First publ. in: Mathematische Annalen 328 (2004), 4, pp. 545-583

SOME NEW THOUGHTS ON OLD RESULTS OF R.T. SEELEY

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ABSTRACT. In this paper we show that the L_p-realization of a vector-valued elliptic boundary value problem $(\mathcal{A}, \mathcal{B}_j)$ admits a bounded $\mathcal{H}^\infty$-calculus on $L_p(G; E)$, $1 < p < \infty$, provided the top-order coefficients of $\mathcal{A}$ are Hölder continuous. Here G denotes a domain in $\mathbb{R}^n$ with compact C^{2m}-boundary and E a Banach space of class $\mathcal{HT}$. Our proof is based on an abstract perturbation result for operators admitting bounded $\mathcal{H}^\infty$-calculus and kernel estimates for the solution of $(\mathcal{A}, \mathcal{B}_j)$.

1. Introduction

In 1968-1971, Seeley [See68], [See69], [See71] developed in a series of papers a representation of the resolvent of operators A_B associated to elliptic boundary value problems of order $2m$ subject to general boundary conditions. This representation combined with the theory of pseudo-differential operators allowed him in particular to prove a celebrated result on the boundedness of the imaginary powers A_B^{it} of A_B in L^p-spaces. As a consequence one obtains also that the domain $D(A_B^\alpha)$ of the fractional powers A_B^α of A_B coincides with the complex interpolation space of order α between L_p and $D(A_B)$; see [Tri78]. A priori estimates for solutions of general boundary value problems were known since the celebrated work of Agmon, Douglis and Nirenberg [ADN59] and Solonnikov [Sol65], [Sol66].

Whereas Seeley's original motivation mainly was inspired by the theory of pseudo-differential operators, his result gained new interest in 1987 and 2001, when Dore and Venni [DV87] and lateron Kalton and Weis [KW01] proved that the boundedness of the imaginary powers of A_B, the existence of a bounded $\mathcal{H}^\infty$-calculus for A_B, respectively, is closely related to the so-called "maximal-regularity-problem" for parabolic evolution equations. In fact, roughly speaking, bounded imaginary powers of A_B of a certain angle imply already maximal L_p-regularity for the corresponding parabolic problem. Thus combining the result of Seeley with the one of Dore and Venni one obtains optimal $L_p - L_q$-estimates for the solution of the parabolic problem.

Aiming, however, for estimates which are useful in nonlinear problems, one is forced to look for minimal smoothness assumptions on the coefficients. For this reason quite a few papers dealt with this problem during the last years (or the related problem of A_B having a bounded $\mathcal{H}^\infty$-calculus). First, after McIntosh's [McI86] invention of the $\mathcal{H}^\infty$-calculus, Duong [Duo90] generalized Seeley's original result to the case of bounded $\mathcal{H}^\infty$-calculi in the context of C^∞-coefficients. For further results dealing with the case of non-smooth coefficients (being measured in various ways) and various boundary conditions, we refer to [Duo89], [PS93], [Pru93], [AHS94], [Ama95], [CDMY96], [DM96], [DR96], [DS97], [ST98], [HP98] and the references therein.

In 2001, the half-space problem in the case of constant coefficients was solved independently by Denk, Hieber and Prüss [DHP01] via L_p-estimates for kernel operators and by Dore and Venni [DV01] by the method of functional calculi in several variables.

The main problem in this context, namely the existence of a bounded $\mathcal{H}^\infty$-calculus for A_B under minimal smoothness assumptions on the coefficients and under general boundary conditions on general domains, however, remained open in all these papers.

1991 *Mathematics Subject Classification.* 35J45, 35K55.

Key words and phrases. elliptic boundary value problems, H^∞-calculus, perturbation theory.

The maximal regularity problem for the parabolic equation associated to A_B however was recently solved by Denk, Hieber and Prüss [DHP01] in the general case: indeed, using the concept of $\mathcal{R}$-boundedness and a recent theorem due to Weis [Wei01], maximal regularity for the solution was proved, even in the case of operator-valued coefficients, provided the top-order coefficients of A_B are bounded and uniformly continuous. The approach in [DHP01] is based on the representation of the resolvent of A_B by kernel operators and by so-called Poisson estimates for the involved kernels.

This representation is also the key for the main result of this paper: existence of a bounded $\mathcal{H}^\infty$-calculus for A_B subject to general boundary conditions in the case of Hölder continuous top-order coefficients of $\mathcal{A}$. More precisely, the combination of the representation of the resolvent $(\lambda + A_B)^{-1}$ of A_B with an abstract perturbation result for the class $\mathcal{H}^\infty(X)$, with the above mentioned result for the half-space and with the concept of $\mathcal{R}$-boundedness enables us to treat even the case of operator-valued coefficients as in [DHP01].

Let us mention here too, that via the perturbation theorem, one obtains a new result on the $\mathcal{H}^\infty$-calculus for the Stokes operator; see [NS02] for domains and [DHP01a] for the half-space. This is interesting since the Stokes system does not fit into our framework.

This paper is organized as follows. In Section 2 we state the main result and collect the material needed lateron on the $\mathcal{H}^\infty$-calculus, spaces of class $\mathcal{HT}$, $\mathcal{R}$-bounded families of operators. We also recall the Lopatinskii-Shapiro condition, as well as the representation of the resolvent of A_B given in [DHP01]. Then, in Section 3 we prove an abstract perturbation result for the class $\mathcal{H}^\infty(X)$, which will be applied in Section 4 to small perturbations of operators acting on $\mathbb{R}^n$. Sections 5 and 6 present the heart of the matter: perturbations of boundary conditions. In Section 7 we combine the results from Sections 4 and 5 to obtain a result on small perturbations for the complete boundary value problem. Finally, in Section 8, we use the localization technique to prove the main result for boundary value problems on domains in $\mathbb{R}^n$ having a compact C^{2m}-boundary.

2. Preliminaries and Main Result

In this section we state our main result and introduce the notation being used throughout this article and collect certain known properties of sectorial operators, operators with bounded $\mathcal{H}^\infty$-calculus, with bounded imaginary powers, and $\mathcal{R}$-sectorial operators. These will be used frequently in the subsequent sections. A general reference for the material presented in this section is the paper by Denk, Hieber and Prüss [DHP01].

In the second half of this section we also recall the notion of parameter-elliptic differential operators with operator-valued coefficients and the Lopatinskii-Shapiro condition which is indispensible for elliptic boundary value problems.

We remark that by C, M and c we denote various constants which may differ from line to line, but which are always independent of the free variables.

If X and Y are Banach spaces, $\mathcal{B}(X,Y)$ denotes the space of all bounded, linear operators from X to Y; moreover, $\mathcal{B}(X) := \mathcal{B}(X,X)$. The spectrum of a linear operator A in X is denoted by $\sigma(A)$, its resolvent set by $\rho(A)$. As usual, domain, range and kernel of an operator A are denoted by $D(A), R(A)$ and $N(A)$, respectively.

Let X be a complex Banach space, and A be a closed linear operator in X. Then A is called *sectorial* if $\overline{D(A)} = X$, $\overline{R(A)} = X$, $(-\infty, 0) \subset \rho(A)$ and

$$|t(t+A)^{-1}| \leq M, \qquad t > 0,$$

for some $M < \infty$. We denote the class of sectorial operators in X by $\mathcal{S}(X)$. $\Sigma_\theta \subset \mathbb{C}$ means the open sector with vertex 0, opening angle 2θ, which is symmetric with respect to the positive halfaxis $\mathbb{R}_+$, i.e.

$$\Sigma_\theta = \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| < \theta\}.$$

If $A \in \mathcal{S}(X)$ then $\rho(-A) \supset \Sigma_\theta$, for some $\theta > 0$ and $\sup\{|\lambda(\lambda+A)^{-1}| : |\arg\lambda| < \theta\} < \infty$.

Therefore, we may define the *spectral angle* ϕ_A of $A \in \mathcal{S}(X)$ by

$$\phi_A = \inf\{\phi : \rho(-A) \supset \Sigma_{\pi-\phi}, \sup_{\lambda\in\Sigma_{\pi-\phi}} |\lambda(\lambda+A)^{-1}| < \infty\}.$$

Evidently, we have $\phi_A \in [0,\pi)$ and $\phi_A \geq \sup\{|\arg\lambda| : \lambda \in \sigma(A)\}$. For $\phi \in (0,\pi]$ we define the space of holomorphic functions on Σ_ϕ by $\mathcal{H}(\Sigma_\phi) = \{f : \Sigma_\phi \to \mathbb{C} \text{ holomorphic}\}$, and

$$\mathcal{H}^\infty(\Sigma_\phi) = \{f : \Sigma_\phi \to \mathbb{C} \text{ holomorphic and bounded}\}.$$

The space $\mathcal{H}^\infty(\Sigma_\phi)$ with norm $|f|_\infty^\phi = \sup\{|f(\lambda)| : |\arg\lambda| < \phi\}$ forms a Banach algebra. We also set $\mathcal{H}_0(\Sigma_\phi) := \bigcup_{\alpha,\beta<0} \mathcal{H}_{\alpha,\beta}(\Sigma_\phi)$, where $\mathcal{H}_{\alpha,\beta}(\Sigma_\phi) := \{f \in \mathcal{H}(\Sigma_\phi) : |f|_{\alpha,\beta}^\phi < \infty\}$, and $|f|_{\alpha,\beta}^\phi := \sup_{|\lambda|\leq 1} |\lambda^\alpha f(\lambda)| + \sup_{|\lambda|\geq 1} |\lambda^{-\beta} f(\lambda)|$. Given $A \in \mathcal{S}(X)$, fix any $\phi \in (\phi_A, \pi]$ and let $\Gamma = (\infty, 0]e^{i\psi} \cup [0,\infty)e^{-i\psi}$. Then

$$f(A) = \frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda - A)^{-1} d\lambda, \quad f \in \mathcal{H}_0(\Sigma_\phi),$$

defines via $\Phi_A(f) = f(A)$ a functional calculus $\Phi_A : \mathcal{H}_0(\Sigma_\phi) \to \mathcal{B}(X)$ which is an algebra homomorphism. Following McIntosh [McI86], we say that a sectorial operator A admits a *bounded* $\mathcal{H}^\infty$*-calculus* if there are $\phi > \phi_A$ and a constant $K_\phi < \infty$ such that

$$|f(A)| \leq K_\phi |f|_\infty^\phi, \quad \text{for all } f \in \mathcal{H}_0(\Sigma_\phi). \tag{2.1}$$

The class of sectorial operators A which admit an $\mathcal{H}^\infty$-calculus will be denoted by $\mathcal{H}^\infty(X)$ and the $\mathcal{H}^\infty$*-angle* of A is defined by

$$\phi_A^\infty = \inf\{\phi > \phi_A : \text{(2.1) is valid}\}.$$

If this is the case, the functional calculus for A on $\mathcal{H}_0(\Sigma_\phi)$ extends uniquely to $\mathcal{H}^\infty(\Sigma_\phi)$.

We consider next another subclass of $\mathcal{S}(X)$, namely operators with bounded imaginary powers. More precisely, a sectorial operator A in X is said to admit *bounded imaginary powers* if $A^{is} \in \mathcal{B}(X)$ for each $s \in \mathbb{R}$ and there is a constant $C > 0$ such that $|A^{is}| \leq C$ for $|s| \leq 1$. The class of such operators will be denoted by $\mathcal{BIP}(X)$. Since A^{is} has the group property, it is clear that A admits bounded imaginary powers if and only if $\{A^{is} : s \in \mathbb{R}\}$ forms a strongly continuous group of bounded linear operators in X. The growth bound θ_A of this group, i.e.

$$\theta_A = \overline{\lim}_{|s|\to\infty} \frac{1}{|s|} \log |A^{is}|$$

will be called the *power angle* of A. Since the functions f_s defined by $f_s(z) = z^{is}$ belong to $\mathcal{H}^\infty(\Sigma_\phi)$, for any $s \in \mathbb{R}$ and $\phi \in (0,\pi)$, we obviously have the inclusions

$$\mathcal{H}^\infty(X) \subset \mathcal{BIP}(X) \subset \mathcal{S}(X),$$

and the inequalities

$$\phi_A^\infty \geq \theta_A \geq \phi_A \geq \sup\{|\arg\lambda| : \lambda \in \sigma(A)\}.$$

Let Y be another Banach space. A family of operators $\mathcal{T} \subset \mathcal{B}(X,Y)$ is called $\mathcal{R}$*-bounded*, if there is a constant $C > 0$ and $p \in [1,\infty)$ such that for each $N \in \mathbb{N}$, $T_j \in \mathcal{T}$, $x_j \in X$ and for all independent, symmetric, $\{-1,1\}$-valued random variables ε_j on a probability space $(\Omega, \mathcal{M}, \mu)$ the inequality

$$|\sum_{j=1}^N \varepsilon_j T_j x_j|_{L_p(\Omega;Y)} \leq C |\sum_{j=1}^N \varepsilon_j x_j|_{L_p(\Omega;X)}$$

is valid. The smallest such C is called $\mathcal{R}$*-bound* of $\mathcal{T}$, we denote it by $\mathcal{R}(\mathcal{T})$.

The concept of $\mathcal{R}$-bounded families of operators leads immediately to the notion of $\mathcal{R}$-sectorial operators. Indeed, a sectorial operator is called $\mathcal{R}$-*sectorial* if

$$\mathcal{R}_A(0) := \mathcal{R}\{t(t+A)^{-1} : t > 0\} < \infty.$$

The $\mathcal{R}$-*angle* ϕ_A^R of A is defined by means of

$$\phi_A^R := \inf\{\theta \in (0,\pi) : \ \mathcal{R}_A(\pi - \theta) < \infty\},$$

where

$$\mathcal{R}_A(\theta) := \mathcal{R}\{\lambda(\lambda + A)^{-1} : |\arg\lambda| \leq \theta\}.$$

The following fundamental result, due to Clément and Prüss [ClPr01], states that the class of operators with bounded imaginary powers is contained in the class of $\mathcal{R}$-sectorial operators, at least in the case when the underlying Banach space X belongs to the class $\mathcal{HT}$. A Banach space X is said to be of class $\mathcal{HT}$, if the Hilbert transform is bounded on $L_p(\mathbb{R};X)$ for some (and then all) $p \in (1,\infty)$. Here the Hilbert transform H of a function $f \in \mathcal{S}(\mathbb{R};X)$, the Schwartz space of rapidly decreasing X-valued functions, is defined by

$$Hf := \frac{1}{\pi} PV(\frac{1}{t}) * f = \frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{\varepsilon\leq|s|\leq 1/\varepsilon} f(t-s)\frac{ds}{s}.$$

These spaces are also called UMD Banach spaces, where the UMD stands for *unconditional martingale difference property*. It is a well known theorem that the set of Banach spaces of class $\mathcal{HT}$ coincides with the class of UMD spaces; cf. Burkholder [Bur86].

Theorem 2.1. *Let X be a Banach space of class $\mathcal{HT}$ and suppose that $A \in \mathcal{BIP}(X)$ with power angle θ_A. Then A is $\mathcal{R}$-sectorial and $\phi_A^R \leq \theta_A$.*

We are now turning our attention to vector-valued elliptic boundary value problems of the form

$$\begin{aligned} \lambda u + \mathcal{A}(x,D)u &= f \text{ in } G \\ \mathcal{B}_j(x,D)u &= g_j \text{ on } \partial G, \ j = 1\ldots m, \end{aligned}$$

where $G \subset \mathbb{R}^{n+1}$ is an open connected set with compact C^{2m}-boundary ∂G. $\mathcal{A}$ is a differential operator of order $2m$ and $\mathcal{B}_j$ are boundary operators of order $m_j < 2m$. More precisely, let E be a Banach space and $m, n, m_1, \ldots, m_m$ be natural numbers with $m_j < 2m$ $(j = 1,\ldots,m)$, and let $\mathcal{A}(x,D) = \sum_{|\alpha|\leq 2m} a_\alpha(x)D^\alpha$ and $\mathcal{B}_j(x,D) = \sum_{|\beta|\leq m_j} b_{j\beta}(x)D^\beta$ with variable $\mathcal{B}(E)$-valued coefficients $a_\alpha(x)$ and $B_{j\beta}(x)$.

Let $\mathcal{A}(\cdot)$ be a $\mathcal{B}(E)$-valued polynomial on $\mathbb{R}^n$ which is homogeneous of degree $m \in \mathbb{N}$, i.e.

$$\mathcal{A}(\xi) = \sum_{|\alpha|=m} a_\alpha \xi^\alpha, \quad \xi \in \mathbb{R}^n.$$

We call such a homogeneous $\mathcal{B}(E)$-valued polynomial $\mathcal{A}(\cdot)$ of degree $m \in \mathbb{N}$ *parameter-elliptic* (see [Ama01],[DHP01]) if there is an angle $\phi \in [0,\pi)$ such that the spectrum $\sigma(\mathcal{A}(\xi))$ satisfies

$$\sigma(\mathcal{A}(\xi)) \subset \Sigma_\phi \quad \text{for all } \xi \in \mathbb{R}^n, \ |\xi| = 1. \tag{2.2}$$

We then call

$$\phi_{\mathcal{A}} := \inf\{\phi : \ (2.2) \text{ holds}\} = \sup_{|\xi|=1} |\arg\sigma(\mathcal{A}(\xi))|$$

the *angle of ellipticity* of $\mathcal{A}$. For $D = -i(\partial_1,\ldots,\partial_n)$ we call $\mathcal{A}(D) = \sum_{|\alpha|=m} a_\alpha D^\alpha$ parameter elliptic, if its symbol $\mathcal{A}(\xi)$ is parameter-elliptic.

For fixed $p \in (1,\infty)$, we will assume the following conditions, where $C^k(\overline{G};\mathcal{B}(E))$ stands for the space of all functions $G \to \mathcal{B}(E)$ for which all derivatives of order $\leq k$ exist, are continuous and can be extended continuously to $\overline{G}$.

(RS) Smoothness Condition.

(i) $a_\alpha \in C(\overline{G}, \mathcal{B}(E))$ for each $|\alpha| = 2m$, and $\lim_{|x|\to\infty, x\in G} a_\alpha(x) = a_\alpha(\infty)$ exist;

(ii) $a_\alpha \in [L_\infty + L_{r_k}](G, \mathcal{B}(E))$ for each $|\alpha| = k < 2m$ with $r_k \geq p$ and $2m - k > n/r_k$;

(iii) $b_{j\beta} \in C^{2m-m_j}(\partial G, \mathcal{B}(E))$ for each j, β.

(E) Ellipticity Condition.

There exists $\phi_{\mathcal{A}} \in [0, \pi)$ such that the following assertions hold.

(i) The principal symbol

$$\mathcal{A}_\#(x, \xi) = \sum_{|\alpha|=2m} a_\alpha(x)\xi^\alpha$$

is parameter-elliptic with angle of ellipticity $\leq \phi_{\mathcal{A}}$ for each $x \in \overline{G}$ and for $x = \infty$ in case G is unbounded.

(ii) (Lopatinskii-Shapiro Condition.) Set

$$\mathcal{B}_{j\#}(x, D) := \sum_{|\beta|=m_j} b_{j\beta}(x)D^\beta,$$

$B_\# := (B_{1\#}, \dots, B_{m\#})$, and let $\nu(x)$ denote the outer normal of G in $x \in \partial G$. For each $x_0 \in \partial G$ and each ξ' in the tangent space of ∂G at x_0, the ODE-problem in $\mathbb{R}_+$

$$\begin{aligned} (\lambda + \mathcal{A}_\#(x_0, \xi' - \nu(x_0)D_y))v(y) &= 0 \quad y > 0, \\ \mathcal{B}_{j\#}(x_0, \xi' - \nu(x_0)D_y)v(0) &= h_j, \quad j = 1, \dots, m \end{aligned}$$

has a unique solution $v \in C_0(\mathbb{R}_+; E)$ for each $(h_1, \dots, h_m) \in E^m$ and each $\lambda \in \overline{\Sigma}_{\pi-\phi_{\mathcal{A}}}$ with $|\xi'| + |\lambda| \neq 0$.

Assume that E is a Banach space of class $\mathcal{HT}$. Let G be a domain in $\mathbb{R}^{n+1}$ with compact C^{2m}-boundary ∂G. Suppose that for $\phi_{\mathcal{A}} \in [0, \pi)$ the boundary value problem $(\mathcal{A}, \mathcal{B}_1, \dots, \mathcal{B}_m)$ satisfies smoothness and ellipticity conditions (RS) and (E) above. Let A_B denote the realization of $\mathcal{A}(x, D)$ in $X = L_p(G; E)$ with domain

$$D(A_B) = \{u \in H_p^{2m}(G; E) : \mathcal{B}_j(x, D)u = 0, \quad j = 1, \dots, m\}. \tag{2.3}$$

Then the following result is proved in [DHP01], Theorem 8.2.

Theorem 2.2. *Assume* (RS) *and* (E). *Then for each $\phi > \phi_{\mathcal{A}}$ there exists $\omega_\phi \geq 0$ such that $\omega_\phi + A_B$ is $\mathcal{R}$-sectorial with $\phi_{\omega_\phi + A_B} \leq \phi$.*

In particular, if $\phi_{\mathcal{A}} < \frac{\pi}{2}$ then the parabolic initial-boundary value problem

$$\begin{aligned} \partial_t u + (A_B + \omega_\phi)u = f, \quad t > 0, \\ u(0) = 0, \end{aligned}$$

has the property of maximal regularity in $L_q(\mathbb{R}_+; L_p(G; E))$ for each $q \in (1, \infty)$.

To state our main result we have to introduce another smoothness conditions on the coefficients of $\mathcal{A}$.

(H) Smoothness Conditions:

$a_\alpha \in BUC^\rho(\overline{G}, \mathcal{B}(E))$ for some $\rho \in (0, 1)$ and each α with $|\alpha| = 2m$, $a_\alpha(\infty) = \lim_{|x|\to\infty} a_\alpha(x)$ exists if G is unbounded and

$$|a_\alpha(x) - a_\alpha(\infty)| \leq c|x|^{-\rho}, \qquad x \in G \quad \text{with} \quad |x| \geq 1.$$

The main result of this paper reads as follows.

Theorem 2.3. *Let E be a Banach space of class $\mathcal{HT}$, $n, m \in \mathbb{N}$ and $1 < p < \infty$. Let G be a domain in $\mathbb{R}^{n+1}$ with compact C^{2m}-boundary ∂G. Suppose that for $\phi_{\mathcal{A}} \in [0, \pi)$ the boundary value problem $(\mathcal{A}, \mathcal{B}_1, \ldots, \mathcal{B}_m)$ satisfies smoothness and ellipticity conditions* (RS), (H) *and* (E) *above.*

Let A_B denote the realization of $\mathcal{A}(x, D)$ in $X = L_p(G; E)$ with domain

$$D(A_B) = \{u \in H_p^{2m}(G; E) : \mathcal{B}_j(x, D)u = 0, \quad j = 1, \ldots, m\}.$$

Then for each $\phi > \phi_{\mathcal{A}}$ there is $\mu_\phi \geq 0$ such that $\mu_\phi + A_B \in \mathcal{H}^\infty(L_p(G; E))$ with $\phi^\infty_{\mu_\phi + A_B} \leq \phi_{\mathcal{A}}$.

We often use the fact that the kernels of various integral operators arising in the study of $(\lambda - A_B)^{-1}$ are pointwise dominated by functions $p^n_{m,k}$ of Poisson type which have been introduced in [DHP01] as follows: for $n, m, k \in \mathbb{N}$ with $n > 1$ the functions $p^n_{m,k} : (0, \infty) \to (0, \infty)$ are defined by

$$p^n_{m,k}(r) = \int_0^\infty \frac{s^{n-2}}{(1+s)^{m-k-1}} e^{-r(1+s)} ds. \tag{2.4}$$

3. An Abstract Perturbation Result

In this section we present a perturbation result for the class $\mathcal{H}^\infty(X)$ where X is a Banach space of class $\mathcal{HT}$. This result will be of crucial importance when considering small perturbations of elliptic boundary value problems in the subsequent sections. We start our considerations with the following lemma. For a similar result see [KW01].

Lemma 3.1. *Let X be Banach space, $A \in \mathcal{H}^\infty(X)$ and $h \in \mathcal{H}^\infty_0(\Sigma_\phi)$ for some $\phi > \phi^\infty_A$. Then there exists a constant $C > 0$ such that*

$$|\sum_{j\in\mathbb{Z}} \varepsilon_j h(2^j rA)|_{\mathcal{B}(X)} \leq C \sup_{j\in\mathbb{Z}} |\varepsilon_j|$$

for all $r > 0$ and $\varepsilon_j \in \mathbb{C}$ such that $\varepsilon_j \neq 0$ only for finitely many ε_j.

Proof. Observe that by assumption there exist constants $c, \beta > 0$ such that

$$|h(z)| \leq c \frac{|z|^\beta}{1 + |z|^{2\beta}}, \qquad z \in \Sigma_\phi.$$

We now set for $r > 0$

$$f(z) := \sum_{j\in\mathbb{Z}} \varepsilon_j h(2^j rz), \qquad z \in \Sigma_\phi.$$

Note that the above series is absolutely convergent since

$$|f(z)| \leq |(\varepsilon_j)|_\infty \sum_{j\in\mathbb{Z}} |h(2^j rz)| \leq C|(\varepsilon_j)|_\infty$$

and since for $t = r|z|$ we have

$$\sum_{j\in\mathbb{Z}} |h(2^j rz)| \leq c \sum_{j\in\mathbb{Z}} \frac{(t2^j)^\beta}{1 + (t2^j)^{2\beta}} \leq \frac{2c}{1 - 2^{-\beta}}.$$

Hence $f \in \mathcal{H}^\infty(\Sigma_\phi)$. By assumption we obtain

$$|\sum_{j\in\mathbb{Z}} \varepsilon_j h(2^j rA)|_{\mathcal{B}(X)} = |f(A)|_{\mathcal{B}(X)} \leq C|f|_{\mathcal{H}^\infty} \leq C|(\varepsilon_j)|_\infty.$$

∎

The following perturbation result on operators admitting a bounded $\mathcal{H}^\infty$-calculus was obtained by J. Prüss in 1994 (see [Pru94]). The development of the concept of $\mathcal{R}$-boundedness during the last few years allows us to give a new and fairly short proof which is included here.

Theorem 3.2. *Let X be a Banach space of class $\mathcal{HT}$ and $A \in \mathcal{H}^\infty(X)$. Let B be a linear operator in X such that $D(B) \supset D(A)$. Assume there exists $\eta > 0$ such that*

$$|Bx| \leq \eta |Ax|, \quad x \in D(A).$$

Suppose further that there exist $\alpha \in (0,1)$ and $C > 0$ such that

$$B(D(A^{1+\alpha})) \subset D(A^\alpha) \qquad \text{and} \qquad |A^\alpha Bx| \leq C|A^{1+\alpha}x| \quad \text{for} \quad x \in D(A^{1+\alpha}).$$

Then $A + B \in \mathcal{H}^\infty(X)$ provided $\eta > 0$ is sufficiently small. Moreover, for each $\phi > \phi_A^\infty$ there is $\eta_0(\phi) > 0$ such that $\phi_{A+B}^\infty \leq \phi$ if $\eta \leq \eta_0(\phi)$.

Remark 3.3. Observe that, by a counterexample due to McIntosh and Yagi [MY90], the assertion of the above theorem is no longer true if merely the first condition above is assumed.

Proof. Denote by ϕ_A the spectral angle of A. Let $\phi > \phi_A$ and $\lambda \in -\Sigma_{\pi-\phi}$. Then

$$(\lambda - (A+B))^{-1} = (\lambda - A)^{-1} \sum_{k=0}^{\infty} [B(\lambda - A)^{-1}]^k,$$

provided $\eta < 1/C_A(\pi - \phi)$, since

$$|B(\lambda - A)^{-1}| \leq \eta |A(\lambda - A)^{-1}| \leq \eta C_A(\pi - \phi).$$

Setting $R_\lambda := A(\lambda - A)^{-1}$ and $K := BA^{-1}$ we obtain

$$\begin{aligned} (I+K)^{-1}(\lambda - (A+B))^{-1}(I+K) &= A(A+B)^{-1}(\lambda - (A+B))^{-1}(A+B)A^{-1} \\ &= A(\lambda - (A+B))^{-1}A^{-1} = \sum_{k=0}^{\infty} [R_\lambda K]^k (\lambda - A)^{-1}, \end{aligned}$$

which yields the following representation of the resolvent of $A + B$

$$(\lambda - (A+B))^{-1} = (I+K)\{\sum_{k=0}^{\infty} [R_\lambda K]^k (\lambda - A)^{-1}\}(I+K)^{-1}.$$

We next consider $g \in \mathcal{H}_0^\infty(\Sigma_\phi)$, choose $\theta \in (\phi_A^\infty, \phi)$ and define Γ to be the contour $\Gamma = (\infty, 0]e^{i\theta} \cup [0, \infty)e^{-i\theta}$. Then

$$\begin{aligned} g(A+B) &= \frac{1}{2\pi i} \int_\Gamma g(\lambda)(\lambda - (A+B))^{-1} d\lambda \\ &= (I+K)\{\frac{1}{2\pi i} \sum_{k=0}^{\infty} \int_\Gamma g(\lambda)[R_\lambda K]^k (\lambda - A)^{-1} d\lambda\}(I+K)^{-1} \\ &= (I+K)\{\frac{1}{2\pi i} \sum_{k=0}^{\infty} G_k\}(I+K)^{-1}. \end{aligned}$$

In the following we estimate the norms of the operators G_k. To this end, we truncate the contour Γ by $\Gamma_N = [\Gamma \cap B_{2^N}(0)] \setminus B_{2^{-N}}(0)$ and observe that

$$\begin{aligned} G_k &= \lim_{N\to\infty} \int_{\Gamma_N} g(\lambda)[R_\lambda K]^k (\lambda - A)^{-1} d\lambda \\ &= -\lim_{N\to\infty} \int_{2^{-N}}^{2^N} g(re^{i\theta})[R_{re^{i\theta}} K]^k (re^{i\theta} - A)^{-1} e^{i\theta} dr \\ &\quad + \lim_{N\to\infty} \int_{2^{-N}}^{2^N} g(re^{-i\theta})[R_{re^{-i\theta}} K]^k (re^{-i\theta} - A)^{-1} e^{-i\theta} dr \\ &= \lim_{N\to\infty} G_{Nk}^- - G_{Nk}^+. \end{aligned}$$

The finite integrals $G_{Nk}^{\pm}$ may be rewritten as

$$\begin{aligned} G_{Nk}^{\pm} &= \sum_{j=-N}^{N-1} \int_{2^j}^{2^{j+1}} g(re^{\pm i\theta})[R_{re^{\pm i\theta}} K]^k (re^{\pm i\theta} - A)^{-1} e^{\pm i\theta} dr \\ &= \int_1^2 \{ \sum_{j=-N}^{N-1} g(2^j re^{\pm i\theta})[R_{2^j re^{\pm i\theta}} K]^k 2^j r (2^j re^{\pm i\theta} - A)^{-1} \} e^{\pm i\theta} dr/r \\ &= \int_1^2 T_{Nk}^{\pm}(r) e^{\pm i\theta} dr/r. \end{aligned}$$

In order to estimate the terms $T_{Nk}^{\pm}$ we use a randomization technique, Lemma 3.1 and the fact that A is $\mathcal{R}$-sectorial (see Theorem 2.1). In fact, for $z \in \Sigma_\phi$ and $\beta \in (0,\alpha)$, set $h_\beta(z) := \frac{z^\beta}{e^{\pm i\theta} - z}$ and observe that $h_\beta \in \mathcal{H}_0^\infty(\Sigma_\psi)$ provided $\phi_A^\infty < \psi < \theta$. Thus Lemma 3.1 applies to these functions. Furthermore, we set $K_\beta := A^\beta K A^{-\beta}$. Choose independent, symmetric $\{-1,1\}$-valued random variables ε_j on a probability space $(\Omega, \mathcal{M}, \mu)$. We then obtain

$$\begin{aligned} &| < T_{Nk}^{\pm}(r)x|x^* > | = | < \sum_{j=-N}^{N-1} \varepsilon_j^2 g(2^j re^{\pm i\theta})[R_{2^j re^{\pm i\theta}} K]^k 2^j r (2^j re^{\pm i\theta} - A)^{-1} x | x^* > | \\ &= | \int_\Omega < \sum_{j=-N}^{N-1} \varepsilon_j^2 g(2^j re^{\pm i\theta}) h_{1-\beta}(A/2^j r)[K_\beta R_{2^j re^{\pm i\theta}}]^{k-1} K_\beta h_\beta(A/2^j r) x | x^* > d\mu | \\ &= | \int_\Omega < \sum_{j=-N}^{N-1} \varepsilon_j g(2^j re^{\pm i\theta})[K_\beta R_{2^j re^{\pm i\theta}}]^{k-1} K_\beta h_\beta(A/2^j r) x | \sum_{j=-N}^{N-1} \varepsilon_j h_{1-\beta}(A^*/2^j r) x^* > d\mu |. \\ &\le | \sum_{j=-N}^{N-1} \varepsilon_j g(2^j re^{\pm i\theta})[K_\beta R_{2^j re^{\pm i\theta}}]^{k-1} K_\beta h_\beta(A/2^j r) x |_{L_2(\Omega;X)} \cdot \\ &\qquad \cdot | \sum_{j=-N}^{N-1} \varepsilon_j h_{1-\beta}(A/2^j r) x^* |_{L_2(\Omega;X^*)} \\ &\le |g|_{\mathcal{H}^\infty} |K_\beta|^k \mathcal{R}\{R_\lambda : \lambda \in \Sigma_{\pi-\phi}\}^{k-1} \cdot | \sum_{j=-N}^{N-1} \varepsilon_j h_\beta(A/2^j r) x |_{L_2(\Omega;X)} \cdot \\ &\qquad \cdot | \sum_{j=-N}^{N-1} \varepsilon_j h_{1-\beta}(A^*/2^j r) x^* |_{L_2(\Omega;X^*)} \\ &\le c^2 |g|_{\mathcal{H}^\infty} |K_\beta|^k \mathcal{R}\{R_\lambda : \lambda \in \Sigma_{\pi-\phi}\}^{k-1} |x||x^*|., \end{aligned}$$

where we used the contraction principle (see e.g. [DHP01], Lemma 3.5), Theorem 2.1, and Lemma 3.1.

Thus we obtain the bounds

$$|G^{\pm}_{Nk}| \leq \sup_{1\leq r\leq 2} |T^{\pm}_{Nk}| \leq c^2 |g|_{\mathcal{H}^\infty} |K_\beta|^k \mathcal{R}\{R_\lambda : \lambda \in \Sigma_\theta\}^{k-1},$$

which are uniform in N. This implies

$$|g(A+B)| \leq |I+K|\{|g(A)| + 2c^2 |g|_{\mathcal{H}^\infty} |K_\beta| \sum_{k=0}^{\infty} |K_\beta|^k \mathcal{R}\{R_\lambda : \lambda \in \Sigma_{\pi-\theta}\}^k\} |(I+K)^{-1}|,$$

which is finite provided

$$|K_\beta| < \frac{1}{\mathcal{R}\{R_\lambda : \lambda \in \Sigma_{\pi-\phi}\}}.$$

Now for $\beta \in (0,\alpha)$ we have by complex interpolation

$$|K_\beta| \leq |K_\alpha|^{\beta/\alpha} |K|^{1-\beta/\alpha} \leq C^{\beta/\alpha} \eta^{1-\beta/\alpha},$$

which tends to zero as η tends to zero. □

4. Small Perturbations for elliptic operators on $\mathbb{R}^n$

Throughout this section let E be a Banach space of class $\mathcal{HT}$, $n, m \in \mathbb{N}$ and $1 < p < \infty$. Consider now a homogeneous differential operator $\mathcal{A}_0(D) = \sum_{|\alpha|=m} a_\alpha D^\alpha$ of order m with operator-valued coefficients $a_\alpha \in \mathcal{B}(E)$ which is parameter-elliptic with angle of ellipticity $\phi_{\mathcal{A}_0}$. We denote by A_0 its realization in $X = L_p(\mathbb{R}^n; E)$ with domain $D(A_0) = H^m_p(\mathbb{R}^n; E)$. Then it was shown in Theorem 5.5 of [DHP01] that $A_0 \in \mathcal{H}^\infty(X)$ with $\mathcal{H}^\infty$-angle $\phi^\infty_{A_0} \leq \phi_{\mathcal{A}_0}$. In the following, we consider small perturbations of the constant coefficient case, i.e. we consider the operator

$$\mathcal{A}_0(D) + \mathcal{A}_1(x, D),$$

where $\mathcal{A}_0(D)$ is the operator defined above and $\mathcal{A}_1$ is a homogeneous differential operator of order m with coefficients $a^1_\alpha \in BUC^\eta(\mathbb{R}^n; \mathcal{B}(E))$, sufficiently small in $L_\infty(\mathbb{R}^n; \mathcal{B}(E))$. Let A_1 be the realization of $\mathcal{A}_1(x, D)$ in $X = L_p(\mathbb{R}^n; E)$ with domain $D(A_1) = H^m_p(\mathbb{R}^n; E)$. The vector-valued version of Mikhlin's theorem in n dimensions (see [Wei01], [SW00], [HHN01] or Theorem 3.25 of [DHP01]) implies that, for $|\alpha| = m$, the operators $D^\alpha A_0^{-1}$ are bounded in $X = L_p(\mathbb{R}^n; E)$, as well as in $H^s_p(\mathbb{R}^n; E)$ for each $s \in \mathbb{R}$. Hence the first condition of the perturbation theorem 3.2 holds provided the L_∞-norms of a^1_α are suffiently small. In order to verify the second assumption of Theorem 3.2, recall that $A_0 \in \mathcal{H}^\infty(X)$ and $\phi^\infty_{A_0} \leq \phi_{\mathcal{A}_0}$ (see Theorem 5.5. of [DHP01]). By classical interpolation arguments (see e.g. [Tri78], pp.103-104) we have

$$\mathcal{D}(A_0^\gamma) = [X, \mathcal{D}(A_0)]_\gamma = H^{m\gamma}_p(\mathbb{R}^n; E).$$

Since functions in $BUC^\eta(\mathbb{R}^n; \mathcal{B}(E))$ are pointwise multipliers for $H^{m\gamma}_p(\mathbb{R}^n; E)$ in the case where $\eta > m\gamma$ (see e.g. [Tri78]), we observe that the second assumption of the perturbation theorem 3.2 is valid as well. Hence we may conclude that $A_0 + A_1$ admits a bounded $\mathcal{H}^\infty$-calculus provided the coefficients of $\mathcal{A}_1(x, D)$ are suffiently small in L_∞-norm.

We reformulate this last assertion in the following way: let $\eta \in (0,1)$ and assume that

$$a_\alpha \in BUC^\eta(\mathbb{R}^n; \mathcal{B}(E)), \qquad |\alpha| = 2m. \tag{4.1}$$

For some fixed $x_0 \in \mathbb{R}^{n+1}_+$ assume that, for given $\varepsilon \in (0,1)$, the coefficients are of uniformly small oscillation in the sense that

$$\sup_{x\in\mathbb{R}^n} \sum_{|\alpha|=2m} |a_\alpha(x) - a_\alpha(x_0)| < \varepsilon. \tag{4.2}$$

Then the following proposition holds.

Theorem 4.1. *For each $\phi > \phi_{\mathcal{A}}$ there exists $\varepsilon_0 = \varepsilon_0(\phi) > 0$ such that for all parameter-elliptic operators $\mathcal{A}(x,D)$ satisfying (E), (4.1) and (4.2) with $\varepsilon < \varepsilon_0$, there exist $C > 0$ such that the L_p-realization A of $\mathcal{A}(x,D)$ admits a bounded $\mathcal{H}^\infty$-calculus on $L_p(\mathbb{R}^n; E)$ with $\phi_A^\infty \le \phi$.*

We remark at this point that in the situation of $E = \mathbb{C}^N$ this result has been obtained earlier by different techniques by Prüss and Sohr [PS93] and also by Amann, Hieber, and Simonett [AHS94]. Duong and Simonett [DS97] proved this result even for $a_\alpha \in L_\infty(\mathbb{R}^n; \mathbb{C}^N)$, however their proof is highly involved, it is based on the so-called T_1-Theorem.

5. Perturbations of boundary conditions

Let $\mathcal{A}(D)$ be a parameter-elliptic operator of order $2m$ with constant coefficients and angle of ellipticity $\phi_{\mathcal{A}}$ on $\mathbb{R}^{n+1}_+$ of the form

$$\mathcal{A}(D) = \sum_{|\alpha|=2m} a_\alpha D^\alpha$$

with $a_\alpha \in \mathbb{C}$. Let further

$$\mathcal{B}_j(x,D) = \sum_{|\beta|=m_j} b_{j\beta}(x) D^\beta, \quad j = 1,\ldots,m$$

be boundary operators with coefficients $b_{j\beta} \in C^{2m-m_j}(\mathbb{R}^{n+1}_+; \mathcal{B}(E))$. Let $\phi > \phi_{\mathcal{A}}$ and assume that E is of class $\mathcal{HT}$. We now fix $x_0 \in \mathbb{R}^{n+1}_+$ and assume that, for given $\varepsilon \in (0,1)$, the coefficients $b_{j\beta}$ are of uniformly small oscillation in the sense that

$$\sup_{x\in\mathbb{R}^{n+1}_+} \sum_{|\beta|=m_j} |b_{j\beta}(x) - b_{j\beta}(x_0)| < \varepsilon. \tag{5.1}$$

For $\lambda \in \Sigma_{\pi-\phi}$ with $\phi > \phi_{\mathcal{A}}$ and $f \in L_p(\mathbb{R}^{n+1}_+; E)$ consider the problem

$$\left\{ \begin{array}{rcll} (\lambda + \mathcal{A}(D))\, u &=& f & \text{in } \mathbb{R}^{n+1}_+, \\ (\mathcal{B}_j(x,D)u)(0) &=& 0 & \text{on } \mathbb{R}^n. \end{array} \right. \tag{5.2}$$

Assume that the Lopatinskii-Shapiro condition is satisfied. It was proved in Section 7.3 of [DHP01] that there exists $\lambda_0 > 0$ such that for $f \in L_p(\mathbb{R}^{n+1}_+; E)$ and $\lambda \in \Sigma_{\pi-\phi}$ with $|\lambda| > \lambda_0$ there exists a unique $u \in H^{2m}_p(\mathbb{R}^{n+1}_+; E)$ satisfying (5.2). Moreover, u is given by

$$u = (\lambda + A_B)^{-1} f = (\lambda + A_B^0)^{-1} f + \sum_{k=1}^{\infty} T(\lambda)^k (\lambda + A_B^0)^{-1} f. \tag{5.3}$$

Here A_B denotes the realization of the boundary value problem (5.2) in $L_p(\mathbb{R}^{n+1}_+; E)$, i.e.

$$\begin{array}{rcl} A_B u &:=& \mathcal{A}(D)u \\ D(A_B) &:=& \{u \in H^{2m}_p(\mathbb{R}^{n+1}_+; E); (\mathcal{B}_j(x',D)u)(x',0) = 0 \quad x' \in \mathbb{R}^n,\ j = 1,\ldots,m\} \end{array}$$

see Section 7.3 of [DHP01] for details. Moreover, A_B^0 denotes the realization of the boundary value problem with frozen constant coefficient boundary conditions, i.e.

$$\begin{aligned} A_B^0 u &:= \mathcal{A}(D)u \\ D(A_B^0) &:= \{u \in H_p^{2m}(\mathbb{R}_+^{n+1};E); (\mathcal{B}_j(x_0,D)u)(x',0)=0 \quad x' \in \mathbb{R}^n,\ j=1,\dots,m\} \end{aligned}$$

and $T(\lambda)$ is defined by

$$T(\lambda) = \sum_{j=1}^{m} [S_j^I(\lambda) D_\lambda^{2m-m_j} + S_j^{II}(\lambda) D_\lambda^{2m-m_j-1}\partial_y](\mathcal{B}_j(x,D) - \mathcal{B}_j(x_0,D)), \tag{5.4}$$

where $D_\lambda = (-\Delta + |\lambda|^{1/m})^{\frac{1}{2}}$. Moreover, $S_j^L(\lambda)$, $L = I, II$, are integral operators on $L_p(\mathbb{R}_+^{n+1};E)$ of the form

$$S_j^L(\lambda) g_j(x',y) = \int_0^\infty \int_{\mathbb{R}^n} k_{j,\lambda}^L(x'-\tilde{x}, y+\tilde{y}) g_j(\tilde{x},\tilde{y})\, d\tilde{x}\, d\tilde{y} \tag{5.5}$$

where the derivatives of order $|\alpha| \le 2m$ of the kernels $k_{j,\lambda}^L$ satisfy a Poisson bound of the form

$$|D^\alpha k_{j,\lambda}^L(x',y)| \le M \cdot |\lambda|^{\frac{n+1-2m+|\alpha|}{2m}} p_{2m,|\alpha|}^{n+1}(c|\lambda|^{\frac{1}{2m}}(|x'|+y)), \quad x' \in \mathbb{R}^n, y > 0, \lambda \in \Sigma_{\pi-\phi}. \tag{5.6}$$

Further note that $(\lambda + \mathcal{A}(D)) S_j^L(\lambda) = 0$, by construction of the operator families $S_j^L(\lambda)$. For detailed proofs of these facts, we refer to Sections 6.6 and 7.3 of [DHP01].

The philosophy in perturbing the boundary conditions is the same as in the abstract perturbation result Theorem 3.2. We cannot directly apply this perturbation result since boundary perturbations change the domain of the underlying operator. However, as we shall see the main idea of our approach still works for boundary perturbations.

It follows from Section 7.3 of [DHP01] that $A_B + \omega$ is a sectorial operator in $L_p(\mathbb{R}_+^{n+1};E)$ with $0 \in \rho(A_B + \omega)$ provided $\omega > 0$ is sufficiently large. Thus, for $h \in \mathcal{H}_0^\infty(\Sigma_\phi)$ with $\phi > \phi_{A_B}$ and $R > 0$ we have

$$h(A_B+\omega) = \frac{1}{2\pi i}\int_{\Gamma_1} h(\lambda)(\lambda - (\omega + A_B))^{-1} d\lambda + \frac{1}{2\pi i}\int_{\Gamma_2} h(\lambda)(\lambda - (\omega + A_B))^{-1} d\lambda,$$

where $\Gamma := \{re^{i\theta}; r > 0\}$ with $\theta \in (\phi_{A_B}, \phi)$, $\Gamma_1 := \{\lambda \in \Gamma; |\lambda| \le R\}$ and $\Gamma_2 := \{\lambda \in \Gamma; |\lambda| \ge R\}$. By a direct estimate, we have

$$|\int_{\Gamma_1} h(\lambda)(\lambda - (\omega + A_B))^{-1} d\lambda|_{\mathcal{B}(L_p(\mathbb{R}_+^{n+1};E))} \le C_R |h|_{\mathcal{H}^\infty(\Sigma_\phi)}, \quad h \in \mathcal{H}_0^\infty(\Sigma_\phi).$$

for some constant $C_R > 0$. Secondly, observe that by (5.3)

$$\begin{aligned} \int_{\Gamma_2} h(\lambda)(\lambda - (\omega + A_B))^{-1} d\lambda &= \int_{-\Gamma_2} h(-\lambda)(\lambda + \omega + A_B^0)^{-1} d\lambda + \\ &\quad \sum_{k=1}^{\infty} \int_{-\Gamma_2} h(-\lambda) T(\lambda+\omega)^k (\lambda + \omega + A_B^0)^{-1} d\lambda, \end{aligned} \tag{5.7}$$

provided the sum on the right hand side of (5.7) converges. By Theorem 7.4 of [DHP01],

$$|\int_{-\Gamma_2} h(-\lambda)(\lambda + \omega + A_B^0)^{-1} d\lambda|_{\mathcal{B}(L_p(\mathbb{R}_+^{n+1};E))} \le C|h|_{\mathcal{H}^\infty(\Sigma_\phi)}, \quad h \in H_0^\infty(\Sigma_\phi).$$

In the following we show that there exists $C_R > 0$ such that for all $k \ge 1$ we have

$$|\int_{-\Gamma_2} h(-\lambda) T(\lambda+\omega)^k (\lambda + \omega + A_B^0)^{-1} d\lambda|_{\mathcal{B}(L_p(\mathbb{R}_+^{n+1};E))} \le C_R^k (|b|_\infty + \frac{1}{R^{1/2m} + \omega^{1/2m}})^k, \tag{5.8}$$

where $|b|_\infty := \sup\{|b_{j,\beta}(x)| : \ x \in \Omega, |\beta| = m_j, j = 1, \ldots, m\}$. This will be shown directly for the case $k = 1$. For $k \geq 2$, let $R > 0$ be such that $2^{j_0} = R$ for some $j_0 \in \mathbb{N}$. Define the path Γ_N by $\Gamma_N := \{\lambda \in \Gamma; R \leq |\lambda| \leq 2^N\}$. We then see that

$$\int_{-\Gamma} h(-\lambda)T(\lambda+\omega)^k(\lambda+\omega+A_B^0)^{-1}d\lambda = \lim_{N\to\infty}\int_{-\Gamma_N} h(-\lambda)T(\lambda+\omega)^k(\lambda+\omega+A_B^0)^{-1}d\lambda = \lim_{N\to\infty} H_{N,k},$$

where $H_{N,k} = H_{N,k}^+ + H_{N,k}^-$ and with $\vartheta = \pi - \theta$

$$\begin{aligned} H_{N,k}^{\pm} &= \int_{2^{j_0}}^{2^N} h(re^{\pm i\vartheta})T(re^{\pm i\vartheta} + \omega)^k(re^{\pm i\vartheta} + \omega + A_B^0)^{-1}e^{\pm i\theta}dr \\ &= \int_1^2 \sum_{j=j_0}^{N-1} h(2^j re^{\pm i\vartheta})T(2^j re^{\pm i\vartheta} + \omega)^k(2^j re^{\pm i\vartheta} + \omega + A_B^0)^{-1}e^{\pm i\theta}2^j r\frac{dr}{r} \\ &= \int_1^2 U_{N,k}(r)\frac{dr}{r}. \end{aligned}$$

Let now (ε_j) be a sequence of independent, symmetric, $\{-1, 1\}$-valued random variables on a probability space $(\Omega, \mathcal{M}, \mu)$. For $x \in X$ and $x^* \in X^*$ we then have

$$< U_{N,k}(r)x, x^* >= \int_\Omega < \sum_{j=j_0}^{N-1} \varepsilon_j^2 h(2^j re^{\pm i\vartheta})T(2^j re^{\pm i\vartheta}+\omega)^k(2^j re^{\pm i\vartheta}+\omega+A_B^0)^{-1}2^j re^{\pm i\theta}x|x^* > .$$

Observe that for $\lambda \in \Sigma_{\pi-\phi}$ the term $T(\lambda)^k$ is the sum of m^k-terms of the form

$$\prod_{l=1}^k S_{j_l}^{L_l}(\lambda)D_\lambda^{2m-m_{j_l}-\delta_{j_l}}(\partial_y)^{\delta_{j_l}}\mathcal{B}_{j_l},$$

where

$$\delta_{j_l} = \begin{cases} 0 & \text{if} \quad L_l = I \\ 1 & \text{if} \quad L_l = II \end{cases}$$

and $j_l \in \{1, \ldots, m\}$ for $l = 1, \ldots, k$. Hence $T(\lambda)^k(\lambda + A_B^0)^{-1}$ is the sum of m^k-terms of the form

$$W(\lambda)V_{k-2}(\lambda)V_k(\lambda),$$

where

$$\begin{aligned} &(5.9) & W(\lambda) &:= J_\lambda S_{j_l}^{L_l}(\lambda) \qquad \text{for some} \quad l \in \{1, \ldots, k\}, \\ &(5.10) & V_{k-2}(\lambda) &:= \prod_{l=1}^{k-2} J_\lambda^{-1} D_\lambda^{2m-m_{j_l}-\delta_{j_l}}\partial_y^{\delta_{jl}}\mathcal{B}_{j_l}J_\lambda S_{j_l}^{L_l}(\lambda), \quad k \geq 1, \\ &(5.11) & V_k(\lambda) &:= J_\lambda^{-1}D_\lambda^{2m-m_j-\delta_j}\partial_y^{\delta_j}\mathcal{B}_j S_{j_k}^{L_k}(\lambda)D_\lambda^{2m-m_{j_k}-\delta_{j_k}}\partial_y^{\delta_{jk}}\mathcal{B}_{j_k}(\lambda + A_B^0)^{-1}. \end{aligned}$$

Here J_λ is defined by $J_\lambda := ((-\Delta)^{\frac{1}{2}} + |\lambda|^{\frac{1}{2m}})$. Thus for $1 < r < 2$, $x \in X$, $x^* \in X^*$ and $\lambda_j = 2^j re^{i\pm\theta} + \omega$, $< U_{N,k}(r)x|x^* >$ is a sum of m^k terms of the form

$$\begin{aligned} & \int_\Omega < \sum_{j=j_0}^{N-1} \varepsilon_j^2 h(\lambda_j - \omega)W(\lambda_j)V_{k-2}(\lambda_j)V_k(\lambda_j)(\lambda_j - \omega)x|x^* > d\mu \\ = \quad & \int_\Omega < \sum_{j=j_0}^{N-1} \varepsilon_j h(\lambda_j - \omega)V_{k-2}(\lambda_j)V_k(\lambda_j)(\lambda_j - \omega)x| \sum_{j=j_0}^{N-1} \varepsilon_j W^*(\lambda_j)x^* > d\mu \end{aligned}$$

By the contraction principle, the latter can be estimated by

$$\leq \quad |\sum_{j=j_0}^{N-1} \varepsilon_j h(\lambda_j-\omega)V_{k-2}(\lambda_j)V_k(\lambda_j)(\lambda_j-\omega)^{1/2m}x|_{L_2(\Omega,X)}\cdot$$
$$\cdot \quad |\sum_{j=j_0}^{N-1} \varepsilon_j W^*(\lambda_j)(\lambda_j-\omega)^{1-\frac{1}{2m}}x^*|_{L_2(\Omega,X^*)}$$
$$\leq \quad C|h|_\infty \mathcal{R}\{V_{k-2}(\lambda_j); j\geq j_0\}|\sum_{j=j_0}^{N-1} \varepsilon_j V_k(\lambda_j)(\lambda_j-\omega)^{1/2m}x|_{L_2(\Omega,X)}\cdot$$
$$\cdot \quad |\sum_{j=j_0}^{N-1} \varepsilon_j W^*(\lambda_j)(\lambda_j-\omega)^{1-\frac{1}{2m}}x^*|_{L_2(\Omega,X^*)}.$$

By the subsequent Lemma 6.5 and by Proposition 6.6, there exists a constant $C>0$ such that

$$\mathcal{R}\{V_{k-1}(\lambda_j); j\geq j_0\}\leq C^{k-1}[|b|_\infty+(R+\omega)^{-1/2m}]^{k-1}.$$

Moreover, it follows from Propositions 6.3 and 6.9 below that $< U_{N,k}(r)x|x^* >$ is a sum of at most m^k terms, each of which can be estimated by

$$C^k|h|_\infty[|b|_\infty+R^{-1/2m}]^k|x||x^*|.$$

Choosing $|b|_\infty$ small enough and ω large enough, we obtain the following result on small perturbations of boundary conditions:

Assume that the coefficients $b_{j\beta}$ of $\mathcal{B}_j(x,D)=\sum_{|\beta|=m_j} b_{j\beta}(x)D^\beta$ satisfy

$$b_{j\beta}\in C^{2m-m_j}(\partial\mathbb{R}^{n+1}_+;\mathcal{B}(E)), \qquad |\beta|=m_j, 1\leq j\leq m. \tag{5.12}$$

Let $x_0\in\mathbb{R}^{n+1}_+$ and assume further that, for given $\varepsilon\in(0,1)$, the coefficients $b_{j\beta}$ are of uniformly small oscillation in the sense that (5.1) is satisfied. Recall that

$$\mathcal{B}_j(x,D) \quad = \quad \mathcal{B}_j(x_0,D)+\mathcal{B}_j{}^{sm}(x,D),$$

where $\mathcal{B}_j(x_0,D):=\sum\limits_{|\beta|=m_j} b_{j\beta}(x_0)D^\beta$ is an operators with constant coefficients and

$$\mathcal{B}_j{}^{sm}(x,D)=\sum_{|\beta|=m_j}(b_{j\beta}(x)-b_{j\beta}(x_0))D^\beta.$$

Then the following result is true.

Theorem 5.1. *For each $\phi>\phi_{\mathcal{A}}$ there is $\varepsilon_0=\varepsilon_0(\phi)>0$ such that for all operators $\mathcal{A}(D)$ and $\mathcal{B}_j(x,D)$ satisfying (E), (5.12) and (5.1) with $\varepsilon<\varepsilon_0$, there exists $\omega>0$ such that $A_B+\omega$ admits a bounded $\mathcal{H}^\infty$-calculus on $L_p(\mathbb{R}^{n+1}_+;E)$ with $\mathcal{H}^\infty$-angle $\phi^\infty_{A_B+\omega}\leq\phi$.*

In order to complete the proof of Theorem 5.1 we need the following results on commutators and kernel estimates for the solution of the perturbed boundary value problem.

6. The hard part of Section 5: commutators, kernels and $\mathcal{R}$-boundedness

In the section we prove the assertions and lemmas cited in the previous section. Let us start with the following observation. Minor modifications of the proofs of Propositions 6.5, 6.6 and 6.9 of [DHP01] yield the following results.

Proposition 6.1. *a) If α is a multiindex such that $0 \le |\alpha| \le 2m$, then $(\lambda + A_B^0)^{-1}$ admits a representation as an integral operator on $L_p(\mathbb{R}_+^{n+1}; E)$ of the form*

$$(\lambda + A_B^0)^{-1} f(x', y) = \int_0^\infty \int_{\mathbb{R}^n} k_\lambda(x' - \tilde{x}, y, \tilde{y}) f(\tilde{x}, \tilde{y}) d\tilde{x} d\tilde{y}, \qquad f \in L_p(\mathbb{R}_+^{n+1}; E),$$

whose kernel k_λ satisfies a Poisson bound of the form

$$|k_\lambda(x', y, \tilde{y})| \le C|\lambda|^{\frac{n+1}{2m}-1} p_{2m,|\alpha|}^{n+1}(c|\lambda|^{1/2m}(|x'| + |y - \tilde{y}|)),$$

for $\lambda \in \Sigma_{\pi-\phi}$ and $x' \in \mathbb{R}^n, y, \tilde{y} > 0$.
b) The kernels $k_{j,\lambda}^L$, $L = I, II$, given in (5.5) satisfy an estimate of the form

$$(6.1) \qquad |D^\beta (-\Delta)^{|\alpha|/2} k_{j,\lambda}^L(x', y)| \le C \cdot |\lambda|^{\frac{n+1-2m+|\alpha|+|\beta|}{2m}} p_{2m,|\alpha|+|\beta|}^{n+1}(c|\lambda|^{\frac{1}{2m}}(|x'| + y)),$$

where $x' \in \mathbb{R}^n, y > 0, \lambda \in \Sigma_{\pi-\phi}$.

Our next lemma will be of crucial importance in proving the central estimate (5.8).

Lemma 6.2. *Let X be a Banach space and let $\lambda \in \Sigma_\phi$ for some $\phi \in (0, \pi)$. Suppose that K_λ is a kernel operator on $L_p(\mathbb{R}_+^{n+1}; E)$ of the form*

$$K_\lambda f(x', y) = \int_0^\infty \int_{\mathbb{R}^n} k_\lambda(x' - \tilde{x}', y, \tilde{y}) f(\tilde{x}', \tilde{y}) d\tilde{x}' d\tilde{y}, \quad x' \in \mathbb{R}^n, y > 0,$$

whose kernel $k_\lambda(\cdot, \cdot, \cdot)$ satisfies a Poisson estimate of the form

$$|k_\lambda(x', y, \tilde{y})| \le C|\lambda|^{\frac{n+1}{2m}-1} p_{2m,l}^{n+1}(\kappa|\lambda|^{1/2m}(|x'| + y + \tilde{y})), \quad \lambda \in \Sigma_\phi,\ x' \in \mathbb{R}^n,\ y, \tilde{y} > 0,$$

where $0 \le l < m$. Then there is a constant $C > 0$ such that

$$|\int_{-\Gamma} h(-\lambda) K_\lambda d\lambda| \le C|h|_\infty, \quad h \in \mathcal{H}_0^\infty(\Sigma_\phi),$$

and

$$|\sum_j \varepsilon_j 2^j r e^{i\theta} K_{2^j r e^{i\theta}}| \le C|\varepsilon|_\infty,$$

where $\varepsilon = (\varepsilon_j) \in l_\infty(\mathbb{Z})$, $r \in (1, 2)$ and $|\theta| \le \pi - \phi$.

Proof. To prove the first part, observe that the kernel of $\int_{-\Gamma} h(-\lambda) K_\lambda d\lambda$ is given by

$$k_h(x, y, \bar{y}) = \int_{-\Gamma} h(-\lambda) k_\lambda(x, y, \bar{y}) d\lambda,$$

which can be estimated by

$$\begin{aligned} |k_h(x, y, \bar{y})| &\le c|h|_\infty \int_0^\infty p_{2m,l}^{n+1}(c\rho^{1/2m}(|x| + y + \bar{y}))\rho^{(n+1)/2m} d\rho/\rho \\ &= C|h|_\infty (|x| + y + \bar{y})^{-(n+1)} \int_0^\infty \frac{s^{n-1} ds}{(1+s)^{2m+n-l}} = C|h|_\infty (|x| + y + \bar{y})^{-(n+1)}. \end{aligned}$$

This is an $L_p(\mathbb{R}_+^{n+1}; E)$-bounded kernel; see e.g. [DHP01], Lemma 7.1.

We estimate the kernel k_Σ of $\sum_j \varepsilon_j 2^j re^{i\theta} K_{2^j re^{i\theta}}$ as follows.

$$\begin{aligned} |k_\Sigma(x',y,\tilde{y})| &\leq |\varepsilon|_\infty \sum_j |k_{2^j re^{i\theta}}(x',y,\tilde{y})| \\ &\leq c|\varepsilon|_\infty \sum_j (2^j r)^{\frac{n+1}{2m}} p^{n+1}_{2m,l}(\kappa(2^j r)^{1/2m}(|x'|+y+\tilde{y})) \\ &\leq c|\varepsilon|_\infty \int_0^\infty \frac{\sigma^{n-1}}{(1+\sigma)^{2m-l-1}} \sum_j (2^j r)^{\frac{n+1}{2m}} e^{-\kappa(1+\sigma)(2^j r)^{1/2m}(|x'|+y+\tilde{y})} d\sigma. \end{aligned}$$

Next, setting $\mu := \kappa r^{1/2m}(1+\sigma)(|x'|+y+\tilde{y}) > 0$ and $a := 2^{1/m}$ the above sum may be estimated as

$$\begin{aligned} \sum_j a^{j(n+1)} e^{-\mu a^j} &\leq \int_{-\infty}^\infty a^{x(n+1)} e^{-[\mu/a]a^x} dx \\ &= \int_0^\infty y^n e^{-[\mu/a]y} dy/\log a = \frac{(n+1)!a^{n+1}}{\mu^{n+1}\log a}. \end{aligned}$$

Since $1 \leq r \leq 2$ and $l < m$ by assumption,

$$\begin{aligned} |k_\Sigma(x',y,\tilde{y})| &\leq c|\varepsilon|_\infty (|x'|+y+\tilde{y})^{-(n+1)} \int_0^\infty \frac{\sigma^{n-1}}{(1+\sigma)^{2m-l+n}} d\sigma \\ &\leq c|\varepsilon|_\infty (|x'|+y+\tilde{y})^{-(n+1)}. \end{aligned} \tag{6.2}$$

Hence the result follows again from [DHP01], Lemma 7.1. □

Proposition 6.3. *Let (ε_j) be a sequence of independent, symmetric, $\{-1,1\}$-valued random variable on a probability space $(\Omega,\mathcal{M},\mu)$ and X be a Banach space. Let $R > 0$ and for $j \in \mathbb{N}$ set $\lambda_j = 2^j re^{i\theta} + \omega$ where $r \in (1,2)$ and $\theta \in$. Choose $j_0 \in \mathbb{N}$ such that $\lambda_{j_0} > R$. Then there exists a constant $C > 0$ such that for $x^* \in X^*$*

$$|\sum_{j\geq j_0} \varepsilon_j W(\lambda_j)^*(\lambda_j-\omega)^{1-\frac{1}{2m}} x^*|_{L_2(\Omega,X^*)} \leq C|x^*|.$$

Proof. Observe that by (6.1) the family of operators

$$\{\lambda^{1-\frac{|\alpha|}{2m}}(-\Delta)^{\frac{|\alpha|}{2}} S_j^L(\lambda), \lambda \in \Sigma_{\pi-\phi}, |\lambda| \geq R\}$$

satisfies for α with $|\alpha| \leq 1$ the assumptions of Lemma 6.2. Thus

$$|\sum_{j\geq j_0} \varepsilon_j W(\lambda_j)^*(\lambda_j-\omega)^{1-\frac{1}{2m}} x^*|_{L_2(\Omega,X^*)} \leq |\sum_{j\geq j_0} \varepsilon_j W(\lambda_j)^*(\lambda_j-\omega)^{1-\frac{1}{2m}}|_{\mathcal{B}(X^*)}|x^*| \leq C|x^*|.$$

□

In the following we consider L_p-mapping properties of the commutator between a multiplication operator with a function in $BUC^{1+\eta}(\mathbb{R}^n,\mathcal{B}(E))$ and $(-\Delta)^{1/2}$. Mapping properties of this kind are well known in the scalar-valued setting even for Lipschitz continuous functions and pseudo-differential operators of order 1 having smooth symbols (see e.g. [Ste93]). In the following we are considering the vector-valued case. Observe also that the symbol of $(-\Delta)^{1/2}$ is not smooth, which means that even in the scalar-valued case the classical results do not apply immediately.

Lemma 6.4. *Let E be a Banach space of class $\mathcal{HT}$ and $b \in BUC^{1+\eta}(\mathbb{R}^n, \mathcal{B}(E))$, with some $\eta \in (0,1)$. Then there exists a constant $C > 0$ such that*

$$|[M_b, (-\Delta)^{1/2}]f|_{L_p(\mathbb{R}^n;E)} \leq C|f|_{L_p(\mathbb{R}^n;E)}, \quad f \in L_p(\mathbb{R}^n; E), \tag{6.3}$$

where M_b denotes the operator of pointwise multiplication with the function $b(x)$.

Proof. Note that

$$\begin{aligned}(-\Delta)^{1/2} = -\Delta(-\Delta)^{-1/2} &= -\frac{1}{\sqrt{\pi}}\Delta\int_0^\infty e^{\Delta t}t^{\frac{-1}{2}}dt \\ &= -\frac{1}{\sqrt{\pi}}(\Delta\int_0^1 e^{\Delta t}t^{\frac{-1}{2}}dt - \Delta\int_1^\infty e^{\Delta t}t^{\frac{-1}{2}}dt) =: -\frac{1}{\sqrt{\pi}}(T_1+T_2),\end{aligned}$$

where $e^{\Delta t}$ denotes the semigroup generated by the Laplacian in $L_p(\mathbb{R}^n;E)$. Thus there is $C > 0$ such that for $f \in L_p(\mathbb{R}^n;E)$ we have

$$\begin{aligned}|M_bT_2f|_p &= |M_b\int_1^\infty \Delta e^{\Delta t}t^{\frac{-1}{2}}fdt|_p \leq C|b|_\infty\int_1^\infty t^{\frac{-3}{2}}dt|f|_p \leq C|f|_p \\ |T_2M_bf|_p &= |\int_1^\infty \Delta e^{\Delta t}t^{\frac{-1}{2}}(bf)dt|_p \leq C|b|_\infty\int_1^\infty t^{\frac{-3}{2}}dt|f|_p \leq C|f|_p.\end{aligned}$$

Hence $|[M_b,T_2]f|_p \leq C|f|_p$.

Consider next $[M_b,T_1]$ and note that for $x, y \in \mathbb{R}^n$ we have

$$b(x) - b(y) = Db(x)(x-y) + h(x,y) \quad \text{where } |h(x,y)| \leq C|x-y|^{1+\eta},$$

since by assumption $b \in BUC^{1+\eta}(\mathbb{R}^n;\mathcal{B}(E))$. Therefore, say with some $f \in C_c(\mathbb{R}^n;E)$,

$$\begin{aligned}[M_b,T_1]f(x) &= \int_0^1\int_{\mathbb{R}^n}(b(x)-b(y))\Delta k_t(x-y)f(y)dyt^{-\frac{1}{2}}dt \\ &= Db(x)\int_{\mathbb{R}^n}\int_0^1(x-y)\Delta k_t(x-y)t^{-\frac{1}{2}}dtf(y)dy + \\ &+ \int_{\mathbb{R}^n}\int_0^1 h(x,y)\Delta k_t(x-y)t^{-\frac{1}{2}}dtf(y)dy \\ &=: Db(x)T_3f(x) + T_4f(x),\end{aligned}$$

where k_t denotes the Gaussian kernel. We observe that

$$\begin{aligned}T_3f &= q * f \quad \text{with} \quad q(x) = \int_0^1 x\Delta k_t(x)t^{-\frac{1}{2}}dt,\ x \in \mathbb{R}^n \\ |T_4f| &\leq r * |f| \quad \text{with} \quad r(x) = C\int_0^1 |x|^{1+\eta}|\Delta k_t(x)|t^{-\frac{1}{2}}dt,\ x \in \mathbb{R}^n.\end{aligned}$$

The Fourier transform of q is given by $\widehat{q}(\xi) = c\int_0^1 \frac{\partial}{\partial\xi}(|\xi|^2e^{-t|\xi|^2})\frac{1}{t^{1/2}}dt$ and we verify that

$$\sup_{|\alpha|\leq(1,\dots,1)}\sup_{\xi\in\mathbb{R}^n} D^\alpha|\xi|^\alpha|\widehat{q}(\xi)| \leq M$$

for some $M < \infty$. It thus follows from Mikhlin's theorem for E-valued functions, E being a space of class $\mathcal{HT}$, and the regularity assumption on b that $|T_3f|_p \leq C|f|_p$. Finally, in order to estimate the term T_4, note that there exist constant $c, C > 0$ such that

$$|\Delta k_t(x-y)| \leq \frac{C}{t^{\frac{n+2}{2}}}e^{-c|x-y|^2/t}, \quad x, y \in \mathbb{R}^n.$$

Thus

$$r(x) \le C \int_0^1 |x|^{1+\eta} \frac{1}{t^{\frac{n+3}{2}}} e^{-c|x|^2/t} dt$$

for $x \in \mathbb{R}^n$. It follows that $|r|_1 \le C \int_0^1 t^{\eta/2-1} dt < \infty$ which implies by Young's inequality and the regularity assumption on b that $|T_4 f|_p \le C|f|_p$. □

Lemma 6.5. *Let $D_\lambda = (\lambda^{1/m} - \Delta)^{1/2}$ and $J_\lambda = \lambda^{1/2m} + (-\Delta)^{1/2}$ on $L_p(\mathbb{R}^n; E)$.*
Then $\mathcal{R}(\{D_\lambda J_\lambda^{-1} : \lambda \in \Sigma_{\pi-\phi}\}) < \infty$, for each angle $\phi \in (0, \pi]$.

Proof. Set $\mu = \lambda^{1/2m}$, $A = (-\Delta)^{1/2}$ in $L_p(\mathbb{R}^n; E)$. Then

$$D_\lambda J_\lambda^{-1} = A^{-1}(\mu^{-2} + A^{-2})^{-1/2} \cdot \mu(\mu + A)^{-1} + A(\mu^2 + A^2)^{-1/2} \cdot A(\mu + A)^{-1};$$

A, A^{-1} as well as A^2 are $\mathcal{R}$-sectorial and admit bounded imaginary powers with angles 0, hence the assertion follows from Proposition 4.14 in [DHP01]. □

Note that this lemma allows us to replace D_λ by J_λ wherever we please.

Proposition 6.6. *There exists a constant $C > 0$ such that for $i \in 1, \dots, m$, $\delta = 0, 1$, we have*

$$\mathcal{R}\{D_\lambda^{2m-m_i-\delta} J_\lambda^{-1} (\partial_y)^\delta \mathcal{B}_i J_\lambda S^\delta ; \lambda \in \Sigma_{\pi-\phi} |\lambda| \ge R\} \le C(|b|_\infty + R)^{-\frac{1}{2m}}).$$

Proof. Notice that $(\partial_y)^\delta \mathcal{B}_i = \sum_{|\beta|=m_i} b_{i\beta}(x) D^\beta (\partial_y)^\delta$. In order to simplify our notation we consider in the following without loss of generality only one term of the form $bD^\beta(\partial_y)^\delta$, $|\beta| = m_i$ in the above sum. We have to distinguish 4 cases.

(a) $2m - m_i - \delta = 0$. Then $m_i + \delta = 2m$. If $\beta + \delta \ne (0, \dots, 0, 2m)$ then D^β contains an x-derivative, say ∂_l. Then we write

$$J_\lambda^{-1} b \partial_y^\delta D^\beta J_\lambda S = \partial_l J_\lambda^{-1} \cdot M_{b_{i,\beta}} \cdot J_\lambda D^\gamma S + J_\lambda^{-1} (\partial_l b_{i\beta}) J_\lambda D^\gamma S,$$

where $|\gamma| = 2m - 1$. This yields

$$\mathcal{R}(\{J_\lambda^{-1} b \partial_y^\delta D^\beta J_\lambda S\} \le C(|b|_\infty + R^{-1/2m}),$$

where

$$C \le \mathcal{R}\{\partial_i J_\lambda^{-1}\} |b|_\infty + |\partial_l b_{i\beta}|_\infty \mathcal{R}\{\lambda^{1/2m} J_\lambda^{-1}\}.$$

If $\partial_y^\delta D^\beta = \partial_y^{2m}$, we use ellipticty and $(\lambda + \mathcal{A}(D))S_\lambda = 0$ to obtain

$$\partial_y^{2m} S_\lambda = -a_0^{-1}(\lambda S_\lambda + \mathcal{A}'(D) S_\lambda),$$

where $\mathcal{A}'(D)$ contains an x-derivative. Hence the same estimates apply.

(b) $2m - m_i - \delta = 1$. Then

$$\mathcal{R}\{D_\lambda J_\lambda^{-1} b \partial_y^\delta D^\beta S(\lambda)\} \le \mathcal{R}\{D_\lambda J_\lambda^{-1}\} |b|_\infty \mathcal{R}\{\partial_y^\delta D^\beta S(\lambda)\} \le C|b|_\infty,$$

by Lemma 6.5 and Proposition 7.6 of [DHP01].

(c) $2m - m_i - \delta =: 2l + 1 \ge 2$ odd. Then D_λ^{2l} is a differential polynomial of order $2l$. Commute it with $b(x)$ to the result

$$[D_\lambda^{2l}, b] = \sum_{\|gamma| + |\beta_1| \le 2l, \gamma \ne 0} c_{2l,\gamma,\beta_1} |\lambda|^{\frac{2l-|\gamma|-|\beta_1|}{2m}} D^\gamma b \cdot D^{\beta_1},$$

where the c_{2l,γ,β_1} are some constants. This implies for $\lambda \in \Sigma_{\pi-\phi}$, $|\lambda| \geq R$,

$$\mathcal{R}\{D_\lambda^{2l+1} J_\lambda^{-1} b \partial_y^\delta J_\lambda S\} \leq \mathcal{R}\{D_\lambda J_\lambda^{-1}\}|b|_\infty \mathcal{R}\{D_\lambda^{2l} \partial_y^\delta J_\lambda S\}$$
$$+R^{-1/2m} \sum_{\gamma,\beta_1} c_{2l,\gamma,\beta_1} |D^\gamma b|_\infty \mathcal{R}\{D_\lambda J_\lambda^{-1}\}\mathcal{R}\{\lambda^{\frac{2l-|\gamma|-|\beta_1|}{2m}} \partial_y^\delta J_\lambda D^{\beta+\beta_1} S(\lambda\}$$
$$\leq C(|b|_\infty + R^{-1/2m},$$

using again Lemma 6.5 the $\mathcal{R}$-boundedness properties of $S(\lambda)$, and the regularity of b.

(d) $2m - m_i - \delta = 2l + 2 \geq 2$ even. Then $D_\lambda^{2m-m_i-\delta-2}$ is such a differential operator of order $2l$; commute it with b as in step (c), and then with J_λ, i.e. with $(-\Delta)^{1/2}$ to the result

$$D_\lambda^{2m-m_i-\delta} J_\lambda^{-1} b = D_\lambda^2 J_\lambda^{-2}(b D_\lambda^{2m-m_i-\delta-2} J_\lambda + [(-\Delta)^{1/2}, b] D_\lambda^{2l}$$
$$- \sum_{\gamma,\beta_1} c_{2l,\gamma,\beta_1} [D^\gamma b, (-\Delta)^{1/2}] |\lambda|^{\frac{2l-|\gamma|-|\beta_1|}{2m}} D^{\beta_1}.$$

Note that, since $D^\gamma b \in BUC^{2m-m_i-|\gamma|} \subset BUC^2$, the commutators $[(-\Delta)^{1/2}, D^\gamma b]$ are bounded by Lemma 6.4. We then may continue as in step (c). □

The next lemma is the crucial one for perturbation of boundary conditions.

Lemma 6.7. *Suppose the kernel $k_\lambda(x,y,\bar{y})$ has a bound according to*

$$|k_\lambda(x,y,\bar{y})| \leq C|\lambda|^{2(n+1)/2m-1} \cdot \tag{6.4}$$
$$\cdot \int_0^\infty \int_{\mathbb{R}^n} p_{2m,k}^{n+1}(|\lambda|^{1/2m}(|x-x'|+y+y'))p_{2m,l}^{n+1}(|\lambda|^{1/2m}(|x'|+|\bar{y}-y'|))dx'dy',$$

where $0 \leq k, l \leq 2m$, $l \neq 2m$. Then there is a constant $C > 0$ such that

$$\int_{\mathbb{R}^n} |\int_{-\Gamma} h(-\lambda) k_\lambda(x,y,\bar{y}) d\lambda| dx \leq C|h|_\infty q(y,\bar{y}), \quad h \in \mathcal{H}_0^\infty(\Sigma_\phi),\ y,\bar{y} > 0,$$

and

$$\int_{\mathbb{R}^n} |\sum_j \varepsilon_j 2^j r e^{i\theta} k_{2^j r e^{i\theta}}(x,y,\bar{y}) d\lambda| dx \leq C|\varepsilon|_\infty q(y,\bar{y}), \quad y,\bar{y} > 0,$$

where the integral operator Q with kernel q is L_p-bounded.

Proof. Set $a = (1+s)(|x-x'|+y+y') + (1+r)(|x'|+|y'-\bar{y}|)$; then

$$\int_\Gamma |\lambda^{2\frac{n+1}{2m}} h(\lambda)| e^{-|\lambda|^{1/2m} a} |d\lambda| \leq C|h|_\infty \int_0^\infty \rho^{2n+2} e^{-\rho a} d\rho/\rho = C|h|_\infty a^{-2n-2}.$$

Integrating this expression w.r.t. x and x' we get

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_\Gamma |\lambda^{2\frac{n+1}{2m}} h(\lambda)| e^{-|\lambda|^{1/2m} a} |d\lambda| dx' dx \leq C|h|_\infty \int_0^\infty \int_0^\infty \frac{\rho^{n-1}\sigma^{n-1} d\rho d\sigma}{a(\rho,\sigma)^{2n+2}},$$

where $a(\rho,\sigma) = (1+s)(\rho+y+y') + (1+r)(\sigma+|y'-\bar{y}|)$. By means of the scalings $\tau = b(1+s)\rho$, $\vartheta = b(1+r)\sigma$, with $b = (1+s)(y+y') + (1+r)|y'-\bar{y}|$, we may continue

$$= \frac{C|h|_\infty}{b^2(1+s)^n(1+r)^n} \int_0^\infty \int_0^\infty \frac{\tau^{n-1}\vartheta^{n-1} d\tau d\vartheta}{(\tau+\vartheta+1)^{2n+2}} = \frac{C|h|_\infty}{b^2(1+s)^n(1+r)^n}.$$

Next we integrate w.r.t. y'; a simple computation yields

$$\int_0^\infty [(1+s)(y+y') + (1+r)|y'-\bar{y}|]^{-2} dy' = \frac{1}{2+s+r}\frac{1}{(1+s)y+(1+r)\bar{y}}(1+2\frac{1+r}{1+s}\frac{\bar{y}}{y+\bar{y}}).$$

Using the definition of the funktions $p^{n+1}_{2m,k}(t)$ we therefore end up with

$$\int_{\mathbb{R}^n} |\int_\Gamma h(\lambda)k_\lambda(x,y,\bar{y})d\lambda|dx \le C|h|_\infty \int_0^\infty \int_0^\infty \frac{r^{n-1}s^{n-1}drds}{(1+s)^{2m+n-k-1}(1+r)^{2m+n-l-1}} \cdot$$
$$\cdot \frac{1}{2+s+r}\frac{1}{(1+s)y+(1+r)\bar{y}}(1+2\frac{1+r}{1+s}\frac{\bar{y}}{y+\bar{y}})$$
$$\le C|h|_\infty \int_1^\infty \int_1^\infty (1+2\frac{r}{s}\frac{\bar{y}}{y+\bar{y}})\frac{1}{r+s}\frac{1}{sy+r\bar{y}}\frac{drds}{r^{2m-k}s^{2m-l}}$$
$$\le \quad C|h|_\infty \int_1^\infty \int_1^\infty \frac{drds}{(r+s)(sy+r\bar{y})}[\frac{1}{r}+\frac{1}{s}],$$

where we used $k \le 2m$ and $l < 2m$. Next we calculate

$$\int_1^\infty \frac{ds}{(s+r)(ys+\bar{y}r)} = \frac{1}{r(\bar{y}-y)}\log[\frac{1+r\bar{y}/y}{1+r}],$$

and then

$$\begin{aligned}\int_1^\infty \int_1^\infty \frac{dsdr}{(s+r)(ys+\bar{y}r)r} &= \frac{1}{\bar{y}-y}\int_1^\infty \log[\frac{1+r\bar{y}/y}{1+r}]\frac{dr}{r^2}\\ &= \frac{1}{\bar{y}-y}\log[1+\frac{\bar{y}-y}{2y}] + \int_1^\infty \frac{dr}{r(1+r)(y+r\bar{y})}\\ &\le \frac{1}{y-\bar{y}}\log[1+\frac{\bar{y}-y}{2y}] + \frac{1}{y+\bar{y}}.\end{aligned}$$

By symmetry we arrive at assertion (i) with

$$q(y,\bar{y}) = \frac{1}{y+\bar{y}} + \frac{1}{y-\bar{y}}\log[1+\frac{y-\bar{y}}{2\bar{y}}] + \frac{1}{\bar{y}-y}\log[1+\frac{\bar{y}-y}{2y}].$$

The estimate leading to (ii) is similar; cp. the proof of Lemma 6.2. In fact, as there we obtain

$$|\sum_j \varepsilon_j \lambda_j^{2(n+1)/2m} e^{-\lambda_j^{1/2m}a}| \le |\varepsilon|_\infty \frac{(n+1)!2^{2(n+1)/2m}}{a^{2n+2}\log 2^{1/m}} = C|\varepsilon|_\infty a^{-2n-2},$$

hence we may continue as above to derive estimate (ii).

It remains to show that the integral operator Q with kernel q is bounded in $L_p(\mathbb{R}_+)$ for each $p \in (1,\infty)$. Since the kernel $1/(y+\bar{y})$ is $L_p(\mathbb{R}_+)$-bounded, by symmetry it is sufficient to consider, say,

$$q_0(y,\bar{y}) = \frac{1}{y-\bar{y}}\log[1+\frac{y-\bar{y}}{2\bar{y}}].$$

Let Q_0 denote the integral operator with kernel q_0 and fix any $p \in (1,\infty)$. Then

$$\begin{aligned}|Q_0 f(y)| &\le |f|_p[\int_0^\infty |q_0(y,\bar{y})|^{p'}d\bar{y}]^{1/p'}\\ &= |f|_p[\int_0^\infty (\frac{1}{y-\bar{y}}\log(1+\frac{y-\bar{y}}{2\bar{y}}))^{p'}d\bar{y}]^{1/p'}\\ &= |f|_p y^{1/p'-1}[\int_0^\infty (\frac{1}{1-t}\log(1+\frac{1-t}{2t}))^{p'}dt]^{1/p'}\\ &\le C|f|_p y^{-1/p}, \quad y>0.\end{aligned}$$

This shows that $Q_0 : L_p(\mathbb{R}_+) \to L_{p,weak}(\mathbb{R}_+)$ is bounded for each $p \in (1,\infty)$, hence the Marcienkiewicz interpolation theorem implies the boundedness of Q_0 in $L_p(\mathbb{R}_+)$, for each $p \in (1,\infty)$. The proof is complete. □

We are now in position to complete the perturbation of boundary conditions. Let us consider the case $k = 1$ first.

Proposition 6.8. *There is a constant $C > 0$ such that*

$$|\int_{-\Gamma_2} h(-\lambda)T(\lambda+\omega)(\lambda+\omega+A_B^0)^{-1}d\lambda| \le C|h|_\infty,$$

for each $h \in \mathcal{H}(\Sigma_\phi)$.

Proof. It is enough to prove the assertion for a term of the form

$$T_1(\lambda) = S^j(\lambda+\omega)D_\lambda^{2m-m_j-\delta}b(x)\partial_y^\delta D^\beta(\lambda+\omega+A_B^0)^{-1}, \quad |\beta| = m_j \le 2m-1.$$

We have to distinguish 3 cases.
(a) $l := m_j + \delta < 2m$. Then $T_1(\lambda)$ has a kernel k_λ with a bound of the form

$$|k_\lambda(x,y,\bar{y})| \le C|b|_\infty|\lambda|^{2(n+1)/2m-1}\cdot$$
$$\cdot\int_0^\infty\int_{\mathbb{R}^n} p_{2m,k}^{n+1}(|\lambda|^{1/2m}(|x-x'|+y+y'))p_{2m,l}^{n+1}(|\lambda|^{1/2m}(|x'|+|y'-\bar{y}|))dx'dy',$$

where $l < 2m$ and $k \le 2m$. Therefore we may apply Lemma 6.7 (i) to see that

$$H_1 = \int_{-\Gamma_2} h(-\lambda)T_1(\lambda)d\lambda$$

is bounded by $C|h|_\infty$.
(b) $l := m_j + \delta = 2m$, $\beta \ne (0,\ldots,0,2m-1)$. Then D^β contains at least one x-derivative, say ∂_i. Commute it with b to the result

$$T_1(\lambda) = \partial_i S^j(\lambda+\omega)b(x)D^\gamma(\lambda+\omega+A_B^0)^{-1} - S^j(\lambda+\omega)\partial_i b(x)D^\gamma(\lambda+\omega+A_B^0)^{-1},$$

where $|\gamma| = 2m-1$. For the first term we use Lemma 6.7 (i) as in (a), while the second term has a decay like $C|\partial_i b|_\infty/|\lambda|^{1+1/2m}$, hence the λ-integral for this term is absolutely convergent. Therefore we see again that H_1 is bounded by $C|h|_\infty$.
(c) $\partial_y D^\beta = \partial_y^{2m}$. Here we use ellipticity to obtain

$$\partial_y^{2m}(\lambda+\omega+A_B^0)^{-1} = a_0^{-1}[1-(\lambda+\omega)(\lambda+\omega+A_B^0)^{-1} - \mathcal{A}'(D)(\lambda+\omega+A_B^0)^{-1}],$$

where $\mathcal{A}'(D)$ is of order $2m$ but contains an x-derivative. This yields the decomposition

$$T_1(\lambda) = S^j(\lambda+\omega)ba_0^{-1} - (\lambda+\omega)S^j(\lambda+\omega)ba_0^{-1}(\lambda+\omega+A_B^0)^{-1} - S^j(\lambda+\omega)ba_0^{-1}\mathcal{A}'(D)(\lambda+\omega+A_B^0)^{-1}.$$

The first term in this decomposition has a kernel bound of type

$$|\lambda|^{(n+1)/2m-1}p_{2m,0}^{n+1}(|\lambda|^{1/2m}(|x|+y+\bar{y})),$$

hence we may apply Lemma 6.2 to this term. The second one is of type (a) above, while the last one is of type (b). So we may again conclude that H_1 is bounded by $C|h|_\infty$, which completes the proof. □

In the last step we show the randomized estimate for $V_k(\lambda)$.

Proposition 6.9. *Let ε_l be independent symmetric $\{-1,1\}$-valued random variables on a probability space $(\Omega,\mathcal{M},\mu)$, and let $V_k(\lambda)$ and $\lambda_l \in \Sigma_{\pi-\phi}$.*

Then there is a constant $C > 0$ such that

$$|\sum_l \varepsilon_l(\lambda_l+\omega)^{1/2m}V_k(\lambda_l)x|_{L_2(\Omega;X)} \le C|x|,$$

for all $x \in X = L_p(\mathbb{R}_+^{n+1};E)$.

Proof. We consider a typical term in $V_k(\lambda)$ of the form

$$T_1(\lambda) = J_\lambda^{-1} D_\lambda^{2m-m_j-\delta_j} b_j \partial_y^{\delta_j} D^{\beta_j} S^k(\lambda+\omega) D_\lambda^{2m-m_k-\delta_k} b_k \partial_y^{\delta_k} D^{\beta_k}(\lambda+\omega+A_B^0)^{-1}.$$

First we use arguments as in the proof of Proposition 6.6 to reduce $T_1(\lambda)$ to a term of the form

$$T_2(\lambda) = D_\lambda^r D^\gamma S^k(\lambda+\omega)^{1/2m} D_\lambda^{2m-m_k-\delta_k} b_k \partial_y^{\delta_k} D^{\beta_k}(\lambda+\omega+A_B^0)^{-1},$$

by means of $\mathcal{R}$-boundedness, where $r+|\gamma| = 2m-1$. Then use the arguments in the proof of the preceeding lemma to to reduce $T_2(\lambda)$ to the form

$$T_3(\lambda) = D_\lambda^r D^\gamma S^k(\lambda+\omega)^{1/2m} b_k D_\lambda^s D^\beta(\lambda+\omega+A_B^0)^{-1},$$

where $r+|\gamma| = 2m$ and $s+|\beta| < 2m$. Then $T_3(\lambda)$ has a kernel k_λ with bound

$$|k_\lambda(x,y,\bar{y})| \le C|\lambda|^{(n+1)/2m-1/2m} \cdot \\ \cdot \int_0^\infty \int_{\mathbb{R}^n} p_{2m,k}^{n+1}(|\lambda|^{1/2m}(|x-x'|+y+y')) p_{2m,l}^{n+1}(|\lambda|^{1/2m}(|x'|+|y'-\bar{y}|)) dx' dy',$$

where $k = 2m$ and $l < 2m$. Then we may apply Lemma 6.7 (ii) to obtain the estimate in question. □

7. Small Perturbations of the complete boundary value problem

In this section, we consider simultaneously perturbations of boundary operators and of the elliptic differential operator acting on a half space. More precisely, for $\lambda \in \Sigma_{\pi-\phi}$ consider the problem

$$\begin{aligned} \lambda u(x) + \mathcal{A}_0(D)u(x) + \mathcal{A}_1(x,D)u(x) &= f(x), \quad x \in \mathbb{R}_+^{n+1}, \\ \mathcal{B}_j(x,D)u(x) &= 0, \qquad x \in \mathbb{R}^n = \partial\mathbb{R}_+^{n+1},\ j = 1,\dots,m. \end{aligned}$$

Here $\mathcal{A}_0(D)$ is a parameter-elliptic, homogeneous differential operator of order $2m$ having constant coefficients, $\mathcal{B}_j(x,D)$, $1 \le j \le m$ are homogeneous boundary operators of order $m_j < 2m$ such that the Lopatinskii-Shapiro condition is satisfied for $(\mathcal{A}_0(D), \mathcal{B}_j(x,D))$. The operator $\mathcal{A}_1(x,D)$ is homogeneous of order 2m with coefficients a_α^1 belonging to $BUC^\eta(\mathbb{R}_+^{n+1}; \mathcal{B}(E))$, $0 < \eta < 1$, which are small in L_∞-norm. We denote by A_1 the L_p-realization of $\mathcal{A}_1(x,D)$ with domain $D(A_1) = H_p^{2m}(\mathbb{R}_+^{n+1}; E)$. Furthermore, A_0^B denotes the L_p-realization of the unperturbed boundary value problem i.e.

$$\begin{aligned} A_0^B u &= \mathcal{A}_0(D)u \\ D(A_0^B) &= \{u \in H_p^{2m}(\mathbb{R}_+^{n+1}; E) : \mathcal{B}_j(x,D)u = 0 \text{ on } \mathbb{R}^n, j = 1,\dots,m\}. \end{aligned}$$

It was shown in Theorem 5.1 that $A_0^B + \omega$ admits a bounded $\mathcal{H}^\infty$-calculus on $X = L_p(\mathbb{R}_+^{n+1}; E)$ provided (5.1) holds and $\omega > 0$ is large enough.

Next we consider the operator $A_0^B + \omega + A_1$ as a perturbation of $A_0^B + \omega$. Observe that the first assumption of the perturbation theorem 3.2 is satisfied in this case because $D(A_1) \supset D(A_0^B)$ by definition and the operators $D^\alpha(A_0^B+\omega)^{-1}$ are bounded in X for $|\alpha| \le 2m$. In order to verify the second assumption of 3.2 we compute the fractional power spaces of A_0^B. Observe that

$${}_0H_p^{2m}(\mathbb{R}_+^{n+1}; E) \subset D(A_0^B) \subset H_p^{2m}(\mathbb{R}_+^{n+1}; E).$$

Thus complex interpolation yields

$${}_0H_p^{2m\gamma}(\mathbb{R}_+^{n+1}; E) \subset D((A_0^B)^\gamma) \subset H_p^{2m\gamma}(\mathbb{R}_+^{n+1}; E)$$

for $0 \le \gamma \le 1$. Since the spaces ${}_0H_p^{2m\gamma}(\mathbb{R}_+^{n+1};E)$ and $H_p^{2m\gamma}(\mathbb{R}_+^{n+1};E)$ coincide for $2m\gamma < 1/p$ we obtain

$$D((A_0^B)^\gamma) = H_p^{2m\gamma}(\mathbb{R}_+^{n+1};E), \quad 2m\gamma < 1/p.$$

Moreover, if follows from the representation of $(A_0^B+\omega)^{-1}$ given in (7.27) of [DHP01] that $(A_0^B+\omega)^{-1}$ maps $L_p(\mathbb{R}^n;E)$ into $H_p^{2m}(\mathbb{R}_+^{n+1};E)$ and also ${}_0H_p^s(\mathbb{R}_+^{n+1};E)$ into $H_p^{2m+s}(\mathbb{R}_+^{n+1};E)$ provided $s < 1/p$. Thus

$$(A_0^B+\omega)^{-1}H_p^{2m\gamma}(\mathbb{R}_+^{n+1};E) \subset H_p^{2m(1+\gamma)}(\mathbb{R}_+^{n+1};E), \qquad \gamma < 1/2mp.$$

Taking into account the fact that functions belonging to $BUC^\eta(\mathbb{R}_+^{n+1};\mathcal{B}(E))$ are pointwise multipliers for $H_p^{2m\gamma}(\mathbb{R}_+^{n+1};E)$ for $\eta > 2m\gamma$, we see that the second assumption of the perturbation theorem 3.2 is satisfied provided $\alpha < \eta/2mp$. Hence, by Theorem 3.2 we have proved the following. Suppose that

$$a_\alpha \in BUC^\eta(\mathbb{R}_+^{n+1};\mathcal{B}(E)), \qquad \eta \in (0,1), |\alpha| = 2m, \tag{7.1}$$

$$b_{j\beta} \in C^{2m-m_j}(\mathbb{R}_+^{n+1};\mathcal{B}(E)), \quad |\beta| = m_j, 1 \le j \le m. \tag{7.2}$$

Moreover, assume that for some $x_0 \in \mathbb{R}_+^{n+1}$ and given $\varepsilon \in (0,1)$, the coefficients a_α and $b_{j\beta}$ are of uniformly small oscillation in the sense that

$$\sup_{x\in\mathbb{R}_+^{n+1}} \sum_{|\alpha|=2m} |a_\alpha(x) - a_\alpha(x_0)| < \varepsilon, \tag{7.3}$$

$$\sup_{x\in\mathbb{R}_+^{n+1}} \sum_{|\beta|=m_j} |b_{j\beta}(x) - b_{j\beta}(x_0)| < \varepsilon. \tag{7.4}$$

We then have $\mathcal{A}(x,D) = \mathcal{A}(x_0,D) + \mathcal{A}^{sm}(x,D)$ and $\mathcal{B}_j(x,D) = \mathcal{B}_j(x_0,D) + {\mathcal{B}_j}^{sm}(x,D)$, where $\mathcal{A}(x_0,D) := \sum_{|\alpha|=2m} a_\alpha(x_0)D^\alpha$ and $\mathcal{B}_j(x_0,D) := \sum_{|\beta|=m_j} b_{j\beta}(x_0)D^\beta$ are operators with constant coefficients and

$$\mathcal{A}^{sm}(x,D) := \sum_{|\alpha|=2m} (a_\alpha(x) - a_\alpha(x_0))D^\alpha$$

and

$${\mathcal{B}_j}^{sm}(x,D) := \sum_{|\beta|=m_j} (b_{j\beta}(x) - b_{j\beta}(x_0))D^\beta.$$

Set

$$\begin{aligned} A_B u &:= (\mathcal{A}(x_0,D) + \mathcal{A}^{sm}(x,D))u \\ D(A_B) &:= \{u \in H_p^{2m}(\mathbb{R}_+^{n+1};E) : (\mathcal{B}_j(x_0,D) + \mathcal{B}_j^{sm}(x,D)u)(x',0) = 0 \text{ for } x' \in \mathbb{R}^n \\ &\qquad \text{and } j = 1\ldots m\}. \end{aligned}$$

Theorem 7.1. *Suppose that $\mathcal{A}(x_0,D)$ is parameter-elliptic with angle of ellipticty $\phi_\mathcal{A}$ and that the Lopatinski-Shapiro condition is satisfied for $(\mathcal{A}(x_0,D),\mathcal{B}_j(x_0,D))$.*

Then, for each $\phi > \phi_\mathcal{A}$ there exists $\varepsilon_0 = \varepsilon_0(\phi) > 0$ such that for all operators $\mathcal{A}(x,D)$ and $\mathcal{B}_j(x,D)$ satisfying (7.3), (7.4) with $\varepsilon < \varepsilon_0$ as well as (7.1), (7.2), there exists $\mu_\phi \ge 0$ such that $A_B + \mu_\phi$ admits a bounded $\mathcal{H}^\infty$-calculus on $L_p(\mathbb{R}_+^{n+1};E)$ with $\phi^\infty_{A_B+\mu_\phi} \le \phi$.

8. Localization: Proof of the Main Result

Throughout this section let $G \subset \mathbb{R}^{n+1}$ be an open connected set with compact C^{2m}-boundary ∂G. It is the aim of this section to combine Theorem 7.1 and Theorem 4.1 with a localization procedure; following this path we obtain our main result on elliptic boundary value problems, namely that $\mu + A_B$ admits a bounded $\mathcal{H}^\infty$ calculus on $L_p(G; E)$, $1 < p < \infty$, provided μ is sufficiently large and the top-order coefficients of $\mathcal{A}(x, D)$ belong to $BUC^\rho(\overline{G})$. Here A_B denotes the L_p-realization of the boundary value problem

$$\begin{aligned} A(x,D)u &= f \text{ in } G \\ B_j(x,D)u &= g_j \text{ on } \partial G,\ j = 1 \dots m, \end{aligned}$$

where A is an operator of order $2m$ and B_j are boundary operators of order $m_j < 2m$. Following the localization procedure described in Section 8 on [DHP01] we subdivide our proof in several steps. For detailed information on local coordinates and admissable coordinate tranfromations we refer to Section 8 of [DHP01].

Step 1: *Coordinate Transformations*

Let $x_0 \in \partial G$ and choose coordinates corresponding to x_0. By definition of a C^{2m}-boundary, there exists an open neighbourhood $U = U_1 \times U_2 \subset \mathbb{R}^{n+1}$ containing $x_0 = 0$ with $U_1 \subset \mathbb{R}^n$ and $U_2 \subset \mathbb{R}$ open and a function $h \in C^{2m}(\overline{U_1})$ satisfying $\partial G \cap U = \{x = (x', y) \in U : y = h(x')\}$ and $G \cap U = \{x \in U : y > h(x')\}$. Setting

$$g(x) := \begin{pmatrix} x' \\ y - h(x') \end{pmatrix} \quad (x \in U) \tag{8.1}$$

we obtain an injection $g \in C^{2m}(\overline{U}, \mathbb{R}^n)$ satisfying $G \cap U = \{x \in U : g_n(x) > 0\}$ and $\partial G \cap U = \{x \in U : g_n(x) = 0\}$. By compactness of ∂G, all derivatives of g and of g^{-1} (defined on $\widetilde{U} := g(U)$) up to order $2m$ may be assumed to be bounded by a constant independent of x_0. We will also need an extension of g. For this we extend $h \in C^{2m}(\overline{U_1})$ to a function $\widetilde{h} \in C^{2m}(\mathbb{R}^n)$ with compact support and set, still using coordinates corresponding to x_0,

$$G_{x_0} := \{x \in \mathbb{R}^{n+1} : y > \widetilde{h}(x')\}.$$

Defining $\widetilde{g}$ again by (8.1) with h being replaced by $\widetilde{h}$, we obtain a C^{2m}-diffeomorphism $\widetilde{g}\colon G_{x_0} \to \mathbb{R}^{n+1}_+$ with $\widetilde{g}|_U = g$. For a function $u\colon U \cap G \to E$ consider the push-forward $v = \mathcal{G}u$ defined on $\widetilde{U} \cap \mathbb{R}^{n+1}_+$ by $v(y) := u(g^{-1}(y))$. The linear transformation $\mathcal{G}$ induces isomorphisms $\mathcal{G}^{(p)}\colon H^j_p(U \cap G; E) \to H^j_p(\widetilde{U} \cap \mathbb{R}^{n+1}_+; E)$ for $p \in [1, \infty]$ and $j = 0, \dots, 2m$. In the same way the linear transformation given by $\widetilde{g}$ induces isomorphisms $\widetilde{\mathcal{G}}^{(p)}\colon H^j_p(G_{x_0}; E) \to H^j_p(\mathbb{R}^{n+1}_+; E)$. The differential operator $\mathcal{A}(x, D)$ is transformed into the operator $\mathcal{A}^g(y, D) := \mathcal{G}\mathcal{A}(x, D)\mathcal{G}^{-1}$. In the same way we define the transformed operators $\mathcal{B}^g_j(y, D)$. Obviously $\mathcal{A}^g$ and $\mathcal{B}^g_j$ are differential operators of order $2m$ and m_j, respectively and act on functions defined on $\widetilde{U} \cap \mathbb{R}^{n+1}_+$. Observe that the smoothness assumptions (H) are satisfied for $\mathcal{A}^g$ and $\mathcal{B}^g_j$ if this is true for $\mathcal{A}$ and $\mathcal{B}_j$. Moreover, the principal symbol of $\mathcal{A}^g$ is given by

$$\mathcal{A}^g_\#(y, \xi) = \mathcal{A}_\#(g^{-1}(y), [Dg(g^{-1}(y))]^T \xi), \quad y \in \widetilde{U} \cap \overline{\mathbb{R}^{n+1}_+},\ \xi \in \mathbb{R}^{n+1}. \tag{8.2}$$

As $Dg(x)$ is an isomorphism of $\mathbb{R}^{n+1}$ for all $x \in U$, (8.2) implies that parameter-ellipticity of $\mathcal{A}_\#$ is preserved under coordinate transformations. Moreover, if g is admissable at x_0 then the transformed boundary value problem $(\mathcal{A}^g, \mathcal{B}^g_1, \dots, \mathcal{B}^g_m)$ satisfies the Shapiro–Lopatinskiii condition at the point $g(x_0)$. We finally remark that $u \in H^{2m}_p(U \cap G; E)$ satisfies $\mathcal{A}(x, D)u = f$ in $U \cap G$ and $\mathcal{B}_j(x, D)u = g_j$ $(j = 1, \dots, m)$ on $U \cap \partial G$ iff the transformed function u^g satisfies $\mathcal{A}^g(x, D)u^g = f^g$ in $\widetilde{U} \cap \mathbb{R}^{n+1}_+$ and $\mathcal{B}^g_j(x, D)u^g = g^g_j$ $(j = 1, \dots, m)$ on $\widetilde{U} \cap \mathbb{R}^n$.

Step 2: *Local Operators*

For an arbitrary $x_0 \in \partial G$ there exist C^{2m}-coordinates $g = g_{x_0} : U_{x_0} \to \mathbb{R}^{n+1}$ defined in an open neigbourhood U_{x_0} of x_0 with the above properties. As parameter-ellipticity of the principal symbol and Lopatinskii-Shapiro condition are preserved under such transformations, we may apply Theorem 7.4 of [DHP01] to the L_p-realization of the transformed problem $(\mathcal{A}^g(y_0, D), \mathcal{B}^g(y_0, D))$ in $\mathbb{R}^{n+1}_+$ with coefficients frozen at $y_0 := g(x_0)$. We then obtain that this realization admits a bounded $\mathcal{H}^\infty$-calculus on $L_p(\mathbb{R}^{n+1}_+; E)$ with an $\mathcal{H}^\infty$-angle not greater than $\phi_{\mathcal{A}}$. Note that

$$\mathcal{A}^g_\#(y_0, \xi) = \mathcal{A}_\#(x_0, \xi) \quad \text{for } \xi \in \mathbb{R}^n,$$

and that the analog statement for the principal symbols of the boundary operators is true, too. It follows that all $L_p(\mathbb{R}^{n+1}_+; E)$-realizations of $((\mathcal{A}^g_\#(y_0, D), \mathcal{B}^g_\#(y_0, D))$ admit a bounded $\mathcal{H}^\infty$-calculus with uniform bound, and there exists an $\varepsilon_0 > 0$, independent of $x_0 \in \partial G$, such that the statements of Proposition 4.1 and Theorem 7.1 hold. Now let us consider the transformed operators $\mathcal{A}^g(y, D)$ and $\mathcal{B}^g_j(y, D)$ with variable coefficients, which we write in the form

$$\begin{aligned} \mathcal{A}^g(y, D) &= \sum_{|\alpha| \le 2m} \tilde{a}_\alpha(y) D^\alpha, \\ \mathcal{B}^g_j(y, D) &= \sum_{|\beta| \le m_j} \tilde{b}_{\beta j}(y) D^\beta, \quad j = 1, \dots, m, \quad y \in g(U_{x_0}) \cap \overline{\mathbb{R}^{n+1}_+}. \end{aligned}$$

As the smoothness properties (H) are preserved under coordinate transformations of the above type there exists $r(x_0) > 0$ such that $g^{-1}_{x_0}(B_{2r(x_0)}(y_0)) \subset U_{x_0}$ and

$$\begin{aligned} \sum_{|\alpha| = 2m} |\tilde{a}_\alpha(y) - \tilde{a}_\alpha(y_0)| &< \varepsilon_0, \\ \sum_{|\beta| = m_j} |\tilde{b}_{j\beta}(y) - \tilde{b}_{j\beta}(y_0)| &< \varepsilon_0, \quad y \in B_{2r(x_0)}(y_0). \end{aligned}$$

We obtain an open covering $\partial G \subset \bigcup_{x_0 \in \partial G} g^{-1}_{x_0}\big(B_{r(x_0)}(y_0)\big)$ and choose a finite subcovering $\partial G \subset \bigcup_{k=1}^N U_k$ with $U_k := g^{-1}_{x_k}\big(B_{r(x_k)}(y_k)\big)$ for $k = 1, \dots, N$. Here we have set $y_k := g_{x_k}(x_k)$. Now we extend the coefficients of $\mathcal{A}^{g_{x_k}}$ and $\mathcal{B}^{g_{x_k}}_j$ from $B_{r(x_k)}(y_k) \cap \overline{\mathbb{R}^{n+1}_+}$ to the whole half-space $\overline{\mathbb{R}^{n+1}_+}$. For the coefficients of $\mathcal{A}^{g_{x_k}}$ we can use the reflection method, i.e., as in [DHP01] Section 8, we define

$$a^k_\alpha(y) := \begin{cases} \widetilde{a}_\alpha(y), & y \in \overline{B_{r_k}(y_k)} \cap \overline{\mathbb{R}^{n+1}_+}, \\ \widetilde{a}_\alpha(y_k + r_k^2 \, \frac{y - y_k}{|y - y_k|^2}), & y \in \overline{\mathbb{R}^{n+1}_+} \setminus \overline{B_{r_k}(y_k)}. \end{cases}$$

For the coefficients of the boundary operators we fix $\chi \in C_0^\infty(R^{n+1})$ with $\chi(x) \equiv 1$ for $|x| \le 1$ and $\chi(x) \equiv 0$ for $|x| \ge 2$ and set for $k = 1, \dots, N$, $j = 1, \dots, m$ and all $|\beta| = m_j$:

$$b^k_{j\beta}(y) := \tilde{b}_{j\beta}\Big(y_k + \chi\Big(\frac{y - y_k}{r_k}\Big)(y - y_k)\Big), \quad y \in \overline{\mathbb{R}^{n+1}_+}.$$

By construction, the extended coefficients a^k_α and $b^k_{j\beta}$ satisfy the assumptions of Proposition 4.1 and of Theorem 7.1.

Step3: *Small Perturbations without Boundary Conditions*

By Proposition 4.1 and a compactness argument, there exists $\eta > 0$, independent of $x \in \overline{G}$ [or of $x_0 \in \overline{G} \cup \{\infty\}$ in the case of G unbounded] such that the L_p-realization of $\mathcal{A}_\#(x_0, D)$ admits a bounded $\mathcal{H}^\infty$-caluclus and, moreover, the L_p-realization of $\mathcal{A}_\#(x_0, D) + \mathcal{A}_1(x, D)$ for any operator $\mathcal{A}_1(x, D)$ with

$$\mathcal{A}_1(x, D) = \sum_{|\alpha| = 2m} a^1_\alpha(x) D^\alpha, \quad \|a^1_\alpha\|_{L_\infty(G;E)} < \eta,$$

admits a bounded $\mathcal{H}^\infty$-caluclus, too, with bounds uniform in $x_0 \in \overline{G}$ [$x_0 \in \overline{G} \cup \{\infty\}$]. If G is unbounded, we choose a large ball $B_{r_{N+1}}(0)$ such that

$$\bigcup_{k=1}^{N} U_k \subset B_{r_{N+1}}(0)$$

and such that for all $|\alpha| = 2m$

$$|a_\alpha(x) - a_\alpha(\infty)|_{\mathcal{B}(E)} \leq \eta \quad \text{for all } x \in \mathbb{R}^{n+1} \text{ with } |x| \geq r_{N+1}$$

holds. We set $U_{N+1} := G \backslash \overline{B_{r_{N+1}}(0)}$. If G is bounded, we set $U_{N+1} := \emptyset$. Now we cover the compact set $\overline{G} \backslash \bigcup_{k=1}^{N+1} U_k$ by finitely many balls $U_k = B_{r_k}(x_k)$, $k = N+2, \ldots, M$, such that

$$|a_\alpha(x) - a_\alpha(x_k)|_{\mathcal{B}(E)} \leq \eta \quad \text{for all } |x - x_k| \leq r_k,\ |\alpha| = 2m,\ k = N+2, \ldots, M.$$

Finally, define coefficients a_α^j of local operators $\mathcal{A}_j$, $j = N+1, \ldots, M$ by reflection as i.e.

$$a_\alpha^{N+1}(x) = \begin{cases} a_\alpha(x) & : \ x \notin \overline{B}_{r_{N+1}}(0) \\ a_\alpha(r_{N+1}^2 \frac{x}{|x|^2}) & : \ x \in \overline{B}_{r_{N+1}}(0) \end{cases},$$

and

$$a_\alpha^j(x) = \begin{cases} a_\alpha(x) & : \ x \in \overline{B}_{r_j}(x_j) \\ a_\alpha(x_j + r_j^2 \frac{x - x_j}{|x - x_j|^2}) & : \ x \notin \overline{B}_{r_j}(x_j) \end{cases}$$

for $j = N+2, \ldots, M$.

Step 4: *Small Perturbations with Boundary Conditions*

For $f \in L_p(G; E)$ consider the boundary value problem

$$\text{(8.3)} \qquad \begin{aligned} \lambda u + \mathcal{A}(x, D)u &= f \quad \text{in } G, \\ \mathcal{B}_j(x, D)u &= 0 \quad \text{on } \partial G, j = 1, \ldots, m. \end{aligned}$$

We choose a partition of unity $\varphi_k \in C^\infty(\mathbb{R}^n)$, $k = 1, \ldots, M$, with $0 \leq \varphi_k \leq 1$ and $\operatorname{supp} \varphi_k \subset U_k$. Then u is a solution of (8.3) iff for $k = 1, \ldots, M$

$$\text{(8.4)} \qquad \lambda(\varphi_k u) + \varphi_k \mathcal{A}(x, D)u = \varphi_k f \quad \text{in } G \cap U_k,$$

$$\text{(8.5)} \qquad \varphi_k \mathcal{B}_j(x, D)u = 0 \quad \text{on } \partial G \cap U_k.$$

For $k = N+1, \ldots, M$ boundary conditions (8.5) do not appear, and we rewrite (8.4) in the form

$$\text{(8.6)} \qquad \lambda(\varphi_k u) + \mathcal{A}_\#(x, D)(\varphi_k u) = \varphi_k f + [\mathcal{A}_\#(x, D), \varphi_k]u - \varphi_k \mathcal{A}'(x, D)u$$

where we have set $\mathcal{A}'(x, D) := \mathcal{A}(x, D) - \mathcal{A}_\#(x, D)$. We denote the resolvent of the $L_p(\mathbb{R}^{n+1}; E)$-realization A_k of the operator

$$\mathcal{A}_k(x, D) = \sum_{|\alpha| = 2m} a_\alpha^k(x) D^\alpha$$

by $R^{(k)}(\lambda)$. Employing $R^{(k)}(\lambda)$ to equation (8.6), we get

$$\text{(8.7)} \qquad \varphi_k u = R^{(k)}(\lambda)\varphi_k f + R^{(k)}(\lambda)\mathcal{C}_k(x, D)u, \quad k = N+1, \ldots, M,$$

where

$$\mathcal{C}_k(x, D) := [\mathcal{A}_\#(x, D), \varphi_k] - \varphi_k \mathcal{A}'(x, D)$$

is an operator of order not greater than $2m - 1$. For $k = 1, \ldots, N$ we use the local coordinates g_{x_k} defined above. Assuming g_{x_k} to be extended to G_{x_k}, the transformation $u \mapsto u^{g_{x_k}}$ induces an isomorphism

$$G_k \colon H_p^j(G_{x_k}; E) \to H_p^j(\mathbb{R}_+^{n+1}; E).$$

We set

$$\mathcal{A}_k(y,D) := \sum_{|\alpha|=2m} a^k_\alpha(y) D^\alpha,$$
$$\mathcal{B}_{jk}(y,D) := \sum_{|\beta|=m_j} b^k_{j\beta}(y) D^\beta, \quad j=1,\dots,m$$

with a^k_α and $b^k_{j\beta}$ being defined in Step 3 of the proof. For each $k=1,\dots,N$ we obtain a boundary value problem in $\mathbb{R}^{n+1}_+$:

$$\lambda G_k(\varphi_k u) + \mathcal{A}_k(y,D) G_k(\varphi_k u) = G_k(\varphi_k f) + G_k \mathcal{C}_k(x,D)u \quad \text{in } \mathbb{R}^{n+1}_+, \tag{8.8}$$
$$\mathcal{B}_{jk}(y,D) G_k(\varphi_k u) = G_k \mathcal{D}_{jk}(x,D)u, \quad \text{on } \mathbb{R}^n, j=1,\dots,m, \tag{8.9}$$

where we have set

$$\mathcal{C}_k(x,D) := [\mathcal{A}(x,D), \varphi_k] - G_k^{-1}\big(\mathcal{A}^{g_{x_k}}(y,D) - \mathcal{A}_k(y,D)\big) G_k \varphi_k$$

and

$$\mathcal{D}_{jk}(x,D) := [\mathcal{B}_j(x,D), \varphi_k] - G_k^{-1}\big(\mathcal{B}_j^{g_{x_k}}(y,D) - \mathcal{B}_k(y,D)\big) G_k \varphi_k$$

($j=1,\dots,m$). Note that $\mathcal{C}_k(x,D)$ and $\mathcal{D}_{jk}(x,D)$ are partial differential operators of order $\leq 2m-1$ and $\leq m_j - 1$, respectively. It was proved in Propositions 7.8 and 7.9 of [DHP01] that the resolvent of the L_p-realization of $(\mathcal{A}_k(y,D), \mathcal{B}_k(y,D))$ exists for large ω and that the solution operators of the boundary value problem

$$(\lambda + \mathcal{A}_k(y,D))u = 0 \quad \text{in } \mathbb{R}^{n+1}_+ \tag{8.10}$$
$$\mathcal{B}_{jk}(y,D)u = g_j \quad \text{on } \mathbb{R}^n, j=1,\dots,m, \tag{8.11}$$

exist also in the sense of Proposition 7.9 of [DHP01] for large ω. We will denote the resolvent by $\widetilde{R}^{(k)}(\lambda)$ and the solution operators of (8.10)–(8.11) by $\widetilde{S}^{j,k}_\lambda$, $j=1,\dots,m$. Applying these solution operators to (8.8)–(8.9), we get for $k=1,\dots,N$

$$\varphi_k u = R^{(k)}(\lambda)\varphi_k f + \Big[R^{(k)}(\lambda)\mathcal{C}_k(x,D) + \sum_{j=1}^m S^{j,k}_\lambda \mathcal{D}_{jk}(x,D)\Big]u \tag{8.12}$$

where we have set

$$R^{(k)}(\lambda) := G_k^{-1}\widetilde{R}^{(k)}(\lambda) G_k,$$
$$S^{j,k}_\lambda := G_k^{-1}\widetilde{S}^{j,k}_\lambda G_k, \quad j=1,\dots,m.$$

Summing (8.7) and (8.12) we obtain

$$\begin{aligned}(\lambda + A_B)^{-1} = &\sum_{j=1}^M R^{(k)}(\lambda)\varphi_k \\ &+ \Big[\sum_{k=1}^M R^{(k)}(\lambda)\mathcal{C}_k(x,D) + \sum_{k=1}^N\sum_{j=1}^m S^{j,k}_\lambda \mathcal{D}_{jk}(x,D)\Big](\lambda + A_B)^{-1}.\end{aligned} \tag{8.13}$$

The fact that $\operatorname{ord}\mathcal{C}_k(x,D) \leq 2m-1$ and $\operatorname{ord}\mathcal{D}_{jk}(x,D) \leq m_j - 1$ implies that there exists a constant $\omega > 0$ such that $(\lambda + \omega + A_B)$ is continuously invertible for $\lambda \in \Sigma_{\pi-\phi}$. Moreover, $\mu + A_B$ is sectorial for $\mu \geq \omega$, and $\phi_{\mu + A_B} \leq \phi$.

Step 5: $\mathcal{H}^\infty$*-calculus*

Choose now $h \in \mathcal{H}_0^\infty(\Sigma_\theta)$ where $\theta \in (\phi, \pi)$ and set $\Gamma := \{se^{-i\psi}, r \leq s < \infty\} \cup \{re^{i\alpha}, -\psi \leq \alpha \leq \psi\} \cup \{se^{i\psi}, r \leq s < \infty\}$ where $\phi_{\mu+A_B} < \psi < \phi$. It follows from Proposition 4.1 that there exist constants $C, \omega > 0$ such that for $k = N+1, \ldots, M$

$$|\int_{-\Gamma} h(-\lambda)R^{(k)}(\lambda+\omega)\varphi_k \, d\lambda|_{L_p(G\cap U_k;E)} \leq C|h|_\infty. \tag{8.14}$$

Moreover, by Theorem 7.1, there exist constants $C, \omega > 0$ such that for $k = 1, \ldots, N$

$$|\int_{-\Gamma} h(-\lambda)R^{(k)}(\lambda+\omega)\varphi_k \, d\lambda|_{L_p(G_{x_k};E)} \leq C|h|_\infty. \tag{8.15}$$

Observe next that for a differential operator $C(x,D)$ of order $2m-1$, by Theorem 8.2 of [DHP01] we have

$$|(\lambda+\omega)^{1/2m}C(x,D)(\lambda+\omega+A_B)^{-1}|_{L_p(G;E)} \leq C, \qquad \lambda \in \Sigma_{\pi-\phi}.$$

Thus by Theorem 5.7 (for $k = N+1, \ldots, M$) and Proposition 7.8 (for $k = 1, \ldots, M$) of [DHP01] we have

$$\begin{aligned} (8.16)\quad & |\int_{-\Gamma} h(-\lambda)R^{(k)}(\lambda+\omega)C_k(x,D)(\lambda+\omega+A_B)^{-1}d\lambda|_{L_p(G\cap U_k;E)} \qquad k = N+1, \ldots, M \\ & \leq C|h|_\infty |\int_{-\Gamma} |\lambda+\omega|^{-1-\frac{1}{2m}}d\lambda \leq C|h|_\infty, \\ (8.17)\quad & |\int_{-\Gamma} h(-\lambda)R^{(k)}(\lambda+\omega)C_k(x,D)(\lambda+\omega+A_B)^{-1}d\lambda|_{L_p(G_{x_k};E)} \qquad k = 1, \ldots, N \\ & \leq C|h|_\infty |\int_{-\Gamma} |\lambda+\omega|^{-1-\frac{1}{2m}}d\lambda \leq C|h|_\infty. \end{aligned}$$

Finally, observe that

$$S_\lambda^{j,k}\mathcal{D}_{jk}(x,D) = T_\lambda^{j,k}\mathcal{D}'_{jk}(\lambda) + R_\lambda^{j,k}\mathcal{D}''_{jk}(\lambda)$$

where

$$\begin{aligned} R_\lambda^{j,k} &:= G_k^{-1}\widetilde{R}_\lambda^{j,k}G_k, \\ T_\lambda^{j,k} &:= G_k^{-1}\widetilde{T}_\lambda^{j,k}G_k, \\ \mathcal{D}'_{jk}(\lambda) &:= G_k^{-1}\Big[(-\Delta)^{\frac{2m-m_j}{2}} + |\lambda|^{\frac{2m-m_j}{2m}}\Big]G_k\mathcal{D}_{jk}(x,D), \\ \mathcal{D}''_{jk}(\lambda) &:= G_k^{-1}\Big[(-\Delta)^{\frac{2m-m_j-1}{2}} + |\lambda|^{\frac{2m-m_j-1}{2m}}\Big]\partial_{n+1}G_k\mathcal{D}_{jk}(x,D). \end{aligned}$$

and where the operators $\widetilde{T}_\lambda^{j,k}$ and $\widetilde{R}_\lambda^{j,k}$ satisfy

$$|\tilde{T}_\lambda^{j,k}| + |\tilde{R}_\lambda^{j,k}| \leq C/|\lambda|, \quad |\lambda| \geq \lambda_0, \; \lambda \in \Sigma_{\pi-\phi}.$$

It follows that

$$\begin{aligned} (8.18)\qquad & |\int_{-\Gamma} h(-\lambda)S_{\lambda+\omega}^{j,k}\mathcal{D}_{j,k}(x,D)(\lambda+\omega+A_B)^{-1}d\lambda|_{L_p(G_{x_k};E)} \\ & \leq C|h|_\infty |\int_\Gamma |\lambda+\omega|^{-1-\frac{1}{2m}}d\lambda \leq C|h|_\infty, \qquad 1 \leq k \leq M, 1 \leq j \leq m. \end{aligned}$$

Step 6: *Lower Order Perturbations*

This can be done as in the proof of Theorem 8.2 of [DHP01] by Gagliardo-Nirenberg's inequality. Finally, summing up all the terms in the representation (8.13) of $(\lambda + A_B)^{-1}$, the proof is complete. □

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