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Johannes Emmerling

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WAVE EQUATIONS WITH TIME-DEPENDENT COEFFICIENTS

JOHANNES EMMERLING

ABSTRACT. We study the damped wave equation with time-dependent coefficients $u_{tt}(t,x) - a(t)^2 \Delta u(t,x) - \frac{b'}{b}(t) u_t(t,x) = 0$ in $\mathbb{R}^n$ and prove energy estimates for a new class of coefficients.

1. Introduction

In this paper we prove energy estimates for the damped wave equation with time-dependent coefficients, i.e.

$$u_{tt}(t,x) - a(t)^2 \Delta u(t,x) - \tfrac{b'}{b}(t) u_t(t,x) = 0 \tag{1.1}$$

for $t \in [0,\infty)$ and $x \in \mathbb{R}^n$, where $a \in C([0,\infty),\mathbb{R})$ and $b \in C^1([0,\infty),\mathbb{R})$ are strictly positive functions. This is a simple transformation of

$$u_{tt}(t,x) - a(t)^2 \Delta u(t,x) + \beta(t) u_t(t,x) = 0$$

with $b(t) := \exp(-\int_0^t \beta(\tau)\, d\tau)$.
If u is a solution of (1.1) such that $u_t(t,\cdot) \in L^2(\mathbb{R}^n)$ and $\nabla u(t,\cdot) \in \big(L^2(\mathbb{R}^n)\big)^n$ for every $t \geq 0$, then we call

$$E_u(t) := \int_{\mathbb{R}^n} u_t(t,x)^2 + a(t)^2 |\nabla u(t,x)|^2 \, dx \tag{1.2}$$

the *energy* of u at time $t \geq 0$. Here, $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$.
We deal with solutions u of (1.1) such that

$$u \in C^0\big([0,\infty), H^2(\mathbb{R}^n)\big) \cap C^1\big([0,\infty), H^1(\mathbb{R}^n)\big) \cap C^2\big([0,\infty), L^2(\mathbb{R}^n)\big).$$

These u shall be called *admissible solutions* of (1.1).
Suppose a and $-\frac{b'}{b}$ are constant and positive, then

$$E_u(t) \leq E_u(0)$$

for each admissible solution u of (1.1). As can be seen from the Fourier representation, the estimate can only be improved by restrictions on the solutions. Whereas in this L^2-L^2-estimate the "damping term" $-\frac{b'}{b}$ seems to have no effect, its influence can be clearly seen in the L^p-L^q-estimate:
If $\frac{b'}{b} = 0$, then for every $2 \leq q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $N_p > n\big(1 - \frac{2}{q}\big)$ there exists a constant $C > 0$ such that

$$\big\| \big(u_t(t,\cdot), \nabla u(t,\cdot)\big) \big\|_{L^q} \leq C(1+t)^{-\frac{n-1}{2}\left(1-\frac{2}{q}\right)} \big\| \big(u_t(0,\cdot), \nabla u(0,\cdot)\big) \big\|_{H^{N_p,p}}$$

for every $t \geq 0$, see [7]. But if $-\frac{b'}{b} = 1$, then for every $2 \leq q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $N_p > n\big(1 - \frac{2}{q}\big)$ there exists a constant $C > 0$ such that

$$\big\| \big(u_t(t,\cdot), \nabla u(t,\cdot)\big) \big\|_{L^q} \leq C(1+t)^{-\frac{n}{2}\left(1-\frac{2}{q}\right)} \big\| \big(u_t(0,\cdot), \nabla u(0,\cdot)\big) \big\|_{H^{N_p,p}}$$

for every $t \geq 0$, c.f. [6]. Here, the damping term improves the rate of decay, see also [2]. First results for wave equations with time-dependent damping term have been proved by Matsumura [3] and Uesaka [13]: For every $\mu \geq 0$ there exists a constant $C > 0$ such that for each admissible solution u of

$$u_{tt}(t,x) + \Delta u(t,x) + \mu(1+t)^{-1}u_t(t,x) = 0,$$

whose initial values $u(0,\cdot)$ and $u_t(0,\cdot)$ have compact support, the following inequality holds:

$$E_u(t) \leq C(1+t)^{-\min\{\mu,2\}}\big(E_u(0) + \|u(0,\cdot)\|_{L^2}^2\big).$$

Note that both energy and damping term decay with time and that the estimate needs more than the initial energy to give an upper bound. Recent works on the wave equation with time-dependent damping term are due to Reissig ([9],[10], [11], [12]) and Wirth ([12],[15]), for a good survey see [9]. In [15], the result above with $\mu \in [0,1)$ is generalized to a broader class of coefficients. This result will be discussed in section 5, subsequent to Corollary 5.2. In that work, estimates for larger, so called "effective" damping terms can be found, which we will not consider here.
A time-dependent propagation speed a can cause many difficulties. This is shown in [11] by means of the equation

$$u_{tt}(t,x) - (2+\sin t)\Delta u(t,x) = 0.$$

For an admissible solution u of this equation it follows immediately that

$$E_u(t) \leq Ce^{ct}E_u(0)$$

with constants $c, C > 0$, see for example the following Lemma 2.1. Reissig and Yagdjian show that this energy estimate cannot be substantially improved, even if L^p-L^q estimates are considered. Thus oscillating coefficients have a deteriorating effect on energy estimates. In [10], this effect is examined more closely for propagation speeds of the form

$$a(t)^2 = 2 + \sin\big((\log(t+30))^\alpha\big)$$

where $\alpha > 0$. Compare example 4 in section 6, where a function a is given, which just does not damage the energy estimate. Equations with two oscillating coefficients can be dealt with, too, see [8]. There, the equation

$$u_{tt}(t,x) - \varphi(t)^2\omega(t)^2\Delta u(t,x) + \alpha\varphi(t)\omega(t)u_t(t,x) = 0$$

is considered, where α is a positive constant and φ, ω are smooth, strictly positive functions, φ increasing and ω oscillating.
In Theorem (5.1) *of this paper we extend the theorem from* [15] *mentioned above, showing that*

$$E_u(t) \leq C\,a(t)b(t)\left(E_u(0) + \|u(0,\cdot)\|_{L^2}^2\right)$$

under considerably weaker assumptions on a and b. In particular, the case $\mu \in [1,\infty)$ *is included and oscillating damping terms are possible, even with decaying energy, see Example 2 in section 6. However, Theorem* (5.1) *does not cover* $a \in L^1([0,\infty))$. *This is different for Theorem 5.3, which gives energy estimates for those functions under a smallness condition for* $|a-b|$. *Here, the estimate*

$$E_u(t) \leq C\,a(t)b(t)\,E_u(0)$$

is possible, where only the initial energy is needed. Again, oscillating coefficients are possible, see Example 5 in section 6.

The methods developed for the proof of these estimates are also applicable to the viscoelastic equation

$$u_{tt}(t,x) - b(t)\Delta u_t(t,x) - a(t)\Delta u(t,x) = 0, \quad t \in [0,\infty),\, x \in \mathbb{R}^n,$$

c.f. [1]. This will be subject of a forthcoming paper.
Our paper is organized as follows: In Section 2 we will prove elementary estimates for the energy which will serve both as motivation and as benchmark for the final estimates. In Section 3 we will derive approximate solutions for equation (1.1). These approximate solutions will be turned into Fourier space estimates in Section 4 and glued together to form energy estimates in Section 5. We conclude with examples in Section 6.

2. Preliminaries

Now we turn to estimates which can be obtained directly form equation (1.1).

Lemma 2.1. *Let $a, b \in C^1([0,\infty),\mathbb{R})$ be strictly positive functions.*
(a) For $t \geq 0$ let $M(t) := \max\left\{\frac{a'}{a}(t), \frac{b'}{b}(t)\right\}$ and $m(t) := \min\left\{\frac{a'}{a}(t), \frac{b'}{b}(t)\right\}$. Then

$$\exp\left(2\int_0^t m(\tau)\,d\tau\right)E_u(0) \leq E_u(t) \leq \exp\left(2\int_0^t M(\tau)\,d\tau\right)E_u(0) \tag{2.1}$$

for each admissible solution u of (1.1) and every $t \geq 0$.
(b) Suppose $\frac{a'}{a} - \frac{b'}{b}$ is absolutely integrable on $[0,\infty)$, then there exists a constant $C > 0$ such that

$$E_u(t) \leq Ca(t)b(t)E_u(0). \tag{2.2}$$

for each admissible solution u of (1.1) and every $t \geq 0$.

Proof. Suppose u is an admissible solution of (1.1). Partial integration and substituting the differential equation leads to

$$E_u'(t) = 2\int_{\mathbb{R}^n} \tfrac{b'}{b}(t)u_t(t,x)^2 + \tfrac{a'}{a}(t)\ a(t)^2|\nabla u(t,x)|^2\,dx.$$

(a) Then we have

$$2m(t)E_u(t) \leq E_u'(t) \leq 2M(t)E_u(t),$$

for every $t \geq 0$, and with a suitable differential inequality, e. g. [14], Theorem XI, we obtain (a).
(b) We write

$$E_u'(t) \leq \left(\tfrac{a'}{a}(t) + \tfrac{b'}{b}(t)\right)E_u(t) + \left|\tfrac{a'}{a}(t) - \tfrac{b'}{b}(t)\right|E_u(t).$$

Now we can apply the same differential inequality or Gronwall's Lemma to obtain

$$\begin{aligned} E_u(t) &\leq \exp\left(\int_0^t \tfrac{a'}{a}(\tau) + \tfrac{b'}{b}(\tau) + \left|\tfrac{a'}{a}(\tau) - \tfrac{b'}{b}(\tau)\right|\,d\tau\right)E_u(0) \\ &\leq Ca(t)b(t)E_u(0). \end{aligned}$$

□

From the estimates of Lemma 2.1 we can draw several conclusions:
First, if $f \in C^1([0,\infty),\mathbb{R})$ is a strictly positive function, then there always exists an equation of the form (1.1), where the energy of each admissible solution behaves like f. In particular, exponential or polynomial stability can be obtained by choosing suitable coefficients.

Suppose $\frac{a'}{a}(t) \le \frac{b'}{b}(t)$ or $\frac{b'}{b}(t) \le \frac{a'}{a}(t)$ for every $t \ge 0$, then by Lemma 2.1 (a) the inequalities

$$a(t)^2 E_u(0) \le E_u(t) \le b(t)^2 E_u(0) \quad \text{or}$$
$$b(t)^2 E_u(0) \le E_u(t) \le a(t)^2 E_u(0)$$

hold for each admissible solution u of (1.1) and for every $t \ge 0$.
If $\frac{a'}{a} - \frac{b'}{b}$ is monotone and if there are contants $c, C > 0$ such that

$$ca(t) \le b(t) \le Ca(t)$$

for every $t \ge 0$, then there exists a constant $C' > 0$ such that

$$E_u(t) \le C' a(t) b(t) E_u(0)$$

for each admissible solution u of (1.1) and for every $t \ge 0$. Suppose $a = b$ or $\frac{a'}{a} - \frac{b'}{b} = 0$, then $M = m$ and we can determine the energy exactly:

$$E_u(t) = \frac{a(t)b(t)}{a(0)b(0)} E_u(0).$$

Hence we can understand Lemma 2.1 as a stability result: As long as a and b are close together, for example if $\frac{a'}{a} - \frac{b'}{b}$ is absolutely integrable or $\frac{a'}{a} - \frac{b'}{b}$ is monotone and there exist constants $c, C > 0$ such that

$$ca(t) \le b(t) \le Ca(t),$$

then the energy estimate of the case $a = b$ is preserved. The question is how far this region of stability can be expanded. At least a little bit, as the example of Matsumura shows: Here we have $a = 1$ and $b(t) = (1+t)^{-\mu}$, hence there is no constant $c > 0$ such that $ca(t) \le b(t)$ for every $t \ge 0$. And since

$$\frac{a'}{a}(t) - \frac{b'}{b}(t) = \mu(1+t)^{-1}$$

the function $\frac{a'}{a} - \frac{b'}{b}$ is not absolutely integrable. Nevertheless, the energy of each admissible solution decays like $a \cdot b$.
Now we can begin with the proof of energy estimates. To this, end we transform the equation: Suppose u is an admissible solution of equation (1.1) and $\hat{u}(t, \cdot)$ is the Fourier transform of $u(t, \cdot)$ for $t \ge 0$. Then $\hat{u}(\cdot, \xi) \in C^2([0, \infty]), \mathbb{C})$ satisfies for every $\xi \in \mathbb{R}^n$ the ordinary differential equation

$$\hat{u}_{tt}(t, \xi) + a(t)^2 |\xi|^2 \hat{u}(t, \xi) - \frac{b'}{b}(t) \hat{u}_t(t, \xi) = 0. \tag{2.3}$$

In this equation, only $|\xi|$ appears as a parameter, thus the ξ-dependence of $\hat{u}(t, \xi)$ is only due to the initial values $\hat{u}(0, \xi)$ and $\hat{u}_t(0, \xi)$. Therefore, we determine a fundamental system of the equation

$$v''(t, \lambda) + a(t)^2 \lambda^2 v(t, \lambda) - \frac{b'}{b}(t) v'(t, \lambda) = 0 \tag{2.4}$$

for every $\lambda \ge 0$.

3. Approximate Solutions

In order to get good energy estimates, we should know the solutions of (2.4) as completely as possible. But even homogeneous linear differential equations of second order like equation (2.4) cannot be solved explicitly in general. Therefore we use approximate solutions of the equation, and these can be derived from the following model equations:

$$y''(t,\lambda) + \lambda^2 y(t,\lambda) = f(t,\lambda)\, y(t,\lambda), \tag{3.1}$$

$$y''(t,\lambda) = f(t,\lambda)\, y(t,\lambda), \tag{3.2}$$

where $\lambda > 0$ and $f(\cdot,\lambda)$ is a continuous function. If $f = 0$, then the equations have explicit solutions, which can serve as approximations of the general case. But when we replace the true solutions by approximations, we make an error, and this error has to be estimated.
For equations of type (3.1) this has been done by F. W. J. Olver, see [5]. The approximate solutions covered in the next theorem are also known as WKB-solutions.

Theorem 3.1 (**Olver**). *Suppose $\alpha \le \gamma \le \beta$, $\lambda > 0$ and $f(\cdot,\lambda) \in C([\alpha,\beta],\mathbb{C})$. Then the differential equation* (3.1) *has a fundamental system $y_+(\cdot,\lambda), y_-(\cdot,\lambda) \in C^2([\alpha,\beta],\mathbb{C})$ such that*

$$y_\pm(t,\lambda) = e^{\pm i\lambda t}\big(1 + r_\pm(t,\lambda)\big),$$

where

$$\big|r_\pm(t,\lambda)\big|, \big|\tfrac{1}{i\lambda} r'_\pm(t,\lambda) \pm r_\pm(t,\lambda)\big| \le \exp\bigg(\frac{F(t,\lambda)}{\lambda}\bigg) - 1,$$

and

$$F(t,\lambda) := \left| \int_\gamma^t |f(\tau,\lambda)|\, d\tau \right|. \tag{3.3}$$

Moreover, the interval $[\alpha,\beta]$ and the value of γ may be infinite provided the integral (3.3) exists.

Now we prove a lemma for equations of type (3.2). To simplify notation, we write $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty,\infty\}$ and we agree that intervals $[\alpha,\beta]$ with $\alpha,\beta \in \overline{\mathbb{R}}$ include neither ∞ nor $-\infty$. For functions $f\colon [\alpha,\infty] \to \mathbb{C}$ we mean by $f(\infty) = c$ that the limit $\lim_{t\to\infty} f(t)$ exists and is equal to c. In the same way we define $f(-\infty)$ for $f\colon (-\infty,\beta] \to \mathbb{C}$.
In order to get error bounds, we use the following variation-of-constants formula:

Lemma 3.2 (**Variation of constants**). *Let $\alpha,\beta \in \overline{\mathbb{R}}$, $\gamma \in [\alpha,\beta]$ and $f,g,h \in C([\alpha,\beta],\mathbb{C})$. Then there exists a function $x \in C^2([\alpha,\beta],\mathbb{C})$ solving*

$$x'' + fx' + gx = h$$

with initial values $x(\gamma) = x'(\gamma) = 0$. If $x_1, x_2 \in C^2([\alpha,\beta],\mathbb{C})$ is a fundamental system of solutions of the corresponding homogeneous equation, then

$$x(t) = \int_\gamma^t \frac{x_1(\tau)x_2(t) - x_1(t)x_2(\tau)}{x_1(\tau)x_2'(\tau) - x_1'(\tau)x_2(\tau)}\, h(\tau)d\tau$$

for every $t \in [\alpha,\beta]$. The interval $[\alpha,\beta]$ and the value of γ may be infinite, provided that the associated integral exists.

The next step is a Lemma of Gronwall type, which will turn the representation of the previous Lemma into error bounds. It allows for inequalities of the form

$$x(t) \leq \int_0^t k(t,\tau)\big(x(\tau)+h(\tau)\big)\,d\tau,$$

which contain an integral kernel $(t,\tau)\mapsto k(t,\tau)$ instead of a function $\tau \mapsto g(\tau)$.

Lemma 3.3. *Let $\alpha,\beta \in \overline{\mathbb{R}}$, let $k \in C\big([\alpha,\beta]^2,\mathbb{R}\big)$ be differentiable with respect to the first variable and let $x,h \in C\big([\alpha,\beta],\mathbb{R}\big)$. Suppose $k(t,\tau), \partial_1 k(t,\tau) \geq 0$ for every $\alpha \leq \tau \leq t \leq \beta$ and $x(t), h(t) \geq 0$ for every $t \in [\alpha,\beta]$. Assume also that $K(t) := \int_\alpha^t k(t,\tau)\,d\tau$ converges. If*

$$x(t) \leq \int_\alpha^t k(t,\tau)\big(x(\tau)+h(\tau)\big)\,d\tau$$

for every $t\in[\alpha,\beta]$, then

$$x(t) \leq \big(\exp(K(t))-1\big)\sup_{\tau\in[\alpha,t]} h(\tau)$$

for every $t\in[\alpha,\beta]$.

Proof. Let $[\gamma,\delta]$ be a finite subinterval of $[\alpha,\beta]$. Then

$$P\colon C([\gamma,\delta],\mathbb{R}) \longrightarrow C([\gamma,\delta],\mathbb{R}) \quad \text{defined by} \quad (Py)(t) = \int_\gamma^t k(t,\tau)y(\tau)\,d\tau$$

is a continuous, linear, positive operator with spectral radius < 1 (see [16], p. 38). Since

$$x(t) \leq (Ph)(t) + (Px)(t),$$

the Abstract Gronwall Lemma ([16], p. 81) implies

$$x(t) \leq \big((\mathrm{id}-P)^{-1}Ph\big)(t) = \sum_{n=0}^{\infty}(P^{n+1}h)(t).$$

Using $H(t) = \max_{\tau\in[\gamma,t]} h(\tau)$, we obtain

$$\begin{aligned}(Ph)(t) &= \int_\gamma^t k(t,\tau)h(\tau)\,d\tau \leq \int_\gamma^t k(t,\tau)H(t)\,d\tau \leq K(t)H(t)\\ (P^2h)(t) &\leq \int_\gamma^t k(t,\tau)K(\tau)H(\tau)\,d\tau \leq \bigg(\int_\gamma^t k(t,\tau)K(\tau)\,d\tau\bigg)H(t).\end{aligned}$$

We estimate the integral

$$I(t) := \int_\gamma^t k(t,\tau)K(\tau)\,d\tau$$

by differentiation:

$$\begin{aligned}I'(t) &= k(t,t)K(t) + \int_\gamma^t \partial_1 k(t,\tau)K(\tau)\,d\tau\\ &\leq \bigg(k(t,t) + \int_\gamma^t \partial_1 k(t,\tau)\,d\tau\bigg)K(t) = K'(t)K(t).\end{aligned}$$

This estimate is possible because $\partial_1 k(t,\tau)$ is always nonnegative and K is monotonically increasing. We conclude that

$$(P^2h)(t) \le \Big(\int_\gamma^t K'(\tau)K(\tau)\,d\tau\Big)H(t) = \tfrac{1}{2}K(t)^2H(t).$$

By induction, we get

$$(P^n h)(t) \le \tfrac{1}{n!}K(t)^n H(t),$$

hence

$$x(t) \le \big(\exp(K(t)) - 1\big)H(t)$$

for every $t \in [\gamma,\delta]$. Since $\gamma,\delta \in [\alpha,\beta]$ were arbitrary, we can extend this inequality to every $t \in [\alpha,\beta]$. □

Now we can state the approximation lemma for equations of type (3.2).

Lemma 3.4. *Let $\alpha \in \mathbb{R}$, $\beta \in \overline{\mathbb{R}}$, and $f \in C([\alpha,\beta),\mathbb{C})$. Then the differential equation*

$$y''(t) = f(t)y(t)$$

has a fundamental system $y_1, y_2 \in C^2([\alpha,\beta),\mathbb{C})$ such that

$$y_1(t) = 1 + r_1(t), \qquad y_2(t) = (t-\alpha)\big(1 + r_2(t)\big),$$

where

$$|r_1(t)|, |r_2(t)| \le \exp\big(F(t)\big) - 1,$$
$$|r_1'(t)|, |r_2'(t)| \le F'(t)\exp\big(F(t)\big)$$

and

$$F(t) := \int_\alpha^t (t-\tau)|f(\tau)|\,d\tau.$$

Proof. We can deal with both solutions at the same time. For that purpose, let $\varphi\colon [\alpha,\beta] \to \mathbb{R}$ and suppose $\varphi = 1$ or $\varphi(t) = t - \alpha$ for every $t \in [\alpha,\beta]$. Now we determine a function $r \in C^2([\alpha,\beta],\mathbb{C})$ such that

$$y(t) := \varphi(t) + r(t)$$

defines a solution of the differential equation. Such a function r satisfies the equation

$$r''(t) = y''(t) - \varphi''(t) = f(t)\big(\varphi(t) + r(t)\big). \tag{3.4}$$

By Lemma 3.2, there exists a solution $r \in C^2([\alpha,\beta],\mathbb{C})$ of this equation such that

$$|r(t)| \le \int_\alpha^t (t-\tau)|f(\tau)|\big(\varphi(\tau) + |r(\tau)|\big)\,d\tau.$$

Now we can apply Lemma 3.3, and since φ is monotonically increasing, we get

$$|r(t)| \le \varphi(t)\big(\exp\big(F(t)\big) - 1\big).$$

By integrating equation (3.4), we obtain

$$r'(t) = \int_\alpha^t f(\tau)\big(\varphi(\tau) + r(\tau)\big)\,d\tau.$$

This leads to the estimate

$$|r'(t)| \leq \int_\alpha^t |f(\tau)|\big(\varphi(\tau) + |r(\tau)|\big)\, d\tau \leq \int_\alpha^t |f(\tau)|\varphi(\tau)\exp\big(F(\tau)\big)\, d\tau$$
$$\leq \varphi(t)\exp\big(F(t)\big)\int_\alpha^t |f(\tau)|\, d\tau = \varphi(t)\exp\big(F(t)\big)F'(t).$$

Put $q(t) := \frac{r(t)}{\varphi(t)}$ for $t \geq \alpha$ and $q(\alpha) = 0$. It follows that

$$|q(t)|, |q'(t)| \leq \big(\exp\big(F(t)\big) - 1\big)$$

for $t \geq \alpha$, hence $q \in C^2([\alpha, \beta], \mathbb{C})$. Thus $y(t) = \varphi(t)\big(1 + q(t)\big)$ is a solution of the original equation. The solutions y_1 and y_2 belonging to $\varphi(t) = 1$ and $\varphi(t) = t - \alpha$ satisfy $y_1(\alpha) = 1$, $y_2(\alpha) = 0$ and $y_1'(\alpha) = 0$, $y_2'(\alpha) = 1$, thus y_1, y_2 is even a fundamental system. □

In the following we give approximate solutions of equation (2.4), sorted by validity for increasing λ. We frequently use the abbreviations

$$A(t) := \int_0^t a(\tau)\, d\tau, \qquad B(t) := \int_0^t b(\tau)\, d\tau.$$

If λ is small, then solutions of (2.4) behave like solutions of the equation

$$w''(t, \lambda) = \tfrac{b'}{b}(t)w'(t, \lambda).$$

A fundamental system is given by the functions 1 and B, which we take as a model for the following approximate solutions.

Lemma 3.5. *Let $\lambda \geq 0$, let $a \in C^0([0, \infty), \mathbb{R})$, $b \in C^1([0, \infty), \mathbb{R})$ such that $a(t) \geq 0$ and $b(t) > 0$ for every $t \in [0, \infty)$. Then equation (2.4) has a fundamental system $v_1(\cdot, \lambda), v_2(\cdot, \lambda) \in C^2([0, \infty), \mathbb{C})$ such that*

$$v_1(t, \lambda) = 1 + \rho_1(t, \lambda), \qquad v_2(t, \lambda) = B(t)\big(1 + \rho_2(t, \lambda)\big),$$

and

$$|\rho_1(t, \lambda)|, |\rho_2(t, \lambda)| \leq \exp\big(\lambda^2 R(t)\big) - 1,$$
$$|\rho_1'(t, \lambda)|, |\rho_2'(t, \lambda)| \leq \lambda^2 R'(t)\exp\big(\lambda^2 R(t)\big),$$

where

$$R(t) := \int_0^t \frac{B(t) - B(\tau)}{b(\tau)}\, a(\tau)^2 d\tau.$$

Proof. Since b is strictly positive, B is strictly increasing. Thus $s = B(t)$ or $\sigma = B(\tau)$ define a transformation of variables. Suppose $w(s, \lambda) = v(t, \lambda)$, then w satisfies the differential equation

$$w_{ss}(s, \lambda) + \frac{a(t)^2\lambda^2}{b(t)^2} w(s, \lambda) = 0.$$

Put

$$f(s, \lambda) := -\frac{a(t)^2\lambda^2}{b(t)^2}.$$

According to Lemma 3.4, the differential equation

$$w_{ss}(s, \lambda) = f(s, \lambda)w(s, \lambda)$$

has a fundamental system

$$w_1(\cdot,\lambda),\ w_2(\cdot,\lambda) \in C^2\big(B([0,\infty)),\mathbb{C}\big)$$

such that

$$w_1(s,\lambda) = 1 + r_1(s,\lambda), \qquad w_2(s,\lambda) = s\,\big(1 + r_2(s,\lambda)\big),$$

where

$$|r_1(s,\lambda)|, |r_2(s,\lambda)| \le \exp\big(F(s,\lambda)\big) - 1,$$
$$|(r_1)_s(s,\lambda)|, |(r_2)_s(s,\lambda)| \le F_s(s,\lambda)\exp\big(F(s,\lambda)\big)$$

and

$$F(s,\lambda) = \int_0^s (s-\sigma)|f(\sigma,\lambda)|\,d\sigma.$$

Put $R(t) = \lambda^{-2}F(s,\lambda)$. It follows that

$$R(t) = \int_0^s (s-\sigma)\frac{a(\tau)^2}{b(\tau)^2}\,d\sigma = \int_0^t \big(B(t)-B(\tau)\big)\frac{a(\tau)^2}{b(\tau)}\,d\tau,$$

and $R'(t) = \lambda^{-2}b(t)F_s(s,\lambda)$. Thus the original differential equation possesses a fundamental system $v_1(\cdot,\lambda)$, $v_2(\cdot,\lambda) \in C^2\big([0,\infty),\mathbb{C}\big)$ such that

$$v_1(t,\lambda) = 1 + \rho_1(t,\lambda), \qquad v_2(t,\lambda) = B(t)\,\big(1 + \rho_2(s,\lambda)\big),$$

where

$$\rho_1(t,\lambda) = r_1(s,\lambda), \qquad \rho_2(t,\lambda) = r_2(s,\lambda),$$
$$\rho_1'(t,\lambda) = b(t)(r_1)_s(s,\lambda), \quad \rho_2'(t,\lambda) = b(t)(r_2)_s(s,\lambda).$$

We obtain the inequalities

$$|\rho_1(t,\lambda)|, |\rho_2(t,\lambda)| \le \exp\big(\lambda^2 R(t)\big) - 1,$$
$$|\rho_1'(t,\lambda)|, |\rho_2'(t,\lambda)| \le \lambda^2 R'(t)\exp\big(\lambda^2 R(t)\big).$$

□

The notation $R(t) = \int_0^t \frac{B(t)-B(\tau)}{b(\tau)}\,a(\tau)^2 d\tau$ will be used again in the next section. In order to find good estimates for $R(t)$, we use the following condition:

(R) $\left\{\begin{array}{l} \text{Let } \varepsilon < 1 \text{ and } c > 0 \text{ such that} \\ \quad \dfrac{b(t)}{b(\tau)} \le c\dfrac{a(t)}{a(\tau)}\dfrac{\big(1+A(t)\big)^\varepsilon}{\big(1+A(\tau)\big)^\varepsilon} \\ \text{for every } 0 \le \tau \le t. \end{array}\right.$

This condition implies

$$R'(t) = b(t)\int_0^t \frac{a(\tau)^2}{b(\tau)}\,d\tau = a(t)\int_0^t \frac{b(t)}{b(\tau)}\frac{a(\tau)^2}{a(t)}\,d\tau$$
$$\le c\,a(t)\big(1+A(t)\big)^\varepsilon \int_0^t a(\tau)\big(1+A(\tau)\big)^{-\varepsilon}\,d\tau \le \frac{c}{\varepsilon+1}\,a(t)\big(1+A(t)\big),$$

thus

$$R(t) \le \frac{c}{2(\varepsilon+1)}\,\big(1+A(t)\big)^2,$$

bearing in mind that $-\varepsilon > -1$.

Approximate solutions as in the previous lemma can be found for every second order ordinary differential equation with a parameter λ. This implies, though, that they

do not respect the special form of the equation. This is different for the following approximation:
We assume that a and b are almost equal. Then we can use solutions of the equation

$$w''(t,\lambda) + b(t)^2\lambda^2 w(t,\lambda) - \tfrac{b'}{b}(t)w'(t,\lambda) = 0$$

as approximations. This equation has the same structure as (2.4) and can be solved explicitly. When we regroup (2.4) in this way, the term $\big(b(t)^2 - a(t)^2\big)\lambda^2 v(t,\lambda)$ remains. Like the remainder term in the previous lemma, this term is quadratic in λ.

Lemma 3.6. *Let $\lambda > 0$ and $\alpha \geq 0$. Suppose $a \in C^0([0,\infty],\mathbb{R})$, $b \in C^1([0,\infty),\mathbb{R})$ are functions such that $a(t) \geq 0$ and $b(t) > 0$ for every $t \in [0,\infty)$. Then the differential equation (2.4) has the fundamental system $v_+(\cdot,\lambda), v_-(\cdot,\lambda) \in C^2([0,\infty),\mathbb{C})$ such that*

$$v_\pm(t,\lambda) = \exp\big(\pm i\lambda B(t)\big)\big(1+\rho_\pm(t,\lambda)\big)$$

where

$$\big|\rho_\pm(t,\lambda)\big|, \big|\tfrac{1}{i\lambda b(t)}\rho'_\pm(t,\lambda) \pm \rho_\pm(t,\lambda)\big| \leq \exp\big(\lambda\,|Q(t)-Q(\alpha)|\big) - 1$$

and

$$Q(t) := \int_0^t \frac{|a(\tau)^2 - b(\tau)^2|}{b(\tau)}\,d\tau.$$

Proof. Since b is strictly positive, B is strictly increasing. Thus $s = B(t)$ or $\sigma = B(\tau)$ define a transformation of variables. Suppose $w(s,\lambda) = v(t,\lambda)$, then w satisfies the differential equation

$$w_{ss}(s,\lambda) + \frac{a(t)^2\lambda^2}{b(t)^2}w(s,\lambda) = 0.$$

Put

$$f(s,\lambda) := \lambda^2 - \frac{a(t)^2\lambda^2}{b(t)^2},$$

then we can write

$$w_{ss}(s,\lambda) + \lambda^2 w(s,\lambda) = f(s,\lambda)w(s,\lambda)$$

and apply Theorem 3.1. Hence this equation has a fundamental system

$$w_+(\cdot,\lambda), w_-(\cdot,\lambda) \in C^2(B([0,\infty)),\mathbb{C})$$

such that

$$w_\pm(s,\lambda) = \exp(\pm i\lambda s)\big(1 + r_\pm(s,\lambda)\big),$$

where

$$\big|r_\pm(s,\lambda)\big|, \big|\tfrac{1}{i\lambda}r'_\pm(s,\lambda) \pm r_\pm(s,\lambda)\big| \leq \exp(\lambda^{-1}F(s,\lambda)) - 1$$

and

$$F(s,\lambda) = \left|\int_{B(\alpha)}^s |f(\sigma,\lambda)|\,d\sigma\right|.$$

Expressing F in terms of t gives

$$F(s,\lambda) = \lambda^2\left|\int_{B(\alpha)}^s \frac{|b(\tau)^2 - a(\tau)^2|}{b(\tau)^2}\,d\sigma\right| = \lambda^2\left|\int_\alpha^t \frac{|b(\tau)^2 - a(\tau)^2|}{b(\tau)}\,d\tau\right|.$$

Thus the original equation has a fundamental system $v_+(\cdot,\lambda), v_-(\cdot,\lambda) \in C^2\big([0,\infty),\mathbb{C}\big)$ such that

$$v_\pm(t,\lambda) = \exp\big(\pm i\lambda B(t)\big)\big(1+\rho_\pm(t,\lambda)\big),$$

where

$$\rho_\pm(t,\lambda) = r_\pm(s,\lambda), \quad \rho'_\pm(t,\lambda) = b(t)(r_\pm)_s(s,\lambda).$$

Using the estimates for r, we obtain

$$\big|\rho_\pm(t,\lambda)\big|, \big|\tfrac{1}{i\lambda b(t)}\rho_\pm(t,\lambda) \pm \rho_\pm(t,\lambda)\big| \le \exp\big(\lambda|Q(t)-Q(\alpha)|\big) - 1$$

where

$$Q(t) := \int_0^t \frac{|a(\tau)^2 - b(\tau)^2|}{b(\tau)}\, d\tau.$$

□

The notation Q will be used again in the next section. We can estimate Q easily using the following assumption:

(Q) $\left\{\begin{array}{l} \text{There exist a nonnegative, monotone function } \varphi_Q \in C^1([0,\infty),\mathbb{R}) \text{ and constants } c_1, c_2 > 0 \text{ such that} \\ \qquad |a(t)-b(t)| \le c_1|\varphi'_Q(t)|, \\ \qquad a(t), b(t) \ge c_2|\varphi'_Q(t)| \\ \text{for every } t \ge 0. \end{array}\right.$

Using φ_Q is advantageous, because we do not have to assume that a or b are monotone. Under condition (Q), we obtain the following estimates

$$\begin{aligned} |a(t)^2 - b(t)^2| &\le c_1|\varphi'_Q(t)|\big(a(t)+b(t)\big) \le c_1|\varphi'_Q(t)|\big(b(t) + c_1|\varphi'_Q(t)| + b(t)\big) \\ &\le c_3\,|\varphi'_Q(t)| b(t). \end{aligned}$$

Since φ_Q is monotone, φ'_Q does not change sign, hence

$$Q(t) - Q(\alpha) \le c_3 \int_\alpha^t |\varphi'_Q(\tau)| d\tau = c_3\,|\varphi_Q(t) - \varphi_Q(\alpha)|. \tag{3.5}$$

Moreover, $\frac{a}{b}$ and $\frac{b}{a}$ are bounded, for

$$\left|\frac{a(t)}{b(t)}\right| \le \frac{b(t) + c_1|\varphi'_Q(t)|}{b(t)} \le 1 + \frac{c_1}{c_2}, \quad \left|\frac{b(t)}{a(t)}\right| \le \frac{a(t) + c_1|\varphi'_Q(t)|}{a(t)} \le 1 + \frac{c_1}{c_2}.$$

Thus there exist constants $c, C > 0$ such that

$$c\,a(t)b(t) \le a(t)^2 \le C\,a(t)b(t), \qquad c\,a(t)b(t) \le b(t)^2 \le C\,a(t)b(t). \tag{3.6}$$

For large λ, we can apply Theorem 3.1 after a suitable transformation. For this purpose, the following function will prove to be useful:

$$\psi := \frac{1}{2}\frac{1}{a}\left(\frac{a'}{a} - \frac{b'}{b}\right). \tag{3.7}$$

We will use this function frequently later on.

Lemma 3.7. *Let $\lambda > 0$ and $\alpha \geq 0$, let $a \in C^2([0,\infty),\mathbb{R})$, $b \in C^2([0,\infty),\mathbb{R})$ such that $a(t), b(t) > 0$ for every $t \in [0,\infty)$. Then there exists a fundamental system $v_+(\cdot,\lambda), v_-(\cdot,\lambda) \in C^2([0,\infty),\mathbb{C})$ for the differential equation (2.4) such that*

$$v_\pm(t,\lambda) = \sqrt{\tfrac{b}{a}}(t)\exp\big(\pm i\lambda A(t)\big)\big(1+\rho_\pm(t,\lambda)\big)$$

and

$$\big|\rho_\pm(t,\lambda)\big|, \big|\tfrac{1}{i\lambda a(t)}\rho'_\pm(t,\lambda) \pm \rho_\pm(t,\lambda)\big| \leq \exp\big(\lambda^{-1}|S(t)-S(\alpha)|\big) - 1,$$

where

$$S(t) := \int_0^t |\psi'(\tau) + a(\tau)\psi(\tau)^2|\, d\tau.$$

The error term S will reappear in the next section.

Proof. In order to use Theorem 3.1, we have to transform equation (2.4). For this purpose, put

$$\varphi := \sqrt{\tfrac{b}{a}}.$$

Since $\psi = \frac{1}{2}\frac{1}{a}\big(\frac{a'}{a} - \frac{b'}{b}\big)$, it follows that $\varphi' = -a\psi\varphi$. Since a is strictly positive, A is strictly increasing. Thus $s = A(t)$ and $\sigma = A(\tau)$ define a transformation of variables. Define $w(\cdot,\lambda) \in C^2\big(A([\alpha,\infty)),\mathbb{C}\big)$ by $w(s,\lambda) := v(t,\lambda)\varphi(t)^{-1}$. We obtain the following equations

$$\begin{aligned} v(t,\lambda) &= \varphi(t)w(s,\lambda),\\ v'(t,\lambda) &= -a(t)\psi(t)\varphi(t)w(s,\lambda) + a(t)\varphi(t)w_s(s,\lambda),\\ v''(t,\lambda) &= \big(-a'(t)\psi(t) - a(t)\psi'(t) + a(t)^2\psi(t)^2\big)\varphi(t)w(s,\lambda)\\ &\quad + \big(-2a(t)^2\psi(t) + a'(t)\big)\varphi(t)w_s(s,\lambda) + a(t)^2\varphi(t)w_{ss}(s,\lambda). \end{aligned}$$

Substituting v in the differential equation leads to

$$\begin{aligned} 0 &= a(t)^2 w_{ss}(s,\lambda) + \Big(-2a(t)^2\psi(t) + a'(t) - \tfrac{b'}{b}(t)\Big)w_s(s\lambda)\\ &\quad - \Big(a'(t)\psi(t) + a(t)\psi'(t) - a(t)^2\psi(t)^2 - \tfrac{b'}{b}(t)a(t)\psi(t) - a(t)^2\lambda^2\Big)w(s,\lambda)\\ &= a(t)^2 w_{ss}(s,\lambda) + a(t)^2\lambda^2 w(s,\lambda)\\ &\quad - a(t)^2\Big(a(t)^{-1}\tfrac{a'}{a}(t)\psi(t) + a(t)^{-1}\psi'(t) - \psi(t)^2 - a(t)^{-1}\tfrac{b'}{b}(t)\psi(t)\Big)w(s,\lambda)\\ &= a(t)^2 w_{ss}(s,\lambda) + a(t)^2\lambda^2 w(s,\lambda) - a(t)^2\Big[\psi(t)^2 + a(t)^{-1}\psi'(t)\Big]w(s,\lambda). \end{aligned}$$

For $s \in A([\alpha,\infty))$ put $f(s) := [\psi(t)^2 + a(t)^{-1}\psi'(t)]$. With this notation, it follows that

$$w_{ss}(s,\lambda) + \lambda^2 w(s,\lambda) = f(s)w(s). \tag{3.8}$$

By Theorem 3.1 there exists a fundamental system

$$w_+(\cdot,\lambda), w_-(\cdot,\lambda) \in C^2\big(A([0,\infty)),\mathbb{C}\big)$$

for this equation such that

$$w_\pm(s,\lambda) = \exp(\pm i\lambda s)\big(1 + r_\pm(s,\lambda)\big),$$

where

$$\big|r_\pm(s,\lambda)\big|, \big|\tfrac{1}{i\lambda a(t)}r'_\pm(s,\lambda) \pm r_\pm(s,\lambda)\big| \leq \exp\bigg(\frac{F(s)}{\lambda}\bigg) - 1,$$

and

$$F(s) = \left| \int_{A(\alpha)}^{s} |f(\sigma)| \, d\sigma \right|.$$

It follows that the original equation has a fundamental system

$$v_+(\cdot,\lambda), v_-(\cdot,\lambda) \in C^2\big[0,\infty),\mathbb{C}\big)$$

such that

$$v_\pm(t,\lambda) = \sqrt{\varphi(t)}\exp\big(\pm i\lambda A(t)\big)\big(1+\rho_\pm(t,\lambda)\big),$$

where

$$\rho_\pm(t,\lambda) = r_\pm(s,\lambda), \quad \rho'_\pm(t,\lambda) = a(t)(r_\pm)_s(s,\lambda).$$

Substituting $r_\pm$ we obtain the estimates

$$\big|\rho_\pm(t,\lambda)\big|, \big|\tfrac{1}{ia(t)\lambda}\rho_\pm(t,\lambda) \pm \rho_\pm(t,\lambda)\big| \leq \exp\big(\lambda^{-1}|S(t)-S(\alpha)|\big) - 1$$

where

$$S(t) := \int_0^s |f(\sigma)| \, d\sigma = \int_0^t a(t)|f(\sigma)| \, d\tau = \int_0^t |\psi'(\tau) + a(\tau)\psi(\tau)^2| \, d\tau.$$

□

The following assumption will help in estimating S:

(S) There exists a nonnegative, monotone function $\varphi_S \in C^1([0,\infty),\mathbb{R})$ and a constant $c > 0$ such that

$$|\psi(t)| \leq c\,\varphi_S(t),$$
$$|\psi'(t) + a(t)\psi(t)^2| \leq c\,|\varphi'_S(t)|$$

for every $t \geq 0$.

The function φ_S helps to avoid conditions on the monotonicity of more important functions, in this case we avoid special assumptions on ψ. We assume (S). Since φ'_S does not change sign,

$$S(t) - S(\alpha) \leq c\,|\varphi_S(t) - \varphi_S(\alpha)| \tag{3.9}$$

for every $t \geq \alpha \geq 0$.

In the following, we assume that $a, b \in C^2([0,\infty),\mathbb{R})$ are strictly positive.

4. Fourier space estimates

Now we use the approximate fundamental systems of equation (2.4) to prove bounds for the energy density

$$e_u(t,\xi) := |\hat{u}_t(t,\xi)|^2 + a(t)^2|\xi|^2|\hat{u}(t,\xi)|^2.$$

of admissible solutions u of (1.1). To this end we use the Euclidean matrix norm

$$\big|(a_{ij})\big| := \sqrt{\sum_{i,j=1}^{n} |a_{ij}|^2},$$

which is compatible with the Euclidean norm for vectors. Let $\lambda > 0$. Suppose $v_1(\cdot,\lambda), v_2(\cdot,\lambda) \in C^2([0,\infty),\mathbb{C})$ is a fundamental system of equation (2.4) and

$$V(\cdot,\lambda) := \begin{pmatrix} v_1(\cdot,\lambda) & v_2(\cdot,\lambda) \\ v'_1(\cdot,\lambda) & v'_2(\cdot,\lambda) \end{pmatrix} \tag{4.1}$$

the associated fundamental matrix. For an admissible solution u of equation (1.1) and for every $\xi \in \mathbb{R}^n$ put

$$U(\cdot,\xi) := \begin{pmatrix} \hat{u}(\cdot,\xi) \\ \hat{u}_t(\cdot,\xi) \end{pmatrix}.$$

For every $s, t \in [0,\infty)$ and $\xi \in \mathbb{R}^n \setminus \{0\}$ we can write

$$U(t,\xi) = V(t,|\xi|)V(s,|\xi|)^{-1}U(s,\xi).$$

For every $\mathcal{M}(\cdot,\lambda)\colon [0,\infty) \to \mathrm{GL}(2,\mathbb{C})$, $\mathcal{N}(\cdot,\lambda)\colon [0,\infty) \to \mathrm{GL}(2,\mathbb{C})$ and for every $t, s \in [0,\infty)$ we define

$$F_{\mathcal{M}\mathcal{N}}(t,s,\lambda) := \big|\mathcal{M}(t,\lambda)V(t,\lambda)V(s,\lambda)^{-1}\mathcal{N}(s,\lambda)^{-1}\big|^2.$$

This definition is independent of the choice of the fundamental matrix $V(\cdot,\lambda)$: If X, Y are fundamental matrices of the same linear differential equation, then $X(t)X(s)^{-1} = Y(t)Y(s)^{-1}$. Suppose $\mathcal{L}(\cdot,\lambda)\colon [0,\infty) \to \mathrm{GL}(2,\mathbb{C})$ is another mapping. Since the Euclidian matrix norm is submultiplicative, it follows that

$$F_{\mathcal{L}\mathcal{N}}(t,r,\lambda) \le F_{\mathcal{L}\mathcal{M}}(t,s,\lambda)\, F_{\mathcal{M}\mathcal{N}}(s,r,\lambda)$$

for every $t \ge s \ge r$. In the following, let

$$\mathcal{A}(t,\lambda) := \mathrm{diag}\big(a(t)\lambda, 1\big), \quad \mathcal{B}(t,\lambda) := \mathrm{diag}\big(a(t)\langle\lambda\rangle, 1\big)$$

where $\langle\lambda\rangle = \sqrt{1+\lambda^2}$. We observe that

$$|\mathcal{A}(t,|\xi|)U(t,\xi)|^2 = e_u(t,\xi), \qquad |\mathcal{B}(t,|\xi|)U(t,\xi)|^2 = e_u(t,\xi) + a(t)^2|\hat{u}(t,\xi)|^2,$$

for every $\xi \in \mathbb{R}^n \setminus \{0\}$. Hence

$$e_u(t,\xi) = \big|\mathcal{A}(t,|\xi|)V(t,|\xi|)V(0,|\xi|)^{-1}\mathcal{A}(0,|\xi|)^{-1}\mathcal{A}(0,|\xi|)U(0,\xi)\big|^2 \quad \le F_{\mathcal{A}\mathcal{A}}(t,0,|\xi|)e_u(0,\xi),$$

and

$$e_u(t,\xi) \le F_{\mathcal{A}\mathcal{B}}(t,0,|\xi|)\big(e_u(0,\xi) + a(0)^2|\hat{u}(0,\xi)|^2\big).$$

If we are able to proof an estimate of the form

$$F_{\mathcal{A}\mathcal{A}}(t,0,\lambda) \le d(t) \quad \text{or} \quad F_{\mathcal{A}\mathcal{B}}(t,0,\lambda) \le d(t)$$

for every $t \ge 0$ and $\lambda > 0$, then

$$E_u(t) = \int_{\mathbb{R}^n} e_u(t,\xi)\,d\xi \le \int_{\mathbb{R}^n} F_{\mathcal{A}\mathcal{A}}(t,0,|\xi|)e_u(0,\xi)\,d\xi \le d(t)\,E_u(0)$$

or

$$\begin{aligned} E_u(t) &= \int_{\mathbb{R}^n} e_u(t,\xi)\,d\xi \le \int_{\mathbb{R}^n} F_{\mathcal{A}\mathcal{B}}(t,0,|\xi|)\big(e_u(0,\xi) + a(t)^2|\hat{u}(0,\xi)|^2\big)\,d\xi \\ &\le d(t)\,\big(E_u(0) + a(0)^2\|u(0,\cdot)\|_{L^2}^2\big), \end{aligned}$$

respectively. We will now prove such estimates.

Lemma 4.1. *Let $c > 0$. Then there exists a constant $C > 0$, depending only on c, such that*

$$F_{\mathcal{A}\mathcal{B}}(t,0,\lambda) \le C(a(0)^{-2} + b(0)^{-2})\big(a(t)^2(1+B(t))^2\lambda^2 + b(t)^2\big)$$

for every $t \ge 0$ and $\lambda > 0$ satisfying $\lambda^2 R(t) \le c$ and $\lambda R'(t) \le ca(t)$.

Proof. Let v_1, v_2 be the fundamental system as in Lemma 3.5, and let $t \geq 0$, $\lambda > 0$ such that $\lambda^2 R(t) \leq c$ and $\lambda R'(t) \leq c$. Then

$$|v_1(t,\lambda)| \leq e^c, \qquad |v_2(t,\lambda)| \leq e^c B(t),$$
$$|v_1'(t,\lambda)| \leq ce^c \lambda a(t), \qquad |v_2'(t,\lambda)| \leq e^c b(t) + ce^c \lambda\, a(t)B(t).$$

It follows that

$$|\mathcal{A}(t,\lambda)V(t,\lambda)|^2 \leq c_1^2\big(2a(t)^2\lambda^2 + a(t)^2\lambda^2 B(t)^2 + (b(t) + a(t)B(t)\lambda)^2\big)$$
$$\leq c_2\big(a(t)^2\big(1 + B(t)\big)^2\lambda^2 + b(t)^2\big).$$

Note that the constants c_1, c_2 depend only on c. The initial values are

$$v_1(0,\lambda) = 1, \qquad v_2(0,\lambda) = 0,$$
$$v_1'(0,\lambda) = 0, \qquad v_2'(0,\lambda) = b(0),$$

hence $|V(0,\lambda)^{-1}\mathcal{B}(0,\lambda)^{-1}|^2 = a(0)^{-2}\langle\lambda\rangle^{-2} + b(0)^{-2} \leq a(0)^{-2} + b(0)^{-2}$. Combining this with the last inequality completes the proof. □

Corollary 4.2. *Let $c > 0$. If condition (R) is satisfied, then there exists a constant $C > 0$ such that*

$$F_{\mathcal{A}\mathcal{B}}(t,0,\lambda) \leq C\big(a(t)^2\big(1 + B(t)^2\big)\lambda^2 + b(t)^2\big)$$

for every $t \geq 0$ and every $\lambda > 0$ satisfying $\big(1 + A(t)\big)\lambda \leq c$. The constant $C > 0$ depends only on c, $a(0)$, $b(0)$ and the constant appearing in (R).

Proof. Let $t \geq 0$ and $\lambda > 0$ such that $\lambda \leq c\,(1 + t)^{-1+\ell}$. By the remark on condition (R), see p. 9, it follows that

$$\lambda^2 R(t) \leq c'\lambda^2\big(1 + A(t)\big)^2 \leq c'c, \qquad \lambda R'(t) \leq c'\lambda\, a(t)\big(1 + A(t)\big) \leq c'ca(t).$$

Lemma 4.1 completes the proof. □

Lemma 4.3. *Let $c > 0$. Then there exists a constant $C > 0$, depending only on c, such that*

$$F_{\mathcal{A}\mathcal{A}}(t,\alpha,\lambda) \leq C\,\big(a(t)^2 + b(t)^2\big)\big(a(\alpha)^{-2} + b(\alpha)^{-2}\big)$$

for every $\alpha \geq 0$, $t \geq \alpha$ and every $\lambda > 0$ satisfying $\lambda\big(Q(t) - Q(\alpha)\big) \leq c$.

Proof. Let v_+, v_- be the fundamental system as in Lemma 3.6 and $V(t,\lambda)$ the associated fundamental matrix as in (4.1). Suppose $t \geq \alpha \geq 0$, $\lambda > 0$ such that $\lambda\big(Q(t) - Q(\alpha)\big) \leq c$. Then

$$|a(t)\lambda v_\pm(t,\lambda)| \leq a(t)\lambda e^c,$$
$$|v_\pm'(t,\lambda)| = \big|\exp\big(\pm i\lambda B(t)\big)\big(\pm i\lambda b(t)\big(1 + \rho_\pm(t,\lambda)\big) + \rho_\pm'(t,\lambda)\big)\big| \leq b(t)\lambda e^c,$$

hence

$$|\mathcal{A}(t,\lambda)V(t,\lambda)|^2 \leq 2\,e^{2c}\,\big(a(t)^2 + b(t)^2\big)\lambda^2.$$

The initial values are

$$v_\pm(\alpha,\lambda) = a(\alpha)\lambda\exp\big(\pm i\lambda B(\alpha)\big), \quad v_\pm'(\alpha,\lambda) = \pm ib(\alpha)\lambda\exp\big(\pm i\lambda B(\alpha)\big).$$

We compute the determinant

$$\det\big(V(\alpha,\lambda)^{-1}\mathcal{A}(\alpha,\lambda)^{-1}\big) = -2ia(\alpha)b(\alpha)\lambda^2,$$

and since $|M^{-1}| = \frac{|M|}{|\det M|}$ for every matrix $M \in \mathbb{C}^{2\times 2}$, we have

$$|V(\alpha,\lambda)^{-1}\mathcal{A}(\alpha,\lambda)^{-1}|^2 \le 2e^{2c}\frac{\big(a(\alpha)^2+b(\alpha)^2\big)\lambda^2}{4a(\alpha)^2b(\alpha)^2\lambda^4} = \frac{e^{2c}}{2}\big(a(\alpha)^{-2}+b(\alpha)^{-2}\big)\lambda^{-2}.$$

This completes the proof. □

Corollary 4.4. *Assume condition (Q) and suppose $c > 0$.*

(a) If φ_Q is increasing, then there exists a constant $C > 0$ such that

$$F_{\mathcal{A}\mathcal{A}}(t,0,\lambda) \le C\frac{a(t)b(t)}{a(0)b(0)}$$

for every $t \ge 0$ and every $\lambda > 0$ such that $\lambda\varphi_Q(t) \le c$.

(b) If φ_Q is decreasing, then there exists a constant $C > 0$ such that

$$F_{\mathcal{A}\mathcal{A}}(t,\alpha,\lambda) \le C\frac{a(t)b(t)}{a(\alpha)b(\alpha)}$$

for every $\alpha \ge 0$, for every $t \ge \alpha$ and $\lambda > 0$ such that $\lambda\varphi_Q(\alpha) \le c$.

The constant $C > 0$ depends only on c and on the constants in (Q).

Proof. By inequality (3.5), there exists a constant $c' > 0$ such that

$$Q(t) - Q(\alpha) \le c'\,|\varphi_Q(t) - \varphi_Q(\alpha)|$$

for every $\alpha \ge 0$ and every $t \ge \alpha$. Suppose φ_Q is increasing, $t \ge 0$ and $\lambda > 0$ such that $\lambda\varphi_Q(t) \le c$. Then

$$\lambda\big(Q(t) - Q(0)\big) \le c'\,\lambda\varphi_Q(t) \le c'c.$$

By Lemma 4.3,

$$F_{\mathcal{A}\mathcal{A}}(t,0,\lambda) \le C\,\big(a(t)^2+b(t)^2\big)\big(a(0)^{-2}+b(0)^{-2}\big) \le C'\frac{a(t)b(t)}{a(0)b(0)}.$$

In the last step we used that under assumption (Q) there are $k, K > 0$ such that

$$k\,a(t)b(t) \le a(t)^2 \le K\,a(t)b(t), \qquad k\,a(t)b(t) \le b(t)^2 \le K\,a(t)b(t),$$

see inequality (3.6). If φ_Q is decreasing, $t \ge \alpha \ge 0$ and $\lambda > 0$ such that $\lambda\varphi_Q(\alpha) \le c$, then

$$\lambda\big(Q(t) - Q(\alpha)\big) \le c'\,\lambda\varphi_Q(\alpha) \le c'c,$$

and we obtain

$$F_{\mathcal{A}\mathcal{A}}(t,\alpha,\lambda) \le C\frac{a(t)b(t)}{a(\alpha)b(\alpha)}$$

by Lemma 4.3. □

Lemma 4.5. *Let $c > 0$. Then there exists a constant $C > 0$ depending only on c, such that*

$$F_{\mathcal{A}\mathcal{A}}(t,\alpha,\lambda) \le C\frac{a(t)b(t)}{a(\alpha)b(\alpha)},$$
$$F_{\mathcal{A}\mathcal{B}}(t,\alpha,\lambda) \le C\frac{\lambda^2}{\langle\lambda\rangle^2}\frac{a(t)b(t)}{a(\alpha)b(\alpha)}$$

for every $\alpha \ge 0$, $t \ge \alpha$ and every $\lambda > 0$ such that $S(t) - S(\alpha) \le c\lambda$ and $|\psi(t)| \le c\lambda$.

Proof. Let v_+, v_- be the fundamental system as in Lemma 3.7 and let $V(t,\lambda)$ be the associated fundamental matrix as in (4.1). Suppose $t \ge \alpha \ge 0$ and $\lambda > 0$ such that $S(t) - S(\alpha) \le c\lambda$ and $|\psi(t)| \le c\lambda$. Then

$$|a(t)\lambda v_\pm(t,\lambda)| \le a(t)\lambda\sqrt{\tfrac{b}{a}}(t)e^c \le c_1\sqrt{a(t)b(t)}\lambda$$

and

$$\begin{aligned} v'_\pm(t,\lambda) &= \pm i\lambda\sqrt{a(t)b(t)}\exp\big(\pm i\lambda A(t)\big)\big(1+\rho_\pm(t,\lambda) \mp \tfrac{1}{i\lambda}\rho'_\pm(t,\lambda)\big) \\ &\quad + \Big(\sqrt{\tfrac{b}{a}}\Big)'(t)\exp\big(\pm i\lambda A(t)\big)\big(1+\rho_\pm(t,\lambda)\big). \end{aligned}$$

Since

$$\Big|\Big(\sqrt{\tfrac{b}{a}}\Big)'(t)\Big| = a(t)|\psi(t)|\sqrt{\tfrac{b}{a}}(t) \le c\lambda\sqrt{a(t)b(t)},$$

we obtain

$$|v'_\pm(t,\lambda)| \le \sqrt{a(t)b(t)}\lambda e^c + c\sqrt{a(t)b(t)}\lambda e^c \le c_1\sqrt{a(t)b(t)}\lambda,$$

where $c_1 := (1+c)e^c$. Hence

$$|\mathcal{A}(t,\lambda)V(t,\lambda)|^2 \le 4\,c_1\,a(t)b(t)\lambda^2.$$

The initial values are

$$\begin{aligned} v_\pm(\alpha,\lambda) &= \sqrt{\tfrac{b}{a}}(\alpha)\lambda\exp\big(\pm iA(\alpha)\big), \\ v'_\pm(\alpha,\lambda) &= \Big(\pm i\sqrt{a(\alpha)b(\alpha)}\lambda + \Big(\sqrt{\tfrac{b}{a}}\Big)'(\alpha)\Big)\exp\big(\pm iA(\alpha)\big), \end{aligned}$$

thus

$$\det\big(V(\alpha,\lambda)^{-1}\mathcal{A}(\alpha,\lambda)^{-1}\big) = -2ia(\alpha)b(\alpha)\lambda^2.$$

Since $|M^{-1}| = \frac{|M|}{|\det M|}$ for every matrix $M \in \mathbb{C}^{2\times 2}$, it follows that

$$|V(\alpha,\lambda)^{-1}\mathcal{A}(\alpha,\lambda)^{-1}|^2 \le 4c_1\,\frac{a(\alpha)b(\alpha)\lambda^2}{4a(\alpha)^2b(\alpha)^2\lambda^4} = c_1\frac{1}{a(\alpha)b(\alpha)\lambda^2}$$

and

$$|V(\alpha,\lambda)^{-1}\mathcal{B}(\alpha,\lambda)^{-1}|^2 \le 4c_1\,\frac{a(\alpha)b(\alpha)\lambda^2}{4a(\alpha)^2b(\alpha)^2\langle\lambda\rangle^2\lambda^2} = c_1\frac{1}{a(\alpha)b(\alpha)\langle\lambda\rangle^2}. \qquad \square$$

Corollary 4.6. *Assume (S) and let $c > 0$.*

(a) If φ_S is increasing, then there exists a constant $C > 0$ such that

$$F_{\mathcal{A}\mathcal{A}}(t,0,\lambda) \le C\frac{a(t)b(t)}{a(0)b(0)}, \qquad F_{\mathcal{A}\mathcal{B}}(t,0,\lambda) \le C\frac{\lambda^2}{\langle\lambda\rangle^2}\frac{a(t)b(t)}{a(0)b(0)}$$

for every $t \ge 0$ and every $\lambda > 0$ such that $\varphi_S(t) \le c\,\lambda$.

(b) If φ_S is decreasing, then there exists a constant $C > 0$ such that

$$F_{\mathcal{A}\mathcal{A}}(t,\alpha,\lambda) \le C\frac{a(t)b(t)}{a(\alpha)b(\alpha)}, \qquad F_{\mathcal{A}\mathcal{B}}(t,\alpha,\lambda) \le C\frac{\lambda^2}{\langle\lambda\rangle^2}\frac{a(t)b(t)}{a(\alpha)b(\alpha)}$$

for every $t \ge \alpha \ge 0$ and every $\lambda > 0$ mit $\varphi_S(\alpha) \le c\,\lambda$.

The constant $C > 0$ depends only on c and on the constant appearing in (S).

Proof. Let φ_S be increasing, let $t \ge 0$ and $\lambda > 0$ such that $\varphi_S(t) \le c\,\lambda$. By condition (S) and inequality (3.9), there exists a constant c' such that

$$S(t) - S(0) \le c'\varphi_S(t) \le cc'\,\lambda$$

and

$$|\psi(t)| \le c'\varphi_S(t) \le cc'\,\lambda.$$

Lemma 4.5 implies (a).
If φ_S is increasing, $t \ge \alpha \ge 0$ and $\lambda > 0$ such that $\varphi_S(\alpha) \le c\,\lambda$, then

$$S(t) - S(\alpha) \le c'\varphi_S(\alpha) \le cc'\,\lambda$$

and

$$|\psi(t)| \le c'\varphi_S(t) \le c'\varphi_S(\alpha) \le cc'\,\lambda.$$

As above, (b) is an immediate implication of Lemma 4.5. □

5. Energy inequalities

Now we glue together the estimates of the previous section in order to get energy estimates. As before, we consider only admissible solutions of equation

$$u_{tt}(t,x) - a(t)^2\Delta u(t,x) - \tfrac{b'}{b}(t)u_t(t,x) = 0, \tag{1.1}$$

i.e. solutions $u \in C^0\big([0,\infty), H^2(\mathbb{R}^n)\big) \cap C^1\big([0,\infty), H^1(\mathbb{R}^n)\big) \cap C^2\big([0,\infty), L^2(\mathbb{R}^n)\big)$.
Under the conditions of the following theorem, the existence of nonvanishing admissible solutions is guaranteed, see Theorem 5.4 below.
To state the main theorem, we only need the following abbreviations introduced in the previous sections:

$$A(t) := \int_0^t a(\tau)\,d\tau, \quad B(t) := \int_0^t b(\tau)\,d\tau \quad \text{and} \quad \psi := \frac{1}{2}\frac{1}{a}\left(\frac{a'}{a} - \frac{b'}{b}\right).$$

Theorem 5.1. *Let $a, b \in C^2([0,\infty),\mathbb{R})$ be strictly positive, and suppose $a \notin L^1([0,\infty))$. Let $\varepsilon < 1$ and $c_1, c_2, c_3 > 0$ such that*

$$\frac{b(t)}{b(\tau)} \le c_1\frac{a(t)}{a(\tau)}\frac{\big(1+A(t)\big)^{\varepsilon}}{\big(1+A(\tau)\big)^{\varepsilon}}, \tag{5.1}$$

$$|\psi(t)| \le c_2\big(1+A(t)\big)^{-1}, \tag{5.2}$$

$$|\psi'(t)| \le c_3\,a(t)\big(1+A(t)\big)^{-2} \tag{5.3}$$

for every $0 \le \tau \le t$. Let $M\colon [0,\infty) \to [0,\infty)$ be a function such that

$$\frac{M(t)}{M(\tau)} \ge \frac{a(t)b(t)}{a(\tau)b(\tau)}, \qquad M(t) \ge a(t)^2\frac{\big(1+B(t)\big)^2}{\big(1+A(t)\big)^2} + \, b(t)^2 \tag{5.4}$$

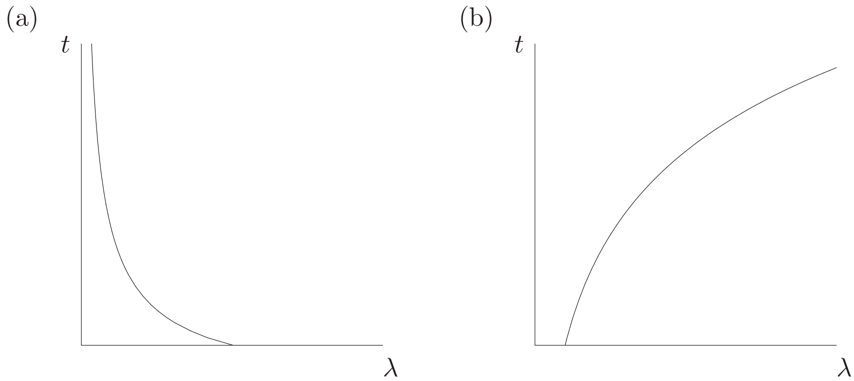

FIGURE 1. Sketch to the proof of Theorem 5.1 and to the proof of Theorem 5.3 showing (a) the curve defined by $\varphi_S(t) = \lambda$ and (b) the curve defined by $\varphi_Q(t) = \frac{1}{\lambda}$.

for every $0 \le \tau \le t$*. Then there exists a constant* $C > 0$ *such that*

$$E_u(t) \le C\, M(t) \left(E_u(0) + \|u(0,\cdot)\|_{L^2}^2\right)$$

for each admissible solution of equation (1.1) and every $t \ge 0$*.*

Proof. By (5.1), the functions a and b satisfy condition (R), and by (5.2) and (5.3), they satisfy condition (S) with

$$\varphi_S(t) := \big(1 + A(t)\big)^{-1}.$$

The curve defined by $\varphi_S(t) = \lambda$ is sketched in Figure 1 (a). Since a is strictly positive, A is strictly increasing, and since a is not integrable, A is unbounded. Hence, φ_S is strictly decreasing and tends to 0.

1. Suppose $\lambda \ge \varphi_S(0)$. By Corollary 4.6,

$$F_{\mathcal{AB}}(t, 0, \lambda) \le c_4 \frac{a(t)b(t)}{a(0)b(0)} \le c_5\, M(t)$$

   for every $t \ge 0$.
2. Suppose $\lambda \le \varphi_S(0)$ and put $\alpha := \varphi_S^{-1}(\lambda)$. Here, the estimates for $t \le \alpha$ and for $t \ge \alpha$ have to be glued together.
   (a) Let $0 \le t \le \alpha$. Then

$$\lambda = \varphi_S(\alpha) = \big(1 + A(\alpha)\big)^{-1} \le \big(1 + A(t)\big)^{-1},$$

   and by Corollary 4.2 it follows that

$$F_{\mathcal{AB}}(t, 0, \lambda) \le c_6 \big(a(t)^2 \big(1 + B(t)^2\big)\big(1 + A(t)\big)^{-2} + b(t)^2\big) \le c_6\, M(t).$$

   (b) Let $t \ge \alpha$. By Corollary 4.6 and since $\varphi_S(\alpha) = \lambda$ the inequality

$$F_{\mathcal{AA}}(t, \alpha, \lambda) \le c_7 \frac{a(t)b(t)}{a(\alpha)b(\alpha)}$$

   holds. For $t \ge \alpha$ it follows that

$$F_{\mathcal{AB}}(t, 0, \lambda) \le F_{\mathcal{AA}}(t, \alpha, \lambda) F_{\mathcal{AB}}(\alpha, 0, \lambda) \le c_8 \frac{a(t)b(t)}{a(\alpha)b(\alpha)} M(\alpha) m \le c_8 M(t).$$

Now it has been proven that

$$F_{\mathcal{AB}}(t,0,\lambda) \le c_9 M(t)$$

for every $t \ge 0$ and every $\lambda > 0$, and thus

$$\begin{aligned} E_u(t) &\le \int_{\mathbb{R}^n} F_{\mathcal{AB}}(t,0,|\xi|)\big(e_u(0,\xi) + |u(0,\xi)|^2\big)\,d\xi \\ &\le CM(t)\left(E_u(0) + \|u(0,\cdot)\|_{L_2}^2\right). \end{aligned}$$

for each admissible solution u of (1.1) and every $t \ge 0$. □

As can be seen from the proof, (5.3) could be replaced by the slightly weaker, but more cumbersome condition

$$|\psi'(t) + a\psi(t)^2| \le c\,a(t)\big(1 + A(t)\big)^{-2}.$$

The conditions of Theorem 5.1 can be simplified when only solutions with energy behaving like $a \cdot b$ are considered:

Corollary 5.2. *Let $a, b \in C^2([0,\infty),\mathbb{R})$ be strictly positive and suppose $a \notin L^1([0,\infty))$. Let $\varepsilon < 1$ and $c_1,\dots,c_4 > 0$ such that*

$$|\psi(t)| \le c_1\big(1 + A(t)\big)^{-1}, \qquad |\psi'(t)| \le c_2\; a(t)\big(1 + A(t)\big)^{-2},$$

$$\frac{b(t)}{b(\tau)} \le c_3 \frac{a(t)}{a(\tau)} \frac{\big(1 + A(t)\big)^{\varepsilon}}{\big(1 + A(\tau)\big)^{\varepsilon}}, \qquad \frac{b(t)}{a(t)} \ge c_4 \left(\frac{\big(1 + B(t)\big)^2}{\big(1 + A(t)\big)^2} + \frac{b(t)^2}{a(t)^2}\right)$$

for every $t \ge \tau \ge 0$. Then there exists a constant $C > 0$ such that

$$E_u(t) \le C\,a(t)b(t)\left(E_u(0) + \|u(0,\cdot)\|_{L^2}^2\right)$$

for each admissible solution of equation (1.1) and for every $t \ge 0$.

In [15], Wirth proves an analogous result to Theorem 5.1 for $a = 1$, where the following conditions are used:

(A1) $\frac{b'}{b}(t) \le 0$ for every $t \ge 0$,
(A2) $\big(\frac{b'}{b}\big)'(t) > 0$ for every $t \ge 0$,
(A3) $\big(\frac{b'}{b}\big)^2(t) \le c\big(\frac{b'}{b}\big)'(t)$ for every $t \ge 0$,
(A4) $\big|\big(\frac{b'}{b}\big)^{(k)}(t)\big| \le c(1+t)^{-1-k}$ for $k = 0, 1$ and every $t \ge 0$.
(C1) $\limsup_{t\to\infty} tb(t) < 1$.

Wirth shows the theorem under conditions (A1) – (A3) and (C1) or under conditions (A1), (A4), and (C1). Of these, Theorem 5.1 only needs condition (A4), which is equivalent to (A3) together with

$$\big(\tfrac{b'}{b}\big)'(t) > c(1+t)^{-2}$$

for every $t \ge 0$. Condition (C1) is replaced by the much weaker condition (5.4). In particular, Theorem 5.1 needs no restrictions on the sign of the coefficient $\frac{b'}{b}$. As the second example of section 6 shows, a decay of the energy is possible even if $\frac{b'}{b}$ changes sign infinitely often

Theorem 5.1 can be reached via an alternative route: First, Theorem 5.1 has to be proved for the special case $a = 1$. For an arbitrary strictly positive function

$$a \in C^2([0,\infty),\mathbb{R}) \setminus L^1([0,\infty))$$

the change of variables $s = A(t)$ leads to

$$w_{ss}(s,x) - \Delta w(s,x) + 2\psi(t) w_s(s,x) = 0 \quad \text{for } s \in A([0,\infty)).$$

Since a is not integrable over $[0,\infty)$, it follows that $A([0,\infty)) = [0,\infty)$, and the estimate for the special case $a = 1$ can be carried over to the general case.
This is not possible if $a \in L^1([0,\infty))[0,\infty)$ and ψ is unbounded: Put $\beta(s) = 2\psi(t)$, then β is unbounded as well and hence has a singularity at the right boundary of $A([0,\infty))$. A continuous extension of β on $[0,\infty)$ is then impossible. But those cases are covered be the following theorem:

Theorem 5.3. *Let $c > 0$. Suppose $a, b \in C^2([0,\infty),\mathbb{R})$ are strictly positive functions such that conditions (Q) and (S) are satisfied. Assume that φ_Q or φ_S is strictly decreasing and tends to 0, and suppose*

$$\varphi_Q(t)\varphi_S(t) \le c$$

for every $t \ge 0$. Then there exists a constant $C > 0$ such that

$$E_u(t) \le C\, a(t)b(t)E_u(0)$$

for each admissible solution of equation (1.1) and every $t \ge 0$.

Proof. First suppose φ_Q is strictly decreasing and tends to 0. In Figure 1 (b) the curve defined by $\varphi_Q(t) = \frac{1}{\lambda}$ is shown for the example $\varphi_Q = e^{-\frac{1}{2}t}$.

1. Suppose $\frac{1}{\lambda} \ge \varphi_Q(0)$. Then, by Corollary 4.4,

$$F_{\mathcal{AA}}(t,0,\lambda) \le C' \frac{a(t)b(t)}{a(0)b(0)}$$

   for every $t \ge 0$.
2. Suppose $\frac{1}{\lambda} \le \varphi_Q(0)$ and put $\alpha := \varphi_Q^{-1}\left(\frac{1}{\lambda}\right)$. Here, the estimates for $t \le \alpha$ and for $t \ge \alpha$ must be glued together.
   (a) For every $0 \le t \le \alpha$ the inequality

$$\tag{5.5} \tfrac{1}{\lambda}\varphi_S(t) = \varphi_Q(\alpha)\varphi_S(t) \le \varphi_Q(t)\varphi_S(t) \le c$$

   holds. If φ_S is increasing, this is exactly the inequality we need for Corollary 4.6 (a). Suppose φ_S is decreasing. Then we can apply Corollary 4.6 (b), if $\varphi_S(0) \le c\lambda$, which is only a special case of (5.5). Thus, by Corollary 4.6,

$$F_{\mathcal{AA}}(t,0,\lambda) \le C' \frac{a(t)b(t)}{a(0)b(0)}$$

   for every $0 \le t \le \alpha$.
   (b) Suppose $t \ge \alpha$. By Corollary 4.4 and since $\lambda\varphi_Q(\alpha) = 1$ it follows that

$$F_{\mathcal{AA}}(t,\alpha,\lambda) \le C' \frac{a(t)b(t)}{a(\alpha)b(\alpha)}.$$

   Thus

$$F_{\mathcal{AA}}(t,0,\lambda) \le F_{\mathcal{AA}}(t,\alpha,\lambda)F_{\mathcal{AA}}(\alpha,0,\lambda) \le C' \frac{a(t)b(t)}{a(0)b(0)}.$$

This shows

$$F_{\mathcal{AA}}(t,0,\lambda) \le C' \frac{a(t)b(t)}{a(0)b(0)}$$

for every $t \ge 0$ and every $\lambda > 0$, therefore

$$E_u(t) \le \int_{\mathbb{R}^n} F_{\mathcal{AA}}(t,0,|\xi|) e_u(0,\xi)\, d\xi \le C a(t)b(t)\, E_u(0)$$

for each admissible solution u of equation (1.1) and every $t \ge 0$. The case of strictly decreasing φ_S can be proven in the same way, only the roles of φ_S and φ_Q and of λ and $\frac{1}{\lambda}$ have to be reversed. □

These results can be carried over to more general equations:

Theorem 5.4. *Let H be a separable Hilbert space and $T\colon D(T) \subset H \to H$ a self-adjoint operator. Let $a \in C([0,\infty),\mathbb{R})$, let $b \in C^1([0,\infty),\mathbb{R})$ be strictly positive. Suppose there exists a function $d \in C([0,\infty),[0,\infty))$ such that for every $\lambda \in \sigma(T)$ each solution $v(\cdot,\lambda) \in C^2([0,\infty),\mathbb{C})$ of*

$$v''(t,\lambda) + a(t)^2\lambda^2 v(t,\lambda) - \tfrac{b'}{b}(t) v'(t,\lambda) = 0$$

satisfies

$$e_v(t,\lambda) \le d(t) e_v(0,\lambda) \quad \text{or} \quad e_v(t,\lambda) \le d(t)\big(e_v(0,\lambda) + |v(0,\lambda)|^2\big).$$

Then the equation

$$u''(t) + a(t)^2 T^2 u(t) - \tfrac{b'}{b}(t) u'(t) = 0$$

has exactly one solution

$$u \in C^0\big([0,\infty), D(T^2)\big) \cap C^1\big([0,\infty), D(T)\big) \cap C^2\big([0,\infty), H\big)$$

such that $u(0) = u_0$ and $u'(0) = u_1$ for $u_0 \in D(T^2)$ and $u_1 \in D(T)$. The corresponding energy satisfies

$$E_u(t) \le d(t) E_u(0) \quad \text{or} \quad E_u(t) \le d(t)\big(E_u(0) + \|u(0)\|_H^2\big),$$

respectively.

For the proof, see [1].

6. Examples

The following examples are applications of Theorems 5.1 and 5.3. In this section c denotes a positive constant, which may vary from inequality to inequality.

1. Let $\ell > -1$ and $m < 1 + 2\ell$. For $t \in [0,\infty)$ put

$$a(t) = (1+t)^\ell, \qquad b(t) = (1+t)^m.$$

Condition (5.1) is satisfied with $\varepsilon = \frac{m-\ell}{\ell+1} < 1$, since

$$\frac{(1+t)^m}{(1+\tau)^m} = \frac{(1+t)^{\ell+\varepsilon(\ell+1)}}{(1+\tau)^{\ell+\varepsilon(\ell+1)}}$$

for every $0 \le \tau \le t$. Furthermore

$$\psi(t) = \tfrac{\ell+m}{2}(1+t)^{-1-\ell},$$

$$\big(1 + A(t)\big)^{-1} \ge c(1+t)^{-1-\ell}, \qquad a(t)\big(1+A(t)\big)^2 \ge c(1+t)^{2-2\ell}$$

for every $t \in [0, \infty)$, which implies (5.2) and (5.3). Now we have to determine a function $M \colon [0, \infty) \to [0, \infty)$ as in Theorem 5.1. We can estimate

$$B(t) = \int_0^t (1+\tau)^m \, d\tau \le c \begin{cases} (1+t)^{m+1} & \text{if } m > -1, \\ \log(1+t) & \text{if } m = -1, \\ 1 & \text{if } m < -1, \end{cases}$$

and it follows that

$$a(t)^2 \frac{\big(1+B(t)\big)^2}{\big(1+A(t)\big)^2} \le c \begin{cases} (1+t)^{2m}, & \text{if } m > -1, \\ \log(1+t)^2(1+t)^{-2}, & \text{if } m = -1, \\ (1+t)^{-2}, & \text{if } m < -1. \end{cases}$$

Hence, for sufficiently large $\mu > 0$, this expression is bounded by

$$M(t) := \mu(1+t)^{\max\{\ell+m, 2m, -2\}}.$$

For the case $m = -1$, note that $2\ell > -1 + m = 2m$, or $\ell + m > -2$. Thus condition (5.3) is satisfied and

$$E_u(t) \le C(1+t)^{\max\{\ell+m, 2m, -2\}}\big(E_u(0) + \|u(0,\cdot)\|_{L^2}^2\big)$$

for each admissible solution of (1.1).

2. According to Theorem 5.1, the sign of the "damping term" $-\frac{b'}{b}$ is not very important, it is only the sign of the integral

$$\int_0^t \frac{b'}{b}(\tau)\, d\tau$$

which matters. In the following example $\frac{b'}{b}$ changes sign infinitely often, and thus violates condition (A1) from [15]. Let $m_1, m_2 \in \mathbb{R}$. For $t \ge 0$, let

$$b(t) := \exp\big(-m_1 \cos\big(\log(1+t)\big)\big)(1+t)^{m_2}.$$

Then

$$\tfrac{b'}{b}(t) = \big(m_1 \sin\big(\log(1+t)\big) + m_2\big)(1+t)^{-1},$$

in particular

$$\tfrac{b'}{b}(t) \le (m_1 + m_2)(1+t)^{-1},$$

and

$$\Big(\tfrac{b'}{b}\Big)'(t) = \big(m_1 \cos\big(\log(1+t)\big) - m_1 \sin\big(\log(1+t)\big) - m_2\big)(1+t)^{-2}.$$

Hence b satisfies the required conditions. As in the first example, b can be combined with a suitable function a of the form

$$a(t) = (1+t)^\ell,$$

such that $m_1 + m_2 < 1 + 2\ell$. Then

$$\frac{b(t)}{b(\tau)} \le \frac{(1+t)^{m_1+m_2}}{(1+\tau)^{m_1+m_2}} \le c \frac{a(t)}{a(\tau)} \frac{\big(1+A(t)\big)^\varepsilon}{\big(1+A(\tau)\big)^\varepsilon}$$

where $\varepsilon = \frac{m_1+m_2-\ell}{\ell+1}$. For $m_1 = \frac{5}{2}$ and $m_2 = -2$, for example, $\frac{b'}{b}$ is positive in $e^{\frac{\pi}{2}+2\pi N}$ and negative in $e^{\pi N}$ for every $N \in \mathbb{N}$. If $a = 1$, then the energy of the corresponding solution decays like $(1+t)^{-2}$, in spite of the oscillations of $\frac{b'}{b}$.

3. Let $\ell \geq 0$, $m < 2\ell$ and
$$a(t) = e^{\ell t}, \qquad b(t) = e^{mt}$$
for $t \in [0,\infty)$. If $\varepsilon = \frac{m-\ell}{\ell} < 1$, then
$$e^{m(t-\tau)} \leq e^{\ell(t-\tau)+\varepsilon\ell(t-\tau)}$$
for every $0 \leq \tau \leq t$, and this implies (5.1). Conditions (5.2) and (5.3) are satisfied since
$$\psi(t) = \tfrac{\ell+m}{2}e^{-\ell t}, \qquad 1 + A(t) = 1 - \ell + \ell e^{\ell t}.$$
Then
$$E_u(t) \leq Ce^{\max\{0,\ell+m,2m\}}\big(E_u(0) + \|u(0,\cdot)\|_{L^2}^2\big)$$
for each admissible solution u of (1.1) and every $t \geq 0$.
4. The function a may oscillate, too. For $t \geq 0$, choose
$$a(t) = e^t(2 + \sin t), \qquad b(t) = e^{-t}.$$
Then
$$c_1e^t \leq a(t) \leq c_2e^t, \qquad c_3e^t \leq \big(1 + A(t)\big) \leq c_4e^t$$
for suitable constants $c_1, \dots, c_4 > 0$, and condition (5.1) can be checked as in the previous example. Since
$$\psi(t) = \frac{1}{2}e^{-t}\left[\frac{3}{2+\sin t} + \frac{\cos t}{(2+\sin t)^2}\right],$$
and the derivatives of the term in square brackets are bounded, conditions (5.2) and (5.3) are satisfied. Hence,
$$E_u(t) \leq C\big(E_u(0) + \|u(0,\cdot)\|_{L^2}^2\big)$$
for each admissible solution u of (1.1) and every $t \geq 0$.
5. The next example shows that Theorem 5.3 is actually an improvement of the simple estimate (2.1) of Lemma 2.1. Put
$$a(t) = e^{-\frac{1}{2}t}, \qquad b(t) = e^{-\frac{1}{2}t}e^{\cos t}$$
for $t \geq 0$. Then
$$|a(t) - b(t)| = e^{-\frac{1}{2}t}|1 - e^{\cos t}| \leq e\, e^{-\frac{1}{2}t}, \qquad a(t), b(t) \geq e^{-1}e^{-\frac{1}{2}t}$$
for every $t \geq 0$, and condition (Q) is satisfied with $\varphi_Q(t) = e^{-\frac{1}{2}t}$. Furthermore
$$\tfrac{a'}{a}(t) = -\tfrac{1}{2}, \qquad \tfrac{b'}{b}(t) = -\tfrac{1}{2} - \sin t,$$
and thus
$$\psi(t) = \tfrac{1}{2}e^{\frac{1}{2}t}\sin t, \qquad \psi'(t) = \tfrac{1}{2}e^{\frac{1}{2}t}\big(\tfrac{1}{2}\sin t + \cos t\big).$$
Since
$$|\psi(t)|,\ a(t)^2|\psi(t)|,\ |\psi'(t)| \leq e^{\frac{1}{2}t},$$
condition (S) is satisfied with $\varphi_S(t) = e^{\frac{1}{2}t}$. The product $\varphi_Q(t)\varphi_S(t)$ is obviously bounded, and by Theorem 5.3 there exists a constant $C > 0$, such that for each admissible solution of
$$u_{tt}(t,x) - e^{-\frac{1}{2}t}\Delta u(t,x) + (\tfrac{1}{2} + \sin t)u_t(t,x) = 0$$
the estimate
$$E_u(t) \leq Ce^{-t}E_u(0)$$

holds. This cannot be obtained with the simple estimate (2.1): For if $N = \lfloor \frac{t}{2\pi} \rfloor$, then

$$\int_0^t \max\Big\{ \tfrac{a'}{a}(\tau), \tfrac{b'}{b}(\tau) \Big\}\, d\tau = -\tfrac{1}{2}t + \sum_{k=1}^{N} \int_0^{\pi} \sin\tau \, d\tau = -\tfrac{1}{2}t + 2N,$$

and the simple estimate gives only

$$E_u(t) \leq \exp\Big(-t + 4 \big\lfloor \tfrac{t}{2\pi} \big\rfloor \Big) E_u(0).$$

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Johannes Emmerling, Fachbereich Mathematik und Statistik, Fach D 187, Universität Konstanz, D-78457 Konstanz, Germany