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A Priori Estimates for a Singularly Perturbed Mixed-Order Boundary Value Problem

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Abstract—In this paper we study mixed-order (Douglis–Nirenberg) boundary value problems that depend on a real parameter but are not elliptic with parameter in the sense of Agmon–Agranovich–Vishik. Using the method of the Newton polygon, we prove *a priori* estimates for the solutions of such problems in the corresponding Sobolev spaces. For the related singularly perturbed problem, the boundary layer structure of the solutions is described. As an application of the *a priori* estimate, we obtain new estimates for the transmission problem studied by Faierman [7].

1. INTRODUCTION

The aim of this paper is to study mixed-order systems of partial differential operators with block structure of the form

$$A(x,D,\lambda) = \begin{pmatrix} A_{11}(x,D) & A_{12}(x,D) \\ A_{21}(x,D) & A_{22}(x,D)-\lambda I \end{pmatrix} \tag{1.1}$$

depending on the real parameter λ and acting on a compact manifold M with boundary ∂M. We assume that system (1.1) admits a Douglis–Nirenberg structure. In what follows, it is supplied with general boundary conditions. Note that the operator (1.1) can be regarded as a mixture of a parameter-independent Douglis–Nirenberg system and a λ-dependent system. If the matrix $A_{11}(x,D)$ were replaced by $A_{11}(x,D)-\lambda I$, we would obtain the parameter-dependent mixed-order system treated in [4, 12, 13] and other papers.

There are several reasons to study the (nonstandard) operator matrix (1.1). First of all, this study is related to the investigation of the resolvent of a Douglis–Nirenberg system on a manifold with boundary. In general, it is impossible to assign a parameter-dependent quasi-homogeneous principal symbol to such a system. For instance, for the 2×2 Douglis–Nirenberg system

$$\begin{pmatrix} A_{11}(x,D)-\lambda & A_{12}(x,D) \\ A_{21}(x,D) & A_{22}(x,D)-\lambda \end{pmatrix} \tag{1.2}$$

with $\operatorname{ord} A_{11} > \operatorname{ord} A_{22}$, there is no definite "weight" for the parameter λ. Therefore, we cannot apply the theory of parameter ellipticity developed by Agmon [1] and Agranovich–Vishik [3]. The aim of the study of boundary value problems for system (1.2) is to prove the unique solvability and to obtain *uniform* estimates (with respect to λ) for the solution for large λ. For scalar equations and for systems of constant order, the corresponding results follow from the Agmon–Agranovich–Vishik theory. For general mixed-order systems, this problem seems to be still open (see [4, 12, 13] for partial results).

To address this problem in its full generality, one can use the concept of Newton polygon, which proves to be very fruitful both in the theory of parameter-dependent Douglis–Nirenberg systems on closed manifolds and in the theory of singularly perturbed operator pencils (see, e.g., [4, 5, 6]). The main idea of the approach based on the notion of Newton polygon is to assign various weights

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to the parameter λ and to obtain, for each weight, a quasi-homogeneous symbol of the operator. If in (1.2) we assign to λ the weight $\operatorname{ord} A_{22}$, then the quasi-homogeneous symbol with respect to this weight becomes

$$\begin{pmatrix} A_{11}(x,\xi) & A_{12}(x,\xi) \\ A_{21}(x,\xi) & A_{22}(x,\xi)-\lambda \end{pmatrix},$$

which is exactly the principal quasi-homogeneous symbol of (1.1). For each weight of the parameter, we obtain an operator with quasi-homogeneous principal symbol. Thus, we face the following question: Under what conditions can we prove uniform estimates for the solutions of (1.1)?

Another reason to study operators of the form (1.1) lies in the close connection between the parameter-dependent operator (1.1) and singularly perturbed boundary value problems (cf. [8, 14, 16]). Let us consider the equation $A(x,D,\lambda)u = 0$ with A given by (1.1) and endow it with appropriate boundary conditions. By setting $\lambda = \varepsilon^{-1}$ and multiplying the second block row in (1.1) by ε, we obtain the system

$$\begin{aligned} A_{11}(x,D)u_1 + A_{22}(x,D)u_2 &= 0\,, \\ \varepsilon\big(A_{21}(x,D)u_1 + A_{22}(x,D)u_2\big) - u_2 &= 0\,. \end{aligned} \tag{1.3}$$

The problem is to find conditions on the boundary operators under which estimates for the solution can be *uniform* with respect to ε as $\varepsilon \to 0$. Moreover, it is desirable to find an asymptotic expansion in ε for the solution and to construct solutions of boundary-layer type. Since equations (1.3) are only a reformulation of the equation $A(x,D,\lambda)u = 0$, it follows that the results concerning (1.1) immediately imply results on singularly perturbed boundary value problems of the form (1.3).

Finally, we can mention the question of spectral asymptotics for boundary value problems and transmission problems with indefinite weight functions (see [2] and the references therein). For the case in which the weight function vanishes identically in some subdomain, the model transmission problem that arises was recently studied by Faierman [7]. One of the main goals is again to obtain uniform estimates for the solutions of the transmission problem because this finally leads to the spectral asymptotics [2].

Summing up, we can see that, in all these applications, the question is to find conditions (in particular, on the boundary operators) ensuring uniform estimates (with respect to the parameter) for the solutions of the corresponding problems. Moreover, it is desirable that these conditions be similar to the classical conditions (parameter-independent or parameter-dependent in the sense of Agmon–Agranovich–Vishik) of Shapiro–Lopatinskii type. As in the theory of parameter ellipticity, the estimates must work in appropriate parameter-dependent Sobolev spaces, and hence the additional question of introducing spaces suitable for operators of the form (1.1) arises. To be more exact, we endow the standard Sobolev space with a parameter-dependent norm adjusted to our problems. Here the main difficulty is in the definition of the boundary Sobolev spaces (boundary parameter-dependent norms).

The first result in this direction is given in the paper [6], which treats scalar operator pencils such that their dependence on the parameter λ is polynomial and the leading term with respect to λ contains an operator of positive order. (Note that, roughly speaking, the determinant of the symbol of system (1.1) is of this form.) The case of systems of the form (1.1) is more complicated than that of scalar pencils with polynomial dependence on λ because, in our case, a solution consists of several components belonging to different parameter-dependent Sobolev spaces and connected by boundary conditions and by the operator $A(x,D,\lambda)$.

The present paper solves the above problem for system (1.1) by defining the notion of weak parameter-ellipticity and by proving an *a priori* estimate for the corresponding weakly parameter-elliptic boundary value problems (with general boundary conditions). The notion of weak parameter ellipticity and the norms in the *a priori* estimates are defined in terms of the Newton polygon corresponding to the boundary value problem, continuing the program originating from [4–6]. It seems that the approach based on the notion of Newton polygon provides an appropriate tool to study and understand nonstandard boundary value problems, including those of the form (1.1).

The paper is organized as follows. In Section 2 we formulate the precise assumptions on the structure of the operator (1.1) and of the boundary operators and define the notion of weak parameter-ellipticity for related boundary value problems. In Section 3 we prove the basic ODE estimate for the so-called fundamental system of solutions of the corresponding model problem in the half-space. These estimates are rewritten in terms of the Newton polygon in Section 4, where the basic definitions for Sobolev spaces associated with the Newton polygon can also be found. The ODE estimates form a substantial step in the proof of *a priori* estimates presented in Section 5. In Section 6 we apply these results to transmission problems, and in Section 7 we discuss the formal asymptotic solution for related singularly perturbed problems.

2. NEWTON POLYGON AND WEAKLY PARAMETER-ELLIPTIC BOUNDARY VALUE PROBLEMS

As in the introduction, let $A(x,D,\lambda) = (a_{ij}(x,D))_{i,j=1,\dots N}$ be an $N\times N$ operator matrix acting on a compact manifold M with boundary ∂M. We assume that this operator has Douglis–Nirenberg structure, i.e., there are nonnegative integers s_j and t_j, $j=1,\dots,N$, such that

$$\operatorname{ord} a_{ij}(x,D) \le s_i + t_j \quad (i,j=1,\dots,N)\,. \tag{2.1}$$

The operator $a_{ij}(x,D)$ is assumed to be of the form

$$a_{ij}(x,D) = \sum_{|\alpha|\le s_i+t_j} a_{ij\alpha}(x)D^\alpha$$

with infinitely smooth scalar coefficients. The principal symbol of $a_{ij}(x,D)$ is defined as

$$a_{ij}^{(0)}(x,\xi) = \sum_{|\alpha|= s_i+t_j} a_{ij\alpha}(x)\xi^\alpha\,.$$

Note that $a_{ij}^{(0)} = 0$ if the order of a_{ij} is less than s_i+t_j. Here and in the following, we use the standard multi-index notation

$$D^\alpha = (-i)^{|\alpha|}(\frac{\partial}{\partial x_1})^{\alpha_1}\cdots(\frac{\partial}{\partial x_n})^{\alpha_n}, \qquad \xi^\alpha = \xi_1^{\alpha_1}\cdot\ldots\cdot\xi_n^{\alpha_n}.$$

We assume that there exist numbers N_1 and N_2 with $N_1+N_2=N$ for which

$$s_1+t_1 = s_2+t_2 = \cdots = s_{N_1}+t_{N_1} = 2\mu\,, \quad s_{N_1+1}+t_{N_1+1} = \cdots = s_N+t_N = 2m-2\mu, \tag{2.2}$$

with integers $m>\mu>0$. Accordingly, we represent the operator matrix $A(x,D,\lambda)$ in the form

$$A(x,D,\lambda) = \begin{pmatrix} A_{11}(x,D) & A_{12}(x,D) \\ A_{21}(x,D) & A_{22}(x,D)-\lambda I_{N_2}\end{pmatrix},$$

where

$$A_{11} = (a_{ij}(x,D))_{i,j=1,\dots,N_1}, \qquad A_{12} = (a_{ij}(x,D))_{i=1,\dots,N_1,\, j=N_1+1,\dots,N},$$
$$A_{21} = (a_{ij}(x,D))_{i=N_1+1,\dots,N,\, j=1,\dots,N_1}, \qquad A_{22} = (a_{ij}(x,D))_{i,j=N_1+1,\dots,N}.$$

Here I_{N_2} stands for the $N_2\times N_2$ identity matrix. The principal symbol of the operator $A(x,D,\lambda)$ is given by

$$A^{(0)}(x,\xi,\lambda) := \begin{pmatrix} A_{11}^{(0)}(x,\xi) & A_{12}^{(0)}(x,\xi) \\ A_{21}^{(0)}(x,\xi) & A_{22}^{(0)}(x,\xi)-\lambda I_{N_2}\end{pmatrix}$$

where $A_{ij}^{(0)}$ stand for the principal symbols of A_{ij}, respectively. In what follows, to make expressions homogeneous, we replace λ by $q^{2m-2\mu}$. For brevity we write $A(x,D,q)$ instead of $A(x,D,q^{2m-2\mu})$.

Assume that

$$B(x,D) = \Big(b_{ij}(x,D)\Big)_{\substack{i=1,\dots,N_1\mu+N_2(m-\mu)\\ j=1,\dots,N}} \tag{2.3}$$

is a matrix of boundary operators of the form

$$b_{ij}(x,D)u = \Big(\sum_{|\beta|\le m_i+t_j} b_{ij\beta}(x)D^\beta u\Big)\Big|_{\partial M}$$

of order $\le m_i + t_j$ with coefficients $b_{ij\beta} \in C^\infty(\overline{M})$. Here the m_i are integers. We also assume that

$$m_1 \le m_2 \le \cdots \le m_{N_1\mu} < m_{N_1\mu+1} \le \cdots \le m_{N_1\mu+N_2(m-\mu)}\,. \tag{2.4}$$

In the following, denote the number of boundary conditions by

$$R := N_1\mu + N_2(m-\mu)\,. \tag{2.5}$$

We also use the abbreviations

$$R_1 := N_1\mu, \quad R_2 := N_2(m-\mu)\,. \tag{2.6}$$

The aim of this paper is to find *a priori* estimates for the solutions

$$u = \begin{pmatrix} u_1\\ \vdots\\ u_N\end{pmatrix}$$

of the boundary value problem

$$A(x,D,q)u = f = \begin{pmatrix} f_1\\ \vdots\\ f_N\end{pmatrix}, \quad B(x,D)u = g = \begin{pmatrix} g_1\\ \vdots\\ g_R\end{pmatrix}. \tag{2.7}$$

Assume that this boundary value problem satisfies an ellipticity condition, the so-called weak parameter ellipticity. We first introduce this notion for the operator A.

Definition 2.1. A symbol $A(x^0,\xi,q)$ is said to be *weakly parameter elliptic* (*with parameter* $q\in[0,\infty)$) at a point $x^0\in\overline{M}$ if the inequality

$$|\det A^{(0)}(x^0,\xi,q)| \ge C\,|\xi|^{2R_1}\,(q+|\xi|)^{2R_2} \quad (\xi\in\mathbb{R}^n\,,\ q\in[0,\infty)) \tag{2.8}$$

holds with some constant C that does not depend on ξ and q. An operator $A(x,D,q)$ and its symbol are said to be weakly parameter elliptic in $\overline{M}$ if (2.8) holds for every $x^0\in\overline{M}$. A similar definition is related to operators acting on $\mathbb{R}^n$.

Remark 2.2. a) If A is weakly parameter elliptic in $\overline{M}$, then we may assume that the constant C in (2.8) does not depend on x^0 as well (by the continuity and compactness).

b) The right-hand side of (2.8) is closely related to the so-called Newton polygon corresponding to the symbol $\det A(x^0,\xi,q)$. This polygon is defined as the convex hull of the set $\{(0,0),\,(0,2R_2),\,(2R_1,2R_2),\,(2R_1+2R_2,0)\}$ (cf. Figure 1). The points $(2R_1,2R_2)$ and $(2R_1+2R_2,0)$ of this polygon correspond to the expressions $|\xi|^{2R_1}q^{2R_2}$ and $|\xi|^{2R_1+2R_2}$. The sum of these two terms is equivalent to the right-hand side of (2.8). We discuss this point of view below in Section 4.

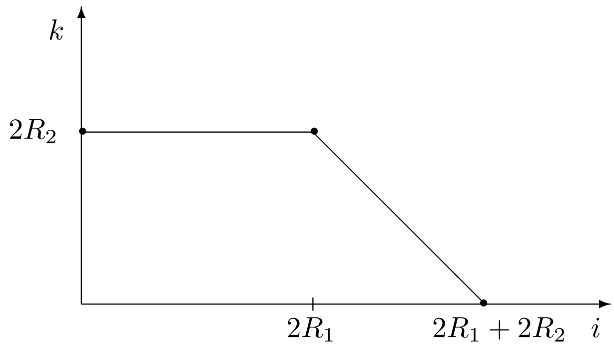

Fig. 1. The Newton polygon $N_{(2R_1,2R_2)}$.

c) Write

$$P(x,\xi,q) := \det A^{(0)}(x,\xi,q)\,.$$

For a chosen x, this is a polynomial in (ξ,q) of order $2R = 2\mu N_1 + 2(m-\mu)N_2 = 2R_1 + 2R_2$ that is of order $2R_2$ with respect to q. We can rewrite this polynomial in the form

$$P(x,\xi,q) = P_{2R}(x,\xi) + q\,P_{2R-1}(x,\xi) + \cdots + q^{2R_2}\,P_{2R_1}(x,\xi),$$

where P_j are polynomials in ξ of order j. It follows from the definition of the determinant that

$$P_{2R}(x,\xi) = \det A^{(0)}(x,\xi,0), \quad P_{2R_1}(x,\xi) = (-1)^{N_2}\det A_{11}^{(0)}(x,\xi)\,.$$

Lemma 3.2 of [5] leads to the following equivalent conditions for the weak parameter ellipticity.

Lemma 2.3. *An operator A is weakly parameter elliptic in $\overline{M}$ if and only if the following conditions hold*:

(i) *For any $x^0 \in \overline{M}$, the matrix operator $A(x,D,0)$ is elliptic in the sense of Douglis-Nirenberg.*
(ii) *For any $x^0 \in \overline{M}$, the matrix operator $A_{11}(x,D)$ is elliptic in the sense of Douglis-Nirenberg.*
(iii) $\det A^{(0)}(x^0,\xi,q) \neq 0$ *for all $x^0 \in \overline{M}$, $\xi \in \mathbb{R}^n \setminus \{0\}$, and $q \in (0,\infty)$.*

If A is weakly parameter elliptic and if a point $x^0 \in \partial M$ is chosen, then we can rewrite A in local coordinates corresponding to x^0, i.e., we choose a coordinate system in a neighborhood of x^0 such that, locally, ∂M is given by the equation $x_n = 0$ and M is given by the inequality $x_n > 0$. We write $x = (x', x_n)$ for the variables and (ξ', τ) for the dual variables. Due to 2.3 (iii), the polynomial $\det A^{(0)}(x^0,\xi',\tau,q)$ has no real roots τ for $\xi' = (\xi_1,\dots,\xi_{n-1}) \neq 0$. Hence, if $n > 2$, then the polynomial has exactly R roots in $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ with R introduced in (2.5). For $n = 2$, this condition is our additional assumption in what follows. In the local coordinates corresponding to $x^0 \in \partial M$ we set (cf. [4])

$$Q(x^0,\tau) := \tau^{-2R_1}\det A^{(0)}(x^0,0,\tau,1) \quad (\tau \in \mathbb{R})\,. \tag{2.9}$$

As was mentioned in the introduction, the boundary value problem (2.7) can equivalently be formulated as a singularly perturbed boundary value problem. The above polynomial Q can seem to be somewhat unusual in the elliptic theory, but it is a basic notion in the singular perturbation theory. The condition of the next definition first appeared in the paper of Vishik–Lyusternik [16] on singular perturbations under the title of the condition of regular degeneration. We stress that this is an example showing how useful it is to change the point of view between large parameter problems and singular perturbations. Note that, according to (2.8), the polynomial (2.9) has no real roots, and its roots belong to $\mathbb{C}_+$ and $\mathbb{C}_-$.

Definition 2.4. We say that $A(x,\xi',\tau,q)$ satisfies the *Vishik–Lyusternik condition* if, for each $x^0 \in \partial M$, the polynomial $Q(x^0,\tau)$ has exactly R_2 roots in $\mathbb{C}_+$.

Remark 2.5. For the important special case in which $N = 2$ and $N_1 = N_2 = 1$, the condition of Definition 2.4 is satisfied automatically if A is weakly parameter elliptic. Indeed, in this case, $A^{(0)}_{ij}(x^0,0,\tau)$ is a (scalar) homogeneous polynomial in τ of degree $s_i + t_j$. Thus, we can write $A^{(0)}_{ij}(x^0,0,\tau) = a_{ij}\tau^{s_i+t_j}$ with $a_{ij} \in \mathbb{C}$. Due to the weak parameter ellipticity, we have

$$\det A^{(0)}(x^0,0,\tau,1) \neq 0 \quad \text{for all} \quad \tau \in \mathbb{R}\setminus\{0\}\,.$$

Therefore, $Q(x^0,\tau) = (a_{11}a_{22} - a_{12}a_{21})\tau^{2R_2} - a_{11}$ has exactly R_2 roots in $\mathbb{C}_+$ and no real roots.

Now let us formulate an analog of the standard Shapiro–Lopatinskii condition for the boundary value problem (1.1), (2.3) in which we have a more delicate dependence on the parameter than in the classical case of the Agmon–Agranovich–Vishik theory (for results on the parameter elliptic Douglis–Nirenberg systems, see also [15]). To this end, we consider $x^0 \in \partial M$ and introduce the corresponding local reference system. According to the block form of the matrix A, we write

$$v(t) = \begin{pmatrix} v^{(1)}(t) \\ v^{(2)}(t) \end{pmatrix},$$

where $v^{(1)}$ consists of the first N_1 components of the vector v and $v^{(2)}$ consists of the other N_2 components. We respectively write

$$B(x,D) = \begin{pmatrix} B_{11}(x,D) & B_{12}(x,D) \\ B_{21}(x,D) & B_{22}(x,D) \end{pmatrix},$$

where B_{ij} are matrix differential operators of sizes $R_1 \times N_1$, $R_1 \times N_2$, $R_2 \times N_1$, and $R_2 \times N_2$, respectively. We also set

$$B_1(x,D) := \big(B_{11}(x,D)\,, \quad B_{12}(x,D)\big)\,, \qquad B_2(x,D) := \big(B_{21}(x,D)\,, \quad B_{22}(x,D)\big)\,.$$

Definition 2.6. The boundary value problem (A,B) of the form (1.1), (2.3) is said to be *weakly parameter elliptic* (with parameter in $[0,\infty)$) if the following conditions are satisfied.

(i) $A(x,D,q)$ is weakly parameter elliptic in $\overline{M}$.

(ii) For any $x^0 \in \partial M$, $\xi' \in \mathbb{R}^{n-1}\setminus\{0\}$, $q \in [0,\infty)$, and $g = (g_1,\dots,g_R) \in \mathbb{C}^R$, the problem

$$A^{(0)}(x^0,\xi',D_t,q)\begin{pmatrix} w^{(1)}(t) \\ w^{(2)}(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (t>0)\,, \tag{2.10}$$

$$B^{(0)}(x^0,\xi',D_t)\begin{pmatrix} w^{(1)}(t) \\ w^{(2)}(t) \end{pmatrix}\Bigg|_{t=0} = g\,, \tag{2.11}$$

$$w^{(i)}(t) \to 0 \quad \text{as} \quad t\to\infty \quad (i=1,2),$$

where $D_t = -i\partial/\partial t$, has a unique solution.

(iii) For any $x^0 \in \partial M$, $\xi' \in \mathbb{R}^{n-1}\setminus\{0\}$, and $h^{(1)} \in \mathbb{C}^{R_1}$, the problem

$$A^{(0)}_{11}(x^0,\xi',D_t)w^{(1)}(t) = 0 \quad (t>0)\,, \tag{2.12}$$

$$B^{(0)}_{11}(x^0,\xi',D_t)w^{(1)}(t)\Big|_{t=0} = h^{(1)}\,, \tag{2.13}$$

$$w^{(1)}(t) \to 0 \quad \text{as} \quad t\to\infty,$$

has a unique solution.

(iv) For any $x^0 \in \partial M$ and any vector $h^{(2)} \in \mathbb{C}^{R_2}$, the problem

$$A^{(0)}(x^0, 0, D_t, 1) \begin{pmatrix} v^{(1)}(t) \\ v^{(2)}(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (t > 0)\,, \tag{2.14}$$

$$B_2^{(0)}(x^0, 0, D_t) \begin{pmatrix} v^{(1)}(t) \\ v^{(2)}(t) \end{pmatrix} \Bigg|_{t=0} = h^{(2)}, \tag{2.15}$$

$$v^{(i)}(t) \to 0 \quad \text{as} \quad t \to \infty \quad (i = 1, 2),$$

has a unique solution.

Remark 2.7. a) Condition (iii) in Definition 2.6 is the ordinary Shapiro–Lopatinskii condition for the Douglis–Nirenberg system $A_{11}(x, D)$ with boundary operators given by $B_{11}(x, D)$. Similarly, setting $q = 0$ in 2.6 (ii), we obtain the ordinary (parameter-independent) Shapiro–Lopatinskii condition for the entire system $(A(x, D, 0), B(x, D))$.

b) For $\xi' = 0$, condition 2.6 (ii) fails in general (even for positive q). Thus, our definition substantially differs from the Agmon–Agranovich–Vishik definition of ellipticity with parameter.

Due to the homogeneity, it suffices to consider (ii) on the hemisphere $\{(\xi', q) \in \mathbb{R}^{n-1} \times [0, \infty) : |\xi'|^2 + q^2 = 1\}$. In contrast to the Agmon–Agranovich–Vishik condition (which is regarded at the points of the closed hemisphere), condition 2.6 (ii) deals with the punctured hemisphere with deleted point $(0, 1)$. In a sense, conditions (iii) and (iv) give a completion for this point. Let us discuss this point of view.

Consider the transformation

$$(\xi', q) \longmapsto \left(\frac{\xi'}{(|\xi'|^2 + q^2)^{1/2}}\,, \ \frac{q}{(|\xi'|^2 + q^2)^{1/2}} \right),$$

which maps every point $(\xi', q) \in \mathbb{R}^{n-1} \times [0, \infty)$ with $|\xi'| + q \neq 0$ to a point of our hemisphere. Under this transformation, the point $(0, 1)$ is not only the image of a point $(0, q)$ for finite $q > 0$ but also the limit of the images of points (ξ', q) as $q \to +\infty$.

Condition 2.6 (iv), where $\xi' = 0$, corresponds to the first group of points, i.e., to the images of the points $(0, q)$. Now let $q \to \infty$. By setting $\lambda = \varepsilon^{-1}$ and considering the homogeneous right-hand side, we come to system (1.3) with small parameter. Passing to the limit as $\varepsilon \to 0$, we obtain $u_2 = 0$ in (1.3). Thus, we must solve the system $A_{11}(x, D)u_1 = 0$ with an overdetermined system of boundary conditions. If we take only the first R_1 boundary conditions, then we obtain a problem for which condition 2.6 (iii) guarantees solvability. To satisfy all boundary conditions, we need condition 2.6 (iv). Due to the Vishik–Lyusternik theory, this condition allows us to add a combination of boundary layers to the above solution.

We continue the study of the relationship between condition (iv) and the existence of boundary layers in Section 7.

c) Conditions 2.6 (ii)–(iv) can be formulated algebraically, cf. [17]. For instance, 2.6 (iv) is equivalent to the following condition.

(iv′) Choose $x^0 \in \partial M$. Let $\gamma^{(2)}$ be a contour in $\mathbb{C}_+$ enclosing all zeros of $Q(x^0, \cdot)$ whose imaginary parts are positive. Then the rank of the matrix

$$\int_{\gamma^{(2)}} B_2^{(0)}(x^0, 0, \tau) \Big[A^{(0)}(x^0, 0, \tau, 1) \Big]^{-1} \, \big(I_N, \tau I_N, \ldots, \tau^{2m-1} I_N \big) \, d\tau$$

is equal to R_2 (and thus the rank of this matrix is the largest possible).

3. AN ESTIMATE FOR THE SOLUTIONS OF THE SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

The aim of this section is to find estimates for the solutions of (2.10)–(2.11) under the assumption of weak parameter ellipticity. Thus, throughout this section we assume that A satisfies the Vishik–Lyusternik condition and that (A, B) satisfies the conditions of Definition 2.6. We choose $x^0 \in \overline{M}$ and write the boundary value problem in coordinates corresponding to x^0. We write $A(\xi, q) := A^{(0)}(x^0, \xi, q)$, $B(\xi) := B^{(0)}(x^0, \xi)$, and $Q(\tau) := Q(x^0, \tau)$.

Lemma 3.1. *Let $\gamma^{(1)}$ be a contour in $\mathbb{C}_+$ enclosing all the zeros of* $\det A_{11}(\xi', \cdot)$ *for* $|\xi'| = 1$ *whose imaginary parts are positive and let $\gamma^{(2)}$ be a contour in $\mathbb{C}_+$ enclosing all the zeros of Q in $\mathbb{C}_+$. Then there exists a $q_0 > 0$ and an enumeration $\tau_1(\xi', q), \ldots, \tau_R(\xi', q)$ of the zeros of* $\det A(\xi', \cdot, q)$ *with positive imaginary parts such that, for all $q \geq q_0$ and $|\xi'| = 1$, the following conditions hold*:

(i) $\gamma^{(1)}$ *encloses* $\tau_1(\xi', q), \ldots, \tau_{R_1}(\xi', q)$,
(ii) $\gamma^{(2)}$ *encloses* $q^{-1}\, \tau_{R_1+1}(\xi', q), \ldots, q^{-1}\, \tau_R(\xi', q)$.

Proof. As was noted in Remark 2.2 c), the determinant $P(\xi', \tau, q) = \det A(\xi', \tau, q)$ is weakly parameter elliptic in the sense of [5]. The behavior of the zeros of this polynomial is described in Lemma 3.5 of [5]. As a result, under an appropriate enumeration, the set $\{\tau_1(\xi', q), \ldots, \tau_{R_1}(\xi', q)\}$ tends to the set $\{\tau_1^0(\xi'), \ldots, \tau_{R_1}^0(\xi', q)\}$ of all zeros of $\det A_{11}(\xi', \cdot)$ with positive imaginary parts, and $q^{-1}\tau_j(\xi', q) \to \tau_j^1$ for $j = R_1 + 1, \ldots, R$, where $\tau_{R_1+1}^1, \ldots, \tau_R^1$ are the zeros of Q in $\mathbb{C}_+$. This implies (i) and (ii) as above.

Denote by $\mathfrak{M}(\xi', q)$ the finite-dimensional space of solutions of the homogeneous equation

$$A(\xi', D_t, q)v(t) = 0 \quad (t > 0)\,,$$

and let $\mathfrak{M}_+(\xi', q)$ be the subspace of $\mathfrak{M}(\xi', q)$ consisting of solutions tending to zero as t tends to $+\infty$. The dimension of this space is R. By Lemma 3.1, this space can be represented as the direct sum $\mathfrak{M}_+(\xi', q) = \mathfrak{M}_+^{(1)}(\xi', q) \dotplus \mathfrak{M}_+^{(2)}(\xi', q)$, where the first space on the right-hand side corresponds to the zeros bounded as $q \to \infty$ and the other subspace corresponds to the other zeros. Note that $\dim \mathfrak{M}_+^{(1)}(\xi', q) = R_1$ and $\dim \mathfrak{M}_+^{(2)}(\xi', q) = R_2$ for $q \geq q_0$.

Lemma 3.2. *Suppose that condition* (iii) *of Definition* 2.6 *is satisfied. Then there is a value $q_0 > 0$ such that, for $|\xi'| = 1$ and $q \geq q_0$, the following assertions hold.*

(i) *The problem on the half-line,*

$$A(\xi', D_t, q)v(t) = 0 \quad (t > 0)\,, \tag{3.1}$$

$$B_1(\xi', D_t)v(0) = h^{(1)}\,, \tag{3.2}$$

$$v(t) \in \mathfrak{M}_+^{(1)}(\xi', q) \tag{3.3}$$

is uniquely solvable for arbitrary $h^{(1)} \in \mathbb{C}^{R_1}$.

(ii) *There exists an $N \times R_1$ rectangular matrix $N^{(1)}(\xi', \tau, q)$, polynomial with respect to τ and bounded for $|\xi'| = 1$, $q \geq q_0$, and $\tau \in \gamma^{(1)}$, for which*

$$\frac{1}{2\pi i} \int_{\gamma^{(1)}} B_1(\xi', \tau) A^{-1}(\xi', \tau, q) N^{(1)}(\xi', \tau, q) d\tau = I_{R_1}\,,$$

where $\gamma^{(1)}$ is the same contour as in Lemma 3.1.

Proof. (i). As in condition (iv′) of Remark 2.7, a necessary and sufficient condition for the unique solvability of problem (3.1)–(3.2) can be written in the following form [17]: the rank of the matrix

$$\Lambda(\xi',q) := \frac{1}{2\pi i}\int_{\gamma^{(1)}} B_1(\xi',\tau)A^{-1}(\xi',\tau,q)(I_N,\tau I_N,\dots,\tau^{2m-1}I_N)d\tau$$

is equal to R_1. Let us show first that

$$A^{-1}(\xi',\tau,q) = \begin{pmatrix} A_{11}^{-1}(\xi',\tau) & 0 \\ 0 & 0 \end{pmatrix} + O(q^{-2(m-\mu)}). \tag{3.4}$$

To prove (3.4), we rewrite the matrix $A(\xi,q)$ in the form

$$A(\xi,q) = \begin{pmatrix} A_{11}(\xi) & 0 \\ 0 & -q^{m-\mu}I_{N_2} \end{pmatrix} \Big[I_N + H(\xi,q)\Big] \begin{pmatrix} I_{N_1} & 0 \\ 0 & q^{m-\mu}I_{N_2} \end{pmatrix},$$

where

$$H(\xi,q) := \begin{pmatrix} 0 & q^{-(m-\mu)}A_{11}^{-1}(\xi)A_{12}(\xi) \\ -q^{-(m-\mu)}A_{21}(\xi) & -q^{-2(m-\mu)}A_{22}(\xi) \end{pmatrix}.$$

It follows from this relation and from the condition $m > \mu$ that the inverse matrix $A^{-1}(\xi,q) =: G(\xi,q)$ exists for q sufficiently large and is equal to

$$\begin{aligned} G(\xi,q) &= \begin{pmatrix} I_{N_1} & 0 \\ 0 & q^{-(m-\mu)}I_{N_2} \end{pmatrix} \Big[I_N - H + O\big(q^{-2(m-\mu)}\big)\Big] \begin{pmatrix} A_{11}^{-1}(\xi) & 0 \\ 0 & -q^{-(m-\mu)}I_{N_2} \end{pmatrix} \\ &= \begin{pmatrix} A_{11}^{-1}(\xi) & 0 \\ 0 & 0 \end{pmatrix} + O(q^{-2(m-\mu)}). \end{aligned}$$

It follows from (3.4) that

$$\lim_{q\to\infty} \Lambda(\xi',q) = \frac{1}{2\pi i}\int_{\gamma^{(1)}} \big(B_{11}(\xi',\tau)A_{11}^{-1}(\xi',\tau),0\big)(I_N,\tau I_N,\dots,\tau^{2m-1}I_N)d\tau.$$

Denote by $M(\xi',q)$ the $R_1\times R_1$ submatrix of $\Lambda(\xi',q)$ that consists of the columns of $\Lambda(\xi',q)$ with the indices $kN+j$ ($k=0,\dots,\mu-1$; $j=1,\dots,N_1$). Then

$$M(\xi') := \lim_{q\to\infty} M(\xi',q) = \frac{1}{2\pi i}\int_{\gamma^{(1)}} B_{11}(\xi',\tau)A_{11}^{-1}(\xi',\tau)(I_{N_1},\dots,\tau^{\mu-1}I_{N_1})d\tau.$$

By condition (iii) of Definition 2.6, the matrix $M(\xi')$ is nonsingular. It follows from (3.4) that the matrix $M(\xi',q)$ is also nonsingular for $q\ge q_0$, where q_0 is large enough, which proves (i).

To prove (ii), it suffices to define (cf. [17]) an $N\times R_1$-dimensional matrix $N^{(1)}(\xi',\tau,q)$ by the formula

$$N^{(1)}(\xi',\tau,q) := \begin{pmatrix} N_1^{(1)}(\xi',\tau,q) \\ 0 \end{pmatrix}$$

with the $N_1\times R_1$-dimensional matrix $N_1^{(1)}(\xi',\tau,q)$ given by

$$N_1^{(1)}(\xi',\tau,q) = (I_{N_1},\tau I_{N_1},\dots,\tau^{\mu-1}I_{N_1})M^{-1}(\xi',q).$$

Lemma 3.3. *Suppose that condition* (iv) *of Definition* 2.6 *is satisfied. Then there exists a* $q_0 > 0$ *such that, for* $|\xi'| = 1$ *and* $q \geq q_0$*, the following statements hold.*

(i) *The problem on the half-line*

$$A(\xi', D_t, q)\, v(t) = 0 \quad (t > 0)\,, \tag{3.5}$$

$$B_2(\xi', D_t)\, v(0) = h^{(2)}\,, \tag{3.6}$$

$$v(t) \in \mathfrak{M}_+^{(2)}(\xi', q) \tag{3.7}$$

is uniquely solvable for arbitrary $h^{(2)} \in \mathbb{C}^{R_2}$.

(ii) *There exists an* $N \times R_2$ *rectangular matrix* $N^{(2)}(\xi', \tau, q)$ *polynomial in* τ *and bounded for all* $|\xi'| = 1, q \geq q_0$ *and* $\tau \in \gamma^{(2)}$ *for which*

$$\frac{1}{2\pi i}\int_{\gamma^{(2)}} B_2\Big(\frac{\xi'}{q}, \tau\Big) A^{-1}\Big(\frac{\xi'}{q}, \tau, 1\Big) N^{(2)}(\xi', \tau, q) d\tau = I_{R_2},$$

where $\gamma^{(2)}$ *is the same contour as in Lemma* 3.1.

Proof. (i). As in Lemma 3.2, we use the criterion in [17] for the unique solvability of the problem on the half-line. This criterion can trivially be modified, namely, the matrices $\tau^l I_N$ can be replaced by $(c\tau)^l I_N$ with arbitrary real $c \neq 0$.

To prove (i), note that the rank of the matrix

$$\frac{1}{2\pi i}\int_{\gamma^{(2)}(q)} B_2(\xi', \tau) A^{-1}(\xi', \tau, q)\Big(I_N, \frac{\tau}{q} I_N, \ldots, \Big(\frac{\tau}{q}\Big)^{2m-1} I_N\Big)\, d\tau$$

is equal to R_2, where $\gamma^{(2)}(q) := \{q\tau : \tau \in \gamma^{(2)}\}$. As in the proof of Lemma 3.2, write $G(\xi, q) = A^{-1}(\xi, q)$. The element $g_{ij}(\xi, q)$ of the matrix G is a homogeneous function in (ξ, q) of degree $-s_j - t_i$. Therefore, making the change of variables $\tau = q\tilde{\tau}$ in the above integral, we obtain the matrix

$$\begin{pmatrix} q^{m_{R_1+1}} & & \\ & \ddots & \\ & & q^{m_R} \end{pmatrix} \Lambda(\xi', q) \begin{pmatrix} q^{1-s_1} & & \\ & \ddots & \\ & & q^{1-s_N} \end{pmatrix},$$

where

$$\Lambda(\xi', q) = \frac{1}{2\pi i}\int_{\gamma^{(2)}} B_2\Big(\frac{\xi'}{q}, \tau\Big) A^{-1}\Big(\frac{\xi'}{q}, \tau, 1\Big)(I_N, \tau I_N, \ldots, \tau^{2m-1} I_N) d\tau\,.$$

Obviously, we must show that, in a neighborhood of $q = +\infty$, the rank of this matrix is R_2. By condition (iv) of Definition 2.6, the rank of the matrix

$$\Lambda(0, 1) := \frac{1}{2\pi i}\int_{\gamma^{(2)}} B_2(0, \tau) A^{-1}(0, \tau, 1)(I_N, \tau I_N, \ldots, \tau^{2m-1} I_N)\, d\tau$$

is equal to R_2. For $\tau \in \gamma^{(2)}$ and $|\xi'| = 1$, the matrix-valued function $B_2(\xi'/q, \tau)\ A^{-1}(\xi'/q, \tau, 1)$ uniformly tends to $B_2(0, \tau) A^{-1}(0, \tau, 1)$ as $q \to +\infty$. This proves (i).

To prove (ii), we choose an invertible submatrix of $\Lambda(0, 1)$ of size $R_2 \times R_2$. Let the indices of the columns of this submatrix be $j_1, \ldots, j_{R_2}$. Then, for large q, the submatrix of $\Lambda(\xi', q)$ consisting of the same columns $j_1, \ldots, j_{R_2}$ is invertible as well. In other words, there exists a $2mN \times R_2$ matrix $\tilde{N}^{(2)}(\xi', q)$ such that $\Lambda(\xi', q)\tilde{N}^{(2)}(\xi', q) = I_{R_2}$. Now it remains to set

$$N^{(2)}(\xi', \tau, q) := (I_N, \tau I_N, \ldots, \tau^{2m-1} I_N)\tilde{N}^{(2)}(\xi', q)\,.$$

Lemma 3.4. *Let*

$$w(t) = \begin{pmatrix} w^{(1)}(t,\xi',q) \\ w^{(2)}(t,\xi',q) \end{pmatrix}$$

be the solution of (2.10)–(2.11). *Then, for* $|\xi'| = 1$ *and sufficiently large* q, *the function* w *can be written in the form*

$$\begin{aligned} w(t,\xi',q) &= \frac{1}{2\pi i}\int_{\gamma^{(1)}} A^{-1}(\xi',\tau,q)N^{(1)}(\xi',\tau,q)e^{it\tau}\,d\tau\cdot\psi^{(1)}(\xi',q) \\ &\quad + \begin{pmatrix} q^{-t_1} & & \\ & \ddots & \\ & & q^{-t_N} \end{pmatrix} \frac{1}{2\pi i}\int_{\gamma^{(2)}} A^{-1}\Big(\frac{\xi'}{q},\tau,1\Big)N^{(2)}(\xi',\tau,q)e^{itq\tau}d\tau\cdot\psi^{(2)}(\xi',q) \end{aligned} \tag{3.8}$$

where $\gamma^{(1)}$ *and* $\gamma^{(2)}$ *are as in Lemma* 3.1 *and* $N^{(1)}$, $N^{(2)}$ *are as in Lemmas* 3.2 *and* 3.3, *respectively. The vectors* $\psi^{(1)}(\xi',q) \in \mathbb{C}^{R_1}$ *and* $\psi^{(2)}(\xi',q) \in \mathbb{C}^{R_2}$ *are given by the formula*

$$\begin{pmatrix} \psi^{(1)}(\xi',q) \\ \psi^{(2)}(\xi',q) \end{pmatrix} = M(\xi',q)\; g$$

for some matrix $M(\xi',q) = \Big(M_{ij}(\xi',q)\Big)_{i,j=1,\ldots,R}$ *such that*

$$|M_{ij}(\xi',q)| \le \begin{cases} C\,, & i \le R_1\,,\quad j \le R_1\,, \\ C\, q^{m_{R_1}-m_j}\,, & i \le R_1\,,\quad j > R_1\,, \\ C\, q^{-m_{R_1+1}}\,, & i > R_1\,,\quad j \le R_1\,, \\ C\, q^{-m_j}\,, & i > R_1\,,\quad j > R_1\,. \end{cases} \tag{3.9}$$

Proof. We are following the scheme of [6], Lemmas 3.1–3.3, which uses an idea of Frank [8]. We define w by (3.8) with unknown vectors $\psi^{(i)}(\xi',q)$. First we note that the ordinary differential operator $A(\xi',D_t,q)$ sends each of the two integrals in the right-hand side of (3.8) to zero. For the first integral, this is obvious. As for the other integral, we note that

$$A(\xi',D_t,q)\begin{pmatrix} q^{-t_1} & & \\ & \ddots & \\ & & q^{-t_N} \end{pmatrix} = \begin{pmatrix} q^{s_1} & & \\ & \ddots & \\ & & q^{s_N} \end{pmatrix} A\Big(\frac{\xi'}{q},\frac{1}{q}D_t,1\Big)\,.$$

To compute $\psi^{(i)}$, we apply the boundary operators to (3.8) and obtain

$$B_1(\xi',D_t)w(t,\xi',q) = \psi^{(1)}(\xi',q) + \Delta_1\frac{1}{2\pi i}\int_{\gamma^{(2)}} B_1\Big(\frac{\xi'}{q},\tau\Big)A^{-1}\Big(\frac{\xi'}{q},\tau,1\Big)N^{(2)}(\xi',\tau,q)\,d\tau\;\psi^{(2)}(\xi',q)$$

with

$$\Delta_1 := \begin{pmatrix} q^{m_1} & & \\ & \ddots & \\ & & q^{m_{R_1}} \end{pmatrix}.$$

Here we used Lemma 3.2 and the homogeneity of b_{ij}. In the same way, using Lemma 3.3, we obtain

$$B_2(\xi',D_t)w(t,\xi',q) = \frac{1}{2\pi i}\int_{\gamma^{(1)}} B_2(\xi',\tau)A^{-1}(\xi',\tau,q)N^{(1)}(\xi',\tau,q)d\tau\;\psi^{(1)}(\xi',q) + \Delta_2\,\psi^{(2)}(\xi',q)$$

with

$$\Delta_2 := \begin{pmatrix} q^{m_{R_1+1}} & & \\ & \ddots & \\ & & q^{m_R} \end{pmatrix}.$$

Therefore, condition (2.11) leads to the system of linear equations

$$\begin{pmatrix} I_{R_1} & \Delta_1 H_{12} \\ H_{21} & \Delta_2 \end{pmatrix} \begin{pmatrix} \psi^{(1)} \\ \psi^{(2)} \end{pmatrix} = g\,, \tag{3.10}$$

where we used the notation

$$\begin{aligned} H_{12} &:= \frac{1}{2\pi i}\int_{\gamma^{(2)}} B_1\Big(\frac{\xi'}{q},\tau\Big)A^{-1}\Big(\frac{\xi'}{q},\tau,1\Big)N^{(2)}(\xi',\tau,q)\,d\tau\,,\\ H_{21} &:= \frac{1}{2\pi i}\int_{\gamma^{(1)}} B_2(\xi',\tau)A^{-1}(\xi',\tau,q)N^{(1)}(\xi',\tau,q)\,d\tau\,. \end{aligned}$$

As was shown in the proof of Lemma 3.3 in [6], for large q, the matrix in (3.10) is invertible (here we use the inequality $m_{R_1} < m_{R_1+1}$, see (2.4)), and its inverse is given by

$$M = \begin{pmatrix} G_1 & -G_1\Delta_1 H_{12}\Delta_2^{-1} \\ -G_2\Delta_2^{-1}H_{21} & G_2\Delta_2^{-1} \end{pmatrix}$$

with $G_1 := (I_{R_1} - \Delta_1 H_{12}\Delta_2^{-1}H_{21})^{-1}$ and $G_2 := (I_{R_2} - \Delta_2^{-1}H_{21}\Delta_1 H_{12})^{-1}$. Since the matrices G_1 and G_2 are bounded for sufficiently large q, the estimate (3.9) follows. (Note that in [6], in the situation of Lemma 3.3, the additional condition $m_j \ge 0$ was imposed. However, it readily follows from the proof of this lemma that the additional condition is not needed, and the condition $m_{R_1} < m_{R_1+1}$ is substantial.)

For $j = 1, \dots, R$, denote by

$$w_j = \begin{pmatrix} w_{j1}(t,\xi',q) \\ \vdots \\ w_{jN}(t,\xi',q) \end{pmatrix}$$

the solution of (2.10)–(2.11) in which g is replaced by the jth unit vector $e_j \in \mathbb{C}^R$.

Theorem 3.5. a) *For $i = 1, \dots, N_1$, $j = 1, \dots, R$, and $l = 0, 1, 2, \dots,$ we have*

$$\|D_t^l w_{ji}(\cdot,\xi',q)\|_{L_2(\mathbb{R}_+)} \le C \begin{cases} |\xi'|^{l-m_j-t_i-1/2}, & l \le m_{R_1+1}+t_i,\ \ j \le R_1,\\ |\xi'|^{m_{R_1+1}-m_j}(q+|\xi'|)^{l-m_{R_1+1}-t_i-1/2}, & l > m_{R_1+1}+t_i,\ \ j \le R_1,\\ |\xi'|^{l-m_{R_1}-t_i-1/2}(q+|\xi'|)^{m_{R_1}-m_j}, & l \le m_{R_1}+t_i,\ \ j > R_1,\\ (q+|\xi'|)^{l-m_j-t_i-1/2}, & l > m_{R_1}+t_i,\ \ j > R_1, \end{cases} \tag{3.11}$$

for all $\xi' \in \mathbb{R}^{n-1}\setminus\{0\}$ and $q \in [0,\infty)$.

b) *For $i = N_1+1, \dots, N$, $j = 1, \dots, R$, and $l = 0, 1, 2, \dots,$ we have*

$$\|D_t^l w_{ji}(\cdot,\xi',q)\|_{L_2(\mathbb{R}_+)} \le C \begin{cases} |\xi'|^{l-m_j-1/2}(q+|\xi'|)^{-t_i}, & l \le m_{R_1+1},\ \ j \le R_1,\\ |\xi'|^{m_{R_1+1}-m_j}(q+|\xi'|)^{l-m_{R_1+1}-t_i-1/2}, & l > m_{R_1+1},\ \ j \le R_1,\\ |\xi'|^{l-m_{R_1}-1/2}(q+|\xi'|)^{m_{R_1}-m_j-t_i}, & l \le m_{R_1},\ \ j > R_1,\\ (q+|\xi'|)^{l-m_j-t_i-1/2}, & l > m_{R_1},\ \ j > R_1, \end{cases} \tag{3.12}$$

for all $\xi' \in \mathbb{R}^{n-1} \setminus \{0\}$ *and* $q \in [0, \infty)$.

Proof. a) We assume first that $i \le N_1$. Due to the unique solvability of (2.10)–(2.11), for $\xi' \neq 0$ and $q \in [0, \infty)$ we have

$$w_{ji}(t, \xi', q) = |\xi'|^{-m_j - t_i} w_{ji}\left(|\xi'|t, \frac{\xi'}{|\xi'|}, \frac{q}{|\xi'|}\right),$$

and hence

$$\|D_n^l w_{ji}(\cdot, \xi', q)\|_{L_2(\mathbb{R}_+)} = |\xi'|^{l - m_j - t_i - 1/2} \Big\| D_n^l w_{ji}\Big(\cdot, \frac{\xi'}{|\xi'|}, \frac{q}{|\xi'|}\Big)\Big\|_{L_2(\mathbb{R}_+)} .$$

Therefore, it suffices to show that, for $|\xi'| = 1$ and for all $q \in [0, \infty)$, the following estimate holds:

$$\|D_t^l w_{ji}(\cdot, \xi', q)\|_{L_2(\mathbb{R}_+)} \le C \begin{cases} 1\,, & l \le m_{R_1+1} + t_i\,, \quad j \le R_1\,, \\ \tilde{q}^{\,l - m_{R_1+1} - t_i - 1/2}\,, & l > m_{R_1+1} + t_i\,, \quad j \le R_1\,, \\ \tilde{q}^{\,m_{R_1} - m_j}\,, & l \le m_{R_1} + t_i\,, \quad j > R_1\,, \\ \tilde{q}^{\,l - m_j - t_i - 1/2}\,, & l > m_{R_1} + t_i\,, \quad j > R_1\,, \end{cases} \tag{3.13}$$

where $\tilde{q} := \max\{1, q\}$. For $|\xi'| = 1$ and $q \in [0, q_0]$ this is true indeed due to continuity and compactness (the proof uses condition (ii) of Definition 2.6). Thus, let q be sufficiently large. We use the representation (3.8) for $g = e_j$. Considering $D_t^l w_j$ instead of w_j, we obtain an additional factor τ^l in the integral over $\gamma^{(1)}$ and an additional factor $q^l \tau^l$ in the integral over $\gamma^{(2)}$. Integrating $D_n^l w_j$ with respect to t, we obtain

$$\|D_t^l w_{ji}\|_{L_2(\mathbb{R}_+)} \le C\left(|\psi^{(1)}(\xi', q)| + q^{l - t_i - 1/2} |\psi^{(2)}(\xi', q)|\right) \tag{3.14}$$

where we used the fact that we may choose $\gamma^{(1)}$ and $\gamma^{(2)}$ at a positive distance from the real axis, and $|\cdot|$ stands for the standard norm in the corresponding space ($\mathbb{C}^{R_1}$ or $\mathbb{C}^{R_2}$).

Now inequalities (3.9) lead to the relation

$$\|D_t^l w_{ji}\|_{L_2(\mathbb{R}_+)} \le C \begin{cases} O(1) + O(q^{l - t_i - 1/2 - m_{R_1+1}})\,, & j \le R_1\,, \\ O(q^{m_{R_1} - m_j}) + O(q^{l - t_i - 1/2 - m_j})\,, & j > R_1\,, \end{cases}$$

which implies (3.13).

b) The proof for the case $i > N_1$ is similar to that above with only one difference. In part a), we used the fact that the elements of $G(\xi', \tau, q) = A^{-1}(\xi', \tau, q)$ are $O(1)$ for a chosen (ξ', τ) and for large values of q. By (3.4), for $i > N_1$, the elements $g_{ij}(\xi', \tau, q)$ of this matrix are $O(q^{-2(m-\mu)}) = O(q^{-2(s_i + t_i)})$ and, for such indices i, we can replace (3.14) by the stronger estimate

$$\|D_t^l w_{ji}\|_{L_2(\mathbb{R}_+)} \le C\left(|q^{-t_i} \psi^{(1)}(\xi', q)| + q^{l - t_i - 1/2} |\psi^{(2)}(\xi', q)|\right),$$

which leads to the desired estimate.

4. NEWTON POLYGON AND THE CORRESPONDING SOBOLEV SPACES

As was already noticed in Remark 2.2 b), the definition of weak parameter ellipticity is closely connected to the Newton polygon that corresponds to the determinant of the symbol of A. Moreover, the *a priori* estimates below use Sobolev spaces that arise by using the Newton polygon approach. In the present section, we briefly recall the main concepts and results of this approach and rewrite the estimate of Theorem 3.5 in terms of the Newton polygon. For a more detailed exposition, the reader is referred to [9] and to [4] and [5].

Let $\mathbf{r} = (r_1, r_2)$ be a pair of reals. Define a weight function $\Xi_{\mathbf{r}}\colon \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ as follows:

$$\Xi_{\mathbf{r}}(\xi, q) := (1 + |\xi|)^{r_1} \, (q + |\xi|)^{r_2} \,. \tag{4.1}$$

Similarly, define a function $\Phi_{\mathbf{r}}$, $\mathbf{r} \in \mathbb{R}^2$, by

$$\Phi_{\mathbf{r}}(\xi, q) := |\xi|^{r_1} \, (q + |\xi|)^{r_2} \,. \tag{4.2}$$

If r_1 and r_2 are positive integers, then these functions allow a geometric interpretation. Let us describe it. For positive integers r_1 and r_2, define the Newton polygon $N_{\mathbf{r}}$ as the convex hull of the set $\{(0,0), (0, r_2), (r_1, r_2), (r_1 + r_2, 0)\}$ (cf. Figure 1). We can readily see that there is an equivalence

$$\Xi_{\mathbf{r}}(\xi, q) \approx \sum_{(i,k) \in N_{\mathbf{r}} \cap \mathbb{Z}^2} |\xi|^i q^k,$$

where the sign $\approx$ means that the ratio of the left-hand side to the right-hand side is bounded above and below by positive constants that do not depend on ξ and q. Similarly, $\Phi_{\mathbf{r}}(\xi, q) \approx \sum_{(i,k)} |\xi|^i q^k$, where the sum is now taken over all integral points on the edge connecting (r_1, r_2) and $(r_1 + r_2, 0)$. (In a sense, this edge is the leading part of the Newton polygon $N_{\mathbf{r}}$.)

Now we again assume that r_1 and r_2 are arbitrary real numbers. In connection with the weight function $\Xi_{\mathbf{r}}$, we endow the (classical) Sobolev space $H^{r_1+r_2}(\mathbb{R}^n)$ with a parameter-dependent norm $\|\cdot\|_{\mathbf{r}}$ given by the formula

$$\|u\|_{\mathbf{r}} := \|F^{-1} \Xi_{\mathbf{r}} F u\|_{L_2(\mathbb{R}^n)} \,, \tag{4.3}$$

where F stands for the Fourier transform. We write $H_{\mathbf{r}}(\mathbb{R}^n)$ for $H^{r_1+r_2}(\mathbb{R}^n)$ endowed with this norm, omitting the symbol q for brevity in the notation for $H_{\mathbf{r}}$ and $\|\cdot\|_{\mathbf{r}}$. If $\mathbb{R}^n$ is replaced by $\mathbb{R}^{n-1} = \{(x', x_n) \in \mathbb{R}^n : x_n = 0\}$, then we use the weight function $\Xi_{\mathbf{r}}(\xi', q) := \Xi_{\mathbf{r}}(\xi', 0, q)$ in the definition of the norm $\|\cdot\|_{\mathbf{r}, \mathbb{R}^{n-1}}$ on the space $H^{r_1+r_2}(\mathbb{R}^{n-1})$. On the half-space $\mathbb{R}^n_+ := \{x = (x', x_n) \in \mathbb{R}^n : x_n > 0\}$, on the manifold M, and on its boundary ∂M, similar norms can be defined in the standard way (cf. [18]); denote the resulting norms by $\|\cdot\|_{\mathbf{r}, \mathbb{R}^n_+}$, etc.

Let us consider the traces of functions in $H_{\mathbf{r}}(\mathbb{R}^n_+)$ on the boundary $\mathbb{R}^{n-1}$. The question is how to define parameter-dependent norms on the Sobolev spaces on the boundary so that the trace operator $\gamma_0\colon u \mapsto u(\cdot, 0)$ is continuous and its norm is bounded by a constant independent of q. To answer this question, we define the functions

$$\Xi^{(-a)}_{\mathbf{r}}(\xi, q) = \begin{cases} (1 + |\xi|)^{r_1 - a} (q + |\xi|)^{r_2} \,, & a \le r_1 \,, \\ (q + |\xi|)^{r_1 + r_2 - a} \,, & a > r_1 \,, \end{cases} \tag{4.4}$$

$$\Phi^{(-a)}_{\mathbf{r}}(\xi, q) = \begin{cases} |\xi|^{r_1 - a} (q + |\xi|)^{r_2} \,, & a \le r_1 \,, \\ (q + |\xi|)^{r_1 + r_2 - a} \,, & a > r_1 \,, \end{cases} \tag{4.5}$$

for $\mathbf{r} = (r_1, r_2) \in \mathbb{R}^2$ and $a \in \mathbb{R}$. Denote the norm related to the function $\Xi^{(-a)}_{\mathbf{r}}$ by $\|\cdot\|^{(-a)}_{\mathbf{r}}$. The space $H^{r_1+r_2-a}$ endowed with this norm is denoted by $H^{(-a)}_{\mathbf{r}}$.

The functions in (4.4) and (4.5) can be interpreted geometrically as well. For positive integers r_1 and r_2 and for $0 \le a \le r_1 + r_2$, the function $\Xi^{(-a)}_{\mathbf{r}}$ corresponds to the Newton polygon that is constructed from $N_{\mathbf{r}}$ by the left shift by a in parallel to the horizontal axis.

The following theorem describes the trace spaces of the Sobolev space $H_{\mathbf{r}}(\mathbb{R}^n_+)$ in the above sense. This is one of the main results of the theory of Sobolev spaces defined by Newton polygons; for the case in which r_1 and r_2 are positive integers, this description follows from Theorem 2.9 in [5]. However, we can readily see that the proof in [5] works for arbitrary reals r_1 and r_2 (with $r_1 + r_2 > 1/2$) as well.

Theorem 4.1. *Let* $\mathbf{r} \in \mathbb{R}^2$ *and let* $r_1 + r_2 > 1/2$. *For any* $q_0 > 0$ *and* $l \in \mathbb{Z}$ *such that* $0 \le l \le r_1 + r_2 - 1/2$, *there is a constant* $C > 0$ *that does not depend on* u *and* q *and for which*

$$\|\gamma_0 D_n^l u\|^{(-l-1/2)}_{\mathbf{r},\mathbb{R}^{n-1}} \le C\|u\|_{\mathbf{r},\mathbb{R}^n_+}$$

holds for all $u \in H_{\mathbf{r}}(\mathbb{R}^n_+)$ *and* $q \ge q_0$.

Now we are going to rewrite the ODE estimate in Theorem 3.5 in terms of the Newton polygon. In the *a priori* estimates below, the structure of the norms of the ith components for $i \le N_1$ differs from that for $i > N_1$. To unify the notation in these cases, we introduce the tuple

$$\mathbf{e}_i := \begin{cases} (1,0) & \text{for} \quad 1 \le i \le N_1, \\ (0,1) & \text{for} \quad N_1 + 1 \le i \le N. \end{cases} \tag{4.6}$$

In this notation, we obtain

$$\Phi_{\mathbf{r}+t_i\mathbf{e}_i}(\xi, q) = \begin{cases} |\xi|^{r_1+t_i}(q + |\xi|)^{r_2} & \text{for} \quad i \le N_1, \\ |\xi|^{r_1}(q + |\xi|)^{r_2+t_i} & \text{for} \quad i > N_1. \end{cases} \tag{4.7}$$

Theorem 4.2. *Assume that* A *satisfies the Vishik–Lyusternik condition and that* (A, B) *is weakly parameter elliptic. Let* r_1 *and* r_2 *be real numbers such that* $r_1 + r_2 \ge m_R + 1/2$ *and* $r_1 \in [m_{R_1} + 1/2, m_{R_1+1} + 1/2]$. *Then, for* $j = 1, \dots, R$ *and* $l = 0, 1, 2, \dots,$ *the inequality*

$$\|D_t^l w_{ji}(\cdot, \xi', q)\|_{L_2(\mathbb{R}_+)} \le C\, \frac{\Phi_{\mathbf{r}}^{(-m_j-1/2)}(\xi', q)}{\Phi^{(-l)}_{\mathbf{r}+t_i\mathbf{e}_i}(\xi', q)} \tag{4.8}$$

holds for all $\xi' \in \mathbb{R}^{n-1} \setminus \{0\}$ *and* $q \in [0, \infty)$.

Proof. We assume first that $i \le N_1$ and $r_1 < m_{R_1+1} + 1/2$. We must compare the right-hand side of (4.8) to the right-hand side of (3.11). Using the definition (4.5) and the fact that $m_j + t_i + 1/2 \le r_1 + t_i$ if and only if $j \le R_1$, we obtain

$$\frac{\Phi_{\mathbf{r}}^{(-m_j-1/2)}(\xi', q)}{\Phi^{(-l)}_{\mathbf{r}+t_i\mathbf{e}_i}(\xi', q)} = \begin{cases} |\xi'|^{l-m_j-t_i-1/2}, & l \le r_1 + t_i, j \le R_1, \\ |\xi'|^{r_1-m_j-1/2}(q + |\xi|)^{l-t_i-r_1}, & l > r_1 + t_i, j \le R_1, \\ |\xi'|^{l-t_i-r_1}(q + |\xi|)^{r_1-m_j-1/2}, & l \le r_1 + t_i, j > R_1, \\ (q + |\xi'|)^{l-m_j-t_i-1/2}, & l > r_1 + t_i, j > R_1, \end{cases} \tag{4.9}$$

Let $j \le R_1$. If $l \le r_1 + t_i$, then $l \le m_{R_1+1} + t_i$, and the first line in (3.11) must be used. For $l > r_1 + t_i$, the ratio of the right-hand sides of (3.11) and (4.9) is equal to

$$\begin{cases} \left(\dfrac{|\xi'|}{q + |\xi'|}\right)^{l-t_i-r_1}, & l \le m_{R_1+1} + t_i\,, \\ \left(\dfrac{|\xi'|}{q + |\xi'|}\right)^{1/2+m_{R_1+1}-r_1}, & l > m_{R_1+1} + t_i\,. \end{cases}$$

In any case, this ratio does not exceed one because of the condition on r_1.

Similarly, for $j > R_1$, the ratio of the right-hand sides of (3.11) and (4.9) is equal to

$$\begin{cases} \left(\dfrac{|\xi'|}{q + |\xi'|}\right)^{r_1-m_{R_1}-1/2}, & l \le m_{R_1} + t_i\,, \\ \left(\dfrac{|\xi'|}{q + |\xi'|}\right)^{r_1+t_i-l}, & m_{R_1} + t_i < l \le r_1 + t_i\,, \\ 1\,, & l > r_1 + t_i \end{cases}$$

which proves (4.8) for $r_1 \in [m_{R_1}+1/2, m_{R_1+1}+1/2)$.

Now let us consider the case in which $i \le N$ and $r_1 = m_{R_1+1}+1/2$. Define the index k by the condition

$$m_{R_1+1} = m_{R_1+2} = \cdots = m_k < m_{k+1}\,.$$

Then $m_j + 1/2 \le r_1$ if and only if $j \le k$. Computing the right-hand side of (4.8), we see that, for $j \le k$, we obtain the first and second lines of the right-hand side of (4.9). However, for $j = R_1+1,\dots,k$ we have $m_j+1/2 = r_1$, and hence, for these values of j, there is no difference between the first and the third lines and between the second and the fourth lines in (4.9). Therefore, (4.9) holds for $r_1 = m_{R_1+1}+1/2$ as well. The remaining part of the proof is just like that in the case $r_1 < m_{R_1+1}+1/2$.

Finally, the proof for $i > N_1$ is similar. We must use Theorem 3.5 b) instead of Theorem 3.5 a).

5. *A PRIORI* ESTIMATES

In this section we prove *a priori* estimates for operators with constant coefficients and without lower-order terms of the form (1.1) and (2.3) in $\mathbb{R}^n$ and $\mathbb{R}^n_+$, respectively (model problems). Throughout this section, we choose a pair $\mathbf{r} = (r_1, r_2)$ of real numbers such that $r_1 + r_2 \ge m_R + 1$ and $r_1 \in [m_{R_1}+1/2, m_{R_1+1}+1/2]$. For simplicity, let us assume in addition that $r_1 + r_2$ is an integer; in this case, we can obtain an easy description (see (5.13) below) for the norms entering in the *a priori* estimate. For the estimates treated below, we must keep in mind definitions (4.1), (4.4), and (4.6).

To begin with, let $A = A(D,q)$ be a model operator of the form (1.1) acting on $\mathbb{R}^n$.

Lemma 5.1. *For $q \ge q_0 > 0$, the operator*

$$A(D,q)\colon \prod_{i=1}^N H_{\mathbf{r}+t_i\mathbf{e}_i}(\mathbb{R}^n) \to \prod_{i=1}^N H_{\mathbf{r}-s_i\mathbf{e}_i}(\mathbb{R}^n)$$

is continuous. Here the norm in the space $H_{\mathbf{r}}(\mathbb{R}^n)$ is defined by (4.3) *and* (4.1), *and the continuity means that A is a bounded operator between these spaces and the norm of A can be estimated by a constant $C = C(q_0)$ independent of q.*

Proof. Due to the Douglis–Nirenberg structure, the parameter-independent operator

$$A_{ij}(D,0)\colon H_{\mathbf{r}+t_j\mathbf{e}_j}(\mathbb{R}^n) \to H_{\mathbf{r}-s_i\mathbf{e}_i}(\mathbb{R}^n) \tag{5.1}$$

is continuous. Indeed, if $j \le N_1$, then the left-hand space is $H_{r_1+t_j,r_2}$, and it is mapped into $H_{r_1-s_i,r_2}$. According to the obvious embedding

$$H_{r_1-a,r_2}(\mathbb{R}^n) \subset H_{r_1,r_2-a}(\mathbb{R}^n) \quad (a \ge 0), \tag{5.2}$$

the operator (5.1) is bounded. For $j > N_1$, the left-hand space is H_{r_1,r_2+t_j}, which is mapped into $H_{r_1-s_i-t_j,r_2+t_j}$. Using (5.2) again, we prove the continuity of (5.1). The same holds for the multiplication operator $q^{2m-2\mu}$.

Theorem 5.2. *Let $A(D,q)$ be weakly parameter elliptic in the sense of Definition* 2.1. *Then, for $q \ge q_0 > 0$, the function u satisfies the a priori estimate*

$$\sum_{i=1}^N \|u_i\|_{\mathbf{r}+t_i\mathbf{e}_i} \le C\Big(\sum_{i=1}^N \|f_i\|_{\mathbf{r}-s_i\mathbf{e}_i} + \sum_{i=1}^{N_1} q^{r_2}\|u_i\|_{L_2} + \sum_{i=N_1+1}^N q^{r_2+t_i}\|u_i\|_{L_2}\Big),$$

where $f = A(D,q)u$.

Proof. Let us take an arbitrary vector $c \in \mathbb{C}^N$, set $d = A(\xi,q)c$, and prove the inequality

$$\sum_{i=1}^N \Phi_{\mathbf{r}+t_i\mathbf{e}_i}(\xi,q)\,|c_i| \le C\sum_{j=1}^N \Phi_{\mathbf{r}-s_j\mathbf{e}_j}(\xi,q)\,|d_j|. \tag{5.3}$$

As in the proof of Lemma 3.2, we set $G(\xi,q)=(g_{ij}(\xi,q))=A^{-1}(\xi,q)$. We already know that $g_{ij}(\xi,q)$ is homogeneous in (ξ,q) of degree $-s_j-t_i$. Write

$$g_{ij}(\xi,q)=(-1)^{i+j}(\det A(\xi,q))^{-1}\det\left[A(\xi,q)^{ji}\right],$$

where the $(N-1)\times(N-1)$ matrix A^{ji} is obtained by deleting the jth row and the ith column from the matrix A. Using the condition of weak parameter ellipticity and the homogeneity of the elements of the matrix $A(\xi,q)$, we readily obtain the following estimate (cf. the proof of Proposition 3.10 in [4]):

$$|g_{ij}(\xi,q)|\le C\begin{cases}|\xi|^{-s_j-t_i}, & i,j\le N_1\,,\\ |\xi|^{t_j-t_i}\,(q+|\xi|)^{-s_j-t_j}, & i\le N_1, j>N_1\,,\\ |\xi|^{s_i-s_j}\,(q+|\xi|)^{-s_i-t_i}, & i>N_1, j\le N_1\,,\\ |\xi|^{s_i+t_j}(q+|\xi|)^{-s_i-t_i-s_j-t_j}, & i,j>N_1, i\ne j\,.\\ (q+|\xi|)^{-s_j-t_j}, & i=j>N_1.\end{cases}\tag{5.4}$$

Let us first consider the case $i\le N_1$, in which the ith term in the sum on the left-hand side of (5.3) is equal to $|\xi|^{r_1+t_i}(q+|\xi|)^{r_2}|c_i|$. Using (5.4), we see that the ith term is not greater than

$$\begin{aligned}&\sum_{j=1}^{N}|\xi|^{r_1+t_i}(q+|\xi|)^{r_2}|g_{ij}(\xi,q)|\,|d_j|\\ &\qquad\le C\Big(\sum_{j=1}^{N_1}|\xi|^{r_1-s_j}(q+|\xi|)^{r_2}|d_j|+\sum_{j=N_1+1}^{N}|\xi|^{r_1+t_j}(q+|\xi|)^{r_2-s_j-t_j}|d_j|\Big)\\ &\qquad\le C\sum_{j=1}^{N}\Phi_{\mathbf{r}-s_j\mathbf{e}_j}(\xi,q)|d_j|\,.\end{aligned}$$

In the same way we can see that, for $i>N_1$, the ith term in the sum on the left-hand side of (5.3) is not greater than

$$\begin{aligned}&\sum_{j=1}^{N}|\xi|^{r_1}(q+|\xi|)^{r_2+t_i}|g_{ij}(\xi,q)|\,|d_j|\\ &\le C\Big(\sum_{j=1}^{N_1}|\xi|^{r_1+s_i-s_j}(q+|\xi|)^{r_2-s_i}|d_j|+\sum_{j=N_1+1}^{N}|\xi|^{r_1+s_i+t_j}(q+|\xi|)^{r_2-s_i-s_j-t_j}|d_j|\Big)\\ &\le C\Big(\sum_{j=1}^{N_1}|\xi|^{r_1-s_j}(q+|\xi|)^{r_2}|d_j|+\sum_{j=N_1+1}^{N}|\xi|^{r_1}(q+|\xi|)^{r_2-s_j}|d_j|\Big)\\ &=C\sum_{j=1}^{N}\Phi_{\mathbf{r}-s_j\mathbf{e}_j}(\xi,q)|d_j|\,,\end{aligned}$$

which completes the proof of (5.3).

For $|\xi|\ge 1$ we have $|\xi|\approx 1+|\xi|$, and we may replace Φ by Ξ in (5.3). For $|\xi|\le 1$ we have

$$\Xi_{\mathbf{r}+t_i\mathbf{e}_i}(\xi,q)\le C\begin{cases}q^{r_2} & \text{for}\quad i\le N_1,\\ q^{r_2+t_i} & \text{for}\quad i>N_1.\end{cases}$$

Therefore, for all $\xi\in\mathbb{R}^n$, we obtain the inequality

$$\sum_{i=1}^{N}\Xi_{\mathbf{r}+t_i\mathbf{e}_i}(\xi,q)\,|c_i|\le C\Big(\sum_{j=1}^{N}\Xi_{\mathbf{r}-s_j\mathbf{e}_j}(\xi,q)\,|d_j|+\sum_{i=1}^{N_1}q^{r_2}|c_i|+\sum_{i=N_1+1}^{N}q^{r_2+t_i}|c_i|\Big)\,.\tag{5.5}$$

Now it remains to replace the numbers c_i in (5.5) by $Fu_i(\xi)$ and d_j by $Ff_j(\xi)$, square the resulting relation, and integrate with respect to $\xi \in \mathbb{R}^n$ to complete the proof of the theorem. (Here and in the following, we tacitly use the fact that all norms on finite-dimensional spaces are equivalent, and thus we may replace terms of the form $(\sum_i |c_i|)^2$ by $\sum_i |c_i|^2$, etc.)

Now let us consider the model problem in the half-space $\mathbb{R}^n_+$, i.e., take operators A and B of the form (1.1) and (2.3), respectively, with constant coefficients and without lower-order terms.

Lemma 5.3. *The operator*

$$(A,B)\colon \prod_{i=1}^{N} H_{\mathbf{r}+t_i\mathbf{e}_i}(\mathbb{R}^n_+) \to \prod_{i=1}^{N} H_{\mathbf{r}-s_i\mathbf{e}_i}(\mathbb{R}^n_+) \times \prod_{j=1}^{R} H_{\mathbf{r}}^{(-m_j-1/2)}(\mathbb{R}^{n-1})$$

is continuous.

Proof. We choose an extension operator E from $\mathbb{R}^n_+$ to $\mathbb{R}^n$ that is continuous from $L_2(\mathbb{R}^n_+)$ to $L_2(\mathbb{R}^n)$ and from $H_{\mathbf{r}+t_i\mathbf{e}_i}(\mathbb{R}^n_+)$ to $H_{\mathbf{r}+t_i\mathbf{e}_i}(\mathbb{R}^n)$. (Such an operator can be defined by using the standard Hestenes–Seeley construction.) Let

$$u \in \prod_{i=1}^{N} H_{\mathbf{r}+t_i\mathbf{e}_i}(\mathbb{R}^n_+), \qquad f := Au.$$

Write $Ef := AEu$. By Lemma 5.1 and by the definition of the norm related to the half-space, we obtain the inequality

$$\sum_{i=1}^{N} \|f_i\|_{\mathbf{r}-s_i\mathbf{e}_i,\mathbb{R}^n_+} \le C \sum_{i=1}^{N} \|(Ef)_i\|_{\mathbf{r}-s_i\mathbf{e}_i,\mathbb{R}^n} \le C \sum_{i=1}^{N} \|(Eu)_i\|_{\mathbf{r}+t_i\mathbf{e}_i,\mathbb{R}^n} \le C \sum_{i=1}^{N} \|u_i\|_{\mathbf{r}+t_i\mathbf{e}_i,\mathbb{R}^n_+}\,.$$

This proves the continuity of the operator A.

To prove the continuity of B, we repeat the argument of Lemma 5.1 and obtain

$$B_{kj}(D)\big(H_{\mathbf{r}+t_j\mathbf{e}_j}(\mathbb{R}^n_+)\big) \subset H_{\mathbf{r}+t_j\mathbf{e}_j-(t_j+m_k)\mathbf{e}_1}(\mathbb{R}^n_+) \subset H_{\mathbf{r}-m_k\mathbf{e}_1}(\mathbb{R}^n_+).$$

By Theorem 4.1,

$$\gamma_0 B_{kj}(D)\big(H_{\mathbf{r}+t_j\mathbf{e}_j}(\mathbb{R}^n_+)\big) \subset H^{(-1/2)}_{\mathbf{r}-m_k\mathbf{e}_1}(\mathbb{R}^{n-1})\,,$$

and, by definition, the space on the right-hand side coincides with the space $H_{\mathbf{r}}^{(-m_k-1/2)}(\mathbb{R}^{n-1})$.

Theorem 5.4. *Let $A(D,q)$ and $B(D)$ be of the form* (1.1) *and* (2.3)*, respectively, with constant coefficients and without lower-order terms which act on functions on the half-space $\mathbb{R}^n_+$. Assume that A satisfies the Vishik–Lyusternik condition and that (A,B) is weakly parameter elliptic. Then, for $q \ge q_0 > 0$, the* a priori *estimate*

$$\sum_{i=1}^{N} \|u_i\|_{\mathbf{r}+t_i\mathbf{e}_i,\mathbb{R}^n_+} \le C\Bigg(\sum_{i=1}^{N} \|f_i\|_{\mathbf{r}-s_i\mathbf{e}_i,\mathbb{R}^n_+} + \sum_{j=1}^{R} \|g_j\|^{(-m_j-1/2)}_{\mathbf{r},\mathbb{R}^{n-1}} + \sum_{i=1}^{N_1} q^{r_2}\|u_i\|_{L_2(\mathbb{R}^n_+)} + \sum_{i=N_1+1}^{N} q^{r_2+t_i}\|u_i\|_{L_2(\mathbb{R}^n_+)}\Bigg) \tag{5.6}$$

holds, where $Au = f$ and $Bu = g$.

Proof. We follow the standard scheme from elliptic theory (cf. [5]).

(i) *Reduction to the case* $f = 0$. Choosing a cut-off function $\psi \in C^\infty(\mathbb{R}^n)$ such that $\psi(\xi) = 1$ for $|\xi| \le 1$ and $\psi(\xi) = 0$ for $|\xi| \ge 2$, we represent u in the form

$$u = u' + u'' + v, \qquad u' := [\psi(D)Eu]\Big|_{\mathbb{R}^n_+}, \qquad u'' := [(1-\psi(D))G(D,q)Ef]\Big|_{\mathbb{R}^n_+}. \tag{5.7}$$

Here E is the extension operator used in the proof of Lemma 5.3 and $G(\xi,q) := A^{-1}(\xi,q)$; we also use the pseudodifferential notation $\psi(D) := F^{-1}\psi(\xi)F$.

To estimate u', we note that

$$\begin{aligned}\sum_{i=1}^N \|u'_i\|_{\mathbf{r}+t_i\mathbf{e}_i,\mathbb{R}^n_+} &\le C\sum_{i=1}^N \|\psi(D)(Eu)_i\|_{\mathbf{r}+t_i\mathbf{e}_i,\mathbb{R}^n} \\ &\le C\Big(\sum_{i=1}^{N_1} q^{r_2}\|(Eu)_i\|_{L_2(\mathbb{R}^n)} + \sum_{i=N_1+1}^N q^{r_2+t_i}\|(Eu)_i\|_{L_2(\mathbb{R}^n)}\Big) \\ &\le C\Big(\sum_{i=1}^{N_1} q^{r_2}\|u_i\|_{L_2(\mathbb{R}^n_+)} + \sum_{i=N_1+1}^N q^{r_2+t_i}\|u_i\|_{L_2(\mathbb{R}^n_+)}\Big)\end{aligned} \tag{5.8}$$

for $q \ge q_0$, where we used the fact that $\psi(D)$ is infinitely smoothing.

To estimate u'', it suffices to consider the sum

$$\sum_{i=1}^N \|(1-\psi(D))(G(D,q)Ef)_i\|_{\mathbf{r}+t_i\mathbf{e}_i,\mathbb{R}^n}.$$

To this end, we use relation (5.3) with $d = F(Ef)(\xi)$ and $c = G(\xi,q)d$ and note that we can replace $\Phi(\xi,q)$ in (5.3) by $\Xi(\xi,q)$ because $1-\psi(\xi)$ vanishes for $|\xi| \le 1$. Integrating with respect to ξ, we obtain

$$\sum_{i=1}^N \|u''_i\|_{\mathbf{r}+t_i\mathbf{e}_i,\mathbb{R}^n_+} \le C\sum_{i=1}^N \|(Ef)_i\|_{\mathbf{r}-s_i\mathbf{e}_i,\mathbb{R}^n} \le C\sum_{i=1}^N \|f_i\|_{\mathbf{r}-s_i\mathbf{e}_i,\mathbb{R}^n}. \tag{5.9}$$

It remains to estimate $v = u - u' - u''$, which is a solution of the problem

$$A(D,q)\,v = 0 \quad \text{in} \quad \mathbb{R}^n_+, \qquad B(D)\,v = h \quad \text{on} \quad \mathbb{R}^{n-1} \tag{5.10}$$

with $h := g - B(u'+u'')$. Because of Lemma 5.3, the norm of $B(u'+u'')$ in the space

$$\prod_{j=1}^R H_{\mathbf{r}}^{(-m_j-1/2)}(\mathbb{R}^{n-1})$$

is not greater than a constant multiple of the sum

$$\sum_{i=1}^N \big(\|u'_i\|_{\mathbf{r}+t_i\mathbf{e}_i,\mathbb{R}^n_+} + \|u''_i\|_{\mathbf{r}+t_i\mathbf{e}_i,\mathbb{R}^n_+}\big).$$

Moreover,

$$\begin{aligned}\sum_{i=1}^{N_1} q^{r_2}\|v_i\|_{L_2(\mathbb{R}^n_+)} + \sum_{i=N_1+1}^N q^{r_2+t_i}\|v_i\|_{L_2(\mathbb{R}^n_+)} &\le \sum_{i=1}^{N_1} q^{r_2}\|u_i\|_{L_2(\mathbb{R}^n_+)} \\ &+ \sum_{i=N_1+1}^N q^{r_2+t_i}\|u_i\|_{L_2(\mathbb{R}^n_+)} + \sum_{i=1}^N \|u'_i\|_{\mathbf{r}+t_i\mathbf{e}_i,\mathbb{R}^n_+} + \sum_{i=1}^N \|u''_i\|_{\mathbf{r}+t_i\mathbf{e}_i,\mathbb{R}^n_+},\end{aligned}$$

and we can see from (5.8) and (5.9) that we must prove the inequality

$$\sum_{i=1}^{N} \|v_i\|_{\mathbf{r}+t_i\mathbf{e}_i,\mathbb{R}^n_+} \le C\Big(\sum_{j=1}^{R} \|h_j\|^{(-m_j-1/2)}_{\mathbf{r},\mathbb{R}^{n-1}} + \sum_{i=1}^{N_1} q^{r_2}\|v_i\|_{L_2(\mathbb{R}^n_+)} + \sum_{i=N_1+1}^{N} q^{r_2+t_i}\|v_i\|_{L_2(\mathbb{R}^n_+)}\Big)\,, \tag{5.11}$$

i.e., we may assume without loss of generality that $f=0$.

(ii) *Proof for the case* $f=0$. Let v be a solution of (5.10). We have the equivalences

$$\Xi_{\mathbf{r}+t_i\mathbf{e}_i}(\xi,q) \approx \Phi_{\mathbf{r}+t_i\mathbf{e}_i}(\xi,q) + \begin{cases} q^{r_2} & \text{for} \quad i\le N_1,\\ q^{r_2+t_i} & \text{for} \quad i>N_1. \end{cases} \tag{5.12}$$

$$\Phi_{\mathbf{r}+t_i\mathbf{e}_i}(\xi,q) \approx \Big[\sum_{l=0}^{r_1+r_2+t_i} \xi_n^{2l}\big(\Phi^{(-l)}_{\mathbf{r}+t_i\mathbf{e}_i}(\xi',q)\big)^2\Big]^{1/2}$$

(cf. [5], Section 2). Therefore,

$$\begin{aligned}\sum_{i=1}^{N} \|v_i\|_{\mathbf{r}+t_i\mathbf{e}_i,\mathbb{R}^n_+} \approx{}& \sum_{i=1}^{N_1} q^{r_2}\|v_i\|_{L_2(\mathbb{R}^n_+)} + \sum_{i=N_1+1}^{N} q^{r_2+t_i}\|v_i\|_{L_2(\mathbb{R}^n_+)}\\ &+\sum_{i=1}^{N}\Big[\sum_{l=0}^{r_1+r_2+t_i}\int_{\mathbb{R}^{n-1}}\|D_n^l(F'v_i)(\xi',\cdot)\|^2_{L_2(\mathbb{R}_+)}\big(\Phi^{(-l)}_{\mathbf{r}+t_i\mathbf{e}_i}(\xi',q)\big)^2 d\xi'\Big]^{1/2}.\end{aligned} \tag{5.13}$$

In (5.10) we take the partial Fourier transform with respect to $\xi'\in\mathbb{R}^{n-1}$ and use the unique solvability for $\xi'\neq 0$, see condition 2.6 (ii). We obtain

$$(F'v)(x_n,\xi',q) = \sum_{j=1}^{R}(F'h_j)(\xi')w_j(x_n,\xi',q)\,.$$

By Theorem 4.2 we have

$$\|D_n^l(F'v_i)(\cdot,\xi',q)\|^2_{L_2(\mathbb{R}_+)}\big(\Phi^{(-l)}_{\mathbf{r}+t_i\mathbf{e}_i}(\xi',q)\big)^2 \le C\sum_{j=1}^{R}|(F'h_j)(\xi')|^2\big(\Phi_{\mathbf{r}}^{(-m_j-1/2)}(\xi',q)\big)^2\,.$$

Integrating this inequality with respect to $\xi'\in\mathbb{R}^{n-1}$ and using (5.13), we obtain the inequality

$$\sum_{i=1}^{N} \|v_i\|_{\mathbf{r}+t_i\mathbf{e}_i,\mathbb{R}^n_+} \le C\Big(\sum_{i=1}^{N_1} q^{r_2}\|v_i\|_{L_2(\mathbb{R}^n_+)} + \sum_{i=N_1+1}^{N} q^{r_2+t_i}\|v_i\|_{L_2(\mathbb{R}^n_+)} + \sum_{j=1}^{R} \|h_j\|^{(-m_j-1/2)}_{\mathbf{r},\mathbb{R}^{n-1}}\Big)$$

which completes the proof of the theorem.

Note that, by Lemma 5.1 and Lemma 5.3, the *a priori* estimates in Theorems 5.2 and 5.4 (respectively) are two-sided.

Let us return to operators of the form (1.1) with boundary operators (2.3) acting on a smooth compact manifold M with smooth boundary ∂M as described in Section 2. Choosing finitely many coordinate systems, we obtain (in local coordinates) operators of the same form that act on $\mathbb{R}^n$ and $\mathbb{R}^n_+$, respectively. Roughly speaking, the fact that the *a priori* estimates established above for the model operators are two-sided allows us to obtain *a priori* estimates for operators with variable coefficients. We thus obtain the following theorem for which the proof needs no new ideas (cf. the proof of Theorem 5.6 in [5]).

Theorem 5.5. *Let the operators A and B of the form* (1.1) *and* (2.3)*, respectively, act on a smooth compact manifold M with smooth boundary ∂M. Assume that the operator $A(x,D,q)$ satisfies the Vishik–Lyusternik condition and that the boundary value problem (A,B) is weakly parameter elliptic in the sense of Definition* 2.6*. Then there exist constants q_0 and C, $q_0>0$ and $C>0$, that do not depend on q and u and for which the* a priori *estimate*

$$\begin{aligned}\sum_{i=1}^{N}\|u_i\|_{\mathbf{r}+t_i\mathbf{e}_i,M} \le C\Bigg(&\sum_{i=1}^{N}\|f_i\|_{\mathbf{r}-s_i\mathbf{e}_i,M}+\sum_{j=1}^{R}\|g_j\|_{\mathbf{r},\partial M}^{(-m_j-1/2)}\\ &+\sum_{i=1}^{N_1}q^{r_2}\|u_i\|_{L_2(M)}+\sum_{i=N_1+1}^{N}q^{r_2+t_i}\|u_i\|_{L_2(M)}\Bigg)\end{aligned}\tag{5.14}$$

holds, where $f=Au$ and $g=Bu$.

Remark 5.6. It follows from the proofs of the above results that the assumptions can be weakened as follows: the operator $A_{22}(x,D)-q^{2m}I_{N_2}$ can be replaced by an operator $A_{22}(x,D,q)$ which is elliptic with parameter in the sense of Agmon–Agranovich–Vishik. Note that this makes sense here because all entries on the diagonal of the Douglis–Nirenberg system A_{22} are of the same order, and thus the parameter has a definite weight in this operator matrix.

In this case the above results remain valid and their proofs are almost literally the same.

6. APPLICATION TO TRANSMISSION PROBLEMS

In the theory of elliptic boundary value problems with indefinite weight functions, a transmission problem arises in which the operator is elliptic in one part of the domain and elliptic with parameter in the other part. Such a problem was recently treated by Faierman [7]; we refer the reader to his paper for further references. In this section we show how the results of Section 4 can be applied to such problems. The *a priori* estimates obtained here differ from those in [7] and seem to be new. Moreover, in [7], only the canonical transmission conditions are treated (for the definition, see below), while our results imply the validity of *a priori* estimates under general transmission conditions. For simplicity, we restrict ourselves to the case of differential operators with constant coefficients that act on $\mathbb{R}^n$ and have no lower-order terms. The case of variable coefficients can be treated by the well-known method of localization, see also [7], where even nonsmooth coefficients are handled.

We thus assume that a transmission problem of the form

$$A_1(D)u_1(x)=f_1(x)\quad (x_n<0)\,,\tag{6.1}$$

$$A_2(D,q)u_2(x)=f_2(x)\quad (x_n>0)\tag{6.2,}$$

$$B_{j1}(D)u_1(x',0)+B_{j2}(D)u_2(x',0)=g_j(x')\quad (j=1,\dots,2m)\,,\tag{6.3}$$

is given Here A_1 and A_2 are scalar partial differential operators of the same order $2m$ with constant coefficients and B_{j1} and B_{j2} are boundary differential operators of order m_j for $j=1,\dots,2m$. We assume that these operators coincide with their principal parts and that

$$m_1\le\dots\le m_m<m_{m+1}\le\dots\le m_{2m}\,.\tag{6.4}$$

Assume that the following conditions hold:

(A1) The operator $A_1(D)$ is elliptic, i.e., $A_1(\xi)\neq 0$ for $\xi\in\mathbb{R}^n\setminus\{0\}$.

(A2) The operator $A_2(D,q)$ is elliptic with parameter in the sense of Agmon–Agranovich–Vishik along the ray $[0,\infty)$, i.e., $A_2(\xi,q)\neq 0$ for $(\xi,q)\in\mathbb{R}^n\times[0,\infty)$ with $|\xi|+q>0$.

(A3) For any $\xi' \in \mathbb{R}^{n-1} \setminus \{0\}$, $q \in [0, \infty)$, and $g = (g_1, \dots, g_{2m}) \in \mathbb{C}^{2m}$, the transmission problem on the real line

$$
\begin{aligned}
A_1(\xi', D_t) w_1(t) &= 0 \quad (t < 0)\,, \\
A_2(\xi', D_t, q) w_2(t) &= 0 \quad (t > 0)\,, \\
\Big[B_{j1}(\xi', D_t) w_1(t) + B_{j2}(\xi', D_t) w_2(t) \Big]\Big|_{t=0} &= g_j \quad (j = 1, \dots, 2m)\,, \\
w_1(t) \to 0 \quad (t \to -\infty)\,, \qquad w_2(t) &\to 0 \quad (t \to +\infty)
\end{aligned} \tag{6.5}
$$

has a unique solution.

(A4) The boundary value problem $(A_1, B_{11}, \dots, B_{1m})$ in $\mathbb{R}^n_-$ satisfies the classical Shapiro–Lopatinskii condition.

(A5) For any $q \in [0, \infty)$ and $(h_{m+1}, \dots, h_{2m}) \in \mathbb{C}^m$, the system of ordinary differential equations on $\mathbb{R}_+$

$$
\begin{aligned}
A_2(0, D_t, 1) v_2(t) = 0 \quad (t > 0)\,, \\
B_{j2}(0, D_t) v_2(t)\Big|_{t=0} = h_j \quad (j = m+1, \dots, 2m)\,, \\
v_2(t) \to 0 \quad (t \to +\infty)
\end{aligned} \tag{6.6}
$$

has a unique solution.

The following theorem is an analog of Theorem 5.4 for such transmission problems.

Theorem 6.1. *Suppose that assumptions* (A1)–(A5) *hold. Let* $\mathbf{r} = (r_1, r_2)$ *be a pair of real numbers for which* $r_1 \in [m_m + 1/2, m_{m+1} + 1/2]$ *and* $r_1 + r_2 \geq m_{2m} + 1$ *is an integer. Then there exist constants* q_0 *and* C, $q_0 > 0$, *independent of* q *and* u *and such that, for* $q \geq q_0$ *and for any solution* u_1, u_2 *of* (6.1)–(6.3), *we have the following* a priori *estimate*:

$$
\begin{aligned}
\|u_1\|_{\mathbf{r}, \mathbb{R}^n_-} + \|u_2\|_{\mathbf{r}, \mathbb{R}^n_+} \leq C \Big[& \|f_1\|_{(r_1 - 2m, r_2), \mathbb{R}^n_-} + \|f_2\|_{(r_1, r_2 - 2m), \mathbb{R}^n_+} \\
& + \sum_{j=1}^{2m} \|g_j\|^{(-m_j - 1/2)}_{\mathbf{r}, \mathbb{R}^{n-1}} + q^{r_2} \big(\|u_1\|_{L_2(\mathbb{R}^n_-)} + \|u_2\|_{L_2(\mathbb{R}^n_+)} \big) \Big].
\end{aligned} \tag{6.7}
$$

Proof. We set $v_1(x) := u_1(x', -x_n)$ and $v_2(x) := u_2(x)$ for $x \in \mathbb{R}^n_+$. Then $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is a solution of

$$
\begin{pmatrix} A_1(D', -D_n) & 0 \\ 0 & A_2(D', D_n, q) \end{pmatrix} \begin{pmatrix} v_1(x) \\ v_2(x) \end{pmatrix} = \begin{pmatrix} f_1(x', -x_n) \\ f_2(x) \end{pmatrix} \qquad (x \in \mathbb{R}^n_+),
$$

$$
\begin{pmatrix} B_{11}(D', -D_n) & B_{12}(D) \\ \vdots & \vdots \\ B_{2m,1}(D', -D_n) & B_{2m,2}(D) \end{pmatrix} \begin{pmatrix} v_1(x', 0) \\ v_2(x', 0) \end{pmatrix} = \begin{pmatrix} g_1(x') \\ \vdots \\ g_{2m}(x') \end{pmatrix} \qquad (x' \in \mathbb{R}^{n-1})\,.
$$

This problem is of the form (1.1), (2.3) with $N_1 = N_2 = 1$, $s_j = 2m$, and $t_j = 0$, where (μ, m) in (2.2) must now be replaced by $(m, 2m)$. Due to Remark 2.5, the Vishik–Lyusternik condition is satisfied. The conditions of Definition 2.6 follow from assumptions (A1)–(A5). Indeed, 2.6 (i) is a consequence of (A1) and (A2), 2.6 (ii) follows from (A3), and 2.6 (iii) coincides with (A4). To show that 2.6 (iv) holds, we must consider the following ODE on the half-line:

$$
\begin{aligned}
A_1(0, -D_t) v_1(t) &= 0 \quad (t > 0)\,, \\
A_2(0, D_t, 1) v_2(t) &= 0 \quad (t > 0)\,, \\
B_{j1}(0, -D_n) v_1(t) + B_{j2}(0, D_t) v_2(t)\Big|_{t=0} &= h_j \quad (j = m+1, \dots, 2m)\,, \\
v_1(t),\, v_2(t) &\to 0 \quad (t \to +\infty)\,.
\end{aligned} \tag{6.8}
$$

Since $A_1(0,-D_t) = \text{const}\, D_t^{2m}$, it follows that the function $v_1(t)$ is a polynomial in t whose degree does not exceed $2m-1$. Since $v_1(t) \to 0$ as $t \to +\infty$, it follows that this polynomial is identically zero. Thus, (6.8) is reduced to (6.6).

The canonical transmission conditions are given by

$$D_n^{j-1}u_1(x',0) - D_n^{j-1}u_2(x',0) = 0 \quad (j = 1,\dots,2m).$$

This case is treated in [7]. More generally, we consider the conditions

$$D_n^{j-1}u_1(x',0) - D_n^{j-1}u_2(x',0) = g_j(x') \quad (j = 1,\dots,2m). \tag{6.9}$$

For these transmission conditions, conditions (A1)–(A5) hold under very natural assumptions on A_1 and A_2. Moreover, in the *a priori* estimate, the $\|u_2\|_{L_2}$-term can be omitted, which follows from the result of Theorem 6.2 below. Here we restrict ourselves to the case in which $\mathbf{r} = (r_1, r_2) = (m, m)$ (this seems to be the most natural choice). Certainly, the corresponding results remain valid for more general parameters $\mathbf{r}$ (instead of (m,m)).

Theorem 6.2. *Suppose that the operator $A_1(D)$ is elliptic without parameter and that $A_2(D,q)$ is elliptic with parameter in the sense of Agmon–Agranovich–Vishik. Then there exist constants q_0 and C, $q_0 > 0$ and $C > 0$, that do not depend on $q \geq q_0$ and u and for which, for every solution u of the transmission problem* (6.1), (6.2), (6.9)*, the following* a priori *estimate holds*:

$$\begin{aligned}
&\|(1+|D|)^m(q+|D|)^m u_1\|_{L_2(\mathbb{R}^n_-)} + \|(1+|D|)^m(q+|D|)^m u_2\|_{L_2(\mathbb{R}^n_-)} \\
&\leq C\Bigg[\|(1+|D|)^{-m}(q+|D|)^m f_1\|_{L_2(\mathbb{R}^n_-)} + \|(1+|D|)^m(q+|D|)^{-m} f_2\|_{L_2(\mathbb{R}^n_+)} \\
&\qquad + \sum_{j=1}^{m} \|(1+|D'|)^{m-j+1/2}(q+|D'|)^m g_j\|_{L_2(\mathbb{R}^{n-1})} \\
&\qquad + \sum_{j=m+1}^{2m} \|(q+|D'|)^{2m-j+1/2} g_j\|_{L_2(\mathbb{R}^{n-1})} + q^m \|u_1\|_{L_2(\mathbb{R}^n_-)}\Bigg].
\end{aligned} \tag{6.10}$$

For $g_1 = \cdots = g_{2m} = 0$ in (6.10), we recover the estimate obtained by Igor' Fedotov (unpublished).

Proof. a) Let us show first that conditions (A1)–(A5) are satisfied for the transmission problem (6.1), (6.2), (6.9). Since conditions (A1) and (A2) hold by assumption and condition (A4) means that the Dirichlet problem for the scalar elliptic operator $A_1(D)$ satisfies the Shapiro–Lopatinskii condition, it remains to consider (A3) and (A5).

Starting from (A5), we note that $A(0,\tau,1)$ has m roots in the upper half-plane of the complex plane. Therefore, problem (6.6) (with $B_{2j} = D_t^{j-1}$ for $j = m+1,\dots,2m$) has a unique solution.

To verify condition (A3), we use an argument in [2], Section 7. Denote by $\mathfrak{M}_1$ the m-dimensional subspace of all solutions of equation $A_1(\xi', D_t)z(t) = 0$ that tend to zero as $t \to -\infty$ and choose a basis $z_1,\dots,z_m$ in $\mathfrak{M}_1$. Similarly, we choose a basis $z_{m+1},\dots,z_{2m}$ of the space $\mathfrak{M}_2$ of all solutions of $A_2(\xi', D_t, q)z(t) = 0$ that tend to zero as $t \to \infty$. In this case, it is obvious that $\{z_1,\dots,z_{2m}\}$ is a set of solutions of the following ordinary differential equation on the entire line:

$$A_{1,-}(\xi', D_t)A_{2,+}(\xi', D_t, q)z(t) = 0 \tag{6.11}$$

where $A_{1,-}(\xi',\tau)$ and $A_{2,+}(\xi',\tau,q)$ stand for the products

$$A_{1,-}(\xi',\tau) := \prod_{j=m+1}^{2m}(\tau - \tau_{1j}(\xi')) \quad \text{and} \quad A_{2,+}(\xi',\tau,q) := \prod_{j=1}^{m}(\tau - \tau_{2j}(\xi',q)),$$

respectively, τ_{1j} $(j = m+1, \dots, 2m)$ stand for the zeros of $A_1(\xi', \cdot)$ with negative imaginary parts and τ_{2j} $(j = 1, \dots, m)$ for the zeros of $A_2(\xi', \cdot, q)$ with positive imaginary parts.

Moreover, the set $\{z_1, \dots, z_{2m}\}$ is linearly independent, i.e., it forms a fundamental system for the ODE (6.11), which is of order $2m$. Indeed, in the case of linear dependence, there would exist constants $c_1, \dots, c_{2m} \in \mathbb{C}$ such that

$$\sum_{j=1}^{m} c_j z_j = \sum_{j=m+1}^{2m} c_j z_j \,, \tag{6.12}$$

where both sides of (6.12) are nontrivial. However, since $z_j(t) \to 0$ as $t \to -\infty$ provided that $j \le m$ and $z_j(t) \to 0$ as $t \to +\infty$ provided that $j > m$, it would follow that both sides of (6.12) are bounded on the entire line, i.e., that (6.11) would have a nontrivial bounded (stable) solution. But this is impossible because all roots of A_1 and A_2 have nonvanishing imaginary part.

Now let v_1, v_2 be a solution of (6.5). Then there exist constants $d_1, \dots, d_{2m} \in \mathbb{C}$ such that

$$v_1 = \sum_{j=1}^{m} d_j z_j \quad \text{and} \quad v_2 = -\sum_{j=m+1}^{2m} d_j z_j \,.$$

The transmission conditions

$$D_t^{j-1}(v_1(t) - v_2(t))\Big|_{t=0} = g_j \qquad (j = 1, \dots, 2m)$$

are equivalent to the system

$$\begin{pmatrix} z_1(0) & \cdots & z_{2m}(0) \\ \vdots & & \vdots \\ D_t^{2m-1} z_1(0) & \cdots & D_t^{2m-1} z_{2m}(0) \end{pmatrix} \begin{pmatrix} d_1 \\ \vdots \\ d_{2m} \end{pmatrix} = \begin{pmatrix} g_1 \\ \vdots \\ g_{2m} \end{pmatrix} . \tag{6.13}$$

Since the Wronskian of the fundamental system $\{z_1, \dots, z_{2m}\}$ is nonzero, it follows that the linear system (6.13) is uniquely solvable, which completes the proof of (A3).

b) It follows from part a) of the proof that Theorem 6.1 can be applied to the transmission problem (6.1), (6.2), (6.9). For $\mathbf{r} = (m, m)$ we obtain an estimate of the form (6.10) with the additional term $q^m \|u_2\|_{L_2(\mathbb{R}^n_+)}$ on the right-hand side. To complete the proof, we must estimate this term. To this end, we note that u_2 is a solution of the problem

$$A_2(D, q) u_2(x) = f_2(x) \quad (x_n > 0) \,, \tag{6.14}$$

$$D_n^{j-1} u_2(x', 0) = -g_j(x') + D_n^{j-1} u_1(x', 0) \quad (j = 1, \dots, m) \,. \tag{6.15}$$

By assumption, the operator $A_2(D, q)$ is elliptic with parameter. Since the Dirichlet boundary conditions $\{D_n^{j-1} : j = 1, \dots, m\}$ are absolutely elliptic, we can apply the Agmon–Agranovich–Vishik theory to problem (6.14)–(6.15). We thus obtain (see [3], Theorem 3.1)

$$\begin{aligned} \|(q + |D|)^{2m} u_2\|_{L_2(\mathbb{R}^n_+)} \le C \Bigg(& \|f_2\|_{L_2(\mathbb{R}^n_+)} + \sum_{j=1}^{m} \|(q + |D'|)^{2m-j+1/2} g_j\|_{L_2(\mathbb{R}^{n-1})} \\ & + \sum_{j=1}^{m} \|(q + |D'|)^{2m-j+1/2} D_n^{j-1} u_1(x', 0)\|_{L_2(\mathbb{R}^{n-1})} \Bigg) . \end{aligned}$$

By [3], Proposition 3.1, we have

$$\sum_{j=1}^{m} \|(q + |D'|)^{2m-j+1/2} D_n^{j-1} u_1(x', 0)\|_{L_2(\mathbb{R}^{n-1})} \le C \|(q + |D|)^{2m} u_1\|_{L_2(\mathbb{R}^n_-)} \,.$$

It follows from the last two inequalities that

$$
\begin{aligned}
q^m\|u_2\|_{L_2(\mathbb{R}^n_+)} \le C\Bigg(& \|q^{-m}f_2\|_{L_2(\mathbb{R}^n_+)} + \sum_{j=1}^{m} \|q^{-m}(q+|D'|)^{2m-j+1/2}g_j\|_{L_2(\mathbb{R}^{n-1})} \\
& + \|q^{-m}(q+|D|)^{2m}u_1\|_{L_2(\mathbb{R}^n_-)}\Bigg) .
\end{aligned}
\tag{6.16}
$$

Now let us estimate the right-hand side of the last inequality via the right-hand side of (6.10). Note that

$$
q^{-m}(q+|\xi|)^m(1+|\xi|)^{-m} = \Big(\frac{q+|\xi|}{q+q|\xi|}\Big)^m \le 1
$$

for $q \ge q_0 > 1$ and $\xi \in \mathbb{R}^n$, and the first term on the right-hand side of (6.16) can be estimated via the second term on the right-hand side of (6.10). In the same way we obtain

$$
\frac{q^{-m}(q+|\xi'|)^{2m-j+1/2}}{(1+|\xi'|)^{m-j+1/2}(q+|\xi'|)^m} = q^{-j+1/2}\Big(\frac{q+|\xi'|}{q+q|\xi'|}\Big)^{m-j+1/2} \le q^{-j+1/2} < 1,
$$

and thus the sum on the right-hand side of (6.16) can be estimated via the third term in (6.10).

Let us show that, for each $q_0 > 1$, there is a constant $C(q_0)$ such that the inequality

$$
\|q^{-m}(q+|D|)^{2m}u_1\|_{L_2(\mathbb{R}^n_-)} \le \|(1+|D|)^m(q+|D|)^m u_1\|_{L_2(\mathbb{R}^n_-)} + C(q_0)q^m\|u_1\|_{L_2(\mathbb{R}^n_-)}
$$

holds for $q \ge q_0$. To prove this inequality, we first find a constant $C(q_0)$ such that

$$
q^{-m}(q+|\xi|)^{2m} \le (1+|\xi|)^m(q+|\xi|)^m + C(q_0)q^m \tag{6.17}
$$

for $q \ge q_0 > 1$ and $\xi \in \mathbb{R}^n$. To this end, note that

$$
\lim_{|\xi|\to\infty} \frac{q^{-m}(q+|\xi|)^{2m}}{(1+|\xi|)^m(q+|\xi|)^m} = q^{-m} \le q_0^{-m} < 1 .
$$

Hence, there exists $R = R(q_0)$ such that the left-hand side of (6.17) does not exceed the first term on the right-hand side of (6.17) for $|\xi| \ge R(q_0)$. For $|\xi| \le R(q_0)$, the left-hand side of (6.17) is not greater than the second term on the right-hand side if $C(q_0) = (1 + R(q_0)/q_0)^{2m}$.

Therefore, all terms on the right-hand side of (6.16) can be estimated by the right-hand side of (6.10), which completes the proof of the theorem.

Remark 6.3. The proof of Theorem 6.2 can also be performed by repeating the proof of Theorem 5.4 for the diagonal system

$$
\begin{pmatrix} A_1(D', -D_n) & 0 \\ 0 & A_2(D', -D_n, q) \end{pmatrix} .
$$

In this case, one must replace the smoothing operator $\psi(D)I_2$, which appears in the decomposition (5.7), by the operator

$$
\begin{pmatrix} \psi(D) & 0 \\ 0 & 1 \end{pmatrix} .
$$

This replacement immediately yields inequality (6.10), i.e., the desired estimate without the term $q^m\|u_2\|_{L_2(\mathbb{R}^n_+)}$.

7. REMARKS ON SINGULAR PERTURBATION PROBLEMS

Now we return to the singularly perturbed problem (1.3). As in the previous section, we assume for simplicity that A_{ij} is a scalar differential operator of order $2m$ with constant coefficients and without lower-order terms that acts on the half-space $\mathbb{R}^n_+$. To make the expressions homogeneous, we replace ε by ε^{2m} in (1.3). Then we obtain the system

$$\begin{aligned} A_{11}(D)\,u_1 + A_{12}(D)\,u_2 &= 0 \quad \text{in} \quad \mathbb{R}^n_+, \\ \varepsilon^{2m}(A_{21}(D)\,u_1 + A_{22}(D)\,u_2) - u_2 &= 0 \quad \text{in} \quad \mathbb{R}^n_+. \end{aligned} \tag{7.1}$$

We assume that the boundary conditions are of the form

$$b_{j1}(D)u_1 + b_{j2}(D)u_2 = g_j \quad (j = 1, \dots, 2m) \quad \text{on} \quad \mathbb{R}^{n-1}. \tag{7.2}$$

We define $B(D)$ by (2.3) for $N_1 = N_2 = 1$ and replace (μ, m) in (2.3) by $(m, 2m)$. We again assume that $B(D)$ has constant coefficients and no lower-order terms.

Definition 7.1. *The boundary value problem* (7.1)–(7.2) *is said to be weakly parameter elliptic if* $(A(D,\lambda), B(D))$ *is weakly parameter elliptic in the sense of Definition* 2.6*, where* $A(D,\lambda)$ *is given by* (1.1).

By Remark 2.5, the Vishik–Lyusternik condition is automatically satisfied in this case. As was mentioned above, all results of the previous sections can equivalently be reformulated in the context of boundary value problems with small parameter. In particular, weak parameter ellipticity implies uniform *a priori* estimates for the solutions u_1, u_2 of (7.1)–(7.2) (these estimates are uniform as $\varepsilon \to 0$). Since the required modifications in the assertions are obvious, we do not formulate these results explicitly. The aim of the present section is to show the connection between the conditions of weak parameter ellipticity and the existence and structure of boundary layers for the singularly perturbed problem (7.1)–(7.2), see also Remark 2.7 b).

Let $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ be a solution of problem (7.1)–(7.2). Multiplying the second equation in (7.1) by ε^{2m}, we see that $(F'u)(t,\xi')$ is a solution of problem (2.10)–(2.11) for $q = \varepsilon^{-1}$ and $g := (F'h)(\xi')$. By Lemma 3.4, for sufficiently small ε, this solution is a sum of two integrals. Since the second integral in (3.8) contains the exponential term $e^{itq\tau} = e^{it\tau/\varepsilon}$, it follows that this integral describes a boundary layer. The existence of this boundary layer term is a consequence of condition 2.6 (iv), which can be seen from Lemma 3.3. On the other hand, the first integral in (3.8) is described in Lemma 3.2, which uses condition 2.6 (iii). Therefore, the above considerations show that conditions (iii) and (iv) in Definition 2.6 (which are in a sense nonstandard) lead to a splitting of the solution of the singularly perturbed problem into a regular part and a boundary-layer part.

The remainder of this section is devoted to the construction of a formal asymptotic solution (FAS) of the boundary value problem (7.1)–(7.2), i.e., of a formal series

$$u(x,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k u^{(k)}(x,\varepsilon)\,, \quad u^{(k)}(x,\varepsilon) = \begin{pmatrix} u_1^{(k)}(x,\varepsilon) \\ u_2^{(k)}(x,\varepsilon) \end{pmatrix}$$

for which the partial sums $\sum_{k=0}^{N} \varepsilon^k u^{(k)}$ satisfy (7.1)–(7.2) up to terms of order $O(\varepsilon^N)$. We claim that conditions 4.2 (iii) and (iv) are again substantial for this construction.

Let us use terminology similar to that in [14]. We say that $(A_{11}(D), B_{11}(D))$ (see (2.12)–(2.13)) is the first limit problem and (2.14)–(2.15) is the second limit problem. Under the condition of weak parameter ellipticity, the first limit problem is elliptic. To avoid technical difficulties, let us also assume that it is uniquely solvable. If this is not the case, one must work with kernels and cokernels of the operator related to this boundary value problem.

Following Vishik–Lyusternik [16] (see also [11]), we seek the solution in the form $u(x,\varepsilon) = v(x,\varepsilon) + w(x,\varepsilon)$, where

$$v(x,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k v^{(k)}(x), \quad v^{(k)}(x) = \begin{pmatrix} v_1^{(k)}(x) \\ v_2^{(k)}(x) \end{pmatrix} \tag{7.3}$$

is the so-called exterior expansion and

$$w(x,\varepsilon)=\sum_{k=0}^{\infty}\varepsilon^{m+k}w^{(k)}\Big(x',\frac{x_n}{\varepsilon}\Big),\quad w^{(k)}(x)=\begin{pmatrix}w_1^{(k)}(x)\\ w_2^{(k)}(x)\end{pmatrix}\tag{7.4}$$

is the so-called interior expansion or boundary layer. Our objective is to find partial differential equations and boundary conditions determining the functions $v^{(k)}$ and $w^{(k)}$.

We begin with the equations in the interior of the domain $\mathbb{R}^n_+$. In the following, we set

$$A(D):=A(D,0)=\begin{pmatrix}A_{11}(D) & A_{12}(D)\\ A_{21}(D) & A_{22}(D)\end{pmatrix}.$$

(i) *Differential equations for* $v^{(k)}$. Substituting (7.3) into (7.1), we obtain

$$\begin{aligned}&\left[\begin{pmatrix}1 & 0\\ 0 & \varepsilon^{2m}\end{pmatrix}A(D)-\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\right]\sum_{k=0}^{\infty}\varepsilon^k\begin{pmatrix}v_1^{(k)}\\ v_2^{(k)}\end{pmatrix}\\ &=\sum_{k=0}^{\infty}\varepsilon^k\left[\begin{pmatrix}A_{11}(D) & A_{12}(D)\\ 0 & -1\end{pmatrix}\begin{pmatrix}v_1^{(k)}\\ v_2^{(k)}\end{pmatrix}+\begin{pmatrix}0 & 0\\ A_{21}(D) & A_{22}(D)\end{pmatrix}\begin{pmatrix}v_1^{(k-2m)}\\ v_2^{(k-2m)}\end{pmatrix}\right],\end{aligned}$$

where

$$v^{(k)}:=0\quad\text{for}\quad k=-2m,-2m+1,\dots,-1.\tag{7.5}$$

Thus, we obtain the recurrence relations

$$A_{11}(D)v_1^{(k)}=-A_{12}(D)\Big(A_{21}(D)\quad A_{22}(D)\Big)v^{(k-2m)}\quad(k=0,1,2,\dots),\tag{7.6}$$

$$v_2^{(k)}=\Big(A_{21}(D)\quad A_{22}(D)\Big)v^{(k-2m)}\quad(k=0,1,2,\dots).\tag{7.7}$$

In order to determine $v^{(k)}$ (with the initial values (7.5)), we must impose m boundary conditions on $v_1^{(k)}$, see below.

(ii) *Differential equations for* $w^{(k)}$. To find the corresponding equations for $w^{(k)}$, we note that

$$A(D)\Big[w^{(k)}\Big(x',\frac{x_n}{\varepsilon}\Big)\Big]=\varepsilon^{-2m}\Big[A(\varepsilon D',D_n)w^{(k)}\Big]\Big(x',\frac{x_n}{\varepsilon}\Big)$$

because of the homogeneity. Substituting (7.4) into (7.1), we obtain

$$\begin{aligned}&\left[\begin{pmatrix}1 & 0\\ 0 & \varepsilon^{2m}\end{pmatrix}A(D)-\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\right]\sum_{k=0}^{\infty}\varepsilon^{k+m}w^{(k)}\Big(x',\frac{x_n}{\varepsilon}\Big)\\ &=\begin{pmatrix}\varepsilon^{-m} & 0\\ 0 & \varepsilon^m\end{pmatrix}\sum_{k=0}^{\infty}\varepsilon^k\left(\left[A(\varepsilon D',D_n)-\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\right]w^{(k)}\right)\Big(x',\frac{x_n}{\varepsilon}\Big).\end{aligned}\tag{7.8}$$

Now we expand $A(\varepsilon D',D_n)$ in a Taylor series with respect to ε,

$$A(\varepsilon D',D_n)=\sum_{\ell=0}^{2m}\varepsilon^\ell A^{(\ell)}(D',D_n),$$

where $A^{(0)}(D',D_n)=A(0,D_n)$ and $A^{(2m)}(D',D_n)$ is a constant complex 2×2 matrix.

Substituting this expansion into the last sum in (7.8), we see that this sum is equal to

$$\sum_{k=0}^{\infty}\varepsilon^k\left[\left(A^{(0)}(D)-\begin{pmatrix}0&0\\0&1\end{pmatrix}\right)w^{(k)}+\sum_{\ell=1}^{2m}A^{(\ell)}(D)w^{(k-\ell)}\right],$$

where

$$w^{(j)}:=0\quad(j=-2m,-2m+1,\dots,-1)\,. \tag{7.9}$$

Therefore, we obtain the recurrence relations

$$\left[A^{(0)}(D)-\begin{pmatrix}0&0\\0&1\end{pmatrix}\right]w^{(k)}=-\sum_{\ell=1}^{2m}A^{(\ell)}(D)w^{(k-\ell)}\quad(k=0,1,2,\dots) \tag{7.10}$$

(iii) *Boundary conditions.* Now let us find boundary conditions for the functions $v_1^{(k)}$ and $w^{(k)}$ for $k=0,1,\dots$. By setting $b_j(D):=\begin{pmatrix}b_{j1}(D)&b_{j2}(D)\end{pmatrix}$ $(j=1,\dots,2m)$, we immediately see that

$$b_j(D)\sum_{k=0}^{\infty}\varepsilon^k v^{(k)}(x',0)=\sum_{k=0}^{\infty}\varepsilon^k b_j(D)v^{(k)}(x',0)$$

and, by homogeneity,

$$b_j(D)w^{(k)}\Big(x',\frac{x_n}{\varepsilon}\Big)\Big|_{x_n=0}=\varepsilon^{-m_j}b_j(\varepsilon D',D_n)w^{(k)}(x',x_n)\Big|_{x_n=0}\,.$$

Therefore,

$$\begin{aligned}B(D)&\left[\sum_{k=0}^{\infty}\varepsilon^k v^{(k)}+\sum_{k=0}^{\infty}\varepsilon^{m+k}w^{(k)}\Big(x',\frac{x_n}{\varepsilon}\Big)\right]\Bigg|_{x_n=0}\\&=\sum_{k=0}^{\infty}\varepsilon^k B(D)v^{(k)}(x',0)+\sum_{k=0}^{\infty}\begin{pmatrix}\varepsilon^{-m_1}&&\\&\ddots&\\&&\varepsilon^{-m_{2m}}\end{pmatrix}\varepsilon^{m+k}B(\varepsilon D',D_n)w^{(k)}(x',0)\,.\end{aligned}$$

We use the Taylor expansion with respect to ε again,

$$b_j(\varepsilon D',D_n)=\sum_{\ell=0}^{m_j}\varepsilon^\ell b_j^{(\ell)}(D',D_n)\,,$$

where $b_j^{(0)}(D',D_n)=b_j(0,D_n)$. Thus,

$$\begin{aligned}b_j(D)(v+w)&=\sum_{k=0}^{\infty}\left[\varepsilon^k b_j(D)v^{(k)}+\varepsilon^{-m_j+m+k}\sum_{\ell=0}^{m_j}\varepsilon^\ell b_j^{(\ell)}(D)w^{(k)}\right]\\&=\sum_{k=\min\{m-m_j,0\}}\varepsilon^k\left[b_j(D)v^{(k)}+\sum_{\ell=0}^{m_j}b_j^{(\ell)}(D)w^{(k+m_j-m-\ell)}\right].\end{aligned} \tag{7.11}$$

Note that negative powers of ε can occur here; for negative values of k in $v^{(k)}$ and $w^{(k)}$, we use (7.5) and (7.9).

The conditions $b_j(v+w) = g_j$ for $j = 1, \ldots, 2m$ lead to the boundary conditions

$$(k,j) \qquad b_j(D)v^{(k)} + \sum_{\ell=0}^{m_j} b_j^{(\ell)}(D) w^{(k+m_j-m-\ell)} = \delta_{k0}\, g_j$$
$$\text{for} \quad k = \min\{m-m_j, 0\}, \min\{m-m_j, 0\}+1, \ldots \quad \text{and} \quad j = 1, \ldots, 2m.$$

This set of boundary conditions is enumerated by using the indices k and j. At the first glance, it can seem unclear how conditions (k,j) determine the functions $v^{(k)}$ and $w^{(k)}$. This is the subjectmatter of the following theorem in which, for simplicity, we restrict ourselves to the special case

$$\text{ord}\, B_j = j-1 \quad (j = 1, \ldots, 2m)\,. \tag{7.12}$$

Theorem 7.2. *Let the boundary value problem* (7.1)–(7.2) *be weakly parameter elliptic and let the first limit problem be uniquely solvable. Assume further that* (7.12) *holds. Then the recursion formulas* (7.5)–(7.7), (7.9)–(7.10) *with boundary conditions* (k,j) *uniquely determine the functions* $v^{(k)}$ *and* $w^{(k)}$ *for* $k = 0, 1, 2 \ldots.$ *The boundary conditions for* $v_1^{(k)}$ *and* $w^{(k)}$ *have the form*

$$B_{11}(D)v_1^{(k)}(x', 0) = g^{(k)}(x') \tag{7.13}$$

$$B_2(0, D_n)w^{(k)}(x', 0) = h^{(k)}(x')\,, \tag{7.14}$$

respectively. The right-hand side of (7.13) *contains only* $v^{(\ell)}$ *and* $w^{(\ell)}$ *for* $\ell < k$*, while the right-hand side of* (7.14) *contains only* $v^{(\ell)}$ *for* $\ell \le k$ *and* $w^{(\ell)}$ *for* $\ell < k$*. In particular, the right-hand sides can be determined recursively.*

Note that the case in which (7.12) fails can be treated by using the same idea, but it is somewhat more delicate.

Proof. By the recursion formulas (7.6)–(7.7) and (7.10) with initial values (7.5) and (7.9), we must only show that the boundary condition (k,j) determines $v_1^{(k)}$ and $w^{(k)}$.

For each pair of indices (k,j) that occurs in the above boundary condition, Figure 2 shows a certain function for which this condition is used. (The index pairs filled with vertical lines do not occur at all). The main question is in the order in which the formula (k,j) must be applied.

$k \downarrow$ \ $j \longrightarrow$	1	2	$\cdots$	m	$m{+}1$	$\cdots$	$\cdots$	$2m$
$-m{+}1$								$w^{(0)}$
$\vdots$							$w^{(0)}$	$w^{(1)}$
-1						$w^{(0)}$	$w^{(1)}$	$\vdots$
0	$v^{(0)}$	$v^{(0)}$	$v^{(0)}$	$v^{(0)}$	$w^{(0)}$	$w^{(1)}$	$\vdots$	
1	$v^{(1)}$	$v^{(1)}$	$v^{(1)}$	$v^{(1)}$	$w^{(1)}$	$\vdots$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$			

Fig. 2. The usage of formula (k,j).

We shall use the formulas (k, j) with $j \le m$ as boundary conditions for $v^{(k)}$ and the formulas $(k - j + 1 - m, j)$ with $j = m + 1, \dots, 2m$ as boundary conditions for $w^{(k)}$. We must prove the statement concerning the right-hand sides of (7.13) and (7.14).

Let us assume that, at some step k, we already know the functions $v^{(\ell)}$ and $w^{(\ell)}$ for $\ell < k$. We intend to find $v^{(k)}$ and $w^{(k)}$. Let $j \le m$. In this case, condition (k, j) contains the functions

$$v^{(k)}, w^{(k+j-m-1)}, w^{(k+j-m-2)}, \dots, w^{(k-m)}\,.$$

Since $k + j - m - 1 < k$ and since $v_2^{(k)}$ is defined by (7.7), which contains only $v^{(k-2m)}$, it follows that the only unknown function in condition (k, j) is $v_1^{(k)}$. We obtain (7.13) in which

$$g_j^{(k)} := \delta_{k0}\, g_j - b_{j2}(D) u_2^{(k)} - \sum_{\ell=0}^{j-1} b_j^{(\ell)}(D) w^{(k+j-1-m-\ell)} \quad (j = 1, \dots, m)\,.$$

Since the first limit problem is uniquely solvable by assumption, it follows that the function $v_1^{(k)}$ is uniquely determined by (7.6) and (7.13).

Let $j > m$. Then the boundary condition $(k - j + 1 - m, j)$ contains the functions

$$w^{(k)}, w^{(k-1)}, \dots, w^{(k-j+1)} \quad \text{and} \quad v^{(k-j+m+1)}.$$

Since $k - j + m + 1 \le k$ and since we already know the functions $v^{(\ell)}$ for $\ell \le k$, this gives m boundary conditions for $w^{(k)}$. Since the second limit problem is uniquely solvable by assumption, it follows that the function $w^{(k)}$ is determined by these boundary conditions, and we can pass to step $k + 1$. The boundary conditions for $w^{(k)}$ are of the form (7.14) with

$$h_j^{(k)} = \delta_{k-j+m+1,0}\, g_j - b_j(D) v^{(k-j+m+1)} - \sum_{\ell=1}^{j-1} b_j^{(\ell)}(D) w^{(k-\ell)} \quad (j = m + 1, \dots, 2m)\,.$$

Summing up, we see that, if we take formulas (k, j) as boundary conditions for $v^{(k)}$ and $w^{(k)}$ as is indicated in Figure 2, then the right-hand sides of (7.13) and (7.14) can be determined recursively.

Remark 7.3. By Theorem 7.2, the terms in the asymptotic expansion are defined as solutions of the first and second limit problem. In particular, condition 2.6 (iv), i.e., the unique solvability of the second limit problem, leads to boundary layer terms, while condition 2.6 (iii), i.e., the unique solvability of the first limit problem, corresponds to the exterior expansion. This shows that conditions (iii) and (iv) are very natural in a sense (namely, from the point of view of singular perturbations).

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