Automorphism groups of Hahn groups and Hahn fields

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Abstract

Let k be a field and G a totally ordered abelian group. Let $\mathbb{K} = k((G))$ be the field of generalised power series with coefficients in k and exponents in G equipped with the canonical valuation v [Hah07]. The main subject of this thesis (Chapter 3) is the study of the group $v\text{-}\operatorname{Aut} K$ of valuation preserving automorphisms of a subfield $k(G) \subseteq K \subseteq \mathbb{K}$, which we call a *Hahn field*. The interest in Hahn fields stems from the fact that they are, in some sense, universal for both valued [Kap42] and real closed fields [MR93]. Moreover, generalised power series play an important role in automata theory [BR88].
Our main results on automorphisms of Hahn fields are the following:

1. We introduce a first notion of *lifting property*. This is the possibility to lift a pair of automorphisms $(\rho, \tau) \in \operatorname{Aut} k \times o\text{-}\operatorname{Aut} G$ to an automorphism $\sigma \in v\text{-}\operatorname{Aut} K$. For a Hahn field with the lifting property we decompose $v\text{-}\operatorname{Aut} K$ into the semi-direct product of its subgroups of *internal* and *external* automorphisms, thereby generalising a result of [Hof91].

2. We introduce a second notion of lifting property, i.e. the possibility to lift a homomorphism $x \in \operatorname{Hom}(G, k^\times)$ to an internal automorphism of K. For a Hahn field with the second lifting property we further decompose the group of internal automorphisms into the semi-direct product of its subgroups of *1-automorphisms* and *G-exponentiations*.

3. We define a *canonical version* of both the aforementioned lifting properties. We study the important class of *Rayner fields*, prove that they all satisfy the second canonical lifting property, and characterise those that satisfy the first canonical lifting property.

4. Combining the results mentioned so far, we establish a general structure theorem that presents the group $v\text{-}\operatorname{Aut} K$ as a 4-factor semi-direct product, for a Hahn field K satisfying both lifting properties. This generalises a result obtained by [Des05] for the field of Puiseux series.

5. A *strongly additive* automorphism of K is one that commutes with infinite sums. For such automorphisms we find an even more precise decomposition. We describe the group of strongly additive automorphisms purely in terms of the valuation invariants of K: the residue field k, the value group

G, the homomorphisms $\mathrm{Hom}(G, k^\times)$ between them, and the homomorphisms $\mathrm{Hom}(G, 1 + I_K)$ of G into the group of *1-units* of the valuation ring of K.

A more general notion, related to that of a Hahn field, is the concept of *Hahn group*. Like Hahn fields are universal among valued and real closed fields, a result of [Hah07] shows that Hahn groups are universal among ordered abelian groups. For Hahn groups we obtain some analogue results to those listed above. A similar notion of lifting property allows to obtain a semi-direct product decomposition of the group $v\text{-}\operatorname{Aut} G$ of valuation preserving automorphisms of a Hahn group G. We introduce *Rayner groups* and prove a criterion for them to satisfy the lifting property. For the special case of *Hahn sums*, we represent the group of *order preserving automorphisms* as a special group of matrices, extending results of [Con58; DG97]. This will also play a role in the explicit description of the group of strongly additive automorphisms of some special Hahn fields.

Zusammenfassung (German Summary)

Seien k ein Körper und G eine total angeordnete abelsche Gruppe. Sei $\mathbb{K} = k((G))$ der Körper der verallgemeinerten formalen Potenzreihen mit Koeffizienten aus k und Exponenten aus G zusammen mit der kanonischen Bewertung v [Hah07]. Der Fokus dieser Dissertation (Kapitel 3) ist die Untersuchung der Gruppe $v\text{-}\operatorname{Aut} K$ der bewertungstreuen Automorphismen eines Teilkörpers der Form $k(G) \subseteq K \subseteq \mathbb{K}$, den wir *hahnscher Körper* nennen. Das Interesse an hahnschen Körpern begründet sich darin, dass sie „universell" sind, sowohl als bewertete Körper [Kap42] als auch als reell abgeschlossene Körper [MR93]. Darüber hinaus spielen verallgemeinerte formale Potenzreihen eine wichtige Rolle in der Automatentheorie [BR88].
Für hahnsche Körper erhalten wir folgende Hauptergebnisse:

1. Wir führen den Begriff der *ersten Hebungseigenschaft* ein. Dies ist die Möglichkeit, ein beliebiges Paar $(\rho, \tau) \in \operatorname{Aut} k \times o\text{-}\operatorname{Aut} G$ zu einem Automorphismus $\sigma \in v\text{-}\operatorname{Aut} K$ zu heben. Für einen hahnschen Körper mit der ersten Hebungseigenschaft geben wir eine Zerlegung von $v\text{-}\operatorname{Aut} K$ als semi-direktes Produkt ihrer Teilgruppen der *internen* and *externen* Automorphismen an. Damit wird ein Satz von [Hof91] verallgemeinert.

2. Die *zweite Hebungseigenschaft* ist die Möglichkeit, einen beliebigen Homomorphismus $x \in \mathrm{Hom}(G, k^\times)$ zu einem internen Automorphismus von K zu heben. Mit dieser Eigenschaft zerlegen wir die Gruppe der internen Automorphismen als semi-direktes Produkt ihrer Teilgruppen der sogenannten *1-Automorphismen* und *G-Exponentiationen*.

3. Eine *kanonische Version* beider obengenannten Hebungseigenschaften wird definiert. Wir untersuchen die Klasse der *raynerschen Körper* und zeigen, dass sie die kanonische zweite Hebungseigenschaft immer erfüllen. Ferner geben wir eine Charakterisierung für die raynerschen Körper an, die die zweite Hebungseigenschaft erfüllen.

4. Mithilfe aller obengenannten Resultate beweisen wir einen allgemeinen Struktursatz: Wir stellen die Gruppe v- $\mathrm{Aut}\, K$ als ein 4-Faktor-semi-direktes Produkt für einen hahnschen Körper K dar, der beiden Hebungseigenschaften erfüllt. Wir erhalten damit eine Verallgemeinerung eines Satzes von [Des05] über den Körper der Puiseux-Reihen.

5. Ein *streng additiver* Automorphismus von K ist ein Automorphismus, der mit unendlichen Summen kommutiert. Für solche Automorphismen ergibt sich eine genauere Zerlegung. Die Gruppe der streng additiven Automorphismen wird in Abhängigkeit von der Bewertungsinvarianten von K dargestellt: Der Restklassenkörper k, die Wertegruppe G, die Gruppe der Homomorphismen $\mathrm{Hom}(G, k^\times)$, und die Gruppe $\mathrm{Hom}(G, 1 + I_K)$ der Homomorphismen von G in die Gruppe der *1-Einheiten* des Bewertungsrings von K.

Hahnsche Gruppen verallgemeinern den Begriff eines hahnschen Körper. Ähnlich wie hahnsche Körper sind nach einem Satz von [Hah07] hahnsche Gruppen „universell" für angeordnete abelsche Gruppen. Ein ähnlicher Begriff von Hebungseigenschaft erlaubt eine Darstellung als semi-direktes Produkt der Gruppe v- $\mathrm{Aut}\, G$ der bewertungstreuen Automorphismen einer hahnschen Gruppe G. Wir führen *raynersche Gruppen* ein und geben ein Kriterium an, damit sie die Hebungseigenschaft erfüllen. Im speziellen Fall einer *hahnschen Summe* stellen wir die Gruppe der *ordnungstreuen Automorphismen* als eine spezielle Matrixgruppe dar. Diese Darstellung wird eine wichtige Rolle in der Beschreibung der Gruppe der streng additiven Automorphismen mancher spezieller hahnscher Körper spielen.

Sommario (Italian summary)

Sia k un campo e sia G un gruppo abeliano totalmente ordinato. Sia $\mathbb{K} = k((G))$ il campo delle serie di potenze formali generalizzate a coefficienti in k ed esponenti in G dotato della valutazione canonica v [Hah07]. L'obiettivo principale di questa tesi (Capitolo 3) è lo studio del gruppo v- Aut K degli automorfismi (che preservano la valutazione) di un campo $k(G) \subseteq K \subseteq \mathbb{K}$, che chiameremo *campo di Hahn*. L'importanza dei campi di Hahn risiede nel fatto che essi siano "universali" sia tra i campi con valutazione [Kap42] che tra i campi realmente chiusi [MR93]. Inoltre, le serie di potenze generalizzate giocano un ruolo importante nella teoria degli automi [BR88].

I risultati chiave di questo lavoro sono i seguenti.

1. Introduciamo una *prima proprietà di sollevamento*, ossia, la possibilità di sollevare una coppia di automorfismi $(\rho, \tau) \in \operatorname{Aut} k \times o\text{-}\operatorname{Aut} G$ ad un automorfismo $\sigma \in v\text{-}\operatorname{Aut} K$. Per un campo di Hahn con la prima proprietà di sollevamento, scomponiamo il gruppo v- Aut K nel prodotto semidiretto dei suoi sottogruppi di automorfismi *interni* ed *esterni*, generalizzando così un risultato di [Hof91].

2. La *seconda proprietà di sollevamento* consiste nel poter sollevare un omomorfismo $x \in \operatorname{Hom}(G, k^{\times})$ ad un automorfismo di K. Per un campo di Hahn K con la seconda proprietà di sollevamento scomponiamo il gruppo degli automorfismi interni nel prodotto semidiretto dei suoi sottogruppi degli *1-automorfismi* e delle *G-esponenziazioni*.

3. Per ciascuna proprietà di sollevament definiamo una versione *canonica*. Studiamo poi i *campi di Rayner* e dimostriamo che essi godono della seconda proprietà di sollevamento canonica. Forniamo inoltre una caratterizzazione dei campi di Rayner che soddisfano la prima proprietà di sollevamento canonica.

4. Combinando i risultati sopracitati dimostriamo un teorema di struttura generale per v- Aut K che descriviamo come un prodotto semidiretto a quattro fattori, per un campo di Hahn K che soddisfi entrambe le proprietà di sollevamento. In tal modo generalizziamo un risultato di [Des05] per il campo delle serie di Puiseux

5. Un automorfismo è detto *fortemente additivo* se rispetta le somme infinite. Per tali automorfismi otteniamo delle decomposizioni ancora più precise.

> Il gruppo degli automorfismi fortemente lineari viene descritto in termini degli invarianti di valutazione di K: il campo dei residui k, il gruppo dei valori G, il gruppo degli omomorfismi fra essi $\mathrm{Hom}(G, k^{\times})$, e quello degli omomorfismi tra G e il gruppo delle *1-unità* dell'anello di valutazione di K: $\mathrm{Hom}(G, 1 + I_K)$.

Il concetto di *gruppo di Hahn* generalizza quello di campo di Hahn sopra descritto. Un teorema di [Hah07] mostra che, analogamente ai campi di Hahn, anche i gruppi di Hahn sono oggetti universali, per i gruppi abeliani ordinati. Risultati analoghi a quelli descritti per i campi di Hann sono ottenuti per i gruppi di Hahn. Una simile nozione di proprietà di sollevamento consente una scomposizione del gruppo v- Aut G degli automorpfismi di un gruppo di Hahn G che rispettano la valutazione. Introduciamo anche i cosiddetti *gruppi di Rayner* e forniamo un criterio affinché soddisfino la proprietà di sollevamento. Nel caso particolare delle *somme di Hahn*, forniamo una rappresentazione del *gruppo degli isomorfismi d'ordine di G* come uno speciale gruppo di matrici, generalizzando così risultati di [Con58; DG97]. Tale scomposizione giocherà un ruolo importante nel calcolo esplicito del gruppo degli automorfismi di alcuni casi particolari di campi di Hahn.

Contents

Introduction

General overview and literary background

Let k be a field, G a totally ordered abelian group and consider the set $\mathbb{K} := k((G))$ consisting of all the formal expressions of the form $a = \sum_{g \in G} a_g t^g$ such that the *support* $\operatorname{supp} a := \{g \in G : a_g \neq 0\}$ is a well ordered subset of G. It is a classical result of Hahn [Hah07] that $\mathbb{K}$, endowed with term-by-term addition and multiplication given by the usual convolution formula, forms a field. If $k[G]$ is the subring of $\mathbb{K}$ consisting of elements with finite support, let $k(G)$ be the fraction field of $k[G]$. A *Hahn field* is any field K such that $k(G) \subseteq K \subseteq \mathbb{K}$ (Definition 3.1.4). Hahn fields are further endowed with the *canonical valuation* v, which has value group G and is given by $v(a) = \min \operatorname{supp} a$ for all $a \in K^\times$. Well studied examples of Hahn fields are the field $\mathbb{L} = k((t))$ of Laurent series and the field $\mathbb{P} \subsetneq k((\mathbb{Q}))$ of Puiseux series.

Hahn fields are important objects of study in several areas of mathematics because of their universality. Kaplansky [Kap42; Kap45] showed that every equi-characteristic valued field is isomorphic to a suitable Hahn field (see Theorem 3.1.12). Mourgues and Ressayre [MR93] proved that every real closed field can be embedded into a maximal Hahn field, in such a way that the image is truncation closed (Definition *A*.1.6).

Various authors studied automorphisms of Hahn fields in special cases. In [Sch44], Schilling studied the automorphisms of the field $\mathbb{L} = k((t))$ of Laurent series. He proved that the group $\operatorname{Aut}_k \mathbb{L}$ of all k-automorphisms of $\mathbb{L}$ is isomorphic to the group $U_{\mathbb{L}}$ of units of the valuation ring of $\mathbb{L}$, provided one endows $U_{\mathbb{L}}$ with a suitable group operation. The automorphisms of Puiseux series over the complex numbers were studied by Webb in his thesis [Web68]. He provided a matrix representation for those automorphisms that are continuous in a certain topology[1]. Hofberger [Hof91] considered maximal Hahn fields of the form $k((G))$ and introduced the notions of *internal* and *external* automorphisms. He showed that the group $v\text{-}\operatorname{Aut} k((G))$ is given by the semi-direct product of its subgroups of internal and external automorphisms. Deschamps [Des05] also studies the valuation preserving automorphisms of a field $\mathbb{P}$ of Puiseux series over an algebraically closed field. For this field he refines Hofberger's decomposition splitting also the group of internal automorphisms into a semi-direct product of what we will call 1-*automorphisms* and G-*exponentiations*.

The notion of a Hahn field generalises to that of a Hahn group. Let Γ be a chain (i.e. a totally ordered set) and, for each $\gamma \in \Gamma$, let A_γ be an abelian group. We call $[\Gamma; \{A_\gamma : \gamma \in \Gamma\}]$ an *ordered system of abelian groups*. Consider the cartesian product $\Pi := \prod_{\gamma\in\Gamma} A_\gamma$ with the usual pointwise addition and let the *support* of an element $a = (a_\gamma)_{\gamma\in\Gamma} \in \Pi$ be $\operatorname{supp} a = \{\gamma \in \Gamma : a_\gamma \neq 0\}$. The subgroups $\coprod_{\gamma\in\Gamma} A_g = \{a \in \Pi : \operatorname{supp} a \text{ is finite}\}$ and $\mathbf{H}_{\gamma\in\Gamma} A_\gamma = \{a \in \Pi : \operatorname{supp} a \text{ is well ordered}\}$ are called the *Hahn sum* and the *Hahn product* of the ordered system $[\Gamma; \{A_\gamma : \gamma \in \Gamma\}]$. A *Hahn group* is a group G such that $\coprod A_\gamma \leq G \leq \mathbf{H}\, A_\gamma$. By *Hahn's embedding theorem* [Hah07] every ordered abelian group is isomorphic to a suitable Hahn group. In particular, a group G appearing as the exponent group of a Hahn field $K \subseteq k((G))$ is a Hahn group. On a Hahn group G we can define a valuation v, with value set Γ, similar to the one defined on Hahn fields: $v(a) = \min \operatorname{supp} a$ for all $a \in G \setminus \{0\}$. Moreover, if each A_γ is an ordered group, we can order G lexicographically and v is compatible with the lexicographic order.

Conrad [Con58] studied the group $o\text{-}\operatorname{Aut} G$ of order preserving automorphisms of a Hahn sum $G = \coprod A_\gamma$. He described those automorphisms that *induce the identity on* Γ (see Subsection 2.2.1) as infinite $\Gamma \times \Gamma$ triangular matrices, whose entry $\sigma_{\gamma\delta}$ is a homomorphism $\sigma_{\gamma\delta} \colon A_\gamma \to A_\delta$ and whose diagonal elements are order preserving automorphisms: $\sigma_{\gamma\gamma} \in o\text{-}\operatorname{Aut} A_\gamma$. Droste and Göbel [DG97] built on Conrad's results. For a *balanced Hahn sum* G (i.e., $A_\gamma = A$ for all γ and a given abelian group A) they expressed the group $o\text{-}\operatorname{Aut} G$ as a semi-direct product of Conrad's matrix group and the group of automorphisms that are *lifts* of automorphisms of Γ .

[1]The product topology when regarding Puiseux series as infinite tuples of complex numbers.

Outline of the thesis' structure and main results

The objective of this thesis is to provide a comprehensive study of the groups of valuation preserving automorphisms of Hahn groups and Hahn fields. In the general overview we talked about Hahn fields first and then moved to Hahn groups. However, the content and results of this thesis will benefit from an exposition carried out in the opposite order. Here is how the work will be organised.

In Chapter 1 we provide the reader with the background that will be needed to follow the rest of the dissertation. We start with establishing some basic notation and terminology that will be used throughout the whole thesis and recall some fundamental facts on Algebra, in particular on divisible groups and semidirect products (Sections 1.1 and 1.2). The remainder of Chapter 1 is devoted to laying the foundations on valued and ordered fields and modules.

Chapter 2 is devoted to the study of automorphisms of Hahn groups.

In Section 2.1 we introduce the notion of Hahn group, define the canonical valuation $v = v_{\min}$ and recall some fundamental properties thereof. We then begin the actual study of the group $v\text{-}\operatorname{Aut} G$ of valuation preserving automorphisms of a Hahn group G. An automorphism $\sigma \in v\text{-}\operatorname{Aut} G$ induces an automorphism $\sigma_{S(G)}$ on the *skeleton* $S(G)$ of G (Definition 1.3.9) and, in particular, one on the *rank* Γ of G (Definition 1.4.1), in case G is ordered.

In Section 2.2 we study two notions of *lifting property*.
In Subsection 2.2.1 we introduce the lifting property with respect to the rank, for an ordered Hahn group G. We characterise automorphisms of the rank that *lift* to valuation preserving automorphisms of G (Definition 2.2.1) in terms of the *principal convex subgroups of* G (Theorem 2.2.6). Moreover, we show that if two Hahn groups G_1, G_2 have the lifting property w.r.t. their rank, this needs not be true for $G_1 \amalg G_2$; however, we give sufficient conditions on an automorphism of the rank of $G_1 \amalg G_2$ in order to lift (Proposition 2.2.8).
A stronger lifting property is studied in Subsection 2.2.2. First we introduce the important normal subgroup $\operatorname{Int}\operatorname{Aut} G \trianglelefteq v\text{-}\operatorname{Aut} G$ of *internal automorphisms of* G. In order to find a complement of $\operatorname{Int}\operatorname{Aut} G$ we give the central definition: we say that G *has the lifting property with respect to the skeleton* if every automorphism $\tau \in \operatorname{Aut} S(G)$ lifts to an automorphism of G (Definition 2.2.12). If G has the lifting property w.r.t. the skeleton we can identify the important subgroup $\operatorname{Ext}\operatorname{Aut} G$ of external automorphisms.

Subsection 2.2.3 contains the main result of this chapter. Theorem 2.2.17 shows that if G has the lifting property w.r.t. the skeleton then the group $v\text{-}\operatorname{Aut} G$ is the semi-direct product $\operatorname{Int}\operatorname{Aut} G \rtimes \operatorname{Ext}\operatorname{Aut} G$. This generalises to Hahn groups a result of Hofberger on maximal Hahn fields [Hof91, Satz 2.2].
The interesting class of *Rayner groups* is studied in Subsection 2.2.4. These are Hahn groups obtained by taking all the elements of a Hahn product whose support satisfies certain conditions (Definition 2.2.21). In Proposition 2.2.24 we characterise Rayner groups that satisfy the lifting property w.r.t. the skeleton. This allows us to obtain several examples of Hahn groups with the lifting property.

Section 2.3 focusses on the special case of Hahn sums. These particular Hahn groups have the advantage of admitting *valuation bases* (Definition 1.3.16).
In Subsection 2.3.1 we show how valuation bases can be used to determine whether an automorphism of G is internal, or whether an automorphism of Γ lifts to G (Propositions 2.3.4 and 2.3.6).
Subsection 2.3.2 is devoted to the study of order preserving automorphisms of an ordered Hahn sum $G = \coprod A_\gamma$. Building on results of [Con58] we identify some important subgroups of $o\text{-}\operatorname{Aut} G$ with matrix groups. In particular, we find matrix descriptions for the group $\operatorname{Int} o\text{-}\operatorname{Aut} G$ of internal order preserving automorphisms (Lemma 2.3.14) and for the group $\operatorname{Ext} o\text{-}\operatorname{Aut}_\Gamma G$ of external order preserving automorphisms that induce the identity on Γ (Lemma 2.3.17). This allows us to present the group $o\text{-}\operatorname{Aut}_\Gamma G$, of order preserving automorphisms that induce the identity on Γ as a semi-direct product of two groups of matrices (Theorem 2.3.19): we thereby improve on a result of [DG97] on balanced Hahn sums. The results of this subsection will be applied in Subsections 3.5.1 and 3.5.4 to provide explicit descriptions of the group of some automorphism groups of special Hahn fields (more on this below).

We conclude Chapter 2 with a brief section on *strong additivity* (Section 2.4), which is the property of an automorphism of commuting with infinite sums (Definition 2.4.3). We will study this notion in much greater detail in Section 3.4, in the field case. Here we will just prove that the canonical lifts (w.r.t. the skeleton) are strongly additive (Proposition 2.4.4) as well as all the valuation preserving automorphisms of a Hahn product whose value group is $\mathbb{Z}$ (Corollary **??**).

The above results will also appear in [KS21a] (currently in preparation).

Chapter 3 is concerned with the study of valuation preserving automorphisms of Hahn fields. This is where the main results of this thesis are to be found.

In Section 3.1 we explain how Hahn fields are constructed, we introduce the canonical valuation v and establish some general properties. We also recall an important characterisation of real closed Hahn fields (Corollary 3.1.15).

In Section 3.2 we introduce the main objects of study: the groups $v\text{-}\operatorname{Aut} K$ (resp. $v\text{-}\operatorname{Aut}_{(k)} K$ and $v\text{-}\operatorname{Aut}_k K$) of valuation preserving automorphisms (resp. k-stable automorphisms and k-automorphisms) of a Hahn field K (we introduced Hahn fields above, or see Definition 3.1.4).

In Section 3.3 we introduce two notions of lifting property, which allow us to obtain several decomposition theorems.

Subsection 3.3.1 is devoted to the study of the first lifting property (Definition 3.3.7). For an arbitrary Hahn field K we introduce the normal subgroup $\operatorname{Int}\operatorname{Aut} K$ of $v\text{-}\operatorname{Aut} K$ (Definition 3.3.4). If, moreover, K satisfies the first lifting property, we can define the complement $\operatorname{Ext}\operatorname{Aut} K$ of $\operatorname{Int}\operatorname{Aut} K$ (Definition 3.3.10).

The main result of Subsection 3.3.2 is Theorem 3.3.13, which generalises Hofberger's decomposition. For any Hahn field K with the first lifting property we can decompose the group of valuation preserving automorphisms as $v\text{-}\operatorname{Aut} K = \operatorname{Int}\operatorname{Aut} K \rtimes \operatorname{Ext}\operatorname{Aut} K$.

In Subsection 3.3.3 we introduce the second lifting property. For an arbitrary Hahn field K we introduce the normal subgroup $1\text{-}\operatorname{Aut} K$ of 1-automorphisms of $\operatorname{Int}\operatorname{Aut} K$ (Definition 3.3.18). If, moreover, K satisfies the second lifting property, we can define the complement $G\text{-}\operatorname{Exp} K$ which we call the group of G-exponentiations (Definition 3.3.21).

In Subsection 3.3.4, Theorem 3.3.27 provides a decomposition of $\operatorname{Int}\operatorname{Aut} K$ into a semi-direct product analogue to Theorem 3.3.13. For a Hahn field K satisfying the second lifting property we have $\operatorname{Int}\operatorname{Aut} K = 1\text{-}\operatorname{Aut} K \rtimes G\text{-}\operatorname{Exp} K$.

Subsection 3.3.5 is dedicated to a special version of the first lifting property: the canonical first lifting property. This allows to simplify the decomposition of the group $v\text{-}\operatorname{Aut}_{(k)} K$ of k-stable automorphisms, as will be shown in Subsection 3.3.7.

In Subsection 3.3.6 we present the class of Rayner fields, first introduced in [Ray68]. All these fields satisfy the (canonical) second lifting property and include, for example, the κ-*bounded* Hahn fields (Example 3.3.45) introduced in [All62] and, in particular, the maximal Hahn fields. The main result of this section is a characterisation of Rayner fields with the (canonical) first lifting property (Proposition 3.3.47). This ensures that the results that we prove for Hahn fields satisfying one of (or both) the lifting properties, actually apply to a broad spectrum of Hahn fields[2].

In Subsection 3.3.7, combining results from the previous sections, we obtain two decompositions into a 4-factor semi-direct product. One for the groups $v\text{-}\operatorname{Aut} K$ and $v\text{-}\operatorname{Aut}_k K$ for a Hahn field K with the first and (canonical) second lifting property (Theorem 3.3.52) and one for $v\text{-}\operatorname{Aut}_{(k)} K$ under the further assumption, that K has the canonical first lifting property (Proposition 3.3.53).

Section 3.4 focuses on strong additivity.

In Subsection 3.4.1 we show that Hofberger's decomposition also holds if we restrict to the group $v\text{-}\operatorname{Aut}^+ K$ of strongly additive automorphisms of K (Proposition 3.4.13 and Proposition 3.4.15). This way we obtain a detailed description of $v\text{-}\operatorname{Aut}^+ K$ and its subgroups $v\text{-}\operatorname{Aut}^+_{(k)} K$, $v\text{-}\operatorname{Aut}^+_k K$ (Theorem 3.4.16), which is the main result of this subsection.

Subsection 3.4.2 is devoted to a deeper investigation of the factor $1\text{-}\operatorname{Aut}^+_k K$ appearing in Theorem 3.4.16. To do this we introduce the *K-summable homomorphisms* $\operatorname{Hom}^+(G, 1 + I_K)$ (Definition 3.4.19) from the value group into the group of 1-units of the valuation ring. The main result of the section, Theorem 3.4.24, provides a decomposition of $v\text{-}\operatorname{Aut}^+_{(k)} K$ and $v\text{-}\operatorname{Aut}^+_k K$ purely in terms of the valuation invariants of K.

Section 3.5 is devoted to the explicit description of the automorphism groups in some special cases.

In Subsection 3.5.1 we study the case of a Hahn field $K \subseteq k((G))$ with a finitely generated exponent group, of the form $G = \mathbb{Z}^n$. In this case, if K has the canonical second lifting property, we can explicitly describe $G\text{-}\operatorname{Exp} K$ in terms of k and the number n of generators of G (Theorem 3.5.2). If $G = \mathbb{Z}^n$ is ordered lexicographically, we can moreover represent $o\text{-}\operatorname{Aut} G$ as a group of matrices, appealing to results from Section 2.3. Applying this, we give a description of the groups $v\text{-}\operatorname{Aut}^+_{(k)} K$ and $v\text{-}\operatorname{Aut}^+_k K$, which depends solely on k, $1 + I_K$ and n (Theorem 3.5.6).

In Subsection 3.5.2 we apply the results from Subsection 3.5.1 to the field $\mathbb{L}$ of Laurent series. We notice that $v\text{-}\operatorname{Aut} \mathbb{L} = v\text{-}\operatorname{Aut}^+ \mathbb{L}$ and obtain a precise description of $v\text{-}\operatorname{Aut}_{(k)} \mathbb{L}$ in terms of $1 + I_{\mathbb{L}}$, $k^\times$ and $\operatorname{Aut} k$ (Theorem 3.5.10). Schilling's result on the k-automorphisms of $\mathbb{L}$ can be derived as a special case (Corollary 3.5.12). We also provide a sharpening of Theorem 3.5.10 for the case of an ordered coefficient field k to characterise the group $o\text{-}\operatorname{Aut} \mathbb{L}$ of order preserving automorphisms.

[2] For a deeper analysis of Rayner structures see [KKS21], also summarised in Appendix A.

Subsection 3.5.3 is also dedicated to the case $G = \mathbb{Z}$, but focusses on the field $k(\mathbb{Z})$. The group $\mathrm{Cr}_1(k) := \mathrm{Aut}_k\, k(\mathbb{Z})$ is the *Cremona group in dimension* 1 *over* k. It is of fundamental importance in algebraic geometry, and well understood. We identify the groups $v\text{-}\mathrm{Aut}_k\, k(\mathbb{Z})$ and $1\text{-}\mathrm{Aut}_k\, k(\mathbb{Z})$ as subgroups of $\mathrm{Cr}_1(k)$.
In Subsection 3.5.4 we study the case of a Hahn field $K \subseteq k((G))$ where G is a totally ordered, divisible, abelian group, of finite dimension d over $\mathbb{Q}$ and k a real closed field. In analogy to Subsection 3.5.1, we provide a description of $o\text{-}\mathrm{Aut}\, G$ as a group of rational matrices and, under the extra assumption that K is henselian, we give an explicit description of the groups $v\text{-}\mathrm{Aut}^+_{(k)} K$ and $v\text{-}\mathrm{Aut}^+_k K$, depending only on k, $1 + I_K$ and d (Theorem 3.5.20).

The above results will appear in [KS21b], which is close to submission.
In Subsection 3.5.5 we consider the field $\mathbb{P}$ of Puiseux series. After observing that $v\text{-}\mathrm{Aut}\,\mathbb{P} = v\text{-}\mathrm{Aut}^+\,\mathbb{P}$ (Proposition 3.5.21), we apply our general results and retrieve decompositions analogous to those appearing in [Des05].

Appendix A contains results from the joint work [KKS21] with L. S. Krapp and S. Kuhlmann. It consists of a further analysis of the conditions defining a Rayner group (resp. a Rayner field) and on the implications of each of them. For a maximal Hahn field $\mathbb{K} = k((G))$ and a family $\mathcal{F}$ of well ordered subsets of G, the *khull* $k((\mathcal{F}))$ of $\mathcal{F}$ is the set of all power series $\mathbb{K}$ whose support belongs to $\mathcal{F}$. Rayner [Ray68] proved that some conditions (see Page 90) are sufficient for $k((\mathcal{F}))$ to be a subfield of $\mathbb{K}$. We provide *necessary and sufficient* conditions in order that $k((\mathcal{F}))$ is a subgroup, a subring, a subfield or indeed a Hahn field.

Chapter 1

Preliminaries

In this chapter we provide the reader with the background that will be necessary to develop the further treatment. This is not meant to be an exhaustive survey of the treated topics: the only aim here is to keep this work self contained. References to the relevant literature will be given whenever needed.

1.1 Basic notation and terminology

In this section we establish some basic conventions about notation and terminology that we adopt all throughout the thesis, unless specified otherwise.

1.1.1 Mathematics

- *Ordering* will always mean *total ordering*.
- *Ring* will always mean *ring with unity*.
- $0 \in \mathbb{N}$. The set of positive integers will be denoted by $\mathbb{Z}^{>0}$ or $\mathbb{N}^{>0}$, depending on the context.
- Composition of functions will be denoted by juxtaposition: $\sigma\tau := \sigma \circ \tau$.

1.1.2 Typography

- We will number chapters, sections and subsections: for example here we are in Chapter 1, Section 1.1, Subsection 1.1.2.

- The statements will be numbered by 3 digits according to section. So "Statement $x.y.z$" will mean Section $x.y$, Statement z.
- The important formulas will be numbered by two digits, in brackets, according to chapter. So "$(x.y)$" will denote the y-th formula in Chapter x.
- The end of a proof, an example, a remark, or a statement whose proof is omitted will be marked with a blank square: $\square$.
- Sometimes a long proof will be split into "claims". The end of the proof of a claim will be marked with a black diamond: $\blacklozenge$.

1.2 Algebraic preliminaries

In this section we recall some notions and results from general Algebra, mostly group theory, that we will use in the sequel. No claim of completeness or originality is made.

Definition 1.2.1. Let $(G,*)$ and $(H,\star)$ be groups and denote by e the neutral element of H. We denote by $\mathrm{Hom}(G,H)$ the group of homomorphisms of G into H with pointwise multiplication: for $\eta,\vartheta \in \mathrm{Hom}(G,H)$ the *$\star$-product* $\eta \star \vartheta$ is the homomorphism defined by $(\eta \star \vartheta)(g) := \eta(g) \star \vartheta(g)$ for all $g \in G$. The neutral element of $\mathrm{Hom}(G,H)$ is the map $\mathbf{e}$ defined by $\mathbf{e}(g) = e$ for all $g \in G$.

Often H will be an additive (resp. multiplicative) group. In these cases we will denote the operation $\star$ by $+$ (resp. $\cdot$) and talk about the *sum* (resp. *product*) of η and ϑ.

Proposition 1.2.2. *Let G, I, J be groups, on which we use the additive notation.*

$$\mathrm{Hom}(G, I \times J) = \mathrm{Hom}(G,I) \times \mathrm{Hom}(G,J).$$

Proof. Let us write $A := \mathrm{Hom}(G,I), B := \mathrm{Hom}(G,J)$ and $C := \mathrm{Hom}(G, I \times J)$. Define a map

$$\xi \colon A \times B \to C,\ (\sigma,\tau) \mapsto \rho$$

where $\rho \colon G \to I \times J$ is defined by $\rho(g) = (\sigma(g),\tau(g))$ for all $g \in G$. Then ξ is a group homomorphism: let $(\sigma,\tau),(\sigma',\tau') \in A \times B$ and let $g \in G$. Then we have

$$\begin{aligned}\xi((\sigma,\tau)+(\sigma',\tau'))(g) &= \xi(\sigma+\sigma',\tau+\tau')(g)\\ &= ((\sigma+\sigma')(g),(\tau+\tau')(g))\\ &= (\sigma(g)+\sigma'(g),\tau(g)+\tau'(g))\\ &= (\sigma(g),\tau(g))+(\sigma'(g),\tau'(g))\\ &= \xi(\sigma,\tau)(g)+\xi(\sigma',\tau')(g).\end{aligned}$$

The homomorphism ξ is surjective. Indeed, let $\rho \in C$ and let $i\colon G \to I$ and $j\colon G \to J$ be the projections of ρ on I and J respectively, so that, for all $g \in G$, we have $\rho(g) = (i(g), j(g))$. Then we obviously have $\xi(i,j) = \rho$. Hence ξ is surjective.
Finally, we have $\ker\xi = \{(0,0)\}$. Indeed, if $\xi(\sigma,\tau)(g) = (\sigma(g),\tau(g)) = (0,0)$ for all $g \in G$, then by definition both σ and τ are zero. Therefore ξ is an isomorphism. □

1.2.1 Divisible groups

Definition 1.2.3 (Divisible group)**.** Let G be an abelian group and let us use the additive notation on it. We say that G is *divisible* if, for all $g \in G$ and for all $n \in \mathbb{N}$ with $n \neq 0$ there exists $h \in G$ such that $nh = g$. Moreover, G is *uniquely divisible* if for all $g \in G$ and for all $n \in \mathbb{N}$ with $n \neq 0$ there exists **a unique** $h \in G$ such that $nh = g$. Such an element h will be denoted by g/n.

Remark 1.2.4. A uniquely divisible group is a vector space over the field $\mathbb{Q}$. For all $m/n \in \mathbb{Q}$ and all $g \in G$ the multiplication by scalars is defined as $(m/n)g := m(g/n)$. □

Proposition 1.2.5. *Let G be an abelian group, denoted additively. Then G is uniquely divisible if and only if it is divisible and torsion-free.*

Proof. Let G be uniquely divisible. In particular, G is divisible, so we need to show that G is torsion-free. Let $h \in G$ and $n \in \mathbb{N}$ be such that $n \neq 0$ and $nh = 0$. By unique divisibility (applied to $g = 0$) we must have $h = 0$.
Let G be divisible and torsion free. Let $g, h_1, h_2 \in G$ and $n \in \mathbb{N}$ be such that $n \neq 0$ and $g = nh_1 = nh_2$. Then $n(h_1 - h_2) = 0$. Since G is torsion-free, this implies $h_1 = h_2$, so G is uniquely divisible. □

Proposition 1.2.6. *Let G and H be uniquely divisible groups and $\vartheta \in \operatorname{Hom}(G, H)$ a group homomorphism. Then ϑ is a linear map of $\mathbb{Q}$-vector spaces.*

Proof. By Remark 1.2.4, G and H are $\mathbb{Q}$-vector spaces. Let $g, g' \in G$ and let $q \in \mathbb{Q}$. Since ϑ is a group homomorphism, it follows that $\vartheta(g + g') = \vartheta(g) + \vartheta(g')$. It remains to show that $\vartheta(qg) = q\vartheta(g)$. Write $q = m/n$ for some $m, n \in \mathbb{Z}$. Then $\vartheta(qg) = m\vartheta\left(\frac{1}{n}g\right)$. Now, $n\vartheta\left(\frac{1}{n}g\right) = \vartheta(g)$ and $\vartheta\left(\frac{1}{n}g\right)$ is the unique element of H with this property, because H is uniquely divisible. Thus $\vartheta\left(\frac{1}{n}g\right) = \frac{1}{n}\vartheta\left(g\right)$ and so $\vartheta(qg) = \frac{m}{n}\vartheta(g) = q\vartheta(g)$. □

Corollary 1.2.7. *Let G be a uniquely divisible group, finite dimensional as a $\mathbb{Q}$-vector space, and let $d = \dim_{\mathbb{Q}} G$. Let H be a uniquely divisible group. Then we have* $\operatorname{Hom}(G, H) \simeq H^d := \prod_{i=1}^{d} H$.

Proof. Let $\{g_1, \ldots, g_n\}$ be a basis of G over $\mathbb{Q}$. Define a map

$$\begin{array}{rccc} \Theta\colon & \operatorname{Hom}(G, H) & \longrightarrow & H^d \\ & \vartheta & \longmapsto & (\vartheta(g_1), \ldots, \vartheta(g_d)). \end{array}$$

Then Θ is a group homomorphism. Indeed, for $\eta, \vartheta \in \operatorname{Hom}(G, H)$ we have

$$\begin{aligned} \Theta(\eta + \vartheta) &= ((\eta + \vartheta)(g_1), \ldots, (\eta + \vartheta)(g_d)) \\ &= (\eta(g_1) + \vartheta(g_1), \ldots, \eta(g_d) + \vartheta(g_d)) \\ &= (\eta(g_1), \ldots, \eta(g_d)) + (\vartheta(g_1), \ldots, \vartheta(g_d)) \\ &= \Theta(\eta) + \Theta(\vartheta). \end{aligned}$$

Moreover, Θ is injective. Indeed, if $\eta, \vartheta \in \operatorname{Hom}(G, H)$ coincide on the basis $\{g_1, \ldots, g_n\}$ then, by $\mathbb{Q}$-linearity, they coincide on G.
Finally, Θ is surjective. Indeed, for every tuple $(h_1, \ldots, h_d) \in H^d$ and for every $i \in \{1, \ldots, d\}$ we can define a map ϑ setting $\vartheta(g_i) := h_i$ and extend by $\mathbb{Q}$-linearity to a linear map of G into H. In particular, ϑ is a group homomorphism. □

For $d = 1$ we obtain the following special case.

Corollary 1.2.8. *Let H be a uniquely divisible group. Then* $\operatorname{Hom}(\mathbb{Q}, H) \simeq H$. □

The next result will only be used to state Theorem 3.1.12, which we only quote for motivation.

Theorem 1.2.9 ([Gri70, Theorem 17]). *Every abelian group H can be embedded in a divisible group, called the* divisible closure *or* divisible hull *of H. It is the smallest divisible group containing H.* □

1.2.2 Semi-direct products

In the sequel, we will often use a classical result on (inner) semi-direct products of groups (Lemma 1.2.12). In this subsection we recall the relevant definitions and provide a proof. For convenience, in this subsection we will use multiplicative notation on groups.

Definition 1.2.10 (Semi-direct product). Let G be a group and N, H two subgroups with N a normal subgroup of G and such that $N \cap H = \{1\}$ and $NH = G$. Then we say that G is the *(inner) semi-direct product of N and H* and we write $G = N \rtimes H$.
Now let H and N be two arbitrary groups and let φ be an action of H on N. In other words, $\varphi\colon H \to \operatorname{Aut} N$ is a homomorphism of H into the automorphism group $\operatorname{Aut} N$ of N. It can be showed that the cartesian product $N \times H$ endowed with the operation $\bullet$ defined by $(n_1, h_1) \bullet (n_2, h_2) = (n_1\varphi(h_1)(n_2), h_1h_2)$ forms a group, called the *(outer) semi-direct product of N and H with respect to φ* and denoted by $N \rtimes_\varphi H$.

Remark 1.2.11. Let $G = N \rtimes_\varphi H$, as above. The groups N and H can be canonically embedded into $N \rtimes_\varphi H$ via $n \mapsto (n, 1)$ and $h \mapsto (1, h)$. Let us denote by $\bar{N}$ and $\bar{H}$ the images of N and H via the aforementioned embeddings. It can be showed that $\bar{N}$ is normal in $N \rtimes_\varphi H$, that $\bar{N} \cap \bar{H} = \{1\}$ and that $N \rtimes_\varphi H = \bar{N} \cdot \bar{H}$.

Vice versa, assume that G is a group and N, H two subgroups with N a normal subgroup of G and such that $N \cap H = \{1\}$ and $NH = G$. Then H acts on N by conjugation, that is, there is a group homomorphism

$$\begin{array}{rccc} \varphi\colon & H & \longrightarrow & \operatorname{Aut} N \\ & h & \longmapsto & \varphi(h) \end{array}$$

where $\varphi(h)$ is conjugation by h, that is $\varphi(h)(n) = hnh^{-1}$ for all $n \in N$. Then it can be proved that $G \simeq N \rtimes_\varphi H$. □

The following lemma will be used later to prove several decomposition results.

Lemma 1.2.12. *Let*

$$N \xrightarrow{f} G \xrightarrow{g} H$$

be an exact sequence of groups, which means $\operatorname{im} f = \ker g$*. Assume* g *admits a section, that is, an injective homomorphism* $s : H \to G$ *such that* $gs = \mathrm{id}_H$ *(in particular,* g *is surjective). Then we have*

$$G = \operatorname{im} f \rtimes \operatorname{im} s.$$

Proof. We will show that

(i) $\operatorname{im} f \trianglelefteq G$

(ii) $\operatorname{im} f \cap \operatorname{im} s = \{1\}$

(iii) $\operatorname{im} f \cdot \operatorname{im} s = G$.

Since the sequence is exact, we have $\operatorname{im} f = \ker g \trianglelefteq G$. Hence (i). For (ii), let $x \in \operatorname{im} f \cap \operatorname{im} s$. Then $x = f(n) = s(h)$ for some $n \in N$, $h \in H$. Then

$$h = \mathrm{id}_H(h) = g(s(h)) = g(f(n)) = 1$$

because, by exactness, gf is the trivial homomorphism. So $x = s(1) = 1$. Finally let us prove (iii). Let $x \in G$ and define $z := s(g(x))$ and $y := xz^{-1}$. Then clearly $x = yz$. Now, we have $g(z) = g(s(g(x))) = g(x)$ from which we get

$$g(y) = g(xz^{-1}) = g(x)g(x)^{-1} = 1$$

so $y \in \ker g = \operatorname{im} f$, and clearly $z \in \operatorname{im} s$. So the proof is complete. □

1.3 Valued modules

In this and the next sections we recall some fundamental facts about valued and ordered modules. We refer the reader to [Kuh00b, Chapter 0] for more details.

Let R be a commutative ring. Throughout this section, all modules will be considered over a fixed ring R, unless specified otherwise.

Definition 1.3.1. Let M be an R-module and Γ a chain (i.e., a totally ordered set). A *valuation on* M *with value set* Γ is a surjective map $v : M \to \Gamma \cup \{\infty\}$ such that, for all $x, y \in M$ we have

(i) $v(x) = \infty \Leftrightarrow x = 0$;

(ii) $v(rx) = v(x)$ for all $r \in R \setminus \{0\}$;

(iii) $v(x - y) \geq \min\{v(x), v(y)\}$.

An R-module M equipped with a valuation v will be called a *valued R-module* and will be often denoted as a pair (M, v).

Remark 1.3.2. Let A be an abelian group. Then A is canonically a $\mathbb{Z}$-module. For all $n \in \mathbb{Z}$ and all $a \in A$ the multiplication by scalars is given by

$$na = \underbrace{a + \ldots + a}_{n \text{ times}}$$

Hence, all the statements about R-modules made in this section apply to abelian groups, regarded as $\mathbb{Z}$-modules. □

Definition 1.3.3. If A is an abelian group, Γ a chain, and $v \colon A \to \Gamma \cup \{\infty\}$ a function, we say that (A, v) is a *valued abelian group* if v is a valuation on A as a $\mathbb{Z}$-module.

Definition 1.3.4. Let $(\Gamma_1, <_1)$ and $(\Gamma_2, <_2)$ be chains. A function $f \colon \Gamma_1 \to \Gamma_2$ is *order preserving* if, for all $\gamma, \delta \in \Gamma_1$ we have that $\gamma <_1 \delta$ implies $f(\gamma) <_2 f(\delta)$. An order preserving bijection of chains will also be called an *order preserving isomorphism*. For a chain Γ, an order preserving bijection of Γ onto itself is called an *order preserving automorphism*. The group of order preserving automorphisms of Γ (under composition) will be denoted by $o\text{-}\operatorname{Aut}\Gamma$.

Definition 1.3.5. Let (M_i, v_i) be valued modules with value set Γ_i, for $i = 1, 2$. Let $f \colon M_1 \to M_2$ be a homomorphism. We say that f is *valuation preserving* if there exists an order preserving map $\tilde{f} \colon \Gamma_1 \to \Gamma_2$ such that for all $x \in M_1$, we have $v_2(f(x)) = \tilde{f}(v_1(x))$.
If f and $\tilde{f}$ are bijective then f is called a *valuation preserving isomorphism*. If, moreover, $M_1 = M_2$, $\Gamma_1 = \Gamma_2$ and $v_1 = v_2$, then we call f a *valuation preserving automorphism*. We call $\tilde{f}$ the *order preserving automorphism of* Γ *induced by* f and we denote it by f_Γ.

Lemma 1.3.6. *Let (M, v) be a value group with value set Γ. The set of valuation preserving automorphisms of a valued module (M, v) is a group under composition.*

Proof. We know that the set $\operatorname{Aut} M$ of all automorphisms of M is a group. Let $f, g \in \operatorname{Aut} M$ be valuation preserving. We need to show that fg and f^{-1} are valuation preserving. Let f_Γ, g_Γ be the order preserving automorphisms of Γ induced by f and g, respectively. Then $f_\Gamma g_\Gamma$ is an order preserving automorphism of Γ and we have

$$v(fg(x)) = f_\Gamma(v(g(x))) = f_\Gamma(g_\Gamma(v(x))) = f_\Gamma g_\Gamma(v(x))$$

so fg is valuation preserving with associated order preserving automorphism given by $f_\Gamma g_\Gamma$. Similarly we have

$$f_\Gamma^{-1}(v(f(x))) = f_\Gamma^{-1}(f_\Gamma(v(x))) = v(x).$$

□

Notation 1.3.7. *Let (M, v) be a valued module. We showed in Lemma 1.3.6 that the set of valuation preserving automorphisms forms a group under composition. We will denote this group by v- $\operatorname{Aut} M$.*

Now we are going to define an important invariant of a valued module: the skeleton. First we need the following definition.

Definition 1.3.8. (i) An *ordered system of modules* is a pair $[\Gamma; \{A_\gamma : \gamma \in \Gamma\}]$ where Γ is a chain and $\{A_\gamma : \gamma \in \Gamma\}$ a family of modules indexed by Γ.

(ii) An *isomorphism of ordered systems*

$$\bar{h} = [\tilde{h}; \{h_\gamma : \gamma \in \Gamma\}] : [\Gamma; \{A_\gamma : \gamma \in \Gamma\}] \to [\Delta; \{D_\delta : \delta \in \Delta\}]$$

consists of an isomorphism of chains $\tilde{h} : \Gamma \to \Delta$ and for each $\gamma \in \Gamma$ an isomorphism $h_\gamma : A_\gamma \to D_{\tilde{h}(\gamma)}$ of modules.

Let (M, v) be a valued module with value set Γ. For $\gamma \in \Gamma$ define

$$\begin{aligned} M^\gamma &:= \{x \in M : v(x) \geq \gamma\}; \\ M_\gamma &:= \{x \in M : v(x) > \gamma\}. \end{aligned} \tag{1.1}$$

Then M^γ, M_γ are submodules satisfying $M_\gamma \subseteq M^\gamma \subseteq M$. We define $B(M, \gamma) := M^\gamma / M_\gamma$. We will write B_γ instead of $B(M, \gamma)$ if the context is clear. Moreover, for all $\gamma \in \Gamma$ we define the *coefficient map*

$$\begin{array}{rccc} \pi_\gamma : & M^\gamma & \longrightarrow & B_\gamma \\ & x & \longmapsto & x + M_\gamma. \end{array}$$

Definition 1.3.9 (Skeleton)**.** The *skeleton* of (M, v) is the ordered system $S(M) := [\Gamma; \{B_\gamma : \gamma \in \Gamma\}]$. The chain Γ is called the *spine* of (M, v) and the B_γ's are called the *ribs*.

Every isomorphism of valued modules canonically induces an isomorphism of the skeletons:

Lemma 1.3.10. *Let (M_1, v_1) and (M_2, v_2) be valued modules with skeletons $S(M_i) := [\Gamma_i; \{B_{i\gamma} : \gamma \in \Gamma_i\}]$, for $i = 1, 2$. Let $h : (M_1, v_1) \to (M_2, v_2)$ be an isomorphism of valued modules and let $\tilde{h} : \Gamma_1 \to \Gamma_2$ be given by $\tilde{h}(v_1(x)) = v_2(h(x))$. Moreover, for every $\gamma \in \Gamma_1$ let*

$$\begin{array}{rccc} h_\gamma : & B_{1,\gamma} & \longrightarrow & B_{2,\tilde{h}(\gamma)} \\ & \pi_{1,\gamma}(x) & \longmapsto & \pi_{2,\tilde{h}(\gamma)}(h(x)). \end{array}$$

Then $\bar{h} = [\tilde{h} : \{h_\gamma : \gamma \in \Gamma\}] : S(M_1) \to S(M_2)$ is an isomorphism of ordered systems.

Proof. The map $\tilde{h}$ is well defined and an isomorphism of chains because h is valuation preserving, by assumption. Let $\gamma \in \Gamma_1$ and let $\tilde{\gamma} := \tilde{h}(\gamma)$. Let $x, y \in M_1$ with $\pi_{1,\gamma}(x) = \pi_{1,\gamma}(y)$. This means that $x - y \in M_{1,\gamma}$, that is $v_1(x - y) > \gamma$. Since $\tilde{h}$ is an isomorphism of chains, it follows that

$$v_2(h(x) - h(y)) = v_2(h(x - y)) = \tilde{h}(v_1(x - y)) > \tilde{h}(\gamma) = \tilde{\gamma}$$

hence $h(x) - h(y) \in M_{2,\tilde{\gamma}}$ i.e., $\pi_{2,\tilde{\gamma}}(h(x)) = \pi_{2,\tilde{\gamma}}(h(y))$. So h_γ is well defined. Moreover, it is surjective, indeed, let $\tilde{\gamma} \in \Gamma_2$ and $\gamma \in \Gamma_1$ such that $\tilde{h}(\gamma) = \tilde{\gamma}$. Let $\bar{z} \in B_{2,\tilde{\gamma}}$ and take $z \in M_2$ such that $\pi_{2,\tilde{\gamma}}(z) = \bar{z}$. Then $v_2(z) \geq \tilde{\gamma}$. Since h is an isomorphism there is $x \in M$ with $z = h(x)$ and since $\tilde{h}$ is a chain isomorphism $\tilde{\gamma} \leq v_2(z) \Rightarrow v_1(x) \geq \gamma$ so $x \in M_{1,\gamma}$. Hence $h_\gamma(\pi_{1,\gamma}(x)) = \bar{z}$.

Finally, h_γ is injective. Indeed we have:

$$\begin{aligned} h_\gamma(\pi_{1,\gamma}(x)) = \pi_{2,\tilde{\gamma}}(h(x)) = 0 &\Leftrightarrow h(x) \in M_{2,\tilde{\gamma}} \\ &\Leftrightarrow v_2(h(x)) > \tilde{\gamma} \\ &\Leftrightarrow v_1(x) > \gamma \\ &\Leftrightarrow x \in M_{1,\gamma} \\ &\Leftrightarrow \pi_{1,\gamma}(x) = 0. \end{aligned}$$

Hence h_γ is an isomorphism of modules. □

The skeleton is thus an invariant of a valued module.

Remark 1.3.11. If we denote by $\operatorname{Aut} S(M)$ the set of all automorphisms of the skeleton $S(M) = [\Gamma; \{B_\gamma : \gamma \in \Gamma\}]$, we can easily see that this forms a group under composition. Indeed, let $h, g \in \operatorname{Aut} S(M)$. Then $h_\Gamma g_\Gamma$ is an automorphism of the ordered set Γ. For all $\gamma \in \Gamma$ we define the isomorphism $(hg)_\gamma$ as

$$h_{g_\Gamma(\gamma)} g_\gamma : B_\gamma \longrightarrow B_{h_\Gamma g_\Gamma(\gamma)}.$$

Clearly the identity $[\mathrm{id}_\Gamma; \{\mathrm{id}_{B_\gamma} : \gamma \in \Gamma\}]$ is the neutral element and $[h_\Gamma^{-1}; \{h_\gamma^{-1} : \gamma \in \Gamma\}]$ is the inverse of $[h_\Gamma; \{h_\gamma : \gamma \in \Gamma\}]$. □

Notation 1.3.12. *By Lemma 1.3.10 an automorphism $\sigma \in v\text{-}\operatorname{Aut} M$ induces an automorphism $\bar{\sigma} = [\sigma_\Gamma; \{\sigma_\gamma : \gamma \in \Gamma\}] \in \operatorname{Aut} S(M)$.*
Instead of $\bar{\sigma}$ we will denote by $\sigma_{S(M)}$ the automorphism induced by σ on the skeleton. Thus

$$\sigma_{S(M)} = [\sigma_\Gamma; \{\sigma_\gamma : \gamma \in \Gamma\}]. \tag{1.2}$$

Proposition 1.3.13. *Let M be a valued module and let $S(M)$ be its skeleton. The map*

$$\begin{array}{rccc} \Phi_M\colon & v\text{-}\operatorname{Aut} M & \to & \operatorname{Aut} S(M) \\ & \sigma & \mapsto & \sigma_{S(M)} \end{array} \tag{1.3}$$

is a group homomorphism.

Proof. For $\sigma, \tau \in v\text{-}\operatorname{Aut} M$ we have

$$\Phi_M(\tau\sigma) = [(\tau\sigma)_\Gamma; (\tau\sigma)_\gamma : \gamma \in \Gamma]$$

and

$$v((\tau\sigma)(x)) = \tau_\Gamma(v(\sigma(x))) = \tau_\Gamma(\sigma_\Gamma(v(x))) = \tau_\Gamma\sigma_\Gamma(v(x)).$$

So $(\tau\sigma)_\Gamma = \tau_\Gamma\sigma_\Gamma$. Moreover, for all $\gamma \in \Gamma$ we have

$$\begin{aligned} (\tau\sigma)_\gamma(\pi_\gamma(x)) &= \pi_{(\tau\sigma)_\Gamma(\gamma)}(\tau\sigma)(x) \\ &= \pi_{\tau_\Gamma(\sigma_\Gamma(\gamma))}(\tau(\sigma(x))) \\ &= \tau_{\sigma_\Gamma(\gamma)}(\pi_{\bar{\sigma}(\gamma)}(\sigma(x))) \\ &= \tau_{\sigma_\Gamma(\gamma)}(\sigma_\gamma(\pi_\gamma(x))) \\ &= (\tau_{\sigma_\Gamma(\gamma)}\sigma_\gamma)(\pi_\gamma(x)). \end{aligned}$$

Thus we also have $(\tau\sigma)_\gamma = (\tau_{\sigma_\Gamma(\gamma)}\sigma_\gamma)$. So Φ_M is a group homomorphism. $\square$

Corollary 1.3.14. *The map*

$$\begin{array}{rccc} \Phi'_M: & v\text{-}\operatorname{Aut} M & \to & o\text{-}\operatorname{Aut}\Gamma \\ & \sigma & \mapsto & \sigma_\Gamma. \end{array}$$

is a group homomorphism.

Proof. Indeed, in Proposition 1.3.13 we proved that for $\sigma, \tau \in v\text{-}\operatorname{Aut} M$ we have $(\sigma\tau)_\Gamma = \sigma_\Gamma\tau_\Gamma$. $\square$

The map Φ_M will play an important role in the study of the automorphisms of Hahn groups in Chapter 2.

Definition 1.3.15 (Valuation independence)**.** Let M be a valued R-module and M_0 a submodule of M. Let $\{x_i : i \in I\}$ be a set of non-zero elements of M, for an index set I. We say that the family $\{x_i : i \in I\}$ is *valuation independent over* M_0 if, for all $x_0 \in M_0$ and for all family $\{r_i : i \in I\}$ of elements of R of which all but finitely many are zero, we have

$$v\left(\sum_{i\in I} r_i x_i + x_0\right) = \min_{i:\, r_i \neq 0} \{v(x_i), v(x_0)\}.$$

We simply say that $\{x_i : i \in I\}$ is *valuation independent* if it is valuation independent over $\{0\}$.

Definition 1.3.16 (Valuation basis)**.** If M is a vector space (i.e., R is a field), a basis of M which is valuation independent will be called a *valuation basis*.

1.4 Ordered modules

We consider an ordered module $(M, <)$ over an ordered commutative ring $(R, <_R)$. A submodule N is *convex* if for all $x, y \in N$ and $z \in M$ we have $x < z < y \Rightarrow z \in N$. Then, for all $x \in M$ we define the following two submodules:

$$C_x = \bigcap\{\text{convex submodules containing } x\}$$
$$D_x = \bigcup\{\text{convex submodules not containing } x\}$$

Note that D_x is a submodule of C_x and so we can set

$$B_x := C_x / D_x.$$

This is again an ordered module that we call the *component of x in M*. For $x \in M$ we set $|x| := \min\{x, -x\}$ and define an equivalence relation by:

$$x \sim y \Leftrightarrow \exists r \in R \text{ s.t. } r|x| \geq |y| \wedge r|y| \geq |x|. \tag{1.4}$$

If $x \sim y$ we say that x and y are *R-equivalent*. We say that x is *infinitely smaller* than y, and write $x \ll y$, if for all $r \in R$ we have $r|x| < |y|$. If we denote by Γ the set of R-equivalence classes of non-zero elements of M, we can order Γ by $[x]_\sim < [y]_\sim \Leftrightarrow x \ll y$. If Γ consists of only one class, or equivalently, if M contains no pair (x, y) where y is infinitely smaller than x, we say that M is *archimedean*. The map

$$\begin{array}{rccc} v^R\colon & M & \longrightarrow & \Gamma \cup \{\infty\} \\ & x & \longmapsto & [x]_\sim \end{array}$$

is a valuation on the ordered module M, called the (R-)*natural valuation*. It is compatible with the order, in the sense that if $x, y > 0$ and $v(x) < v(y)$ then $x > y$.

Definition 1.4.1 (Archimedean rank)**.** The set Γ will be called the *(archimedean) rank* of M, denoted by $\operatorname{rk} M$ and its elements are called the *archimedean clsses of M*.

Lemma 1.4.2. *A homomorphism $\varphi\colon (M, <) \to (M', <')$ of ordered modules preserves the natural valuation. In other words, if v, v' are the natural valuations on M and M' respectively, then for all $x, y \in M$ we have $v(x) = v(y) \Rightarrow v'(\varphi(x)) = v'(\varphi(y))$.*

Proof. Denote by $\sim$ and $\sim'$ the relations of R-equivalence on M and M' respectively. Let x, y be such that $v(x) = v(y)$. As remarked above, this is equivalent to $x \sim y$ which means (cf. (1.4)) that for some positive $r \in R$ we have $r|x| \geq |y| \wedge r|y| \geq |x|$. Since φ is order preserving this implies $r|\varphi(x)| \geq' |\varphi(y)| \wedge r|\varphi(y)| \geq' |\varphi(x)|$ that is $\varphi(x) \sim' \varphi(y)$ which is $v'(\varphi(x)) = v'(\varphi(y))$. □

1.5 Valued fields

In this section we recall the definition of a valued field and some basic properties that will be relevant in the sequel. No claim of completeness is made: for a general treatment of valued fields see, for example, [EP05].

Definition 1.5.1. Let K be a field, $(G, <)$ a totally ordered abelian group and ∞ an element larger than any element of G. A *valuation* on K with *value group G* is a surjective map $v : K \twoheadrightarrow G \cup \{\infty\}$ such that

(i) $v(x) = \infty \Leftrightarrow x = 0$;

(ii) $v(xy) = v(x) + v(y)$ for all $x, y \in K$;

(iii) $v(x + y) \geq \min\{v(x), v(y)\}$.

A field K equipped with a valuation v will be called a *valued field* and we will often write it as a pair (K, v).

To a valued field we associate the following important invariants.

Notation 1.5.2. *(i) The set* $R_{K,v} := \{x \in K \text{ s.t. } v(x) \geq 0\}$ *is a subring of K called the* valuation ring of (K, v). *It is a local ring.*

(ii) We denote by $I_{K,v} := \{x \in K \text{ s.t. } v(x) > 0\} \subset R_{K,v}$ *the unique maximal ideal of* $R_{K,v}$. *We call* $I_{K,v}$ *the* valuation ideal of (K, v).

(iii) The quotient $\bar{K}_v := R_{K,v}/I_{K,v}$ *is a field called the* residue field *of* (K, v).

(iv) We denote by $U_K := R_K^\times$ *the group of units of the valuation ring.*

(v) Let $\mathrm{res}\colon R_K \to \bar{K}, x \mapsto x + I_K$ *be the quotient map. We will denote the image* $\mathrm{res}(x)$ *of an element* $x \in R_K$ *by* $\bar{x} := x + I_K$.

(vi) An important subgroup of U_K *is the group* $1 + I_K := \mathrm{res}^{-1}(1)$, *called the group of* 1-units.

If there is no risk of confusion we will omit the subscript v and from the notation and simply write R_K, I_K and $\bar{K}$ for the valuation ring, valuation ideal and residue field, respectively.

Definition 1.5.3. Let (K_1, v_1), (K_2, v_2) be two valued fields with value groups G_1 and G_2 respectively. An isomorphism of valued fields $f : (K_1, v_1) \to (K_2, v_2)$ is an isomorphism of the fields K_1 and K_2 that *preserves the valuation*, that is, there is an isomorphism of ordered groups $\tilde{f} : G_1 \to G_2$ such that, for all $x \in K_1$, we have $v_2(f(x)) = \tilde{f}(v_1(x))$.
In particular, if (K, v) is a valued field with value group G, a *valuation preserving*

automorphism of (K, v) is an automorphism σ of K such that there exists an order preserving automorphism σ_G of G such that, for all $a \in K$, we have $v(\sigma(a)) = \sigma_G(v(a))$. We call σ_G the *automorphism of G induced by* σ.

Lemma 1.5.4. *Let (K_i, v_i) be valued fields, for $i = 1, 2$ and let $f\colon K_1 \to K_2$ be a field isomorphism. Then f preserves the valuation if and only if, for all $a, b \in K_1$ we have*

$$v_1(a) < v_1(b) \Leftrightarrow v_2(f(a)) < v_2(f(b)). \tag{1.5}$$

Proof. The forward direction is clear from Definition 1.5.3.
Let us assume condition (1.5) holds. First we notice that, for all $a, b \in K$ we have $v_1(a) = v_1(b) \Leftrightarrow v_2(f(a)) = v_2(f(b))$. Indeed, since G_1 and G_2 are totally ordered, we have

$$\begin{aligned} v_1(a) \neq v_1(b) &\Leftrightarrow v_1(a) \lesseqgtr v_1(b) \\ &\Leftrightarrow v_2(f(a)) \lesseqgtr v_2(f(b)) \qquad \text{by (1.5)} \\ &\Leftrightarrow v_2(f(a)) \neq v_2(f(b)). \end{aligned}$$

Then the map $\tilde{f}\colon G_1 \to G_2$ by $\tilde{f}(v_1(a)) := v_2(f(a))$ is well defined.
We need to show that $\tilde{f}$ is an order preserving group isomorphism.

- The fact that it is order preserving follows directly from condition (1.5).
- It is a group homomorphism: let $a, b \in K_1$, then

$$\begin{aligned} \tilde{f}(v_1(a) + v_1(b)) &= \tilde{f}(v_1(ab)) \\ &= v_2(f(ab)) \\ &= v_2(f(a)f(b)) \\ &= v_2(f(a)) + v_2(f(b)) \\ &= \tilde{f}(v_1(a)) + \tilde{f}(v_1(b)). \end{aligned}$$

- It is injective:

$$\tilde{f}(v_1(a)) = \tilde{f}(v_1(b)) \Leftrightarrow v_2(f(a)) = v_2(f(b)) \Leftrightarrow v_1(a) = v_1(b)$$

- It is surjective: let $v_2(b) \in G_2$. Then, since f is a field isomorphism there exists $a \in K_1$ such that $b = f(a)$ and therefore $v_2(b) = v_2(f(a)) = \tilde{f}(v_1(a))$.

□

Notation 1.5.5. *The set of valuation preserving automorphisms of a valued field* (K, v) *will be denoted by* v- Aut K.

Lemma 1.5.6. *The set* v- Aut K *forms a group under composition.*

Proof. Since we know that the set Aut K of all automorphisms of K forms a group under composition, all we need to observe is that, if σ, τ are valuation preserving automorphisms of K, then so are σ^{-1} and $\sigma\tau$. Let $a, b \in K$. Then, applying Lemma 1.5.4, we have

$$v(a) < v(b) \Leftrightarrow v(\sigma(a)) = v(\sigma(b)) \Leftrightarrow v(\tau(\sigma(a))) < v(\tau(\sigma(b)))$$

where the first equivalence comes from the fact that σ is valuation preserving and the second equivalence from the fact that τ is.

Again, since σ is valuation preserving, by Lemma 1.5.4, we get

$$v(\sigma^{-1}(a)) < v(\sigma^{-1}(b)) \Leftrightarrow v(\sigma(\sigma^{-1}(a))) < v(\sigma(\sigma^{-1}(b))) \Leftrightarrow v(a) < v(b).$$

□

The next theorem gives several characterisations of valuation preserving automorphisms.

Theorem 1.5.7 ([KMP17, Theorem 4.2])**.** *Let* (K, v) *be a valued field and let* $\sigma \in$ Aut K. *The following are equivalent*

(i) $\sigma \in v$- Aut K;

(ii) $\sigma^{-1} \in v$- Aut K;

(iii) $\sigma(R_K) = R_K$;

(iv) $\sigma(I_K) = I_K$;

(v) $\sigma(U_K) = U_K$. □

Lemma 1.5.8. *For* $i = 1, 2$ *let* (K_i, v_i) *be a valued field with value group* G_i *and residue field* $\bar{K}_i$. *Let* $f : (K_1, v_1) \to (K_2, v_2)$ *be an isomorphism of valued fields. Define a map* $\bar{f} : \bar{K}_1 \to \bar{K}_2$ *by* $\bar{f}(\bar{x}) := f(x) \pmod{I_{K_2}}$ *for* $\bar{x} \in \bar{K}_1$ *and* $x \in R_{K_1}$ *a representative. Then* $\bar{f}$ *is an isomorphism of the residue fields.*

Proof. Let $\tilde{f} : G_1 \to G_2$ be the isomorphism of value groups induced by f. We define $\bar{f}$ as follows. Let $\bar{x} \in \bar{K}_1$ and let $x \in R_{K_1}$ be a representative: $\bar{x} = \mathrm{res}(x)$. Then set $\bar{f}(\bar{x}) := f(x) \pmod{I_{K_2}}$. This is well defined because if $x, y \in K_1$ and $x - y \in I_{K_1}$ then, since $v_2(f(x-y)) = \tilde{f}(v_1(x-y))$ we have $f(x-y) \in I_{K_2}$. The fact that $\bar{f}$ is a field homomorphism follows from the fact that f is and that passing to the quotient respects the field operations: for all $x, y \in R_{K_1}$ we have

$$\begin{aligned}\bar{f}(\bar{x}+\bar{y}) &= \bar{f}(\overline{x+y}) \\ &= f(x+y) \pmod{I_{K_2}} \\ &= (f(x) \pmod{I_{K_2}}) + (f(y) \pmod{I_{K_2}}) \\ &= \bar{f}(\bar{x}) + \bar{f}(\bar{y})\end{aligned}$$

and

$$\begin{aligned}\bar{f}(\bar{x}\bar{y}) &= \bar{f}(\overline{xy}) \\ &= f(xy) \pmod{I_{K_2}} \\ &= f(x)f(y) \pmod{I_{K_2}} \\ &= \bar{f}(\bar{x})\bar{f}(\bar{y}).\end{aligned}$$

With an identical argument as the one we gave for $\bar{f}$ we see that $\bar{g}\colon \bar{K}_2 \to \bar{K}_1$, $\bar{g}(\bar{y}) = f^{-1}(y) \pmod{I_{K_1}}$, for any $\bar{y} \in \bar{K}_2$ and any representative $y \in R_{K_2}$, is a well defined field homomorphism, and clearly $\bar{g}\bar{f} = \mathrm{id}_{\bar{K}_1}$. Swapping the roles of f and g if follows that also $\bar{f}\bar{g} = \mathrm{id}_{\bar{K}_2}$, so the proof is complete. □

Definition 1.5.9. By Lemma 1.5.8 a valuation preserving automorphism $\sigma \in v\text{-}\operatorname{Aut} K$ induces an isomorphism $\bar{\sigma} \in \operatorname{Aut} \bar{K}$. We call $\bar{\sigma}$ *the automorphism of $\bar{K}$ induced by σ.*

Corollary 1.5.10. *Let (K, v) be a valued field with value group G and residue field $\bar{K}$. Then the maps*

$$\Phi_{K,G}\colon v\text{-}\operatorname{Aut} K \to o\text{-}\operatorname{Aut} G,\ \sigma \mapsto \sigma_G$$

and

$$\bar{\Phi}_K\colon v\text{-}\operatorname{Aut} K \to \operatorname{Aut} \bar{K},\ \sigma \mapsto \bar{\sigma}$$

where σ_G and $\bar{\sigma}$ are defined as in Definitions 1.5.3 and 1.5.9 respectively, are group homomorphisms.

Proof. Let $\sigma, \tau \in v\text{-}\operatorname{Aut} K$ and $g \in G$. Then $g = v(x)$ for some $x \in K$. We already noticed that both σ_G and τ_G are order preserving. We compute

$$\begin{aligned}\Phi_{K,G}(\sigma\tau)(g) &= (\sigma\tau)_G(v(x))\\ &= v(\sigma(\tau(x)))\\ &= \Phi_{K,G}(\sigma)(v(\tau(x)))\\ &= \Phi_{K,G}(\sigma)(\Phi_{K,G}(\tau)(x)).\end{aligned}$$

So $\Phi_{K,G}$ is a homomorphism. Now let $\bar{x} \in \bar{K}$. Then

$$\begin{aligned}\bar{\Phi}_K(\sigma\tau)(\bar{x}) &= \overline{\sigma\tau}(\bar{x})\\ &= \sigma\tau(x) \pmod{I_K}\\ &= \bar{\sigma}(\overline{\tau(x)}) \pmod{I_K}\\ &= \bar{\sigma}(\bar{\tau}(\bar{x}))\\ &= (\bar{\Phi}_K(\sigma)\bar{\Phi}_K(\tau))(\bar{x})\end{aligned}$$

and $\bar{\Phi}_K$ is also a homomorphism. □

Corollary 1.5.11. *If (K, v) is a valued field with value group G, there is a group homomorphism*

$$\begin{array}{rcc} v\text{-}\operatorname{Aut} K & \longrightarrow & \operatorname{Aut} \bar{K} \times o\text{-}\operatorname{Aut} G \\ \sigma & \longmapsto & (\bar{\sigma}, \sigma_G) \end{array}$$

where the group structure on the right-hand side is the usual direct product of groups.
□

Remark 1.5.12. One can define a map from the set $\operatorname{Iso}((K_1, v_1), (K_2, v_2))$ of valued field isomorphisms between (K_1, v_1) and (K_2, v_2) into the set $\operatorname{Iso}(G_1, G_2)$ of order preserving isomorphisms between G_1 and G_2 given by $f \mapsto \tilde{f}$; and another map of $\operatorname{Iso}((K_1, v_1), (K_2, v_2))$ into the set $\operatorname{Iso}(\bar{K}_1, \bar{K}_2)$ of isomorphisms between $\bar{K}_1$ and $\bar{K}_2$ given by $f \mapsto \bar{f}$. These together give a function

$$\begin{array}{rcc} \operatorname{Iso}((K_1, v_1), (K_2, v_2)) & \longrightarrow & \operatorname{Iso}(G_1, G_2) \times \operatorname{Iso}(\bar{K}_1, \bar{K}_2) \\ f & \longmapsto & (\tilde{f}, \bar{f}). \end{array}$$

□

1.5.1 The induced topology

Let (K, v) be a valued field with residue field k and value group G. Then for all $g \in G$ and for all $a \in K$ we define the set

$$\mathcal{U}_g(a) := \{x \in K : v(x-a) > g\}. \tag{1.6}$$

For a fixed $a \in K$ the family $\{\mathcal{U}_g(a) : g \in G\}$ fulfils the properties of a basis of open neighbourhoods of a. Hence v induces a topology on K, with respect to which the field operations are continuous, thus making K into a topological field.

Proposition 1.5.13. *Let $\sigma \in v\text{-}\operatorname{Aut} K$ be an automorphism of the valued field (K, v). Then σ is continuous for the topology induced by v.*

Proof. It suffices to show that σ is an open map, because σ is an automorphism and the same will hold for σ^{-1}. Let σ_G be an automorphism of the ordered group G such that, for all $x \in K$, we have $v(\sigma(x)) = \sigma_G(v(x))$. Let $a \in K$ and $g \in G$. Then $\sigma(\mathcal{U}_g(a)) = \{\sigma(x) : v(x-a) > g\}$. Now, for all $x \in K$ we have $v(x-a) > g \Leftrightarrow \sigma_G(v(x-a)) > \sigma_G(g) \Leftrightarrow v(\sigma(x-a)) = v(\sigma(x) - \sigma(a)) > \sigma_G(g)$ and therefore

$$\begin{aligned}\sigma(\mathcal{U}_g(a)) &= \{\sigma(x) \in K : v(\sigma(x) - \sigma(a)) > \sigma_G(g)\} \\ &= \{y \in K : v(y - \sigma(a)) > \sigma_G(g)\} \\ &= \mathcal{U}_{\sigma_G(g)}(\sigma(a)).\end{aligned}$$

This shows that the image of an open neighbourhood is again an open neighbourhood, and hence the map is open. Now, since every valuation preserving automorphism is open, the inverse of every valuation preserving automorphism is continuous, and hence they are all continuous. □

Definition 1.5.14. Let K be a field and let v_1, v_2 be two valuations on K. Let R_i be the valuation ring of v_i, $i = 1, 2$. We say that v_1 is *finer than* v_2 (and that v_2 is *coarser than* v_1) if $R_1 \subseteq R_2$. Equivalently, v_1 is *finer than* v_2 if the topology induced by v_1 on K is finer than the one induced by v_2.

1.5.2 Henselian valued fields

In this subsection we recall the notion of a henselian valued field. The main purpose is to show that for a henselian field K with residue field of characteristic 0, the multiplicative group $1 + I_K$ of 1-units is uniquely divisible (Corollary 1.5.18).

Definition 1.5.15 (Henselian field)**.** Let (K, v) be a valued field with valuation ring R and residue field $\bar{K}$. We say that (K, v) is *henselian* if the following holds:

$$\begin{array}{c}\text{For every } P \in R[X] \text{ and for every } y \in R \text{ such that } \bar{P}(\bar{y}) = 0 \\ \text{and } \bar{P}'(\bar{y}) \neq 0 \text{ there exists } x \in R \text{ such that } \bar{x} = \bar{y} \text{ and } P(x) = 0.\end{array} \qquad (\mathcal{H})$$

For a field F, denote by $\mu(F)$ the multiplicative group of roots of unity of F, i.e., the torsion subgroup of $(F^\times, \cdot)$.

Remark 1.5.16. Let (F, w) be a valued field with value group H and valuation ring R_F. Then $\mu(F) \subseteq R_F$. Indeed, if $x \in \mu(F)$ and, for some positive $n \in \mathbb{N}$ we have $x^n = 1$, then $0 = w(x^n) = nw(x)$. Since H is ordered and hence torsion-free, this implies $v(x) = 0$. □

Proposition 1.5.17. *Let (K, v) be a Henselian valued field such that* $\operatorname{char} \bar{K} = 0$*. Then the map* $\operatorname{res}' \colon (\mu(K), \cdot) \to (\mu(\bar{K}), \cdot)$, $x \mapsto \bar{x}$ *is well defined and a group isomorphism.*

Proof. By Remark 1.5.16 the restriction $\operatorname{res}|_{\mu(K)} \colon \mu(K) \to \bar{K}$ makes sense. Let $x \in \mu(K)$. Then x has finite order in $K^\times$ and, since res is a ring homomorphism, $\bar{x}$ has finite order in $\bar{K}^\times$. So $\bar{x} \in \mu(\bar{K})$. So the codomain of the restriction $\operatorname{res}|_{\mu(K)}$ is contained in $\mu(\bar{K})$. The map res′ is thus precisely the restriction of res to the multiplicative group $\mu(K)$ into $\mu(\bar{K})$. Hence it is a well defined group homomorphism. Now we show it is an isomorphism.
Let $\bar{y} \in \mu(\bar{K})$ and n be the minimal positive integer such that $\bar{y}^n = 1$. So $\bar{y}$ is a root of $P(X) = X^n - 1$ in $\bar{K}$. Since $\operatorname{char} \bar{K} = 0$ we have $P'(\bar{y}) = n\bar{y}^{n-1} \neq 0$, because $y^{n-1} \neq 0$. Since K is Henselian, there exists $x \in K$ such that $P(x) = x^n - 1 = 0$ and $\bar{x} = \bar{y}$. So res is surjective.
To prove injectivity, assume that $\ker(\operatorname{res}')$ is non-trivial. Then, $\ker(\operatorname{res}')$ is a non-trivial torsion group. Hence it must contain some element x of prime order, say p. So x is a primitive p-th root of unity with $\bar{x} = 1$. But then $\bar{x}^m = 1$ for all $m \in \mathbb{N}$. Since x is a primitive p-th root of unity, it is a root of the p-th cyclotomic polynomial, i.e., $x^{p-1} + x^{p-2} + \ldots + x + 1 = 0$. Taking residues now yields $\bar{0} = \bar{1} + \bar{x} + \ldots + \bar{x}^{p-1} = \bar{1}$. A contradiction. So res′ is injective which completes the proof. □

Corollary 1.5.18. *Let (K, v) be a Henselian valued field such that* $\operatorname{char} \bar{K} = 0$*. Then the multiplicative group $(1 + I_K, \cdot)$ is uniquely divisible.*

Proof. Let $a \in 1 + I_K$ and let $n \in \mathbb{N}$. We want to show the existence and the uniqueness of a $b \in 1 + I_K$ such that $b^n = a$. Consider the polynomial $P = X^n - a \in R_K[X]$. Because $a \in 1 + I_K$ we have $\bar{a} = 1$. The polynomial $\bar{P} = X^n - 1 \in \bar{K}[X]$ has the root $x = 1$ in $\bar{K}$, which is simple because $\bar{P}'(1) = n$ and $\operatorname{char} \bar{K} = 0$. Since (K, v) is Henselian there exists $b \in R_K$ such that $P(b) = 0$ and $\bar{b} = \bar{a} = 1$. So $b^n = a$ and $b \in 1 + I_K$. This proves the existence. To prove uniqueness, let $c \in 1 + I_K$ be such that $c^n = a$. Then $b^n = c^n$ and so $(b/c)^n = 1$. Therefore, $d := b/c \in 1 + I_K$ is an n-th root of unity and $\bar{d} = 1$. But 1 is also an n-th root of unity with $\bar{1} = 1$. By Proposition 1.5.17 we must have $d = 1$ and therefore $b = c$. □

1.6 Ordered fields

Let $(K, +, \cdot, 0, 1, <)$ be an ordered field and denote by $K^{>0}$ the set of positive elements of K. Then $(K, +, 0, <)$ and $(K^{>0}, \cdot, 1, <)$ are ordered abelian groups and the former is divisible. In particular they are both ordered $\mathbb{Z}$-modules. Note that an ordered field has necessarily characteristic 0, i.e., its prime field is $\mathbb{Q}$.

Definition 1.6.1. (i) A subring $R \subseteq K$ is *convex* if for all $r \in R$ and all $a \in K$ we have

$$0 < a < r \Rightarrow a \in R.$$

(ii) Let S be a subring of K. The *convex closure* or *convex hull* of S is the smallest convex subring of K that contains S.

(iii) Let w be a valuation on K with value group H. We say that w is *convex* or *compatible with the ordering on* K if, for all $a, b \in K$ we have

$$0 < a \leq b \Rightarrow w(a) \geq w(b).$$

Lemma 1.6.2. *A convex subring R of K contains the convex closure of $\mathbb{Q}$ in K.*

Proof. Since $1 \in R$ and R is a ring, it follows that $\mathbb{Z} \subseteq R$. Let $q \in \mathbb{Q}$ with $q > 0$ and let $n = \lceil q \rceil$ be the smallest integer greater or equal to q. Then $0 < q \leq n \in R$ and, since R is convex, it follows $q \in R$. So R is a convex subring containing $\mathbb{Q}$, hence it contains the convex closure of $\mathbb{Q}$. □

Lemma 1.6.3. *Let w be a valuation on K with valuation ring R. Then w is a convex valuation if and only if R is a convex subring of K.*

Proof. Let w be convex and let $r \in R$, $a \in K$ be such that $0 < a < r$. Since w is convex, this implies $w(a) \geq w(r) \geq 0$. Thus $w(a) \geq 0$, that is, $a \in R$.
Vice versa, let R be convex and let $a, b \in K$ be such that $0 < a \leq b$. Then $0 < a/b \leq 1 \in R$, and since R is convex we have $a/b \in R$. Thus $v(a/b) \geq 0$, that is $v(a) \geq v(b)$ as required. $\square$

Let v_{nat} be the natural valuation on the ordered $\mathbb{Z}$-module $(K, +, 0, <)$ defined as in Section 1.4, and let G be the corresponding (ordered) value set. Since K is a field we can endow G with an operation "+" defined by

$$[x] + [y] := [xy].$$

Lemma 1.6.4. *$(G, +, [1], <)$ is an ordered abelian group.*

Proof. It is obvious that $(G, +, [1])$ is an abelian group. The order $<$ on G is given, as in Section 1.4, by $[x] < [y] \Leftrightarrow y \ll x$. The operation $+$ is compatible with the order: let $[x], [y], [z] \in G$ with $[x] < [y]$. Then for all $n \in \mathbb{Z}$ we have $|x| > n|y|$ which implies $|xz| > n|yz|$ so $[x] + [z] < [y] + [z]$. $\square$

Proposition 1.6.5. *The map $v_{\mathrm{nat}} \colon K \to G \cup \{\infty\}$, $x \mapsto [x]$ is a valuation on K.*

Proof. The properties (i) and (iii) of Definition 1.5.1 are inherited by the fact that v_{nat} is the natural valuation on the ordered group $(K, +, 0, <)$.

Let $x, y \in K$. Then $v_{\mathrm{nat}}(xy) = [xy] = [x] + [y] = v_{\mathrm{nat}}(x) + v_{\mathrm{nat}}(y)$ so the remaining property is also verified. $\square$

Definition 1.6.6 (Natural valuation)**.** The valuation v_{nat} introduced in Proposition 1.6.5 is called the *natural valuation on the ordered field K*.

Proposition 1.6.7. *The natural valuation v_{nat} is the finest convex valuation on the ordered field K.*

Proof. First note that v_{nat} is convex. Indeed, let $a, b \in K$ with $0 < a \leq b$. Then $a \not\gg b$, so $[a] \leq [b]$ and thus $v_{\mathrm{nat}}(a) \geq v_{\mathrm{nat}}(b)$. Let R be the valuation ring of v_{nat}. By Lemma 1.6.3, R is convex in K. We will show that R coincides with the convex closure $C(\mathbb{Q})$ of $\mathbb{Q}$ in K. By Lemma 1.6.2, it suffices to show that $R \subseteq C(\mathbb{Q})$. Let $r \in R^{>0}$. Then $v_{\mathrm{nat}}(r) \geq 0$ and so $[r] \sim [1]$ or $[r] < [1]$. Either way there exists $n \in \mathbb{N}$ such that $r \leq n \in C(\mathbb{Q})$ and, since $C(\mathbb{Q})$ is convex, it follows $r \in C(\mathbb{Q})$ which proves $R = C(\mathbb{Q})$. $\square$

Definition 1.6.8. Let $(K_1, <_1)$ and $(K_2 <_2)$ be ordered fields. *A homomorphism of ordered fields* is a homomorphism of fields $\sigma\colon K_1 \to K_2$ such that, for all $a \in K_1$ we have $a >_1 0 \Rightarrow \sigma(a) >_2 0$. An *automorphism of the ordered field* $(K, <)$ or an *order preserving automorphism* of $(K, <)$ is an automorphism of K which is also a homomorphism of ordered fields.

Notation 1.6.9. *The set of order preserving automorphisms of an ordered field* $(K, <)$ *will be denoted by* $o\text{-}\operatorname{Aut} K$.

Lemma 1.6.10. *Let K be an ordered field. The set of automorphisms of K preserving the ordering forms a subgroup of* $\operatorname{Aut} K$.

Proof. Let $\sigma, \tau \in \operatorname{Aut} K$ be order preserving, that is, for all $a \in K$ we have $a > 0 \Leftrightarrow \sigma(a), \sigma(b) > 0$. Then $a > 0 \Leftrightarrow \sigma(a) > 0 \Leftrightarrow \tau(\sigma(a)) > 0$ hence $\tau\sigma$ is order preserving. Moreover, let $a > 0$ and consider $\sigma^{-1}(a)$. Let $b \in K$ be such that $\sigma(b) = a$. Then $0 < b = \sigma^{-1}\sigma(b) = \sigma^{-1}(a)$. □

Proposition 1.6.11. *Let $\sigma\colon (K_1, <_1) \to (K_2, <_2)$ be an isomorphism of ordered fields and let $w_i := v_{\mathrm{nat},i}$ be the natural valuation on K_i, for $i = 1, 2$. Then σ preserves the natural valuation, that is, for all $x, y \in K_1$ if $w_1(x) = w_1(y)$ then $w_2(\sigma(x)) = w_2(\sigma(y))$.*

Proof. Let $w_1(x) = w_1(y)$. By definition there exists $n \in \mathbb{Z}$ such that $n|x| \geq_1 |y|$ and $n|y| \geq_1 |x|$. Applying σ and recalling that it preserves the ordering we get $n|\sigma(x)| \geq_2 |\sigma(y)|$ and $n|\sigma(y)| \geq_2 |\sigma(x)|$ which, by definition, means $w_2(\sigma(x)) = w_2(\sigma(y))$. □

1.6.1 Real closed fields

Some of the examples that we will present at the end of the thesis will rely on real closed fields (see Subsection 3.5.4). For this reason and to keep this work self contained we will briefly recall the definition and the main properties of real closed fields that will be relevant for the sequel.

If $(K, <)$ is an ordered field and L/K is a field extension, we say that *the ordering $<$ extends to L* if there exists an ordering $<'$ such that $(L, <')$ is an ordered field and for all $x \in K$ we have $x > 0 \Leftrightarrow x >' 0$.

The following lemma consists of classical results of Artin and Schreier. A proof can be found in [AS27] or in [PD01, Lemma 1.2.9, Theorem 1.2.10].

Lemma 1.6.12. *Let $(K, <)$ be an ordered field. The following conditions are equivalent* *(i) The ordering $<$ is the unique ordering making $(K, <)$ an ordered field and if L/K is a proper algebraic extension then $<$ does not extend to L;*

(ii) every positive element of K is a square and every polynomial $p \in K[X]$ of odd degree has a root in K;

(iii) $K(\sqrt{-1})$ is algebraically closed and $K \neq K(\sqrt{-1})$. □

Definition 1.6.13 (Real closed field)**.** A field K is said to be *real closed* if it satisfies any (and hence all) of the conditions of Lemma 1.6.12.

Remark 1.6.14. Observe that, by part (ii) of Lemma 1.6.12, it follows that every positive element of a real closed field K has an n-th root in K, for all $n \in \mathbb{N} \setminus \{0\}$. □

Chapter 2

Automorphisms of Hahn groups

2.1 Construction of Hahn groups

Fix an ordered system of abelian groups $[\Gamma; \{A_\gamma : \gamma \in \Gamma\}]$ (Definition 1.3.8). On the A_γ we use additive notation and we use the same symbol 0 to denote the neutral element in each of the A_γ's. Consider the cartesian product $\Pi = \prod_{\gamma\in\Gamma} A_\gamma$ with the usual component-wise addition: if $a = (a_\gamma)_{\gamma\in\Gamma}$ and $(b_\gamma)_{\gamma\in\Gamma}$ are elements of Π then

$$a + b = (a_\gamma + b_\gamma)_{\gamma\in\Gamma}.$$

Definition 2.1.1. (i) For an element $a = (a_\gamma)_{\gamma\in\Gamma} \in \Pi$ the set $\operatorname{supp} a = \{\gamma \in \Gamma : a_\gamma \neq 0\}$ is called the *support of* a.

(ii) The *Hahn sum of the ordered system* $[\Gamma; \{A_\gamma : \gamma \in \Gamma\}]$ is the subgroup of Π, denoted by $\coprod_{\gamma\in\Gamma} A_\gamma$, consisting of elements with finite support:

$$\coprod_{\gamma\in\Gamma} A_\gamma = \{a \in \Pi : \operatorname{supp} a \text{ is finite}\}. \tag{2.1}$$

(iii) The *Hahn product of the ordered system* $[\Gamma; \{A_\gamma : \gamma \in \Gamma\}]$ is the subgroup $\mathbf{H}_{\gamma\in\Gamma} A_\gamma$ of Π consisting of elements with well ordered support:

$$\mathop{\mathbf{H}}_{\gamma\in\Gamma} A_\gamma = \{a \in \Pi : \operatorname{supp} a \text{ is well ordered}\}. \tag{2.2}$$

If we fixed an ordered system, we will denote the Hahn product of the given system by $\mathbb{G}$.

Notation 2.1.2. *To ease the notation, especially in in-line formulas, we will often write* $\mathbf{H}_\Gamma A_\gamma$ *for* $\mathbf{H}_{\gamma\in\Gamma} A_g$ *and similarly* $\coprod_\Gamma A_\gamma$ *for* $\coprod_{\gamma\in\Gamma} A_\gamma$.

As finite subsets of an ordered set are well ordered, we have

$$\coprod_{\gamma\in\Gamma} A_\gamma \leq \mathop{\mathbf{H}}_{\gamma\in\Gamma} A_\gamma \leq \Pi.$$

Definition 2.1.3 (Hahn group)**.** A *Hahn group* is any group G comprised between the Hahn sum and the Hahn product, of the same ordered system:

$$\coprod_{\gamma\in\Gamma} A_\gamma \leq G \leq \mathop{\mathbf{H}}_{\gamma\in\Gamma} A_\gamma. \tag{2.3}$$

If for some abelian group A and for all $\gamma \in \Gamma$ we have $A_\gamma \simeq A$, we say that G is a *balanced Hahn group*.
Note that a Hahn group is necessarily abelian.

Let us now fix a Hahn group $G \leq \mathbf{H}_{\gamma\in\Gamma} A_\gamma$.

Notation 2.1.4. *Let* $a = (a_\gamma)_{\gamma\in\Gamma} \in G$. *Denote by* $a_\gamma \mathbb{1}_\gamma$ *the tuple having* a_γ *in position* γ *and zero everywhere else. We will also call it a* monomial. *We will use the notation*

$$a = \sum_{\gamma\in\Gamma} a_\gamma \mathbb{1}_\gamma. \tag{2.4}$$

For the elements of a Hahn group G *we will use both the first letters of the alphabet* $a, b, \ldots$, *when we also give a name to the coefficients, as in* (2.4); *or we will use letters like* $g, h, \ldots$ *if we want to remind of the membership* $g \in G$.

On G we can define a valuation as follows.

Definition 2.1.5 (Canonical valuation)**.** Let $v_{\min}\colon G \to \Gamma \cup \{\infty\}$ be defined, for all $a \in G$, by

$$v_{\min}(a) = \begin{cases} \min \operatorname{supp} a & \text{if } a \neq 0 \\ \infty & \text{if } a = 0. \end{cases} \tag{2.5}$$

We will show (Lemma 2.1.6) that $v_{\min}$ is a valuation on the Hahn group G and we call it *the canonical valuation*. If there is no risk of confusion, we will drop the subscript and just write v instead of $v_{\min}$.

Lemma 2.1.6. *The map $v = v_{\min}$ defined in (2.5) is a valuation.*

Proof. We need to verify conditions (i)–(iii) from Definition 1.3.1. Let $a \in G$ and let $n \in \mathbb{Z}$, $n \neq 0$. Then $v(a) = \infty \Leftrightarrow a = 0$ holds by definition, which shows (i). Assume now $a \neq 0$ and let $\delta = v(a) = \min \operatorname{supp} a$. Write $a = \sum_{\gamma \geq \delta} a_\gamma \mathbb{1}_\gamma$. Then $na = \sum_{\gamma \geq \delta} na_\gamma \mathbb{1}_\gamma$ and $\delta = \min \operatorname{supp}(na) = v(na)$. So (ii) is proven. Now let $b = \sum b_\gamma \mathbb{1}_\gamma$ so that $a - b = \sum (a_\gamma - b_\gamma)\mathbb{1}_\gamma$. Let $\gamma_0 := \min\{v(a), v(b)\}$. For $\gamma < \gamma_0$ we have $a_\gamma - b_\gamma = 0 - 0 = 0$, hence $v(a - b) \geq \gamma_0$. This proves (iii) and finishes the proof. □

Now we determine the skeleton (Definition 1.3.9) of a Hahn group.

Theorem 2.1.7. *Let $[\Gamma; \{A_\gamma : \gamma \in \Gamma\}]$ be an ordered system of abelian groups and let $G \leq \mathbf{H}_\Gamma A_\gamma$ be a Hahn group. Let $v = v_{\min}$ be the canonical valuation on G. The skeleton $S(G)$ of (G, v) is isomorphic to the ordered system $[\Gamma; \{A_\gamma : \gamma \in \Gamma\}]$.*

Proof. By definition of v the value set is Γ, which is therefore the spine of (G, v). Let $\gamma \in \Gamma$ and recall the notation (1.1), which applied to the current situation becomes

$$G^\gamma := \{a \in G : v(a) \geq \gamma\};$$
$$G_\gamma := \{a \in G : v(a) > \gamma\}.$$

We need to show that $G^\gamma / G_\gamma \simeq A_\gamma$. We therefore consider the map

$$\begin{array}{rccc} f_\gamma \colon & G^\gamma & \to & A_\gamma \\ & a & \mapsto & a_\gamma. \end{array}$$

Because of the way addition is defined on G this is a group homomorphism. Moreover, f_γ is surjective, because for all $x \in A_\gamma$ we have $x\mathbb{1}_\gamma \in G^\gamma$ and $f_\gamma(x\mathbb{1}_\gamma) = x$. Finally, the kernel of f_γ is precisely G_γ: for all $a \in G^\gamma$ we have $f_\gamma(a) = 0$ if and only if $a_\gamma = 0$ i.e., $v(a) > \gamma$. Hence, by the first isomorphism theorem for groups, we have $G^\gamma / G_\gamma \simeq A_\gamma$ as required. □

Notation 2.1.8. *Let $[\Gamma; \{A_\gamma : \gamma \in \Gamma\}]$ be an ordered system of abelian groups and let $G \leq \mathbf{H}_\Gamma A_\gamma$ be a Hahn group. We will identify $S(G)$ with the ordered system $[\Gamma; \{A_\gamma : \gamma \in \Gamma\}]$. If G is balanced, say $A_\gamma \simeq A$ for all $\gamma \in \Gamma$, we will shorten the notation for the skeleton and write $S(G) = [\Gamma; A]$.*

Remark 2.1.9. (i) Let G be a Hahn group with skeleton $S(G) = [\Gamma; \{A_\gamma : \gamma \in \Gamma\}]$. An element $a = \sum_{\gamma \in \Gamma} a_\gamma \mathbb{1}_\gamma \in G$ is a tuple $(a_\gamma)_{\gamma \in \Gamma}$ indexed by Γ where

$a_\gamma \in A_\gamma$. Thus, we can also interpret a as a function of Γ into the union $\bigcup_\gamma A_\gamma$, as follows:

$$\begin{array}{cccc} g\colon & \Gamma & \longrightarrow & \bigcup_\gamma A_\gamma \\ & \gamma & \longmapsto & a_\gamma \in A_\gamma. \end{array}$$

(ii) Notice that, in general, $\mathbb{1}_\gamma$ is not an element of G. If the A_γ's are isomorphic rings (it will be of great importance the case where they all equal a given field, e.g., $A_\gamma = \mathbb{R}$) and we denote by 1 the unity of all of them, then $\mathbb{1}_\gamma$ can be seen as the tuple having 1 in position γ, so it is indeed an element of G, and $g_\gamma \mathbb{1}_\gamma$ is the multiplication by scalars in the A_γ-module G. □

Lemma 2.1.10. *Let G be a Hahn group with skeleton $S(G) = [\Gamma; \{A_\gamma : \gamma \in \Gamma\}]$. Assume all the A_γ's to be archimedean ordered abelian groups and that G be ordered lexicographically, so that $\Gamma = \operatorname{rk} G$ (Definition 1.4.1). Then the valuation $v = v_{\min}$ on G is equivalent to the natural valuation on G as an ordered module.*

Proof. Let $\sim$ be the equivalence relation defined in (1.4) on Page 20. We will prove that

$$x \sim y \Leftrightarrow v(x) = v(y).$$

Let $v(x) = v(y) = \gamma$. Since A_γ is archimedean, there exists $r \in R$ such that $r|x_\gamma| \geq |y_\gamma|$ and $r|y_\gamma| \geq |x_\gamma|$. As G is ordered lexicographically this means precisely that $r|x| \geq |y|$ and $r|y| \geq |x|$.
Vice versa, let $v(x) \neq v(y)$, say $v(x) < v(y)$. Then $r|x| > |y| \Leftrightarrow r > 0$, so there exists no $r \in R$ for which both $r|x| \geq |y|$ and $r|y| \geq |x|$ hold.

Moreover

$$[x] < [y] \Leftrightarrow r|x| < |y| \text{ for all } r \in R \Leftrightarrow v(y) < v(x).$$

So $v(x) = v(y) \Leftrightarrow x \sim y \Leftrightarrow v_{\text{nat}}(x) = v_{\text{nat}}(y)$ thus the two valuations are equivalent. □

2.2 The lifting property

In sections 1.3 and 1.4 we introduced valued and ordered modules. We saw that an automorphism of a valued (resp. ordered) module induces an automorphism of the skeleton (resp. the rank). In this section we introduce the *lifting property*, i.e., the property of a Hahn group that, given an automorphism of the skeleton (resp. rank) we can find an automorphism of the valued group that induces it. We will give precise definitions in a moment.

2.2.1 Lifting property with respect to the rank

Let G be a Hahn group with skeleton $S(G) = [\Gamma; \{A_\gamma : \gamma \in \Gamma\}]$ and let $\mathbb{G} = \mathbf{H}_\Gamma A_\gamma$ be the corresponding Hahn product. Assume, moreover, that all the A_γ are ordered archimedean groups, and that G is ordered lexicographically. Then Γ is also (isomorphic to) the archimedean rank of G: $\Gamma = \operatorname{rk} G$ (Definition 1.4.1). As we noticed, every automorphism $\sigma \in v\text{-}\operatorname{Aut} G$ induces an automorphism σ_Γ of the chain Γ defined by $\sigma_\Gamma(v(g)) = v(\sigma(g))$. This gives rise to a group homomorphism (see Corollary 1.3.14)

$$\Phi' := \Phi'_G : v\text{-}\operatorname{Aut} G \longrightarrow o\text{-}\operatorname{Aut} \Gamma$$

(we drop the subscript G to make the notation lighter).

Definition 2.2.1 (Lifting property w.r.t. the rank)**.** We say that an automorphism $\tau \in o\text{-}\operatorname{Aut}\Gamma$ *lifts to* G if there exists an automorphism $\sigma \in \operatorname{Aut} G$ such that $\Phi'(\sigma) = \tau$.
If Φ' admits a section, that is, an injective morphism $\Psi' \colon o\text{-}\operatorname{Aut}\Gamma \hookrightarrow v\text{-}\operatorname{Aut} G$ such that $\Psi' \circ \Phi' = \mathrm{id}_{o\text{-}\operatorname{Aut}\Gamma}$, we say that G *has the lifting property with respect to* Γ *and* Ψ'.

Definition 2.2.2 (Canonical lifting property w.r.t. the rank)**.** If $\mathbb{G}$ is balanced, say $A_\gamma \simeq A$ for all γ, then $\Phi' \colon v\text{-}\operatorname{Aut}\mathbb{G} \to o\text{-}\operatorname{Aut}\Gamma$ admits the *canonical section* Ψ'_c defined, for all $\tau \in o\text{-}\operatorname{Aut}\Gamma$, by

$$\Psi'_c(\tau)\left(\sum_\gamma g_\gamma \mathbb{1}_\gamma\right) = \sum_\gamma g_\gamma \mathbb{1}_{\tau(\gamma)}. \tag{2.6}$$

If $\Psi'_c(\tau)(G) = G$ we say that τ *lifts canonically to an automorphism of* G *and call* $\Psi'_c(\tau)|_G$ *the* canonical lift *of* τ *to* G*. If all automorphisms of* Γ *lift canonically to* G *we say that* G *has the* canonical lifting property with respect to Γ.

Examples 2.2.3. Let $[\Gamma; \{A_\gamma : \gamma \in \Gamma\}]$ be an ordered system of abelian groups.

(i) The Hahn product $\mathbb{G} = \mathbf{H}_{\gamma\in\Gamma} A_\gamma$ has the canonical lifting property with respect to Γ, by definition.

(ii) The Hahn sum $\coprod_{\gamma\in\Gamma} A_\gamma$ also has the canonical lifting property. Indeed, let $\tau \in o\text{-}\operatorname{Aut}\Gamma$ and let $\sigma := \Psi'_c(\tau) \in v\text{-}\operatorname{Aut}\mathbb{G}$ be defined as in (2.6). Let $a = \sum_{\gamma\in I} a_\gamma \mathbb{1}_\gamma \in \coprod_{\gamma\in\Gamma} A_\gamma$, for a finite set $I \subseteq \Gamma$ and let $b = \sigma(a)$. Then $b = \sum_{\gamma\in I} a_\gamma \mathbb{1}_{\tau(\gamma)}$, that is, $\operatorname{supp} b = \tau(\operatorname{supp} b)$ is finite. Thus $b \in \coprod_{\gamma\in\Gamma} A_\gamma$. So $\sigma\left(\coprod_{\gamma\in\Gamma} A_\gamma\right) \subseteq G$. With the same argument we can show that $\sigma^{-1}\left(\coprod_{\gamma\in\Gamma} A_\gamma\right) \subseteq \coprod_{\gamma\in\Gamma} A_\gamma$. Hence $\sigma\left(\coprod_{\gamma\in\Gamma} A_\gamma\right) = \coprod_{\gamma\in\Gamma} A_\gamma$, as required.

(iii) Further examples of Hahn groups satisfying a stronger lifting property (which implies the one at hand) will be given in Section 2.2.4.

In Section 2.2.2 we will study the stronger *lifting property with respect to the skeleton* and use it to obtain a decomposition of the group $v\text{-}\operatorname{Aut} G$ of valuation preserving automorphisms of G. Here we give a characterisation of the automorphisms of the rank that lift to a valuation preserving automorphism of G by means of the principal convex subgroups (defined just below). We will do this in Proposition 2.2.6, after proving a preliminary lemma.

Definition 2.2.4. For an element $\gamma \in \Gamma$, the *principal convex subgroup* of G associated to γ is the subgroup $C_\gamma = \{x \in G : v(x) \geq \gamma\}$. For $g \in G$ we will also write $C_g := C_{v(g)}$.

Lemma 2.2.5. *Let $\sigma \in v\text{-}\operatorname{Aut} G$ and let $g \in G$ and $C_g = \{x \in G :\ v(x) \geq v(g)\}$. Then we have:*

$$\sigma(C_g) = C_{\sigma(g)} = \{x :\ v(x) \geq v(\sigma(g))\}.$$

Proof. Let σ_Γ be the automorphism of the chain Γ induced by σ. Let $x \in C_g$. Then

$$v(x) \geq v(g) \Rightarrow v(\sigma(x)) = \sigma_\Gamma(v(x)) \geq \sigma_\Gamma(v(g)) = v(\sigma(g))$$

so $\sigma(x) \in C_{\sigma(g)}$, that is $\sigma(C_g) \subseteq C_{\sigma(g)}$.

Vice versa, let $x \in C_{\sigma(g)}$. Then $v(x) \geq v(\sigma(g))$ implies

$$v(\sigma^{-1}(x)) = \sigma_\Gamma^{-1}(v(x)) \geq \sigma_\Gamma^{-1}(v(\sigma(g))) = v(\sigma^{-1}(\sigma(g))) = v(g)$$

because $\sigma_\Gamma^{-1} \in o\text{-}\operatorname{Aut}\Gamma$. So $\sigma^{-1}(x) \in C_g$, that is $x \in \sigma(C_g)$. Hence $\sigma(C_g) \supseteq C_{\sigma(g)}$, which completes the proof. □

Theorem 2.2.6. *Let $\tau \in o\text{-}\operatorname{Aut}\Gamma$. Then τ lifts to an automorphism $\sigma \in v\text{-}\operatorname{Aut} G$ if and only if, for all $\gamma \in \Gamma$ there is an isomorphism*

$$\vartheta_\gamma \colon C_\gamma \xrightarrow{\sim} C_{\tau(\gamma)}$$

such that

$$\gamma > \delta \Rightarrow \vartheta_\gamma|_{C_\delta} = \vartheta_\delta. \tag{2.7}$$

Proof. Assume that τ lifts to $\sigma \in v\text{-}\operatorname{Aut} G$. Let $\gamma \in \Gamma$ and choose $g \in G$ such that $v(g) = \gamma$. Then we have $v(\sigma(g)) = \tau(v(g))$. Then Lemma 2.2.5 gives

$$\sigma(C_\gamma) = \sigma(C_g) = C_{\sigma(g)} = C_{\tau(\gamma)}.$$

Because $\sigma \in v\text{-}\operatorname{Aut} G$ condition (2.7) is satisfied.
Vice versa, suppose that for all $\gamma \in \Gamma$ there is an isomorphism

$$\vartheta_\gamma \colon C_\gamma \xrightarrow{\sim} C_{\tau(\gamma)}$$

such that $\gamma > \delta \Rightarrow \vartheta_\gamma(C_\gamma) \subset \vartheta_\delta(C_\delta)$. We want to construct a lift $\sigma \in v\text{-}\operatorname{Aut} G$ of τ. We define it as follows: for $x \in G$ with $v(x) = \gamma$, set $\sigma(x) := \vartheta_\gamma(x)$. First let us show that σ is a well defined group homomorphism. Let $x, y \in G$. We have two cases:
Case 1. $v(x) \neq v(y)$, and we can assume $v(x) < v(y)$. Then $v(x + y) = v(x) =: \gamma$. So we have

$$\sigma(x - y) = \vartheta_\gamma(x - y) = \vartheta_\gamma(x) - \vartheta_\gamma(y) = \sigma(x) - \vartheta_{v(y)}(y) = \sigma(x) - \sigma(y)$$

where in the second-last equality we used the condition (2.7).
Case 2. $v(x) = v(y) = \delta$. Then $v(x + y) =: \gamma \geq \delta$. Then

$$\sigma(x - y) = \vartheta_\gamma(x - y) = \vartheta_\delta(x - y) = \vartheta_\delta(x) - \vartheta_\delta(y) = \sigma(x) - \sigma(y).$$

Again, for the second equality we used condition (2.7).

To show that σ is an automorphism, we define its inverse. Similarly to what we did to define σ we define σ^{-1} as follows: for $x \in G$ with $v(x) = \tau(\gamma)$ we set $\sigma^{-1}(x) = \vartheta_\gamma^{-1}(x)$. Checking that this is a well defined group morphism is done the same way as it was done for σ. It is then easy to see that it is actually the inverse of σ. Indeed, let $x \in G$ with $v(x) = \gamma$. Recall that $\vartheta_\gamma(x) \in C_{\tau(\gamma)}$ and $\vartheta_\gamma^{-1} : C_{\tau(\gamma)} \to C_\gamma$. So

$$\sigma^{-1}(\sigma(x)) = \sigma^{-1}(\vartheta_\gamma(x)) = \vartheta_\gamma^{-1}(\vartheta_\gamma(x)) = x$$

and similarly $\sigma(\sigma^{-1}(x)) = x$. So σ is an automorphism of G. □

In the following example we apply Lemma 2.2.5 to show that the lifting property is not preserved after taking direct sums.

Example 2.2.7. Let $\Gamma = \mathbb{Q}$ and $G = (\mathbf{H}_{q<0}\mathbb{Q}) \amalg (\coprod_{q\geq 0}\mathbb{Q})$. So $\Gamma = \operatorname{rk} G = \mathbb{Q}$. By Example 2.2.3 $(\mathbf{H}_{q<0}\mathbb{Q})$ and $(\coprod_{q\geq 0}\mathbb{Q})$ have the (canonical) lifting property with respect to their rank. Consider the automorphism of the chain $\mathbb{Q}$ defined by $\bar{\sigma}(q) = q+2$ and assume that $\bar{\sigma}$ lifts to an automorphism σ of G. Let $g \in G$ be such that $v(g) = -1/2$. So $v(\sigma(g)) = \bar{\sigma}(-\frac{1}{2}) = \frac{3}{2} > 1$. By Lemma 2.2.5 we have $C_g \simeq C_{\sigma(g)}$. Now $C_{\sigma(g)} = \{x : v(x) \geq \frac{3}{2}\} \subseteq \coprod_{q>0}\mathbb{Q}$ is countable. But $C_g = \{x : v(x) \geq -\frac{1}{2}\}$ is uncountable. To prove this, we show that the uncountable set $\{0,1\}^{\mathbb{N}}$ of functions $f : \mathbb{N} \to \{0,1\}$ can be embedded into C_g. We construct the embedding as follows: to each function $f : \mathbb{N} \to \{0,1\}$ we associate the element

$$f^* = \sum_{q<0} f^*_q \mathbb{1}_q \in \mathop{\mathbf{H}}_{q<0}\mathbb{Q} \qquad \text{where} \quad f^*_q = \begin{cases} f(n) & \text{if } q = -\frac{1}{2+n} \\ 0 & \text{otherwise.} \end{cases}$$

First, notice that, for all $f \in \{0,1\}^{\mathbb{N}}$, the element f^* belongs to C_g: indeed we have $\operatorname{supp} f^* \subseteq \{-\frac{1}{2+n} : n \in \mathbb{N}\}$ and this last set is well ordered, so f^* is a member of $\mathbf{H}_{q<0}\mathbb{Q}$ and $v(f^*) \geq -\frac{1}{2}$. Moreover, the map $f \mapsto f^*$ is injective, for if $f \neq g$ then there exists $n \in \mathbb{N}$ such that

$$f^*_{-\frac{1}{2+n}} = f(n) \neq g(n) = g^*_{-\frac{1}{2+n}}.$$

□

The last example also shows that the lexicographic sum of groups with the lifting property might not inherit the lifting property. However, we have the following result, which characterises the automorphisms of the rank of a lexicographic sum that admit a lift.

Proposition 2.2.8. *Let G_1, G_2 be two groups with the lifting property (with respect to the rank) and consider $G = G_1 \amalg G_2$. Then $\operatorname{rk} G = \operatorname{rk} G_1 + \operatorname{rk} G_2$ (ordinal sum). Let τ be an automorphism of $\operatorname{rk} G$ such that $\tau|_{\operatorname{rk} G_i} \in o\text{-}\operatorname{Aut}(\operatorname{rk} G_i)$ for $i = 1,2$. Then τ lifts to an automorphism of G.*

Proof. Let $\Gamma := \operatorname{rk} G$ and $\Gamma_i := \operatorname{rk} G_i$ for $i = 1,2$. Let $\tau \in o\text{-}\operatorname{Aut}\Gamma$ be such that $\tau_i := \tau|_{\Gamma_i} \in o\text{-}\operatorname{Aut}\Gamma_i$ for $i = 1,2$. Since both G_i's have the lifting property, let $\sigma_i \in v\text{-}\operatorname{Aut} G_i$ be a lift of τ_i and define $\sigma\colon G \to G$ through $\sigma(g_1\mathbb{1}_1 + g_2\mathbb{1}_2) := \sigma_1(g_1)\mathbb{1}_1 + \sigma_2(g_2)\mathbb{1}_2$. It is straightforward to verify that this is a valuation preserving automorphism of G. □

2.2.2 Lifting property with respect to the skeleton

Let G be a Hahn group with skeleton $S(G) = [\Gamma; \{A_\gamma : \gamma \in \Gamma\}]$. In this section we study a stronger lifting property: the lifting property with respect to the skeleton. In Section 1.3 we defined what an isomorphism of skeletons is, and showed that a valuation preserving automorphism of valued groups induces an isomorphism of the skeletons (cf. Lemma 1.3.10). In particular, every valuation preserving automorphism $\sigma \in v\text{-}\operatorname{Aut} G$ induces an automorphism $\sigma_{S(G)} = [\sigma_\Gamma; \{\sigma_\gamma : \gamma \in \Gamma\}] \in \operatorname{Aut} S(G)$. This gives rise to a group homomorphism

$$\begin{array}{rccc} \Phi_G\colon & v\text{-}\operatorname{Aut} M & \to & \operatorname{Aut} S(G) \\ & \sigma & \mapsto & \sigma_{S(G)}. \end{array} \tag{2.8}$$

Whenever the context is clear we will drop G from the notation and write $\Phi := \Phi_G$ and $\sigma_S := \sigma_{S(G)}$.

Definition 2.2.9 (Internal automorphisms)**.** The kernel of the map Φ defined in (2.8) will be called the group of *internal automorphisms of* G and denoted by $\operatorname{Int}\operatorname{Aut} G$.

The following proposition establishes some fundamental properties of internal automorphisms.

Proposition 2.2.10. *The following hold.*

(i) $\operatorname{Int}\operatorname{Aut} G \trianglelefteq v\text{-}\operatorname{Aut} G$.

(ii) Let $\sigma \in v\text{-}\operatorname{Aut} G$. *Then* $\sigma \in \operatorname{Int}\operatorname{Aut} G$ *if and only if, for all* $g \in G$ *we have*

$$v(g) = v(\sigma(g)) \quad \text{and} \quad g_{v(g)} = \sigma(g)_{v(g)}. \tag{2.9}$$

Proof. (i) Follows from the definition: $\operatorname{Int}\operatorname{Aut} G = \ker \Phi$.

(ii) Assume $\sigma \in \operatorname{Int}\operatorname{Aut} G$ and fix $g \in G$. Let $\gamma = v(g)$. Then σ induces the identity on Γ, that is $\sigma_\Gamma = \mathrm{id}_\Gamma$, and so $\sigma_\Gamma(v(g)) = v(\sigma(g)) = v(g)$, so the first condition is fulfilled. Moreover, the induced automorphism on A_γ is the identity: $\sigma_\gamma = \mathrm{id}_{A_\gamma}$ and so

$$\pi_\gamma(\sigma(g)) = \sigma_\gamma(\pi_\gamma(g)) = \pi_\gamma(g).$$

Vice versa, assume (2.9) holds for all $g \in G$. The first condition implies that the automorphism σ_Γ induced on Γ is the identity. Fix $\delta \in \Gamma$ and let

$g_\delta \in A_\delta$. Define $g \in G$ by $\pi_\delta(g) = g_\delta$ and $\pi_\gamma(g) = 0$ for all $\gamma \neq \delta$. Then $v(g) = \delta$ and, by assumption, $\sigma_\delta(g_\delta) = \pi_\delta(\sigma(g)) = \pi_\delta(g) = g_\delta$. Hence $\sigma_\delta = \mathrm{id}_{A_\delta}$ for all $\delta \in \Gamma$.

□

Remark 2.2.11. In Chapter 3 we will study two notions of lifting property for Hahn fields. We warn the reader that, in the context of Hahn fields, internal automorphisms will only satisfy the first of the two properties of (2.9). Both properties will only be satisfied by a special subgroup of internal automorphisms, that we will call 1-automorphisms. □

The group $\mathrm{Int\,Aut}\, G$ is an important normal subgroup of v- $\mathrm{Aut}\, G$. Aiming at a decomposition of v- $\mathrm{Aut}\, G$ we now want to determine a complement of $\mathrm{Int\,Aut}\, G$. To this end we introduce the following definition.

Definition 2.2.12 (Lifting property w.r.t. the skeleton)**.** (i) We say that an automorphism $\tau \in \mathrm{Aut}\, S(G)$ *lifts to* G if there exists an automorphism $\sigma \in v$- $\mathrm{Aut}\, G$ such that $\tau = \Phi(\sigma)$.

(ii) If there exists a section $\Psi_G\colon \mathrm{Aut}\, S(G) \to v$- $\mathrm{Aut}\, G$ i.e., an injective homomorphism such that $\Phi_G \Psi_G = \mathrm{id}_{\mathrm{Aut}\, S(G)}$ we say that G *has the lifting property with respect to* $S(G)$ *and* Ψ_G.
Again, if no confusion can arise, we will omit the subscript G from Ψ.

Let $\mathbb{G}$ be the Hahn product over $S(G)$. Then $\Phi_{\mathbb{G}}$ admits the *canonical section*

$$\begin{array}{cccc} \Psi_{\mathbb{G},c}\colon & \mathrm{Aut}\, S(\mathbb{G}) & \to & v\text{- Aut}\, \mathbb{G} \\ & \tau = [\tau_\Gamma : \{\tau_\gamma : \gamma \in \Gamma\}] & \mapsto & \tilde{\tau} \end{array} \tag{2.10}$$

where $\tilde{\tau} \in v$- $\mathrm{Aut}\, \mathbb{G}$ is defined by

$$\tilde{\tau}\left(\sum g_\gamma \mathbb{1}_\gamma\right) = \sum \tau_\gamma(g_\gamma) \mathbb{1}_{\tau_\Gamma(\gamma)}.$$

Definition 2.2.13 (Canonical lifting property w.r.t. the skeleton)**.** (i) We call $\tilde{\tau}$ the *the canonical lift of* τ *to* $\mathbb{G}$.

(ii) If the restriction $\tilde{\tau}|_G$ is an automorphism of G we say that τ *lifts canonically to* G and call $\tilde{\tau}|_G$ *the canonical lift of* τ *to* G.

(iii) If

$$\begin{array}{cccc} \Psi_{G,c}\colon & \mathrm{Aut}\, S(G) & \to & v\text{- Aut}\, G \\ & \tau = [\tau_\Gamma : \{\tau_\gamma : \gamma \in \Gamma\}] & \mapsto & \tilde{\tau}|_G \end{array}$$

is a section of Φ_G we say that G has the *canonical lifting property with respect to* $S(G)$. In particular, every automorphism of $S(G)$ lifts canonically to G.

Remark 2.2.14. (i) Let G be a balanced Hahn group, say with skeleton $S(G) = [\Gamma; A]$ and assume that G has the lifting property w.r.t. the skeleton. Then G has the lifting property w.r.t. the rank. Indeed, let $\tau_\Gamma \in o\text{-}\operatorname{Aut}\Gamma$. Then the automorphism $\tau_S = [\tau_\Gamma; \{\mathrm{id}_A : \gamma \in \Gamma\}] \in \operatorname{Aut} S(G)$ lifts to an automorphism $\sigma \in v\text{-}\operatorname{Aut} G$ such that $\Phi'_G(\sigma) = \tau_\Gamma$. In other words, the map

$$\begin{array}{rccc} \Psi'_G\colon & o\text{-}\operatorname{Aut}\Gamma & \to & v\text{-}\operatorname{Aut} G \\ & \tau_\Gamma & \mapsto & \Psi_G\left([\tau_\Gamma; \{\mathrm{id}_A : \gamma \in \Gamma\}]\right) \end{array}$$

is a section of Φ'_G.

(ii) It is worth noting that not all automorphisms of the rank can always occur in automorphisms of the skeleton. Consider the ordered system $S = [\mathbb{Z}; \{A_n : n \in \mathbb{Z}\}]$ where $A_{2n} = \mathbb{Q}$ and $A_{2n+1} = \mathbb{R}$. Then the map $n \mapsto n+1$ is an automorphism of the chain $\mathbb{Z}$, but it cannot occur in an automorphism of S: indeed, this would imply the existence on an isomorphism $A_0 = \mathbb{Q} \simeq \mathbb{R} = A_1$, which is impossible.

□

Definition 2.2.15 (External automorphisms)**.** Let G be a Hahn group with the lifting property w.r.t. its skeleton $S(G)$ and let Ψ be a section of the map Φ_G defined in (2.8). The subgroup $\Psi\text{-}\operatorname{Ext}\operatorname{Aut} G := \operatorname{im}\Psi$ is called the group of Ψ*-external automorphisms of* G.

Remark 2.2.16. (i) Because Ψ is an injective homomorphism it follows that $\Psi\text{-}\operatorname{Ext}\operatorname{Aut} G \simeq \operatorname{Aut} S(G)$, independent of the choice of the section Ψ. Therefore, whenever G has the lifting property, we will assume to have fixed a section Ψ and omit it from the notation and terminology.

(ii) If G has the lifting property w.r.t. $S(G)$ and Ψ, the homomorphism Φ_G is surjective and every $\tau \in \operatorname{Aut} S(G)$ lifts to an automorphism $\sigma = \Psi(\tau) \in v\text{-}\operatorname{Aut} G$. The first isomorphism theorem then yields

$$\operatorname{Aut} S(G) \simeq \frac{v\text{-}\operatorname{Aut} G}{\operatorname{Int}\operatorname{Aut} G}.$$

In particular, the set of lifts of some $\tau \in \operatorname{Aut} S(G)$ is the coset $\{\sigma\sigma' : \sigma' \in \operatorname{Int}\operatorname{Aut} G\}$, where σ is any given lift of τ. □

2.2.3 A decomposition theorem

We fix a Hahn group G with skeleton $S(G) = [\Gamma : \{A_\gamma : \gamma \in \Gamma\}]$. We showed that $v\text{-}\operatorname{Aut} G$ has the normal subgroup $\operatorname{Int}\operatorname{Aut} G$. Assume that G has the lifting property with respect to the skeleton and to a given section Ψ. Then we also have another subgroup of $v\text{-}\operatorname{Aut} G$: the group $\operatorname{Ext}\operatorname{Aut} G$ of external automorphisms. The following is a generalisation to Hahn groups of [Hof91, Satz 2.2].

Theorem 2.2.17. *Let G be a Hahn group with the lifting property w.r.t. the skeleton. Then we have*

$$v\text{-}\operatorname{Aut} G = \operatorname{Int}\operatorname{Aut} G \rtimes \operatorname{Ext}\operatorname{Aut} G.$$

Proof. Let us consider the sequence

$$\operatorname{Int}\operatorname{Aut} G \xrightarrow{\iota} v\text{-}\operatorname{Aut} G \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \operatorname{Aut} S(G)$$

where we denote by ι the canonical embedding. Then, by definition of internal automorphism, we have $\operatorname{im}\iota = \ker\Phi$, so the sequence is exact. Moreover, sice G has the lifting property we have $\Phi\Psi = \operatorname{id}_{S(G)}$. So Lemma 1.2.12 applies, yielding

$$v\text{-}\operatorname{Aut} G = \operatorname{im}\iota \rtimes \operatorname{im}\Psi = \ker\Phi \rtimes \operatorname{im}\Psi = \operatorname{Int}\operatorname{Aut} G \rtimes \operatorname{Ext}\operatorname{Aut} G.$$

□

Applying Remark 2.2.16 we obtain the following corollary to Theorem 2.2.17.

Corollary 2.2.18. *Let G be a Hahn group with the lifting property w.r.t. the skeleton. Then we have*

$$v\text{-}\operatorname{Aut} G \simeq \operatorname{Int}\operatorname{Aut} G \rtimes \operatorname{Aut} S(G).$$

□

We introduced the lifting property with respect to the skeleton and showed that, under this condition, the group $v\text{-}\operatorname{Aut} G$ can be decomposed into the semidirect product $\operatorname{Int}\operatorname{Aut} G \rtimes \operatorname{Ext}\operatorname{Aut} G$ of its subgroups of internal and external automorphisms, provided G satisfies the lifting property w.r.t. the skeleton. In the next section we show that there is a sufficient supply of Hahn groups satisfying the lifting property w.r.t. the skeleton.

2.2.4 Rayner groups

Let $\mathbb{G} = \mathbf{H}_\Gamma A_\gamma$ be the Hahn product over the skeleton $[\Gamma; \{A_\gamma : \gamma \in \Gamma\}]$.

Definition 2.2.19 (Rayner group family)**.** A family $\mathcal{F}$ of subsets of Γ is said to be a *Rayner group family with respect to* Γ (cf. [Ray68, Paragraph 2]) if the following properties are satisfied:

(RG1) the elements of $\mathcal{F}$ are well ordered subsets of Γ;

(RG2) $B, C \in \mathcal{F} \Rightarrow B \cup C \in \mathcal{F}$;

(RG3) $B \in \mathcal{F} \wedge C \subseteq B \Rightarrow C \in \mathcal{F}$.

The following proposition is part of [Ray68, Theorem 1]. Since the statement is not the same, we include a proof.

Proposition 2.2.20. *If $\mathcal{F}$ is a Rayner group family then the set $\mathbb{G}(\mathcal{F})$ of elements of $\mathbb{G}$ whose supports belong to $\mathcal{F}$ is a subgroup of $\mathbb{G}$.*

Proof. By property (i) all elements of $\mathcal{F}$ are well ordered so they can occur as supports of elements of $\mathbb{G}$. Since for all $g \in \mathbb{G}$ we have $\operatorname{supp} g = \operatorname{supp}(-g)$ then $g \in \mathbb{G}(\mathcal{F})$ implies $-g \in \mathbb{G}(\mathcal{F})$. If $g, h \in \mathbb{G}(\mathcal{F})$, then $\operatorname{supp}(g + h) \subseteq \operatorname{supp} g \cup \operatorname{supp} h$: the latter is in $\mathcal{F}$ by (ii) and hence by (iii) the former is too. So $g + h \in \mathbb{G}(\mathcal{F})$. It follows then that also $0 \in \mathbb{G}(\mathcal{F})$ so $\mathbb{G}(\mathcal{F})$ is indeed a subgroup of $\mathbb{G}$. $\square$

Definition 2.2.21 (Rayner group)**.** Groups obtained as in Proposition 2.2.20 will be called *Rayner groups*.

Proposition 2.2.22. *Let $\mathcal{F}$ be a Rayner group family and let $G = \mathbb{G}(\mathcal{F})$ be the corresponding Rayner group. Then G is a Hahn group if and only if $\mathcal{F}$ satisfies*

(RG4) $\{\gamma\} \in \mathcal{F}$ *for all* $\gamma \in \Gamma$ *for which* $A_\gamma \neq \{0\}$.

Proof. Let G be a Hahn group. Then $\coprod_{\gamma\in\Gamma} A_\gamma \leq G$. For all $\gamma \in \Gamma$ such that $A_\gamma \neq \{0\}$ let $a_\gamma \in A_\gamma \setminus \{0\}$. Then $a_\gamma \mathbb{1}_\gamma \in G$ and $\{\gamma\} = \operatorname{supp}(a_\gamma \mathbb{1}_\gamma)$. So, by definition of $\mathbb{G}(\mathcal{F})$ we must have $\{\gamma\} \in \mathcal{F}$.
Vice versa, assume $\mathcal{F}$ satisfies (RG4) and let $a := \sum_{\gamma\in I} a_\gamma \mathbb{1}_\gamma \in \coprod_{\gamma\in\Gamma} A_\gamma$ for a finite set $I \subseteq \Gamma$. So $\operatorname{supp}(a) \subseteq I$. Because $\mathcal{F}$ is a Rayner group family, by (RG2) we have $I \in \mathcal{F}$. Now by (RG3) we have $\operatorname{supp}(a) \in \mathcal{F}$ and therefore $a \in \mathbb{G}(\mathcal{F}) = G$. $\square$

Rayner groups generalise the analogous notion of Rayner fields, introduced by Rayner in [Ray68] and which we will study in Subsection 3.3.6. Further generalisations (Rayner structures) will be studied in Appendix A, based on the joint work [KKS21] with L. S. Krapp and S. Kuhlmann.

Example 2.2.23. Let $\mathbb{G} = \mathbf{H}_\Gamma A_\gamma$.

(i) The families $\mathcal{F}_0$ of all finite subsets of Γ and $\mathcal{W}$ of all well ordered subsets clearly satisfy the conditions of a Rayner group family. Hence $\mathbb{G}(\mathcal{F}_0) = \coprod_\Gamma A_\gamma$ and $\mathbb{G}(\mathcal{W}) = \mathbb{G}$ are special cases of Rayner subgroups.

(ii) Let κ be any cardinal. The family $\mathcal{F}_\kappa$ of well ordered subsets of Γ with cardinality smaller than κ is a Rayner group family. Indeed, let $A, B \subseteq \Gamma$ be such that $|A|, |B| < \kappa$. Then also $|A \cup B| < \kappa$, and every subset of A has cardinality smaller than κ. So $\mathcal{F}_\kappa$ satisfies (RG1)–(RG3).

By Proposition 2.2.20 $\mathbb{G}_\kappa := \mathbb{G}(\mathcal{F}_\kappa)$ is a group that we call the *κ-bounded subgroup* of $\mathbb{G}$. It consists of all elements of $\mathbb{G}$ whose support has cardinality less then κ.

In particular, the Hahn sum and product of (i) are also special cases of κ-bounded subgroups: $\mathbb{G}_{\aleph_0} = \coprod_\Gamma B_\gamma$ and $\mathbb{G}_{|\Gamma|^+} = \mathbb{G}$.

(iii) Notice that all the examples given in parts (i) and (ii) are Hahn groups, as the corresponding group families all satisfy condition (RG4). □

The next proposition gives a necessary and sufficient condition for a Rayner group to satisfy the canonical lifting property.

Proposition 2.2.24. *Let $\mathbb{G}$ be the Hahn product over the skeleton $[\Gamma; \{A_\gamma : \gamma \in \Gamma\}]$. Let $\mathcal{F}$ be a Rayner group family satisfying (RG4). Then the Hahn group $G = \mathbb{G}(\mathcal{F})$ has the canonical lifting property with respect to the skeleton if and only if $\mathcal{F}$ is stable under the action of $\operatorname{Aut} S(G)$, that is, for all $[\tau_\Gamma; \{\tau_\gamma : \gamma \in \Gamma\}] \in \operatorname{Aut} S(G)$ and for all $T \in \mathcal{F}$ we have $\tau_\Gamma(T) = T$.*

Proof. Assume that $\mathcal{F}$ is stable under o-$\operatorname{Aut}\Gamma$ and let $a = \sum a_\gamma \mathbb{1}_\gamma \in G$. Let also $[\tau_\Gamma; \{\tau_\gamma : \gamma \in \Gamma\}] \in \operatorname{Aut} S(G)$. Then we have $\operatorname{supp}\left(\sum \tau_\gamma(a_\gamma)\mathbb{1}_{\tau_\Gamma(\gamma)}\right) = \{\tau_\Gamma(\gamma) : \gamma \in \operatorname{supp} a\} = \tau_\Gamma(\operatorname{supp} a)$ which, by our assumption, belongs to $\mathcal{F}$. Thus $\sum \tau_\gamma(a_\gamma)\mathbb{1}_{\tau_\Gamma(\gamma)} \in G$ and so G has the canonical lifting property with respect to the skeleton.

Vice versa, assume that G has the canonical lifting property and let $T \in \mathcal{F}$. Let $a = \sum a_\gamma \mathbb{1}_\gamma \in G$ be such that $\operatorname{supp} a = T$. Then, for all $[\tau_\Gamma; \{\tau_\gamma : \gamma \in \Gamma\}] \in \operatorname{Aut} S(G)$ we have $b := \sum \tau_\gamma(a_\gamma)\mathbb{1}_{\tau_\Gamma(\gamma)} \in G$, so in particular $\operatorname{supp} b \in \mathcal{F}$. So $\mathcal{F}$ is stable under action of $\operatorname{Aut} S(G)$. □

Corollary 2.2.25. *Let $\mathbb{G} = \mathbf{H}_\Gamma A_\gamma$ and let κ be an infinite regular cardinal. Then $\mathbb{G}_\kappa$ has the canonical lifting property. In particular, $\mathbb{G}$ and $\coprod_\Gamma A_\gamma$ have the canonical lifting property.*

Proof. By Proposition 2.2.24 it suffices to show that the family $\mathcal{F}_\kappa$ of well ordered subsets of Γ is stable under $\operatorname{Aut} S(G)$. Let $[\tau_\Gamma; \{\tau_\gamma : \gamma \in \Gamma\}] \in \operatorname{Aut} S(G)$ and let T be a well ordered subset of Γ of cardinality less than κ. Then $\tau_\Gamma(T)$ will have the same cardinality as T, because τ_Γ is an automorphism of Γ. Since it is also order preserving then $\tau_\Gamma(T)$ will also be well ordered, hence $\tau_\Gamma(T) \in \mathcal{F}_\kappa$. □

We can revisit Example 2.2.7 from the point of view of Rayner groups, which provides us with a much shorter argument to exclude the canonical lifting property.

Example 2.2.26. Let $\Gamma = \mathbb{Q}$ and let $G = (\mathbf{H}_{q<0}\,\mathbb{Q}) \amalg \mathbb{Q} \amalg (\coprod_{q>0} \mathbb{Q})$. Then we can view G as the subgroup $\mathbb{G}(\mathcal{F})$ of $\mathbb{G} = \mathbf{H}_\mathbb{Q}\,\mathbb{Q}$ where

$$\mathcal{F} = \{A \subseteq \mathbb{Q} : A \cap \mathbb{Q}^{<0} \text{ is well ordered and } A \cap \mathbb{Q}^{\geq 0} \text{ is finite}\}.$$

It is clear that $\mathcal{F}$ is a family of well ordered subsets of $\mathbb{Q}$ and it is closed under taking subsets and finite unions, so it is indeed a Rayner group family. Nevertheless, it is not stable under the action of $o\text{-}\operatorname{Aut}\mathbb{Q}$ (recall that, here, $\mathbb{Q}$ is only seen as a chain). For example, if we take the order preserving automorphism $\tau \in o\text{-}\operatorname{Aut}\mathbb{Q}$ defined by $\tau(q) = q + 2$ and the set $A = \{-\frac{1}{n} : n \in \mathbb{N}^{>0}\} \in \mathcal{F}$ we have $\tau(A) = \{2 - \frac{1}{n} : n \in \mathbb{N}^{>0}\} \notin \mathcal{F}$ because $\tau(A)$ is infinite and contained in $\mathbb{Q}^{>0}$. So, by Proposition 2.2.24, G can't have the canonical lifting property. □

2.3 Hahn sums

In this section we consider the special case of Hahn sums. These offer the considerable advantage of admitting valuation bases, which simplify the computations as we will see in Subsection 2.3.1. In Subsection 2.3.2 we will focus on the case of an ordered Hahn sum G. We will describe the group $o\text{-}\operatorname{Aut} G$ of order preserving automorphisms of G as a group of (infinite) matrices and provide a semidirect product decomposition of said group, thereby generalising and improving results of Conrad [Con58] and Droste and Göbel [DG97]. The results of Subsection 2.3.2 will be applied in Section 3.5 to get explicit descriptions of the groups of valuation preserving automorphisms of some special Hahn fields.

2.3.1 Valuation bases

Let G be a Hahn group that is also a vector space over $\mathbb{Q}$ (e.g. G is uniquely divisible). Recall (Definition 1.3.16) that a valuation basis is a basis of the vector space which is also valuation independent. The existence of a valuation basis can be characterised as follows.

Lemma 2.3.1 ([Kuh00b, Corollary 0.5]). *A valued vector space admits a valuation basis if an only if it is isomorphic to the Hahn sum over its skeleton.* □

For the remainder of this section we will assume that $G = \coprod_{\gamma\in\Gamma} A_\gamma$ is a Hahn sum.

Proposition 2.3.2. *A bijection of valuation bases extends to a valuation preserving automorphism.*

Proof. Let $\mathcal{B}$ and $\mathcal{C}$ be two valuation bases of G and let $\vartheta\colon \mathcal{B} \to \mathcal{C}$ be a bijection. Note that two valuation bases are, in particular, bases of the vector space G, so they are equipotent. For all $g \in G$ write $g = \sum_{b\in\mathcal{B}} g_b b$. The map

$$\begin{array}{rccc} \hat{\vartheta}\colon & G & \to & G \\ & \sum_{b\in\mathcal{B}} g_b b & \mapsto & \sum_{b\in\mathcal{B}} g_b \vartheta(b) \end{array}$$

is an automorphism of G, by general linear algebra. It remains to show that it is valuation preserving. Let $h = \sum_{b\in\mathcal{B}} h_b b$ be such that $v(g) = v(h)$. That is $\min_{g_b\neq 0}\{v(b)\} = \min_{h_b\neq 0}\{v(b)\}$. Then we have

$$\min_{g_b\neq 0}\{v(\hat{\vartheta}(b))\} = \min_{h_b\neq 0}\{v(\hat{\vartheta}(b))\}$$

which means exactly $v(\hat{\vartheta}(g)) = v(\hat{\vartheta}(h))$. □

If we have a valuation basis $\mathcal{B}$, to characterise internal automorphisms it is enough to impose condition (2.9) from page 41 on the elements of $\mathcal{B}$. We will make use of the following lemma:

Lemma 2.3.3 ([Kuh00b, Corollary 0.12]). *A subset $\mathcal{B} \subseteq G \setminus \{0\}$ is a valuation basis if and only if, for all $\gamma \in \Gamma$, the set*

$$\mathcal{B}_\gamma := \{b_\gamma : b \in \mathcal{B} \text{ and } v(b) = \gamma\}$$

is a basis of A_γ. □

Proposition 2.3.4. *Let $\mathcal{B}$ be a valuation basis of G and $\sigma \in v\text{-}\operatorname{Aut} G$. Then σ is internal if and only if, for all $b \in \mathcal{B}$, we have*

$$v(b) = v(\sigma(b)) \quad \text{and} \quad b_{v(b)} = \sigma(b)_{v(b)}. \tag{2.11}$$

Proof. Assume σ is internal. Then $\sigma_\Gamma = \operatorname{id}_\Gamma$ and $\sigma_\gamma = \operatorname{id}_{A_\gamma}$. By definition of σ_γ, for all $b \in \mathcal{B}$ we have

$$\sigma(b)_\gamma = \sigma(b)_{\sigma_\Gamma(\gamma)} = \sigma_\gamma(b_\gamma) = b_\gamma$$

from which it also follows that $v(b) = v(\sigma(b))$.
Vice versa, suppose (2.11) holds for all $b \in \mathcal{B}$. We remark that, since $\sigma \in v\text{-}\operatorname{Aut} G$ then $\sigma(\mathcal{B})$ is also a valuation basis of G. By hypothesis we have $b \in \mathcal{B} \Rightarrow v(b) = v(\sigma(b))$, that is $\sigma_\Gamma|_{v(\mathcal{B})} = \operatorname{id}_\Gamma$. Now $\mathcal{B}$ is a basis, so every $a \in G$ can be expressed as $a = \sum_{b \in \mathcal{B}} a_b b$ and so $\sigma(x) = \sum_{b \in \mathcal{B}} a_b \sigma(b)$ (in these sums all but finitely many coefficients are zero). Therefore:

$$\begin{aligned} v(\sigma(x)) &= v\left(\sum_{b \in \mathcal{B}} a_b \sigma(b)\right) \\ &= \min\{v(\sigma(b))\} \quad [\sigma(\mathcal{B}) \text{ is a valuation basis}] \\ &= \min\{v(b)\} \quad [\text{by hypothesis}] \\ &= v\left(\sum_{b \in \mathcal{B}} a_b b\right) \quad [\mathcal{B} \text{ is a valuation basis}] \\ &= v(x). \end{aligned}$$

Hence, for all $a \in G$ we have $\sigma_\Gamma(v(a)) = v(\sigma(a)) = v(a)$ thus $\sigma_\Gamma = \operatorname{id}_\Gamma$. Now let us fix $\gamma \in \Gamma$. Since $\mathcal{B}$ is a valuation basis, then $\mathcal{B}_\gamma = \{b_\gamma : b \in \mathcal{B} \text{ and } v(b) = \gamma\}$ is a basis of A_γ. By definition of σ_γ, for all $b_\gamma \in \mathcal{B}_\gamma$, we have $\sigma_\gamma(b_\gamma) = \sigma(b)_\gamma = b_\gamma$ hence σ_γ is the identity on a basis of A_γ and thus on the whole of it. □

Let us give an easy example of a non-trivial internal automorphism constructed by means of a valuation basis.

Example 2.3.5. Consider $\mathbb{Q} \amalg \mathbb{Q}$ ordered lexicographically. Take the two bases $\mathcal{B} = \{b_1 = (1,1), b_2 = (0,1)\}$ and $\mathcal{B}' = \{b'_1 = (1,0), b'_2 = (0,1)\}$. Mapping one onto the other produces a non-trivial internal automorphism, namely the one that maps the element (q,r) to the element $(q, q+r)$, for all $q, r \in \mathbb{Q}$. Indeed, the resulting automorphism σ is non-trivial because it is not the identity on the first basis-element. It is internal by virtue of the forgoing proposition, because $v(b_i) = v(\sigma(b_i))$ and $(b_i)_{v(b_i)} = (\sigma(b_i))_{v(b_i)}$, $i = 1, 2$.

Using valuation bases we can give an alternative proof that Hahn sums have the lifting property.

Proposition 2.3.6. *Let $\mathcal{B}$ be a valuation basis of G. An automorphism $[\sigma_\Gamma; \{\sigma_\gamma : \gamma \in \Gamma\}] \in \operatorname{Aut} S(G)$ lifts to an automorphism $\sigma \in v\text{-}\operatorname{Aut} G$.*

Proof. We will prove this by explicitly defining $\sigma(\mathcal{B})$. For $b \in \mathcal{B}$ define

$$\sigma(b)_{\sigma_\Gamma(\gamma)} := \sigma_\gamma(\pi_\gamma(b)).$$

Now, $\sigma(\mathcal{B})$ is a valuation basis, indeed, for all $\gamma \in \Gamma$, we have

$$\begin{aligned}
(\sigma(\mathcal{B}))_{\tilde{h}(\gamma)} &= \{\pi_{\sigma_\Gamma(\gamma)}(\sigma(b)) \mid b \in \mathcal{B} \text{ and } v(\sigma(b)) = \sigma_\Gamma(\gamma)\} \\
&= \{\sigma_\gamma(\pi_\gamma(b)) \mid b \in \mathcal{B} \text{ and } v(b) = \gamma\} \\
&= \sigma_\gamma(\mathcal{B}_\gamma).
\end{aligned}$$

Since $\mathcal{B}$ is a valuation basis, then $\mathcal{B}_\gamma$ is a basis of A_γ for all $\gamma \in \Gamma$; since the σ_γ's are isomorphisms, $\sigma_\gamma(\mathcal{B}_\gamma) = (\sigma(\mathcal{B}))_{\sigma_\Gamma(\gamma)}$ is a basis of $A_{\sigma_\Gamma(\gamma)}$ and therefore, by Lemma 2.3.3, $\sigma(\mathcal{B})$ is a valuation basis of G. So σ maps a valuation basis onto another, hence by Proposition 2.3.2 we have $\sigma \in v\text{-}\operatorname{Aut} G$. □

2.3.2 Automorphism groups as groups of matrices

In this section we are going to generalise and sharpen results of Conrad [Con58] and Droste and Göbel [DG97]. The first describes the group of order preserving automorphisms of a Hahn sum as a group of (infinite) triangular matrices. The latter obtain a semidirect product decomposition, which is different from the one we gave in Theorem 2.2.17. Our main goal here is to study the group

o- Aut G of order preserving automorphisms of an ordered Hahn sum G, which we want to describe as a special group of matrices.

Let $G = \coprod_{\gamma\in\Gamma} A_\gamma$ be the Hahn sum over the skeleton $[\Gamma; \{A_\gamma : \gamma \in \Gamma\}]$. Let us introduce some notation.

Notation 2.3.7. *(i) For an arbitrary group $(H, +)$ let* End H *be the ring of endomorphisms of H, endowed with the operations of pointwise addition and composition of functions.*

(ii) For all $\alpha, \beta \in \Gamma$ let $H_{\alpha\beta} = \mathrm{Hom}(A_\alpha, A_\beta)$ be the group of homomorphisms from A_α into A_β (with pointwise addition).

(iii) Let Δ be the set of all $\Gamma \times \Gamma$-matrices $(\sigma_{\alpha\beta})$ where

(a) $\sigma_{\alpha\alpha} \in \mathrm{End}\, A_\alpha$;

(b) $\sigma_{\alpha\beta} \in H_{\alpha\beta}$;

(c) for every $\alpha \in \Gamma$ and for all $x \in A_\alpha$ we have $\sigma_{\alpha\beta}(x) = 0$ for all but finitely many β.

The set Δ forms a ring under the usual operations of sum and multiplication of matrices.

Lemma 2.3.8. *There is a ring isomorphism between* End G *and Δ.*

Proof. Let $(\sigma_{\alpha\beta}) \in \Delta$. Then, for $a = \sum a_\gamma \mathbb{1}_\gamma \in G$ and $(\sigma_{\alpha\beta})_{\alpha,\beta\in\Gamma} \in \Delta$ we can consider the row vector $(a_\gamma)_{\gamma\in\Gamma}$ and multiply it on the right to get

$$(\sigma_{\alpha\beta})(a_\gamma) = \left(\sum_{\alpha\in\mathrm{supp}(a)} \sigma_{\alpha\beta}(a_\alpha) \right)_{\beta\in\Gamma} =: (b_\beta)_{\beta\in\Gamma} \tag{2.12}$$

where the sum in the middle term is finite because supp(a) is, and $(b_\beta)_{\beta\in\Gamma}$ has finite support because of the condition (c) satisfied by $\sigma_{\alpha\beta}$. Then the map

$$\begin{array}{rccc} \sigma\colon & G & \to & G \\ & \sum a_\gamma \mathbb{1}_\gamma & \mapsto & \sum b_\gamma \mathbb{1}_\gamma \end{array} \tag{2.13}$$

is an endomorphism of G induced by $(\sigma_{\alpha\beta})$. Hence we get a map

$$\begin{array}{rccc} \zeta\colon & \Delta & \to & \mathrm{End}\, G \\ & (\sigma_{\alpha\beta}) & \mapsto & \sigma \end{array}$$

where σ is defined as in (2.13).

Vice versa, if $\sigma \in \operatorname{End} G$ and $a = \sum a_\gamma \mathbb{1}_\gamma \in G$ let $b = \sum b_\gamma \mathbb{1}_\gamma = \sigma(a)$. Then for all $\alpha \in \Gamma$ define $\sigma_{\alpha\beta}(a_\alpha) := b_\beta$. Since $b \in G$ then $\operatorname{supp} b$ is finite and therefore $\sigma_{\alpha\beta}$ satisfies condition (c) above. Hence $(\sigma_{\alpha\beta}) \in \Delta$. Thus we get another map

$$\begin{array}{rccc} \eta\colon & \operatorname{End} G & \to & \Delta \\ & \sigma & \mapsto & (\sigma_{\alpha\beta}). \end{array}$$

Claim 1. The maps ζ and η are mutual inverses.
Let $(\sigma_{\alpha\beta}) \in \Delta$, let $\tau = \zeta((\sigma_{\alpha\beta}))$ and let $(\tau_{\alpha\beta}) = \eta(\tau)$. We want to show that $(\sigma_{\alpha\beta}) = (\tau_{\alpha\beta})$. Let $a_\alpha \in A_\alpha$ for some $\alpha \in \Gamma$. Then, by the definition of η, for all $\beta \in \Gamma$ we have $\tau_{\alpha\beta}(a_\alpha) = b_\beta$ where $b = \tau(a_\alpha \mathbb{1}_\alpha)$. Since $\tau = \zeta((\sigma_{\alpha\beta}))$ we have $b = \zeta((\sigma_{\alpha\beta}))(a_\alpha \mathbb{1}_\alpha)$ hence $b_\beta = \sigma_{\alpha\beta}(a_\alpha)$. We proved that, for all $\alpha, \beta \in \Gamma$ and all $a_\alpha \in A_\alpha$ we have $\sigma_{\alpha\beta}(a_\alpha) = \tau_{\alpha\beta}(a_\alpha)$ and thus $(\sigma_{\alpha\beta}) = (\tau_{\alpha\beta})$ which shows $\eta\zeta = \mathrm{id}_\Delta$.
Now let $\sigma \in \operatorname{End} G$ and let $(\sigma_{\alpha\beta}) = \eta(\sigma)$. Moreover, let $\tau = \zeta((\sigma_{\alpha\beta})) = \zeta\eta(\sigma)$. We want to show that $\sigma = \tau$. Let $a = \sum a_\gamma \mathbb{1}_\gamma \in G$. Then $\tau(a) = \sum b_\gamma \mathbb{1}_\gamma$ where b_γ is defined by (2.13). So we immediately get $\tau(a) = \sigma(a)$ which is what we required to show $\eta\zeta = \mathrm{id}_{\operatorname{End} G}$. ♦

Claim 2. The maps ζ and η are ring homomorphisms.
Recall that the operations on Δ are addition and multiplication of matrices and on $\operatorname{End} G$ we take pointwise addition and composition of functions. Consider two matrices $(\sigma_{\alpha\beta}), (\tau_{\alpha\beta}) \in \Delta$ and the corresponding endomorphisms $\sigma := \zeta((\sigma_{\alpha\beta}))$ and $\tau := \zeta((\tau_{\alpha\beta}))$. Now the sum $(\sigma_{\alpha\beta}) + (\tau_{\alpha\beta}) = (\sigma_{\alpha\beta} + \tau_{\alpha\beta})$ and $\zeta((\sigma_{\alpha\beta} + \tau_{\alpha\beta}))$ is the endomorphism $\rho \in \operatorname{End} G$ defined by $\rho\left(\sum a_\gamma \mathbb{1}_\gamma\right) = \sum b_\gamma \mathbb{1}_\gamma$ where

$$\begin{aligned} b_\beta &= \sum (\sigma_{\alpha\beta} + \tau_{\alpha\beta})(a_\alpha) \\ &= \sum (\sigma_{\alpha\beta}(a_\alpha) + \tau_{\alpha\beta}(a_\alpha)) \\ &= \sum \sigma_{\alpha\beta}(a_\alpha) + \sum \tau_{\alpha\beta}(a_\alpha) \\ &= (\zeta((\sigma_{\alpha\beta}))(a))_\beta + (\zeta((\tau\alpha\beta))(a))_\beta. \end{aligned}$$

Thus $\zeta((\sigma_{\alpha\beta} + \tau_{\alpha\beta})) = \zeta((\sigma_{\alpha\beta})) + \zeta((\tau_{\alpha\beta}))$.
Next we consider the product $(\sigma_{\alpha\beta})(\tau_{\alpha\beta}) = (\rho_{\alpha\beta})$ where (noticing that $\sigma_{\alpha\beta}$ is

applied first)

$$\rho_{\alpha\beta} = \sum_{\gamma\in\Gamma} \tau_{\gamma\beta}\sigma_{\alpha\gamma}.$$

Let $\rho \in \operatorname{End} G$ be $\rho = \zeta((\rho_{\alpha\beta}))$ and let $a = \sum_{\gamma\in\Gamma} a_\gamma \mathbb{1}_\gamma \in G$. Then $\rho(a) = \sum_{\gamma\in\Gamma} b_\gamma \mathbb{1}_\gamma$ where

$$\begin{aligned} b_\beta &= \sum_{\alpha\in\operatorname{supp}(a)} \rho_{\alpha\beta}(a_\alpha) \\ &= \sum_{\alpha\in\operatorname{supp}(a)} \sum_{\gamma\in\Gamma} \tau_{\gamma\beta}\sigma_{\alpha\gamma}(a_\alpha) \\ &= \sum_{\gamma\in\Gamma} \tau_{\gamma\beta}\left(\sum_{\alpha\in\operatorname{supp}(a)} \sigma_{\alpha\gamma}(a_\alpha)\right) \\ &= \sum_{\gamma\in\Gamma} \tau_{\gamma\beta}(\sigma_{\alpha\gamma}(a_\alpha)) \\ &= (\sigma_{\alpha\beta})(\tau_{\alpha\beta})(a_\alpha). \end{aligned}$$

So ζ is a ring homomorphism. Showing that η is a ring homomorphism is done in a similar way. ♦

From the two Claims it follows that ζ and ϑ are mutually inverse ring isomorphisms, so the proof is complete. □

The group $(\operatorname{Aut} G, \circ)$ of automorphisms of G is the subset of $(\operatorname{End} G, +, \circ)$ consisting of elements that are invertible with respect to $\circ$. Hence we have the following.

Corollary 2.3.9. *The group* Aut G *is isomorphic to the multiplicative group of units of* Δ. □

We remark that, in Corollary 2.3.9, by Aut G we mean *all* automorphisms of G as an abelian group, order (resp. valuation) preserving or otherwise. As announced at the beginning of this subsection, we intend to study the group o-Aut G of an ordered Hahn sum G. From now on, let us assume that all the A_γ are ordered archimedean groups and that, consequently, $G = \coprod_{\gamma\in\Gamma} A_\gamma$ is ordered lexicographically. Recall that the canonical valuation $v = v_{\min}$ and the natural valuation v^{nat} coincide on G. Corollary 2.3.9 identifies the group Aut G of all automorphisms of G with the multiplicative group of units of the ring Δ. Now we want to identify the subgroup of units of Δ that corresponds to o-Aut G. Lemma 2.3.11 and Corollary 2.3.12 are part of [Con58].

Notation 2.3.10. *Let $T(\Delta)$ be the multiplicative monoid of all matrices $(\sigma_{\alpha\beta}) \in \Delta$ (invertible or not) such that*

(i') $(\sigma_{\alpha\beta})$ is upper triangular;

(ii') for all $\alpha \in \Gamma$, $\sigma_{\alpha\alpha} \in o\text{-}\operatorname{Aut} A_\alpha$.

Moreover, denote by $UT(\Delta)$ the group of invertible matrices in $T(\Delta)$.

Lemma 2.3.11. *A matrix $(\sigma_{\alpha\beta}) \in T(\Delta)$ induces, via the correspondence just described, an order preserving endomorphism σ on G. In particular, an invertible matrix in $T(\Delta)$ induces an order preserving automorphism of G.*

Proof. Let $a = \sum_\gamma a_\gamma \mathbb{1}_\gamma > 0$ with $v(a) = \delta$. Then

$$\begin{aligned}\sigma(a) = \sum_{\gamma\in\Gamma}((a)_\alpha(\sigma_{\alpha\gamma}))\mathbb{1}_\gamma &= \sum_{\gamma\in\Gamma}\left(\sum_{\alpha\geq\delta}\sigma_{\alpha\gamma}(a_\alpha)\right)\mathbb{1}_\gamma \\ &= \sum_{\gamma\geq\delta}\left(\sum_{\alpha\geq\delta}\sigma_{\alpha\gamma}(a_\alpha)\right)\mathbb{1}_\gamma \\ &= \sigma_{\delta\delta}(a_\delta)\mathbb{1}_\delta + \sum_{\gamma>\delta}\left(\sum_{\alpha\geq\delta}\sigma_{\alpha\gamma}(a_\alpha)\right)\mathbb{1}_\gamma{}' > 0\end{aligned}$$

where the third equality holds because $\sigma_{\alpha\beta}$ is upper triangular and the final inequality is due to the fact that, by (ii'), $\sigma_{\delta\delta}$ is order preserving and, since $a_\delta > 0$ then also $\sigma_{\delta\delta}(a_\delta) > 0$. □

As an immediate consequence we get that

Corollary 2.3.12. *$UT(\Delta)$ embeds into $o\text{-}\operatorname{Aut} G$.* □

We want to identify exactly which order preserving automorphisms of G correspond to the elements of $UT(\Delta)$.

Let $UT(\Delta)^1$ be the subgroup of $UT(\Delta)$ given by all the *unitriangular* matrices of $T(\Delta)$, that is, those matrices that belong to $T(\Delta)$ and have all the diagonal entries equal to 1. When we multiply two upper triangular matrices, the elements on the diagonal of the product are the products of the corresponding diagonal entries. It follows that if A is upper triangular and invertible with $B = A^{-1}$ then $b_{ii} = a_{ii}^{-1}$ and therefore, for all $A \in UT(\Delta)$, $B \in UT(\Delta)^1$ the matrix ABA^{-1} has all diagonal entries equal to 1. Hence $UT(\Delta)^1$ is a normal subgroup of $UT(\Delta)$.

Consider the group $UT(\Delta)^d$ of diagonal matrices in $UT(\Delta)$.

Lemma 2.3.13. *$UT(\Delta)^d$ is a complement of $UT(\Delta)^1$ in $UT(\Delta)$ and we have*

$$UT(\Delta) = UT(\Delta)^1 \rtimes UT(\Delta)^d.$$

Proof. Let $A \in UT(\Delta)$ and define $D \in UT(\Delta)^d$ by

$$D_{\alpha\beta} = \begin{cases} A_{\alpha\beta} & \alpha = \beta; \\ 0 & \alpha \neq \beta. \end{cases}$$

Then $D \in UT(\Delta)^d$ and $AD^{-1} \in UT(\Delta)^1$ are such that $A = (AD^{-1})D$. So $UT(\Delta)^d$ is a complement of $UT(\Delta)^1$ in $UT(\Delta)$. Obviously, $UT(\Delta)^1 \cap UT(\Delta)^d = I$ (we are denoting by I the identity matrix). We already noted that $UT(\Delta)^1 \trianglelefteq UT(\Delta)$ and so, by Definition 1.2.10, we have $UT(\Delta) = UT(\Delta)^1 \rtimes UT(\Delta)^d$. □

Now we want to identify the subgroups of $UT(\Delta)$ that we just introduced with subgroups of $o\text{-}\operatorname{Aut} G$. Recall that all the order preserving automorphisms are valuation preserving and that G, as a Hahn sum, has the canonical lifting property with respect to the skeleton. So we will use the notation

$$\begin{cases} \operatorname{Int} o\text{-}\operatorname{Aut} G := \operatorname{Int}\operatorname{Aut} G \cap o\text{-}\operatorname{Aut} G; \\ \operatorname{Ext} o\text{-}\operatorname{Aut} G := \operatorname{Ext}\operatorname{Aut} G \cap o\text{-}\operatorname{Aut} G. \end{cases} \tag{2.14}$$

Lemma 2.3.14. *Under the correspondence established in Lemma 2.3.8 we have*

$$\operatorname{Int} o\text{-}\operatorname{Aut} G \simeq UT(\Delta)^1.$$

Proof. We know that ζ is a bijective ring homomorphism between $\operatorname{Aut} G$ and Δ. Therefore, to prove the claim it suffices to show that ζ maps $UT(\Delta)^1$ to $\operatorname{Int} o\text{-}\operatorname{Aut} G$ and that its inverse η maps $\operatorname{Int} o\text{-}\operatorname{Aut} G$ to $UT(\Delta)^1$.
From the proof of Lemma 2.3.11 it is clear that all o-automorphisms of G corresponding to the elements $(\sigma_{\alpha\beta}) \in UT(\Delta)$ induce the identity on Γ. From the formula in the proof of Lemma 2.3.11 we see that, for all $(\sigma_{\alpha\beta}) \in UT(\Delta)$, we have

$$\zeta((\sigma_{\alpha\beta})) \in \operatorname{Int}\operatorname{Aut} G \Leftrightarrow \sigma_{\alpha\alpha} = 1 \text{ for all } \alpha \in \Gamma. \tag{2.15}$$

That is, if and only if $((\sigma_{\alpha\beta})) \in UT(\Delta)^1$. So $\zeta(UT(\Delta)^1) \subseteq \operatorname{Int} o\text{-}\operatorname{Aut} G$.
Now let $\sigma \in \operatorname{Int} v\text{-}\operatorname{Aut} G$ and let $(\sigma_{\alpha\beta}) = \eta(\sigma)$. On the diagonal of $(\sigma_{\alpha\beta})$, for all $\gamma \in \Gamma$ there is the automorphism $\sigma_\gamma \in o\text{-}\operatorname{Aut} A_\gamma$ such that $\sigma_\gamma(a_\gamma) = \sigma(a_\gamma \mathbb{1}_\gamma)_\gamma$

and, since $\sigma \in \operatorname{Int} v\text{-}\operatorname{Aut} G$ this is $\sigma(a_\gamma \mathbb{1}_\gamma)_\gamma = a_\gamma$ for all $a_\gamma \in A_\gamma$. So $\sigma_\gamma = 1$ for all $\gamma \in \Gamma$. The entry $\sigma_{\alpha\beta}$ is such that $\sigma_{\alpha\beta}(a_\alpha) = \sigma(a_\alpha \mathbb{1}_\alpha)_\beta$ which is 0 for all $\beta < \alpha$, because σ is internal, hence valuation fixing. Thus $(\sigma_{\alpha\beta}) \in UT(\Delta)^1$ and so $\eta(\operatorname{Int} o\text{-}\operatorname{Aut} G) \subseteq UT(\Delta)^1$. Because ζ and η are mutual inverses the statement is proven. □

As G has the canonical lifting property w.r.t. the skeleton we identify the group $\operatorname{Ext}\operatorname{Aut} G$ (and its subgroup $\operatorname{Ext} o\text{-}\operatorname{Aut} G$ of order preserving external automorphisms) with the canonical lifts of automorphisms of the skeleton. We will distinguish a special type of canonical lifts:

Notation 2.3.15. *Denote by* $\operatorname{Ext} o\text{-}\operatorname{Aut}_\Gamma G$ *the set of external order preserving automorphisms of G that are lifts of automorphisms of* $S(G)$ *of the form* $[\mathrm{id}_\Gamma; \{\sigma_g : \gamma \in \Gamma\}]$. *That is*

$$\operatorname{Ext} o\text{-}\operatorname{Aut}_\Gamma G := \{\sigma \in \operatorname{Ext} o\text{-}\operatorname{Aut} G : \sigma_\Gamma = \mathrm{id}_G\}.$$

Remark 2.3.16. The set $\operatorname{Ext} o\text{-}\operatorname{Aut}_\Gamma G$ is a normal subgroup of $\operatorname{Ext} o\text{-}\operatorname{Aut} G$. Indeed, if $\sigma, \tau \in \operatorname{Ext} o\text{-}\operatorname{Aut}_\Gamma G$, since Φ_G is a group homomorphism then $(\sigma\tau)_\Gamma = \sigma_\Gamma \tau_\Gamma$ and $(\sigma_\Gamma)^{-1} = (\sigma^{-1})_\Gamma$. So $\operatorname{Ext} o\text{-}\operatorname{Aut}_\Gamma G$ is a subgroup of $\operatorname{Ext} o\text{-}\operatorname{Aut} G$. Now let $\rho \in \operatorname{Ext} o\text{-}\operatorname{Aut} G$ be arbitrary. Then $(\rho\sigma\rho^{-1})_\Gamma = \rho_\Gamma \,\mathrm{id}_\Gamma \,\rho_\Gamma^{-1} = \mathrm{id}_\Gamma$ thus $\rho\sigma\rho^{-1} \in \operatorname{Ext} o\text{-}\operatorname{Aut}_\Gamma G$. Hence $\operatorname{Ext} o\text{-}\operatorname{Aut}_\Gamma G \trianglelefteq \operatorname{Ext} o\text{-}\operatorname{Aut} G$. □

Lemma 2.3.17. *Under the correspondence established in Lemma 2.3.8 we have*

$$\operatorname{Ext} o\text{-}\operatorname{Aut}_\Gamma G \simeq UT(\Delta)^d.$$

Proof. Like in Lemma 2.3.14, we just need to show the inclusions $\zeta(UT(\Delta)^d) \subseteq \operatorname{Ext} o\text{-}\operatorname{Aut}_\Gamma G$ and $\eta(\operatorname{Ext} o\text{-}\operatorname{Aut}_\Gamma G) \subseteq UT(\Delta)^d$. Let $(\sigma_{\alpha\beta}) \in UT(\Delta)^d$, let $\sigma = \zeta((\sigma_{\alpha\beta}))$ and let $a = \sum_{\gamma\in\Gamma} a_\gamma \mathbb{1}_\gamma$. Then $\sigma(a) = \sum_{\gamma\in\Gamma} b_\gamma \mathbb{1}_\gamma$ and $b_\gamma = \sigma_{\gamma\gamma}(a_\gamma)$, because for $\alpha \neq \beta$ we have $\sigma_{\alpha\beta} = 0$. Hence $\sigma(a) = \sum_{\gamma\in\Gamma} \sigma_{\gamma\gamma}(a_\gamma)\mathbb{1}_\gamma$, that is σ is the canonical lift of $[\mathrm{id}_\Gamma; \{\sigma_{\gamma\gamma} : \gamma \in \Gamma\}]$ and, therefore, $\sigma \in \operatorname{Ext} o\text{-}\operatorname{Aut}_\Gamma G$. This proves $\zeta(UT(\Delta)^d) \subseteq \operatorname{Ext} o\text{-}\operatorname{Aut}_\Gamma G$.
Now let $\sigma \in \operatorname{Ext} o\text{-}\operatorname{Aut}_\Gamma G$ and let $(\sigma_{\alpha\beta}) = \eta(\sigma)$. Since σ is the canonical lift of an automorphism of $S(G)$ of the form $[\mathrm{id}_\Gamma; \{\sigma_\gamma : \gamma \in \Gamma\}]$ it follows that $\sigma_{\gamma\gamma} = \sigma_\gamma$ for all $\gamma \in \Gamma$ and that $\sigma_{\alpha\beta} = 0$ for all $\alpha \neq \beta$. Therefore $(\sigma_{\alpha\beta}) \in UT(\Delta)^d$, which prove the second inclusion and completes the proof. □

Lemmas 2.3.14 and 2.3.17 allow us to identify an important subgroup of $o\text{-}\operatorname{Aut} G$ that we can represent as a group of matrices:

Definition 2.3.18. Let $o\text{-}\operatorname{Aut}_\Gamma G$ be the group consisting of all order preserving automorphisms σ of G such that $\sigma_\Gamma = \mathrm{id}_G$.

Theorem 2.3.19. *Let $G = \coprod_{\gamma\in\Gamma} A_\gamma$ be an ordered Hahn sum, where the A_γ are all ordered archimedean groups. Then*

$$o\text{-}\operatorname{Aut}_\Gamma G \simeq UT(\Delta) \simeq UT(\Delta)^1 \rtimes UT(\Delta)^d \simeq \operatorname{Int} o\text{-}\operatorname{Aut} G \rtimes \operatorname{Ext} o\text{-}\operatorname{Aut}_\Gamma G. \tag{2.16}$$

□

To achieve a complete description of $o\text{-}\operatorname{Aut} G$ we need to consider the lifts of automorphisms $[\sigma_\Gamma; \{\sigma_\gamma : \gamma \in \Gamma\}] \in \operatorname{Aut} S(G)$ such that $\sigma_\Gamma \neq \mathrm{id}_\Gamma$. These do not form a subgroup of $o\text{-}\operatorname{Aut} G$, as they are not closed under composition, and we are not able to find a complement to $o\text{-}\operatorname{Aut}_\Gamma G$ in $o\text{-}\operatorname{Aut} G$.

Nevertheless, if the only order preserving automorphism of Γ is the identity, since $o\text{-}\operatorname{Aut}_\Gamma G = o\text{-}\operatorname{Aut} G$ we retrieve the following result of Conrad as a corollary.

Corollary 2.3.20 ([Con58, Corollary to Lemma 1]). *If $o\text{-}\operatorname{Aut}\Gamma = \{\mathrm{id}_\Gamma\}$ then we have an isomorphism $o\text{-}\operatorname{Aut} G \simeq UT(\Delta)$.* □

Theorem 2.3.19 improves [DG97, Corollary 3.2] in two ways: it generalises from the balanced to the general case and sharpens the decomposition by identifying the internal automorphisms within the group $UT(\Delta)$.

2.4 Summability and strong additivity

Let $\mathbb{G}$ be the Hahn product over the skeleton $S(\mathbb{G}) = [\Gamma; \{A_\gamma : \gamma \in \Gamma\}]$ and let $\mathcal{F}$ be a (possibly infinite) family of elements of G. We define the *support of $\mathcal{F}$* as the union of the supports of all members of $\mathcal{F}$ and denote it by

$$\operatorname{Supp}\mathcal{F} = \bigcup_{a\in\mathcal{F}} \operatorname{supp}(a).$$

Definition 2.4.1. We say that $\mathcal{F}$ is *summable* if

(i) $\operatorname{Supp}\mathcal{F}$ is well ordered

(ii) For every $\gamma \in \operatorname{Supp}\mathcal{F}$ there are only finitely many $a \in \mathcal{F}$ such that $\gamma \in \operatorname{supp}(a)$.

Then we can define the *sum of* $\mathcal{F}$ to be the element

$$\Sigma\mathcal{F} := \sum_{\gamma\in \operatorname{Supp}\mathcal{F}} \left(\sum_{a\in\mathcal{F}:\gamma\in\operatorname{supp}(a)} a_\gamma \right) \mathbb{1}_\gamma.$$

Notice that $\Sigma\mathcal{F}$ is well defined as condition (i) ensures that the support be well ordered and by condition (ii) the sum in brackets is finite for all γ.

Example 2.4.2. Let $\mathbb{G} = \mathbf{H}_{\mathbb{N}}\,\mathbb{Q}$ and consider the family $\mathcal{F} = \{\sum_{n=m}^{\infty} \mathbb{1}_n : m \in \mathbb{N}\}$. Then $\operatorname{Supp}\mathcal{F} = \mathbb{N}$ is well ordered and for every $r \in \mathbb{N}$ we have $r \in \operatorname{supp}(\sum_{n=m}^{\infty} \mathbb{1}_n)$ if and only if $m \leq r$. So $\mathcal{F}$ is summable and its sum is the series

$$\Sigma\mathcal{F} = \sum_{n\in\mathbb{N}} n\mathbb{1}_n.$$

Definition 2.4.3. An automorphism $\sigma \in \operatorname{Aut}\mathbb{G}$ is called *summable* if, for all $a = \sum_\Gamma a_\gamma \mathbb{1}_\gamma \in \mathbb{G}$ the family $\mathcal{F} = \{\sigma(a_\gamma\mathbb{1}_\gamma) : \gamma \in \operatorname{supp}(a)\}$ is summable and $\sigma(a) = \Sigma\mathcal{F} = \sum_\Gamma \sigma(a_\gamma\mathbb{1}_\gamma)$.
Moreover, we say that σ is *strongly additive* if both σ and σ^{-1} are summable.

In the following chapter we will study strongly additive automorphisms much more extensively, in the context of Hahn fields, where we will require a (possibly) stronger definition. Here we will only show that canonical lifts are always strongly additive, and that all valuation preserving automorphisms are strongly additive, provided the value set of our Hahn group is $\mathbb{Z}$.

Proposition 2.4.4. *Canonical lifts are always strongly additive.*

Proof. Let $\mathbb{G} = \mathbf{H}_\Gamma\, A_\gamma$, let $G \leq \mathbb{G}$ and let $\sigma \in \operatorname{Ext}\operatorname{Aut}\mathbb{G}$. Then there is an automorphism $\tau \in \operatorname{Aut} S(G)$ such that σ is the restriction to G of the canonical lift $\tilde{\tau} \in v\text{-}\operatorname{Aut}\mathbb{G}$. Let $\tau = [\tau_\Gamma; \{\tau_\gamma : \gamma \in \Gamma\}]$ and let $a = \sum a_\gamma \mathbb{1}_\gamma \in G$ be an arbitrary element. Then the family $\mathcal{F} = \{\sigma(a_\gamma\mathbb{1}_\gamma) = \tau_\gamma(a_\gamma)\mathbb{1}_{\tau_\Gamma(\gamma)}\}$ is summable. Indeed, the support of $\mathcal{F}$ is $\tau_\Gamma(\operatorname{supp}(a)) = \{\tau_\Gamma(\gamma) : \gamma \in \operatorname{supp}(a)\}$ which is well ordered, as it is the image of a well ordered set under an order preserving automorphism; moreover, for each $\tau_\Gamma(\gamma) \in \operatorname{Supp}\mathcal{F}$ there is exactly one element of $\mathcal{F}$, namely $\tau_\gamma(a_\gamma)\mathbb{1}_{\tau_\Gamma(\gamma)}$ in whose support $\tau_\Gamma(\gamma)$ appears. Finally, we have

$$\sigma\left(\sum a_\gamma \mathbb{1}_\gamma\right) = \sum \tau_\gamma(a_\gamma)\mathbb{1}_{\tau_\Gamma(\gamma)} = \sum \sigma(a_\gamma\mathbb{1}_\gamma).$$

□

The next proposition shows, that if G is a balanced Hahn product, then also all the internal automorphisms are strongly additive.

Proposition 2.4.5. *Let $G = \mathbf{H}_{\mathbb{Z}} A$, for a fixed abelian group A and let $\sigma \in \operatorname{Int}\operatorname{Aut} G$. Then σ is strongly additive.*

Proof. Let $a = \sum a_n \mathbb{1}_n \in G$ and $m := v(a)$. Consider the family $\mathcal{F} = \{\sigma(a_n \mathbb{1}_n) : n \in \operatorname{supp}(a)\}$. Then $\operatorname{Supp} \mathcal{F} \subseteq \mathbb{Z}^{\geq m}$ and therefore it is well ordered. Moreover, every $p \geq m$ can only appear in the support of $\sigma(a_n \mathbb{1}_n)$ for $m \leq n \leq p$, hence for finitely many n. So the family is indeed summable. The sum of this family is

$$\Sigma\mathcal{F} = \sum_{n=m}^{\infty} \sigma(a_n \mathbb{1}_n)$$

and if we consider its d-th coefficient, for a $d \geq m$, it is

$$\left(\sum_{n=m}^{\infty} \sigma(a_n \mathbb{1}_n) \right)_d = \sum_{n=m}^{\infty} (\sigma(a_n \mathbb{1}_n))_d = \sum_{n=m}^{d} (\sigma(a_n \mathbb{1}_n))_d.$$

Indeed, from the second sum we can discard all the elements of index $n > d$, because σ is internal and so $v(\sigma(a_n \mathbb{1}_n)) = n > d$ so $(\sigma(a_n \mathbb{1}_n))_d = 0$ for all $n > d$.

On the other hand, we have

$$\sigma \left(\sum_{n=m}^{\infty} a_n \mathbb{1}_n \right)_d = \sigma \left(\sum_{n=m}^{d} a_n \mathbb{1}_n + \sum_{n=d+1}^{\infty} a_n \mathbb{1}_n \right)_d = \sigma \left(\sum_{n=m}^{d} a_n \mathbb{1}_n \right)_d$$

the second equality because $v\left(\sum_{n=d+1}^{\infty} a_n \mathbb{1}_n\right) > d$, so this part does not contribute to the d-th component of $\sigma(a)$. So

$$\sigma \left(\sum_{n=m}^{\infty} a_n \mathbb{1}_n \right)_d = \sum_{n=m}^{d} \sigma\left(a_n \mathbb{1}_n\right)_d = (\Sigma\mathcal{F})_d. \tag{2.17}$$

Hence σ is summable. Since σ was arbitrary, the same holds for σ^{-1}, hence σ is strongly additive. □

Chapter 3

Automorphisms of Hahn fields

3.1 Construction of Hahn fields

Let us consider a totally ordered abelian group G, a field k and the Hahn product $k((G)) := \mathbf{H}_G\, k$ (see Definition 2.1.1). We already know that this is an abelian group, where addition is defined component-wise. On this group we can define a second operation as follows. Let $a = \sum a_g \mathbb{1}_g$ and $b = \sum b_g \mathbb{1} g$ be two elements of $k((G))$ and let us define

$$c := a \cdot b = \sum_G c_g \mathbb{1}_g \quad \text{where } c_g = \sum_{\substack{(r,s)\in \operatorname{supp} a \times \operatorname{supp} b \\ r+s=g}} a_r b_s \tag{3.1}$$

and $a_r b_s$ is the multiplication in k.

Lemma 3.1.1. *The operation in* (3.1) *is well defined.*

Proof. First, the sum defining c_g is finite. Indeed, assume there were infinitely many pairs $(r, s) \in \operatorname{supp}(a) \times \operatorname{supp}(b)$ such that $r + s = g$. Then there would be infinitely many different r's (or s's, then we just swap the names). They are totally ordered, so we can extract an infinite sequence $r_1 < r_2 < r_3 < \dots$ But to it, an infinite descending sequence $s_1 > s_2 > s_3 > \dots$ corresponds, which is impossible since $\operatorname{supp}(b)$ is well ordered. So the set $\{(r, s) : r \in \operatorname{supp}(a),\ s \in \operatorname{supp}(b),\ r + s = g\}$ is finite.

Moreover, the support $\operatorname{supp} c$ is well ordered. Indeed, call $S = \operatorname{supp} a \times \operatorname{supp} b$ and assume, for a contradiction, that there is an infinite decreasing sequence $g_1 > g_2 > \dots \in \operatorname{supp} c$. For each g_i there is a finite set $T_i = \{(r, s) \in S :$

$r + s = g_i\}$. Let $(r_i, s_i) \in T_i$ be such that r_i is minimal (and therefore s_i is maximal). Moreover, let r_I be the minimum among the r_i's. Now, for every $i > I$ we have $g_i = r_i + s_i$ and $r_i \leq r_I$. Since $g_i > g_{i+1}$ and $r_i, r_{i+1} \leq r_I$ it follows that $s_i > s_{i+1}$. Hence we have an infinite decreasing sequence $s_I > s_{I+1} > \dots$ which contradicts the well order of $\operatorname{supp} b$. □

In 1907, Hahn proved that $(k((G)), +, \cdot)$ forms a field [Hah07]. Since for all $g, h \in G$ we have $1\mathbb{1}_g \cdot 1\mathbb{1}_h = 1\mathbb{1}_{g+h}$ it is convenient to write the indices as exponents: we will write t^g for $\mathbb{1}_g$ and use power series notation for an element $a \in k((G))$:

$$a = \sum_{g \in G} a_g t^g := \sum_{g \in G} a_g \mathbb{1}_g.$$

Remark 3.1.2. Just as it was the case for the Hahn groups, a power series $a = \sum a_g t^g \in k((G))$ can be seen as a function

$$\begin{aligned} a\colon \quad G &\longrightarrow k \\ g &\longmapsto a_g. \end{aligned}$$

□

Definition 3.1.3 (Maximal Hahn field)**.** Let k be a field and G an ordered abelian group. The field $k((G))$ will be called *the maximal Hahn field over k with exponents in G* and will be denoted by $\mathbb{K} := k((G))$.

We proceed further in analogy with Hahn groups. Recall that a balanced Hahn product $\mathbf{H}_\Gamma A$ always has the corresponding Hahn sum $\coprod_\Gamma A$ as a subgroup and we defined a Hahn group to be any group comprised between the Hahn sum and product over the same skeleton. If $\mathbb{K} = k((G))$ is a maximal Hahn field, it also admits a subfield analogous to the Hahn sum: let $k[G]$ be the set of elements of $\mathbb{K}$ with finite support. We call $k[G]$ the *group ring of G over k*. In general this is not a field (take for example the ring of Laurent polynomials $k[\mathbb{Z}]$), but it is an integral domain, so we can take its field of fractions $k(G)$ which is a subfield of $\mathbb{K}$. Sometimes $k(G)$ will be referred to as *the minimal Hahn field over k with exponents in G*.

Definition 3.1.4 (Hahn field)**.** Let k be a field and G and ordered abelian group. A *Hahn field* is any field K such that $k(G) \subseteq K \subseteq \mathbb{K}$.

Lemma 3.1.5. *Let K be a Hahn field and define a map $v_{\min}\colon K \to G \cup \{\infty\}$ as follows:*

$$\begin{cases} v_{\min}(a) = \min \operatorname{supp} a, \text{ for } a \in K^\times \\ v_{\min}(0) = \infty \end{cases}$$

Then $v_{\min}$ defines a valuation on K.

Proof. Verifying the axioms of a valuation given in Definition 1.5.1 is immediate, but we do it for completeness. Let $a = \sum_{g \in G} a_g t^g$ and $b = \sum_{g \in G} b_g t^g$ with $g_a := v_{\min}(a)$ and $g_b := v_{\min}(b)$

(i) $v_{\min}(a) = \infty \Leftrightarrow a = 0$ holds by definition.

(ii) By the way multiplication is defined in K we have $v_{\min}(ab) = \min\{r + s : r \in \operatorname{supp} a, s \in \operatorname{supp} b\} = g_a + g_b = v_{\min}(a) + v_{\min}(b)$.

(iii) For all $g < \min\{g_a, g_b\}$ we have $a_g + b_g = 0 + 0 = 0$ so $v_{\min}(a + b) = \min \operatorname{supp}(a + b) \geq \min\{g_a, g_b\}$.

Moreover, thanks to the condition $k(G) \subseteq K$, $v_{\min}$ is surjective: for all $g \in G$ we have $t^g \in G$ and $g = v_{\min}(t^g)$. □

Definition 3.1.6. We call $v_{\min}$ *the canonical valuation on K*. Unless confusion can arise, we will remove the subscript from the notation and write v instead of $v_{\min}$.

Terminology 3.1.7. *Let $a = \sum_{g \in G} a_g t^g \in k((G))$ and $h = v(a)$. The term a_0 will be referred to as the* constant term *of a. The term $a_h t^h$ (resp. the coefficient a_h) will be called* the first term *(resp.* the first coefficient*) of a.*

Remark 3.1.8. Let $a, b \in K$ and let $g_a = v(a)$ and $g_b = v(b)$. Then $v(ab) = g_a + g_b$ and $(ab)_{g_a + g_b} = a_{g_a} b_{g_b}$. □

Notation 3.1.9. *Throughout this chapter, K will denote a Hahn field, v its canonical valuation and $G = v(K^\times)$ its value group. We denote by t^G the multiplicative group of monic monomials $t^G := \{t^g : g \in G\}$. The valuation ring R_K is $k\big((G^{\geq 0})\big) \cap K$ (elements with non-negative value) and the valuation ideal I_K is $k\big((G^{>0})\big) \cap K$ (elements with positive value). The residue field is $\bar{K} = R_K / I_K$. The group of units of R_K will be denoted by $U_K := R_K^\times$. It consists of all elements $u \in R_K$ with $u_0 \neq 0$. An important subgroup will be the group of* 1-units $1 + I_K := \{u \in U_K : u_0 = 1\}$.

The next two lemmas establish some useful properties of the objects we just described.

Lemma 3.1.10. *Let K be a Hahn field. Then $U_K \simeq (1 + I_K) \times k^\times$ and thus we have*

$$(\mathrm{Hom}(G, U), \cdot) \simeq (\mathrm{Hom}(G, 1 + I_K), \cdot) \times (\mathrm{Hom}(G, k^\times), \cdot) \tag{3.2}$$

Proof. Let $u \in U$ and let $u' := u/u_0$. Then $u' = 1 + \sum_{g>0} \frac{u_g}{u_0} t^g$ and clearly we have $u = u_0 u'$. The map

$$\begin{array}{rccc} \vartheta\colon & U & \longrightarrow & (1 + I_K) \times k^\times \\ & u & \longmapsto & (u', u_0) \end{array}$$

is a group homomorphism. Indeed, it is surjective because for all $(u, \alpha) \in (1 + I_K) \times k^\times$ the element $\alpha u \in U$ is such that $\vartheta(\alpha u) = (u, \alpha)$. It is also injective: let $(u_1', u_{10}) = (u_2', u_{20}) \in (1 + I_K) \times k^\times$. Then $u_1' u_{10} = u_2' u_{20}$. Finally, ϑ is a group homomorphism. Indeed, let $u_1, u_2 \in U$. Then, by Remark 3.1.8 we have $(u_1 u_1)_0 = u_{10} u_{20}$ and, therefore, also $(u_1 u_2)' = (u_1 u_2)/u_{10} u_{20} = u_1' u_2'$. We proved $U_K \simeq (1 + I_K) \times k^\times$. Equation (3.2) now follows from Proposition 1.2.2. □

Lemma 3.1.11. *Let K be a Hahn field. The map*

$$\begin{array}{rccc} f_c\colon & \bar{K} & \to & k \\ & a + I_K & \mapsto & a_0 \end{array} \tag{3.3}$$

is an isomorphism and every other isomorphism $f\colon \bar{K} \to k$ factors through f_c, i.e., there exists a uniquely determined automorphism $\rho_f \in \mathrm{Aut}\, k$ such that $f = \rho_f f_c$:

$$\begin{array}{ccc} \bar{K} & \xrightarrow{f_c} & k \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \rho_f} \\ & & k \end{array}$$

Proof. Consider the map

$$\begin{array}{rccc} f_c'\colon & R_K & \to & k \\ & a = \sum_{g \geq 0} a_g t^g & \mapsto & a_0. \end{array}$$

This is clearly a ring homomorphism. It is surjective, as for every $\alpha \in k$, taking $a = \alpha$ we have $f'_c(a) = a_0 = \alpha$. Moreover, we have $f'_c(a) = 0 \Leftrightarrow a_0 = 0 \Leftrightarrow a \in I_K$, that is $\ker f'_c = I_K$. The quotient map $f_c \colon \bar{K} \to k$, $a + I_K \mapsto a_0$ is thus a field isomorphism. Now let $f \colon \bar{K} \to k$ be another isomorphism and define $\rho_f := f f_c^{-1}$. As both f and f_c^{-1} are isomorphisms, it follows that ρ_f is an automorphism of k and we have $f = \rho_f f_c$.

For the uniqueness of ρ_f, note that if $\rho'_f \in \operatorname{Aut} k$ is such that $f = \rho_f f_c = \rho'_f f_c$, composing on the right with f_c^{-1} gives $\rho_f = \rho'_f$. □

One of the reasons why Hahn fields are interesting is the fact that they are *universal as valued fields* [Mac39]. By this we mean that every valued field is isomorphic to a suitable Hahn field. More precisely, we have this important result of Kaplansky[1]:

Theorem 3.1.12 (Kaplansky's embedding theorem, [Kap42]). *Let (F, w) be a valued field with value group H and residue field $\bar{F}$ such that $\operatorname{char} F = \operatorname{char} \bar{F} = 0$ and $\bar{F}$ is either real closed or algebraically closed. Let $\Delta(H)$ be the divisible closure of H. Then there is a valuation preserving embedding*

$$(F, w) \hookrightarrow (\bar{F}((\Delta(H))), v_{\min}).$$

□

3.1.1 Ordered Hahn fields

Suppose that $(k, <_k)$ be an ordered field. Then a Hahn field $k(G) \subseteq K \subseteq \mathbb{K}$ can be ordered lexicographically: for a power series $a \in K$, set

$$a >_{\text{lex}} 0 \Leftrightarrow a_{v(a)} >_k 0 \tag{3.4}$$

(the meaning of the subscripts is self-explanatory and they will be omitted in the sequel). On the ordered field $(K, <_{\text{lex}})$ we have the natural valuation v_{nat}. The next proposition shows that the natural valuation is equivalent to the canonical one, if the ordered residue field $(k, <_k)$ is archimedean.

Proposition 3.1.13. *Let $(k, +, \cdot, <)$ be archimedean. Then the natural valuation v_{nat} and the canonical valuation $v_{\min}$ are equivalent on $\mathbb{K} = k((G))$. In particular, they are equivalent on every Hahn field K such that $k(G) \subseteq K \subseteq \mathbb{K}$.*

[1] The restrictions on the residue field can be dropped, introducing so called *factor sets*. We do not go into this here and refer the interested reader to, e.g., [For06].

Proof. Let $\sim$ be the relation on $k((G))$ defined in (1.4) and let $a, b \in k((G))^{\times}$. We have $v_{\text{nat}}(a) = v_{\text{nat}}(b) \Leftrightarrow a \sim b$. This means that there exists $n \in \mathbb{Z}$ such that $na \geq_{\text{lex}} b$ and $nb \geq_{\text{lex}} a$. This happens if and only if $v_{\min}(a) = v_{\min}(b) = g$ and $na_g \geq b_g \wedge nb_g \geq a_g$. The second condition is always satisfied, because k is archimedean. Hence we have

$$v_{\text{nat}}(a) = v_{\text{nat}}(b) \Leftrightarrow v_{\min}(a) = v_{\min}(b).$$

□

We include the following result, without proof, as we will make use of it in Example 3.2.4.

Theorem 3.1.14 ([EP05, Theorem 4.3.7])**.** *Let $(K, <)$ be an ordered field and v a non-trivial convex valuation (Definition 1.6.1) on K with value group G and residue field $\bar{K}$. Then K is real closed (Definition 1.6.13) if and only if the following three conditions are satisfied:*

(i) $\bar{K}$ is a real closed field;

(ii) G is a divisible group (Definition 1.2.3);

(iii) (K, v) is henselian (Definition 1.5.15). □

For easier reference, we provide a translation of Theorem 3.1.14 for Hahn fields.

Corollary 3.1.15. *Let k be an ordered field, G a totally ordered abelian group and K a Hahn field with $k(G) \subseteq K \subseteq \mathbb{K}$. Then $(K, <_{\text{lex}})$ is real closed if and only if all of the following hold:*

(i) k is a real closed field;

(ii) G is a divisible group;

(iii) (K, v) is henselian. □

3.2 Automorphisms of Hahn fields

Let $\mathbb{K} = k((G))$ and let $k(G) \subseteq K \subseteq \mathbb{K}$ be a Hahn field. Let $\sigma \in \operatorname{Aut} K$ be an automorphism of K. Unless specified otherwise, we will always consider

K equipped with its canonical valuation $v_{\min}$, which, therefore, we will simply denote by v. Recall (Definition 1.5.3) that $\sigma \in \operatorname{Aut} K$ is *valuation preserving* if there is an order preserving automorphism σ_G of G such that, for all $a \in K$, we have $v(\sigma(a)) = \sigma_G(v(a))$. The group of valuation preserving automorphisms will be denoted by $v\text{-}\operatorname{Aut} K$ (Notation 1.5.5).

Definition 3.2.1. (i) An automorphism $\sigma \in \operatorname{Aut} K$ such that $\sigma|_k = \mathrm{id}_k$ will be called a *k-automorphism*. We will denote the groups of k-automorphisms by

$$o\text{-}\operatorname{Aut}_k K,\ v\text{-}\operatorname{Aut}_k K \leq \operatorname{Aut}_k K. \tag{3.5}$$

(ii) An automorphism $\sigma \in \operatorname{Aut} K$ such that $\sigma(k) = k$ will be called a *k-stable automorphism*. We will denote the groups of k-stable automorphisms by

$$o\text{-}\operatorname{Aut}_{(k)} K,\ v\text{-}\operatorname{Aut}_{(k)} K \leq \operatorname{Aut}_{(k)} K. \tag{3.6}$$

Remark 3.2.2. If K is ordered, we can define on it the natural valuation induced by the ordering (see Proposition 1.6.5). By Proposition 1.6.11, an order preserving automorphism preserves the natural valuation, because it fixes the archimedean classes: $o\text{-}\operatorname{Aut} K \leq v_{\mathrm{nat}}\text{-}\operatorname{Aut} K$.
If, moreover, k is archimedean, by Proposition 3.1.13, the canonical valuation and the natural valuation coincide. Hence we have a chain of subgroups

$$o\text{-}\operatorname{Aut} K \leq v\text{-}\operatorname{Aut} K \leq \operatorname{Aut} K \tag{3.7}$$

and similarly, by taking intersections with $\operatorname{Aut}_k K$ and $\operatorname{Aut}_{(k)} K$, we get

$$\begin{aligned} o\text{-}\operatorname{Aut}_k K \leq v\text{-}\operatorname{Aut}_k K &\leq \operatorname{Aut}_k K; \\ o\text{-}\operatorname{Aut}_{(k)} K \leq v\text{-}\operatorname{Aut}_{(k)} K &\leq \operatorname{Aut}_{(k)} K. \end{aligned} \tag{3.8}$$

□

Example 3.2.3 below shows that the containments in (3.7) and (3.8) can be strict; Example 3.2.4 then shows that the assumption that k be archimedean is necessary in order for the inclusion $o\text{-}\operatorname{Aut} K \leq v\text{-}\operatorname{Aut} K$ to hold.

Example 3.2.3. On the maximal Hahn field $\mathbb{K} = \mathbb{Q}((\mathbb{Z}))$ consider the automorphism $\sigma\colon \sum a_n t^n \mapsto \sum a_n(-t)^n$. Then $\sigma \in v\text{-}\operatorname{Aut}\mathbb{K}$, as we will prove later that multiplying t by a unit of the valuation ring always yields a valuation preserving automorphism. Also $\sigma \notin o\text{-}\operatorname{Aut}\mathbb{K}$, indeed $\sigma(t) = -t < t = \sigma(-t)$. It is interesting to notice that σ is not order reversing either, as for example it is the identity on constants. □

The following example shows that Proposition 3.1.13 fails if we drop the assumption that k is archimedean. The reason is that, in general, the natural valuation on K, seen as an ordered field (with the lexicographic order), is not equivalent to the canonical one, when we see K as a Hahn field over k. Because the main interest is in relation to Proposition 3.1.13 we decided to include the example here, although it uses some notions that will only be developed in Subsection 3.3.5. Precise reference to the relevant statements will be given.

Example 3.2.4. Let $A = \coprod_{\mathbb{Z}^{<0}} \mathbb{Q}$, $B = \coprod_{\mathbb{Z}^{\geq 0}} \mathbb{Q}$ and $G = A \amalg B = \coprod_{\mathbb{Z}} \mathbb{Q}$. The group G is a *Hahn group* with valuation $v_G\colon G^{\neq 0} \to \mathbb{Z}$ given by the formula $v_G(\sum_{n\in\mathbb{Z}} a_n \mathbb{1}_n) = \min\{n : a_n \neq 0\}$.

Consider the ordered Hahn fields $k = \mathbb{R}((B))$, $K = k((A))$ and $F = \mathbb{R}((G))$. The canonical valuation $v^K_{\min}$ on K has value group $v^K_{\min}(K^\times) = A$ and non-archimedean residue field $k = \mathbb{R}((B))$.

The canonical valuation $v^F_{\min}$ has value group $v^F_{\min}(F^\times) = G$ and residue field $\bar{F} = \mathbb{R}$. Recall that this is the finest convex valuation on F. The field F also admits a coarser valuation w whose value group is $w(F^\times) = A = v^K_{\min}(K^\times)$ and residue field is $\bar{F}^w = \mathbb{R}((B)) = k$. The valuation w is defined (for non-zero elements) by

$$w\left(\sum_{(a,b)\in G} \alpha_{(a,b)} t^{(a,b)}\right) = \min\{a \in A : \exists b \in B,\ \alpha_{(a,b)} \neq 0\}. \tag{3.9}$$

The map

$$\xi\colon (F, w) \to (K, v^K_{\min}), \qquad \sum_{(a,b)\in G} \alpha_{(a,b)} t^{(a,b)} \mapsto \sum_a \left(\sum_b \alpha_{(a,b)} x^b\right) y^a$$

is an isomorphism of valued fields i.e., ξ is a field isomorphism and, for all $\alpha \in F$, we have $w(\alpha) = v^K_{\min}(\xi(\alpha))$. Indeed

- ξ respects the field operations: consider two elements $\alpha, \beta \in F$ and write $\alpha = \sum_{(a,b)\in G} \alpha_{(a,b)} t^{(a,b)}$ and $\beta = \sum_{(a,b)\in G} \beta_{(a,b)} t^{(a,b)}$. Then

$$\begin{aligned}
\xi(\alpha+\beta) &= \sum_a \left(\sum_b (\alpha_{(a,b)} + \beta_{(a,b)}) x^b \right) y^a \\
&= \sum_a \sum_b (\alpha_{(a,b)} x^b y^a + \beta_{(a,b)} x^b y^a) \\
&= \sum_a \sum_b \alpha_{(a,b)} x^b y^a + \sum_a \sum_b \beta_{(a,b)} x^b y^a \\
&= \xi(\alpha) + \xi(\beta)
\end{aligned}$$

and

$$\begin{aligned}
\xi(\alpha)\xi(\beta) &= \left(\sum_a \left(\sum_b \alpha_{(a,b)} x^b \right) y^a \right) \left(\sum_a \left(\sum_b \beta_{(a,b)} x^b \right) y^a \right) \\
&= \sum_a \left(\sum_{a_1+a_2=a} \left(\sum_b \alpha_{(a_1,b)} x^b \right) \left(\sum_b \beta(a_2,b) x^b \right) \right) y^a \\
&= \sum_a \left(\sum_{a_1+a_2=a} \left(\sum_b \left(\sum_{b_1+b_2=b} \alpha_{(a_1,b_1)} \beta(a_2,b_2) \right) x^b \right) \right) y^a \\
&= \sum_a \sum_r \left(\sum_{(a_1,b_1)+(a_2,b_2)=(a,b)} \alpha_{(a_1,b_1)} \beta_{(a_2,b_2)} x^b \right) y^a \\
&= \xi(\alpha\beta).
\end{aligned}$$

- Bijectivity: we construct the inverse. Let $\alpha \in K$, so $\alpha = \sum_{a\in A} \alpha_a y^a$ with $\alpha_a \in k = \mathbb{R}((B))$. Write $\alpha_a = \sum_{b\in B} \alpha_{a,b} x^b$. Then $\alpha = \sum_{a\in A} \left(\sum_{b\in B} \alpha_{a,b} x^b\right) y^a$. The inverse of ξ is then given by

$$\xi^{-1} \colon (K, v_{\min}^K) \longrightarrow (F, w), \quad \sum_{a\in A} \left(\sum_{b\in B} \alpha_{a,b} x^b \right) y^a \longmapsto \sum_{(a,b)\in G} \alpha_{(a,b)} t^{(a,b)}.$$

- ξ is valuation preserving:

$$\begin{aligned} w(\alpha) &= \min\{a \in A : \exists b \in B,\ \alpha_{(a,b)} \neq 0\} \\ &= \min \left\{ a \in A : \exists b \in B,\ \sum_b \alpha_{(a,b)} x^b \neq 0 \right\} \\ &= v^K_{\min}(\xi(\alpha)). \end{aligned}$$

We will construct a $\sigma \in o\text{-}\operatorname{Aut} F$ that does not preserve w and therefore $\xi\sigma\xi^{-1} \in o\text{-}\operatorname{Aut} K$ does not preserve $v^K_{\min}$.

Consider the automorphism ρ of the chain $(\mathbb{Z}, <)$ given by $n \mapsto n+1$. Since G has the canonical lifting property as a Hahn group (see Corollary 2.2.25), the map ρ lifts to an order preserving automorphism $\rho_G \in o\text{-}\operatorname{Aut} G$ given by $\rho_G\left(\sum_{n\in\mathbb{Z}} q_n \mathbb{1}_n\right) = \sum_{n\in\mathbb{Z}} q_n \mathbb{1}_{\rho(n)}$. Now, since F is a maximal Hahn field, by Example 3.3.48 it has the canonical first lifting property, so ρ_G lifts to an automorphism $\sigma \in v^F_{\min}\text{-}\operatorname{Aut} F$. Since G is divisible and $\mathbb{R}$ is real closed, then F is also real closed (Corollary 3.1.15). By Lemma 1.6.12 F admits a unique ordering and thus all its automorphisms are necessarily order-preserving, so $\sigma \in o\text{-}\operatorname{Aut} F$. Now we show that σ does not preserve w.

Set $U_w = \{\alpha \in F^\times : w(\alpha) = 0\}$, the group of units of the valuation ring of w and $G_w = v^F_{\min}(U_w)$. From (3.9), we can verify that $G_w \simeq B$:

$$\begin{aligned} \alpha \in U_w \Leftrightarrow w(\alpha) &= \min\{a \in A : \exists b \in B,\ \alpha_{(a,b)} \neq 0\} = 0 \\ &\Leftrightarrow \exists b_0 \in B : \begin{cases} \alpha_{(0,b_0)} \neq 0 \\ (a,b) < (0,b_0) \Rightarrow \alpha_{(a,b)} = 0 \end{cases} \end{aligned}$$

so $\alpha \in U_w \Leftrightarrow v^F_{\min}(\alpha) = (0, b_0)$ thus $G_w =^F_{\min} (U_w) = \{(0,b) : b \in B\} \simeq B$.

Write $\Gamma_w = v_G(G_w^{\neq 0}) = \mathbb{Z}^{\geq 0}$. Then [KMP17, Theorem 4.7] implies that σ preserves w if and only if the induced chain automorphism ρ of $\mathbb{Z}$ preserves Γ_w. But we have $\Gamma_w = \mathbb{Z}^{\geq 0} \neq \mathbb{Z}^{>0} = \rho(\Gamma_w)$ thus σ does not preserve w. □

Remark 3.2.5. Keeping the notation of Example 3.2.4, it is worth noticing that the automorphism る $:= \xi\sigma\xi^{-1}$ is not a k-automorphism ("る" is a japanese character pronounced "ru"). Indeed, let $\alpha = x^{1\mathbb{1}_1} \in k$. We write $\alpha = \left(x^{1\mathbb{1}_1}\right) y^0$ in order to make computations clearer when we view α as an element of K. Then we have

$$x^{1\mathbb{1}_1} = \left(x^{1\mathbb{1}_1}\right) y^0 \overset{\xi^{-1}}{\longmapsto} t^{(0,1\mathbb{1}_1)} \overset{\sigma}{\longmapsto} t^{\rho((0,1\mathbb{1}_1))} = t^{\rho((0,1\mathbb{1}_2))} \overset{\xi}{\longmapsto} \left(x^{1\mathbb{1}_2}\right) y^0 = x^{1\mathbb{1}_2}$$

hence $\text{る}(\alpha) \neq \alpha$ and so $\text{る}|_k \neq \mathrm{id}_k$.

We also remark that る is strongly additive (strong additivity will be defined in Definition 3.4.1). Indeed, let $\alpha = \sum_{a\in A}\left(\sum_{b\in B}\alpha_{a,b}x^b\right) y^a$ be a general element of K. For $a \in A$ and $b \in B$ write $a = \sum_{n<0} a_n \mathbb{1}_n$ and $b = \sum_{n\geq 0} b_n \mathbb{1}_n$. Let us compute $\text{る}(y^a)$ and $\text{る}\left(\sum_{b\in B}\alpha_{a,b}x^b\right)$ separately:

$$y^a \overset{\xi^{-1}}{\longmapsto} t^{(a,0)} \overset{\sigma}{\longmapsto} t^{\rho((a,0))} = t^{\left(\sum_{n<0} a_{n-1}\mathbb{1}_n, a_{-1}\mathbb{1}_0\right)} \overset{\xi}{\longmapsto} x^{a_{-1}\mathbb{1}_0} y^{\sum_{n<0} a_{n-1}\mathbb{1}_n} \tag{3.10}$$

and

$$\begin{aligned}\sum_{b\in B}\alpha_{a,b}x^b \overset{\xi^{-1}}{\longmapsto} \sum_{b\in B}\alpha_{a,b}t^{(0,b)} \overset{\sigma}{\longmapsto} \sum_{b\in B}\alpha_{a,b}t^{\rho((0,b))} = \sum_{b\in B}\alpha_{a,b}t^{\left(0,\sum\limits_{n\geq 1} b_{n-1}\mathbb{1}_n\right)} \\ \overset{\xi}{\longmapsto} \sum_{b\in B}\alpha_{a,b}x^{\sum_{n\geq 1} b_{n-1}\mathbb{1}_n}.\end{aligned} \tag{3.11}$$

Now we compute $\text{る}(\alpha)$:

$$\begin{aligned}\sum_{a\in A}\left(\sum_{b\in B}\alpha_{a,b}x^b\right) y^a &\overset{\xi^{-1}}{\longmapsto} \sum_{a\in A, b\in B}\alpha_{a,b}t^{(a,b)} \\ &\overset{\sigma}{\longmapsto} \sum_{a\in A, b\in B}\alpha_{a,b}t^{\rho((a,b))} \\ &= \sum_{a\in A, b\in B}\alpha_{a,b}t^{\left(\sum\limits_{n<0} a_{n-1}\mathbb{1}_n,\, a_{-1}\mathbb{1}_0 + \sum\limits_{n\geq 1} b_{n-1}\mathbb{1}_n\right)} \\ &\overset{\xi}{\longmapsto} \sum_{a\in A}\left(\sum_{b\in B}\alpha_{a,b}x^{\sum_{n\geq 1} b_{n-1}\mathbb{1}_n} x^{a_{-1}\mathbb{1}_0}\right) y^{\sum_{n<0} a_{n-1}\mathbb{1}_n} \\ &= \sum_{a\in A}\left(\sum_{b\in B}\alpha_{a,b}x^{\sum_{n\geq 1} b_{n-1}\mathbb{1}_n}\right) x^{a_{-1}\mathbb{1}_0} y^{\sum_{n<0} a_{n-1}\mathbb{1}_n} \\ &= \sum_{a\in A}\text{る}\left(\sum_{b\in B}\alpha_{a,b}x^b\right)\text{る}(y^a)\end{aligned} \tag{3.12}$$

which shows that $\mathfrak{z}$ is strongly additive. □

3.3 Decomposition theorems

3.3.1 The first lifting property

Let $\mathbb{K} = k((G))$ and let $k(G) \subseteq K \subseteq \mathbb{K}$ be a Hahn field, endowed with the canonical valuation $v = v_{\min}$. Recall (Corollary 1.5.11) that a valuation preserving automorphism $\sigma \in v\text{-}\operatorname{Aut} K$ induces automorphisms $\bar{\sigma} \in \operatorname{Aut} \bar{K}$ and $\sigma_G \in o\text{-}\operatorname{Aut} G$ given by:

$$\sigma_G(v(a)) = v(\sigma(a))$$
$$\bar{\sigma}(\bar{a}) = \overline{\sigma(a)}$$

for all $a \in K$. Note that σ_G is well defined because $\sigma \in v\text{-}\operatorname{Aut} K$. This defines a map

$$\Phi_K \colon v\text{-}\operatorname{Aut} K \longrightarrow \operatorname{Aut} \bar{K} \times o\text{-}\operatorname{Aut} G, \qquad \sigma \longmapsto (\bar{\sigma}, \sigma_G). \tag{3.13}$$

Lemma 3.3.1. *The map Φ_K defined in (3.13) is a group homomorphism.*

Proof. Let $\sigma, \tau \in v\text{-}\operatorname{Aut} K$ and $a \in K$. Then

$$(\sigma\tau)_G(v(a)) = v(\sigma(\tau(a))) = \sigma_G(v(\tau(a))) = \sigma_G(\tau_G(v(a))) = (\sigma_G\tau_G)(v(a)).$$

Next we need to show that we have $(\sigma_G)^{-1} = (\sigma^{-1})_G$. We thus compute

$$\sigma_G(\sigma^{-1})_G(v(a)) = \sigma_G(v(\sigma^{-1}(a))) = v(\sigma\sigma^{-1}(a)) = v(a).$$

In a similar way, we can compute

$$\overline{\sigma\tau}(\bar{a}) = \overline{\sigma\tau(a)} = \bar{\sigma}(\overline{\tau(a)}) = \bar{\sigma}\bar{\tau}(\bar{a})$$

and

$$\bar{\sigma}\overline{\sigma^{-1}}(\bar{a}) = \bar{\sigma}(\overline{\sigma^{-1}(a)}) = \overline{\sigma\sigma^{-1}(a)} = \bar{a}.$$

□

Lemma 3.3.2. *Let $\sigma \in v\text{-}\operatorname{Aut} K$ and let $f \colon \bar{K} \to k$ be an isomorphism (note that, by Lemma 3.1.11 such an isomorphism exists). Then σ induces an automorphism $\sigma_{k,f} \in$*

$\operatorname{Aut} k$ on the base field k. In particular, $\sigma_k := \sigma_{k,f_c}$ is defined by

$$\sigma_k(\alpha) = \sigma(\alpha)_0,$$

for all $\alpha \in k$. Moreover, if $\sigma \in v\text{-}\operatorname{Aut}_{(k)} K$, then σ_{k,f_c} coincides with the restriction $\sigma|_k$ of σ to k.

Proof. The map $\sigma_{k,f} = f\bar{\sigma}f^{-1} \colon k \to k$ is the composition of three isomorphisms, hence an isomorphism. Now, for all $\alpha \in k$ we obviously have $\alpha_0 = \alpha$ (on the right hand side of the last equation we see α as an element of K), thus

$$f_c\bar{\sigma}f_c^{-1}(\alpha) = f_c\bar{\sigma}(\alpha + I_K) = f_c((\sigma\alpha)_0 + I_K) = \sigma(\alpha)_0 = \sigma_k(\alpha).$$

If $\sigma \in v\text{-}\operatorname{Aut}_{(k)} K$ the second part of the statement follows immediately as $\sigma(\alpha)_0 = \sigma(\alpha)$. In other words, the diagram

$$\begin{array}{ccc} \bar{K} & \xrightarrow{\bar{\sigma}} & \bar{K} \\ \downarrow & & \downarrow \\ k & \xrightarrow{\sigma|_k} & k \end{array}$$

is commutative. □

Let us fix an isomorphism $f \colon \bar{K} \to k$. By Lemma 3.3.2 we have a map

$$\Phi_{K,f} \colon v\text{-}\operatorname{Aut} K \longrightarrow \operatorname{Aut} k \times o\text{-}\operatorname{Aut} G, \qquad \sigma \longmapsto (\sigma_{k,f}, \sigma_G). \tag{3.14}$$

Lemma 3.3.3. *The map $\Phi_{K,f}$ defined in (3.14) is a group homomorphism.*

Proof. Consider the map

$$\begin{aligned} f^* \colon \operatorname{Aut} \bar{K} \times o\text{-}\operatorname{Aut} G &\to \operatorname{Aut} k \times o\text{-}\operatorname{Aut} G \\ (\xi, \tau) &\mapsto (f\xi f^{-1}, \tau). \end{aligned} \tag{3.15}$$

where the operation on all the components is composition of function and we take the usual direct product of groups. Note that on the second component f^* is the identity. Then f^* is a group isomorphism, induced by the isomorphism $f \colon \bar{K} \to k$ and thus $\Phi_{K,f} = f^*\Phi$ is also an isomorphism. □

Whenever the context is clear we will omit K, f from the notation and write Φ instead of $\Phi_{K,f}$.

Definition 3.3.4. The kernel $\ker \Phi$ of the map (3.14) is a normal subgroup of $v\text{-}\operatorname{Aut} K$ that we call the subgroup of *internal automorphisms of* K. We will denote it by $\operatorname{Int}\operatorname{Aut} K$. We write $\operatorname{Int}\operatorname{Aut}_{(k)} K := \operatorname{Int}\operatorname{Aut} K \cap v\text{-}\operatorname{Aut}_{(k)} K$ as well as $\operatorname{Int}\operatorname{Aut}_k K := \operatorname{Int}\operatorname{Aut} K \cap v\text{-}\operatorname{Aut}_k K$. Notice that, since $\sigma_{k,f} = f\bar{\sigma}f^{-1}$, then $\sigma_{k,f} = \mathrm{id}_k \Leftrightarrow \bar{\sigma} = \mathrm{id}_{\bar{K}}$. The definition of $\operatorname{Int}\operatorname{Aut} K$ is therefore independent of our choice of f.

The following is a characterisation of the internal automorphisms.

Proposition 3.3.5. *Let $\sigma \in v\text{-}\operatorname{Aut} K$ Then $\sigma \in \operatorname{Int}\operatorname{Aut} K$ if and only if the following two conditions hold*

(i) $v(a) = v(\sigma(a))$, for all $a \in K$;

(ii) if $a \in R_K$ then $\sigma(a)_0 = a_0$;

Proof. Let $\sigma \in \operatorname{Int}\operatorname{Aut} K$. Then

(i) holds by definition, since $\sigma_G = \mathrm{id}_G$;

(ii) Let $a \in R_K$ and $\bar{\sigma} \in \operatorname{Aut}\bar{K}$ the induced automorphism on $\bar{K}$ (Definition 1.5.9). Since $\bar{\sigma} = \mathrm{id}_{\bar{K}}$ we have

$$a_0 + I_K = a + I_K = \bar{\sigma}(a + I_K) = \sigma(a) + I_K = \sigma(a)_0 + I_K$$

which, since $a_0, \sigma(a)_0 \in k$ implies $a_0 = \sigma(a)_0$.

Vice versa, let $\sigma \in v\text{-}\operatorname{Aut} K$ satisfy (i) and (ii). From (i) if follows that $\sigma_G = \mathrm{id}_G$. Let $a \in R_K$ and compute

$$\bar{\sigma}(a + I_K) = \sigma(a) + I_K = \sigma(a)_0 + I_K \overset{(ii)}{=} a_0 + I_K$$

thus $\bar{\sigma} = \mathrm{id}_{\bar{K}}$ and so $\sigma \in \operatorname{Int}\operatorname{Aut} K$. □

Corollary 3.3.6. $\operatorname{Int}\operatorname{Aut}_{(k)} K = \operatorname{Int}\operatorname{Aut}_k K$.

Proof. The inclusion $\operatorname{Int}\operatorname{Aut}_k K \subseteq \operatorname{Int}\operatorname{Aut}_{(k)} K$ is obvious. Let $\sigma \in \operatorname{Int}\operatorname{Aut}_k K$ and let $a_0 \in k$. Then we have

$$a_0 = \sigma(a_0)_0 = \sigma(a_0)$$

where the first equality follows from Proposition 3.3.5 and the second from the fact that $\sigma \in \operatorname{Int}\operatorname{Aut}_{(k)} K$. Hence $\sigma \in \operatorname{Int}\operatorname{Aut}_k K$ and the proof is complete. □

We will study the normal subgroup $\operatorname{Int}\operatorname{Aut} K$ in more detail in Subsection 3.3.3. Before doing that, our next goal is to find a complement of $\operatorname{Int}\operatorname{Aut} K$ in $v\text{-}\operatorname{Aut} K$. We therefore give the following definition.

Definition 3.3.7 (First lifting property)**.** We say that a pair $(\rho, \tau) \in \operatorname{Aut} k \times o\text{-}\operatorname{Aut} G$ *lifts to* K if there exists an automorphism $\sigma \in v\text{-}\operatorname{Aut} K$ such that we have $\Phi_{K,f}(\sigma) = (\rho, \tau)$. In this case we call σ a *lift of* (ρ, τ). If $\Phi_{K,f}$ admits a section, i.e., an injective group homomorphism $\Psi\colon \operatorname{Aut} k \times o\text{-}\operatorname{Aut} G \to v\text{-}\operatorname{Aut} K$ such that $\Phi_{K,f}\Psi = \mathrm{id}$, then every pair $(\rho, \tau) \in \operatorname{Aut} k \times o\text{-}\operatorname{Aut} G$ lifts to an automorphism $\Psi(\rho, \tau)$ of K and we say that K has the *first lifting property with respect to* Ψ.

Example 3.3.8. (i) The maximal Hahn field $\mathbb{K}$ has the first lifting property. Indeed, for every pair $(\rho, \tau) \in \operatorname{Aut} k \times o\text{-}\operatorname{Aut} G$ the map $\sigma\colon \sum a_g t^g \mapsto \sum \rho(a_g) t^{\tau(g)}$ is an automorphism of $\mathbb{K}$ such that $\Phi(\sigma) = (\rho, \tau)$ (see Corollary 3.3.48).

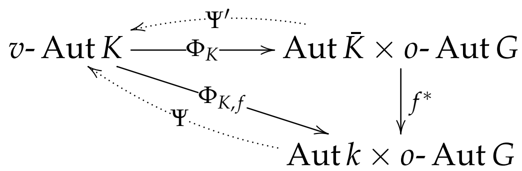
(ii) The field $k(G)$ has the first lifting property. The map σ of part (i) restricts to an automorphism of $k(G)$.

(iii) A large class of Hahn fields with the first lifting property will be described in Subsection 3.3.6.

□

Lemma 3.3.9. *The map $\Phi_{K,f}$ admits a section if and only if Φ_K does. In particular, whether $\Phi_{K,f}$ admits a section is independent of the choice of f.*

Proof. Let f^* be the isomorphism defined in (3.15). If Ψ is a section of $\Phi_{K,f}$ then Ψf^* is a section of Φ_K and, similarly, if Ψ' is a section of Φ_K then $f^{*-1}\Psi'$ is a section of $\Phi_{K,f}$



Notice that Ψ and $f^{*-1}\Psi'$ (resp. Ψ' and Ψf^*) need not coincide. □

Let $f\colon \bar{K} \to k$ be an arbitrary isomorphism.

Definition 3.3.10 (External automorphisms)**.** Assume that K has the first lifting property with respect to a fixed section Ψ of $\Phi_{K,f}$. The subgroup $\Psi(\operatorname{Aut} k \times$

$o\text{-}\operatorname{Aut} G$) of $v\text{-}\operatorname{Aut} K$ will be called the subgroup of *Ψ-external automorphisms of* K and denoted by $\Psi\text{-}\operatorname{Ext}\operatorname{Aut} K$. Therefore, for every $\sigma \in \Psi\text{-}\operatorname{Ext}\operatorname{Aut} K$ and every $(\rho, \tau) \in \operatorname{Aut} k \times o\text{-}\operatorname{Aut} G$ we have

$$\Phi_{K,f}(\sigma) = (\rho, \tau) \Leftrightarrow \Psi(\rho, \tau) = \sigma.$$

Therefore, for any section Ψ we have $\Psi\text{-}\operatorname{Ext}\operatorname{Aut} K \simeq \operatorname{Aut} k \times o\text{-}\operatorname{Aut} G$.
We will also use the notations $\Psi\text{-}\operatorname{Ext}\operatorname{Aut}_{(k)} K := \Psi\text{-}\operatorname{Ext}\operatorname{Aut} K \cap v\text{-}\operatorname{Aut}_{(k)} K$ and $\Psi\text{-}\operatorname{Ext}\operatorname{Aut}_k K := \Psi\text{-}\operatorname{Ext}\operatorname{Aut} K \cap v\text{-}\operatorname{Aut}_k K$.

Lemma 3.3.11. *Let Ψ_1, Ψ_2 be two sections of Φ_K and let $(\xi, \tau) \in \operatorname{Aut} \bar{K} \times o\text{-}\operatorname{Aut} G$. Then*

$$\Psi_1(\xi, \tau) \equiv \Psi_2(\xi, \tau) \pmod{\operatorname{Int}\operatorname{Aut} K}.$$

Proof. Let $\sigma_i := \Psi_i(\xi, \tau)$. We need to show that $\sigma_1 \sigma_2^{-1} \in \operatorname{Int}\operatorname{Aut} K$. We compute

$$\begin{aligned}\Phi_K(\sigma_1\sigma_2^{-1}) &= \Phi_K(\Psi_1(\xi,\tau)\Psi_2(\xi,\tau)^{-1}) \\ &= \Phi_K(\Psi_1(\xi,\tau))\Phi_K(\Psi_2(\xi,\tau))^{-1} \\ &= (\xi,\tau)(\xi,\tau)^{-1} \\ &= (\mathrm{id}_{\bar{K}}, \mathrm{id}_G).\end{aligned}$$

□

Remark 3.3.12. (i) If K has the first lifting property with respect to Ψ, then $\Phi_{K,f}$ is surjective and every pair $(\rho, \tau) \in \operatorname{Aut} k \times o\text{-}\operatorname{Aut} G$ lifts to an automorphism $\sigma = \Psi(\tau) \in v\text{-}\operatorname{Aut} K$. The first isomorphism theorem then yields

$$\operatorname{Aut} k \times o\text{-}\operatorname{Aut} G \simeq \frac{v\text{-}\operatorname{Aut} K}{\operatorname{Int}\operatorname{Aut} K}.$$

In particular, the set of lifts of some pair $(\rho, \tau) \in \operatorname{Aut} k \times o\text{-}\operatorname{Aut} G$ is the coset $\{\sigma\sigma' : \sigma' \in \operatorname{Int}\operatorname{Aut} K\}$, where σ is any given lift of (ρ, τ).

(ii) Notice also that a k-automorphism is not necessarily internal. Let $\mathrm{id}_G \neq \tau \in o\text{-}\operatorname{Aut} G$, then the pair (id_k, τ) lifts to an automorphism $\sigma \in v\text{-}\operatorname{Aut}_k K \setminus \operatorname{Int}\operatorname{Aut} K$. □

The first lifting property will reveal very powerful in the study of the automorphism group, as having a section of the map Φ of Definition 3.3.4 will allow us to split the automorphism group of K into a semi-direct product (see Theorem 3.3.13 below).

From now on we will fix the isomorphism $f_c\colon \bar{K} \to k$, $a \mapsto a_0$ and, correspondingly, the homomorphism $\Phi := \Phi_{f_c}$. We will also assume that all the Hahn fields under consideration have the first lifting property with respect to a given section Ψ of Φ, which we will omit from the notation.

3.3.2 The first decomposition theorem

In [Hof91, Satz 2.2] Hofberger shows that $v\text{-}\operatorname{Aut}\mathbb{K}$ can be decomposed into a semi-direct product of the groups of internal and external automorphisms (with respect to a specific section – see Definition 3.3.37). We generalise Hofberger's result to a Hahn field $K \subseteq \mathbb{K}$ which has the first lifting property with respect to an arbitrary section.

Theorem 3.3.13. *Assume that $K \subseteq \mathbb{K}$ is a Hahn field that has the first lifting property with respect to a section Ψ of Φ_K. Then the group $v\text{-}\operatorname{Aut} K$ admits an inner[2] semi-direct product decomposition:*

$$v\text{-}\operatorname{Aut} K = \operatorname{Int}\operatorname{Aut} K \rtimes \operatorname{Ext}\operatorname{Aut} K \tag{3.16}$$

$$v\text{-}\operatorname{Aut}_{(k)} K = \operatorname{Int}\operatorname{Aut}_k K \rtimes \operatorname{Ext}\operatorname{Aut}_{(k)} K \tag{3.17}$$

$$v\text{-}\operatorname{Aut}_k K = \operatorname{Int}\operatorname{Aut}_k K \rtimes \operatorname{Ext}\operatorname{Aut}_k K \tag{3.18}$$

Proof. Let us consider the sequence

$$\operatorname{Int}\operatorname{Aut} K \xrightarrow{\iota} v\text{-}\operatorname{Aut} K \xrightarrow{\Phi_K} \operatorname{Aut} k \times o\text{-}\operatorname{Aut} G$$
$$\Psi\colon \operatorname{Aut} k \times o\text{-}\operatorname{Aut} G \dashrightarrow v\text{-}\operatorname{Aut} K$$

where ι is the canonical embedding. By definition of $\operatorname{Int}\operatorname{Aut} K$ we have $\operatorname{im}\iota = \ker\Phi_K$ so the sequence is exact. Therefore (see [Con, Theorem 3.3]) we have:

$$v\text{-}\operatorname{Aut} K = \operatorname{im}\iota \rtimes \operatorname{im}\Psi = \operatorname{Int}\operatorname{Aut} K \rtimes \operatorname{Ext}\operatorname{Aut} K.$$

So (3.16) is established. Equations (3.17) and (3.18) are obtained from (3.16) by taking intersections with $v\text{-}\operatorname{Aut}_{(k)} K$ and $v\text{-}\operatorname{Aut}_k K$ respectively and applying Corollary 3.3.6. □

[2] For $\alpha_i \in \operatorname{Int}\operatorname{Aut} K$ and $\beta_i \in \operatorname{Ext}\operatorname{Aut} K$, $i = 1, 2$, we have $(\alpha_1, \beta_1)(\alpha_2\beta_2) = (\beta_1^{-1}\alpha_1\beta_1\alpha_2, \beta_1\beta_2)$.

Remark 3.3.14. By Theorem 3.3.13, describing v-Aut K consists of two tasks: describing the normal subgroup Int Aut K and the subgroup Ext Aut K. By Remark 3.3.12, we have Ext Aut $K \simeq$ Aut $k \times o$-Aut G. We studied o-Aut G, in some special cases, in Subsection 2.3.2 and we will see applications of those results and of the ones we just proved in Section 3.5. In the next section we investigate the structure of Int Aut K. □

3.3.3 The second lifting property

In this section we study Int Aut K in more detail and provide a decomposition into a semi-direct product of two notable subgroups.

Definition 3.3.15. Let $\text{Hom}(G, k^\times)$ be the set of all homomorphisms of the additive group $(G, +)$ into the multiplicative group $(k^\times, \cdot)$. Let $x \in \text{Hom}(G, k^\times)$. We will denote the image of a $g \in G$ under x by $x^g := x(g)$, so, for all $g, h \in G$ we have $x^{g+h} = x^g x^h$. Let $\mathbf{1}\colon (G, +) \to (k^\times, \cdot)$, $g \mapsto 1$ be the trivial morphism. Then the set $\text{Hom}(G, k^\times)$ forms a group under the pointwise multiplication defined by $(xy)^g := x^g y^g$. The inverse of x is the morphism $g \mapsto x^{-g}$ and $\mathbf{1}$ is the neutral element.

We intend to proceed in an analogous way to Subsection 3.3.1 in order to find a semi-direct product decomposition of Int Aut K. We therefore define the following map.

Definition 3.3.16. For a Hahn field K define

$$\begin{array}{rccc} X_K\colon & \text{Int Aut}\, K & \to & \text{Hom}(G, k^\times) \\ & \sigma & \mapsto & x_\sigma \end{array} \tag{3.19}$$

where

$$x_\sigma(g) := x_\sigma^g := \sigma(t^g)_g \tag{3.20}$$

for all $g \in G$.
If there is no risk of confusion we will omit the subscript and just write X instead of X_K.

Lemma 3.3.17. *The map X_K of Definition 3.3.16 is a group homomorphism.*

Proof. Let $\sigma \in$ Int Aut K and $g, h \in G$. The formula (3.20) defines an element of $\text{Hom}(G, k^\times)$. Indeed, by Proposition 3.3.5, $g = v(t^g) = v(\sigma(t^g))$, so $x_\sigma(g) =$

$\sigma(t^g)_g \in k^\times$. Moreover, we have

$$x_\sigma(g+h) = \sigma(t^{g+h})_{g+h} = (\sigma(t^g)\sigma(t^h))_{g+h} = \sigma(t^g)_g\sigma(t^h)_h = x_\sigma(g)x_\sigma(h).$$

So X is a well defined map. To show that it is also a group homomorphism, let $\sigma, \tau \in \operatorname{Int Aut} K$. Let $\alpha := \tau(t^g)_g$. Then

$$x_{\sigma\tau}(g) = (\sigma\tau(t^g))_g = \sigma(\alpha t^g)_g = \alpha\sigma(t^g)_g = x_\sigma(t^g)x_\tau(t^g).$$

□

Definition 3.3.18 (1-automorphisms)**.** The kernel $\ker X_K$ will be called the group of *1-automorphisms of* K and denoted by $1\text{-}\operatorname{Aut} K$. Hence $1\text{-}\operatorname{Aut} K \trianglelefteq \operatorname{Int Aut} K$. We will also use the notations $1\text{-}\operatorname{Aut}_{(k)} K := 1\text{-}\operatorname{Aut} K \cap \operatorname{Aut}_{(k)} K$ and $1\text{-}\operatorname{Aut}_k K := 1\text{-}\operatorname{Aut} K \cap \operatorname{Aut}_k K$.

Lemma 3.3.19. *Let K be a Hahn field. The following hold:*

(i) Let $\tau \in \operatorname{Int Aut} K$. Then $\tau \in 1\text{-}\operatorname{Aut} K$ if and only if, for all $a \in K^\times$ we have $\tau(a)_{v(a)} = a_{v(a)}$;

(ii) $1\text{-}\operatorname{Aut}_{(k)} K = 1\text{-}\operatorname{Aut}_k K$.

Proof. (i) Assume $\tau \in 1\text{-}\operatorname{Aut} K$. By definition of $1\text{-}\operatorname{Aut} K$, if $\tau \in 1\text{-}\operatorname{Aut} K$ and $a \in K^\times$ with $v(a) = h$ we have $\tau(a) = \tau\left(a_h t^h + b\right)$ with $v(b) > h$. Therefore $\tau(a) = \tau(a_h t^h) + \tau(b) = a_h t^h + c + \tau(b)$ for some $b, c \in K$ with $v(\tau(b)) > h$ and $v(c) > h$.

Vice versa, assume that for all $a \in K^\times$ we have $\tau(a)_{v(a)} = a_{v(a)}$. Then, with the notation of Definition 3.3.16, for all $g \in G$ we have

$$x_\tau(g) = \tau(t^g)_g = 1$$

because $v(t^g) = g$ and, clearly, $(t^g)_g = 1$. Thus $\tau \in \ker X$, i.e., $\tau \in 1\text{-}\operatorname{Aut} K$.

(ii) An automorphism that fixes the first coefficient of every series and keeps k invariant is necessarily trivial on k.

□

We determined an important normal subgroup of the group $\operatorname{Int Aut} K$ of internal automorphisms. In analogy to what we did above for $\operatorname{Int Aut} K$ inside the full group $v\text{-}\operatorname{Aut} K$, we are going to determine a complement of $1\text{-}\operatorname{Aut} K$ inside $\operatorname{Int Aut} K$. We therefore introduce the following definition.

Definition 3.3.20 (Second lifting property)**.** We say that a Hahn field K satisfies the *second lifting property with respect to* P (to be read as a capital "ρ") if there exists an injective homomorphism

$$P\colon \operatorname{Hom}(G,k^{\times}) \hookrightarrow \operatorname{Int}\operatorname{Aut} K \tag{3.21}$$

such that $X_K P = \mathrm{id}_{\operatorname{Hom}(G,k^{\times})}$.

Definition 3.3.21. Let K satisfy the second lifting property with respect to P. The subgroup $\operatorname{im} P \leq \operatorname{Int}\operatorname{Aut} K$ is called the group of *G_P-exponentiations on K* and denoted by $G_P\text{-}\operatorname{Exp} K$.
Clearly we have $G_P\text{-}\operatorname{Exp} K \simeq \operatorname{Hom}(G,k^{\times})$, hence it only depends on G and $k^{\times}$. If the context is clear we will drop P from the notation and terminology and write $G\text{-}\operatorname{Exp} K$, which we will simply call the group of *G-exponentiations*.

Consider the maximal Hahn field $\mathbb{K}$ and the following map

$$\begin{array}{rccc} P_c\colon & \operatorname{Hom}(G,k^{\times}) & \to & \operatorname{Int}\operatorname{Aut}\mathbb{K} \\ & x & \mapsto & \rho_x \end{array} \tag{3.22}$$

where ρ_x is given by

$$\rho_x\left(\sum a_g t^g\right) = \sum a_g x^g t^g. \tag{3.23}$$

Lemma 3.3.22. *The map P_c defined in (3.22) is an injective group homomorphism and we have $X_{\mathbb{K}} P_c = \mathrm{id}_{\operatorname{Hom}(G,k^{\times})}$.*

Proof. Let $x, y \in \operatorname{Hom}(G,k^{\times})$ and let $a = \sum a_g t^g \in K$. Moreover, let $\rho_x = P_c(x)$ and $\rho_y = P_c(y)$. We show first that P_c is a homomorphism. Let $\rho_{xy} = P_c(xy)$. Then we have

$$\rho_{xy}(a) = \sum a_g (xy)^g t^g = \sum a_g x^g y^g t^g = \rho_x\left(\sum a_g y^g t^g\right) = \rho_x\left(\rho_y(a)\right)$$

thus $\rho_{xy} = \rho_x \rho_y$. So P_c is a homomorphism.
If $\rho_x = \rho_y$ then, for all $g \in G$ we have $x^g t^g = \rho_x(t^g) = \rho_y(t^g) = y^g t^g$, which implies $x^g = y^g$ for all g and so $x = y$. So P is injective.
Finally, for all $g \in G$ we have

$$XP(x)(g) = X(\rho_x)(g) = (\rho_x(t^g))_g = (x^g t^g)_g = x^g,$$

thus $XP(x) = x$ which proves $XP = \mathrm{id}_{\operatorname{Hom}(G,k^{\times})}$ and, in particular, X is surjective. □

Definition 3.3.23 (Canonical second lifting property)**.** Let K be a Hahn field. Suppose that for all $x \in \operatorname{Hom}(G, k^\times)$ the map ρ_x defined in (3.23) restricts to an automorphism of K: $\rho_x|_K \in \operatorname{Int}\operatorname{Aut} K$. In other words, the map

$$\begin{array}{rccc} P_c\colon & \operatorname{Hom}(G,k^\times) & \to & \operatorname{Int}\operatorname{Aut}\mathbb{K} \\ & x & \mapsto & \rho_x|_K \end{array}$$

is a section of X_K. Then we say that K satisfies the *canonical second lifting property*.

Example 3.3.24. (i) By Lemma 3.3.22 the maximal Hahn field $\mathbb{K}$ satisfies the canonical second lifting property.

(ii) The field $k(G)$ satisfies the (canonical) second lifting property. Indeed, let $a = c/d \in k(G)$ for $c, d \in \mathbb{K}$ with finite support. Let $x \in \operatorname{Hom}(G, k^\times)$ and consider ρ_x as an automorphism of $\mathbb{K}$. Then

$$\rho_x\left(\frac{c}{d}\right) = \frac{\rho_x(c)}{\rho_x(d)} \in k(G)$$

because $\operatorname{supp}(c) = \operatorname{supp}(\rho_x(c))$ and $\operatorname{supp}(d) = \operatorname{supp}(\rho_x(d))$ are finite.

(iii) Further examples of Hahn fields satisfying the (canonical) second lifting property will be given in Subsection 3.3.6.

The next proposition establishes some further properties of G_{P_c}-exponentiations.

Proposition 3.3.25. *Let K be a Hahn field satisfying the canonical second lifting property. The following assertions hold.*

(i) For all $\rho \in G_{P_c}\text{-}\operatorname{Exp} K$ and for all $a \in K$ we have $\operatorname{supp}\rho(a) = \operatorname{supp} a$.

(ii) The group $G_{P_c}\text{-}\operatorname{Exp} K$ is abelian.

(iii) The inverse of ρ_x is given by

$$\rho_x^{-1}\left(\sum a_g t^g\right) = \sum a_g x^{-g} t^g. \tag{3.24}$$

(iv) All G_{P_c}-exponentiations are trivial on k, so we have $G_{P_c}\text{-}\operatorname{Exp} K \leq \operatorname{Aut}_k K$.

Proof. (i) Let $a = \sum_{g\in G} a_g t^g$. By (3.23) we have

$$\rho_x(a) = \sum_{g\in G} a_g x^g t^g$$

and for all g we have $x^g \in k^z$. Thus $a_g x^g \neq 0$ if and only if $a_g \neq 0$. Therefore $\operatorname{supp} \rho_x(a) = \operatorname{supp} a$.

(ii) Let $x, y \in \operatorname{Hom}(G, k^\times)$ and let ρ_x, ρ_y be the corresponding elements of G_{P_c}- Exp K. Let also $a = \sum a_g t^g \in K$. Then we have

$$\rho_x \rho_y(a) = \sum a_g x^g y^g t^g = \sum a_g y^g x^g t^g = \rho_y \rho_x(a).$$

(iii) Let $\tau \in \operatorname{Int} \operatorname{Aut} G$ be defined by

$$\tau(a) = \sum a_g x^{-g} t^g.$$

Because $g \mapsto x^{-g}$ is the inverse of x in $\operatorname{Hom}(G, k^\times)$ it follows that $\tau \in G_{P_c}$- Exp K. Computing that it is in fact the inverse of ρ_x is immediate:

$$\tau\left(\rho_x(a)\right) = \tau\left(\sum_{g \in G} a_g x^g t^g\right) = \sum_{g \in G} a_g x^g x^{-g} t^g = \sum_{g \in G} a_g t^g = a.$$

It is evident that the same holds for the composition in the reversed order: $\rho_x \tau = \mathrm{id}_K$.

(iv) Let $a_0 \in k$ and identify it with the element $a_0 t^0 \in K$. Let $x \in \operatorname{Hom}(G, k^\times)$. Then

$$\rho_x(a_0) = a_0 g^0 t^0 = a_0.$$

□

Remark 3.3.26. We do not know whether the inclusion 1- $\operatorname{Aut}_k K \trianglelefteq$ 1- Aut K is strict. It will follow from Theorem 3.3.52 below that this is the case if and only if the inclusion v- $\operatorname{Aut}_{(k)} K \leq v$- Aut K is strict. □

3.3.4 The second decomposition theorem

The next proposition gives a decomposition of Int Aut K that will be used to further refine Theorem 3.3.13.

Theorem 3.3.27. *Let K satisfy the second lifting property w.r.t. to P. Then the group* Int Aut K *admits the following semi-direct product decomposition:*

$$\operatorname{Int} \operatorname{Aut} K = 1\text{- } \operatorname{Aut} K \rtimes G_P\text{- } \operatorname{Exp} K. \tag{3.25}$$

If, moreover, K has the canonical second lifting property, then

$$\operatorname{Int}\operatorname{Aut}_k K = \operatorname{Int}\operatorname{Aut}_{(k)} K = 1\text{-}\operatorname{Aut}_k K \rtimes G_{P_c}\text{-}\operatorname{Exp} K. \tag{3.26}$$

Proof. Consider the sequence

$$1\text{-}\operatorname{Aut} K \overset{\iota}{\hookrightarrow} \operatorname{Int}\operatorname{Aut} K \overset{X_K}{\twoheadrightarrow} \operatorname{Hom}(G, k^\times), \qquad \operatorname{Hom}(G, k^\times) \overset{P}{\dashrightarrow} \operatorname{Int}\operatorname{Aut} K$$

where ι is the canonical embedding. By Lemma 3.3.19, $\ker X = 1\text{-}\operatorname{Aut} K = \operatorname{im} \iota$ so the sequence is exact and, by definition of second lifting property, P is a section of X. Hence (3.25) follows.
Now let K satisfy the canonical second lifting property. By Proposition 3.3.25 we have $G_{P_c}\text{-}\operatorname{Exp} K \leq v\text{-}\operatorname{Aut}_k K$ and Lemma 3.3.19 gives $1\text{-}\operatorname{Aut}_{(k)} K = 1\text{-}\operatorname{Aut}_k K$, so (3.26) follows. □

Notation 3.3.28. *From now on we will only consider the canonical second lifting property. We will omit the subscript P_c and simply talk about G-exponentiations, whose group will be denoted by $G\text{-}\operatorname{Exp} K$. We will also write X for X_K.*

Theorem 3.3.27 implies that every $\sigma \in \operatorname{Int}\operatorname{Aut} K$ can be expressed as a product $\sigma = \rho\tau$ for some $\rho \in G\text{-}\operatorname{Exp} K$ and $\tau \in 1\text{-}\operatorname{Aut} K$. The following lemma shows that this is a characterisation of internal automorphisms; most interestingly, the proof shows how to compute ρ and τ explicitly from σ.

Lemma 3.3.29. *Let K have the canonical second lifting property and $\sigma \in v\text{-}\operatorname{Aut} K$. Then $\sigma \in \operatorname{Int}\operatorname{Aut} K$ if and only if there exist $\rho \in G\text{-}\operatorname{Exp} K$ and $\tau \in 1\text{-}\operatorname{Aut} K$ such that $\sigma = \rho\tau$.*

Proof. Since $\operatorname{Int}\operatorname{Aut} K$ is a group, composition of internal automorphisms is internal. So if there exist ρ, τ as in the statement, then in particular $\rho, \tau \in \operatorname{Int}\operatorname{Aut} K$, therefore $\sigma = \rho\tau \in \operatorname{Int}\operatorname{Aut} K$.
Conversely, let $\sigma \in \operatorname{Int}\operatorname{Aut} K$. For all $g \in G$ let $x^g := \sigma(t^g)_g$ be the first coefficient of $\sigma(t^g)$. Notice that the elements $\{x^g : g \in G\}$ have the property that

$$x^g x^h = x^{g+h} \tag{3.27}$$

indeed x^{g+h} is the first coefficient of $\sigma(t^{g+h}) = \sigma(t^g t^h) = \sigma(t^g)\sigma(t^h)$ and the first coefficient of the last series is the product of the first coefficients of the factors. Hence the map $x \colon G \to k^\times$, $g \mapsto x^g$ is an element of $\operatorname{Hom}(G, k^\times)$, and the corresponding ρ_x defined as in (3.23) is a G-exponentiation on K. Set $\rho = \rho_x$. Now let $\tau := \rho^{-1}\sigma$. Obviously we have $\sigma = \rho\tau$, so we just need to show that $\tau \in 1\text{-}\operatorname{Aut} K$. Let $a \in K$ and let $h = v(a)$. Then we have

$$\begin{aligned}
\tau(a)_h &= (\rho^{-1}\sigma(a))_h \\
&= (\rho^{-1}\sigma(a_h t^h))_h \\
&= (\rho^{-1}\sigma(a_h) \cdot \rho^{-1}\sigma(t^h))_h \\
&= (\rho^{-1}\sigma(a_h))_0 \cdot (\rho^{-1}\sigma(t^h))_h \\
&= a_h (x^h)^{-1}\sigma(t^h)_h \\
&= a_h
\end{aligned}$$

So $\tau \in 1\text{-}\operatorname{Aut} K$ and the proof is complete. □

It is worth noticing that the semi-direct product of $1\text{-}\operatorname{Aut} K$, $G\text{-}\operatorname{Exp} K$ and $\operatorname{Ext}\operatorname{Aut} K$ is associative, as the following lemma shows.

Lemma 3.3.30. *Let K satisfy the first and canonical second lifting property. Then we have*

$$\begin{aligned}
v\text{-}\operatorname{Aut} K &= (1\text{-}\operatorname{Aut} K \rtimes G\text{-}\operatorname{Exp} K) \rtimes \operatorname{Ext}\operatorname{Aut} K \\
&= 1\text{-}\operatorname{Aut} K \rtimes (G\text{-}\operatorname{Exp} K \rtimes \operatorname{Ext}\operatorname{Aut} K).
\end{aligned}$$

Proof. The first equality was already established. We will prove

$$v\text{-}\operatorname{Aut} K = 1\text{-}\operatorname{Aut} K \rtimes (G\text{-}\operatorname{Exp} K \rtimes \operatorname{Ext}\operatorname{Aut} K).$$

To show that $1\text{-}\operatorname{Aut} K$ is normal in $v\text{-}\operatorname{Aut} K$, let $\sigma \in v\text{-}\operatorname{Aut} K$ and $\tau \in 1\text{-}\operatorname{Aut} K$. Let $a \in K$ and $b := \sigma\tau\sigma^{-1}(a)$. First we show that $v(a) = v(b)$. Indeed, let $h := v(a)$. Then $v(b) = v(\sigma(\tau\sigma^{-1}(a)))$. Since $\tau \in \operatorname{Int}\operatorname{Aut} K$, then $v(\tau\sigma^{-1}(a)) = v(\sigma^{-1}(a))$, hence $v(b) = v(\sigma\sigma^{-1}(a)) = v(a)$. Now let us prove that $a_h = b_h$. We have

$$\begin{aligned}
b_h &= \sigma\tau\sigma^{-1}(a)_h = \sigma\tau\sigma^{-1}(a_h t^h)_h = \sigma\tau(a'_{h'} t^{h'} + \varepsilon)_h \\
&= \sigma(a'_{h'} t^{h'} + \varepsilon')_h = (a_h + \varepsilon)_h \\
&= a_h.
\end{aligned}$$

□

3.3.5 The canonical first lifting property

Recall that, by Lemma 3.1.11, for any Hahn field K there is a distinguished isomorphism $f_c \colon \bar{K} \to k$ defined by $f_c(a + I_K) = a_0$, for all $a \in R_K$. We call f_c the *canonical* or *coefficient isomorphism* between $\bar{K}$ and k. Lemma 3.1.11 also stipulates that every other isomorphism f between $\bar{K}$ and k factors through f_c. For a given isomorphism $f \colon \bar{K} \to k$, the homomorphism $\Phi_{K,f}$ defined in (3.14) implicitly depends on the choice of f: from now on we fix this to be the coefficient isomorphism f_c. Then $\Phi_{K,f}$ assumes the special form

$$\Phi_c \colon v\text{-}\operatorname{Aut} K \longrightarrow \operatorname{Aut} k \times o\text{-}\operatorname{Aut} G, \qquad \sigma \longmapsto (\sigma_k, \sigma_G) \tag{3.28}$$

where $\sigma_k = f_c \bar{\sigma} f_c^{-1}$. Computing gives $f_c \bar{\sigma} f_c^{-1}(a_0) = f_c \bar{\sigma}(a_0 + I_K) = f_c(\sigma(a_0) + I_K) = \sigma(a_0)_0$, for all $a_0 \in k$. Thus

$$\sigma_k(a_0) = \sigma(a_0)_0 \quad \text{for all } a_0 \in k. \tag{3.29}$$

Remark 3.3.31. Let $\sigma \in v\text{-}\operatorname{Aut}_{(k)} K$. Then $\sigma|_k = \sigma_k$. Indeed, for $a_0 \in k$, from $\sigma \in v\text{-}\operatorname{Aut}_{(k)} K$ it follows that $\sigma(a_0) \in k$ so Equation (3.29) gives $\sigma_k(a_0) = \sigma(a_0)_0 = \sigma(a_0) = \sigma|_k(a_0)$. Moreover, let $\pi_1 : \operatorname{Aut} k \times o\text{-}\operatorname{Aut} G \to \operatorname{Aut} k$, $(\rho, \tau) \mapsto \rho$ be the projection on the first component. Then the restriction $\pi_1 \Phi_c \colon v\text{-}\operatorname{Aut}_{(k)} K \to \operatorname{Aut} k$ is a homomorphism with kernel $v\text{-}\operatorname{Aut}_k K$. Thus $v\text{-}\operatorname{Aut}_k K \trianglelefteq v\text{-}\operatorname{Aut}_{(k)} K$. □

Lemma 3.3.32. *Let $(\rho, \tau) \in \operatorname{Aut} k \times o\text{-}\operatorname{Aut} G$ and define a map $\widetilde{\rho\tau} \colon \mathbb{K} \to \mathbb{K}$ by*

$$\widetilde{\rho\tau}\left(\sum_{g \in G} a_g t^g\right) = \sum_{g \in G} \rho(a_g) t^{\tau(g)}. \tag{3.30}$$

Then $\widetilde{\rho\tau} \in v\text{-}\operatorname{Aut}_{(k)} \mathbb{K}$ and we have $\Phi_c(\widetilde{\rho\tau}) = (\rho, \tau)$.

Proof. Let $a = \sum_{g \in G} a_g t^g, b = \sum_{g \in G} b_g t^g \in \mathbb{K}$.
Claim: $\widetilde{\rho\tau}$ is well defined.
Because $\rho \in \operatorname{Aut} k$ we have $\rho(a_g) = 0 \Leftrightarrow a_g = 0$ and therefore $\operatorname{supp}(\widetilde{\rho\tau}(a)) = \tau(\operatorname{supp} a)$ is a well ordered subset of G. So $\widetilde{\rho\tau}$ is a well defined map of $\mathbb{K}$ into itself. ♦
Claim: $\widetilde{\rho\tau}$ is an automorphism.

- We have

$$\begin{aligned}\widetilde{\rho\tau}(a-b) &= \sum_{g\in G}\rho(a_g-b_g)t^{\tau(g)}\\ &= \sum_{g\in G}\rho(a_g)t^{\tau(g)} + \sum_{g\in G}\rho(b_g)t^{\tau(g)}\\ &\widetilde{\rho\tau}(a)-\widetilde{\rho\tau}(b).\end{aligned}$$

  so $\widetilde{\rho\tau}$ is a homomorphism of the additive group $(\mathbb{K},+)$.

- Let $c = ab$, so that $c = \sum_{g\in G} c_g t^g$ with $c_g = \sum_{r+s=g} a_r b_s$. Then $\widetilde{\rho\tau}(ab) = \widetilde{\rho\tau}(c) = \sum_{g\in G}\rho(c_g)t^{\tau(g)}$. Now we have

$$\rho(c_g)t^{\tau(g)} = \sum_{r+s=g}\rho(a_r)\rho(b_s)t^{\tau(r)}t^{\tau(s)}$$

  thus

$$\begin{aligned}c &= \sum_{g\in G}\sum_{r+s=g}\rho(a_r)t^{\tau(r)}\rho(b_s)t^{\tau(s)}\\ &= \left(\sum_{g\in G}\rho(a_g)t^{\tau(g)}\right)\left(\sum_{g\in G}\rho(b_g)t^{\tau(g)}\right)\\ &= \widetilde{\rho\tau}(a)\widetilde{\rho\tau}(b).\end{aligned}$$

  So $\widetilde{\rho\tau}$ also respects the multiplication.

- To prove that $\widetilde{\rho\tau}$ is an automorphism it suffices now to exhibit the inverse. Let σ be given by

$$\sigma\left(\sum_{g\in G} a_g t^g\right) = \sum_{g\in G}\rho^{-1}(a_g)t^{\tau^{-1}(g)}.$$

  Then a computation shows

$$\sigma\widetilde{\rho\tau}(a) = \sum_{g\in G}\rho^{-1}\rho(a_g)t^{\tau^{-1}\tau(g)} = \sum_{g\in G} a_g t^g$$

  and similarly for the composition $\widetilde{\rho\tau}\sigma = \mathrm{id}_{\mathbb{K}}$.

Thus $\widetilde{\rho\tau} \in \operatorname{Aut}\mathbb{K}$. ♦

Claim: we have $\Phi_c(\widetilde{\rho\tau}) = (\rho, \tau)$.
This follows immediately from the definition of $\widetilde{\rho\tau}$:

- For all $g \in G$ we have
$$\widetilde{\rho\tau}_G(g) = v(\widetilde{\rho\tau}(t^g)) = v(t^{\tau(g)}) = \tau(g).$$
- For all $a_0 \in k$ we have
$$\widetilde{\rho\tau}_k(a_0) = \widetilde{\rho\tau}(a_0 t^0)_0 = (\rho(a_0)t^0)_0 = \rho(a_0).$$

So this claim is also proven. ♦
Claim: $\widetilde{\rho\tau} \in \operatorname{Ext}\operatorname{Aut}_{(k)} \mathbb{K}$.
That $\widetilde{\rho\tau} \in \operatorname{Ext}\operatorname{Aut}\mathbb{K}$ follows from the last Claim, by definition of external automorphisms. To prove that it is a k-automorphism, let $a_0 \in k$. We already computed that $\widetilde{\rho\tau}(a_0)_0 = a_0$, and we also remarked that $\operatorname{supp}(\widetilde{\rho\tau}(a_0)) = \tau(\operatorname{supp} a_0) = \tau(\{0\}) = \{0\}$. Together this shows that $\widetilde{\rho\tau}(a_0) = a_0$, which proves the claim and completes the proof. □

Definition 3.3.33 (Canonical lift)**.** The automorphism $\widetilde{\rho\tau} \in v\text{-}\operatorname{Aut}_{(k)} \mathbb{K}$ given in (3.30) is a lift of the pair (ρ, τ) to $\mathbb{K}$ that we call *the canonical lift of* (ρ, τ) *to* $\mathbb{K}$.
We will denote by $\tilde{\rho}$ the lift of (ρ, id_G) and simply refer to it as the lift of $\rho \in \operatorname{Aut} k$. Similarly for $\tilde{\tau}$.

Remark 3.3.34. For a canonical lift $\widetilde{\rho\tau}$ we have $\widetilde{\rho\tau} \in v\text{-}\operatorname{Aut}_k K \Leftrightarrow \rho = \mathrm{id}_k$. □

Lemma 3.3.35. *If k is an ordered field, $\rho \in o\text{-}\operatorname{Aut} k$ and we take the induced lexicographic ordering on $\mathbb{K}$, then the canonical lift of a pair $(\rho, \tau) \in o\text{-}\operatorname{Aut} k \times o\text{-}\operatorname{Aut} G$ preserves the lexicographic ordering on $\mathbb{K}$.* □

Proof. Let $a \in \mathbb{K}^{>0}$. Then $g := v(a) \neq 0$ and $a_g > 0$. By (3.30) we have $v(\widetilde{\rho\tau}(a)) = \tau(g)$ and $\widetilde{\rho\tau}(a)_{\tau(g)} = \rho(a_g) > 0$ because $\rho \in o\text{-}\operatorname{Aut} k$. So $\widetilde{\rho\tau} \in o\text{-}\operatorname{Aut}\mathbb{K}$. □

Lemma 3.3.36. *Let K be a Hahn field such that*
$$\forall\, (\rho, \tau) \in \operatorname{Aut} k \times o\text{-}\operatorname{Aut} G : \quad \widetilde{\rho\tau}(K) = K. \tag{3.31}$$
Then the map $\Psi_c \colon \operatorname{Aut} k \times o\text{-}\operatorname{Aut} G \to v\text{-}\operatorname{Aut} K,\ (\rho, \tau) \mapsto \widetilde{\rho\tau}|_K$ is a section of Φ_c.

Proof. Let (ρ, τ) and (ρ', τ') be elements of $\operatorname{Aut} k \times o\text{-}\operatorname{Aut} G$ and let $\widetilde{\rho\tau}, \widetilde{\rho'\tau'}$ be the respective canonical lifts. Then, for all $a = \sum a_g t^g \in K$, we have

$$\begin{aligned}\Psi_c(\rho\rho', \tau\tau')(a) &= \sum \rho(\rho'(a_g))t^{\tau(\tau'(g))} \\ &= \Psi_c(\rho, \tau)\left(\sum \rho'(a_g)t^{\tau'(g)}\right) \\ &= \Psi_c(\rho, \tau)\left(\Psi_c(\rho', \tau')(a)\right).\end{aligned}$$

For injectivity, let $(\rho, \tau) \neq (\rho', \tau')$. Then if $\rho(\alpha) \neq \rho'(\alpha)$ for some $\alpha \in k$ then $\widetilde{\rho\tau}(\alpha) = \rho(\alpha) \neq \rho'(\alpha) = \widetilde{\rho'\tau'}(\alpha)$; similarly, if $\tau(g) \neq \tau'(g)$ for some $g \in G$ then $\widetilde{\rho\tau}(t^g) = t^{\tau(g)} \neq t^{\tau'(g)} = \widetilde{\rho'\tau'}(t^g)$.
Finally, we prove that $\Phi_c\Psi_c = \mathrm{id}_{\operatorname{Aut} k \times o\text{-}\operatorname{Aut} G}$. Let $(\rho, \tau) \in \operatorname{Aut} K \times o\text{-}\operatorname{Aut} G$ and let $\sigma = \Psi_c(\rho, \tau)$. Then, for all $a = \sum a_g t^g \in K$ we have $\sigma(a) = \sum \rho(a_g)t^{\tau(g)}$. Then $\Phi_c(\sigma) = (\sigma_k, \sigma_G)$ where σ_k is defined by $\sigma_k(a_0) = \sigma(a_0)_0 = \rho(a_0)$ and σ_G is defined by $\sigma_G(v(a)) = v(\sigma(a)) = v\left(\sum_{g \geq v(a)} \rho(a_g)t^{\tau(g)}\right) = \tau(v(a))$. So $(\sigma_k, \sigma_G) = (\rho, \tau)$ and thus $\Phi_c\Psi_c = \mathrm{id}$. □

Definition 3.3.37 (Canonical first lifting property)**.** Let K be a Hahn field satisfying (3.31). We say that K has the *canonical first lifting property* and we call Ψ_c the *canonical section (on K)* of Φ_c.

Example 3.3.38. By definition, the maximal Hahn field $\mathbb{K} = k((G))$ has the canonical first lifting property.

In the next proposition we show that also the minimal Hahn field $k(G)$ has the canonical first lifting property.

Proposition 3.3.39. *The Hahn field $k(G)$ has the canonical first lifting property, for every field k and every ordered abelian group G.*

Proof. Let $(\rho, \tau) \in \operatorname{Aut} k \times o\text{-}\operatorname{Aut} G$ and let $a \in k(G)$. Then there exist $p, q \in k[G]$ such that $q \neq 0$ and $a = p/q$. Now $\mathbb{K} = k((G))$ has the canonical first lifting property, so $\widetilde{\rho\tau} \in v\text{-}\operatorname{Aut} \mathbb{K}$. Let us write $p = p_{g_1}t^{g_1} + \ldots + p_{g_m}t^{g_m}$ and $q = q_{h_1}t^{h_1} + \ldots + q_{h_n}t^{h_n}$. Then $\widetilde{\rho\tau}(p) = \sum_{i=1}^{m} \rho(p_{g_i})t^{\tau(g_i)} \in k[G]$ and $\widetilde{\rho\tau}(q) = \sum_{j=1}^{n} \rho(q_{h_j})t^{\tau(h_j)} \in k[G]$ and since $\widetilde{\rho\tau}$ is a field automorphism it follows that $\widetilde{\rho\tau}(a) = \frac{\widetilde{\rho\tau}(p)}{\widetilde{\rho\tau}(q)} \in k(G)$. Hence $\widetilde{\rho\tau}(k(G)) \subseteq k(G)$. So $\widetilde{\rho\tau}|_{k(G)}$ is a field homomorphism of $k(G)$ into itself. Now $\widetilde{\rho\tau}^{-1}$ is the canonical lift of (ρ^{-1}, τ^{-1}) so, by the same argument used for (ρ, τ), the restriction $\widetilde{\rho\tau}^{-1}|_{k(G)}$ is a field homomorphism of $k(G)$ into itself. It follows that $\widetilde{\rho\tau}|_{k(G)}$ is an automorphism of $k(G)$ which, therefore, has the canonical first lifting property. □

Remark 3.3.40. Let K have the canonical first lifting property.

(i) For every pair $(\rho, \tau) \in \operatorname{Aut} k \times o\text{-}\operatorname{Aut} G$ we have $\Psi_c(\rho, \tau) \in v\text{-}\operatorname{Aut}_{(k)} K$.

(ii) Let $f = \rho_f f_c \colon \bar{K} \to k$ be an isomorphism (see Lemma 3.1.11). Then an explicit section $\Psi_{K,f}$ of $\Phi_{K,f}$ is given by the formula:

$$\begin{aligned} \Psi_{K,f}(\rho, \tau) &= \Psi_{K,c}(\rho_f^{-1} \rho \rho_f, \tau) \\ \Psi_{K,f}(\rho, \tau)\left(\sum_{g \in G} a_g t^g\right) &= \sum_{g \in G} \rho_f^{-1} \rho \rho_f(a_g) t^{\tau(g)}. \end{aligned} \tag{3.32}$$

□

Lemma 3.3.41. *Let K be a Hahn field with the canonical first lifting property. Then*

$$v\text{-}\operatorname{Aut}_{(k)} K \simeq v\text{-}\operatorname{Aut}_k K \rtimes \operatorname{Aut} k.$$

Proof. By Remark 3.3.31 we have $v\text{-}\operatorname{Aut}_k K = \ker \pi_1 \Phi_c$, so the sequence

$$v\text{-}\operatorname{Aut}_k K \hookrightarrow v\text{-}\operatorname{Aut}_{(k)} K \overset{\pi_1 \Phi_c}{\twoheadrightarrow} \operatorname{Aut} k$$

is exact. Because Ψ_c is a section of Φ_c it follows that the map $\operatorname{Aut} k \to v\text{-}\operatorname{Aut}_{(k)} K$ defined by $\rho \mapsto \tilde{\rho} = \Psi_c(\rho, \mathrm{id}_G)$ is a section of $\pi_1 \Psi_c$. The statement follows. □

So far we proved several results on the groups of valuation preserving automorphisms of a Hahn field K under the assumption that K satisfies some of the thus far introduced lifting properties. One of the goals of the next section is to show that there is a sufficient supply of Hahn fields satisfying both the canonical lifting properties.

3.3.6 Rayner fields

Now we are going to study a class of Hahn fields that are defined by conditions imposed on the support of their elements. All these fields satisfy the canonical second lifting property, and we are going to characterise those that also satisfy the first canonical lifting property.

Definition 3.3.42 (Rayner field family)**.** Let G be a non-trivial ordered abelian group. A family $\mathcal{F} \neq \emptyset$ of subsets of G is said to be a *Rayner field family (with respect to G)* (see [Ray68, Section 2]) if the following six properties are satisfied:

(RF1) The elements of $\mathcal{F}$ are well ordered subsets of G.

(RF2) The union of the elements of $\mathcal{F}$ generates G as a group.

(RF3) $A, B \in \mathcal{F} \Rightarrow A \cup B \in \mathcal{F}$.

(RF4) $A \in \mathcal{F}$, $B \subset A \Rightarrow B \in \mathcal{F}$.

(RF5) $A \in \mathcal{F}$, $g \in G \Rightarrow A + g \in \mathcal{F}$.

(RF6) if $A \in \mathcal{F}$ and $A \subseteq G^{\geq 0}$ then the set of all finite sums of elements of A belongs to $\mathcal{F}$.

Theorem 3.3.43 ([Ray68, Theorem 1])**.** *If $\mathcal{F}$ is a Rayner field family then the set $k((\mathcal{F}))$ of elements of $\mathbb{K}$ whose support belongs to $\mathcal{F}$ is a subfield of $\mathbb{K}$.*

Definition 3.3.44. The fields $k((\mathcal{F}))$ obtained in Theorem 3.3.43 are Hahn fields[3] that will be called *Rayner fields*.

Examples 3.3.45. (i) A general class of Rayner fields is described in [Ray68, Section 3]. In particular, the field of Puiseux series: let $\mathbb{K} = k((\mathbb{Q}))$ and consider the family $\mathcal{F}$ of sets of the form $\frac{1}{d}A$ where d is a positive integer and A is a well ordered subset of $\mathbb{Z}$. It is clear that $\mathcal{F}$ is a Rayner field family and that the field $k((\mathcal{F}))$ thus obtained is the field $\mathbb{P}$ of Puiseux series (see Section 3.5.5).

(ii) Let κ be an uncountable regular cardinal. The family $\mathcal{F}_\kappa$ of well ordered subsets of G with cardinality smaller than κ is clearly a Rayner field family. The resulting field, denoted by $\mathbb{K}_\kappa$, is called the *κ-bounded subfield* of $\mathbb{K}$. It consists of all elements of $\mathbb{K}$ whose support has cardinality less then κ (see [All62] or [KS05]).

(iii) Consider the set S of finitely generated subgroups of G and let $\mathcal{F}_S$ consist of all well ordered subsets of elements of S. Then $\mathcal{F}_S$ is a Rayner field family and thus $k((\mathcal{F}_S))$ is a Hahn field called the field of *grid based series*. □

[3] See [KKS21, Theorem 3.15].

Lemma 3.3.46. *Let K be a Rayner field. Then K satisfies the (canonical) second lifting property.*

Proof. Let $a \in K$ and $x \in \mathrm{Hom}(G, k^\times)$. By part (i) of Proposition 3.3.25 we have $\mathrm{supp}(a) = \mathrm{supp}(\rho_x(a))$. Since K is a Rayner field this implies $\rho_x(a) \in K$. □

The following proposition characterises Rayner fields with the canonical first lifting property.

Proposition 3.3.47. *Let $\mathcal{F}$ be a Rayner field family and let $F = k((\mathcal{F}))$ be the corresponding Rayner field. Then F has the canonical first lifting property if and only if $\mathcal{F}$ is stable under o-$\mathrm{Aut}\, G$, by which we mean that if $A \in \mathcal{F}$ and $\tau \in o$-$\mathrm{Aut}\, G$ then $\tau(A) \in \mathcal{F}$.*

Proof. Assume that $\mathcal{F}$ be stable under o-$\mathrm{Aut}\, G$, let $a = \sum a_g t^g \in F$ and $(\rho, \tau) \in \mathrm{Aut}\, k \times o$-$\mathrm{Aut}\, G$. Then the support of $\widetilde{\rho\tau}(a) = \sum \rho(a_g) t^{\tau(g)}$ is $\{\tau(g) : g \in \mathrm{supp}(a)\} = \tau(\mathrm{supp}(a)) \in \mathcal{F}$, by assumption (since $\mathrm{supp}(a) \in \mathcal{F}$). Hence $\widetilde{\rho\tau}(a) \in F$. So F has the canonical first lifting property.

Vice versa, assume that F has the canonical first lifting property and let $A \in \mathcal{F}$. Take any element $a = \sum a_g t^g \in F$ such that $\mathrm{supp}(a) = A$. By assumption, for all $(\rho, \tau) \in \mathrm{Aut}\, k \times o$-$\mathrm{Aut}\, G$ we have $b = \widetilde{\rho\tau}(a) \in F$ so, in particular, $\mathrm{supp}(b) = \tau(A) \in \mathcal{F}$ and so $\mathcal{F}$ is stable under o-$\mathrm{Aut}\, G$. □

Corollary 3.3.48. *(i) The field $\mathbb{P}$ of Puiseux series has the canonical first lifting property.*

(ii) The κ-bounded subfields of $\mathbb{K}$ (Example 3.3.45) have the canonical first lifting property.

Proof. (i) Let $\frac{1}{d}A$ be as in Example 3.3.45 above and let τ be an order preserving automorphism of $\mathbb{Q}$. Now o-$\mathrm{Aut}(\mathbb{Q}, +) \simeq (\mathbb{Q}^{>0}, \cdot)$ so τ is multiplication by some positive rational $\frac{m}{n}$. Then it is clear that $\tau(\frac{1}{d}A) = \frac{1}{dn}(mA)$ also belongs to the same Rayner field family.

(ii) An order preserving automorphism of G maps any well ordered subset onto another one of the same cardinality, hence $\mathcal{F}_\kappa$ is stable under o-$\mathrm{Aut}\, G$. □

A non-Rayner field with the canonical first lifting property. We showed in Proposition 3.3.39 that the minimal Hahn field $k(G)$ has the canonical first lifting property for all k and all G. Now we are going to show that, in general, it is not a Rayner field. Let $\mathbb{K} = \mathbb{Q}((\mathbb{Z}))$ be the maximal Hahn field over $\mathbb{Q}$ with value group $\mathbb{Z}$ and let $K = \mathbb{Q}(\mathbb{Z}) = \operatorname{Frac}\mathbb{Q}[t]$. By Proposition 3.3.39 K has the canonical first lifting property. We want to show that K is not a Rayner field. Let us give a definition:

Definition 3.3.49. A *rational linear recurrent sequence* is a sequence $\{q_n\}_{n\geq 0} \in \mathbb{Q}$ for which there exists an $r \in \mathbb{N}$ and coefficients $c_1, \ldots, c_r \in \mathbb{Q}$ such that $q_n = c_1 q_{n-1} + c_2 q_{n-2} + \ldots + c_r q_{n-r}$ for all $n > r$. This last relation is called a recurrence relation.

The next proposition is also referred to as Eisenstein's theorem for algebraic power series.

Proposition 3.3.50 ([Eis75, pp. 765–767])**.** *Let $a \in \mathbb{K}$. If $a \in K$ then the coefficients of a satisfy a recurrence relation. In particular, there can only be finitely many prime numbers occurring in the denominators of the coefficients of a.*

Proof. If $a \in K$ there exist polynomials $p, q \in \mathbb{Q}[t]$ such that $a = \frac{p}{q}$.

If r, s are the least common multiples of the denominators of the coefficients of p and q respectively, then we can write $a = \frac{p}{q} = \frac{s}{r}\frac{\tilde{p}}{\tilde{q}}$ with $\tilde{p}, \tilde{q} \in \mathbb{Z}[t]$. So, up to factoring out $\frac{r}{s}$ from the coefficients of a we can assume, without loss of generality, that $p, q \in \mathbb{Z}[t]$.

Moreover, if $\deg p \geq \deg q$, then we can perform polynomial division by q to get $p = fq + g$ for some $f, g \in \mathbb{Z}[t]$ with $\deg g < \deg q$ and $a = \frac{p}{q} = f + \frac{g}{q}$. So, up to subtracting a polynomial f from a, we can assume that $\deg p < \deg q$.

Now, let $q = q_0 + q_1 t + \ldots + q_m t^m$. We have

$$\begin{aligned} p = qa &= q_0 \sum_{j\geq 0} a_j t^j + \ldots + b_m \sum_{j\geq 0} a_j t^{j+m} \\ &= \sum_{j\geq 0} q_0 a_j t^j + \ldots + \sum_{j\geq m} b_m a_{j-m} t^j \\ &= q^*(t) + \sum_{j>m} (q_0 a_j + \ldots + q_m a_{j-m}) t^j \end{aligned}$$

for a polynomial $q^* \in \mathbb{Z}[t]$ with $\deg q^* \leq m$. Since we assumed that $\deg p < \deg q = m$ then for all $j > m$ the coefficient in the last expression above must be zero, which yields

$$a_j = -\frac{1}{q_0}(q_1 a_{j-1} + \ldots + q_m a_{j-m})$$

which is the recurrence relation we claimed. □

Corollary 3.3.51. *The Hahn field $K = \mathbb{Q}(\mathbb{Z})$ is not a Rayner field.*

Proof. The power series $e^t := \sum_{n=0}^{\infty} \frac{t^n}{n!}$ does not belong to K. Indeed, every prime number will eventually appear as a factor of the denominator of some coefficient, thus contradicting Proposition 3.3.50.
However, the series $\sum_{n=0}^{\infty} t^n = (1-t)^{-1}$ belongs to K, as it is the inverse of a series with finite support, which certainly belongs to K and K is a field. Now $\mathrm{supp}(e^t) = \mathrm{supp}((1-t)^{-1}) = \mathbb{N}$. Hence K contains some but not all the power series with support $\mathbb{N}$. Thus it is not a Rayner field. □

3.3.7 General decomposition theorem

Now that we introduced the (canonical) first and second lifting properties and showed that there exist many Hahn fields satisfying them, we can combine the partial results we obtained so far and get the following, general decomposition theorem.

Theorem 3.3.52. *Let K be a Hahn field with the first and second lifting property. Then*

$$v\text{-}\operatorname{Aut} K \simeq (1\text{-}\operatorname{Aut} K \rtimes \operatorname{Hom}(G, k^\times)) \rtimes (\operatorname{Aut} k \times o\text{-}\operatorname{Aut} G). \tag{3.33}$$

If, moreover, K has the canonical first and second lifting property, then we have

$$v\text{-}\operatorname{Aut}_k K \simeq (1\text{-}\operatorname{Aut}_k K \rtimes \operatorname{Hom}(G, k^\times)) \rtimes o\text{-}\operatorname{Aut} G. \tag{3.34}$$

Proof. The first lifting property gives $v\text{-}\operatorname{Aut} K \simeq \operatorname{Int}\operatorname{Aut} K \rtimes (Autk \times o\text{-}\operatorname{Aut} G)$ and the second lifting property gives $\operatorname{Int}\operatorname{Aut} K \simeq 1\text{-}\operatorname{Aut} K \rtimes \operatorname{Hom}(G, k^\times)$. So Equation (3.33) is established.
To establish (3.34) we need to remark that a canonical lift is a k-automorphism if and only if it is of the form $\tilde{\tau}$ for some $\tau \in o\text{-}\operatorname{Aut} G$ (Definition 3.3.33) and all the G-exponentiations with respect to P_c are k-automorphisms. □

The canonical first lifting property allows us, moreover, to refine some of the results that we proved above under the weaker assumption of a general first lifting property.

Proposition 3.3.53. *Let K be a Hahn field with the canonical first and second lifting property. Then*

(i) $\operatorname{Ext}\operatorname{Aut} K = \operatorname{Ext}\operatorname{Aut}_{(k)} K$

(ii) $v\text{-}\operatorname{Aut}_{(k)} K = \operatorname{Int}\operatorname{Aut}_k K \rtimes \operatorname{Ext}\operatorname{Aut} K$

(iii) $v\text{-}\operatorname{Aut}_{(k)} K \simeq (1\text{-}\operatorname{Aut}_k K \rtimes \operatorname{Hom}(G, k^\times)) \rtimes (\operatorname{Aut} k \times o\text{-}\operatorname{Aut} G)$.

Proof. Part (i) follows immediately from Remark 3.3.40. Parts (ii) and (iii) are analogous to Theorems 3.3.13 and 3.3.52 respectively, for the case where K has the canonical first lifting property: (ii) follows from (3.17) and the fact that all canonical lifts are k-stable (Lemma 3.3.32); (iii) is then an immediate consequence. □

Corollary 3.3.54. *Let $k(G) \subseteq K, F \subseteq \mathbb{K}$ be two Hahn fields with the first and (canonical) second lifting property. Then*

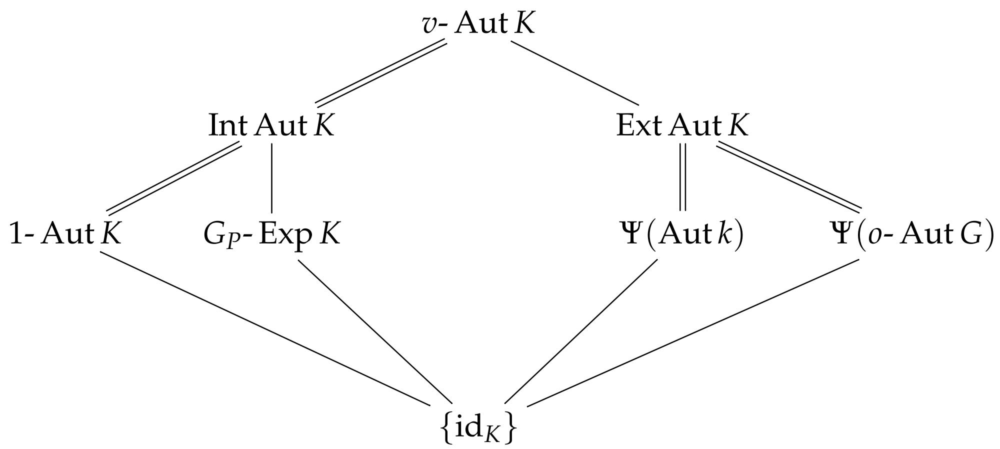
$$v\text{-}\operatorname{Aut} K \simeq v\text{-}\operatorname{Aut} F \Leftrightarrow 1\text{-}\operatorname{Aut} K \simeq 1\text{-}\operatorname{Aut} F; \tag{3.35}$$

$$v\text{-}\operatorname{Aut}_{(k)} K \simeq v\text{-}\operatorname{Aut}_{(k)} F \Leftrightarrow 1\text{-}\operatorname{Aut}_k K \simeq 1\text{-}\operatorname{Aut}_k F; \tag{3.36}$$

$$v\text{-}\operatorname{Aut}_k K \simeq v\text{-}\operatorname{Aut}_k F \Leftrightarrow 1\text{-}\operatorname{Aut}_k K \simeq 1\text{-}\operatorname{Aut}_k F. \tag{3.37}$$

□

The following diagram summarises the information on the group structure of $v\text{-}\operatorname{Aut} K$, for a Hahn field K satisfying the first lifting property w.r.t. a section Ψ and second lifting property w.r.t a section P. The double line means that the smaller group is normal in the larger one and, in general, all the inclusions are strict.



(3.38)

Analogous diagrams hold for $v\text{- Aut}_{(k)} K$ and $v\text{- Aut}_k K$.

3.4 Strongly additive automorphisms

Intuitively, a strongly additive automorphism is one that commutes with infinite sums. In this section we are going to formalise this notion and study in detail the strongly additive automorphisms of a Hahn field.

In his paper [Sch44], Schilling obtains a description of the group of valuation preserving k-automorphisms of the field $\mathbb{L} := k((\mathbb{Z}))$ of Laurent series over a field k. He does this by relating the group of automorphisms to $U_{\mathbb{L}}$, the group of units of the valuation ring of $\mathbb{L}$. In his argument, a crucial role is played by the fact that all the automorphisms of $\mathbb{L}$ are strongly additive (we will provide all the details in Subsection 3.5.2). In this section we study the strongly additive automorphisms (Definition 3.4.6) of a Hahn field K and provide a wide generalisation of Schilling's results. We begin with the definition of summability.

Let k be an arbitrary field and G an arbitrary ordered abelian group.

Definition 3.4.1 (Summable family)**.** Let $A = \{a_{(i)} : i \in I\} \subseteq \mathbb{K}$ be a family of elements of $\mathbb{K}$ indexed by a set I. Let $\operatorname{Supp} A := \bigcup_{i\in I} \operatorname{supp} a_{(i)}$ and for $g \in G$ define $S_g = \{i \in I : g \in \operatorname{supp} a_{(i)}\}$. We say that A is *summable* if both the following conditions are satisfied.

(a) Supp A is well ordered;

(b) for all $g \in \operatorname{Supp} A$ the set S_g is finite.

Lemma 3.4.2. *Let* $A = \{a_{(i)} : i \in I\} \subseteq \mathbb{K}$ *and for all* $g \in \operatorname{Supp} A$ *let* $\mathbf{a}_g := \sum_{i \in S_g} (a_{(i)})_g$. *Then*

(i) A is summable if and only if

$$\mathbf{a} = \sum_{i \in I} a_{(i)} := \sum_{g \in \operatorname{Supp} A} \mathbf{a}_g t^g$$

is a well defined element of $\mathbb{K}$.

(ii) Assume A is summable and set $\nu = \min\{v(a_{(i)}) : i \in I\}$. *Then*

(ii.1) $v(\mathbf{a}) \geq \nu$.

(ii.2) If $|S_\nu| = 1$, *i.e.,* $\exists! j \in I : v(a_{(j)}) = \nu$, *then* $v(\mathbf{a}) = \nu$ *and* $\mathbf{a}_\nu = (a_{(j)})_\nu$.

Proof. (i) Let A be summable. By condition (b) in Definition 3.4.1, for all $g \in \operatorname{Supp} A$, the set S_g is finite, thus $\mathbf{a}_g := \sum_{i \in S_g} (a_{(i)})_g \in k$. Since, by condition (a), $\operatorname{Supp} A$ is well ordered, then $\sum_{g \in \operatorname{Supp} A} \mathbf{a}_g t^g$ is indeed a well defined element of $\mathbb{K}$.

Vice versa, if $\sum_{g \in \operatorname{Supp} A} \mathbf{a}_g t^g \in \mathbb{K}$, by definition of $\mathbb{K}$ this implies that $\operatorname{Supp} A$ is well ordered (thus condition (a)) and that $\mathbf{a}_g := \sum_{i \in S_g} (a_{(i)})_g \in k$, which in turn implies that S_g is finite (hence condition (b)).

(ii) Let A be summable.

(ii.1) Let $h < \nu$. Then $S_h = \{i \in I : h \in \operatorname{supp} a_{(i)}\} = \emptyset$ and thus $\mathbf{a}_h = \sum_{i \in \emptyset} (a_{(i)})_h = 0$.

(ii.2) Write $\mathbf{a} = a_{(j)} + \sum_{i \neq j} a_{(i)}$. Now, clearly $\{a_{(i):i \in I \setminus \{j\}}\}$ is a summable family, hence $\mathbf{b} := \sum_{i \neq j} a_{(i)}$ is a well defined element of $\mathbb{K}$. Since j is the only element of I such that $v(a_{(j)}) = \nu$ it follows that $\min\{v(a_{(i)}) : i \neq j\} > \nu$. By (ii.1) we have $v(\mathbf{b}) > \nu$. Thus

$$v(a_{(j)} + \mathbf{b}) = \min\{v(a_{(i)}), v(\mathbf{b})\} = v(a_{(i)}) = \nu.$$

□

Definition 3.4.3. Let $A = \{a_{(i)} : i \in I\} \subseteq \mathbb{K}$ be a family of elements of $\mathbb{K}$ indexed by a set I.

(i) If A is a summable family we call $\mathbf{a} = \sum_{i\in I} a_{(i)}$ *the sum of* A.

(ii) Let K be a Hahn field and $A = \{a_{(i)} : i \in I\} \subseteq K$ a summable family. We say that A is *K-summable* if $\mathbf{a} \in K$.

(iii) Let $C = \{c_i : i \in I\} \subseteq k$ be a family of coefficients. We define the family $CA = \{c_i a_{(i)} : i \in I\} \subseteq \mathbb{K}$. We call CA the *scalar multiple* of A by C.

(iv) Let $B = \{b_{(i)} : i \in J\} \subseteq \mathbb{K}$ be another family. Assume, without loss of generality, that $I = J$. We define the *sum* $A + B = \{a_{(i)} + b_{(i)} : i \in I\}$ and the *product* $AB = \{a_{(i)} b_{(j)} : i, j \in I\}$.

Proposition 3.4.4. *Let $A = \{a_{(i)} : i \in I\}$ and $B = \{b_{(i)} : i \in I\}$ be summable families in $\mathbb{K}$ and let $C = \{c_i,\ i \in I\} \subseteq k$ be a family of coefficients. Then the families CA, $A + B$ and AB are summable.*

Proof. Let us begin with CA. For every $i \in I$ we have

$$\operatorname{supp}(c_i a_{(i)}) = \begin{cases} \operatorname{supp} a_{(i)} & \text{if } c_i \neq 0; \\ \emptyset & \text{if } c_i = 0. \end{cases} \tag{3.39}$$

In particular, $\operatorname{supp}(c_i a_{(i)}) \subseteq \operatorname{supp} a_{(i)}$ and thus $\operatorname{Supp}(CA) \subseteq \operatorname{Supp} A$, which is well ordered. Hence so is $\operatorname{Supp}(CA)$.
Now let $g \in \operatorname{Supp}(CA) \subseteq \operatorname{Supp} A$. Since A is summable the set S_g^A is finite. By (3.39) we have $S_g^{CA} \subseteq S_g^A$. So S_g^{CA} is also finite and CA is summable.♦

Let us consider $A + B$. As $\operatorname{supp}(a_{(i)} + b_{(i)}) \subseteq \operatorname{supp}(a_{(i)}) \cup \operatorname{supp}(b_{(i)})$ for all $i \in I$, it follows that $\operatorname{Supp}(A + B) \subseteq \operatorname{Supp} A \cup \operatorname{Supp} B$, which is well ordered. So $\operatorname{Supp}(A + B)$ is well ordered too.
Now let $g \in G$. We want to show that S_g^{A+B} is finite. Now $g \in \operatorname{supp}(a_{(i)} + b_{(i)})$ for some $i \in I$ if and only if $g \in \operatorname{supp} a_{(i)}$ or $g \in \operatorname{supp} b_{(i)}$. Therefore $S_g^{A+B} \subseteq S_g^A \cup S_g^B$ which is finite, because both S_g^A and S_g^B are. ♦

Finally we look at AB. By definition, we have the inclusion $\operatorname{Supp}(AB) \subseteq \bigcup_{i,j\in I} \operatorname{supp}(a_{(i)} b_{(j)})$. To show that $\operatorname{Supp}(AB)$ is well ordered, let us assume it contains a strictly decreasing chain $g_1 > g_2 > \ldots$. For all $n \in \mathbb{N}$ there is a pair $(r_n, s_n) \in \operatorname{Supp} A \times \operatorname{Supp} B$ such that $g_n = r_n + s_n$. So we have two sequences $(r_n)_{n\in\mathbb{N}} \subseteq \operatorname{Supp} A$ and $(s_n)_{n\in\mathbb{N}} \subseteq \operatorname{Supp} B$ neither of which can be strictly decreasing, because $\operatorname{Supp} A$ and $\operatorname{Supp} B$ are well ordered. Because $(r_n)_{n\in\mathbb{N}}$ cannot be strictly decreasing there exists a least $m \in \mathbb{N}$ such that $r_m = \min\{r_n : n \in \mathbb{N}\}$. Hence $g_{m+1} = r_{m+1} + s_{m+1} < r_m + s_m$ and $r_{m+1} \geq r_m$

implies $s_{m+1} < s_m$. A similar argument shows $s_{m+1} > s_{m+2} > \ldots$, which contradicts the fact that $(s_n)_{n\in\mathbb{N}}$ cannot be strictly decreasing. Therefore, $\operatorname{Supp}(AB)$ cannot contain a strictly decreasing sequence and it is thus well ordered.
Now let $g \in \operatorname{Supp}(AB)$. Then $S_g^{AB} = \{(i,j) \in I \times I : g \in \operatorname{supp}(a_{(i)}b_{(j)})\}$. Now $g \in \operatorname{supp}(a_{(i)}b_{(j)})$ implies there exist $r_i \in \operatorname{supp} a_{(i)}$ and $s_j \in \operatorname{supp} b_{(j)}$ such that $g = r_i + s_j$. Assume there are infinitely many distinct pairs $(i_n, j_n) \in I \times I$, $n \in \mathbb{N}$ for which there exist r_{i_n}, s_{j_n} such that $g = r_{i_n} + s_{j_n}$. Then one of the two sequences $(r_{i_n})_{n\in\mathbb{N}}$ and $(s_{j_n})_{n\in\mathbb{N}}$ must contain infinitely many distinct elements. Let this be $(r_{i_n})_{n\in\mathbb{N}}$. Since it is contained in the well ordered set $\operatorname{Supp} A$, it must be strictly increasing. But since $r_{i_n} + s_{j_n} = g$ for all n, it follows that $(s_{j_n})_{n\in\mathbb{N}}$ must be strictly decreasing, which is impossible, because it is contained in the well ordered set $\operatorname{Supp} B$. The contradiction shows that the set S_g^{AB} must be finite for all g, which completes the proof of the summability of AB. □

Remark 3.4.5. The maximal Hahn field $\mathbb{K}$ is the only Hahn field that is closed under taking sums of arbitrary summable families. Indeed, if K is a Hahn field such that every summable family $A \subseteq K$ is K-summable, then for all $a = \sum a_g t^g \in \mathbb{K}$ the family $\{a_g t^g : g \in \operatorname{supp} a\}$ is K-summable. Thus $a \in K$ and so $K = \mathbb{K}$. □

Definition 3.4.6. Let K be a Hahn field

(i) A map $\sigma\colon K \to K$ is K-*summable* if, for all K-summable family $\{a_{(i)} : i \in I\} \subseteq K$,

  (a) the family $\{\sigma(a_{(i)}) : i \in I\}$ is K-summable;

  (b) $\sigma(\sum_{i\in I} a_{(i)}) = \sum_{i\in I} \sigma(a_{(i)})$.

(ii) An automorphism $\sigma \in v\text{-}\operatorname{Aut} K$ is *strongly additive* if both σ and σ^{-1} are K-summable maps.

Notation 3.4.7. *The set of strongly additive, valuation preserving automorphisms will be denoted by* $v\text{-}\operatorname{Aut}^+ K$. *We will also use the superscript " + " on the other groups of automorphisms, to denote the corresponding subsets of strongly additive automorphisms: for example* $\operatorname{Int}\operatorname{Aut}^+ K = \operatorname{Int}\operatorname{Aut} K \cap v\text{-}\operatorname{Aut}^+ K$.

Remark 3.4.8. Strongly additive automorphisms need not be k-automorphisms: let $\alpha \in \operatorname{Aut} k$ be a non-trivial automorphism of k (for example, we can choose $k = \mathbb{Q}(\sqrt{2})$ and $\alpha\colon \sqrt{2} \mapsto -\sqrt{2}$) and let $\sigma \in \operatorname{Aut}\mathbb{K}$ be defined by $\sigma(\sum a_g t^g) = \sum \alpha(a_g) t^g$. This is a strongly additive automorphism (as $\sigma \in \operatorname{Ext}\operatorname{Aut}\mathbb{K}$) that does not fix k. □

The following is an example of an automorphism that is not strongly additive. The author is grateful L. S. Krapp for suggesting the idea.

Example 3.4.9. Let ω be the first infinite ordinal, let $G = \coprod_{\omega+1} \mathbb{Q}$ and let $\mathbb{K} = \mathbb{C}((G))$. Elements of G have the form $\sum_{n\in\mathbb{N}} q_n \mathbb{1}_n + q_\omega \mathbb{1}_\omega$ and the set $\{\mathbb{1}_n : n \in \mathbb{N}\} \cup \{\mathbb{1}_\omega\}$ is a $\mathbb{Q}$-*valuation basis* for G (see [Kuh00b, Page 4]).

The set $\{t^{-\mathbb{1}_n} : n \in \mathbb{N}\}$ is algebraically independent over $\mathbb{C}$ (see [EP05, Theorem 3.4.2]), therefore, it extends to a transcendence basis $\mathcal{B}$ of $\mathbb{K}$ over $\mathbb{C}$. The set $\{t^{-\mathbb{1}_n} + t^{\mathbb{1}_\omega} : n \in \mathbb{N}\}$ is also algebraically independent so it also extends to a transcendence basis $\mathcal{B}'$. There exists a bijection $f \colon \mathcal{B} \to \mathcal{B}'$ such that $f(t^{-\mathbb{1}_n}) = t^{-\mathbb{1}_n} + t^{\mathbb{1}_\omega}$ for all $n \in \mathbb{N}$. Since $\mathbb{C}$ is algebraically closed, a bijection of transcendence bases extends, in turn, to an isomorphism of fields: that is, there exists a field automorphism $f \in \operatorname{Aut}\mathbb{K}$ such that $f(t^{-\mathbb{1}_n}) = t^{-\mathbb{1}_n} + t^{\mathbb{1}_\omega}$ for all $n \in \mathbb{N}$.

Now consider the element $a = \sum_{n\in\mathbb{N}} t^{-\mathbb{1}_n}$. Notice that the sequence $(\mathbb{1}_n)_{n\in\mathbb{N}}$ is anti-well ordered in G, so the support $(-\mathbb{1}_n)_{n\in\mathbb{N}}$ of a is well ordered, so $a \in \mathbb{K}$. However, the family $A = \{f(t^{-\mathbb{1}_n}) : n \in \mathbb{N}\}$ is not summable. Indeed $\operatorname{Supp} A = \{-\mathbb{1}_n : n \in \mathbb{N}\} \cup \{\mathbb{1}_\omega\}$ and $\mathbb{1}_\omega \in \operatorname{supp} f(t^{-\mathbb{1}_n})$ for all $n \in \mathbb{N}$ hence $S_{\mathbb{1}_\omega} = \mathbb{N}$ violates condition (b) of Definition 3.4.1. □

Proposition 3.4.10. *Let K be a Hahn field. Then*

(i) v-$\operatorname{Aut}^+ K$ is a subgroup of v-$\operatorname{Aut} K$.

(ii) Assume that K has the canonical first lifting property. Then we have $\operatorname{Ext}\operatorname{Aut} K \leq v\text{-}\operatorname{Aut}^+ K$.

(iii) Assume, moreover, that K satisfies the canonical second lifting property. Then we have $G\text{-}\operatorname{Exp} K \leq \operatorname{Int}\operatorname{Aut}_k^+ K$.

Proof. (i) Let σ and τ be two strongly additive automorphisms. Then, for all K-summable family $\{a_{(i)} : i \in I\}$, applying subsequently the strong additivity of σ and τ we get

$$(\sigma\tau)(\sum a_{(i)}) = \sigma\left(\sum \tau(a_{(i)})\right) = \sum \sigma(\tau(a_{(i)}))$$

so $\sigma\tau$ is strongly additive.

(ii) Let $\sigma \in \operatorname{Ext}\operatorname{Aut} K$. Then there are $\rho \in \operatorname{Aut} k$ and $\tau \in o\text{-}\operatorname{Aut} G$ such that, for all $a = \sum a_g t^g \in K$ we have $\sigma(a) = \sum \rho(a_g) t^{\tau(g)}$. Now let $\mathcal{F} = \{a_{(i)} : i \in I\} \subseteq K$ be a K-summable family.

We have

$$\begin{aligned}\operatorname{Supp}\sigma(\mathcal{F}) &= \bigcup_{i\in I}\operatorname{supp}(\sigma(a_{(i)}))\\ &= \bigcup_{i\in I}\tau(\operatorname{supp}(a_{(i)}))\\ &= \tau\left(\bigcup_{i\in I}\operatorname{supp}(a_{(i)})\right)\\ &= \tau(\operatorname{Supp}\mathcal{F}).\end{aligned}$$

Now $\mathcal{F}$ is summable, so $\operatorname{Supp}\mathcal{F}$ is well ordered and $\tau \in o\text{-}\operatorname{Aut} G$ so $\tau(\operatorname{Supp}\mathcal{F})$ is well ordered too.

Now let $h \in \operatorname{Supp}\sigma(\mathcal{F})$. Then there exists $g \in \operatorname{Supp}\mathcal{F}$ such that $h = \tau(g)$. Let $S_h = \{i \in I : h \in \operatorname{supp}\sigma(a_{(i)})\}$. Because $\operatorname{supp}\sigma(a_{(i)}) = \tau(\operatorname{supp} a_i)$ and τ is an isomorphism, we have $\tau(g) \in \tau(\operatorname{supp} a_{(i)})$ if and only if $g \in \operatorname{supp} a_{(i)}$ and the latter is only true for finitely many $i \in I$. Hence the set S_h is finite, which shows that $\sigma(\mathcal{F})$ is K-summable.

Finally, let us show that $\sigma(\sum a_{(i)}) = \sum \sigma(a_{(i)})$. We have

$$\sum a_{(i)} = \sum_{g\in\operatorname{Supp}\mathcal{F}}(\sum_{i\in S_g} a_{(i)g})t^g$$

and applying σ yields

$$\begin{aligned}\sigma\left(\sum_{i\in I} a_{(i)}\right) &= \sigma\left(\sum_{g\in\operatorname{Supp}\mathcal{F}}\left(\sum_{i\in S_g} a_{(i)g}\right)t^g\right)\\ &= \sum_{g\in\operatorname{Supp}\mathcal{F}}\rho\left(\sum_{i\in S_g} a_{(i)g}\right)t^{\tau(g)}\\ &= \sum_{g\in\operatorname{Supp}\mathcal{F}}\sum_{i\in S_g}\rho\left(a_{(i)g}\right)t^{\tau(g)}\\ &= \sum_{i\in I}\sum_{g\in\operatorname{supp} a_{(i)}}\rho\left(a_{(i)g}\right)t^{\tau(g)}\\ &= \sum_{i\in I}\sigma(a_{(i)}).\end{aligned}$$

So σ is K-summable. Since $\sigma^{-1} \in \operatorname{Ext}\operatorname{Aut} K$, it follows that σ is, in fact, strongly additive.

(iii) Let $\sigma \in G\text{-}\operatorname{Exp} K$ correspond to $x \in \operatorname{Hom}(G, k^\times)$ and let $\mathcal{F} = \{a_{(i)} : i \in I\} \subseteq K$ be a K-summable family. Because, for all $i \in I$ we have $\operatorname{supp} \sigma(a_{(i)}) = \operatorname{supp} a_{(i)}$ (Proposition 3.3.25), it follows that $\operatorname{Supp} \sigma(\mathcal{F}) = \operatorname{Supp} \mathcal{F}$, which is well ordered by assumption. Moreover, for all $g \in \operatorname{Supp} \mathcal{F}$ and all $i \in I$ we have $(\sigma(a_{(i)}))_g = (a_{(i)})_g x^g = 0 \Leftrightarrow (a_{(i)})_g = 0$, hence, for all $g \in \operatorname{Supp} \mathcal{F}$ we have $\{i \in I : g \in \operatorname{supp} \sigma(a_{(i)})\} = \{i \in I : g \in \operatorname{supp} a_{(i)}\}$ and the latter is finite by assumption.

In a similar manner as in (ii) we can compute

$$\begin{aligned}
\sigma\left(\sum_{i\in I} a_{(i)}\right) &= \sigma\left(\sum_{g\in \operatorname{Supp}\mathcal{F}} \left(\sum_{i\in S_g} a_{(i)g}\right) t^g\right) \\
&= \sum_{g\in \operatorname{Supp}\mathcal{F}} x^g \left(\sum_{i\in S_g} a_{(i)g}\right) t^g \\
&= \sum_{g\in \operatorname{Supp}\mathcal{F}} \sum_{i\in S_g} x^g a_{(i)g} t^g \\
&= \sum_{i\in I} \sum_{g\in \operatorname{supp} a_{(i)}} x^g a_{(i)g} t^g \\
&= \sum_{i\in I} \sigma(a_{(i)}).
\end{aligned}$$

This completes the proof that σ is K-summable, and since the inverse of a G-exponentiation is again such, then all G-exponentiations are strongly additive.

□

A remark on our definition of v- Aut$^+$ K. Recall that we defined an automorphism $\sigma \in v\text{-}\operatorname{Aut} K$ to be strongly additive if both σ and σ^{-1} are K-summable. For the maximal Hahn field $\mathbb{K}$, using a result of Aschenbrenner, van den Dries and van der Hoeven, it can be proven that if $\sigma \in v\text{-}\operatorname{Aut} \mathbb{K}$ is summable then so is σ^{-1}.

Lemma 3.4.11 ([ADH05, Corollary 1.4]). *Let k be an infinite field, let $\mathbb{K} = k((G))$*

and let $\varepsilon\colon \mathbb{K} \to \mathbb{K}$ *be a summable*[4] *map with the property that* $v(\varepsilon(\alpha t^g)) > g$ *for all* $g \in G$ *and all* $\alpha \in k$. *Then the summable operator* $\mathrm{id} + \varepsilon$ *is bijective with summable inverse given by*

$$(\mathrm{id} + \varepsilon)^{-1}(a) = \sum_{n=0}^{\infty} (-1)^n \varepsilon^n(a).$$

□

Proposition 3.4.12. *Let* $\sigma \in v\text{-}\operatorname{Aut}\mathbb{K}$ *be summable. Then* σ^{-1} *is summable.*

Proof. The field $\mathbb{K}$ satisfies the canonical first ant second lifting property. By Lemma 3.3.30 we can write σ as a composition $\sigma = \xi\rho\tau$ with $\xi \in 1\text{-}\operatorname{Aut}\mathbb{K}$, $\rho \in G\text{-}\operatorname{Exp}\mathbb{K}$ and $\tau \in \operatorname{Ext}\operatorname{Aut}\mathbb{K}$. Then $\sigma^{-1} = \tau^{-1}\rho^{-1}\xi^{-1}$. Because $\rho^{-1} \in G\text{-}\operatorname{Exp}\mathbb{K}$ and $\tau^{-1} \in \operatorname{Ext}\operatorname{Aut}\mathbb{K}$, by Proposition 3.4.10 we know they are summable. It remains to show that ξ^{-1} is summable. Consider the map $\varphi := \xi - \mathrm{id}_{\mathbb{K}}$. Since $\xi \in 1\text{-}\operatorname{Aut}\mathbb{K}$, for all $a \in \mathbb{K}$ we have $v(\varphi(a)) = v(\xi(a) - a) > v(a)$, because a and $\xi(a)$ have the same valuation and the same first coefficient. So φ fulfils the hypotheses of Lemma 3.4.11 and thus $\xi = \mathrm{id} + \varphi$ has a summable inverse. □

3.4.1 The structure of $v\text{-}\operatorname{Aut}^+ K$

Since normality is preserved by taking intersections, it follows that $\operatorname{Int}\operatorname{Aut}^+ K = \operatorname{Int}\operatorname{Aut} K \cap v\text{-}\operatorname{Aut}^+ K \trianglelefteq v\text{-}\operatorname{Aut}^+ K$. Decomposition results analogous to those obtained in Section 3.3 hold for the group of strongly additive automorphisms and its subgroups:

Proposition 3.4.13. *Let* K *be a Hahn field with the first lifting property. Then*

$$v\text{-}\operatorname{Aut}^+ K = \operatorname{Int}\operatorname{Aut}^+ K \rtimes \operatorname{Ext}\operatorname{Aut} K. \tag{3.40}$$

Proof. Let $\sigma \in v\text{-}\operatorname{Aut}^+ K$. By Theorem 3.3.13 there exist $\tau \in \operatorname{Ext}\operatorname{Aut} K$ and $\rho \in \operatorname{Int}\operatorname{Aut} K$ such that $\sigma = \rho\tau$. Now, by Remark 3.4.10, τ is strongly additive, and since $\rho = \sigma\tau^{-1}$ then ρ is also strongly additive. So $\operatorname{Ext}\operatorname{Aut} K$ and $\operatorname{Int}\operatorname{Aut}^+ K$ generate $v\text{-}\operatorname{Aut}^+ K$. Moreover, $\operatorname{Ext}\operatorname{Aut} K \cap \operatorname{Int}\operatorname{Aut}^+ K \subseteq \operatorname{Ext}\operatorname{Aut} K \cap \operatorname{Int}\operatorname{Aut} K = \{\mathrm{id}_K\}$. Finally, as remarked above, $\operatorname{Int}\operatorname{Aut}^+ K$ is a normal subgroup of $v\text{-}\operatorname{Aut}^+ K$. The statement follows. □

[4] In the paper the expression "strongly linear" is used in place of "summable".

Lemma 3.4.14. *Let K be a Hahn field satisfying the (canonical) second lifting property. The following hold.*

$$\operatorname{Int}\operatorname{Aut}^+ K = 1\text{-}\operatorname{Aut}^+ K \rtimes G\text{-}\operatorname{Exp} K; \tag{3.41}$$
$$\operatorname{Int}\operatorname{Aut}^+_{(k)} K = \operatorname{Int}\operatorname{Aut}^+_k K = 1\text{-}\operatorname{Aut}^+_k K \rtimes G\text{-}\operatorname{Exp} K. \tag{3.42}$$

Proof. Follows from Theorem 3.3.27 taking intersections with v-$\operatorname{Aut}^+ K$ and applying part (iii) of Proposition 3.4.10. □

Applying the decomposition of the subgroup of internal automorphisms we obtain

Proposition 3.4.15. *Let K be a Hahn field with the canonical first and second lifting property. Then*

$$v\text{-}\operatorname{Aut}^+ K = (1\text{-}\operatorname{Aut}^+ K \rtimes G\text{-}\operatorname{Exp} K) \rtimes \operatorname{Ext}\operatorname{Aut} K;$$
$$v\text{-}\operatorname{Aut}^+_{(k)} K = (1\text{-}\operatorname{Aut}^+_k K \rtimes G\text{-}\operatorname{Exp} K) \rtimes \operatorname{Ext}\operatorname{Aut} K;$$
$$v\text{-}\operatorname{Aut}^+_k K = (1\text{-}\operatorname{Aut}^+_k K \rtimes G\text{-}\operatorname{Exp} K) \rtimes \operatorname{Ext}\operatorname{Aut}_k K.$$

Proof. It follows from Proposition 3.4.13 applying Lemma 3.4.14 to $\operatorname{Int}\operatorname{Aut}^+ K$. □

Theorem 3.4.16. *Let K be a Hahn field with the canonical first and second lifting property. Then*

$$v\text{-}\operatorname{Aut}^+ K \simeq (1\text{-}\operatorname{Aut}^+ K \rtimes \operatorname{Hom}(G, k^\times)) \rtimes (\operatorname{Aut} k \times o\text{-}\operatorname{Aut} G);$$
$$v\text{-}\operatorname{Aut}^+_{(k)} K \simeq (1\text{-}\operatorname{Aut}^+_k K \rtimes \operatorname{Hom}(G, k^\times)) \rtimes (\operatorname{Aut} k \times o\text{-}\operatorname{Aut} G);$$
$$v\text{-}\operatorname{Aut}^+_k K \simeq (1\text{-}\operatorname{Aut}^+_k K \rtimes \operatorname{Hom}(G, k^\times)) \rtimes o\text{-}\operatorname{Aut} G.$$

Proof. It is a consequence of Proposition 3.4.15: by Definition 3.3.10 we have $\operatorname{Ext}\operatorname{Aut} K \simeq \operatorname{Aut} k \times o\text{-}\operatorname{Aut} G$ and by Definition 3.3.21 it follows that $G\text{-}\operatorname{Exp} K \simeq \operatorname{Hom}(G, k^\times)$. This gives the first equation. The second one follows from part (iv) of Proposition 3.3.25: with the canonical second lifting property all the G-exponentiations are k-automorphisms. Finally, by Remark 3.3.34, we have $\operatorname{Ext}\operatorname{Aut}_k K \simeq o\text{-}\operatorname{Aut} G$, which yields the third equation. □

From now we will focus on the subgroups $v\text{-}\operatorname{Aut}^+_{(k)} K$ and $v\text{-}\operatorname{Aut}^+_k K$, for which we will be able to obtain yet more precise descriptions. In Theorem 3.4.16 we see that all components, except possibly $1\text{-}\operatorname{Aut}_k K$, only depend on k and G. The next section is devoted to the study of the remaining component: $1\text{-}\operatorname{Aut}_k^+ K$. Recall that, by Lemma 3.3.19, we have $1\text{-}\operatorname{Aut}_{(k)} K = 1\text{-}\operatorname{Aut}_k K$.

3.4.2 Description of $\operatorname{Int}\operatorname{Aut}^+ K$ and $1\text{-}\operatorname{Aut}^+ K$

Schilling [Sch44] describes the group $v\text{-}\operatorname{Aut}_k \mathbb{K}$, for $\mathbb{K} = k((\mathbb{Z}))$, in terms of $U_{\mathbb{K}}$, the group of units of the valuation ring of $\mathbb{K}$. Drawing inspiration from his work, we aim at an explicit description of the groups $v\text{-}\operatorname{Aut}_k^+ K$ and $v\text{-}\operatorname{Aut}_{(k)}^+ K$, for an arbitrary Hahn field K, in terms of the fundamental objects connected to K. Let $U = U_K$ be the group of units of the valuation ring of K. We will further describe the group $\operatorname{Int}\operatorname{Aut}_k^+ K$ in terms of (a subgroup of) the group $\operatorname{Hom}(G, U)$. Then we will deduce a description of $1\text{-}\operatorname{Aut}_{(k)}^+ K$ in terms of a subgroup of $\operatorname{Hom}(G, 1 + I_K)$. In Section 3.5.2 we will retrieve Schilling's result as a special case of ours.

Let K be a Hahn field and $\sigma \in \operatorname{Int}\operatorname{Aut}_k K$. Recall that σ satisfies the following conditions: for all $a \in K$ we have $v(\sigma(a)) = v(a)$ and $\sigma|_k = \mathrm{id}_k$. These properties of σ imply that $\sigma(t^g) = u_\sigma(g)t^g$ for some $u_\sigma(g) \in U$ depending on g. For all $\sigma \in \operatorname{Int}\operatorname{Aut} K$ define $u_\sigma\colon G \to U$ by $g \mapsto t^{-g}\sigma(t^g)$.

Lemma 3.4.17. *Let K be a Hahn field. For all $\sigma \in \operatorname{Int}\operatorname{Aut} K$ the map u_σ is a group homomorphism.*

Proof. Let $\sigma \in \operatorname{Int}\operatorname{Aut} K$ and $g, h \in G$. Then we have

$$\begin{aligned} u_\sigma(g+h) &= t^{-(g+h)}\sigma(t^{g+h}) \\ &= t^{-g}t^{-h}\sigma(t^g)\sigma(t^h) \\ &= t^{-g}\sigma(t^g)t^{-h}\sigma(t^h) \\ &= u_\sigma(g)u_\sigma(h). \end{aligned}$$

□

This gives rise to a map $\mathcal{S}'$ from $\operatorname{Int}\operatorname{Aut} K$ to the set $\operatorname{Hom}(G, U)$ of homomorphisms of G into U defined by $\mathcal{S}'(\sigma) = u_\sigma$. We are interested in the restriction of $\mathcal{S}'$ to $\operatorname{Int}\operatorname{Aut}_{(k)}^+ K = \operatorname{Int}\operatorname{Aut}_k^+ K$, which we will denote by $\mathcal{S}$:

$$\mathcal{S}\colon \operatorname{Int}\operatorname{Aut}_k^+ K \to \operatorname{Hom}(G, U) \quad \sigma \mapsto u_\sigma. \tag{3.43}$$

Lemma 3.4.18. *The map $\mathcal{S}$ defined in (3.43) is injective.*

Proof. Let $\sigma, \tau \in \operatorname{Int}\operatorname{Aut}_k^+ K$ be such that $\mathcal{S}(\sigma) = u_\sigma = u_\tau = \mathcal{S}(\tau)$. Then, for all $g \in G$, we have $t^{-g}\sigma(t^g) = u_\sigma(g) = u_\tau(g) = t^{-g}\tau(t^g)$ which implies $\sigma(t^g) = \tau(t^g)$ and this (since σ and τ are strongly additive k-automorphisms) implies that $\sigma(a) = \tau(a)$ for all $a \in K$, hence $\sigma = \tau$. □

Now we determine the image of $\mathcal{S}$.

Definition 3.4.19 (K-Summable homomorphism)**.** An element $u \in \operatorname{Hom}(G, U)$ is *K-summable* if, for every $a \in K$, the family $\{a_g u(g) t^g : g \in \operatorname{supp} a\}$ is K-summable. Let us denote the set of K-summable elements of $\operatorname{Hom}(G, U)$ by $\operatorname{Hom}^+(G, U)$.

Lemma 3.4.20. *We have* $\operatorname{im}\mathcal{S} = \operatorname{Hom}^+(G, U)$. *Therefore $\mathcal{S}$ corestricts to a bijection*

$$\mathcal{S}\colon \operatorname{Int}\operatorname{Aut}_k^+ K \to \operatorname{Hom}^+(G, U). \tag{3.44}$$

Proof. Let $u \in \operatorname{im}\mathcal{S}$. Then $u = u_\sigma$ for some $\sigma \in \operatorname{Int}\operatorname{Aut}_k^+ K$. Now let $a = \sum a_g t^g \in K$. Since $\sigma \in \operatorname{Int}\operatorname{Aut}_k^+ K$ we have $\sigma(a) = \sum a_g \sigma(t^g) = \sum a_g u_\sigma(g) t^g$, hence the family $\{a_g u_\sigma(g t^g) : g \in \operatorname{supp} a\}$ is K-summable. Therefore $u \in \operatorname{Hom}^+(G, U)$ and so $\operatorname{im}\mathcal{S} \subseteq \operatorname{Hom}^+(G, U)$.

Conversely, let $u \in \operatorname{Hom}^+(G, U)$ and define σ by $\sigma\left(\sum a_g t^g\right) = \sum a_g u(g) t^g$ for all $\sum a_g t^g \in K$. Since u is K-summable, $\sigma \in v\text{-}\operatorname{Aut}_k^+ K$ is well defined. Now we show $\sigma \in \operatorname{Int}\operatorname{Aut}_k K$. For $g \in G$ let $u(g) = u_0 + \varepsilon(g)$ with $u_0 \in k^\times$, $\varepsilon(g) \in I_K$, let $a = \sum a_g t^g \in K$ and set $v(a) = h$. Note that $v(a_g u(g) t^g) = g$ for all $g \in \operatorname{supp} a$. Moreover, $\bar{\sigma}(a_0 + I_K) = \sigma(a_0)_0 + I_K = a_0 u(0) + I_K = a_0 + I_K$ so $\bar{\sigma} = \mathrm{id}$ ($\bar{\sigma}$ was defined in Definition 1.5.9). So $\sigma \in \operatorname{Int}\operatorname{Aut}_k^+ K$ and, by definition of σ we have $\sigma(t^g) = u(g) t^g$ thus $u = u_\sigma \in \operatorname{im}\mathcal{S}$. Hence $\operatorname{Hom}^+(G, 1 + I_K) \subseteq \operatorname{im}\mathcal{S}$, which completes the proof. □

Definition 3.4.21. We define an operation[5]

$$\begin{array}{cccc} \times\colon & \operatorname{Hom}^+(G, U) \times \operatorname{Hom}^+(G, U) & \longrightarrow & \operatorname{Hom}^+(G, U) \\ & (u_\tau, u_\sigma) & \longmapsto & [u_\tau \times u_\sigma : g \mapsto \tau(u_\sigma(g)) u_\tau(g)]. \end{array}$$

Proposition 3.4.22. *The map $\mathcal{S}\colon \operatorname{Int}\operatorname{Aut}_k^+ K \to \operatorname{Hom}^+(G, U)$ defined in (3.44) is a group isomorphism, if we equip $\operatorname{Hom}^+(G, U)$ with the new operation $\times$:*

$$\mathcal{S}\colon (\operatorname{Int}\operatorname{Aut}_k^+ K, \circ) \xrightarrow{\sim} (\operatorname{Hom}^+(G, U), \times). \tag{3.45}$$

[5] This operation corresponds to the crossed representation defined by Schilling for $\operatorname{Aut}_k \mathbb{L}$ where $\mathbb{L} = k((\mathbb{Z}))$. See Section 3.5.2 for more details.

Proof. By Lemmas 3.4.18 and 3.4.20 the map (3.45) is bijective. It remains to show that it is a group homomorphism. Let $\sigma, \tau \in \operatorname{Int}\operatorname{Aut}_k^+ K$ and let $g \in G$. Then we have

$$\begin{aligned} u_{\tau\sigma}(g) &= t^{-g}(\tau\sigma)(t^g) = t^{-g}\tau(\sigma(t^g)) = t^{-g}\tau(u_\sigma(g)t^g) = t^{-g}\tau(u_\sigma(g))\tau(t^g) \\ &= t^{-g}\tau(u_\sigma(g))t^g u_\tau(g) = \tau(u_\sigma(g))u_\tau(g) = (u_\tau \times u_\sigma)(g). \end{aligned}$$

□

Corollary 3.4.23. *Restricting $\mathcal{S}$ we get*

$$(G\text{-}\operatorname{Exp} K, \circ) \simeq \left(\operatorname{Hom}(G, k^\times,), \times\right) = \left(\operatorname{Hom}(G, k^\times), \cdot\right) \tag{3.46}$$

$$(1\text{-}\operatorname{Aut}_k^+ K, \circ) \simeq \left(\operatorname{Hom}^+(G, 1 + I_K), \times\right) \tag{3.47}$$

and thus

$$(\operatorname{Hom}^+(G, U), \times) \simeq (\operatorname{Hom}^+(G, 1 + I_K), \times) \rtimes (\operatorname{Hom}(G, k^\times), \cdot). \tag{3.48}$$

Proof. Let $\rho = \rho_x \in G\text{-}\operatorname{Exp} K$. Then $\mathcal{S}(\rho) = u_\rho = x \in \operatorname{Hom}(G, k^\times)$, so $\mathcal{S}|_{G\text{-}\operatorname{Exp} K} = P^{-1}$, where P is the map given in Definition 3.3.20. We therefore have an isomorphism

$$\mathcal{S} \colon G\text{-}\operatorname{Exp} K \xrightarrow{\sim} (\operatorname{Hom}(G, k^\times), \times). \tag{3.49}$$

To prove (3.46) we notice that, for $x, y \in \operatorname{Hom}(G, k^\times)$, corresponding to $\sigma_x, \sigma_y \in \operatorname{Int}\operatorname{Aut}_k^+ K$ we have

$$(x \times y)(g) = x(g) \cdot \sigma_x(y(g)) = x(g)y(g)$$

because $y(g) \in k^\times$ and $\sigma_x \in G\text{-}\operatorname{Exp} K \leq \operatorname{Aut}_k K$. So $(\operatorname{Hom}(G, k^\times), \times) = (\operatorname{Hom}(G, k^\times), \cdot)$, and (3.46) follows.

Similarly, if $\tau \in 1\text{-}\operatorname{Aut}_k^+ K$ then $u_\tau \in \operatorname{Hom}(G, 1 + I_K)$. So $\operatorname{Hom}^+(G, 1 + I_K) := (\operatorname{Hom}^+(G, U)) \cap \operatorname{Hom}(G, 1 + I_K)$. Then we have

$$\mathcal{S} \colon 1\text{-}\operatorname{Aut}_k^+ K \xrightarrow{\sim} \operatorname{Hom}^+(G, 1 + I_K) \tag{3.50}$$

which proves (3.47). Equation (3.48) now follows from Proposition 3.4.22 applying (3.42), (3.46) and (3.47). □

Now we have the last ingredient to prove the main result of this chapter.

Theorem 3.4.24. *Let K be a Hahn field with the first and (canonical) second lifting property. Then*

$$v\text{- Aut}^+_{(k)} K \simeq \left(\mathrm{Hom}^+(G, 1 + I_K) \rtimes \mathrm{Hom}(G, k^\times)\right) \rtimes (\mathrm{Aut}\, k \times o\text{- Aut}\, G);$$
$$v\text{- Aut}^+_k K \simeq \left(\mathrm{Hom}^+(G, 1 + I_K) \rtimes \mathrm{Hom}(G, k^\times)\right) \rtimes o\text{- Aut}\, G.$$

Proof. It follows from Theorem 3.4.16 using Corollary 3.4.23 to replace $1\text{- Aut}^+_k K$ by $\mathrm{Hom}^+(G, 1 + I_K)$. □

Theorem 3.4.24 provides a decomposition of $v\text{- Aut}^+_{(k)} K$ and $v\text{- Aut}^+_k K$ purely in terms of the valuation invariants of K. In the next section we are going to apply the results obtained so far under some further assumptions on the group G and the field k. This will allow to retrieve results of Schilling [Sch44] on the field of Laurent series and of Deschamps [Des05] on the field of Puiseux series.

3.5 Explicit examples in special cases

3.5.1 Finitely generated exponent group

Let k be an arbitrary field and let G be a totally ordered, finitely generated abelian group. Since G is totally ordered, it must be torsion free. Therefore, by the fundamental theorem on finitely generated abelian groups [DF04, Theorem 3, page 158], we can assume without loss of generality that $G = \mathbb{Z}^n$, for some $n \in \mathbb{N}$.

Lemma 3.5.1. *We have* $\mathrm{Hom}(G, k^\times) \simeq (k^\times)^n$.

Proof. For every $i \in \{1, \ldots, n\}$ let e_i be the element of $\mathbb{Z}^n$ having 1 in position i and 0 elsewhere. Consider the map

$$\begin{array}{rccc} f\colon & \mathrm{Hom}(G, k^\times) & \to & (k^\times)^n \\ & \eta & \mapsto & (\eta(e_1), \ldots, \eta(e_n)). \end{array}$$

Claim: f is a group homomorphism.
Let $\eta, \vartheta \in \mathrm{Hom}(G, k^\times)$. Then for all i we have $(\eta - \vartheta)(e_i) = \eta(e_i) - \vartheta(e_i)$ so $f(\eta - \vartheta) = (\eta(e_1) - \vartheta(e_1), \ldots, \eta(e_n) - \vartheta(e_n)) = f(\eta) - f(\vartheta)$. ♦

Claim: f is injective.
Let $f(\eta) = f(\vartheta)$, that is $\eta(e_i) = \vartheta(e_i)$ for all $i = 1, \dots, n$. Then, for all $g = \sum_{i=1}^{n} m_i e_i \in G$ we have

$$\eta(g) = \sum_{i=1}^{n} m_i \eta(e_i) = \sum_{i=1}^{n} m_i \vartheta(e_i) = \vartheta(g)$$

so $\eta = \vartheta$. ♦
Claim: f is surjective.
Let $(\alpha_1, \dots, \alpha_n) \in (k^\times)^n$ and define $\eta \colon G \to k^\times$ by $\eta\left(\sum m_i e_i\right) = \prod \alpha_i^{m_i}$. We see that in fact $\eta \in \operatorname{Hom}(G, k^\times)$: indeed, for $g = \sum_{i=1}^{n} m_i e_i$ and $g' = \sum_{i=1}^{n} m_i' e_i$ we have $g + g' = \sum (m_i + m_i') e_i$ and thus

$$\eta(g + g') = \prod \alpha_i^{m_i + m_i'} = \prod \alpha_i^{m_i} \prod \alpha_i^{m_i'} = \eta(g)\eta(g').$$

It is then obvious that η satisfies $f(\eta) = (\alpha_1, \dots, \alpha_n)$. The proof is now complete. □

Let $K \subseteq k((G))$ be a Hahn field satisfying the first and second lifting property. The following is a rewriting of Theorem 3.3.52 using the isomorphism established in Lemma 3.5.1.

Theorem 3.5.2. *Let $G = \mathbb{Z}^n$. Let k be a field and $K \subseteq k((G))$ a Hahn field with the first and second lifting property. Then we have*

$$v\text{-}\operatorname{Aut} K \simeq (1\text{-}\operatorname{Aut} K \rtimes (k^\times)^n) \rtimes (\operatorname{Aut} k \times o\text{-}\operatorname{Aut} G)$$
$$v\text{-}\operatorname{Aut}_k K \simeq (1\text{-}\operatorname{Aut}_k K \rtimes (k^\times)^n) \rtimes o\text{-}\operatorname{Aut} G.$$

If, moreover, K satisfies the canonical first lifting property, Proposition 3.3.53 yields

$$v\text{-}\operatorname{Aut}_{(k)} K \simeq (1\text{-}\operatorname{Aut}_k K \rtimes (k^\times)^n) \rtimes (\operatorname{Aut} k \times o\text{-}\operatorname{Aut} G).$$

□

Now we will provide a description of $v\text{-}\operatorname{Aut}^+_{(k)} K$ and $v\text{-}\operatorname{Aut}^+_k K$. For $G = \mathbb{Z}^n$ we have $\operatorname{Hom}(G, 1 + I_K) \simeq (1 + I_K)^n$: the proof is identical to that of Lemma 3.5.1. Let us fix some notation. Let $g_1, \dots, g_n$ be generators of G, let $\bar{u} \in \operatorname{Hom}(G, 1 + I_K)$ and let $u_i := \bar{u}(g_i) \in 1 + I_K$, for $i = 1, \dots, n$. Then

$$\xi \colon \operatorname{Hom}(G, 1 + I_K) \to (1 + I_K)^n, \; \bar{u} \mapsto (u_1, \dots, u_n)$$

is a group isomorphism. Under ξ, a summable automorphism $\bar{u} \in \mathrm{Hom}^+(G, 1 + I_K)$ (Definition 3.4.19) corresponds to a tuple $\xi(\bar{u}) = (u_1, \dots, u_n)$ such that, for all $a \in K$ the family

$$\left\{a_g \left(\sum r_i u_i\right) t^g : r_i \in \mathbb{Z}, \sum r_i g_i = g, \; g \in \operatorname{supp} a\right\}$$

is K-summable. Let us denote by $(1 + I_K)^{n+} := \xi\left(\mathrm{Hom}^+(G, 1 + I_K)\right)$. On $\mathrm{Hom}^+(G, 1 + I_K)$ we defined the operation $\times$ (Definition 3.4.21). We can define an operation on $(1 + I_K)^{n+}$, also denoted by $\times$, by setting $\mathbf{u}_1 \times \mathbf{u}_2 := \xi\left(\xi^{-1}(\mathbf{u}_1) \times \xi^{-1}(\mathbf{u}_2)\right)$, for all $\mathbf{u}_1, \mathbf{u}_1 \in (1 + I_K)^{n+}$. We thus obtain

Lemma 3.5.3. $(\mathrm{Hom}^+(G, 1 + I_K), \times) \simeq ((1 + I_K)^{n+}, \times)$.

Proof. The isomorphism ξ is, in particular, a bijection. We then use this bijection to induce a group structure on the image $(1 + I_K)^{n+} := \xi\left(\mathrm{Hom}^+(G, 1 + I_K)\right)$, which is therefore isomorphic, as a group to $(\mathrm{Hom}^+(G, 1 + I_K), \times)$. □

Now assume that $G = \mathbb{Z}^n$ is equipped with the lexicographic order $<_{\mathrm{lex}}$. Then G is the Hahn sum $\coprod_{\gamma \in \Gamma} \mathbb{Z}$ where $\Gamma = \{1, \dots, n\}$. We can explicitly describe $o\text{-}\operatorname{Aut} G$.

Lemma 3.5.4. *Consider* $(\mathbb{Z}, +, <)$ *as an ordered abelian group, with the usual ordering. Then* $\operatorname{End}(\mathbb{Z}, +) \simeq \mathbb{Z}$ *and* $o\text{-}\operatorname{Aut} \mathbb{Z} = \{\mathrm{id}_{\mathbb{Z}}\}$.

Proof. Consider the map

$$\begin{array}{rccc} \mu: & \mathbb{Z} & \to & \operatorname{End} \mathbb{Z} \\ & r & \mapsto & r^* \end{array}$$

where r^* is defined by $r^*(n) = rn$ for all $n \in \mathbb{Z}$. Then we can check that $(r^* + s^*)(n) = (r + s)n = rn + sn = r^*(n) + s^*(n)$ so addition in $\mathbb{Z}$ corresponds to pointwise addition in $\operatorname{End} \mathbb{Z}$. Similarly $(rs)^*(n) = rsn = r^*(sn) = r^*(s^*(n))$ so multiplication in $\mathbb{Z}$ corresponds to composition in $\operatorname{End} \mathbb{Z}$. So μ is a ring homomorphism. It is injective, for if $r \neq s \in \mathbb{Z}$ then $r^*(1) = r \neq s = s^*(1)$. And it is surjective: for all $\eta \in \operatorname{End} \mathbb{Z}$ let $r := \eta(1)$. Then $\eta(n) = n\eta(1) = nr = r^*(n)$. Hence μ is a ring isomorphism. It follows that $\operatorname{Aut} \mathbb{Z}$ is isomorphic, via μ, to the multiplicative group of units in $\mathbb{Z}$, so $\operatorname{Aut} \mathbb{Z} \simeq \{1, -1\}$. Finally, $-1^* \notin o\text{-}\operatorname{Aut} \mathbb{Z}$ because, for example, $1 > 0$ but $-1^*(1) = -1 < 0$. Hence the only order preserving automorphism of $\mathbb{Z}$ is 1, which finishes the proof. □

Let us identify $\operatorname{End} \mathbb{Z}$ with $\mathbb{Z}$ and $o\text{-}\operatorname{Aut} \mathbb{Z}$ with $\{1\}$. Moreover, let $\mathrm{UUT}_n(\mathbb{Z})$ be the multiplicative group of upper uni-triangular $n \times n$-matrices with integer coefficients. Because Γ is finite, also the only order preserving automorphism of Γ is the identity. Thus Corollary 2.3.20 applies, yielding.

Lemma 3.5.5. $o\text{-}\operatorname{Aut} G \simeq \mathrm{UUT}_n(\mathbb{Z})$. □

Now Lemmas 3.5.3 and 3.5.5 applied to Theorem 3.5.2 provide the following refinement of Theorem 3.4.24.

Theorem 3.5.6. *Let $G = (\mathbb{Z}^n, <_{\mathrm{lex}})$. Let k be a field and $K \subseteq k((G))$ a Hahn field with the first and second (canonical) lifting property. Then we have*

$$v\text{-}\operatorname{Aut}^+_{(k)} K \simeq (((1 + I_K)^{n+}, \times) \rtimes (k^\times)^n) \rtimes (\operatorname{Aut} k \times \mathrm{UUT}_n(\mathbb{Z}))$$
$$v\text{-}\operatorname{Aut}^+_k K \simeq (((1 + I_K)^{n+}, \times) \rtimes (k^\times)^n) \rtimes \mathrm{UUT}_n(\mathbb{Z}).$$

□

In the next two sections we investigate in more detail the case $G = \mathbb{Z}$ and provide a more explicit description of the automorphism group of the field $\mathbb{L} = k((Z))$ of Laurent series and of the function field $k(\mathbb{Z})$.

3.5.2 Laurent series

Let k be a field and let $\mathbb{L} := k((\mathbb{Z}))$ be the field of formal Laurent series with coefficients in k. This is a maximal Hahn field, thus it has the canonical first and second lifting properties. On this field the valuation v has residue field k and value group $\mathbb{Z}$. In [Sch44] Schilling studies the group $v\text{-}\operatorname{Aut}_k \mathbb{L}$ of k-automorphisms of $\mathbb{L}$. In this section we prove Theorem 3.5.10, which is both a generalisation and a refinement of Schilling's result. We also provide a refinement in order to describe the group $o\text{-}\operatorname{Aut} \mathbb{L}$, in the case of k an ordered field (Corollary 3.5.13).

We recall that the group of units is $U := U_{\mathbb{L}} \simeq (1 + I_{\mathbb{L}}) \times k^\times$.

Lemma 3.5.7. *We have* $\operatorname{Hom}(\mathbb{Z}, U) = \operatorname{Hom}^+(\mathbb{Z}, U)$.

Proof. Let $\bar{u} \in \operatorname{Hom}(\mathbb{Z}, U)$, $u = \bar{u}(1)$ and $a \in \mathbb{L}$. By Neumann's Lemma [Pri83, p. 57] the family $\{a_n u^n t^n : n \in \operatorname{supp} a\}$ is $\mathbb{L}$-summable. So $u \in \operatorname{Hom}^+(\mathbb{Z}, U)$. □

By Lemma 3.5.7 we can use the group structure $(\operatorname{Hom}(\mathbb{Z}, U), \times)$ described in Definition 3.4.21 to induce an alternative group structure on U. Denote by

$\vartheta\colon \operatorname{Hom}(\mathbb{Z}, U) \to U$ the isomorphism given by $\vartheta(\bar{u}) = u := \bar{u}(1)$. Set, for all $u_1, u_2 \in U$

$$u_1 \times_s u_2 = \vartheta(\vartheta^{-1}(\bar{u}_1) \times \vartheta^{-1}(\bar{u}_2)). \tag{3.51}$$

Lemma 3.5.8. *We have*

$$(\operatorname{Hom}(\mathbb{Z}, U), \times) \simeq (U, \times_s) \tag{3.52}$$

$$\left(\operatorname{Hom}(\mathbb{Z}, k^\times), \times\right) \simeq \left(\operatorname{Hom}(\mathbb{Z}, k^\times), \cdot\right) \simeq (k^\times, \cdot) \tag{3.53}$$

$$(\operatorname{Hom}(\mathbb{Z}, 1 + I_{\mathbb{L}}), \times) \simeq (1 + I_{\mathbb{L}}, \times_s) \tag{3.54}$$

and thus

$$(\operatorname{Hom}(\mathbb{Z}, U), \times) \simeq (1 + I_{\mathbb{L}}, \times_s) \rtimes (k^\times, \cdot). \tag{3.55}$$

Proof. Equation (3.52) follows immediately from (3.51). Equations (3.53), (3.54) and (3.55) are now special cases of Corollary 3.4.23. □

Next we show that all automorphisms of $\mathbb{L}$ are strongly additive.

Lemma 3.5.9. *We have* $v\text{-}\operatorname{Aut}\mathbb{L} = v\text{-}\operatorname{Aut}^+\mathbb{L}$.

Proof. Let $\sigma \in v\text{-}\operatorname{Aut}\mathbb{L}$. By Lemma 3.5.5 with $n = 1$ it follows that $o\text{-}\operatorname{Aut}\mathbb{Z}$ is trivial so for all $a \in \mathbb{L}$ we have $v(a) = v(\sigma(a))$.

Now let $\mathcal{F} = \{a_{(i)} : i \in I\} \subseteq \mathbb{L}$ be a summable family. In particular, $\operatorname{Supp}\mathcal{F}$ is well ordered which, in this case, means bounded from below: let $m = \min\operatorname{Supp}\mathcal{F}$. Since, for all $i \in I$ we have $v(a_{(i)}) = v(\sigma(a_{(i)}))$, it follows that also $m = \min\operatorname{Supp}\sigma(\mathcal{F})$, which is therefore well ordered. Now, for every $n \in \operatorname{Supp}\sigma(\mathcal{F})$ there can only be finitely many $i \in I$ such that $v(\sigma(a_{(i)})) = v(a_{(i)}) \leq n$, because $\mathcal{F}$ is summable. For all other $j \in I$ we have $v(a_{(j)}) = v(\sigma(a_{(j)})) > n$ thus $n \notin \operatorname{supp}\sigma(a_{(j)})$. So $\sigma(\mathcal{F})$ is summable.

Finally, let us prove that $\sigma\left(\sum a_{(i)}\right) = \sum \sigma(a_{(i)})$. For all $n \in \mathbb{Z}$ let $T_n = \{i \in$

$I : v(a_{(i)}) \leq n\}$. As we noticed, this is a finite set. We have

$$\begin{aligned} v\left(\sigma\left(\sum a_{(i)}\right) - \sum \sigma(a_{(i)})\right) &= v\left(\sigma\left(\sum_{i\in T_n} a_{(i)} + \sum_{i\in I\setminus T_n} a_{(i)}\right) - \sum_{i\in T_n}\sigma(a_{(i)}) - \sum_{i\in I\setminus T_n}\sigma(a_{(i)})\right) \\ &= v\left(\sum_{i\in T_n}\sigma(a_{(i)}) + \sigma\left(\sum_{i\in I\setminus T_n} a_{(i)}\right) - \sum_{i\in T_n}\sigma(a_{(i)}) - \sum_{i\in I\setminus T_n}\sigma(a_{(i)})\right) \\ &= v\left(\sigma\left(\sum_{i\in I\setminus T_n} a_{(i)}\right) - \sum_{i\in I\setminus T_n}\sigma(a_{(i)})\right) > n. \end{aligned}$$

It follows that $v\left(\sigma\left(\sum a_{(i)}\right) - \sum \sigma(a_{(i)})\right) = \infty$ thus $\sigma\left(\sum a_{(i)}\right) = \sum \sigma(a_{(i)})$, and the proof is complete. □

The following theorem is now a consequence of Theorem 3.5.6 and Lemmas 3.5.8 and 3.5.9.

Theorem 3.5.10. *We have*

$$v\text{-}\operatorname{Aut}_{(k)} \mathbb{L} \simeq ((1 + I_{\mathbb{L}}, \times_s) \rtimes (k^{\times}, \cdot)) \rtimes \operatorname{Aut} k.$$

□

Remark 3.5.11. We can now show explicitly how a $\sigma \in v\text{-}\operatorname{Aut}_{(k)} K$ acts. Let $a = \sum_{i\geq m} a_i t^i \in \mathbb{L}$. We know that σ is strongly additive, so $\sigma(a) = \sum \sigma(a_i)\sigma(t)^i$. For all $i \in \operatorname{supp} a$ we have $\sigma(a_i) \in k$. Moreover, because $v(t) = v(\sigma(t)) = 1$ we have $u_\sigma := t^{-1}\sigma(t) \in U$. Then σ is uniquely determined by u_σ and $\sigma|_k$:

$$\sigma\left(\sum_{i\geq m} a_i t^i\right) = \sum_{i\geq m} \sigma|_k(a_i)(u_\sigma t)^i.$$

Conversely, to every unit $u \in U$ and every $\tau \in \operatorname{Aut} k$ we have the corresponding $\sigma_{u,\tau} \in v\text{-}\operatorname{Aut}_{(k)} \mathbb{L}$ defined by

$$\sigma_{u,\tau}\left(\sum_{i\geq m} a_i t^i\right) = \sum_{i\geq m} \tau(a_i)(ut)^i.$$

□

Corollary 3.5.12. *We have* $v\text{-}\operatorname{Aut}_k \mathbb{L} \simeq (U, \times_s) \simeq (1 + I_{\mathbb{L}}, \times_s) \rtimes (k^{\times}, \cdot)$. □

With Corollary 3.5.12 we retrieve Schilling's result [Sch44, Theorem 1]. To conclude this subsection we sharpen Theorem 3.5.10 in the case where k is an ordered field, to characterise the subgroup of U corresponding to the group $o\text{-}\operatorname{Aut}_k \mathbb{L}$ of order preserving k-automorphisms of $\mathbb{L}$.

Corollary 3.5.13. *The k-automorphisms preserving the lexicographic order on $\mathbb{L}$ are exactly those corresponding to positive units: $o\text{-}\operatorname{Aut}_k \mathbb{L} \simeq (U^{>0}, \times_s)$. More precisely, we have*

$$o\text{-}\operatorname{Aut}_{(k)} \mathbb{L} \simeq \left((1 + I_{\mathbb{L}}, \times_s) \rtimes k^{>0} \right) \rtimes o\text{-}\operatorname{Aut} k. \tag{3.56}$$

Proof. Let $a = \sum_{i=m}^{\infty} a_i t^i$ with $v(a) = m \in \mathbb{Z}$ and $u = \sum_{i=0}^{\infty} u_i t^i$ a unit ($u_0 \neq 0$). We can write $a = t^m \sum_{i=0}^{\infty} b_i t^i$ with $b_i = a_{i-m}$. Let us assume $a > 0$, that is $b_0 > 0$. Let us write $\sigma_u := \sigma_{u,\mathrm{id}}$ (see Remark 3.5.11). Then

$$\begin{aligned}
\sigma_u(a) = \sigma_u\left(t^m \sum_{i=0}^{\infty} b_i t^i\right) &= \sigma_u(t^m)\sigma_u\left(\sum_{i=0}^{\infty} b_i t^i\right) = (tu)^m \left(b_0 + \sigma_u\left(\sum_{i=1}^{\infty} b_i t^i\right)\right) \\
&= (tu)^m \left(b_0 + [\text{higher order terms}]\right) \\
&= t^m u_0^m b_0 + [\text{higher order terms}]
\end{aligned}$$

hence $\sigma(a)_m = u_0^m a_m > 0$ if and only if $u_0 > 0$ or m is even. Thus we have $o\text{-}\operatorname{Aut}_k \mathbb{L} \simeq (U^{>0}, \times_s) = (1 + I_{\mathbb{L}}, \times_s) \rtimes k^{>0}$ (the 1-units are all positive). Now (3.56) follows immediately. □

3.5.3 The Cremona group in dimension one

Let k be an arbitrary field and $n \in \mathbb{N}$. A central problem in algebraic geometry is the study of the *Cremona group*. This is the group $\mathrm{Cr}_n(k)$ consisting of all birational transformations of the projective plane $\mathbb{P}^n(k)$ over k. The Cremona group is isomorphic to the group $\operatorname{Aut}_k k(x_1, \ldots, x_n)$ of k-automorphisms of the function field $k(x_1, \ldots, x_n)$.
For $n = 1$ the Cremona group $\mathrm{Cr}_1(k)$ is completely understood: we have $\mathrm{Cr}_1(k) \simeq \mathrm{PGL}_2(k)$ (see, for example, [Can18, § 1.2]).

Now consider the Hahn field $k(\mathbb{Z}) \subseteq \mathbb{L}$. Theorem 3.3.52 applies to this field, so we have

$$v\text{-}\operatorname{Aut}_k k(\mathbb{Z}) = \operatorname{Int}\operatorname{Aut}_k k(\mathbb{Z}) \simeq 1\text{-}\operatorname{Aut} k(\mathbb{Z}) \rtimes k^{\times}. \tag{3.57}$$

Note that $v\text{-}\operatorname{Aut}_k k(\mathbb{Z})$ is a subgroup of $\operatorname{Aut}_k k(\mathbb{Z}) \simeq \mathrm{Cr}_1(k)$. An automorphism

$\sigma \in \mathrm{Cr}_1(k)$ is completely determined by the invertible matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in k^{2\times 2}$ such that $\sigma(t) = (at+b)/(ct+d)$. We characterise $v\text{-}\mathrm{Aut}_k\, k(\mathbb{Z})$ as a subgroup of $\mathrm{Cr}_1(k)$ as follows:

$$\begin{aligned} v\text{-}\mathrm{Aut}_k\, k(\mathbb{Z}) &= \left\{\sigma \in \mathrm{Cr}_1(k) : \sigma(t) = \frac{at}{ct+d},\ a,c,d \in k \text{ with } ad \neq 0\right\} \\ &= \left\{\sigma \in \mathrm{Cr}_1(k) : \sigma(t) = \frac{at}{ct+d},\ \text{ with } v\left(\frac{a}{ct+d}\right) = 0\right\}. \end{aligned} \tag{3.58}$$

Indeed if $\sigma \in \mathrm{Int\,Aut}_k\, k(\mathbb{Z})$ then $1 = v(t) = v(\sigma(t)) = v\left(\frac{at+b}{ct+d}\right) = v(at+b) - v(ct+d)$. This implies $v(at+b) = 1$ and therefore $a \neq 0$ and $b = 0$. Conversely, let $a, c, d \in k$ with $ad \neq 0$. Then $u := \frac{a}{ct+d}$ is a unit in the valuation ring of $\mathbb{L}$, because $v(u) = 0$. Therefore, by Lemma 3.5.9, $t \mapsto ut = \frac{at}{ct+d}$ determines a $\sigma_u \in v\text{-}\mathrm{Aut}_k\,\mathbb{L}$. Thus the restriction $\sigma_u|_{k(\mathbb{Z})} \in v\text{-}\mathrm{Aut}_k\, k(\mathbb{Z})$, as required.

What we just showed implies, in particular, that every $\sigma \in v\text{-}\mathrm{Aut}_k\, k(\mathbb{Z})$ extends to an automorphism in $v\text{-}\mathrm{Aut}_k\,\mathbb{L}$. Moreover, since the group of lower triangular matrices is not normal inside $\mathrm{PGL}_2(k)$, it follows that $v\text{-}\mathrm{Aut}_k\, k(\mathbb{Z})$ is not a normal subgroup of $\mathrm{Cr}_1(k)$.

We also characterise $1\text{-}\mathrm{Aut}_k\, k(\mathbb{Z})$ as a subgroup of $\mathrm{Cr}_1(k)$ as follows:

$$\begin{aligned} 1\text{-}\mathrm{Aut}_k\, k(\mathbb{Z}) &= \left\{\sigma \in \mathrm{Cr}_1(k) : \sigma(t) = \frac{at}{ct+a},\quad a,c \in k \quad \text{with } a \neq 0 \right\} \\ &= \left\{\sigma \in \mathrm{Cr}_1(k) : \sigma(t) = \frac{at}{ct+a},\ \text{ with } v\left(\frac{a}{ct+a} - 1\right) > 0\right\}. \end{aligned} \tag{3.59}$$

Indeed, $\sigma \in v\text{-}\mathrm{Aut}_k\, k(\mathbb{Z})$ (and its extension to $\mathbb{L}$) is defined by $t \mapsto ut$ with $u = \frac{a}{ct+d}$ and $ad \neq 0$. By Corollary 3.5.12 we know that $\sigma \in 1\text{-}\mathrm{Aut}_k\,\mathbb{L}$ if and only if $u \in 1 + I_{\mathbb{L}}$ which is indeed equivalent to the condition $a = d$.

3.5.4 Divisible and finite dimensional exponent group

In this subsection we consider the special case of a Hahn field $K \subseteq k((G))$ where G is uniquely divisible and finite dimensional as a $\mathbb{Q}$-vector space (see Definition 1.2.3 and Remark 1.2.4 on Page 11). We are going to carry out the following three steps.

- If G is ordered lexicographically, we describe $o\text{-}\operatorname{Aut} G$ precisely.
- If k is real closed, we get an explicit description of the group $\operatorname{Hom}(G, k^\times)$.
- If K is henselian of characteristic 0, we explicitly describe $\operatorname{Hom}(G, 1 + I_K)$.

Throughout this subsection, let G be a uniquely divisible, totally ordered, abelian group which is finite dimensional as a vector space over $\mathbb{Q}$. Set $d = \dim_{\mathbb{Q}} G$. Without loss of generality we can assume $G = \mathbb{Q}^d$. By Proposition 1.2.5, if H is another uniquely divisible group then every group then we have $\operatorname{Hom}(G, H) \simeq H^d$.

Lexicographically ordered exponent group

Assume that $G = \mathbb{Q}^d$ is equipped with the lexicographic ordering. Let $\mathrm{UPT}_d(\mathbb{Q})$ be the multiplicative group of upper triangular $d \times d$-matrices over $\mathbb{Q}$ with positive diagonal entries:

$$\mathrm{UPT}_d(\mathbb{Q}) = \left\{ (q_{i,j})_{i,j=1}^d : q_{ij} \in \mathbb{Q} \text{ and } \begin{cases} q_{ij} = 0 \text{ for } i > j \\ q_{ij} > 0 \text{ for } i = j \end{cases} \quad \text{for } i, j = 1, \ldots, d \right\}.$$

Lemma 3.5.14. *We have*

(i) $(\operatorname{End}(\mathbb{Q}, +), +, \circ) \simeq (\mathbb{Q}, +, \cdot)$ *as rings;*

(ii) $(\operatorname{Aut} \mathbb{Q}, \circ) \simeq (\mathbb{Q}^\times, \cdot)$*;*

(iii) $o\text{-}\operatorname{Aut} \mathbb{Q} \simeq (\mathbb{Q}^{>0}, \cdot)$.

Proof. (i) Let $\sigma \in \operatorname{End} \mathbb{Q}$. Consider the map

$$\begin{array}{rcl} \mu \colon \mathbb{Q} & \to & \operatorname{End} \mathbb{Q} \\ q & \mapsto & q^* \end{array}$$

where $q^*(x) := qx$ for all $x \in \mathbb{Q}$. Then q^* is indeed an endomorphism of $(\mathbb{Q}, +)$: for all $x, y \in \mathbb{Q}$ we have $q^*(x+y) = q(x+y) = qx + qy = q^*(x) + q^*(y)$. Moreover, μ is a ring homomorphism: for all $q, r \in \mathbb{Q}$ we have $(q+r)^*(x) = qx + rx = q^*(x) + r^*(x)$ and $(qr)^*(x) = qrx = q^*(rx) = q^*(r^*(r))$. So addition (resp. multiplication) in $\mathbb{Q}$ corresponds to pointwise addition (resp. composition) in $\operatorname{End} \mathbb{Q}$. Further, μ is injective. Indeed, for $r \neq q \in \mathbb{Q}$ $r^*(1) = r \neq q = q^*(1)$. And surjective: for $\sigma \in \operatorname{End} \mathbb{Q}$

let $q := \sigma(1)$. In particular, $\sigma \in \mathrm{Hom}(\mathbb{Q}, \mathbb{Q})$, and, since $\mathbb{Q}$ is uniquely divisible, by Proposition 1.2.6 σ is $\mathbb{Q}$-linear. In particular, for all $r \in \mathbb{Q}$ we have $\sigma(r) = r\sigma(1) = qr = q^*(r)$. So $\sigma = \mu(q)$.

(ii) This follows directly from part (i). Indeed, an automorphism $\sigma \in \mathrm{Aut}\,\mathbb{Q}$ is an invertible element in $\mathrm{End}\,\mathbb{Q}$, which correspond isomorphically via μ to the invertible elements of $\mathbb{Q}$, i.e. to $\mathbb{Q}^\times$.

(iii) Let $r \in \mathbb{Q}$ and $x \in \mathbb{Q}^{>0}$. Then $r^*(x) = rx > 0$ if and only if $r > 0$.

□

If we consider the lexicographic ordering on $G = \mathbb{Q}^d$, then G is the ordered Hahn sum $G = \coprod_{\gamma \in \Gamma} \mathbb{Q}$ where $\Gamma = \{1, \ldots, d\}$. The chain Γ admits the unique order preserving automorphism id_Γ. Then Corollary 2.3.20 applies yielding

Lemma 3.5.15. *We have* o*-* $\mathrm{Aut}\,G \simeq \mathrm{UPT}_d(\mathbb{Q})$. □

Real closed coefficient field

Let k be a real closed field (Definition 1.6.13) and let $k^{>0}$ be the multiplicative subgroup of positive elements of k.

Lemma 3.5.16. *We have* $\mathrm{Hom}((\mathbb{Q}, +), (k^\times, \cdot)) = \mathrm{Hom}((\mathbb{Q}, +), (k^{>0}, \cdot))$.

Proof. Let $\vartheta \in \mathrm{Hom}((\mathbb{Q}, +), (k^\times, \cdot))$. We need to show that $\vartheta(\mathbb{Q}) \subseteq k^{>0}$. Let $q \in \mathbb{Q}$. Then $q = 2\frac{q}{2}$. Therefore

$$\vartheta(q) = \vartheta\left(2\frac{q}{2}\right) = \vartheta\left(\frac{q}{2}\right)^2 > 0.$$

□

Lemma 3.5.17. *The group* $(k^{>0}, \cdot)$ *is uniquely divisible.*

Proof. Let $x \in k^{>0}$ and $n \in \mathbb{N}$ with $n > 0$. Since k is real closed, by Remark 1.6.14 there exists $y \in k^{>0}$ such that $y^n = x$, so $k^{>0}$ is divisible. Moreover, assume that $z \in k^{>0}$ is such that $z^n = x = y^n$ and that we have $y \neq z$. We may assume $y > z$ (the case $y < z$ is identical). Because y and z are both positive, it follows that $y^n > z^n$. A contradiction. So y is unique and the proof is complete. □

Corollary 3.5.18. *We have* $\mathrm{Hom}(G, (k^\times, \cdot)) \simeq (k^{>0}, \cdot)^d$. *In particular, we also have* $\mathrm{Hom}((\mathbb{Q}, +), (k^\times, \cdot)) \simeq (k^{>0}, \cdot)$. □

Henselian Hahn field

Let G be an arbitrary ordered abelian group. Let k be an arbitrary field with $\operatorname{char} k = 0$ and let $K \subseteq k((G))$ be a henselian Hahn field (Definition 1.5.15). By Corollary 1.5.18 the multiplicative group $(1 + I_K, \cdot)$ is uniquely divisible. Assume now that $G = \mathbb{Q}^d$. By Corollary 1.2.7 we have $\operatorname{Hom}((G, +), (1 + I_K, \cdot)) \simeq (1 + I_K)^d$. In particular, $\operatorname{Hom}((\mathbb{Q}, +), (1 + I_K, \cdot)) \simeq 1 + I_K$.

A summable automorphism $u \in \operatorname{Hom}^+(G, 1 + I_K)$ (Definition 3.4.19) corresponds to a tuple $(u_1, \ldots, u_d) \in (1 + I_K)^d$ such that, for all $a \in K$ the family

$$\left\{a_g \left(\sum q_i u_i\right) t^g : q_i \in \mathbb{Q},\ \sum r_i u_i = g,\ g \in \operatorname{supp} a\right\}$$

is K-summable. Let us denote by $(1 + I_K)^{d+}$ the subgroup of $(1 + I_K)^d$ corresponding to $\operatorname{Hom}^+(G, 1 + I_K)$, equipped with the operation $\times$ induced by that on $\operatorname{Hom}^+(G, 1 + I_K)$ (same exact procedure as in Lemma 3.5.3). We thus get the following

Corollary 3.5.19. *We have* $\operatorname{Hom}(G, 1 + I_K) \simeq ((1 + I_K)^{d+}, \times)$. □

Applying directly Lemmas 3.5.15 and 3.5.18 and Corollary 3.5.19 we obtain the following refinement of Theorem 3.4.24.

Theorem 3.5.20. *Let k be a real closed field, $G = (\mathbb{Q}^d, <_{\text{lex}})$ and $K \subseteq k((G))$ a henselian Hahn field satisfying the canonical first and second lifting property. Then*

$$v\text{-}\operatorname{Aut}^+_{(k)} K \simeq (((1 + I_K)^{d+}, \times) \rtimes (k^\times)^d) \rtimes (\operatorname{Aut} k \times \operatorname{UPT}_d(\mathbb{Q}))$$
$$v\text{-}\operatorname{Aut}_k K \simeq (((1 + I_K)^{d+}, \times) \rtimes (k^\times)^d) \rtimes \operatorname{UPT}_d(\mathbb{Q}).$$

□

In the next section we analyse in further detail a special case for $G = \mathbb{Q}$ (i.e., $d = 1$), namely the field $\mathbb{P}$ of Puiseux series.

3.5.5 Puiseux series

Let k be a real closed field and let $\mathbb{P}$ be the field of Puiseux series in the indeterminate t over k. These are power series with coefficients in the field k and exponents in $\mathbb{Q}$ with the restriction that all the exponents of a given power series have a common denominator. A general Puiseux series has the form:

$$a = \sum_{n=m}^{\infty} a_n t^{\frac{n}{n_a}} \tag{3.60}$$

where $m \in \mathbb{Z}$ and $n_a \in \mathbb{Z}^{>0}$ is a positive integer depending on a. The field $\mathbb{P}$ is a subfield of the Hahn field $k((\mathbb{Q}))$ (Example 3.3.45). Therefore it has value group $(\mathbb{Q}, +, <)$. By Lemma 3.3.46 and Corollary 3.3.48 the field $\mathbb{P}$ has the canonical first and second lifting property. Moreover, like in the case of Laurent series, all the valuation preserving automorphisms of $\mathbb{P}$ are strongly additive:

Proposition 3.5.21. *We have* $v\text{-}\operatorname{Aut}\mathbb{P} = v\text{-}\operatorname{Aut}^{+}\mathbb{P}$.

Proof. Let $\sigma \in v\text{-}\operatorname{Aut}_{(k)}\mathbb{P}$. We showed earlier (Proposition 3.4.10) that external automorphisms are always strongly additive. So let us assume σ to be internal. By Proposition 3.3.53 we have $\sigma \in v\text{-}\operatorname{Aut}_k\mathbb{P}$. Since σ is internal, then for all $a \in \mathbb{P}$ we have $v(\sigma(a)) = v(a)$.

Let $\mathcal{F} = \{a_{(i)} : i \in I\}$ be a summable family such that $\sum_{i\in I} a_{(i)} \in \mathbb{P}$. Then there exists $m \in \mathbb{Z}^{>0}$ such that $\operatorname{Supp}\mathcal{F}$ is a well ordered subset of $\{\frac{n}{m} : n \in \mathbb{Z}\}$. In particular, for all $i \in I$ there is $n_i \in \mathbb{Z}$ such that $v(a_{(i)}) = \frac{n_i}{m}$. For all $n \in \mathbb{Z}$ there are only finitely many $i \in I$ such that $v(a_{(i)}) = v(\sigma(a_{(i)})) \leq \frac{n}{m}$. Now let $q \in \bigcup \operatorname{supp}\sigma(a_{(i)})$ and let $n \in \mathbb{N}$ be smallest such that $\frac{n}{m} \geq q$. Then there are finitely many $i \in I$ such that $v(\sigma(a_{(i)})) \leq q$ and, in particular such that $q \in \operatorname{supp}\sigma(a_{(i)})$. For every such i the set $\{r \in \operatorname{supp}\sigma(a_{(i)}) : r \leq q\}$ is finite, because $\sigma(a_{(i)}) \in \mathbb{P}$ and thus $\operatorname{supp}\sigma(a_{(i)})$ is cofinal in $\mathbb{Q}$. Therefore, the set $\{x \in \bigcup \operatorname{supp}\sigma(a_{(i)}) : x \leq q\}$ is finite, for all q, which implies that $\bigcup \operatorname{supp}\sigma(a_{(i)})$ is well ordered. This completes the proof that the family $\{\sigma(a_{(i)}) : i \in I\}$ is summable.

The proof is now similar to that of Lemma 3.5.9. For all $n \in \mathbb{N}$ let $T_n = \{i \in I : v(a_{(i)}) \leq \frac{n}{m}\}$. We have

$$\begin{aligned} v\left(\sigma\left(\sum a_{(i)}\right) - \sum \sigma(a_{(i)})\right) &= v\left(\sigma\left(\sum_{i\in T_n} a_{(i)} + \sum_{i\in I\setminus T_n} a_{(i)}\right) - \sum_{i\in T_n}\sigma(a_{(i)}) - \sum_{i\in I\setminus T_n}\sigma(a_{(i)})\right) \\ &= v\left(\sum_{i\in T_n}\sigma(a_{(i)}) + \sigma\left(\sum_{i\in I\setminus T_n} a_{(i)}\right) - \sum_{i\in T_n}\sigma(a_{(i)}) - \sum_{i\in I\setminus T_n}\sigma(a_{(i)})\right) \\ &= v\left(\sigma\left(\sum_{i\in I\setminus T_n} a_{(i)}\right) - \sum_{i\in I\setminus T_n}\sigma(a_{(i)})\right) > \frac{n}{m}. \end{aligned}$$

Thus $v\left(\sigma\left(\sum a_{(i)}\right) - \sum \sigma(a_{(i)})\right) = \infty$ and so $\sigma\left(\sum a_{(i)}\right) = \sum \sigma(a_{(i)})$. □

The field $\mathbb{P}$ is henselian [Kuh00a, Lemma 10.1], so Theorem 3.5.20 applies. Combining this with Proposition 3.5.21 we get

Theorem 3.5.22. *Let k be a real closed field. Then*

$$v\text{-}\operatorname{Aut}_{(k)} \mathbb{P} \simeq \left((1 + I_{\mathbb{P}}, \times_s) \rtimes k^{\times}\right) \rtimes (\operatorname{Aut} k \times (\mathbb{Q}^{>0}, \cdot)) \tag{3.61}$$
$$v\text{-}\operatorname{Aut}_{k} \mathbb{P} \simeq \left((1 + I_{\mathbb{P}}, \times_s) \rtimes k^{\times}\right) \rtimes (\mathbb{Q}^{>0}, \cdot) \tag{3.62}$$

□

The case where k is an algebraically closed field of characteristic 0 was treated by Deschamps [Des05, Théorème 10]. Under this assumption, he proves that $1\text{-}\operatorname{Aut}_k \mathbb{P}$ and $\mathbb{Q}\text{-}\operatorname{Exp} \mathbb{P}$ can be described as $1\text{-}\operatorname{Aut}_k \mathbb{P} \simeq \varinjlim (1 + I_{\mathbb{P}})$ and $\mathbb{Q}\text{-}\operatorname{Exp} k \simeq \varprojlim k^{\times}$, where the limits are taken over the directed system given by the positive integers with divisibility. The second isomorphisms is given by

$$\begin{array}{ccc} \mathbb{Q}\text{-}\operatorname{Exp} k & \to & \varprojlim k^{\times} \subseteq (k^{\times})^n \\ \sigma & \mapsto & \left(t^{\frac{-1}{n}} \sigma\left(t^{\frac{1}{n}}\right)\right)_n \in (k^{\times})^n \end{array}$$

recalling that σ is uniquely determined by the $\sigma\left(t^{\frac{1}{n}}\right)$, $n \in \mathbb{N}^{>0}$. The isomorphism $1\text{-}\operatorname{Aut}_k \mathbb{P} \simeq \varinjlim (1 + I_{\mathbb{P}})$ is obtained by taking the direct limit of the isomorphisms

$$\begin{array}{cccc} f_n\colon & (1\text{-}\operatorname{Aut}_k \mathbb{P})_n & \to & (1 + I_{\mathbb{P}}, \cdot) \\ & \sigma & \mapsto & \dfrac{\sigma\left(t^{\frac{1}{n}}\right)}{t^{\frac{1}{n}}} \end{array}$$

where $(1\text{-}\operatorname{Aut}_k \mathbb{P})_n = \left\{\sigma|_{k\left(\left(t^{\frac{1}{n}}\right)\right)} : \sigma \in 1\text{-}\operatorname{Aut}_k \mathbb{P}\right\}$.

Appendices

Appendix A

k-Hulls

The content of this Appendix comes from [KKS21], a joint work with L. S. Krapp and S. Kuhlmann. It stems from the notion of Rayner field, discussed in Section 3.3.6 and introduced by Rayner in [Ray68]. The related notion of Rayner group was considered in Section 2.2.4.

Let k be a field, G an additive ordered abelian group and $\mathbb{K} = k((G))$ the maximal Hahn field over k with exponents in G. Moreover, let $\mathcal{F}$ be a family of well-ordered subsets of G. Throughout this appendix we will adopt some conventions. Let $A, B \subseteq G$ and $g \in G$.

- $\mathcal{W}(A)$ will denote the family of well-ordered subsets of A.
- $\langle A \rangle$ will denote the subgroup of G generated by A.
- $A \oplus B := \{a + b \mid a \in A, b \in B\}$.
- $A + g := \{a + g \mid a \in A\}$, the translation of A by g.
- $\bigoplus_{n\in\mathbb{N}} A := \{\sum_{i=1}^{n} a_i \mid n \in \mathbb{N}, a_1, \ldots, a_n \in A\}$, the set of finite sums of elements of A. By convention, $\bigoplus_{n\in\mathbb{N}} \emptyset := \{0\}$.

A.1 Conditions on $\mathcal{F}$

Consider the following conditions on $\mathcal{F}$. Because there are many of them, making it hard to remember what each condition says, we use symbols that will evoke the meaning.

Definition A.1.1. **($C_{\emptyset}$)** $\mathcal{F} \neq \emptyset$;

($C_{\{0\}}$) $\{0\} \in \mathcal{F}$;

($C_{\{g\}}$) $\{g\} \in \mathcal{F}$ for all $g \in G$;

($C_{\subseteq}$) $A \in \mathcal{F}$ and $B \subseteq A$ implies $B \in \mathcal{F}$;

(C_{init}) if $A \in \mathcal{F}$ and B is an *initial* segment of A, then $B \in \mathcal{F}$;

($C_{\cup}$) $A, B \in \mathcal{F}$ implies $A \cup B \in \mathcal{F}$;

($C_{\oplus}$) $A, B \in \mathcal{F}$ implies $A \oplus B \in \mathcal{F}$;

($C_{\geq 0}$) $A \in \mathcal{F}$ and $A \subseteq G^{\geq 0}$ implies $\bigoplus_{n \in \mathbb{N}} A \in \mathcal{F}$;

($C_{\{-g\}}$) if $g \in G$ such that $\{g\} \in \mathcal{F}$, then $\{-g\} \in \mathcal{F}$.

(C_{gen}) $\langle \bigcup_{A \in \mathcal{F}} A \rangle = G$, i.e., the union of the members of $\mathcal{F}$ *generates* G;

(C_{trans}) $A \in \mathcal{F}$ and $g \in G$ implies $A + g \in \mathcal{F}$, i.e., $\mathcal{F}$ is closed under *translations* by elements of G.

Remark A.1.2. (i) Some of the conditions above are the ones that already appeared on Page 90: (C_{gen}) = (RF2); ($C_{\cup}$) = (RF3); ($C_{\subseteq}$) = (RF4); (C_{trans}) = (RF5); ($C_{\geq 0}$) = (RF6).

(ii) It is apparent that certain conditions in Definition A.1.1 imply others. For instance, ($C_{\{g\}}$) implies ($C_{\{0\}}$), ($C_{\{0\}}$) implies ($C_{\emptyset}$), and ($C_{\subseteq}$) implies (C_{init}).

Now we give the central definition of this appendix.

Definition A.1.3 (k-Hull)**.** We call the set

$$k((\mathcal{F})) = \{a \in \mathbb{K} \mid \operatorname{supp}(a) \in \mathcal{F}\} \subseteq \mathbb{K}$$

the *k-hull of $\mathcal{F}$* in $\mathbb{K}$.

We have already encountered some k-hulls in the prequel.

Examples A.1.4. The valuation ring of $\mathbb{K}$ is given by $R_{\mathbb{K}} = k\big((\mathcal{W}(G^{\geq 0}))\big)$; Its maximal ideal by $I_{\mathbb{K}} = k\big((\mathcal{W}(G^{>0}))\big)$. □

Notation A.1.5. *Whenever the family $\mathcal{F}$ is of the form $\mathcal{W}(S)$ for some set $S \subseteq G$, we write $k((S))$ instead of $k((\mathcal{F}))$.*

Notation *A*.1.5 also allows us to use the standard notation $k((G))$ for the maximal Hahn field $\mathbb{K}$ as well as $k((G^{\geq 0}))$ and $k((G^{>0}))$ respectively for the valuation ring and its maximal ideal. We lastly introduce the notions of restriction and truncation closure for k-hulls.

Definition A.1.6. The k-hull $k((\mathcal{F}))$ of $\mathcal{F}$ is called *restriction closed* if $\mathcal{F}$ satisfies ($C_{\subseteq}$). It is called *truncation closed* if $\mathcal{F}$ satisfies (C_{init}).

A.2 Algebraic properties

We start by summarising the sufficient conditions on $\mathcal{F}$ given in [Ray68] in order to ensure that $k((\mathcal{F}))$ has certain algebraic properties as the following theorem (cf. [Ray68, page 147] or Theorem 3.3.43).

Theorem A.2.1. 1. *If $\mathcal{F}$ satisfies ($C_{\emptyset}$), ($C_{\subseteq}$) and ($C_{\cup}$), then $k((F))$ is a subgroup of* $(\mathbb{K}, +)$.

2. *If $\mathcal{F}$ satisfies ($C_{\emptyset}$), ($C_{\subseteq}$), ($C_{\cup}$), ($C_{\oplus}$) and (C_{trans}), then $k((F))$ is a subring (with identity) of* $\mathbb{K}$.

3. *If $\mathcal{F}$ satisfies ($C_{\emptyset}$), ($C_{\subseteq}$), ($C_{\cup}$), ($C_{\geq 0}$), (C_{gen}) and (C_{trans}), then $k((F))$ is a subfield of* $\mathbb{K}$. □

Theorem *A*.2.1 (3) gives rise to the following definition.

Remark A.2.2. By Remark A.1.2, $\mathcal{F}$ is a Rayner field family (Definition 3.3.42) in G if it satisfies conditions ($C_{\emptyset}$), ($C_{\subseteq}$), ($C_{\cup}$), ($C_{\geq 0}$), (C_{gen}) and (C_{trans}). If $\mathcal{F}$ is a Rayner field family in G, then we call the field $k((F))$ a *Rayner field* (Definition 3.3.44).

Rayner is merely interested in sufficient conditions on $\mathcal{F}$ in order to ensure that $k((\mathcal{F}))$ exhibits certain algebraic properties. Therefore some of the conditions he poses may not be necessary. The aim of this work is to carefully analyse further the relations between the conditions given in Definition *A*.1.1 and the properties of $k((\mathcal{F}))$ as an algebraic substructure of $\mathbb{K}$.

In Propositions 2.2.20 and 2.2.22 we showed that $k((\mathcal{F}))$ is a subgroup of $(\mathbb{K}, +)$ if $\mathcal{F}$ satisfies ($C_{\emptyset}$), ($C_{\subseteq}$) and ($C_{\cup}$), and a Hahn group if, moreover, $\mathcal{F}$ satisfies ($C_{\{g\}}$). In [KKS21, Proposition 3.4] we prove that, moreover, the converse holds for $k \neq \mathbb{F}_2$.

We now also consider multiplication on $k((\mathcal{F}))$.

Lemma A.2.3. *If $\mathcal{F}$ satisfies (C$_\emptyset$),(C$_{\{0\}}$), (C$_\subseteq$), (C$_\cup$) and (C$_\oplus$), then $k((F))$ is a subring of $\mathbb{K}$. If, moreover,* $\operatorname{char} k = 0$ *then the converse holds too.*

Proof. By Propositions 2.2.20, $k((\mathcal{F}))$ is an additive subgroup of $(\mathbb{K}, +)$. Now let $a, b \in k((\mathcal{F}))$. We set $A = \operatorname{supp}(a)$, $B = \operatorname{supp}(b)$ and let $c = ab \in \mathbb{K}$. Then by definition of the product, we have $\operatorname{supp}(ab) \subseteq A \oplus B \in \mathcal{F}$. Hence, by (C$_\oplus$) we obtain $\operatorname{supp}(ab) \in \mathcal{F}$ and thus $ab \in k((\mathcal{F}))$. Condition (C$_{\{0\}}$) assures that $1 \in k((\mathcal{F}))$.

Now assume $\operatorname{char} k = 0$ and suppose that $k((\mathcal{F}))$ is a subring of $\mathbb{K}$. By [KKS21, Proposition 3.4], it remains to verify (C$_\oplus$). Let $A, B \in \mathcal{F}$ and set $a = \sum_{g \in A} t^g$ and $b = \sum_{g \in B} t^g$. Then since $\operatorname{char}(k) = 0$, we obtain that $A \cup B = \operatorname{supp}(ab) \in \mathcal{F}$. □

The condition $\operatorname{char}(k) = 0$ in Lemma A.2.3 ensures that in its proof the sums of the coefficients of the power series representing a and b do not cancel in the product ab, whence $\operatorname{supp}(ab) = \operatorname{supp}(a) \oplus \operatorname{supp}(b)$. This can also be ensured by a condition on the cardinality of $|k|$ as the following result shows.

Proposition A.2.4. *Suppose that $|k| > |G|$. Then $k((\mathcal{F}))$ is a subring (possibly without identity) of $\mathbb{K}$ if and only if $\mathcal{F}$ satisfies conditions (C$_\emptyset$), (C$_\subseteq$), (C$_\cup$) and (C$_\oplus$).*

Proof. See [KKS21, Proposition 3.9]. □

The following results are the most interesting, for this thesis' context. We relate conditions on $\mathcal{F}$ to Hahn fields as well as Rayner fields.

Lemma A.2.5. *Suppose that $\mathcal{F}$ satisfies conditions (C$_{\{0\}}$), (C$_\subseteq$), (C$_\cup$), (C$_\oplus$), (C$_{\geq 0}$) and (C$_{\{-g\}}$). Then $k((\mathcal{F}))$ is a field.*

Proof. Lemma A.2.3 implies that $k((\mathcal{F}))$ is a ring. Let $b \in k((\mathcal{F})) \setminus \{0\}$ be arbitrary and let $h = \min \operatorname{supp}(b)$. Then by (C$_\subseteq$) and (C$_{\{-g\}}$), we have $t^{-h} \in k((\mathcal{F}))$ and thus obtain

$$b_h^{-1} t^{-h} b = 1 + \sum_{g \in G^{>0}} b_h^{-1} b_g t^{g-h} \in k((\mathcal{F}))\,.$$

Now set $a = -\sum_{g \in G^{>0}} b_h^{-1} b_g t^{g-h}$ and let $A = \operatorname{supp}(a) \subseteq G^{>0} \in \mathcal{F}$. Then $(1 - a)^{-1} = \sum_{i=0}^{\infty} a^i$ (cf. Neumann's Lemma [Neu49, page 211]). Hence, $\operatorname{supp}(1 - a)^{-1} \subseteq \bigoplus_{n \in \mathbb{N}} A$ and, by (C$_{\geq 0}$) and (C$_\subseteq$), it lies in $\mathcal{F}$. This implies $b_h t^h b^{-1} = \left(b_h^{-1} t^{-h} b\right)^{-1} \in k((\mathcal{F}))$, whence $b^{-1} \in k((\mathcal{F}))$, as required. □

Proposition A.2.6. *Suppose that* $\mathrm{char}(k) = 0$*. Then* $k((\mathcal{F}))$ *is a field if and only if* $\mathcal{F}$ *satisfies conditions* ($C_{\{0\}}$), ($C_{\subseteq}$), ($C_{\cup}$), ($C_{\oplus}$), ($C_{\geq 0}$) *and* ($C_{\{-g\}}$)*.*

Proof. By Lemma A.2.5, only the forward direction needs to be shown. Let $k((\mathcal{F}))$ be a field. Then Lemma A.2.3 implies that $\mathcal{F}$ satisfies conditions ($C_{\{0\}}$), ($C_{\subseteq}$), ($C_{\cup}$) and ($C_{\oplus}$). To prove condition ($C_{\geq 0}$) let $A \in \mathcal{F}$ be such that $A \subseteq G^{\geq 0}$ and let $a = \sum_{g \in A^{>0}} t^g$. Then $\mathrm{supp}(1-a) = A$. By Neumann's Lemma, $(1-a)^{-1} = \sum_{i=0}^{\infty} a^i$. As $\mathrm{char}(k) \neq 0$, the support of $(1-a)^{-1}$ is $\bigoplus_{n \in \mathbb{N}} A$. Since $k((\mathcal{F}))$ is a field, $(1-a)^{-1} \in k((\mathcal{F}))$, establishing ($C_{\geq 0}$). Finally, ($C_{\{-g\}}$) follows easily, as for any monomial $t^g \in k((\mathcal{F}))$ we already have $t^{-g} \in k((\mathcal{F}))$. □

As a corollary, we obtain necessary and sufficient conditions (in the case $\mathrm{char}(k) = 0$) in order that $k((\mathcal{F}))$ is a Hahn field.

Corollary A.2.7. *Suppose that* $\mathrm{char}(k) = 0$*. Then* $k((\mathcal{F}))$ *is a Hahn field if and only if* $\mathcal{F}$ *satisfies conditions* ($C_{\{g\}}$), ($C_{\subseteq}$), ($C_{\cup}$), ($C_{\oplus}$), ($C_{\geq 0}$)*.*

Proof. If $k((F))$ is a Hahn field, then $\mathcal{F}$ clearly satisfies ($C_{\{g\}}$). The other properties follow from Proposition A.2.6. For the converse, note that ($C_{\{g\}}$) implies ($C_{\{0\}}$) and ($C_{\{-g\}}$). The rest follows from Proposition A.2.6. □

Finally, we show that $k((\mathcal{F}))$ is a Hahn field if and only if it is a Rayner field. By Corollary A.2.7, it suffices to show that $\mathcal{F}$ is a Rayner field family if and only if it satisfies ($C_{\{g\}}$), ($C_{\subseteq}$), ($C_{\cup}$), ($C_{\oplus}$), ($C_{\geq 0}$). We first prove that if G is non-trivial, then condition (C_{gen}) in 90 3.3.44 can be removed.

Lemma A.2.8. *Suppose that* $G \neq \{0\}$ *and that* $\mathcal{F}$ *satisfies conditions* ($C_{\subseteq}$) *and* (C_{trans})*. Then*

$$(C_{\emptyset}) \Leftrightarrow (C_{\{0\}}) \Leftrightarrow (C_{\{g\}}) \Leftrightarrow (C_{gen}).$$

Proof. ($C_{\emptyset}$) $\Rightarrow$ ($C_{\{0\}}$): Let $\mathcal{F} \neq \emptyset$ and let $A \in \mathcal{F}$. If $A \neq \emptyset$, then for any $g \in A$, we obtain by ($C_{\subseteq}$) and (C_{trans}) that $\{0\} = \{g\} - g \in \mathcal{F}$. ($C_{\{0\}}$) $\Rightarrow$ ($C_{\{g\}}$) : this follows immediately from (C_{trans}). ($C_{\{g\}}$) $\Rightarrow$ (C_{gen}) and (C_{gen}) $\Rightarrow$ ($C_{\emptyset}$) are obvious. Note that for the latter we need that $G \neq \{0\}$. □

Theorem A.2.9. *Let* $\mathrm{char}(k) = 0$*. Then* $k((\mathcal{F}))$ *is a Hahn field if and only if it is a Rayner field.*

Proof. Suppose that $k((\mathcal{F}))$ is a Rayner field, that is, $\mathcal{F}$ satisfies ($C_{\emptyset}$), ($C_{\subseteq}$), ($C_{\cup}$), ($C_{\geq 0}$), (C_{gen}) and (C_{trans}). By Corollary A.2.7, it remains to verify ($C_{\{g\}}$) and ($C_{\oplus}$). If $G = \{0\}$, then we have $\mathcal{F} = \{\emptyset, \{0\}\}$, which trivially satisfies ($C_{\{g\}}$) and ($C_{\oplus}$).

If $G \neq \{0\}$, then Lemma A.2.8 shows that $\mathcal{F}$ satisfies ($C_{\{g\}}$). We thus only have to show ($C_{\oplus}$). Let $A, B \in \mathcal{F}$ be non-empty. Let $a = \min A$ and $b = \min B$. Then by (C_{trans}), we have $A - a, B - b \in \mathcal{F}$. Note that $A - a, B - b \in G^{\geq 0}$. Hence, by ($C_{\cup}$) and ($C_{\geq 0}$), we obtain

$$\bigoplus_{n \in \mathbb{N}} ((A - a) \cup (B - b)) \in \mathcal{F}.$$

In particular, $(A - a) \oplus (B - b) \in \mathcal{F}$. By ($C_{trans}$) we obtain $A \oplus B = ((A - a) \oplus (B - b)) + (a + b) \in \mathcal{F}$.

Vice versa, suppose that $k((\mathcal{F}))$ is a Hahn field, that is, $\mathcal{F}$ satisfies ($C_{\{g\}}$), ($C_{\subseteq}$), ($C_{\cup}$), ($C_{\oplus}$), ($C_{\geq 0}$). We need to show that ($C_{\emptyset}$), (C_{gen}) and (C_{trans}) hold. Again, if $G = \{0\}$, then $\mathcal{F} = \{\emptyset, \{0\}\}$ and there is nothing to prove. Otherwise, by Lemma A.2.8 it suffices to show (C_{trans}). Let $A \in \mathcal{F}$ and let $g \in G$. Then by ($C_{\{g\}}$), we have $\{g\} \in \mathcal{F}$. Hence, by ($C_{\oplus}$), we obtain $A + g = A \oplus \{g\} \in \mathcal{F}$, as required. □

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