Theorem 4.2.16. Then the line between the corresponding Gram tensors on \( \text{Gram}(f) \) is contained in the boundary of \( \text{Gram}(f) \).

**I.e. if \( \vartheta_1, \vartheta_2 \) are the corresponding Gram tensors, then for every \( \lambda \in [0, 1] \) the point \( \lambda \vartheta_1 + (1 - \lambda) \vartheta_2 \) is on the boundary of \( \text{Gram}(f) \).**

**Proof.** By Lemma 4.3.5 the two Steiner complexes \( S_1, S_2 \) are syzygetic, hence there exist bitangents \( l_1, \ldots, l_4 \) such that \( \{l_1, l_2\}, \{l_3, l_4\} \in S_1 \) and \( \{l_1, l_3\}, \{l_2, l_4\} \in S_2 \). Let \( q \in A_2, \lambda \in C \) such that \( f = \lambda l_1l_2l_3l_4 + q^2 \), which is possible by Theorem 4.2.15. Then \( q \) is contained in the (complex) images of \( \vartheta_1 \) and \( \vartheta_2 \). Hence \( \dim(\text{im}_C(\vartheta_1)) + \text{im}_C(\vartheta_2)) \leq 5 \). Since \( \vartheta_1 \) is real, \( \text{im}_C(\vartheta_1) \) has a real \( C \)-basis \( (i = 1, 2) \), thus it follows that \( \dim(\text{im}(\vartheta_1 + \vartheta_2)) = \dim(\text{im}(\vartheta_1)) + \text{im}(\vartheta_2)) \leq 5 \).
4.3 THE STEINER GRAPH

We now work towards showing that the Steiner graph of a smooth quartic is the disjoint union of two $K_4$. We start with some facts about bitangents.

**Definition 4.3.8.** Let $f \in \mathbb{R}[x]_4$ be smooth, and let $l_1, l_2, l_3$ be three different bitangents of $f$. The triple is called syzygetic if there exists a Steiner complex $\mathcal{S}$ such that $\{l_1, l_2, l_3, l\} \in \mathcal{S}$ for another bitangent $l$ of $f$. Otherwise, the triple is called azygetic. Analogously, a 4-tuple $l_1, \ldots, l_4$ of bitangents is called syzygetic if one (equivalently any) subset consisting of three bitangents in syzygetic.

**Remark 4.3.9.** Three bitangents are syzygetic if and only if there exists $q \in A_2$ such that $\mathcal{V}(l_1 l_2 l_3, f) \subseteq \mathcal{V}(q)$, and for every point $P \in \mathcal{V}(l_1 l_2 l_3, f)$ the intersection multiplicity of $f$ and $q$ is at least the intersection multiplicity of $f$ and $l_1 l_2 l_3$. Indeed, if the bitangents are syzygetic, then by definition of a Steiner complex, there exists such $q$. Conversely, if we have a quadratic form $q$ as above, we have an equation $f = q^2 + l_1 l_2 l_3 l$ after scaling by Theorem 4.2.2 for some $l \in A_1$. This shows that $l$ is another bitangent of $f$ and the representation of $f$ gives rise to a quadratic symmetric determinantal representation $Q = \begin{pmatrix} l_1 l_2 & iq \\ iq & l_3 l \end{pmatrix}$. Therefore, the sets $\{l_1, l_2\}, \{l_3, l\}$ are contained in the Steiner complex corresponding to this quadratic symmetric determinantal representation.

We note that this Steiner complex is not unique, as we could have also considered the quadratic symmetric determinantal representation \left(\begin{smallmatrix} l_1 l_3 & iq \\ iq & l_2 l_3 \end{smallmatrix}\right) or \left(\begin{smallmatrix} l_1 l & iq \\ iq & l_2 l_3 \end{smallmatrix}\right).

**Lemma 4.3.10.** Let $Q$ be a quadratic symmetric determinantal representation with corresponding Steiner complex $\mathcal{S} = \{l_i, l'_i\} : i = 1, \ldots, 6$ and corresponding Gram tensor $\vartheta$. Then $\text{im}_\mathbb{C}(\vartheta) = \text{span}(l_i l'_i, l_j l'_j, l_k l'_k)$ for any pairwise different integers $i, j, k \in \{1, \ldots, 6\}$.

**Proof.** Let $i, j, k \in \{1, \ldots, 6\}$ be pairwise different. We have equations

\[
c f = q^2 + l_i l'_j l'_j \quad \text{and} \quad df = p^2 + l'_i l'_j l'_k
\]

for some $p, q \in A_2$ and $c, d \in \mathbb{C}$. These scalars are actually non-zero since otherwise wlog $p = l_i l'_i$ or $p = l_i l_k$. In the first case $\dim \text{im}_\mathbb{C}(\vartheta) \leq 2$ and in the second case $l'_i l'_k = l_i l_k$ which is not possible, as the bitangents are different. The same argument shows $c \neq 0$.

Subtracting the two equations after scaling we get

\[
l_i l'_i (l_j l'_j - \lambda_k l'_k) = q^2 - \lambda p^2 = (q - \sqrt{\lambda} p)(q + \sqrt{\lambda} p)
\]

for some $\lambda \in \mathbb{C}$. If $l_i$ would divide $(q - \sqrt{\lambda} p)$ and $l'_i$ would divide $(q + \sqrt{\lambda} p)$, then at the intersection point of $l_i$ and $l'_i$ the forms $p$ and $q$ vanish. But this point is then a singular point of $f$. Hence, wlog $l_i l'_i \in \text{span}(q - \sqrt{\lambda} p)$ and $(l_j l'_j - \lambda_k l'_k) \in \text{span}(q + \sqrt{\lambda} p)$.

This shows $p, q \in \text{span}(l_i l'_i, l_j l'_j, l_k l'_k)$. Since $\text{span}(l_i l'_i, l_j l'_j, q)$ has dimension 3, the same holds for $\text{span}(l_i l'_i, l_j l'_j, l_k l'_k)$.

**Lemma 4.3.11.** Let $l_1, l_2, l_3$ be three bitangents. The following are equivalent:

(i) There exist bitangents $l'_1, l'_2, l'_3$ such that $l_1, l_2, l_3, l'_1, l'_2, l'_3$ are pairwise different, and there exists a Steiner complex $\mathcal{S}$ such that $\{l_1, l'_1\}, \{l_2, l'_2\}, \{l_3, l'_3\} \in \mathcal{S}$.

(ii) The bitangents $l_1, l_2, l_3$ are azygetic.

**Proof.** (i)⇒(ii): \cite{Dol12} Lemma 6.1.6. (ii)⇒(i): \cite{Dol12} Proposition 6.1.4.
Lemma 4.3.12. Three azygetic bitangents cannot intersect in a common point.

Proof. Let $l_1, l_2, l_3$ be three azygetic bitangents. By Lemma 4.3.11 we can find a Steiner complex $S$ such that $\{l_1, l'_1\}, \{l_2, l'_2\}, \{l_3, l'_3\} \in S$ for some bitangents $l'_1, l'_2, l'_3$.

Using Lemma 4.3.10 we see that the image of the corresponding Gram tensor is given by $\text{span}(l'_1, l'_2, l'_3)$. If $l_1, l_2, l_3$ would intersect in a point, then this space has a base-point and thus $f$ has a singularity at this point. \qed

Remark 4.3.13. Summarizing some facts about syzygetic and azygetic bitangents and Steiner complexes.

Let $l_1, l_2, l_3$ be three syzygetic bitangents. By Lemma 4.3.11 we can find a Steiner complex $S$ such that $\{l_1, l'_1\}, \{l_2, l'_2\}, \{l_3, l'_3\} \in S$ for some bitangents $l'_1, l'_2, l'_3$.

Furthermore, any two syzygetic Steiner complexes determine a unique third one, defined as above.

Let $l_1, l_2, l_3$ be three azygetic bitangents. By Lemma 4.3.11 we can find a Steiner complex $S$ such that $\{l_1, l'_1\}, \{l_2, l'_2\}, \{l_3, l'_3\} \in S$ for some bitangents $l'_1, l'_2, l'_3$.

Example 4.3.14. Consider the two different Steiner complexes $\{\{l_i, l'_i\}: i = 1, \ldots, 6\}$ and $\{\{l_i, l'_i\}: i = 1, \ldots, 6\}$. Then these are syzygetic and have exactly the six bitangents $l_1, \ldots, l_6$ in common. These could for example be the Steiner complexes corresponding to the subsets $I = 1357$ and $J = 1358$ after fixing a Cayley octad and a linear symmetric determinantal representation.

We need one last simple lemma before turning to the main theorem of this section.

Lemma 4.3.15. Let $n \geq 2$, $d \geq 1$. Let $f \in \Sigma_{n,2d}$ and let $\emptyset \in \text{Gram}(f)$ with corresponding subspace $U = \text{im}(\emptyset)$ of dimension $r$. If $p_1, \ldots, p_r \in U$ form a basis of $U$, then there exist $a_{ij} \in \mathbb{R}$ ($1 \leq i \leq r$, $1 \leq j \leq i$) such that

$$f = \sum_{i=1}^{r} \left( \sum_{j=1}^{i} a_{ij} p_j \right)^2.$$

I.e. the $i$-th summand is a linear combination of only the first $i$ forms $p_1, \ldots, p_i$.

Proof. This can be shown using the QL-decomposition of a matrix. Let $f_1, \ldots, f_r \in U$ be a basis such that $f = \sum_{i=1}^{r} f_i^2$. Let $A$ be the $r \times r$ matrix containing the coordinates of $f_i$ wrt the basis $p_1, \ldots, p_r$ in its $i$-th row. Then

$$f = (f_1, \ldots, f_r)(f_1, \ldots, f_r)^T = (p_1, \ldots, p_r)A^T A(p_1, \ldots, p_r)^T.$$

As $A$ is a rank $r$ square matrix there exists an orthogonal matrix $O \in O(r)$ and a lower triangular $r \times r$ real matrix $L$ such that $A = OL$. This shows

$$f = (p_1, \ldots, p_r)(OL)^T (OL)(p_1, \ldots, p_r)^T = (p_1, \ldots, p_r)L^T L(p_1, \ldots, p_r)^T.$$

Since $L$ is lower triangular, this sos representation of $f$ has the required form. \qed
4.3 THE STEINER GRAPH

We now show the following theorem which is our goal in this section.

**Theorem 4.3.16.** Let \( f \in \Sigma \) be a smooth quartic. Then the Steiner graph of \( f \) is the union of two disjoint \( K_4 \).

We already know that there are two disjoint \( K_4 \) contained in the Steiner graph, hence we need to show that there are no additional edges. We first show that none of those edges can have rank \( \leq 4 \).

For the rest of this section, we fix the following setup. Let \( f \in \Sigma \) smooth and let \( \vartheta_1, \vartheta_2 \in \text{Gram}(f) \) be two real psd rank 3 Gram tensors with corresponding Steiner complexes \( S_1, S_2 \) such that \( S_1 \) and \( S_2 \) are azygetic. By Lemma 4.3.7 these are the Gram tensors left to consider. Equivalently, syzygetic Steiner complexes in the Steiner graph are connected by an edge and azygetic ones are not.

**Proposition 4.3.17.** \( \dim(\text{im}(\vartheta_1) + \text{im}(\vartheta_2)) \geq 5 \).

**Proof.** Since \( S_1 \) and \( S_2 \) are azygetic, we can pick pairs of bitangents such that

\[
\{l_1, l'_1\}, \{l_2, l'_2\}, \{l_3, l'_2\} \in S_1 \text{ and } \{l_1, l''_1\}, \{l_2, l''_2\} \in S_2.
\]

Assume that \( \dim(\text{im}(\vartheta_1) + \text{im}(\vartheta_2)) \leq 4 \). Then the quadratic forms \( l_1 l'_1, l_2 l'_2, l_3 l'_2, l_1 l''_1, l_2 l''_2 \) are linearly dependent. We therefore find \( \lambda_1, \ldots, \lambda_5 \in \mathbb{C} \) not all zero such that

\[
0 = \lambda_1 l_1 l'_1 + \lambda_2 l_2 l'_2 + \lambda_3 l_3 l'_2 + \lambda_4 l_1 l''_1 + \lambda_5 l_2 l''_2
= l_1(\lambda_1 l'_1 + \lambda_4 l''_1) + l_2(\lambda_2 l'_2 + \lambda_5 l''_2) + \lambda_3 l_3 l'_2.
\]

Let \( P \in \mathbb{P}^2 \) be the intersection point \( \nu(l_1, l_2) \). Then either \( l_3 \) or \( l'_2 \) has to vanish at \( P \) if \( \lambda_3 \neq 0 \). But both triples \( l_1, l_2, l_3 \) and \( l_1, l_2, l'_2 \) are azygetic by Lemma 4.3.11 hence cannot intersect in a common point by Lemma 4.3.12. This means that \( \lambda_3 = 0 \) which shows

\[
0 = l_1(\lambda_1 l'_1 + \lambda_4 l''_1) + l_2(\lambda_2 l'_2 + \lambda_5 l''_2).
\]

Since two bitangents cannot be multiples of each other, \( l_1 \in \text{span}(l'_2, l''_2) \) and \( l_2 \in \text{span}(l'_1, l''_1) \). But again the triples \( l_1, l'_2, l''_2 \) and \( l_2, l'_1, l''_1 \) are azygetic by the next lemma and therefore cannot intersect in a common point.

**Lemma 4.3.18.** With everything as in Proposition 4.3.17 the triples \( l_1, l'_2, l''_2 \) and \( l_2, l'_1, l''_1 \) of bitangents are azygetic.

**Proof.** Consider the Steiner complex \( S_1 = \{l_i, l'_i\} : i = 1, \ldots, 6 \) and the four bitangents \( l_1, l'_1, l_2, l'_2 \). Then, there exist Steiner complexes \( T_1, T_2 \) such that \( T_1 \) contains \( \{l_1, l_2\}, \{l'_1, l'_2\} \) and \( T_2 \) contains \( \{l_1, l'_2\}, \{l'_1, l_2\} \). By [Dol12] Theorem 5.4.10, any of the 28 bitangents of \( f \) is contained in one of the three Steiner complexes. Especially, one contains the bitangent \( l'_i \). This bitangent cannot be contained in \( T_2 \); otherwise \( T_2 \) contains \( \{l_1, l'_2\}, \{l'_1, l_2\} \) and \( \{l''_i, L\} \) for some bitangent \( L \). Especially \( l_1, l_2, l''_i \) are azygetic by Lemma 4.3.11. But \( S_2 \) contains \( \{l_1, l''_i\} \) and \( \{l_2, l''_i\} \). Hence \( l_1, l_2, l''_i \) are azygetic, a contradiction.

Therefore \( l''_i \) is contained in the Steiner complex \( T_1 \). Then \( l'_1, l'_i, l_2 \) are contained in \( T_1 \) as in Lemma 4.3.11 hence the triple is azygetic.

An analogous argument shows that \( l_1, l'_2, l''_2 \) form an azygetic triple. \( \square \)
Remark 4.3.19. We first sketch the second part of the proof of Theorem 4.3.16. If 
\[ \dim(\text{im}_C(\vartheta_1) + \text{im}_C(\vartheta_2)) = 5 \] and \( q \in A_2 \) spans the intersection of the images, then we get representations 
\[ p_1p_2 + q_2 = cf \] and \( q_1q_2 + q_2^2 = df \) for some \( p_1, p_2 \in \text{im}_C(\vartheta_1) \), \( q_1, q_2 \in \text{im}_C(\vartheta_2) \) and \( c,d \in \mathbb{C} \).

Let \( P_1, \ldots, P_8 \) be the intersection points of \( \mathcal{V}(q) \) and \( \mathcal{V}(f) \). The equations show that the products \( q_1q_2 \) and \( p_1p_2 \) are tangent to \( f \) at the points \( P_1, \ldots, P_8 \). A form \( g \in \mathbb{P}A_2 \) has 5 degrees of freedom and we require \( \mathcal{V}(g) \) to be tangent at 4 points. Therefore, it seems unlikely to find two different products of two quadratic forms satisfying the conditions.

In the case where the two Steiner complexes \( S_1, S_2 \) are syzygetic, this is easily possible: as in Lemma 4.3.17 there can be four bitangents \( l_1, l_2, l_3, l_4 \in A_1 \) of \( f \) such that \( q_1 = l_1l_2, q_2 = l_3l_4 \) and \( p_1 = l_1l_3, p_2 = l_2l_4 \).

We prepare with an application of Theorem 4.2.2.

Assume we have an equation \( f = p_1p_2 + q^2 \) for some \( p_1, p_2, q \in A_2 \). Let \( \mathcal{V}(f, p_1) = \{ P_1, \ldots, P_4 \} \) and \( \mathcal{V}(f, p_2) = \{ P_5, \ldots, P_8 \} \) with \( P_1, \ldots, P_8 \in \mathbb{P}^2 \) (not necessarily different). For any two different points \( P, Q \in \mathbb{P}^2 \), we write \( \overline{PQ} \) for the line through \( P \) and \( Q \). Let

\[ L_1 = \overline{P_1P_3}, L_2 = \overline{P_1P_4}, L_3 = \overline{P_5P_6}, L_4 = \overline{P_7P_8} \]

if all points are different. If any two points \( P_i, P_j \) are the same, the line \( \overline{P_iP_j} \) should be understood as the tangent line at \( P_i \) of \( \mathcal{V}(p_1) \) or \( \mathcal{V}(p_2) \) depending on whether \( 1 \leq i \leq 4 \) or \( 5 \leq i \leq 8 \). Moreover, for \( i = 1, \ldots, 4 \) write \( l_i \in A_1 \) for the linear form such that \( \mathcal{V}(l_i) = L_i \).

Proposition 4.3.20. In the situation above, if \( q \) is also irreducible, there exists \( h \in \text{span}(p_1, p_2, q) \subseteq A_2 \) and \( 0 \neq \lambda \in \mathbb{C} \) such that

\[ f = \lambda l_1l_2l_3l_4 + qh. \]

Proof. By Theorem 4.2.2 there exist \( \lambda_1, \ldots, \lambda_6 \in \mathbb{C} \) such that

\[ \lambda_1p_1 = \lambda_2l_1l_2 + \lambda_3q, \quad \lambda_4p_2 = \lambda_5l_3l_4 + \lambda_6q \]

by choice of the forms \( l_1, \ldots, l_4 \). If \( \lambda_2\lambda_5 = 0 \), then either \( q = \nu p_1 \) or \( q = \nu p_2 \) for some \( \nu \in \mathbb{C} \). However, the image of the Gram tensor corresponding to the representation \( f = p_1p_2 + q^2 \) has image \( \text{span}(p_1, p_2, q) \). Hence, this Gram tensor has rank 2 which is not possible since \( f \) is smooth. If \( \lambda_2\lambda_5 = 0 \), then \( q \) was reducible, contradicting the assumption.

We therefore see \( \text{span}(p_1, p_2, q) = \text{span}(l_1l_2, l_3l_4, q) \). Multiplying the two equations, we get

\[ \lambda_1\lambda_4p_1p_2 = \lambda_2\lambda_5l_1l_2l_3l_4 + q(\lambda_3\lambda_5l_3l_4 + \lambda_3\lambda_6q + \lambda_2\lambda_6l_1l_2). \]

Substituting with the equation \( f = p_1p_2 + q^2 \), we get

\[ \lambda_1\lambda_4(f - q^2) = \lambda_2\lambda_5l_1l_2l_3l_4 + q(\lambda_3\lambda_5l_3l_4 + \lambda_3\lambda_6q + \lambda_2\lambda_6l_1l_2), \]

and thus

\[ f = \frac{\lambda_2\lambda_5}{\lambda_1\lambda_4}l_1l_2l_3l_4 + q \left( \frac{\lambda_3\lambda_5}{\lambda_1\lambda_4}l_3l_4 + \frac{\lambda_3\lambda_6}{\lambda_1\lambda_4}q + \frac{\lambda_2\lambda_6}{\lambda_1\lambda_4}l_1l_2 + q \right). \]
Proof of Theorem 4.3.16. Let \( \text{span}(q) = \text{im}(\vartheta_1) \cap \text{im}(\vartheta_2) \). By Lemma 4.3.15 there exist \( p_1, p_2 \in \text{im}_C(\vartheta_1), q_1, q_2 \in \text{im}_C(\vartheta_2) \), and \( 0 \neq c, d \in \mathbb{R} \) such that

\[
f = p_1p_2 + (cq)^2, \tag{4.4}
\]

\[
f = q_1q_2 + (dq)^2. \tag{4.5}
\]

If \( c^2 - d^2 \neq 0 \), subtracting equations (4.4) and (4.5) yields \( p_1p_2 = q_1q_2 \). Therefore, either \( \text{im}(\vartheta_1) = \text{im}(\vartheta_2) \) or all these forms are reducible and we can write \( p_1 = l_1l_2, p_2 = l_3l_4 \) and \( q_1 = l_1l_3, q_2 = l_2l_4 \) for some \( l_1, l_2, l_3, l_4 \in A_1 \). But the equations show that \( l_1, \ldots, l_4 \) are bitangents and \( \{l_1, l_2\}, \{l_3, l_4\} \) are in the first Steiner complex and \( \{l_1, l_3\}, \{l_2, l_4\} \) are in the second. Hence the two Steiner complexes are syzygetic, a contradiction.

Assume \( q \) is irreducible. By Proposition 4.3.20 there exist \( h_1 \in \text{im}_C(\vartheta_1), h_2 \in \text{im}_C(\vartheta_2) \), and \( 0 \neq \nu, \lambda \in \mathbb{C} \) such that

\[
f = \lambda l_1l_2l_3l_4 + qh_1, \tag{4.6}
\]

\[
f = \nu l_1l_2l_3l_4 + qh_2. \tag{4.7}
\]

Subtracting the two equations, we get

\[
(\nu - \lambda)l_1l_2l_3l_4 = q(h_1 - h_2). 
\]

Since \( q \) is irreducible, it follows that \( \nu = \lambda \) and \( h_1 = h_2 \). Hence, \( h_1 = h_2 \in \text{im}_C(\vartheta_1) \cap \text{im}_C(\vartheta_2) = \text{span}(q) \) and equations (4.6) and (4.7) now read

\[
f = \lambda l_1l_2l_3l_4 + \lambda' q^2 
\]

\[
f = \nu l_1l_2l_3l_4 + \nu' q^2
\]

for some \( \lambda', \nu' \in \mathbb{C} \). This shows that \( l_1, \ldots, l_4 \) are bitangents of \( f \). Moreover, wlog

\[
\{l_1, l_2\}, \{l_3, l_4\} \in S_1 \quad \text{and} \quad \{l_1, l_3\}, \{l_2, l_4\} \in S_2,
\]

which means that \( S_1 \) and \( S_2 \) are syzygetic, a contradiction.

Now, we may assume \( q \) is reducible. We write \( q = l_1l_2 \) for some \( l_1, l_2 \in A_1 \). If any of the forms \( p_1, p_2, q_1, q_2 \) was divisible by \( l_1 \) or \( l_2 \), the form \( f \) was as well. Thus all forms \( p_1, p_2, q_1, q_2 \) can only share two points with \( l_1 \) and \( l_2 \).

Let \( V(f, l_1) = \{P_1, \ldots, P_4\} \) and \( V(f, l_2) = \{P_5, \ldots, P_8\} \). Then, wlog we may assume

\[
\begin{align*}
V(p_1, f) &= \{P_1, P_2, P_5, P_6\}, \quad V(p_2, f) = \{P_3, P_4, P_7, P_8\}, \\
V(q_1, f) &= \{P_1, P_2, P_7, P_8\}, \quad V(q_2, f) = \{P_3, P_4, P_5, P_6\}.
\end{align*} \tag{4.8}
\]

Firstly, \( V(q_1) \) cannot share more than two points with any of the lines \( V(l_1) \) and \( V(l_2) \) due to the reasoning above. However, it may also not share more than two points with \( V(p_i) \) (\( i = 1, 2 \)) since \( q_1 \) and \( p_i \) are both tangent to \( f \) at these points. Otherwise, by Bézouts Theorem either \( q_1 \in \text{span}(p_i) \) which shows \( \dim(\text{im}_C(\vartheta_1) + \text{im}_C(\vartheta_2)) \leq 4 \) contradicting Proposition 4.3.17, or \( q_1 \) and \( p_i \) are reducible and not multiples of each other. Again this leads to a contradiction as follows: assume wlog \( i = 1, q_1 = h_1h_2 \) and \( p_1 = h_1h_3 \) for some \( h_1, h_2, h_3 \in A_1 \). Then,

\[
f = h_1h_3p_2 + (cq)^2, \\
f = h_1h_2q_2 + (dq)^2.
\]
Subtracting the two equations and plugging in \( q = l_1l_2 \), we get
\[
h_1(h_3p_2 - h_2q_2) = (d^2 - c^2)l_1^2l_2^2.
\]
From the beginning of the proof we know \( d^2 - c^2 \neq 0 \), hence \( h_1 \in \text{span}(l_1) \) or \( h_1 \in \text{span}(l_2) \). In either case, \( f \) is reducible, a contradiction. This shows that the only way to split the points \( P_1, \ldots, P_8 \) is the one given in (4.8) and (4.9).

Again, we use Theorem 4.2.2 and find \( \lambda_1, \ldots, \lambda_4 \in \mathbb{C} \) such that
\[
p_1 = \lambda_1l_1^2 + \lambda_2q_1, \quad p_2 = \lambda_3l_1^2 + \lambda_4q_2.
\]
However, this means
\[
\text{im}_C(\vartheta_1) + \text{im}_C(\vartheta_2) = \text{span}(p_1, p_2, q_1, q_2, q) = \text{span}(l_1^2, q_1, q_2, q),
\]
but by Proposition 4.3.17 this is impossible.

To finish the proof of Theorem 4.3.16, we note that \( \text{im}(\vartheta_1) + \text{im}(\vartheta_2) = \text{im}(\vartheta_1 + \vartheta_2) \) since \( \vartheta_1, \vartheta_2 \) are real psd Gram tensors. Therefore, any tensor \( \lambda \vartheta_1 + (1 - \lambda)\vartheta_2 \) (\( \lambda \in [0,1] \)) has image \( \text{im}(\vartheta_1) + \text{im}(\vartheta_2) \) which has dimension 6.

Remark 4.3.21. We briefly sketch a second possible proof. By slightly adjusting Proposition 4.3.17, we can show that \( \dim(\text{im}(\vartheta_1) + \text{im}(\vartheta_2)) = 5 \) if and only if there are certain linear relations between bitangents. In [Dol12, Theorem 6.1.9] it is shown how to construct all bitangents of a smooth quartic by prescribing seven of them. Most importantly, the coefficients of all other bitangents can be computed by solving linear systems in the coefficients of the given bitangents.

Solving all linear equations symbolically, we get one polynomial equation in the given coefficients such that this equation has a solution if and only if \( \dim(\text{im}(\vartheta_1) + \text{im}(\vartheta_2)) = 5 \). Using a computer, one can show that there are indeed no solutions corresponding to smooth quartics.

We note that this does not give any geometric insight but it does give a proof using different methods.

### 4.4 The Gram spectrahedron

From Example 3.1.24 and Theorem 4.1.2, we know that the Gram spectrahedron of a smooth ternary quartic can only have the following faces. Faces of
- rank 5 and dimension 0 or 2,
- rank 4 and dimension 0 or 1,
- rank 3 and dimension 0.

We already know that there are exactly eight extreme points of rank 3 on the Gram spectrahedron. We therefore turn our attention to rank 4 and rank 5 faces. We determine the dimension of all rank \( r \) points that lie on faces of dimension \( s \) for all possible combinations of rank and dimension as above, if the quartic is chosen generically.
4.4 THE GRAM SPECTRAHEDRON

4.4.1 Rank 4 points

We show that for a generic quartic there are no faces of rank 4 and dimension 1. Those faces only appear on Gram spectrahedra of quartics that are invariant under some automorphism of \(\mathbb{C}^3\) of order 2. Moreover, the dimension of the set of all rank 4 extreme points is 3 for a generic quartic.

We start by considering faces of rank 4 and dimension 1. Since the boundary of any rank 4 face consists of tensors of rank at most 3, it follows that if the quartic is smooth, the rank 4 faces can only occur as line segments between the eight rank 3 tensors. Since we know the Steiner graph, there are at most 12 1-dimensional faces on the Gram spectrahedron.

**Lemma 4.4.1.** Let \(f \in \Sigma\) be a smooth quartic, then the degree of \(f\) in each variable is 4.

**Proof.** The degree in any variable cannot be three, since \(f\) is psd. Assume the degree in \(x\) was 2, then we can write \(f = x^2g_2(y,z) + xg_3(y,z) + g_4(y,z)\) where \(g_2, g_3, g_4 \in \mathbb{R}[y,z]\) are forms of degrees 2, 3, and 4 respectively.

We easily check that \(f\) and all partial derivatives vanish at \((1:0:0)\). But this means that \((1:0:0)\) is a singular point of \(f\). \(\square\)

**Definition 4.4.2.** Let \(f \in A_4\) and \(C = V(f) \subseteq \mathbb{P}^2\). We denote by \(\text{Aut}(f)\) (or \(\text{Aut}(C)\)) the automorphism group of \(f\) (or \(C\)) embedded in \(\text{PGL}_3(\mathbb{C})\).

**Theorem 4.4.3.** Let \(f \in \Sigma\) be a smooth quartic, then the following are equivalent.

(i) The automorphism group of \(f\) has even order.

(ii) There exist four different bitangents of \(f\) that intersect in a common point.

(iii) After a linear change of coordinates \(f = (z^2 - f_2)^2 - 4xy(ax + by)(cx + dy)\) with \(f_2 \in \mathbb{C}[x,y]_2\) and \(a,b,c,d \in \mathbb{C}\).

(iv) There are three pairwise syzygetic Steiner complexes \(S_1,S_2,S_3\) (and corresponding Gram tensors \(\vartheta_1,\vartheta_2,\vartheta_3\)) and four different bitangents \(l_1,\ldots,l_4\) such that

a) \(\{l_1,l_2\}, \{l_3,l_4\} \in S_1\),

b) \(\{l_1,l_3\}, \{l_2,l_4\} \in S_2\),

c) \(\{l_1,l_4\}, \{l_2,l_3\} \in S_3\),

and \(\dim(\text{im}_C(\vartheta_i) + \text{im}_C(\vartheta_j)) \leq 4\) for all \(i,j = 1,2,3\).

**Proof.** (i)⇒(ii),(iii): After a linear change of coordinates, the automorphism operates as \((a,b,c) \mapsto (a,b,-c)\) on \(\mathbb{C}^3\) and therefore no odd powers of \(z\) appear in \(f\). Hence, we can write \(f\) as \(z^4 + z^2f_2(x,y) + f_4(x,y) = (z^2 + g_2(x,y))^2 + g_4(x,y)\) where \(f_i, g_i \in \mathbb{C}[x,y,z]\) for \(i = 2,4\). Since \(g_4\) is a form in 2 variables, it is a product of four linear forms. By definition, those are bitangents of \(f\) and intersect in the point \((0:0:1)\). Moreover, they are no scalar multiples of each other as \(f\) is smooth.

(ii)⇒(iii): After scaling \(f_i\), we may assume that \(f\) is monic in \(z\). By Lemma 4.4.1 the monomial \(z^4\) appears in \(f\). Wlog the common intersection point of the bitangents is \((0:0:1)\),
then the four bitangents $l_1, \ldots, l_4$ lie in $\mathbb{C}[x, y]_1$. As four bitangents can only intersect in a common point if they are syzygetic by Lemma 4.3.12, there exists a Steiner complex containing the pairs $\{l_1, l_2\}, \{l_3, l_4\}$. Hence, there exists $q \in A_2$, $\lambda \in \mathbb{C}$ such that

$$f = q^2 + \lambda l_1 l_2 l_3 l_4.$$

We write $q = z^2 + z f_1(x, y) + f_2(x, y)$ for some $f_1 \in \mathbb{C}[x, y]_1$, $(i = 1, 2)$ and $f_1 = ax + by$ for some $a, b \in \mathbb{C}$. Then, the coordinate change $z \mapsto -\frac{3}{2}x - \frac{b}{y} + z$ yields the desired form in (iii).

(iii)$\Rightarrow$(i): The form $g := (z^2 - f_2)^2 - 4xy(ax + by)(cx + dy)$ with $a, b, c, d \in \mathbb{C}$ is invariant under the action of the automorphism $\sigma$ defined by $(x, y, z) \mapsto (x, y, -z)$. Let $\phi$ be the automorphism of $\mathbb{C}^3$ mapping $f$ to $g$. Since $g$ is invariant under $\sigma$, the form $f$ is invariant under $\phi^{-1}\sigma\phi$.

(ii)$\Rightarrow$(iv): Let $l_1, l_2, l_3, l_4$ be the four bitangents intersecting in a common point, and let $S_1, S_2, S_3$ be the Steiner complexes that contain $\{l_1, l_2\}, \{l_3, l_4\}$ and $\{l_1, l_3\}, \{l_2, l_4\}$ and $\{l_1, l_4\}, \{l_2, l_3\}$ respectively, i.e. the Steiner complexes as in Remark 4.3.13. Let $\vartheta_1, \vartheta_2, \vartheta_3$ be the corresponding Gram tensors.

We prove the statement for $\vartheta_1, \vartheta_2$, the other ones work analogously. Let $q \in A_2$ be the quadratic form such that $f = q^2 + \lambda l_1 l_2 l_3 l_4$ for some $\lambda \in \mathbb{C}$. Then, we have $\text{im}_\mathbb{C}(\vartheta_1) = \text{span}(l_1 l_2, l_3 l_4, q)$, $\text{im}_\mathbb{C}(\vartheta_2) = \text{span}(l_1 l_3, l_2 l_4, q)$, and

$$\text{im}_\mathbb{C}(\vartheta_1) + \text{im}_\mathbb{C}(\vartheta_2) = \text{span}(l_1 l_2, l_3 l_4, l_1 l_3, l_2 l_4, q).$$

By assumption, there are non-zero scalars $\lambda_1, \ldots, \lambda_4 \in \mathbb{C}$ such that

$$l_1 = \lambda_1 l_2 + \lambda_2 l_3 \quad \text{and} \quad l_4 = \lambda_3 l_2 + \lambda_4 l_3.$$ 

This gives

$$0 = l_1 l_4 - l_1 l_4
= l_1(\lambda_3 l_2 + \lambda_4 l_3) - l_4(\lambda_1 l_2 + \lambda_2 l_3)
= -\lambda_1 l_2 l_4 - \lambda_2 l_3 l_4 + \lambda_3 l_1 l_2 + \lambda_4 l_1 l_3.$$

This shows that the products $l_1 l_2, l_3 l_4, l_1 l_3, l_2 l_4$ are linearly dependent and thus

$$\dim \text{span}(l_1 l_2, l_3 l_4, l_1 l_3, l_2 l_4, q) \leq 4.$$ 

(iv)$\Rightarrow$(ii): Again, consider the Steiner complexes $S_1$ and $S_2$, the bitangents, and the quadratic form $q$ as above. Then, we know

$$\text{im}_\mathbb{C}(\vartheta_1) = \text{span}(l_1 l_2, l_3 l_4, q) \quad \text{and} \quad \text{im}_\mathbb{C}(\vartheta_2) = \text{span}(l_1 l_3, l_2 l_4, q).$$

By assumption, there exist $\lambda_1, \ldots, \lambda_5 \in \mathbb{C}$ not all zero such that

$$0 = \lambda_1 l_1 l_2 + \lambda_2 l_3 l_4 + \lambda_3 l_1 l_3 + \lambda_4 l_2 l_4 + \lambda_5 q.$$ 

We claim that $\lambda_5 = 0$. Assume otherwise, then

$$q = -\frac{1}{\lambda_5}(\lambda_1 l_1 l_2 + \lambda_2 l_3 l_4 + \lambda_3 l_1 l_3 + \lambda_4 l_2 l_4).$$
This gives
\[ f = \left( \frac{1}{\lambda_5} (\lambda_1 l_1 l_2 + \lambda_2 l_2 l_4 + \lambda_3 l_3 l_3 + \lambda_4 l_4 l_4) \right)^2 + \lambda_1 l_2 l_3 l_4. \]

But the right-hand side and its derivatives vanish at the intersection point of \( l_1 \) and \( l_4 \) which means that this point is a singular point of \( f \).

Hence, we get the equation
\[ \lambda_1 l_1 l_2 + \lambda_3 l_3 l_3 = -\lambda_2 l_2 l_4 - \lambda_4 l_4 l_4, \]
or equivalently
\[ l_1 (\lambda_1 l_2 + \lambda_3 l_3) = l_4 (-\lambda_2 l_2 - \lambda_4 l_2). \]

Since any two bitangents are no scalar multiple of each other, there exist \( 0 \neq \nu_1, \nu_2 \in \mathbb{C} \) such that
\[ l_1 = \nu_1 (-\lambda_2 l_2 - \lambda_4 l_4) \quad \text{and} \quad l_4 = \nu_2 (\lambda_1 l_2 + \lambda_3 l_3). \]

This means that all four bitangents intersect in a common point, namely \( \mathcal{Y}(l_2, l_3) \). \qed

**Remark 4.4.4.** In the situation of Theorem 4.4.3, assume that \( f \) has an automorphism \( \sigma \) of order 2. Let \( l_1, \ldots, l_4 \) be the four bitangents constructed in (ii). We see that \( \sigma \) fixes the 2-dimensional subspace \( \text{span}(l_1, \ldots, l_4) \subseteq A_1 \). As \( \dim A_1 = 3 \) and \( \sigma \) has order 2, this defines \( \sigma \).

Therefore, the four bitangents are uniquely defined by \( \sigma \). Indeed, Lemma 4.4.5 shows that there is no other bitangent of \( f \) contained in \( \text{span}(l_1, \ldots, l_4) \) and therefore no other bitangent is fixed by \( \sigma \).

**Lemma 4.4.5.** Let \( f \in \mathbb{R}[x]_4 \) be a smooth quartic. There can be at most four bitangents of \( f \) intersecting in a common point.

**Proof.** Assume there are five bitangents \( l_1, \ldots, l_5 \) of \( f \) intersecting in a common point. Since no three azygetic bitangents can intersect in a common point (Lemma 4.3.12), the bitangents \( l_1, \ldots, l_4 \) are syzygetic. Let \( S_1, S_2, S_3 \) be the three syzygetic Steiner complexes containing the bitangents \( l_1, \ldots, l_4 \) in the three different ways. Since any two of them only have these four bitangents in common (Proposition 4.3.3), the three Steiner complexes contain \( 3 \cdot 12 - 8 = 28 \) bitangents in total. Hence, one of them contains \( l_5 \). Assume this is \( S_1 \) and \( \{l_1, l_2\}, \{l_3, l_4\} \in S_1 \). Then, the three bitangents \( l_1, l_3, l_5 \) are azygetic by Lemma 4.3.11, hence cannot intersect in a common point, especially \( l_1, \ldots, l_5 \) do not intersect in a common point. \qed

**Lemma 4.4.6.** The semi-algebraic set of all \( f \in \mathbb{R}[x]_4 \) such that \( \text{Aut}(f) \) contains an element of order 2 has codimension 2 in \( \mathbb{R}[x]_4 \).

**Proof.** Let \( \sigma \) be the element of order 2 in \( \text{GL}_3(\mathbb{C}) \) that acts as follows
\[ \sigma : x \mapsto x, \ y \mapsto y, \ z \mapsto -z. \]

We first show the result over \( \mathbb{C} \). Every quartic \( f \in \mathbb{C}[x, y, z]_4 \) that contains \( \sigma \) in its automorphism group has the form
\[ f = \lambda z^4 + z^2 f_2(x, y) + f_4(x, y) \]
for some \( \lambda \in \mathbb{C} \), \( f_2 \in \mathbb{C}[x, y]_2 \), and \( f_4 \in \mathbb{C}[x, y]_4 \). Denote the vector space of such quartics by \( W \). The dimension of \( W \) is 9.
Wrt the basis $x,y,z$, the automorphism $\sigma$ is given by

$$\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$ 

We consider the set of all automorphisms of order 2, given by

$$S := \{ A\sigma A^{-1} : A \in \mathrm{GL}_3(\mathbb{C}) \}.$$ 

The set $S$ has dimension 4: consider the action of $\mathrm{GL}_3(\mathbb{C})$ on itself via conjugation. The stabilizer of $\sigma$ consists of all matrices $A$ that satisfy $\sigma A = A \sigma$. We easily check that these $A$ have the form

$$\begin{pmatrix} a_1 & a_2 & 0 \\ a_3 & a_4 & 0 \\ 0 & 0 & a_5 \end{pmatrix}$$

for some $a_1, \ldots, a_5 \in \mathbb{C}$. Hence, the stabilizer has dimension 5, and $S$ has dimension 4.

Consider the set

$$X = \{(f, \tau) \in \mathbb{C}[x,y,z]^4 \times \mathrm{GL}_3(\mathbb{C}) : \tau f = f, \tau^2 = \mathrm{id}, \tau \neq \mathrm{id}\}.$$ 

Projecting onto the second component, the image is exactly $S$ and thus has dimension 4. The fiber of any element $A\sigma A^{-1}$ is given by $AW$ and therefore has dimension 9. Especially, the dimension of $X$ is 13. The projection onto the first component has generically finite fibers: indeed, if the form $f \in \mathbb{C}[x,y,z]^4$ is smooth, its automorphism group is finite by [Dol12, §6.5]. Thus $\dim \text{pr}_1(X) = 13$. Hence, the result is true over $\mathbb{C}$.

We can perform the whole proof over $\mathbb{R}$ and see with the same argument that the result also holds over $\mathbb{R}$.

Remark 4.4.7. We note that in the last proof we cannot only do the proof over $\mathbb{R}$ without any additional knowledge, as there might be real forms $f \in \mathbb{R}[x,y,z]^4$ with non-real automorphisms. We would only see that the codimension is at most 2. Performing the same proof over $\mathbb{C}$ shows that the codimension cannot be smaller.

Corollary 4.4.8. Let $f \in \Sigma$ be a smooth quartic.

(i) If the Gram spectrahedron of $f$ has a face of dimension 1, then $f$ has an automorphism of order 2.

(ii) If $f$ has an automorphism of order 2, there are three rank 3 Gram tensors $\vartheta_1, \vartheta_2, \vartheta_3$ such that $\dim (\text{im}_C(\vartheta_i) + \text{im}_C(\vartheta_j)) \leq 4$ for all $i,j = 1,2,3$. If two of those three rank 3 Gram tensors are real and psd, the Gram spectrahedron has a face of dimension 1.

Especially, if $f \in \Sigma$ is chosen generically, its Gram spectrahedron has no faces of rank 4 and dimension 1.

Proof. (i): If $\text{Gram}(f)$ has a face $F$ of dimension 1 and rank 4, the boundary of $F$ consists of two Gram tensors $\vartheta_1, \vartheta_2$ of rank 3, since $f$ is smooth.

As $\dim (\text{im} \vartheta_1 + \text{im} \vartheta_2) = \dim \text{im}(\vartheta_1 + \vartheta_2) = 4$, it follows from Theorem 4.4.3 that there exists an automorphism of $f$ of order 2.

(ii) immediately follows from Theorem 4.4.3.
Next, we give an example of a psd ternary quartic with an automorphism of order 2 such that there is no 1-dimensional face on Gram(f).

**Remark 4.4.9.** Consider the following family of quartics

\[ f_{\alpha,\beta} = (z^2 + \alpha x^2 + \beta y^2)^2 + \prod_{j=1}^{4} (jx + y) \]

and \( \alpha, \beta \in \mathbb{R} \). Then \( f := f_{\alpha,\beta} \) is psd and smooth if \( \alpha, \beta \) are chosen large enough and generic.

Indeed, let \( g := \alpha x^2 + \beta y^2 \) and \( h := \prod_{j=1}^{4} (jx + y) \), then \( f = (z^2 + g)^2 + h \). If \( \alpha, \beta > 0 \), the form \( g \in \mathbb{R}[x,y]^2 \) is strictly positive (on \( \mathbb{R}^2 \)). We write

\[ f = (z^4 + 2z^2 g) + (g^2 + h). \]

Since \( g \) is positive (on \( \mathbb{R}^2 \)), the first summand is psd (on \( \mathbb{R}^3 \)). Since \( g^2 \) is strictly positive (on \( \mathbb{R}^2 \)), the form \( g^2 + h \in \mathbb{R}[x,y]^4 \) is also strictly positive if \( \alpha, \beta \) are chosen large enough. Especially \( f \) is strictly positive on \( \mathbb{R}^3 \).

Moreover, since \( \alpha, \beta \) are chosen generically, this quartic has an automorphism group isomorphic to \( C_2 \) by [Dol12, Theorem 6.5.2.].

The lines \( V(jx + y) \ (j = 1, \ldots, 4) \) are bitangents of \( f \) and intersect in a common point. Let \( S_1, S_2, S_3 \) be the three syzygetic Steiner complexes that contain these four bitangents as in Theorem 4.4.3. Let \( \vartheta_1, \vartheta_2, \vartheta_3 \) be the corresponding Gram tensors. It follows from Theorem 4.4.3 that \( \dim(\text{im}_C(\vartheta_i) + \text{im}_C(\vartheta_j)) \leq 4 \) for all \( i, j = 1, 2, 3 \).

However, since all four of these bitangents are real, it follows from [PSV11] Proposition 6.6. that the Gram tensors \( \vartheta_1, \vartheta_2, \vartheta_3 \) are not real psd. Especially, this automorphism of order 2 does not give rise to a face of dimension 1 on the Gram spectrahedron.

In fact, there is no 1-dimensional face on Gram(f) at all. Assume there are another four bitangents \( l_1, \ldots, l_4 \) intersecting in a common point. By Lemma 4.4.5 they intersect in a different point. From Remark 4.4.4 we know that the non-trivial automorphism fixes the 2-dimensional subspace of \( \mathbb{C}[x,y,z]^1 \) spanned by the four bitangents. This shows that the automorphism cannot fix the first 4 bitangents because then it would fix the whole space. Hence, there are two different automorphisms of order 2, contradicting the fact that \( \text{Aut}(f) \cong C_2 \).

This shows that although four bitangents are intersecting in a common point, there is no 1-dimensional face on the Gram spectrahedron.

**Lemma 4.4.10.** For a generic quartic \( f \in \Sigma \), the Gram spectrahedron Gram(f) has an extreme point of rank 4.

**Proof.** Every face corresponding to an edge in the Steiner graph has rank 5 and dimension 2, as \( f \) is generic. A generic point of the boundary of this face is then a rank 4 extreme point. □

**Remark 4.4.11.** In fact, on the Gram spectrahedron of every smooth quartic \( f \in \Sigma \), there is an extreme point of rank 4. However, to prove this, we need several results we show later on, and for Proposition 4.4.13 this weaker version is enough.

Nonetheless, we now prove it using results from the next sections.
Proposition 4.4.12. For every smooth quartic \( f \in \Sigma \), the Gram spectrahedron \( \text{Gram}(f) \) has an extreme point of rank 4.

Proof. By Corollary 4.4.20 at least six edges in the Steiner graph of \( f \) correspond to faces of rank 5 and dimension 2 on \( \text{Gram}(f) \). Let \( F \) be one such 2-dimensional face. By Theorem 4.5.7 the boundary of \( F \) is not the union of 1-dimensional faces of rank 4. Hence, there exists a 1-dimensional semi-algebraic set of rank 4 extreme points on the boundary of \( F \). \( \square \)

Proposition 4.4.13. There exists an open, dense, semi-algebraic subset \( U \subseteq \Sigma \) such that for every \( f \in U \)

\[
\dim S_f(4,0) = 3.
\]

I.e. the semi-algebraic set of all rank 4 extreme points on \( \text{Gram}(f) \) has dimension 3.

Proof. We use Proposition 3.9.11. The set \( W_{4,0} \subseteq S^+_2 \mathbb{R}[x]^4 \) of all rank 4 Gram tensors \( \vartheta \) such that the map \( S^2 \text{im}(\vartheta) \to \mathbb{R}[x]^4 \) is injective is dense in the set of all rank 4 psd Gram tensors. The condition is open by Proposition 2.3.22 and the set is non-empty since \( x^2 \otimes x^2 + y^2 \otimes y^2 + z^2 \otimes z^2 + (xy + zy) \otimes (xy + zy) \) is contained.

By Lemma 4.4.10 the Gram spectrahedron of a generic psd quartic contains an extreme point of rank 4. Therefore, the image of \( W_{4,0} \) under the multiplication map is dense in \( \Sigma \). By Proposition 3.9.11, the dimension of \( S_f(4,0) \) is therefore given by

\[
4 \cdot \dim \mathbb{R}[x]^2 - \mathbb{R}[x]^4 - \left( \begin{array}{c} 4 \\ 2 \end{array} \right) = 3
\]

for every \( f \) in an open, dense, semi-algebraic set \( U \subseteq \Sigma \). \( \square \)

We have already seen that 1-dimensional faces can only appear as line segments between two rank 3 tensors. However, not all faces represented in the Steiner graph of a smooth quartic can be of rank 4 at the same time.

Proposition 4.4.14. Let \( f \in \Sigma \) be smooth. There can be at most six 1-dimensional faces on \( \text{Gram}(f) \).

Proof. Let \( S_1, S_2 \) be two syzygetic Steiner complexes that appear in the Steiner graph. As they are syzygetic, they contain four common bitangents as in Proposition 4.3.3. If the edge in the Steiner graph connecting the two corresponds to a face of dimension 1 on \( \text{Gram}(f) \), these four bitangents intersect in a common point.

Looking at the Steiner graph, we see that in every case such a 4-tuple of bitangents contains two bitangents \( b_{1j}, b_{2i} \) with \( \{i,j\} \in \{\{3,4\}, \{5,6\}, \{7,8\}\} \). Moreover, every of the six edges in one \( K_4 \) corresponds to a different choice of \( \{i,j\} \), namely

\[
b_{13},b_{21}, \quad b_{14},b_{23}, \quad b_{15},b_{26}, \quad b_{16},b_{25}, \quad b_{17},b_{28}, \quad b_{18},b_{27},
\]

and these are all possibilities.

Assume there are seven or more 1-dimensional faces. Then there exist two 1-dimensional faces corresponding to the 4-tuples of bitangents \( b_{1j}, b_{2i}, l_1, l_2 \) and \( b_{1j}, b_{2i}, l_3, l_4 \) each intersecting in a common point for some \( \{i,j\} \in \{\{3,4\}, \{5,6\}, \{7,8\}\} \) and some bitangents \( l_1, l_2, l_3, l_4 \). However, as \( b_{1j}, b_{2i} \) are contained in both 4-tuples, all bitangents \( b_{1j}, b_{2i}, l_1, l_2, l_3, l_4 \) intersect in a common point which is not possible by Lemma 4.4.5 Note that \( l_1, l_2 \) and \( l_3, l_4 \) cannot be the same bitangents as they correspond to different edges in the Steiner graph. \( \square \)
Remark 4.4.15. It is not clear if there exists a smooth quartic such that \( \text{Gram}(f) \) contains six 1-dimensional faces. Even on the Gram spectrahedron of the Fermat quartic \( x^4 + y^4 + z^4 \), which has a large automorphism group, there are only three 1-dimensional faces.

We now discuss the action of an automorphism of order 2 on the Steiner graph.

Remark 4.4.16. Let \( f \in \Sigma \) be a smooth quartic such that \( \text{Gram}(f) \) has a 1-dimensional face \( F \). The two boundary points of \( F \) are rank 3 Gram tensors \( \vartheta_1, \vartheta_2 \), and there exist four bitangents \( L_1, \ldots, L_4 \) with \( L_i = V(l_i) \) and \( l_i \in A_1 \) for \( i = 1, \ldots, 4 \), intersecting in a common point such that \( l_1l_2, l_3l_4 \in \text{im}_C(\vartheta_1) \) and \( l_1l_3, l_2l_4 \in \text{im}_C(\vartheta_2) \). The automorphism \( \sigma \) corresponding to this face is defined by the fact that it fixes the 2-dimensional subspace in \( A_1 \) spanned by the bitangents \( l_1, \ldots, l_4 \) point-wise.

The automorphism \( \sigma \) acts on the rank 3 Gram tensors of \( f \) and we may ask, what this action is.

First, we observe that \( \sigma \) also acts on the bitangents by permuting them: let \( l \in A_1 \) be a bitangent of \( f \) and \( g \in A_2 \), \( h \in A_3 \) such that

\[
f = g^2 + lh.
\]

Applying \( \sigma \) we get

\[
f = \sigma(g)^2 + \sigma(l)\sigma(h)
\]

and hence \( \sigma(l) \) is also a bitangent of \( f \). Moreover, it also acts on the Steiner complexes of \( f \): let \( S = \{\{l_i, l_j'\} : i = 1, \ldots, 6\} \) be a Steiner complex of \( f \). By definition, for every \( 1 \leq i < j \leq 6 \) there exist \( q \in A_2 \) and \( \lambda \in \mathbb{C} \) such that \( f = \lambda l_i' l_j' + q^2 \). Applying \( \sigma \), we get

\[
f = \lambda \sigma(l_i) \sigma(l_j') \sigma(l_j') + \sigma(q)^2.
\]

Hence, \( \sigma(S) := \{\{\sigma(l_i), \sigma(l_j')\} : i = 1, \ldots, 6\} \) is again a Steiner complex of \( f \).

Let \( S_1 \) and \( S_2 \) be the Steiner complexes corresponding to the Gram tensors \( \vartheta_1 \) and \( \vartheta_2 \). By assumption, \( \{l_1, l_2\}, \{l_3, l_4\} \in S_1 \) and \( \{l_1, l_3\}, \{l_2, l_4\} \in S_2 \). Then, \( \sigma(S_1) \) again contains \( \{l_1, l_3\} \) and \( \{l_2, l_4\} \), but there is only one Steiner complex containing these two sets, which is \( S_1 \). Thus \( \sigma(S_1) = S_1 \). The same holds for \( S_2 \) and therefore the corresponding Gram tensors are also fixed by \( \sigma \). However, the action on the Steiner complexes \( S_1 \) and \( S_2 \) is even more constrained. We may write

\[
f = \lambda l_1l_2l_3l_4 + q^2,
\]

and as \( \text{im}_C(\vartheta_1) \) is spanned by \( l_1l_2, l_3l_4, l_5l_6 \) (Lemma 4.3.10) for every pair \( l_5, l_6 \) of bitangents such that \( \{l_5, l_6\} \in S_1 \) and \( \{l_5, l_6\} \neq \{l_1, l_2\}, \{l_3, l_4\} \), we can write \( q = \nu_1l_1l_2 + \nu_2l_3l_4 + \nu_3l_5l_6 \) for some \( \nu_1, \nu_2, \nu_3 \in \mathbb{C}, \nu_3 \neq 0 \). Applying \( \sigma \) to this representation of \( f \), we get

\[
f = \lambda l_1l_2l_3l_4 + (\nu_1l_1l_2 + \nu_2l_3l_4 + \nu_3\sigma(l_5)\sigma(l_6))^2.
\]

Subtracting the two representations, we get

\[
(\nu_1l_1l_2 + \nu_2l_3l_4 + \nu_3l_5l_6)^2 = q^2 = (\nu_1l_1l_2 + \nu_2l_3l_4 + \nu_3\sigma(l_5)\sigma(l_6))^2
\]

which means that \( q = \nu_1l_1l_2 + \nu_2l_3l_4 + \nu_3\sigma(l_5)\sigma(l_6) \) up to sign. But as the first two summands have the same sign, they are equal. Especially, \( \sigma(l_5)\sigma(l_6) = l_5l_6 \). As mentioned earlier, \( \sigma \) cannot fix any other bitangents than \( l_1, \ldots, l_4 \), as this would mean that the bitangents \( l_1, \ldots, l_4 \) and \( l_5 \) or \( l_6 \) intersect in a common point, but only 4 bitangents can do so by
Therefore, \( \sigma(l_5) = \nu l_6 \) and \( \sigma(l_6) = \nu' l_5 \) for some \( \nu, \nu' \in \mathbb{C} \). Since \( \sigma \) has order 2, it follows that \( \nu = \frac{1}{\nu'} \), hence replacing \( l_5 \) by \( \nu' l_5 \), we can assume that \( \sigma(l_5) = l_6 \) and \( \sigma(l_6) = l_5 \). The same argument works for \( S_2 \) instead of \( S_1 \). Therefore, we know what happens to all bitangents contained in one of the two Steiner complexes. Namely, every element of the Steiner complexes is fixed, but the bitangents are swapped.

We now recall that there is a third Steiner complex \( S_3 \) that forms a syzygetic pair together with \( S_1 \) and together with \( S_2 \), and that contains \( \{l_1, l_4\}, \{l_2, l_3\} \). This Steiner complex gives rise to a rank 3 Gram tensor that is not on the Gram spectrahedron of \( f \). However, the same argument as above also shows that for every set \( \{l_5, l_6\} \in S_3 \) that is not equal to \( \{l_1, l_4\} \) or \( \{l_2, l_3\} \), \( \sigma \) maps \( l_5 \) to \( l_6 \) and vice versa. By [Dol12, Theorem 5.4.10.], we know that any of the 28 bitangents of \( f \) is contained in one of the three Steiner complexes, hence we know the action of \( \sigma \) on the bitangents.

For the next part, we recall the Steiner graph of \( f \) as shown in Fig. 4.1.

![Steiner Graph](image)

**Remark 4.4.17.** Assume there is a face of rank 4 and dimension 1 on Gram(\( f \)). Let \( O_1, \ldots, O_8 \) be our fixed Cayley octad such that complex conjugation acts as in Theorem 4.2.16 i.e. such that \( O_i \) is mapped to \( O_{i+1} \) for \( i = 1, 3, 5, 7 \). We may permute the Cayley octad in the following two ways. Either we swap the sets \( \{O_i, O_{i+1}\} \) and \( \{O_j, O_{j+1}\} \) for some \( i, j \in \{1, 3, 5, 7\} \) or we swap \( O_i \) and \( O_{i+1} \) for some \( i \in \{1, 3, 5, 7\} \). Afterward, complex conjugation still acts on the Cayley octad as required for Theorem 4.2.16. We may therefore assume that the 1-dimensional face is the line segment between the Gram tensors corresponding to 1457 and 1358.

The four bitangents intersecting in a common point that correspond to this face are \( b_{15}, b_{47}, b_{38}, b_{26} \). The third Steiner complex containing these four such that the three Steiner complexes are pairwise syzygetic is 1256, which as we can see from Theorem 4.2.16 corresponds to a real Gram tensor which is not psd.

As shown above, all elements (pairs of bitangents) of these three Steiner complexes are fixed and all bitangents except for \( l_1, \ldots, l_4 \) are not fixed. We first consider the Steiner complex 1457. It contains the elements

\[ \{b_{15}, b_{47}\}, \{b_{14}, b_{57}\}, \{b_{17}, b_{45}\}, \{b_{23}, b_{68}\}, \{b_{28}, b_{36}\}, \{b_{26}, b_{38}\}. \]

The bitangents in the first and the last are fixed, and on the other bitangents, \( \sigma \) acts as follows

\[ b_{14} \leftrightarrow b_{57}, \quad b_{17} \leftrightarrow b_{45}, \quad b_{23} \leftrightarrow b_{68}, \quad b_{28} \leftrightarrow b_{36}. \]

We now consider 1358 and 1256 and find the following action of \( \sigma \) on the bitangents

\[ b_{13} \leftrightarrow b_{58}, \quad b_{18} \leftrightarrow b_{35}, \quad b_{24} \leftrightarrow b_{67}, \quad b_{27} \leftrightarrow b_{46}. \]

\[ b_{12} \leftrightarrow b_{56}, \quad b_{16} \leftrightarrow b_{25}, \quad b_{34} \leftrightarrow b_{78}, \quad b_{37} \leftrightarrow b_{48}. \]
We can now determine what the other psd Gram tensors are mapped to. We do this for 1357. This Steiner complex contains the following elements

\[ \{b_{13}, b_{57}\}, \{b_{15}, b_{37}\}, \{b_{17}, b_{35}\}, \{b_{24}, b_{68}\}, \{b_{26}, b_{48}\}, \{b_{28}, b_{46}\}. \]

These are mapped to

\[ \{b_{14}, b_{58}\}, \{b_{15}, b_{48}\}, \{b_{18}, b_{45}\}, \{b_{23}, b_{67}\}, \{b_{26}, b_{37}\}, \{b_{27}, b_{36}\}. \]

which is the Steiner complex 1458. Doing the same for the other Steiner complexes we find that in fact all Steiner in the \( K_4 \) containing 1457 are fixed by \( \sigma \), and in the other \( K_4 \) the action is as follows

\[ 1357 \leftrightarrow 1458 \quad 1368 \leftrightarrow 1467. \]

This also implies that all points on the supporting faces of two rank 3 tensors in the first \( K_4 \) are fixed by \( \sigma \), and on the other \( K_4 \) two faces are fixed, but not point-wise and all others are not.

Remark 4.4.18. The action on the bitangents can also be described as a permutation on the Cayley octad. We write \( f = \det(Ax + By + Cz) \) for some real symmetric matrices \( A, B, C \). Then the Cayley octad corresponding to this linear symmetric determinantal representation is the zero set of \( uAu^T = uBu^T = uCu^T = 0 \) in \( \mathbb{P}^3 \) where we fix some order \( O_1, \ldots, O_8 \). Applying \( \sigma \), we get

\[ f = A\sigma(x) + B\sigma(y) + C\sigma(z) = A'x + B'y + C'z \]

where \( A', B', C' \) are linear combinations of \( A, B, C \). Especially, the zero set of the corresponding quadratic forms does not change.

However, the action of \( \sigma \) on the bitangents does induce a permutation action on the Cayley octad. In the situation above we get the following action:

\[ O_1 \leftrightarrow O_5, \quad O_2 \leftrightarrow O_6, \quad O_4 \leftrightarrow O_7, \quad O_3 \leftrightarrow O_8. \]

Most importantly, we notice that the set \( \{O_1, O_2\} \) is mapped to \( \{O_5, O_6\} \) and vice versa, and the same holds for the sets \( \{O_3, O_4\} \) and \( \{O_7, O_8\} \). This shows (as we have already proven) that real psd Gram tensors are permuted, as complex conjugation still acts on the Cayley octad as required in Theorem 4.2.16.

4.4.2 Rank 5 points

We have already seen that faces of rank 5 are either extreme points or faces of dimension 2, whenever we restrict to Gram spectrahedra of forms in the interior of the sos cone.

Remark 4.4.19. Let \( f \in \text{int} \Sigma \). Then any interior point of \( \text{Gram}(f) \) has rank 6. The boundary is smooth at a generic point and hence has rank 5 by Lemma 2.1.5.

Furthermore, if \( f \) is smooth and has no automorphism of order 2, then \( \text{Gram}(f) \) has faces of rank 5 and dimension 2, namely the ones in its Steiner graph.

In fact, every smooth quartic has a face of rank 5 and dimension 2.

Corollary 4.4.20. For every smooth \( f \in \Sigma \), the Gram spectrahedron \( \text{Gram}(f) \) has a face of rank 5 and dimension 2.
Proof. At most 6 faces in the Steiner graph have rank 4 by Proposition 4.4.14. Therefore, the other faces have rank 5 and dimension 2.

**Proposition 4.4.21.** There exists an open, dense, semi-algebraic subset $W \subseteq \Sigma$ such that for every $f \in W$

$$\dim S_f(5,0) = 5, \quad \mathrm{and} \quad \dim S_f(5,2) = 4.$$  

Proof. As in Proposition 3.9.11 we consider the semi-algebraic set $W_{5,2} \subseteq S_2^+ \mathbb{R}[x]_2$ of all rank 5 psd tensors such that the multiplication map $S_2 \operatorname{span}(p) \hookrightarrow \mathbb{R}[x]_4$ has a kernel of dimension 2. First, we see that both spaces $S_2 \operatorname{span}(p)$ and $\mathbb{R}[x]_4$ have dimension 15. Moreover, the map never has a kernel of dimension 1 by Proposition 4.1.5. Therefore, $W_{5,2}$ is the subset of all rank 5 tensors in $S_2^+ \mathbb{R}[x]_2$ such that the determinant of the multiplication map vanishes. Especially, $W_{5,2}$ should be a codimension 1 subset of the set of all psd rank 5 tensors in $S_2^+ \mathbb{R}[x]_2$. This is indeed true, and we check this in Lemma 4.4.22. Hence, $\dim W_{5,2} = 19$ as the subset of rank 5 tensors has dimension 20.

Since a generic $f \in \Sigma$ has a face of rank 5 and dimension 2, it follows from Proposition 3.9.11 that $\dim S_f(5,2) = 19 - \dim \mathbb{R}[x]_4 = 4$. As $\dim S_f(5,2) = 4$, and rank 5 tensors are dense in $\partial \operatorname{Gram}(f)$, which has dimension 5, it follows that $\dim S_f(5,0) = 5$.

**Lemma 4.4.22.** Let $W_{5,2} \subseteq S_2^+ \mathbb{R}[x]_2$ be as above. Then $\dim W_{5,2} = 19$.

Proof. Let $q \in W_{5,2}$ and let $t = \operatorname{im}(q)$. Let $\theta \in \mathbb{R}[x]_2$ such that $\dim \operatorname{span}(q) = 4$.

For generic $q \in X$, the fiber is isomorphic to $\mathbb{P}(\mathbb{R}^5 \times O(5))$: Indeed, in terms of matrices, this means that such a matrix has kernel $q$, the eigenvalues are given by the vector in $\mathbb{R}^5$, and as $q$ is generic, the orthogonal matrix used to diagonalize is unique. Hence,

$$\dim k^{-1}(X) = 4 + \binom{5}{2} - 1 = 18.$$  

Let $C = \{q \in S_2^+ \mathbb{R}[x]_2 : \dim \operatorname{ker}(q) = 4\}$. Then, $W_{5,2} = C \cap S_2^+ \mathbb{R}[x]_2$. Since we may slightly perturb any psd tensor of rank 5 inside $C$ and stay psd, the semi-algebraic set $C \cap S_2^+ \mathbb{R}[x]_2$ has the same dimension as $C$, and $\dim C = \dim k^{-1}(X) + 1 = 19$.

Lastly, we show that every smooth quartic has an extreme point of rank 5. This is not surprising as for a generic sos quartic, rank 5 extreme points are even dense in the boundary of its Gram spectrahedron.

**Theorem 4.4.23.** For every smooth quartic $f \in \Sigma$, the semi-algebraic set of rank 5 extreme points is dense in the boundary of $\operatorname{Gram}(f)$.

Especially, every smooth sos quartic has a rank 5 extreme point on its Gram spectrahedron.

**Proof.** Assume there is $f \in \Sigma$ smooth such that rank 5 extreme point are not dense in $\partial \operatorname{Gram}(f)$. As rank 5 points are dense in $\partial \operatorname{Gram}(f)$ by Lemma 2.1.5, but rank 5 extreme point are not, it follows from Theorem 4.1.2 that $\dim S_f(5,2) = 5$.  

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We show that this implies dim $S_f(4,0) = 4$. Consider the semi-algebraic set
\[ X = \{(\vartheta, \delta) \in S_f(5,2) \times S_f(4,0) : \text{im}\ \delta \subseteq \text{im}\ \vartheta \}. \]
The projection onto the first factor has dimension 5 by assumption, and for $\vartheta \in S_f(5,2)$ the
dimension of the fiber is 1, which is given by all tensors $\delta \in S_f(4,0)$ lying on the boundary
of the supporting face of $\vartheta$. We note that as this face of dimension 2 is not polyhedral by
Theorem 4.5.7, the boundary always contains a 1-dimensional semi-algebraic set of rank 4
extreme points. Therefore, dim $X = 6$. We claim that the dimension of the projection onto
the second factor has dimension 4. Equivalently, we may show that for $\delta \in pr_2(X)$ the fiber
has dimension 2.
Let $\delta \in S_f(4,0)$, and $U := \text{im}\ \delta$. Let $W = U^\perp$, then dim $W = 2$. Every 1-dimensional
subspace $W'$ of $W$ gives rise to a subspace $U \oplus W'$ of dimension 5 containing $U$. By
Corollary 4.1.7 and Proposition 4.1.5 the dimension of $(U \oplus W')^2$ is 13 if and only if any
quadratic form spanning $(U \oplus W')^2$ has rank 2. Since $\mathbb{P} W \cong \mathbb{P}^1$ is irreducible, either every
non-zero quadratic form $q \in W$ has rank at most 2 or there are only finitely many of them,
and a generic $q \in W$ has rank 3.
Assume the fiber $pr_2^{-1}(\delta)$ has dimension 3. Then there are infinitely many 2-dimensional
faces of rank 5 containing $\delta$ on their boundaries. Especially, every element in $W$ has rank
at most 2.
We identify every quadratic form in $W$ with its symmetric matrix (wrt the basis $x, y, z$
of $\mathbb{R}[x, y, z]$), and still write $W$ for this space. By Lemma 4.4.24, we may assume that after a
change of coordinates, every element in $W$ has the form
\[
\begin{pmatrix}
0 & 0 & * \\
0 & 0 & * \\
* & * & *
\end{pmatrix}
\]
or every element in $W$ has the form
\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & * & * \\
0 & * & *
\end{pmatrix}.
\]
We note that by Lemma 4.4.24 this is a change of coordinates where the matrices are
considered as quadratic forms and not just linear maps.
In case (a), every element in $W$ has the form $a_1xz + a_2yz + a_3z^2$ for some $a_1, a_2, a_3 \in \mathbb{R}$. As
$U^\perp = W$, we see $x^2, y^2, xy \in U$. Hence, the element $x^2 \otimes y^2 - xy \otimes xy \in S_2U$ is contained
in the kernel of the map $S_2U \to U^2$ which shows dim $U^2 \leq 9$. From Remark 3.1.17, we get
equality.
In case (b), every element of $W$ has the form $a_1y^2 + a_2yz + a_3z^2$ for some $a_1, a_2, a_3 \in \mathbb{R}$. Therefore, $x^2, xy, xz \in U$. There exists $g \in \mathbb{R}[x, y, z]$ such that $U = \text{span}(x^2, xy, xz, g)$ which
shows that $\emptyset \neq \mathcal{V}(x, y) \subseteq \mathcal{V}(U)$.
Hence, the subspace $U$ either has a base-point, or satisfies dim $U^2 = 9$. This implies that
either $f$ is singular or $\delta$ lies in the interior of a face of dimension 1, both of which are not
the case.
Therefore, the fiber $pr_2^{-1}(\delta)$ has dimension 2, and dim $S_f(4,0) = 4$. The dimension of the
normal cone at $\delta$ is 3 (if read inside the affine hull of Gram($f$) which has dimension 6) by
Example 4.6.7, hence dim $S_f(4,0) + \dim \mathcal{N}(\delta) = 7 > 6$ for every $\delta \in S_f(4,0)$. But this is
impossible: we see for example from the proof of Sim15, Theorem 3.8, that dim $S_f(4,0) + \dim \mathcal{N}(\delta) \leq 6$.  \qed
Lemma 4.4.24. Let $W \subseteq \mathbb{S}^3$ be a subspace of dimension 2 such that every matrix in $W$ has rank at most 2. Then there exists $S \in \text{GL}_3(\mathbb{R})$ such that every matrix in $S^TWS = \{S^TAS : A \in W\}$ has the form

$$
(a) \begin{pmatrix}
0 & 0 & *
\end{pmatrix}
$$

or every matrix has the form

$$
(b) \begin{pmatrix}
0 & 0 & 0 \\
* & * & *
\end{pmatrix}
$$

Proof. By [1115] table 4] or by the first lines of the proof of [1115] Theorem 1.1], there exists $i \in \{1, 2, 3\}$ and a subspace $L \subseteq \mathbb{R}^3$ of dimension $i$, as well as a subspace $V \subseteq \mathbb{R}^3$ of dimension $i - 1$, such that every matrix in $W$ maps $L$ to $V$. We note that these results do not require the matrices in $W$ to be symmetric. Let $A, B \in W$ be a basis of $W$.

(i): Let $i = 1$. Then there exists a subspace $L$ of dimension 1, that is mapped to zero by every matrix in $W$, i.e. lies in the kernel of these matrices. Let $S \in \text{GL}_3(\mathbb{R})$ be a matrix that maps $L$ to the span of $e_1 = (1, 0, 0)^T \in \mathbb{R}^3$. Then both matrices $S^TAS$ and $S^TBS$ have the form in (b), and are a basis of $S^TWS$. Therefore, every matrix in $S^TWS$ has the form in (b).

(ii): Let $i = 3$. As the symmetric matrix $A$ has image $V$, which is a 2-dimensional vector space, the kernel of $A$ is given by its orthogonal complement $V^\perp \subseteq \mathbb{R}^3$. The same holds for the matrix $B$. Thus, as in case (i) we map $V^\perp$ to the span of $e_1$ and are again in case (b).

(iii): Let $i = 2$. We write $L = \text{span}(u_1, u_2)$ and $V = \text{span}(v)$. Let $S \in \text{GL}_3(\mathbb{R})$ be the matrix that maps $e_j$ to $u_j$ ($j = 1, 2$) and let $C = \lambda_1A + \lambda_2B$ with indeterminates $\lambda_1, \lambda_2$. For $j = 1, 2$, we have

$$
(S^TCS)e_j = \lambda_1S^T(Au_j) + \lambda_2S^T(Bu_j) = \lambda_1\nu_{1j}(S^Tv) + \lambda_2\nu_{2j}(S^Tv) = (\lambda_1\nu_{1j} + \lambda_2\nu_{2j})(S^Tv)
$$

for some $\nu_{1j}, \nu_{21}, \nu_{12}, \nu_{22} \in \mathbb{R}$. We write $v' = S^Tv = (v_1, v_2, v_3)^T \in \mathbb{R}^3$ and $l_j = \lambda_1\nu_{1j} + \lambda_2\nu_{2j} \in \mathbb{R}[\lambda_1, \lambda_2]_1$ for $j = 1, 2$. With this notation, the matrix $S^TCS$ has the form

$$
\begin{pmatrix}
l_1v_1 & l_2v_1 & a_1 \\
l_1v_2 & l_2v_2 & a_2 \\
l_1v_3 & l_2v_3 & a_3
\end{pmatrix}
$$

for some $a_1, a_2, a_3 \in \mathbb{R}[\lambda_1, \lambda_2]_1$. This matrix is still symmetric, hence

$$
l_1v_2 = l_2v_1, \quad l_1v_3 = a_1, \quad l_2v_3 = a_2.
$$

If $l_1 = 0$ or $l_2 = 0$, we immediately see that we are in case (b). If $l_1, l_2 \neq 0$, but $v_2 = 0$, it follows due to symmetry that $v_1 = 0$, and therefore we are in case (a).

Now assume $l_1, l_2 \neq 0$ and $v_1 \neq 0$. We add $(-\frac{v_2}{v_1})$ times the first column to the second, and $(-\frac{v_3}{v_1})$ times the first row to the second. Because of the equations above, this results in the matrix

$$
\begin{pmatrix}
l_1v_1 & 0 & l_1v_3 \\
0 & 0 & 0 \\
l_1v_3 & 0 & a_3
\end{pmatrix}
$$

Swapping columns one and two, as well as rows one and two, we reach the form in (b). \qed
Remark 4.4.25. Summarizing our findings concerning faces, we know the following. Every smooth ternary quartic $f \in \Sigma$ has extreme points of rank 3, rank 4, and rank 5, as well as 2-dimensional faces of rank 5 (Proposition 4.4.12, Theorem 4.4.23, Corollary 4.4.20) and rank 5 extreme points are dense in the boundary of Gram($f$).

If $f \in \Sigma$ is a generic ternary quartic, extreme points of rank 4 form a 3-dimensional semi-algebraic subset, and rank 5 points that are in the relative interior of 2-dimensional faces form a semi-algebraic set of dimension 4. Furthermore, there are no other faces of positive dimension on Gram($f$) since $f$ is generic. As was already known, there are exactly eight extreme points of rank 3, and exactly twelve line segments between them are contained in the boundary of Gram($f$) as shown in the Steiner graph.

We summarize this in the following table.

<table>
<thead>
<tr>
<th>rank $r$</th>
<th>dimension of face $s$</th>
<th>dimension of $S_f(r, s)$</th>
<th>appearance</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>every smooth $f$</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>3 (generic $f$)</td>
<td>every smooth $f$</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>1</td>
<td>only if 2</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>5</td>
<td>every smooth $f$</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>4 (generic $f$)</td>
<td>every smooth $f$</td>
</tr>
</tbody>
</table>

Example 4.4.26. We consider the Fermat quartic $f = x^4 + y^4 + z^4$. The automorphism group of $f$ has order 96 and can be found in [Dol12, Theorem 6.5.2]. We therefore should expect to find several 1-dimensional faces on Gram($f$). This is true, and we can even visualize a 3-dimensional slice of Gram($f$) where we can see one of the two $K_4$ in the Steiner graph as well as the 2-dimensional faces (see Fig. 4.2).

It is in general not the case that the 3-dimensional space spanned by the four rank 3 Gram tensors in one $K_4$ of its Steiner graph also contains a face of dimension 2 and rank 5.

We consider the 3-dimensional affine slice of Gram($f$) spanned by the four rank 3 Gram tensors in one $K_4$ of the Steiner graph of $f$. More precisely, it is the $K_4$ containing the Gram tensor corresponding to the sos representation $f = (x^2)^2 + (y^2)^2 + (z^2)^2$. In Fig. 4.2 we see the algebraic boundary of this affine slice.

It is given by the determinant of the matrix pencil

$$G(\lambda_1, \lambda_2, \lambda_3) = \vartheta_0 + \lambda_1(\vartheta_1 - \vartheta_0) + \lambda_2(\vartheta_2 - \vartheta_0) + \lambda_3(\vartheta_3 - \vartheta_0)$$

where

$$\vartheta_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \vartheta_1 = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\vartheta_2 = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \vartheta_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. $$
Figure 4.2: The algebraic boundary of the Gram spectrahedron of the Fermat quartic intersected with the 3-dimensional affine space spanned by the four rank 3 psd Gram tensors $\vartheta_0, \ldots, \vartheta_3$.

The part that is contained in $\text{Gram}(f)$ is the inner convex part of the red orthant, or more precisely, the (closure of the) connected component of the complement of the algebraic boundary containing the tensor $\frac{1}{4} \sum_{i=0}^{3} \vartheta_i$. The Gram tensor $\vartheta_0$ corresponding to the representation $f = (x^2)^2 + (y^2)^2 + (z^2)^2$ is the intersection point of the three hyperplanes and lies in the middle of this figure. The other three rank 3 psd tensors in the same $K_4$ in the Steiner graph are $\vartheta_1, \vartheta_2, \vartheta_3$. All of them are connected to $\vartheta_0$ via a rank 4 and dimension 1 face which is the line segment contained in the intersection of two of the hyperplanes. All other line segments connecting two rank 3 tensors are contained in a 2-dimensional face, each contained in one of the three hyperplanes.

Contrary to what we might expect at first, due to the size of the automorphism group, the other $K_4$ in the Steiner graph of $f$ does not contain any 1-dimensional faces at all. We visualize part of the second $K_4$ in Fig. 4.3 it is the yellow cone shape containing the convex hull of the points $\delta_0, \delta_1,$ and $\delta_2$. We note that we can only visualize three of the four rank 3 extreme points if we also want to see a 2-dimensional face. Furthermore, the 1-dimensional faces connecting $\delta_0$ to $\delta_1$ and $\delta_2$ are 2-dimensional faces on the Gram spectrahedron and only have dimension 1 due to visualizing a lower dimensional slice. However, we see the structure of a 'usual' 2-dimensional face with two rank 3 points $\delta_1$ and $\delta_2$ on its boundary.
4.5 Polyhedral faces

Let \( n \in \mathbb{N} \). A semi-algebraic set \( S \subseteq \mathbb{R}^n \) is called polyhedral, if \( S \) is the intersection of finitely many half spaces, i.e. there exist linear forms \( l_1, \ldots, l_r \in \mathbb{R}[x_1, \ldots, x_n] \) such that

\[
S = \{ x \in \mathbb{R}^n : l_i(x) \geq 0, \forall i = 1, \ldots, r \}.
\]

We show that the only polyhedral faces on Gram spectrahedra of ternary quartics are either 0-dimensional or 1-dimensional of rank 4.

As we have already seen, for generic sos quartics there are no faces of dimension 1 and rank 4, hence generically there are no positive dimensional polyhedral faces at all.

For the next proofs, we not only work with Gram tensors but also consider Gram matrices.

**Remark 4.5.1.** We consider \( \mu^{-1}(0) \subseteq S_2 \mathbb{C}[x] \), the set of all Gram tensors of 0. Wrto the monomial basis of \( \mathbb{C}[x] \) ordered as at the beginning of Section 4.2 the set \( \mu^{-1}(0) \) is given by all matrices

\[
\begin{pmatrix}
0 & \lambda_1 & \lambda_2 & 0 & 0 & \lambda_4 \\
\lambda_1 & 0 & \lambda_3 & 0 & \lambda_5 & 0 \\
\lambda_2 & \lambda_3 & 0 & \lambda_6 & 0 & 0 \\
0 & 0 & \lambda_6 & -2\lambda_1 & -\lambda_4 & -\lambda_5 \\
0 & \lambda_5 & 0 & -\lambda_4 & -2\lambda_2 & -\lambda_6 \\
\lambda_4 & 0 & 0 & -\lambda_5 & -\lambda_6 & -2\lambda_3
\end{pmatrix}
\]

with \( \lambda_1, \ldots, \lambda_6 \in \mathbb{C} \).

We read this matrix \( \vartheta \) as a matrix with entries in \( \mathbb{R}[\lambda_1, \ldots, \lambda_6] \) where \( \lambda_1, \ldots, \lambda_6 \) are indeterminates. Calculating its determinants, we find

\[
\det \vartheta = -p^2 \text{ with } p = -4\lambda_1\lambda_2\lambda_3 + \lambda_3\lambda_4^2 + \lambda_2\lambda_5^2 + \lambda_4\lambda_5\lambda_6 + \lambda_1\lambda_6^2.
\]
and every $5 \times 5$-minor of $\vartheta$ is divisible by $p$. This means that there is no rank 5 Gram tensor of 0.

For 2-dimensional faces this means the following: Let $F$ be a 2-dimensional face on a Gram spectrahedron of a ternary quartic with corresponding subspace $U$. Let $\vartheta_1, \vartheta_2$ be a basis of the kernel of the map $S_2 U \rightarrow U^2$. Then, $\dim(\text{im}(\vartheta_1) + \text{im}(\vartheta_2)) = 5$ but every tensor in the kernel has rank 3 or 4.

The following is a reformulation of [Ram98, Theorem 1] adapted to our situation.

**Theorem 4.5.2** ([Ram98, Theorem 1]). Let $U \subseteq \mathbb{R}[x_1, \ldots, x_n]$ be a subspace and $L \subseteq S_2 U$ an affine-linear subspace. Assume that the spectrahedron $S = S_2^+ U \cap L$ is polyhedral and that there exists an interior point $\vartheta \in S$ such that $\vartheta \succ 0$. Then, there exists an integer $r \in \mathbb{N}$ and forms $p_1, \ldots, p_r \in U$ such that $S = \{ \vartheta \in S_2 U \cap L : \vartheta = \sum_{i=1}^r \lambda_i p_i \otimes p_i + \gamma, \lambda_1, \ldots, \lambda_r \geq 0, \gamma \in S_2 V \}$, where $V = \text{span}(p_1, \ldots, p_r) \perp \subseteq U$ is the orthogonal complement of $\text{span}(p_1, \ldots, p_r)$ in $U$.

**Remark 4.5.3.** In terms of matrices Theorem 4.5.2 says the following: If $S$ is defined by the matrix pencil $A(x) = A_0 + \sum_{i=1}^s x_i A_i$ with real symmetric $n \times n$ matrices $A_0, \ldots, A_s$, then there exists an invertible matrix $X \in \text{GL}_n(\mathbb{R})$ such that

$$X^T A(x) X = \begin{pmatrix} l_1(x) & \cdots & l_r(x) \\ \vdots \ & \ddots \ & \vdots \\ 0 \ & \cdots \ & 0 \end{pmatrix}$$

for linear polynomials $l_1, \ldots, l_r \in \mathbb{R}[x_1, \ldots, x_n]$ and some matrix pencil $B(x)$. Moreover, $y \in S$ if and only if $l_1(y), \ldots, l_r(y) \geq 0$.

**Remark 4.5.4.** We show that if there was a polyhedral face of dimension 2 and rank 5 on a Gram spectrahedron of a ternary quartic, then the bottom right block in the matrix above can also be diagonalized.

It is clear that such a polyhedral face is bounded by either three, four or five 1-dimensional faces. In the two latter cases, the matrix is diagonal. In the first case, it might not be a priori. However, the fact that there is no rank 5 tensor of 0 can be used to show that we can pull out one more diagonal entry from $B(x)$, i.e. we can change coordinates such that $B(x)$ maps to

$$\begin{pmatrix} l_4(x) \\ 0 \end{pmatrix}$$

which means that $C(x)$ is a $1 \times 1$ matrix and our matrix can be fully diagonalized.

For the next proof, we use the following notation. Let $\vartheta \in S_2 U$, then we write $\det(\vartheta)$ for the determinant of the bilinear map $b_\vartheta$ defined by $\vartheta$. As a bilinear map does not have a well-defined determinant, we read this in $\mathbb{PR} = \mathbb{P}_1(\mathbb{R})$ where it is well-defined. i.e. it is the determinant of $b_\vartheta$ wrt to some chosen basis basis modulo $\mathbb{R}$. This allows us to keep everything coordinate-free and avoids having to track base-changes.
Lemma 4.5.5. Let \( f \in \text{int}\, \Sigma \). Let \( F \subseteq \text{Gram}(f) \) a 2-dimensional face of rank 5 with corresponding subspace \( U \subseteq \mathbb{R}[x]^2 \). Assume there exist \( p_1, p_2, p_3 \in U \) such that \( F \) is the set of all tensors \( \vartheta \in S_2 U \) with

\[
\vartheta = \sum_{i=1}^{3} \lambda_i p_i \otimes p_i + \gamma
\]

for some \( \gamma \in S_2 V \) where \( V := \text{span}(p_1, p_2, p_3)^\perp \subseteq U \) and \( \lambda_1, \lambda_2, \lambda_3 \geq 0 \). Then, there exist \( p_4, p_5 \in U \) such that \( \text{span}(p_1, \ldots, p_5) = U \), and every tensor in \( F \) has the form

\[
\vartheta = \sum_{i=1}^{5} \lambda_i p_i \otimes p_i
\]

for some \( \lambda_1, \ldots, \lambda_5 \in \mathbb{R} \).

Proof. Let \( \vartheta_1, \vartheta_2 \in S_2 U \) be a basis of the kernel of \( S_2 U \rightarrow U^2 \), and let \( \vartheta \in \text{relint} \, F \). Then \( F \) is the set of all psd Gram tensors of the form \( \vartheta + \nu_1 \vartheta_1 + \nu_2 \vartheta_2 \) with \( \nu_1, \nu_2 \in \mathbb{R} \).

By assumption, if we consider \( \nu_1, \nu_2 \) as indeterminates, the determinant of \( \vartheta + \nu_1 \vartheta_1 + \nu_2 \vartheta_2 \) is a product of three linear polynomials and a polynomial of degree at most 2.

We claim that we can diagonalize all tensors in \( F \) simultaneously. We homogenize wrt to \( \nu_1, \nu_2 \) and get

\( \Delta := \nu_0 \vartheta + \nu_1 \vartheta_1 + \nu_2 \vartheta_2 \).

Furthermore, we write

\[
\vartheta = \sum_{i=1}^{3} \lambda_i p_i \otimes p_i + \gamma, \quad \vartheta_1 = \sum_{i=1}^{3} \lambda_i p_i \otimes p_i + \gamma_1, \quad \vartheta_2 = \sum_{i=1}^{3} \lambda_i p_i \otimes p_i + \gamma_2
\]

for some \( \gamma, \gamma_1, \gamma_2 \in S_2 V \) and \( \lambda_i, \lambda_j \in \mathbb{R} \) (\( i = 1, 2, 3, j = 1, 2 \)). Combined, this gives

\[
\Delta = \sum_{i=1}^{3} c_i p_i \otimes p_i + \nu_0 \gamma + \nu_1 \gamma_1 + \nu_2 \gamma_2
\]

for some linear forms \( c_1, \ldots, c_3 \in \mathbb{R}[\nu_0, \nu_1, \nu_2]^1 \).

Let \( D(\nu_0, \nu_1, \nu_2) := \text{det}(\Delta) \in \mathbb{R}[\nu_0, \nu_1, \nu_2] \). Since there are no rank 5 tensors that are mapped to 0 by \( \mu \), it follows that \( D(0, \nu_1, \nu_2) = 0 \). This shows that \( \nu_0 \) divides \( D(\nu_0, \nu_1, \nu_2) \). On the other hand, for \( \nu_0 = 1 \) three other linear polynomials are dividing \( D(1, \nu_1, \nu_2) \). Therefore, \( D(\nu_0, \nu_1, \nu_2) \) is a product of five linear forms, one being \( \nu_0 \).

Let \( \Delta' := \nu_0 \gamma + \nu_1 \gamma_1 + \nu_2 \gamma_2 \). Then, as just shown, the determinant of \( \Delta' \) is a product of two linear forms \( l, \nu_0 \) with \( l \in \mathbb{R}[\nu_0, \nu_1, \nu_2]^1 \). We claim that the determinant is not divisible by \( \nu_0^3 \). By Lemma 4.5.6, \( \gamma_1 \) and \( \gamma_2 \) are zero. Let \( W := \text{im} \, \vartheta_1 + \text{im} \, \vartheta_2 = \text{span}(p_1, p_2, p_3) \), then the kernel of the multiplication map \( S_2 W \rightarrow W^2 \) has dimension 2. This is impossible by Remark 3.1.17 as \( \dim W = 3 \).

Let \( S \) be the following polyhedron

\[
S = \{ \delta \in S_2 V : \delta = \gamma + \nu_1 \gamma_1 + \nu_2 \gamma_2 : \nu_1, \nu_2 \in \mathbb{R}, \delta \geq 0 \}
\]

The boundary of \( S \) is defined by the linear polynomial \( l(1, \nu_1, \nu_2) \) which shows that \( S \) is a polyhedron. Moreover, since \( \vartheta \) is a relative interior point of \( F \), there exists a positive
definite tensor in $S$. Using [Ram98, Theorem 1], it follows that there exist $p_4, p_5 \in V$ such that every tensor in $S$ can be written as

$$\lambda_4 p_4 \otimes p_4 + \lambda_5 p_5 \otimes p_5$$

for some $\lambda_4, \lambda_5 \in \mathbb{R}$.

Altogether, this shows that every tensor in $F$ can be written as

$$\sum_{i=1}^5 \lambda_i p_i \otimes p_i$$

for some $\lambda_1, \ldots, \lambda_5 \in \mathbb{R}$. \hfill \qed

**Lemma 4.5.6.** Let $U \subseteq \mathbb{R}[x]_2$ be a subspace of dimension 2, and let $\gamma, \gamma_1, \gamma_2 \in S_2 U$. Let $D := \det(\nu_0 \gamma + \nu_1 \gamma_1 + \nu_2 \gamma_2) \in \mathbb{P}[\nu_0, \nu_1, \nu_2]_2$. If there exist $\nu_0, \nu_1, \nu_2 \in \mathbb{R}$ such that $\nu_0 \gamma + \nu_1 \gamma_1 + \nu_2 \gamma_2 \succ 0$, and $\gamma_1, \gamma_2$ are not both zero, then $D$ is not divisible by $\nu_0^2$.

**Proof.** Assume $D$ was divisible by $\nu_0^2$. Fix a basis of $U$ such that

$$\gamma = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} b_1 & b_3 \\ b_2 & b_3 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} c_1 & c_3 \\ c_2 & c_3 \end{pmatrix}.$$  

Then

$$\det(\nu_0 \gamma + \nu_1 \gamma_1 + \nu_2 \gamma_2) = \nu_0^2 a_1 a_2 + \nu_1^2 (b_1 b_2 - b_3^2) + \nu_2 (a_1 b_2 + a_2 b_1).$$

By assumption, $\det(\nu_0 \gamma + \nu_1 \gamma_1 + \nu_2 \gamma_2) = a \nu_0^2$ for some $a \in \mathbb{R}$. Since there exists a positive definite tensor, the scalar $a$ has to be positive, and we may assume that $\gamma \succ 0$.

Comparing coefficients, we get

$$a_1 a_2 = 1, \quad b_1 b_2 - b_3^2 = 0, \quad a_1 b_2 + a_2 b_1 = 0.$$  

We calculate

$$b_3^2 = b_1 b_2 = b_1 \left( -\frac{a_2}{a_1} b_1 \right) = -\frac{a_2}{a_1} \frac{a_2}{a_1} = -(a_1 a_2)^2 a.$$  

As all numbers are real, and $a > 0$, this is only possible if $b_1 = b_2 = 0$. Since $a_1 \neq 0$, it follows from the equation $a_1 b_2 + a_2 b_1 = 0$ that $b_2 = 0$, hence $\gamma_1 = 0$.

The same argument also shows $\gamma_2 = 0$, which is a contradiction. \hfill \qed

**Theorem 4.5.7.** Let $f \in \text{int} \Sigma$, then $\text{Gram}(f)$ does not have any polyhedral faces of dimension 2 or higher.

**Proof.** The only faces of dimension 2 have rank 5. Let $f \in \text{int} \Sigma$ and let $F \subseteq \text{Gram}(f)$ be a face of rank 5 and dimension 2. Let $\vartheta \in F$ be a relative interior point and $U := \text{im} \vartheta$. Assume $F$ is polyhedral. By [Ram98, Theorem 1] there are two possibilities for $F$. Either (1) there exist forms $p_1, p_2, p_3 \in U$ ($V := \text{span}(p_1, p_2, p_3)^\perp \subseteq U$) such that every tensor in $F$ can be written as

$$\gamma = \sum_{i=1}^3 \lambda_i p_i \otimes p_i + \delta$$
with $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ and $\delta \in S_2 V$. Furthermore, a tensor $\gamma$ as above is contained in $F$ if and only if $\mu(\gamma) = f$ and $\lambda_i \geq 0$ for $i = 1, 2, 3$. Or (2) there exist $p_1, \ldots, p_5 \in U$ such that every tensor in $U$ has the form

$$\sum_{i=1}^5 \lambda_i p_i \otimes p_i$$

for some $\lambda_1, \ldots, \lambda_5 \in \mathbb{R}$.

If we are in case (1), we use Lemma 4.5.5 to reduce to case (2). Therefore, we only need to show that case (2) is not possible.

Assume we are in case (2). Since $\dim F = 2$, there exist two linearly independent tensors $\vartheta_1, \vartheta_2$ with $\vartheta_i = \sum_{j=1}^5 \nu_{ij} p_j \otimes p_j \ (i = 1, 2)$ for some $\nu_{ij} \in \mathbb{R}$ and $\mu(\vartheta_1) = \mu(\vartheta_2) = 0$ where $\mu$ denotes the Gram map $S_2 U \to U^2$. For every $j = 1, \ldots, 5$ either $\nu_{1j} \neq 0$ or $\nu_{2j} \neq 0$, otherwise let $W := \text{im}(\vartheta_1) + \text{im}(\vartheta_2)$. Then $\dim W \leq 4$ and the multiplication map $S_2 W \to W^2$ has a kernel of dimension at least two, which is not possible by Example 3.1.24. Now, we choose $C \gg 0$ large enough and consider $\gamma := \vartheta_1 + C\vartheta_2 = \sum_{j=1}^5 (\nu_{1j} + C\nu_{2j}) p_j \otimes p_j$. Then for every $j = 1, \ldots, 5$ the coefficient $\nu_{1j} + C\nu_{2j}$ is non-zero, but $\mu(\gamma) = 0$. This however is not possible by Remark 4.5.1, as this tensor has rank 5.

**Lemma 4.5.8.** Let $f \in \partial \Sigma$, then $\text{Gram}(f)$ does not have any polyhedral faces of dimension 2 or higher.

**Proof.** First assume a relative interior point of $\text{Gram}(f)$ has rank 5, then $\dim \text{Gram}(f) = 3$. The boundary consists of points of rank 4 or less, especially there are no faces of dimension $\geq 2$ of these ranks. Hence, neither $\text{Gram}(f)$ is a polyhedron nor any of its faces has dimension 2 or higher.

Assume a relative interior point has rank 4. Then $\dim \text{Gram}(f) \leq 1$ and we are done. The same holds if relative interior points have a rank of less than 4. \qed

Combined we showed the following theorem.

**Theorem 4.5.9.** Let $f \in \Sigma$, then $\text{Gram}(f)$ has no polyhedral faces of dimension 2 or higher. Whenever $f$ is positive definite and the automorphism group of $f$ contains no element of order 2, there are no positive dimensional polyhedral faces at all. In particular, this is the case if the form $f$ is generic.

### 4.6 Normal cones of Gram spectrahedra

In the next section, we want to understand the dual of Gram spectrahedra. To use results of [Sin15] which relate irreducible components of the boundary of the Gram spectrahedron to irreducible components of the boundary of the dual Gram spectrahedron, we need to understand the dimensions of normal cones.

If $P \subseteq \mathbb{R}^n$ is a polyhedron and $\emptyset \neq F \subseteq P$ is a face, then it is well-known that the dimension of the normal cone is determined by the dimension of the face, namely $n = \dim F + \dim \mathcal{N}_P(F)$. For spectrahedra, this is in general not true. We show that for Gram spectrahedra, the dimension of the normal cone is rather independent of the dimension of the face.
**Definition 4.6.1.** Let \( C \subseteq \mathbb{R}^n \) be a closed convex set and \( F \subseteq C \) a face. The normal cone of \( F \) is the set of all linear functionals that are minimal on \( F \), namely

\[
N_C(F) = \{ l \in (\mathbb{R}^n)^* : \forall y \in C \forall x \in F : l(y) \geq l(x) \}.
\]

If \( \vartheta \in C \), we define the normal cone at \( \vartheta \), \( N_C(\vartheta) \), as the normal cone of \( \text{supp}(\vartheta) \) where \( \text{supp}(\vartheta) \) denotes the supporting face of \( \vartheta \) on \( C \).

We recall the notation from Chapter 2. Let \( R \) be an \( R \)-algebra and \( V \subseteq R \) be a finite-dimensional \( R \)-vector space. Let \( \mu : S_2 V \to R \), \( f \otimes g \mapsto fg \) be the Gram map. For \( f \in R \) we write \( \text{Gram}_V(f) := \mu^{-1}(f) \cap S_2^+ V \) for the Gram spectrahedron wrt \( V \). We want to calculate the dimensions of the normal cones of \( C := \text{Gram}_V(f) \).

For any integer \( r \geq 1 \), let

\[
\phi_r : V^r \to R, \quad (q_1, \ldots, q_r) \mapsto \sum_{i=1}^r q_i^2
\]

be the sum of squares map, and

\[
d\phi_r(p) : V^r \to R, \quad (q_1, \ldots, q_r) \mapsto 2 \sum_{i=1}^r p_i q_i
\]

be its differential at the point \( p = (p_1, \ldots, p_r) \in V^r \). Since the rank of \( d\phi_r(q_1, \ldots, q_r) \) is the same for any basis \( q_1, \ldots, q_r \) of \( \text{span}(p_1, \ldots, p_r) \), we can also write \( \text{rk}(d\phi_r(U)) \) for a subspace \( U \) of \( V \) and define this as the rank at any basis of \( U \).

The following is a reformulation of [DT15, Theorem 2.9.] adapted to our situation.

**Theorem 4.6.2** ([DT15, Theorem 2.9.]). Let \( f \in R \) and \( \vartheta \in C \). Write \( U := \text{im}(\vartheta) \) and \( r := \dim U \). If \( C \) contains a positive definite tensor, then

\[
\dim N_C(\vartheta) = \dim(S_2 V) - \dim(\ker(\mu \cap \text{Sym}(U \otimes V)))
\]

where \( \text{Sym} \) is the symmetrization map \( V \otimes V \to S_2 V \).

**Theorem 4.6.3.** Let \( f \in R \) and \( \vartheta \in C \). Write \( U := \text{im}(\vartheta) \) and \( r := \dim U \). If \( C \) contains a positive definite tensor, then

\[
\dim N_C(\vartheta) = \dim(S_2 V) - \dim(\ker(d\phi_r(U))) + \binom{r}{2}
\]

**Proof.** Using Theorem 4.6.2 we need to show

\[
\dim(\ker(\mu \cap \text{Sym}(U \otimes V))) = \dim(\ker(d\phi_r(U))) - \binom{r}{2}
\]

Since \( UV \) is the image of \( d\phi_r(U) \), we have

\[
\dim UV = r \cdot \dim V - \dim(\ker(d\phi_r(U)))
\]

The dimension of \( \text{Sym}(U \otimes V) \) is given by \( \binom{\dim V + 1}{2} - \binom{\dim V - r + 1}{2} \), since \( \text{Sym}(U \otimes V) \cong S_2 V/S_2(U^{\perp}) \).
The map $\mu|_{\text{Sym}(U \otimes V)} : \text{Sym}(U \otimes V) \to UV$ is surjective, hence we calculate

$$\dim(\ker \mu \cap \text{Sym}(U \otimes V)) = \dim \text{Sym}(U \otimes V) - \dim UV = \left(\dim V + 1\right) - \left(\dim V - r + 1\right) - r \dim V + \dim(d\phi_r(U)) = \dim \ker(d\phi_r(U)) - \binom{r}{2}.$$\[\square\]

**Remark 4.6.4.** (i) The dimension of the normal cone only depends on $\text{im} \vartheta$ and not on $\vartheta$ itself.
(ii) If $p$ can be extended to a regular sequence in $R$, we have $\dim \ker(d\phi_r(p_1, \ldots, p_r)) = \binom{s}{r}$ and we see that $\dim N(\vartheta) = \dim S_2 V$.

**Remark 4.6.5.** If $C$ contains no positive definite element, then we can restrict everything to $V' := \text{im} \vartheta$ where $\vartheta$ is a relative interior point of $C$. We then get the dimension of the normal cone inside the affine hull of $C$.

From now on, we only consider $R = \mathbb{R}[x]/I$ for some homogeneous ideal $I \subseteq \mathbb{R}[x]$.

**Corollary 4.6.6.** Let $R$ be as above. Let $f \in \text{int} \Sigma(R_d)^2$, and $\vartheta \in \text{Gram}(f)$ be a Gram tensor of $f$. Let $U := \text{im} \vartheta$ and $r := \text{rk} \vartheta$. Then the dimension of the normal cone of $\text{Gram}(f)$ at $\vartheta$ is given by

$$\dim N_{\text{Gram}(f)}(\vartheta) = \dim(S_2 R_d) - \dim \ker(d\phi_r(U)) + \binom{r}{2}.$$\[Especially, if the Hilbert function of $\langle U \rangle$ is given by $(h_i)_{i \geq 0}$, then\]

$$\dim N_{\text{Gram}(f)}(\vartheta) = \dim(S_2 R_d) + \binom{r}{2} - r \dim R_d + \dim R_{2d} - h_{2d}.\]

**Proof.** The first part is exactly Theorem 4.6.2. For the second part, we have

$$\dim \ker(d\phi_r(U)) = r \dim R_d - \dim \text{im} d\phi_r(U), \quad \text{and} \quad \dim \text{im} d\phi_r(U) = \dim R_{2d} - h_{2d}.\]

**Example 4.6.7.** Let $f \in \Sigma_{3,4}$ be smooth. At every point of $\text{Gram}(f)$, the normal cone has the maximal possible dimension for a point of this rank in $S_2 \mathbb{R}[x]_2$. More precisely,

(i) $\dim N(\vartheta) = \dim S_2 \mathbb{R}[x]_2 = 21$ if $\text{rk} \vartheta = 3$,
(ii) $\dim N(\vartheta) = \dim S_2 \mathbb{R}[x]_2 = 3 = 18$ if $\text{rk} \vartheta = 4$,
(iii) $\dim N(\vartheta) = \dim S_2 \mathbb{R}[x]_2 = 5 = 16$ if $\text{rk} \vartheta = 5$,
(iv) $\dim N(\vartheta) = \dim S_2 \mathbb{R}[x]_2 = 6 = 15$ if $\text{rk} \vartheta = 6$.\[\square\]
Since $f$ is smooth, there exists a regular sequence contained in im$(\vartheta)$. As the Hilbert function of a regular sequence is $(1, 3, 3, 1)$, it follows that in every case $h_4 = \dim A_4/A_2U = 0$. Therefore, we get the dimension of the normal cone immediately from Corollary 4.6.6.

Example 4.6.8. Let $X \subseteq \mathbb{P}^n$ be a non-degenerate variety such that its real points $X(\mathbb{R})$ are dense. $X$ is called a variety of minimal degree if codim $X + 1 = \deg(X)$, and a variety of almost minimal degree if codim $X + 2 = \deg(X)$.

Using results of [BSV16] about the Hilbert function of $\mathbb{R}[X]$, one can show that if $X$ is a variety of minimal degree and $f \in \Sigma(\mathbb{R}[X])^2$ is non-singular, then at every point $\vartheta$ of its Gram spectrahedron, the dimension of the normal cone is maximal. I.e. the dimension is given by $\dim S_2\mathbb{R}[X]_1 + \binom{r}{2} - r \dim \mathbb{R}[X]_1 + \dim \mathbb{R}[X]_2$ where $r$ is the rank of $\vartheta$.

If $X$ is a variety of almost minimal degree and $\mathbb{R}[X]$ is Cohen-Macaulay, then the same statement holds if and only if the rank of the point is at least two greater than the dimension of $X$.

Another example where all normal cones have the maximal possible dimension is the following.

Corollary 4.6.9. Let $n \geq 3$, $d \geq 3$ (resp. $n \geq 4$, $d = 2$) and let $f \in \Sigma_{n,2d}$ be non-singular. Let $\vartheta \in \text{Gram}(f)$ be a Gram tensor of rank $r$ and let $U := \text{im}\vartheta$. If $\text{codim} U = \dim R_d - r < 3d - 2$ (resp. < 6), then

$$\dim N_{\text{Gram}(f)}(\vartheta) = \dim (S_2R_d) + \binom{r}{2} - r \dim R_d + \dim R_{2d}.$$  

Proof. This follows from Corollary 4.6.6 if we show $UA_d = A_{2d}$. Since $f$ is non-singular, $U$ is base-point-free, and hence $UA_d = A_{2d}$ by Theorem 3.3.6. \hfill $\Box$

Next, we want to look at the connection between the dimension of a face and the dimension of its normal cone.

Remark 4.6.10. Let $f \in \Sigma(R_d)^2$ for some $R = \mathbb{R}[x]/I$, and let $\vartheta \in \text{Gram}(f)$ be a Gram tensor of rank $r$ with $U := \text{im}(U) = \text{span}(p_1, \ldots, p_r)$. We have seen that the dimension of the normal cone at $\vartheta$ is determined by all relations of the form $0 = \sum_{i=1}^{r} p_i q_i$ with $q_1, \ldots, q_r \in R_d$, whereas the dimension of the face is determined by the number of relations of the form $\sum_{1 \leq i < j \leq r} a_{ij} p_i p_j$. Or equivalently, by the kernels of the maps

$$d\phi_\vartheta(p) : R_d^r \to R_{2d}$$

and

$$d\phi_\vartheta(p)|_{U^r} : U^r \to R_{2d}.$$  

This once again is also equivalent to saying that the dimension of the normal cone is determined by the Hilbert function of $(U)$ or specifically by $h_{2d}(U)$, whereas the dimension of the face is determined by $h_{2d}(U^2)$.

In general the Hilbert function of $(U)$ does not determine the Hilbert function of $(U^2)$. For example, all base-point-free subspaces of codimension 1 in $\mathbb{R}[x]_d$ have the same Hilbert function, but the Hilbert function of $(U^2)$ depends on whether or not $(U)$ is Gorenstein by Corollary 4.1.7.
Remark 4.6.11. Writing everything in terms of matrices, we have the kernels of the following maps. Let $p_1, \ldots, p_r$ be a basis of $U$, and extend to a basis $p_1, \ldots, p_r, q_1, \ldots, q_s$ of $R_d$. We write $X = \langle p_1, \ldots, p_r, q_1, \ldots, q_s \rangle$. Consider the following maps:

$$\{A \in \mathbb{S}^n : A_{ij} = 0 \text{ if } i > r \text{ and } j > r \} \to R_{2d}, \quad A \to XAX^T,$$

and

$$\{A \in \mathbb{S}^n : A_{ij} = 0 \text{ if } i > r \text{ or } j > r \} \to R_{2d}, \quad A \to XAX^T.$$

In the first map, we are considering matrices of the form \( \begin{pmatrix} * & * \\ * & 0 \end{pmatrix} \), whereas in the second one we are looking at matrices of the form \( \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix} \).

Hence, the first one corresponds to the dimension of the normal cone and the second one to the dimension of the face. In either case, the dimension is given by the dimension of the kernel modulo the trivial relations.

### 4.7 The Dual Gram spectrahedron

For a compact, convex set $C$, we denote by $\text{Ex}(C)$ the set of extreme points of $C$, and by $\text{Ex}_{a}(C)$ the Zariski closure of $\text{Ex}(C)$. If $Y \subseteq \mathbb{R}^n$ is any set, we denote by $\overline{Y}$ the projective closure of $Y$ in $\mathbb{P}^n$ wrt to the embedding $x \to (1: x)$ of $\mathbb{R}^n$ in $\mathbb{P}^n$. Lastly, we denote by $C^\circ$ the dual convex body or the polar of $C$, defined by

$$C^\circ = \{ \lambda \in (\mathbb{R}^n)^* : \forall x \in C : \lambda(x) \geq -1 \}.$$

**Theorem 4.7.1** ([Sin15, Corollary 3.9.]). Let $C \subseteq \mathbb{R}^n$ be a compact, convex and semialgebraic set with $0 \in \text{int}(C)$. Let $Z$ be an irreducible subvariety of $\text{Ex}_{a}(C)$, and suppose $Z \cap \text{Ex}(C)$ is dense in $Z$. Then the dual variety to $\overline{Z}$ is an irreducible component of $\overline{\partial_{a}C^\circ}$ if and only if

$$\dim(Z) + \dim N_C(\{x\}) = n$$

for a general extreme point $x \in Z \cap \text{Ex}(C)$. Conversely, if $Y$ is an irreducible component of $\partial_{a}C^\circ$, then the dual variety to $\overline{Y}$, denoted by $\overline{Y}^*$, is an irreducible subvariety of $\overline{\text{Ex}_{a}(C)}$, the set $\overline{Y}^* \cap \text{Ex}(C)$ is dense in $\overline{Y}^*$, and the condition on the normal cone is satisfied at a general extreme point.

**Lemma 4.7.2.** Let $f \in \Sigma$ be generic. Then the set of all rank 5 points on $\text{Gram}(f)$ is irreducible.

**Proof.** Let $g = \sum_{\alpha \in \mathcal{Z}_{1}^{2}, |\alpha|=4} c_{\alpha} x^{\alpha}$ where the $c_{\alpha}$ are also indeterminates and consider the matrix

$$G = \begin{pmatrix}
c_{400} & \lambda_1 & \lambda_2 & \frac{1}{2}c_{310} & \frac{1}{2}c_{301} & \lambda_4 \\
\lambda_1 & c_{040} & \lambda_3 & \frac{1}{2}c_{130} & \lambda_5 & \frac{1}{2}c_{031} \\
\lambda_2 & \lambda_3 & c_{004} & \lambda_6 & \frac{1}{2}c_{103} & \frac{1}{2}c_{013} \\
\frac{1}{2}c_{310} & \frac{1}{2}c_{301} & \lambda_6 & c_{220} - 2\lambda_1 & \frac{1}{2}c_{211} - \lambda_4 & \frac{1}{2}c_{121} - \lambda_5 \\
\frac{1}{2}c_{301} & \lambda_5 & \frac{1}{2}c_{103} & \frac{1}{2}c_{211} - \lambda_4 & c_{202} - 2\lambda_2 & \frac{1}{2}c_{112} - \lambda_6 \\
\lambda_4 & \frac{1}{2}c_{031} & \frac{1}{2}c_{013} & \frac{1}{2}c_{121} - \lambda_5 & \frac{1}{2}c_{112} - \lambda_6 & c_{022} - 2\lambda_3
\end{pmatrix}.$$
We check for example with SAGE [The19] that the determinant of $G$ is irreducible (in $\mathbb{C}[\lambda_1, \ldots, \lambda_6, \alpha_1, \ldots, \alpha_6]$). Let $f \in \Sigma$ be generic and write $f = \sum_{\alpha \in \mathbb{Z}_+^3, |\alpha| = 4} a_\alpha x_\alpha$ for some $a_\alpha \in \mathbb{R}$ for all $\alpha \in \mathbb{Z}_+^3, |\alpha| = 4$. As $f$ is generic and the determinant of $G$ is irreducible, so is the determinant of $G'$ (as an element of $\mathbb{C}[\lambda_1, \ldots, \lambda_6]$).

Therefore, this determinant defines the algebraic boundary of $\text{Gram}(f)$, and the rank 5 Gram tensors of $f$ are dense inside.

**Proposition 4.7.3.** Let $f \in \Sigma$ be generic. The boundary of the dual of $\text{Gram}(f)$ is the union of at least 10 irreducible components.

**Proof.** We consider $\text{Gram}(f)$ as a subset of its affine hull, which is isomorphic to $\mathbb{R}^6$. As we have seen earlier, the dimensions of the normal cones only depend on the rank of the tensor. We have

(i) $\dim N(\vartheta) = 1$, if $\text{rk}(\vartheta) = 5$,

(ii) $\dim N(\vartheta) = 3$, if $\text{rk}(\vartheta) = 4$,

(iii) $\dim N(\vartheta) = 6$, if $\text{rk}(\vartheta) = 3$.

First, we note that the rank 5 extreme points are dense in the boundary of $\text{Gram}(f)$ by Proposition 4.4.21. By Theorem 4.7.1 we have to find irreducible subvarieties of the algebraic boundary that satisfy the condition on the normal cones.

Let $Z$ be the algebraic boundary of $\text{Gram}(f)$. Then the rank 5 extreme points are dense and $Z$ is irreducible. Furthermore, we have $\dim Z = 5$ and $\dim N(\vartheta) = 1$ for every rank 5 extreme points, especially $\dim Z + \dim N(\vartheta) = 6$, which means that the dual of $Z'$ is an irreducible component of $\partial_a C$. Moreover, no subvariety $Z'$ of $Z$ with dense rank 5 points can correspond to an irreducible component of the dual since $\dim Z' < \dim Z$ but all rank 5 points have 1-dimensional normal cones.

Now, let $Z$ be the closure of the rank 4 points. Then $\dim Z = 3$ and rank 4 extreme points are dense and have 3-dimensional normal cones. Hence, with the same argument as above, this gives an irreducible component of the dual for every irreducible component of $Z$ that has dimension 3.

The last 8 irreducible components come from the 8 rank 3 points of $\text{Gram}(f)$. Certainly every single one is an irreducible subvariety with dense rank 3 points and satisfies the condition on the normal cones.

**Remark 4.7.4.** We do not know whether the set of all rank 4 extreme points is irreducible for a generic quartic. Even for a fixed quartic, for example, the Fermat quartic, we were not able to certify irreducibility via computer algebra software due to the size of the problem. Neither via Gröbner basis methods nor any numerical methods we know of.

**Remark 4.7.5.** We note that for the dual spectrahedron the existence of faces of rank 4 and dimension 1 on the Gram spectrahedron is unimportant as extreme points are not dense in these faces.
Example 4.7.6. Let $f = x^4 + y^4 + z^4$ be the Fermat quartic. Then $\text{Gram}(f)$ is given all matrices of the form

$$
\begin{pmatrix}
1 & \lambda_1 & \lambda_2 & 0 & 0 & \lambda_4 \\
\lambda_1 & 1 & \lambda_3 & 0 & \lambda_5 & 0 \\
\lambda_2 & \lambda_3 & 1 & \lambda_6 & 0 & 0 \\
0 & 0 & \lambda_6 & -2\lambda_1 & -\lambda_4 & -\lambda_5 \\
0 & \lambda_5 & 0 & -\lambda_4 & -2\lambda_2 & -\lambda_6 \\
\lambda_4 & 0 & 0 & -\lambda_5 & -\lambda_6 & -2\lambda_3
\end{pmatrix}
$$

with $\lambda_1, \ldots, \lambda_6 \in \mathbb{R}$ and such that the matrix is psd. Calculating the determinant and homogenizing, we obtain

$$
p = -16\lambda_1^2\lambda_2^2\lambda_3^2 + 8\lambda_1\lambda_2\lambda_2^2\lambda_4^2 - \lambda_3^2\lambda_4^2 + 8\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5 - 2\lambda_2\lambda_3\lambda_4^2\lambda_5^2 - \lambda_3^2\lambda_4^2 + 8\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5 - 2\lambda_1\lambda_2\lambda_3^2\lambda_5^2 - \lambda_3^2\lambda_5^2 - \\
2\lambda_1\lambda_2\lambda_5\lambda_6 - 2\lambda_2\lambda_4\lambda_5\lambda_6 + 8\lambda_1\lambda_2\lambda_3\lambda_6^2 - 2\lambda_1\lambda_3^2\lambda_6^2 - 2\lambda_2\lambda_3^2\lambda_6^2 - \lambda_4^2\lambda_5^2 - 2\lambda_4\lambda_5\lambda_6 - \\
2\lambda_1\lambda_2\lambda_5\lambda_6 - 2\lambda_2\lambda_4\lambda_5\lambda_6 + 8\lambda_1\lambda_2\lambda_3\lambda_6^2 - 2\lambda_1\lambda_3^2\lambda_6^2 - 2\lambda_2\lambda_3^2\lambda_6^2 - \lambda_4^2\lambda_5^2 - 2\lambda_4\lambda_5\lambda_6 - \\
2\lambda_3^2\lambda_4\lambda_5 - 2\lambda_2\lambda_5\lambda_6 - 2\lambda_2\lambda_5\lambda_6 - 2\lambda_2\lambda_4\lambda_5\lambda_6 - 2\lambda_2\lambda_4\lambda_5\lambda_6 - 2\lambda_2^2\lambda_4\lambda_5\lambda_6 - \\
2\lambda_2^2\lambda_4\lambda_5\lambda_6 - 2\lambda_2^2\lambda_4\lambda_5\lambda_6 - 2\lambda_2^2\lambda_4\lambda_5\lambda_6 - 2\lambda_2^2\lambda_4\lambda_5\lambda_6 - 2\lambda_2^2\lambda_4\lambda_5\lambda_6 - \\
4\lambda_1\lambda_3^2\lambda_5^2 + \lambda_3^4\lambda_5^2 - 4\lambda_2\lambda_3\lambda_5^2\lambda_6^2 + \lambda_3^4\lambda_5^2 - 8\lambda_1\lambda_2\lambda_3\lambda_5^2 - 2\lambda_3^2\lambda_5^2\lambda_6^2 + 2\lambda_2^2\lambda_5^2\lambda_6^2 - \\
2\lambda_4\lambda_5\lambda_6\lambda_6^2 + 2\lambda_1\lambda_2\lambda_5^2\lambda_6^2.
$$

The polynomial $p$ is irreducible, hence by Theorem 4.7.1 the dual variety of $\mathcal{V}(p)$ is one irreducible component of $\partial_6 \text{Gram}(f)^2$.

Any of the eight rank 3 Gram tensors of $f$ gives a hyperplane in the dual space.

In this way, we can determine the dual of $\text{Gram}(f)$ assuming we know a decomposition of the set of all rank 4 tensors on $\text{Gram}(f)$ into irreducible sets or at least know all irreducible components of maximal dimension.
Zusammenfassung auf Deutsch

Sei \( f \in \mathbb{R}[x_1, \ldots, x_n]_{2d} \) ein homogenes Polynom. Das Polynom \( f \) heißt eine Quadratsumme, wenn es \( p_1, \ldots, p_r \in \mathbb{R}[x_1, \ldots, x_n]_d \) gibt, sodass \( f = \sum_{i=1}^{r} p_i^2 \). Im Allgemeinen gibt es viele verschiedene, äquivalente Möglichkeiten \( f \) in einer solchen Form zu schreiben. Das Gram Spektraeder von \( f \) parametriert alle solchen Darstellungen bis auf Äquivalenz. Es ist aber nicht nur eine Menge, sondern hat als konvexe Menge die Struktur eines Spektraeders, d.h. es ist der Schnitt eines affin-linearen Unterraums mit dem Kegel der positiv semidefiniten Matrizen. Wie jede abgeschlossene, konvexe Menge ist auch der (euklidische) Rand des Gram Spektraeders die Vereinigung seiner Seiten. Die Struktur solcher Seiten von Spektraedern wurde schon 1995 von Ramana und Goldman in [RG95] studiert. Sie zeigten, dass Seiten von Spektraedern auf natürliche Weise zu Unterräumen korrespondieren. Wir verwenden aber im Gegensatz dazu eine koordinatenfreie Darstellung, wie sie in [Sch] von Scheiderer eingeführt wurde. Anstatt also Matrizen zu verwenden, sprechen wir über Tensoren. Das hat den Vorteil, dass mehr Struktur zur Verfügung steht und somit mehrere Definitionen natürlicher sind. Wir definieren das Gram Spektraeder von \( f \) als die Menge

\[
\text{Gram}(f) = \{ \vartheta \in S_2^d \mathbb{R}[x_1, \ldots, x_n]_d : \vartheta \succeq 0, \mu(\vartheta) = f \}.
\]

Diese besteht aus allen symmetrischen, positiv semidefiniten Tensoren \( \vartheta \), die unter der Gram Abbildung \( \mu : S_2^d \mathbb{R}[x_1, \ldots, x_n]_d \to \mathbb{R}[x_1, \ldots, x_n]_{2d}, \vartheta = \sum_{i=1}^{r} p_i \otimes q_i \mapsto \sum_{i=1}^{r} p_i q_i \) auf \( f \) abgebildet werden. Ein wichtiges Objekt der semidefiniten Optimierung sind Extremalpunkte von Spektraedern. Extremalpunkte von Gram Spektraedern wurden unter anderem in [PSV11], [Sch17], und [CPSV17] untersucht. Hier wurde insbesondere studiert, welches der kleinstmögliche Rang von Extremalphunkten ist. Unser Fokus liegt im Gegensatz dazu nicht auf Extremalphunkten, sondern auf der gesamten Seitenstruktur von Gram Spektraedern. Für ternäre Quartiken \((n = 3, d = 2)\) sind wir in der Lage die gesamte Struktur zu beschreiben.

Sei \( f \in \Sigma_{n, 2d} \). Wie in [Sch] gezeigt können wir zu jeder Seite \( F \subseteq \text{Gram}(f) \) einen Unterraum \( U \subseteq \mathbb{R}[x]_d \) assozieren, dessen Dimension gerade der Rang jedes Punktes im relativen Inneren von \( F \) ist. Die Dimension von \( F \) ist dann gegeben durch \( \frac{\dim U + 1}{2} - \dim U^2 \), wobei \( U^2 \subseteq \mathbb{R}[x]_{2d} \) der von allen Produkten \( pq \) mit \( p, q \in U \) aufgespannte Unterraum ist. Insbesondere sehen wir, dass die Dimension von \( F \) nur von \( \dim U^2 \) abhängt, solange die Dimension von \( U \) fixiert ist.


Wir untersuchen daher Unterräume $U$ mit $\mathcal{V}(U) = \emptyset$. Im Fall $\text{codim} U \in \{1,2\}$ kann man zeigen, dass es Schranken für $\text{codim} U^2$ gibt, die unabhängig von $n$ und $d$ sind, falls $\mathcal{V}(U) = \emptyset$ gilt (Proposition 3.4.3, Theorem 3.4.8). Hierfür wollen wir eine Verallgemeinerung für größere Kodimensionen finden. Sei $U \subseteq \mathbb{C}[x_1, \ldots, x_n]$ ein Unterraum von Kodimension $k$. Das Hauptresultat in diesem Abschnitt ist Theorem 3.8.12, welches besagt, dass für kleine Kodimensionen $k \leq d-1$ unter der Voraussetzung $\mathcal{V}(U) = \emptyset$ Folgendes gilt:

$$\text{codim } U^2 \leq 2k^2 + \left(\frac{k + 2}{3}\right).$$

Wie gewünscht gibt dies eine obere Schranke für $\text{codim} U^2$, welche unabhängig von $n$ und $d$ ist. Für Gram Spektraeder nicht-singulärer Formen gibt dies eine obere Schranke für die Seitenendimension, wenn die Seite großen Rang hat.

In Kapitel 4 betrachten wir hauptsächlich den Fall $n = 3$, $d = 2$ von ternären Quartiken. Quadratsummendarstellungen ternärer Quartiken wurden bereits 1888 von Hilbert in [Hil88] studiert. Er zeigte, dass jede reelle, positiv semidefinite ternäre Quartik auch eine Summe von Quadraten ist, und dann stets auch eine Summe von drei Quadraten. Später wurde in [PRSS04] gezeigt, dass eine glatte, positiv semidefinite Quartik genau 8 solcher Darstellungen der Länge 3 hat (bis auf orthogonale Äquivalenz).

In [PSV11] verbinden die Autoren diese Darstellungen der Länge 3 mit den 28 Bitangenten der Kurve $\{f = 0\}$ und verwenden die Kombinatorik dieser Bitangenten. Sie bemerken außerdem, dass diese 8 Gram Tensoren von Rang 3 in zwei disjunkte vierelementige Gruppen zerfallen, sodass die Verbindungsstrecken zwischen zwei Tensorfamilien aus derselben Vierergruppe auf dem Rand des Gram Spektraeders liegen. Die stärkere Aussage, dass zusätzlich für je zwei Tensorfamilien aus verschiedenen Gruppen, die Verbindungsstrecke im Inneren des Spektraeders liegt, wird auch für generische Quartiken behauptet, aber nur teilweise bewiesen. Wir vervollständigen diesen Beweis für alle glatten ternären Quartiken (Theorem 4.3.16). Der Graph, der diese 8 Gram Tensoren als Knoten hat, wird der Steiner Graph der Form $f$ genannt.


Anschließend betrachten wir die Struktur der einzelnen Seiten und zeigen, dass außer 0- und 1-dimensionalen Seiten, keine anderen Seiten Polyeder sein können (Theorem 4.5.9). Dies steht im Gegensatz zum Fall binärer Form, in welchem es sehr viele solcher polyedrischer Seiten gibt (May21).


Wir machen noch einige Bemerkungen zu den Beweismethoden. Hier verwenden wir hauptsächlich zwei verschiedene Typen an Argumenten. Um Seitenzüge zu verstehen verwenden wir die Tatsache, dass wir dies rein algebraisch tun können. Sei \( f \in \Sigma_{n,2d} \) und sei \( F \subseteq \text{Gram}(f) \) eine Seite. Sei \( \vartheta \in F \) ein Punkt im relativen Inneren mit Bild \( U \subseteq \mathbb{R}[x_1, \ldots, x_n]_d \). Die Dimension von \( F \) ist nun gegeben durch \( \binom{\dim U + 1}{2} - \dim U \). Es genügt daher Dimensionen von Quadraten von Unterräumen zu untersuchen. Hierzu verwenden wir die folgenden zwei Techniken. Einerseits betrachten wir Unterräume \( (U : l)_{d-1} \) in Abschnitt 3.5, wobei \( l \in \mathbb{C}[x_1, \ldots, x_n]_1 \) eine generische Linearform ist und \( (U : l) \) für das Quotientenideal steht. Andererseits untersuchen wir \( U \cap \mathbb{C}[x_1, \ldots, x_{n-1}]_d \) in Abschnitt 3.7, was ein Unterraum in einer Variablen weniger ist. Beide können wir mit Hilfe von Theoremen über Hilbertfunktionen studieren, etwa von Macaulay, Green oder Gotzmann (siehe Section 3.3). In Kapitel 4 verwenden wir klassische Methoden, wie in [PSV11] und [PRSS04], indem wir Quadratsummendarstellungen glatter Quartiken mit den 28 Bitangenten in Verbindung bringen. Viele Resultate über die Kombinatorik der Bitangenten und deren Verknüpfung mit verschiedenen Determinantendarstellungen ist auch im Buch von Dolgachev [Dol12] enthalten, welches wir regelmäßig als Quelle verwenden. Das ist insbesondere nötig um zusätzliche Eigenschaften des Steiner Graphen zu zeigen (Proposition 4.4.14).

References


REFERENCES


