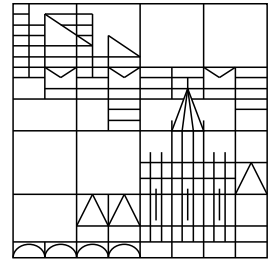


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# Solutions to the Equations of Viscous Quantum Hydrodynamics in Multiple Dimensions

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## Abstract

We study the viscous model of quantum hydrodynamics in a bounded domain of space dimension 1, 2, or 3. This model is a mixed order partial differential system with nonlocal and nonlinear terms for the particle density, current density and electric potential. By a viscous regularization approach, we show existence and uniqueness of local in time solutions.

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*Keywords:* quantum hydrodynamics, existence, uniqueness and persistence of solutions, boundary conditions of Zaremba type.

## 1 Introduction

Depending on the size of a semiconductor device and other physical aspects, there are several different models describing the flow of charged particles. As examples, we mention the (quantum) drift diffusion model, the (quantum) energy transport model, or the (quantum) hydrodynamic model. Derivations of such models can be found in, e.g., [10]. The quantum hydrodynamic model can be derived from the Schrödinger–Poisson system by WKB wave functions ([7]), or from the Wigner equations via the moment method, together with a closure of the system with the thermal equilibrium distribution ([4]). Another derivation exploits the entropy minimization principle ([8]).

Taking into account collisions of the charged particles with the background oscillators, one obtains the *viscous* quantum hydrodynamic model, as it can be derived from the Wigner equation using the Fokker-Planck collision operator:

$$\left\{ \begin{array}{l} \partial_t n - \operatorname{div} J = \nu_0 \Delta n, \\ \partial_t J - \operatorname{div} \left( \frac{J \otimes J}{n} \right) - T \nabla n + n \nabla V + \frac{\varepsilon^2}{2} n \nabla \left( \frac{\Delta \sqrt{n}}{\sqrt{n}} \right) = \nu_0 \Delta J - \frac{J}{\tau}, \\ \lambda^2 \Delta V = n - C(x), \\ (n, J)(0, x) = (n_0, J_0)(x), \end{array} \right. \quad (1.1)$$

where  $(t, x) \in (0, \infty) \times \Omega$ , and  $\Omega \subset \mathbb{R}^d$  ( $d = 1, 2, 3$ ) is a bounded domain with boundary  $\Gamma = \partial\Omega$  of regularity  $C^4$ . The unknown functions are the particle density  $n: \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}_+$ , the current density

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$J: \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}^d$ , and the electrostatic potential  $V: \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ . The function  $C: \Omega \rightarrow \mathbb{R}$  models the given profile of background charges. The (scaled) physical constants are the temperature  $T$ , the Planck constant  $\varepsilon$ , the Debye length  $\lambda$ , and a viscosity constant  $\nu_0$  as well as the momentum relaxation time  $\tau$ , which are related to the collision operator. All these constants are supposed to be positive.

The boundary conditions for the unknowns  $(n, J, V)$  are

$$\begin{cases} \partial_\nu n(t, x) = 0, \\ J(t, x) = J_\Gamma(x), & (t, x) \in (0, \infty) \times \Gamma, \\ (KV)(t, x) = g(x), \end{cases} \quad (1.2)$$

where  $\partial_\nu$  denotes the outward normal derivative, and the last line is an abbreviation for mixed Dirichlet-Neumann conditions as follows: we assume the boundary  $\Gamma$  to be split into two sub-manifolds, where the boundary part  $\Gamma_N$  is allowed to be empty.

$$\Gamma = \overline{\Gamma_D} \cup \overline{\Gamma_N}, \quad \Gamma_D \cap \Gamma_N = \emptyset, \quad \overline{\Gamma_D} \cap \overline{\Gamma_N} = Z,$$

with  $Z$  being a manifold of dimension  $d - 2$  or the empty set. We are given a pair of functions  $g = (g_D, g_N)$ , and the function  $V$  has to satisfy boundary conditions of Zaremba type,

$$\begin{cases} V(t, x) = g_D(x) & : (t, x) \in (0, \infty) \times \Gamma_D, \\ \partial_\nu V(t, x) = g_N(x) & : (t, x) \in (0, \infty) \times \Gamma_N. \end{cases}$$

We require the compatibility conditions

$$\partial_\nu n_0(x) = 0, \quad J_0(x) = J_\Gamma(x), \quad x \in \Gamma, \quad (1.3)$$

$$\inf_{x \in \Omega} n_0(x) > 0. \quad (1.4)$$

For transient quantum hydrodynamic models, only a few analytic results are available. We mention the existence of smooth solutions to the *inviscid* model ( $\nu_0 = 0$ ) and their asymptotic behavior for large time and small initial data, as investigated in [9, 13].

Concerning the transient *viscous* model (1.1), the exponential stability of a constant steady state was proved in [6] for the one-dimensional case, and in [2] for the higher-dimensional case. The local existence and uniqueness of solutions was shown in [2] for a one-dimensional setting with insulating boundary conditions, and for the case of higher dimensions with periodic boundary conditions. It seems that the viscous transient model (1.1) in higher dimensions has been analytically investigated for the first time in [2].

A few remarks about the strategy of the approach of this paper are in order. Putting  $U = (n, J_1, \dots, J_d)^T$  and observing that the quantum correction term (also called Bohm potential term) can be expressed as

$$\frac{\varepsilon^2}{2} n \nabla \left( \frac{\Delta \sqrt{n}}{\sqrt{n}} \right) = \frac{\varepsilon^2}{4} \nabla \Delta n - \varepsilon^2 \operatorname{div} \left( (\nabla \sqrt{n}) \otimes (\nabla \sqrt{n}) \right),$$

we can reformulate the equations for  $n$  and  $J$  from (1.1) as

$$\begin{aligned} \partial_t U + A(\partial_x)U + \begin{pmatrix} 0 \\ G \end{pmatrix} &= 0, \\ A(\partial_x) &= -\nu_0 \Delta I_d + \begin{pmatrix} 0 & -\partial_1 & -\partial_2 & \cdots & -\partial_d \\ \frac{\varepsilon^2}{4} \partial_1 \Delta - T \partial_1 & \tau^{-1} & 0 & \cdots & 0 \\ \frac{\varepsilon^2}{4} \partial_2 \Delta - T \partial_2 & 0 & \tau^{-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\varepsilon^2}{4} \partial_d \Delta - T \partial_d & 0 & 0 & \cdots & \tau^{-1} \end{pmatrix}, \\ G &= -\operatorname{div} \left( \frac{J \otimes J}{n} \right) + n \nabla V - \varepsilon^2 \operatorname{div} ((\nabla \sqrt{n}) \otimes (\nabla \sqrt{n})). \end{aligned} \quad (1.5)$$

It turns out that the matrix  $A$  is an elliptic differential matrix operator of mixed order in the sense of Douglis and Nirenberg, see [1] or [5]. This suggests to derive *a priori* estimates in a similar fashion as for parabolic systems, after having approximated the system by an introduction of a fourth order viscous regularization.

The differences between this paper and [2] are twofold: first, we do no longer require insulating or periodic boundary conditions. And second, we greatly relax the conditions on the doping profile  $C$ , which is now merely in  $L^2(\Omega)$ .

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## 2 Main Results

Our notations are standard:  $L^p$  denote the usual Lebesgue spaces of real-valued functions, and  $H^k(\Omega) := W_2^k(\Omega)$  are the  $L^2$ -based Sobolev spaces, for  $k \in \mathbb{N}_0$ . The expression  $\langle \cdot, \cdot \rangle$  stands for the scalar product in  $L^2(\Omega)$  as well as in  $(L^2(\Omega))^d$ .

**Theorem 2.1.** *Suppose  $n_0 \in H^3(\Omega)$ ,  $J_0 \in H^2(\Omega)$ ,  $J_\Gamma \in H^{3/2}(\Gamma)$ ,  $C \in L^2(\Omega)$ ,  $g_D \in H^{3/2}(\Gamma_D)$ ,  $g_N \in H^{1/2}(\Gamma_N)$  the compatibility condition (1.3), and (1.4).*

*Then the system (1.1) with the boundary conditions (1.2) has a unique local in time solution  $(n, J, V)$  with*

$$\begin{aligned} n &\in L^\infty((0, t_*), H^3(\Omega)), & J &\in L^\infty((0, t_*), H^2(\Omega)), \\ \partial_t n &\in L^2((0, t_*), H^2(\Omega)), & \partial_t J &\in L^2((0, t_*), H^1(\Omega)), \\ (n, \nabla n, J) &\in C([0, t_*] \times \bar{\Omega}), \\ V &\in C([0, t_*], H_{\text{loc}}^2(\Omega)) \cap C([0, t_*], H^1(\Omega)), \\ \partial_t V &\in L^2((0, t_*), H^1(\Omega)). \end{aligned}$$

*This solution persists as long as  $n$  stays positive and  $(n, \nabla n, J)$  are bounded in  $L^\infty(\Omega)$ .*

**Remark 2.2.** *After slight modifications of the proof, time-dependent boundary conditions can be treated, too.*

### 3 Proof of Theorem 2.1

We introduce a viscous regularization term  $\gamma \Delta^2$  into (1.1). Its purpose is to ensure the existence of approximate solutions (to be shown in the appendix); we will never use it for *a priori* estimates:

$$\left\{ \begin{array}{l} \partial_t n_\gamma + \gamma \Delta^2 n_\gamma - \nu_0 \Delta n_\gamma - \operatorname{div} J_\gamma = 0, \\ \partial_t J_\gamma + \gamma \Delta^2 J_\gamma - \nu_0 \Delta J_\gamma + \frac{\varepsilon^2}{4} \nabla \Delta n_\gamma \\ \quad - T \nabla n_\gamma + \frac{1}{\tau} J_\gamma + G_\gamma = 0, \\ \lambda^2 \Delta V_\gamma = n_\gamma - C(x), \\ (n_\gamma, J_\Gamma)(0, x) = (n_{0,\gamma}, J_{0,\gamma})(x), \\ (\partial_\nu n_\gamma, J_\gamma, KV_\gamma)(t, x) = (0, J_{\Gamma,\gamma}(x), g(x)) \quad \text{on } (0, \infty) \times \Gamma, \end{array} \right. \quad (3.1)$$

where  $0 < \gamma < 1$  and

$$G_\gamma = -\operatorname{div} \left( \frac{J_\gamma \otimes J_\gamma}{n_\gamma} \right) + n_\gamma \nabla V_\gamma - \varepsilon^2 \operatorname{div} \left( (\nabla \sqrt{n_\gamma}) \otimes (\nabla \sqrt{n_\gamma}) \right). \quad (3.2)$$

Additionally, we require the boundary conditions

$$\partial_\nu \Delta n_\gamma(t, x) = 0, \quad \Delta J_\gamma(t, x) = 0, \quad (t, x) \in (0, \infty) \times \Gamma. \quad (3.3)$$

The given data  $(n_{0,\gamma}, J_{0,\gamma}, J_{\Gamma,\gamma}) \in H^6(\Omega) \times H^6(\Omega) \times H^4(\Gamma)$  are suitably constructed approximations for  $(n_0, J_0, J_\Gamma)$ ; and, for  $\gamma \rightarrow 0$ , we have convergence

$$n_{0,\gamma} \rightarrow n_0 \quad \text{in } H^3(\Omega), \quad J_{0,\gamma} \rightarrow J_0 \quad \text{in } H^2(\Omega), \quad J_{\Gamma,\gamma} \rightarrow J_\Gamma \quad \text{in } H^{3/2}(\Gamma).$$

We define a function  $J_{D,\gamma} \in H^4(\Omega)$  as solution to the elliptic boundary value problem

$$\left\{ \begin{array}{ll} \Delta J_{D,\gamma}(x) = 0 & : x \in \Omega, \\ J_{D,\gamma}(x) = J_{\Gamma,\gamma}(x) & : x \in \Gamma, \end{array} \right.$$

and get uniform in  $\gamma$  estimates  $\|J_{D,\gamma}\|_{H^2(\Omega)} \leq C$ . Note that  $\Delta J_{D,\gamma} \in C(\overline{\Omega})$  due to the continuous embedding  $H^2(\Omega) \subset C(\overline{\Omega})$ , hence  $\Delta J_{D,\gamma} = 0$  on  $\Gamma$ .

According to Proposition A.1, this regularized system has a unique solution  $(n_\gamma, J_\gamma, V_\gamma)$  with

$$\begin{aligned} (n_\gamma, J_\gamma) &\in C([0, t_\gamma], H^3(\Omega)) \cap L^\infty((0, t_\gamma), H^4(\Omega)), \\ (\partial_t n_\gamma, \partial_t J_\gamma) &\in L^\infty((0, t_\gamma), H^2(\Omega)) \cap L^2((0, t_\gamma), H^4(\Omega)), \\ (V_\gamma, \partial_t V_\gamma) &\in L^\infty((0, t_\gamma), H^1(\Omega)), \end{aligned}$$

and the solution persists as long as  $(n_\gamma, \nabla n_\gamma, J_\gamma)$  stay bounded in  $L^\infty(\Omega)$ . The time derivatives of  $n_\gamma$  and  $J_\gamma$  satisfy the same boundary conditions as  $n_\gamma, J_\gamma - J_{D,\gamma}$ .

Choose a number  $\delta_0 > 0$ , subject to the conditions

$$\delta_0 \leq \frac{1}{2} \inf_{x \in \Omega} n_0(x), \quad 2 \max \left( \|n_0\|_{L^\infty(\Omega)}, \|\nabla n_0\|_{L^\infty(\Omega)}, \|J_0\|_{L^\infty(\Omega)} \right) \leq \delta_0^{-1}.$$

By the continuous embedding  $H^3(\Omega) \subset C^1(\bar{\Omega})$ , we can assume that

$$\delta_0 \leq \inf_{x \in \Omega} n_\gamma(t, x), \quad \max \left( \|n_\gamma(t, \cdot)\|_{L^\infty(\Omega)}, \|\nabla n_\gamma(t, \cdot)\|_{L^\infty(\Omega)}, \|J_\gamma(t, \cdot)\|_{L^\infty(\Omega)} \right) \leq \delta_0^{-1}, \quad (3.4)$$

for  $0 \leq t \leq t_\gamma$ ; otherwise, we shrink the interval  $[0, t_\gamma]$ . In the following, we will prove uniform in  $\gamma$  estimates of  $(n_\gamma, J_\gamma, V_\gamma)$ , which will imply that the life span  $t_\gamma$  can not tend to zero for  $\gamma$  going to zero. Then we will have a uniform existence interval as well as uniform estimates and can prove the convergence of a subsequence  $(n_\gamma, J_\gamma, V_\gamma)_\gamma$  by compactness arguments.

In the sequel,  $C$  will denote a generic constant which may change from line to line and is allowed to depend on  $\delta_0$  and  $\|J_{D,\gamma}\|_{H^2(\Omega)}$ , but not  $\gamma$ .

First of all: because of  $\|g_D\|_{H^{3/2}(\Gamma_D)} \leq C$ ,  $\|g_N\|_{H^{1/2}(\Gamma_N)} \leq C$  and  $\|n_\gamma(t, \cdot) - C(\cdot)\|_{L^2(\Omega)} \leq C$ , we obtain

$$\|V_\gamma(t, \cdot)\|_{H^1(\Omega)} \leq C(1 + \lambda^{-2}), \quad 0 \leq t \leq t_\gamma.$$

For later reference, we remark that the embedding  $H^1(\Omega) \subset L^4(\Omega)$  then yields a uniform estimate of  $\|V_\gamma(t, \cdot)\|_{L^4(\Omega)}$ . Similarly, we have

$$\|\partial_t V_\gamma(t, \cdot)\|_{L^4(\Omega)} \leq C \|\partial_t V_\gamma(t, \cdot)\|_{H^1(\Omega)} \leq C \lambda^{-2} \|\partial_t n_\gamma\|_{L^2(\Omega)}.$$

We multiply the parabolic equation for  $n_\gamma, J_\gamma$ , respectively, with  $n_\gamma$  or  $J_\gamma - J_{D,\gamma}$ , respectively, integrate over  $\Omega$ , and perform partial integration where appropriate:

$$\begin{aligned} & \frac{1}{2} \partial_t \|n_\gamma\|_{L^2(\Omega)}^2 + \nu_0 \|\nabla n_\gamma\|_{L^2(\Omega)}^2 + \gamma \|\Delta n_\gamma\|_{L^2(\Omega)}^2 - \langle \operatorname{div} J_\gamma, n_\gamma \rangle = 0, \\ & \frac{1}{2} \partial_t \|J_\gamma - J_{D,\gamma}\|_{L^2(\Omega)}^2 + \nu_0 \|\nabla(J_\gamma - J_{D,\gamma})\|_{L^2(\Omega)}^2 + \gamma \|\Delta J_\gamma\|_{L^2(\Omega)}^2 + \frac{1}{\tau} \|J_\gamma\|_{L^2(\Omega)}^2 \\ & \quad + T \langle n_\gamma, \operatorname{div}(J_\gamma - J_{D,\gamma}) \rangle - \frac{\varepsilon^2}{4} \langle \Delta n_\gamma, \operatorname{div}(J_\gamma - J_{D,\gamma}) \rangle \\ & \quad - \frac{1}{\tau} \langle J_\gamma, J_{D,\gamma} \rangle + \langle G_\gamma, J_\gamma - J_{D,\gamma} \rangle = 0. \end{aligned}$$

Here,  $\|\nabla J_\gamma\|_{L^2(\Omega)}$  stands for the Frobenius norm:  $\|\nabla J_\gamma\|^2 = \sum_{k,l} \|\partial_k J_{\gamma,l}\|^2$ .

The linear contribution from the quantum term is handled by means of

$$\begin{aligned} \langle \Delta n_\gamma, \operatorname{div} J_\gamma \rangle &= \langle \Delta n_\gamma, \partial_t n_\gamma \rangle - \langle \Delta n_\gamma, \nu_0 \Delta n_\gamma \rangle + \langle \Delta n_\gamma, \gamma \Delta^2 n_\gamma \rangle \\ &= -\frac{1}{2} \partial_t \|\nabla n_\gamma\|_{L^2(\Omega)}^2 - \nu_0 \|\Delta n_\gamma\|_{L^2(\Omega)}^2 - \gamma \|\nabla \Delta n_\gamma\|_{L^2(\Omega)}^2, \end{aligned}$$

which gives us the identity

$$\begin{aligned} & \frac{1}{2} \partial_t \left( \|J_\gamma - J_{D,\gamma}\|_{L^2(\Omega)}^2 + T \|n_\gamma\|_{L^2(\Omega)}^2 + \frac{\varepsilon^2}{4} \|\nabla n_\gamma\|_{L^2(\Omega)}^2 \right) \\ & \quad + \nu_0 \|\nabla(J_\gamma - J_{D,\gamma})\|_{L^2(\Omega)}^2 + T \nu_0 \|\nabla n_\gamma\|_{L^2(\Omega)}^2 + \frac{\varepsilon^2}{4} \nu_0 \|\Delta n_\gamma\|_{L^2(\Omega)}^2 + \frac{1}{\tau} \|J_\gamma\|_{L^2(\Omega)}^2 \\ & \quad + \gamma \left( \|\Delta J_\gamma\|_{L^2(\Omega)}^2 + T \|\Delta n_\gamma\|_{L^2(\Omega)}^2 + \frac{\varepsilon^2}{4} \|\nabla \Delta n_\gamma\|_{L^2(\Omega)}^2 \right) \\ & = T \langle n_\gamma, \operatorname{div} J_{D,\gamma} \rangle - \frac{\varepsilon^2}{4} \langle \Delta n_\gamma, \operatorname{div} J_{D,\gamma} \rangle + \frac{1}{\tau} \langle J_\gamma, J_{D,\gamma} \rangle - \langle G_\gamma, J_\gamma - J_{D,\gamma} \rangle. \end{aligned} \quad (3.5)$$

In system (3.1), we take time derivatives, and define

$$n'_\gamma := \partial_t n_\gamma, \quad J'_\gamma := \partial_t J_\gamma, \quad G'_\gamma := \partial_t G(n_\gamma, J_\gamma, V_\gamma).$$

Then we deduce, by a similar computation as before, that

$$\begin{aligned} & \frac{1}{2} \partial_t \left( \|J'_\gamma\|_{L^2(\Omega)}^2 + T \|n'_\gamma\|_{L^2(\Omega)}^2 + \frac{\varepsilon^2}{4} \|\nabla n'_\gamma\|_{L^2(\Omega)}^2 \right) \\ & + \nu_0 \|\nabla J'_\gamma\|_{L^2(\Omega)}^2 + T \nu_0 \|\nabla n'_\gamma\|_{L^2(\Omega)}^2 + \frac{\varepsilon^2}{4} \nu_0 \|\Delta n'_\gamma\|_{L^2(\Omega)}^2 + \frac{1}{\tau} \|J'_\gamma\|_{L^2(\Omega)}^2 \\ & + \gamma \left( \|\Delta J'_\gamma\|_{L^2(\Omega)}^2 + T \|\Delta n'_\gamma\|_{L^2(\Omega)}^2 + \frac{\varepsilon^2}{4} \|\nabla \Delta n'_\gamma\|_{L^2(\Omega)}^2 \right) \\ & = - \langle G'_\gamma, J'_\gamma \rangle. \end{aligned} \tag{3.6}$$

The scalar products  $\langle G_\gamma, J_\gamma - J_{D,\gamma} \rangle$  and  $\langle G'_\gamma, J'_\gamma \rangle$  have the representations

$$\begin{aligned} \langle G_\gamma, J_\gamma - J_{D,\gamma,l} \rangle &= I_1 + I_2 + I_3 \\ &= \sum_l \int_\Omega \frac{J_{\gamma,l}}{n_\gamma} J_\gamma \nabla (J_{\gamma,l} - J_{D,\gamma,l}) \, dx + \int_\Omega n_\gamma (\nabla V_\gamma) (J_\gamma - J_{D,\gamma}) \, dx \\ &+ \varepsilon^2 \sum_l \int_\Omega (\partial_l \sqrt{n_\gamma}) (\nabla \sqrt{n_\gamma}) \nabla (J_{\gamma,l} - J_{D,\gamma,l}) \, dx, \\ \langle G'_\gamma, J'_\gamma \rangle &= I'_1 + I'_2 + I'_3 \\ &= \sum_l \int_\Omega \left( \partial_t \frac{J_{\gamma,l}}{n_\gamma} J_\gamma \right) (\nabla J'_{\gamma,l}) \, dx + \int_\Omega (\partial_t n_\gamma (\nabla V_\gamma)) J'_\gamma \, dx \\ &+ \varepsilon^2 \sum_l \int_\Omega (\partial_t (\partial_l \sqrt{n_\gamma}) (\nabla \sqrt{n_\gamma})) (\nabla J'_{\gamma,l}) \, dx, \end{aligned}$$

and the items  $I_k$  can be estimated as follows:

$$\begin{aligned} |I_1| &\leq C \|\nabla (J_\gamma - J_{D,\gamma})\|_{L^2(\Omega)} \leq \frac{\nu_0}{4} \|\nabla (J_\gamma - J_{D,\gamma})\|_{L^2(\Omega)}^2 + C \nu_0^{-1}, \\ |I_2| &\leq C \|V_\gamma\|_{H^1(\Omega)} \|J_\gamma - J_{D,\gamma}\|_{L^2(\Omega)} \leq C(1 + \lambda^{-2}) \|J_\gamma - J_{D,\gamma}\|_{L^2(\Omega)} \\ &\leq C(1 + \lambda^{-4}) + \|J_\gamma - J_{D,\gamma}\|_{L^2(\Omega)}^2, \\ |I_3| &\leq C \varepsilon^2 \|\nabla (J_\gamma - J_{D,\gamma})\|_{L^2(\Omega)} \leq \frac{\nu_0}{4} \|\nabla (J_\gamma - J_{D,\gamma})\|_{L^2(\Omega)}^2 + C \varepsilon^4 \nu_0^{-1}. \end{aligned}$$

In treating the other terms  $I'_k$ , note that  $\|n'_\gamma\|_{L^4(\Omega)} \leq C(1 + \|\nabla n'_\gamma\|_{L^2(\Omega)})$  due to the identity  $\int_\Omega n'_\gamma \, dx = \int_{\partial\Omega} J_{\Gamma,\gamma} \cdot \vec{\nu} \, d\sigma$  and the Poincaré-Sobolev inequality:

$$\begin{aligned} |I'_1| &\leq \frac{\nu_0}{6} \|\nabla J'_\gamma\|_{L^2(\Omega)}^2 + C \nu_0^{-1} \left( \|J'_\gamma\|_{L^2(\Omega)}^2 + \|n'_\gamma\|_{L^2(\Omega)}^2 \right), \\ |I'_2| &\leq \|\nabla n'_\gamma\|_{L^2(\Omega)} \|V_\gamma\|_{L^4(\Omega)} \|J'_\gamma\|_{L^4(\Omega)} + \|\nabla n_\gamma\|_{L^\infty(\Omega)} \|\partial_t V_\gamma\|_{L^2(\Omega)} \|J'_\gamma\|_{L^2(\Omega)} \\ &+ \|n'_\gamma\|_{L^4(\Omega)} \|V_\gamma\|_{L^4(\Omega)} \|\nabla J'_\gamma\|_{L^2(\Omega)} + \|n_\gamma\|_{L^\infty(\Omega)} \|\partial_t V_\gamma\|_{L^2(\Omega)} \|\nabla J'_\gamma\|_{L^2(\Omega)} \\ &\leq C \|\nabla n'_\gamma\|_{L^2(\Omega)} (1 + \lambda^{-2}) \|\nabla J'_\gamma\|_{L^2(\Omega)} + C \lambda^{-2} \|n'_\gamma\|_{L^2(\Omega)} \|J'_\gamma\|_{H^1(\Omega)} \\ &\leq \frac{\nu_0}{6} \|\nabla J'_\gamma\|_{L^2(\Omega)}^2 + C(\nu_0^{-1} + \lambda^{-4} \nu_0^{-1}) \|\nabla n'_\gamma\|_{L^2(\Omega)}^2 \\ &+ C(\nu_0^{-1} + \lambda^{-4} \nu_0^{-1} + \lambda^{-4}) \|n'_\gamma\|_{L^2(\Omega)}^2 + \|J'_\gamma\|_{L^2(\Omega)}^2, \end{aligned}$$

$$|I_3| \leq C\varepsilon^2 \|\nabla n'_\gamma\|_{L^2(\Omega)} \|\nabla J'_\gamma\|_{L^2(\Omega)} \leq \frac{\nu_0}{6} \|\nabla J'_\gamma\|_{L^2(\Omega)}^2 + C\varepsilon^4 \nu_0^{-1} \|\nabla n'_\gamma\|_{L^2(\Omega)}^2.$$

Plugging these estimates into (3.5) and (3.6), we then find

$$\begin{aligned} & \partial_t \left( \|J_\gamma - J_{D,\gamma}\|_{L^2(\Omega)}^2 + T \|n_\gamma\|_{L^2(\Omega)}^2 + \frac{\varepsilon^2}{4} \|\nabla n_\gamma\|_{L^2(\Omega)}^2 \right) \\ & \quad + \nu_0 \|\nabla(J_\gamma - J_{D,\gamma})\|_{L^2(\Omega)}^2 + \frac{\varepsilon^2 \nu_0}{4} \|\Delta n_\gamma\|_{L^2(\Omega)}^2 \\ & \leq C(1 + \nu_0^{-1} + \lambda^{-4} + \varepsilon^4 \nu_0^{-1} + T + \tau^{-1}) + \|J_\gamma - J_{D,\gamma}\|_{L^2(\Omega)}^2, \\ & \partial_t \left( \|J'_\gamma\|_{L^2(\Omega)}^2 + T \|n'_\gamma\|_{L^2(\Omega)}^2 + \frac{\varepsilon^2}{4} \|\nabla n'_\gamma\|_{L^2(\Omega)}^2 \right) \\ & \quad + \nu_0 \|\nabla J'_\gamma\|_{L^2(\Omega)}^2 + 2T\nu_0 \|\nabla n'_\gamma\|_{L^2(\Omega)}^2 + \frac{\varepsilon^2 \nu_0}{2} \|\Delta n'_\gamma\|_{L^2(\Omega)}^2 \\ & \leq C(\nu_0^{-1} + 1) \|J'_\gamma\|_{L^2(\Omega)}^2 + C(\nu_0^{-1} + \lambda^{-4} \nu_0^{-1} + \lambda^{-4}) \|n'_\gamma\|_{L^2(\Omega)}^2 \\ & \quad + C(\nu_0^{-1} + \lambda^{-4} \nu_0^{-1} + \varepsilon^4 \nu_0^{-1}) \|\nabla n'_\gamma\|_{L^2(\Omega)}^2 \\ & \leq C(\nu_0^{-1} + 1 + (\nu_0^{-1} + \lambda^{-4} \nu_0^{-1} + \lambda^{-4})T^{-1} + (\nu_0^{-1} + \lambda^{-4} \nu_0^{-1} + \varepsilon^4 \nu_0^{-1})\varepsilon^{-2}) \times \\ & \quad \times \left( \|J'_\gamma\|_{L^2(\Omega)}^2 + T \|n'_\gamma\|_{L^2(\Omega)}^2 + \frac{\varepsilon^2}{4} \|\nabla n'_\gamma\|_{L^2(\Omega)}^2 \right). \end{aligned}$$

We can assume that

$$\begin{aligned} (1 + \nu_0^{-1} + \lambda^{-4} + \varepsilon^4 \nu_0^{-1} + T + \tau^{-1}) t_\gamma & \leq 1, \\ (\nu_0^{-1} + 1 + (\nu_0^{-1} + \lambda^{-4} \nu_0^{-1} + \lambda^{-4})T^{-1} + (\nu_0^{-1} + \lambda^{-4} \nu_0^{-1} + \varepsilon^4 \nu_0^{-1})\varepsilon^{-2}) t_\gamma & \leq 1. \end{aligned}$$

Making use of Gronwall's Lemma, we then conclude that

$$\begin{aligned} & \|J_\gamma - J_{D,\gamma}\|_{L^\infty((0,t),L^2(\Omega))}^2 + T \|n_\gamma\|_{L^\infty((0,t),L^2(\Omega))}^2 + \frac{\varepsilon^2}{4} \|\nabla n_\gamma\|_{L^\infty((0,t),L^2(\Omega))}^2 \\ & \quad + \nu_0 \|\nabla(J_\gamma - J_{D,\gamma})\|_{L^2(Q_t)}^2 + \frac{\varepsilon^2 \nu_0}{4} \|\Delta n_\gamma\|_{L^2(Q_t)}^2 \\ & \leq C \left( 1 + \|J_{0,\gamma} - J_{D,\gamma}\|_{L^2(\Omega)}^2 + T \|n_{0,\gamma}\|_{L^2(\Omega)}^2 + \frac{\varepsilon^2}{4} \|\nabla n_{0,\gamma}\|_{L^2(\Omega)}^2 \right), \\ & \|J'_\gamma\|_{L^\infty((0,t),L^2(\Omega))}^2 + T \|n'_\gamma\|_{L^\infty((0,t),L^2(\Omega))}^2 + \frac{\varepsilon^2}{4} \|\nabla n'_\gamma\|_{L^\infty((0,t),L^2(\Omega))}^2 \\ & \quad + \nu_0 \|\nabla J'_\gamma\|_{L^2(Q_t)}^2 + 2T\nu_0 \|\nabla n'_\gamma\|_{L^2(Q_t)}^2 + \frac{\varepsilon^2 \nu_0}{2} \|\Delta n'_\gamma\|_{L^2(Q_t)}^2 \\ & \leq C \left( \|J'_\gamma(0, \cdot)\|_{L^2(\Omega)}^2 + T \|n'_\gamma(0, \cdot)\|_{L^2(\Omega)}^2 + \frac{\varepsilon^2}{4} \|\nabla n'_\gamma(0, \cdot)\|_{L^2(\Omega)}^2 \right), \end{aligned}$$

where  $0 \leq t \leq t_\gamma$  and  $Q_t := (0, t) \times \Omega$ . The time derivatives at  $t = 0$  can be estimated as follows:

$$\begin{aligned} \|J'_\gamma(0, \cdot)\|_{L^2(\Omega)} & \leq \gamma \|J_{0,\gamma}\|_{H^4(\Omega)} + \nu_0 \|J_{0,\gamma}\|_{H^2(\Omega)} + \frac{\varepsilon^2}{4} \|n_{0,\gamma}\|_{H^3(\Omega)} + C(\delta_0)T \\ & \quad + C \left( \|J_{0,\gamma}\|_{H^1(\Omega)} + \|n_{0,\gamma}\|_{H^1(\Omega)} \right) + C\lambda^{-2} + C\varepsilon^2 \|n_{0,\gamma}\|_{H^2(\Omega)}, \\ \|n'_\gamma(0, \cdot)\|_{H^1(\Omega)} & \leq \gamma \|n_{0,\gamma}\|_{H^5(\Omega)} + \nu_0 \|n_{0,\gamma}\|_{H^3(\Omega)} + \|J_{0,\gamma}\|_{H^2(\Omega)}. \end{aligned}$$



We may assume that the numbers  $\gamma$  are chosen in such a way that

$$\lim_{\gamma \rightarrow 0} \left( \gamma \|J_{0,\gamma}\|_{H^4(\Omega)} + \gamma \|n_{0,\gamma}\|_{H^5(\Omega)} \right) = 0.$$

With this choice of  $\gamma$ , the following uniform in  $\gamma$  estimates have been obtained:

$$\begin{aligned} \|n_\gamma\|_{C([0,t_\gamma] \times \bar{\Omega})} + \|\nabla n_\gamma\|_{C([0,t_\gamma] \times \bar{\Omega})} + \|J_\gamma\|_{C([0,t_\gamma] \times \bar{\Omega})} &\leq C_0, \\ \|J_\gamma\|_{L^2((0,t_\gamma), H^1(\Omega))} + \|n_\gamma\|_{L^2((0,t_\gamma), H^2(\Omega))} &\leq C_0, \\ \|J'_\gamma\|_{L^\infty((0,t_\gamma), L^2(\Omega))} + \|n'_\gamma\|_{L^\infty((0,t_\gamma), H^1(\Omega))} &\leq C_0, \\ \|J'_\gamma\|_{L^2((0,t_\gamma), H^1(\Omega))} + \|n'_\gamma\|_{L^2((0,t_\gamma), H^2(\Omega))} &\leq C_0, \end{aligned}$$

where the constant  $C_0$  is allowed to depend on all physical constants,  $\delta_0$ , and the norms  $\|n_0\|_{H^3(\Omega)}$ ,  $\|J_0\|_{H^2(\Omega)}$ ,  $\|J_{D,\gamma}\|_{H^2(\Omega)}$ , but not  $\gamma$ .

To get more estimates, we consider certain elliptic boundary value problems:

**Definition 3.1.** For  $f \in L^2(\Omega)$  and  $\nu_0, \gamma > 0$ , let  $u = Q_\gamma f \in H^2(\Omega)$  be the solution to the problem

$$\begin{cases} (\nu_0 - \gamma \Delta)u(x) = f(x) & : x \in \Omega, \\ \partial_\nu u(x) = 0, & : x \in \Gamma, \end{cases}$$

**Lemma 3.2.** The operator  $Q_\gamma$  is a bounded endomorphism on  $L^2(\Omega)$  and on  $H^1(\Omega)$ ,

$$\|Q_\gamma f\|_{L^2(\Omega)} \leq \frac{1}{\nu_0} \|f\|_{L^2(\Omega)}, \quad \|\nabla Q_\gamma f\|_{L^2(\Omega)} \leq \frac{1}{\nu_0} \|\nabla f\|_{L^2(\Omega)}.$$

The operator  $Q_\gamma \Delta$ , first defined on  $H^2(\Omega)$ , extends to a continuous endomorphism on  $L^2(\Omega)$ :

$$\|Q_\gamma \Delta f\|_{L^2(\Omega)} \leq \frac{1}{\gamma} \|f\|_{L^2(\Omega)}, \quad f \in L^2(\Omega).$$

Moreover,  $Q_\gamma \Delta$  is non-positive:

$$\langle Q_\gamma \Delta f, f \rangle \leq 0, \quad f \in L^2(\Omega).$$

*Proof.* Write  $u = Q_\gamma f$ . Taking the  $L^2(\Omega)$  scalar product of the equation  $(\nu_0 - \gamma \Delta)u = f$  with  $u$  (with  $\Delta u$ ) and performing appropriate integrations by parts gives the first estimate (second estimate).

Let  $0 \leq \lambda_1 \leq \lambda_2 \leq \dots$  denote the eigenvalues of  $-\Delta$  with homogeneous Neumann boundary conditions on  $\Gamma$ , counted according to multiplicity, and call the associated eigenfunctions  $\varphi_j$ , normalized by  $\|\varphi_j\|_{L^2(\Omega)} = 1$ . Then  $(\varphi_1, \varphi_2, \dots)$  is an orthonormal basis of  $L^2(\Omega)$ , and the space  $X$  of finite linear combinations is a sub-space of  $H^2(\Omega)$  and dense in  $L^2(\Omega)$ . Write  $f \in X$  as  $f = \sum_{j=1}^N \alpha_j \varphi_j$ .

Then we have

$$\begin{aligned} Q_\gamma \Delta f &= \sum_{j=1}^N \frac{-\lambda_j}{\nu_0 + \gamma \lambda_j} \alpha_j \varphi_j, \\ \|Q_\gamma \Delta f\|_{L^2(\Omega)}^2 &= \sum_{j=1}^N \frac{\lambda_j^2}{(\nu_0 + \gamma \lambda_j)^2} |\alpha_j|^2 \leq \frac{1}{\gamma^2} \sum_{j=1}^N |\alpha_j|^2 = \frac{1}{\gamma^2} \|f\|_{L^2(\Omega)}^2, \\ \langle Q_\gamma \Delta f, f \rangle &= \sum_{j=1}^N \frac{-\lambda_j}{\nu_0 + \gamma \lambda_j} |\alpha_j|^2 \leq 0. \end{aligned}$$

A density argument concludes the proof of Lemma 3.2.  $\square$

Then we have, for a.e.  $t \in (0, t_\gamma)$ , the representation

$$(\Delta n_\gamma)(t, x) = Q_\gamma(\partial_t n_\gamma - \operatorname{div} J_\gamma)(t, x).$$

The parabolic equation for  $J_\gamma$  can be expressed as

$$(\nu_0 - \gamma \Delta) \Delta J_\gamma - \frac{\varepsilon^2}{4} \nabla(\Delta n_\gamma) = \partial_t J_\gamma + \frac{1}{\tau} J_\gamma - T \nabla n_\gamma + G_\gamma,$$

from which we deduce that

$$(\nu_0 - \gamma \Delta) \Delta J_\gamma + \frac{\varepsilon^2}{4} \nabla(Q_\gamma(\operatorname{div} J_\gamma)) = \partial_t J_\gamma + \frac{1}{\tau} J_\gamma - T \nabla n_\gamma + G_\gamma + \frac{\varepsilon^2}{4} \nabla Q_\gamma(\partial_t n_\gamma) =: R_\gamma.$$

We take the  $L^2(\Omega)$  scalar product of  $R_\gamma$  with  $\Delta J_\gamma \in H^2(\Omega) \cap H_0^1(\Omega)$ :

$$\begin{aligned} \langle R_\gamma, \Delta J_\gamma \rangle &= \nu_0 \|\Delta J_\gamma\|_{L^2(\Omega)}^2 - \gamma \langle \Delta^2 J_\gamma, \Delta J_\gamma \rangle + \frac{\varepsilon^2}{4} \langle \nabla(Q_\gamma(\operatorname{div} J_\gamma)), \Delta J_\gamma \rangle \\ &= \nu_0 \|\Delta J_\gamma\|_{L^2(\Omega)}^2 + \gamma \|\nabla \Delta J_\gamma\|_{L^2(\Omega)}^2 - \frac{\varepsilon^2}{4} \langle Q_\gamma(\operatorname{div} J_\gamma), \operatorname{div} \Delta J_\gamma \rangle \\ &= \nu_0 \|\Delta J_\gamma\|_{L^2(\Omega)}^2 + \gamma \|\nabla \Delta J_\gamma\|_{L^2(\Omega)}^2 - \frac{\varepsilon^2}{4} \langle \operatorname{div} J_\gamma, Q_\gamma(\Delta \operatorname{div} J_\gamma) \rangle \\ &\geq \nu_0 \|\Delta J_\gamma\|_{L^2(\Omega)}^2 + \gamma \|\nabla \Delta J_\gamma\|_{L^2(\Omega)}^2, \end{aligned}$$

due to Lemma 3.2. Consequently, we arrive at

$$\|\Delta J_\gamma\|_{L^2(\Omega)} \leq \nu_0^{-1} \|R_\gamma\|_{L^2(\Omega)}.$$

A careful analysis of  $R_\gamma$  shows  $R_\gamma \in L^\infty((0, t_\gamma), L^2(\Omega))$  with uniform in  $\gamma$  estimate, whence

$$\|J_\gamma\|_{L^\infty((0, t_\gamma), H^2(\Omega))} \leq C_0.$$

Going back to  $\Delta n_\gamma = Q_\gamma(\partial_t n_\gamma - \operatorname{div} J_\gamma)$  with  $(\partial_t n_\gamma - \operatorname{div} J_\gamma) \in L^\infty((0, t_\gamma), H^1(\Omega))$  we then find  $\Delta n_\gamma \in L^\infty((0, t_\gamma), H^1(\Omega))$ , and therefore

$$\|n_\gamma\|_{L^\infty((0, t_\gamma), H^3(\Omega))} \leq C_0,$$

uniformly in  $\gamma$ .

Eventually, we are in a position to show that  $t_\gamma$  can not tend to zero for  $\gamma \rightarrow 0$ : for  $0 \leq t' \leq t'' \leq t_\gamma$ , we obtain the Hölder estimates

$$\|n_\gamma(t', \cdot) - n_\gamma(t'', \cdot)\|_{H^2(\Omega)} \leq \int_{t'}^{t''} \|n'_\gamma(t, \cdot)\|_{H^2(\Omega)} dt \leq |t' - t''|^{1/2} \|n'_\gamma\|_{L^2((0, t_\gamma), H^2(\Omega))},$$

which enables us to estimate  $n_\gamma$  in  $C^{1/2}([0, t_\gamma], C(\overline{\Omega}))$ . Next, fix a number  $\alpha$  with  $0 < \alpha < \frac{1}{2}$ . By Sobolev's embedding theorem and interpolation,

$$\begin{aligned} \|J_\gamma(t', \cdot) - J_\gamma(t'', \cdot)\|_{C(\overline{\Omega})} &\leq C \|J_\gamma(t', \cdot) - J_\gamma(t'', \cdot)\|_{H^{2-\alpha}(\Omega)} \\ &\leq C \|J_\gamma(t', \cdot) - J_\gamma(t'', \cdot)\|_{L^2(\Omega)}^{\alpha/2} \|J_\gamma(t', \cdot) - J_\gamma(t'', \cdot)\|_{H^2(\Omega)}^{(2-\alpha)/2} \\ &\leq C |t' - t''|^{\alpha/2} \|J'_\gamma\|_{L^\infty((0, t_\gamma), L^2(\Omega))}^{\alpha/2} \|J_\gamma\|_{L^\infty((0, t_\gamma), H^2(\Omega))}^{(2-\alpha)/2}, \\ \|\nabla n_\gamma(t', \cdot) - \nabla n_\gamma(t'', \cdot)\|_{C(\overline{\Omega})} &\leq C |t' - t''|^{\alpha/2} \|n'_\gamma\|_{L^\infty((0, t_\gamma), H^1(\Omega))}^{\alpha/2} \|n_\gamma\|_{L^\infty((0, t_\gamma), H^3(\Omega))}^{(2-\alpha)/2}. \end{aligned}$$

The right-hand sides are bounded uniformly with respect to  $\gamma$ . Then these Hölder estimates give us an estimate from below for the earliest time  $t_*$  at which the solution  $(n_\gamma, J_\gamma)$  is able to violate the conditions (3.4). As a consequence,  $t_\gamma \geq t_* > 0$ , for  $0 < \gamma < 1$ .

Having secured a uniform existence interval, we can now show the convergence of the sequence  $(n_\gamma, J_\gamma, V_\gamma)_\gamma$  for  $\gamma \rightarrow 0$ . We have the uniform bounds

$$\|n_\gamma\|_{L^\infty(0, t_*), H^3(\Omega)} \leq C, \quad \|\partial_t n_\gamma\|_{L^\infty(0, t_*), H^2(\Omega)} \leq C,$$

and the compact embedding  $H^3(\Omega) \subset H^2(\Omega)$ . Then Aubin's Lemma [12, Corollary 4] shows that a subsequence of  $(n_\gamma)_\gamma$  (which we will not relabel) converges to a limit  $n$  in the space  $C([0, t_*], H^2(\Omega))$ . Similarly, we can prove the convergence of a subsequence  $(J_\gamma)_\gamma$  to a limit  $J$  in the space  $C([0, t_*], H^1(\Omega))$ . By interpolation, we then have the convergences

$$\begin{aligned} n_\gamma &\rightarrow n & \text{in} & C([0, t_*], H^{3-\delta}(\Omega)), & \delta > 0, \\ J_\gamma &\rightarrow J & \text{in} & C([0, t_*], H^{2-\delta}(\Omega)), & \delta > 0. \end{aligned}$$

In particular, there is uniform convergence

$$(n_\gamma, \nabla n_\gamma, J_\gamma) \rightarrow (n, \nabla n, J) \quad \text{in} \quad C(\overline{Q_*}), \quad Q_* := (0, t_*) \times \Omega,$$

which guarantees that the limit functions  $n$  and  $J$  satisfy the initial conditions from (1.1) and the boundary conditions from (1.2).

Obviously, we have the following weak convergences, too:

$$\begin{aligned} \partial_t n_\gamma &\rightharpoonup \partial_t n & \text{in} & L^2((0, t_*), H^2(\Omega)), & \partial_t J_\gamma &\rightharpoonup \partial_t J & \text{in} & L^2((0, t_*), H^1(\Omega)), \\ n_\gamma &\rightharpoonup^* n & \text{in} & L^\infty((0, t_*), H^3(\Omega)), & J_\gamma &\rightharpoonup^* J & \text{in} & L^\infty((0, t_*), H^2(\Omega)). \end{aligned}$$

Moreover, a sub-sequence  $(V_\gamma)_\gamma$  converges to a limit  $V$  in the space  $C([0, t_*], H_{\text{loc}}^2(\Omega))$ ; and this limit  $V$  solves the Poisson equation  $\lambda^2 \Delta V = n - C(x)$ .

In a next step, we show that  $(n, J, V)$  solves (1.1). We choose a test function  $\varphi \in C_0^\infty(Q_*)$ , and it follows that

$$\begin{aligned} & \iint_{Q_*} (-\varphi_t n_\gamma + \gamma(\Delta^2 \varphi) n_\gamma - \nu_0(\Delta \varphi) n_\gamma + (\nabla \varphi) J_\gamma) \, dx \, dt = 0, \\ & \iint_{Q_*} \left( -\varphi_t J_\gamma + \gamma(\Delta^2 \varphi) J_\gamma - \nu_0(\Delta \varphi) J_\gamma - \frac{\varepsilon^2}{4}(\nabla \Delta \varphi) n_\gamma + T(\nabla \varphi) n_\gamma + \varphi G_\gamma \right) \, dx \, dt = 0. \end{aligned}$$

Note that  $G_\gamma$  approaches  $G$  in the norm of the space  $C([0, t_*], L^2(\Omega))$ . Sending  $\gamma$  to zero and making use of the uniform convergence of  $(n_\gamma, J_\gamma)_\gamma$  to  $(n, J)$  we deduce that  $(n, J)$  are solutions to (1.1).

It only remains to show the uniqueness of the solutions. Let  $(n^1, J^1, V^1)$  and  $(n^2, J^2, V^2)$  be two solutions of (1.1), (1.2) with regularity as in Theorem 2.1. Define

$$n_\Delta = n^1 - n^2, \quad J_\Delta = J^1 - J^2, \quad V_\Delta = V^1 - V^2, \quad G_\Delta = G^1 - G^2,$$

where  $G^1$  and  $G^2$  are given in (1.5). Then we get the system

$$\left\{ \begin{array}{l} \partial_t n_\Delta - \nu_0 \Delta n_\Delta - \operatorname{div} J_\Delta = 0, \\ \partial_t J_\Delta - \nu_0 \Delta J_\Delta + \frac{\varepsilon^2}{4} \nabla \Delta n_\Delta - T \nabla n_\Delta + \frac{1}{\tau} J_\Delta + G_\Delta = 0, \\ \lambda^2 \Delta V_\Delta = n_\Delta, \\ (n_\Delta, J_\Delta)(0, x) = 0, \\ (\partial_\nu n_\Delta, J_\Delta(t, x), KV_\Delta)(t, x) = 0, \quad (t, x) \in (0, t_*) \times \Gamma. \end{array} \right.$$

Similarly as in (3.5), we can prove

$$\begin{aligned} & \frac{1}{2} \partial_t \left( \|J_\Delta\|_{L^2(\Omega)}^2 + T \|n_\Delta\|_{L^2(\Omega)}^2 + \frac{\varepsilon^2}{4} \|\nabla n_\Delta\|_{L^2(\Omega)}^2 \right) \\ & \quad + \nu_0 \|\nabla J_\Delta\|_{L^2(\Omega)}^2 + T \nu_0 \|\nabla n_\Delta\|_{L^2(\Omega)}^2 + \frac{\varepsilon^2}{4} \nu_0 \|\Delta n_\Delta\|_{L^2(\Omega)}^2 + \frac{1}{\tau} \|J_\Delta\|_{L^2(\Omega)}^2 \\ & = - \langle G_\Delta, J_\Delta \rangle, \end{aligned}$$

and we estimate, after some calculations,

$$\begin{aligned} |\langle G_\Delta, J_\Delta \rangle| & \leq C \left( \|J^j\|_{L^\infty(\Omega)}, \|n^j\|_{L^\infty(\Omega)}, \|\nabla n^j\|_{L^\infty(\Omega)}, \|\nabla V^j\|_{L^\infty(\Omega)} \right) \times \\ & \quad \times \left( \|\nabla J_\Delta\|_{L^2(\Omega)} \|J_\Delta\|_{L^2(\Omega)} + \|J_\Delta\|_{L^2(\Omega)} (\|n_\Delta\|_{L^2(\Omega)} + \|\nabla V_\Delta\|_{L^2(\Omega)}) \right. \\ & \quad \left. + \|\nabla J_\Delta\|_{L^2(\Omega)} \|n_\Delta\|_{H^1(\Omega)} \right). \end{aligned}$$

By Young's inequality, we then have

$$\frac{T}{2} \partial_t \|n_\Delta\|_{L^2}^2 + \frac{\varepsilon^2}{2} \partial_t \|\nabla n_\Delta\|_{L^2}^2 + \frac{1}{2} \partial_t \|J_\Delta\|_{L^2}^2 \leq C \left( \|J_\Delta\|_{L^2}^2 + \|n_\Delta\|_{H^1}^2 \right).$$

Now it suffices to exploit Gronwall's lemma for the conclusion  $n_\Delta \equiv 0$ ,  $J_\Delta \equiv 0$ , which completes the proof of Theorem 2.1.

## A Fourth Order Parabolic Systems with Nonlocal Nonlinearities

In this appendix, we prove the following existence result:

**Proposition A.1.** *Suppose  $\gamma > 0$  and  $n_{0,\gamma}, J_{0,\gamma} \in H^6(\Omega)$ ,  $J_{\Gamma,\gamma} \in H^4(\Gamma)$ , with  $\inf_{x \in \Omega} n_{0,\gamma}(x) > 0$  and the compatibility conditions*

$$\partial_\nu n_{0,\gamma}(x) = 0, \quad J_{0,\gamma}(x) = J_{\Gamma,\gamma}(x), \quad x \in \Gamma.$$

Let  $n'_{0,\gamma} = \partial_t n_\gamma(0, x)$  and  $J'_{0,\gamma} = \partial_t J_\gamma(0, x)$  be formally computed from (3.1), and assume

$$\partial_\nu n'_{0,\gamma}(x) = 0, \quad J'_{0,\gamma}(x) = 0, \quad x \in \Gamma.$$

Finally, let  $g = (g_D, g_N) \in H^{3/2}(\Gamma_D) \times H^{1/2}(\Gamma_N)$  be given.

Then the initial boundary value problem (3.1)–(3.3) has a unique local solution  $(n_\gamma, J_\gamma, V_\gamma)$  with the regularity

$$\begin{aligned} (n_\gamma, J_\gamma) &\in W_2^1((0, t_\gamma), H^4(\Omega)) \cap W_\infty^1((0, t_\gamma), H^2(\Omega)), \\ V_\gamma &\in W_\infty^1((0, t_\gamma), H^1(\Omega)). \end{aligned}$$

The time derivatives satisfy the boundary conditions

$$\begin{aligned} \partial_\nu \partial_t n_\gamma(t, x) &= \partial_\nu \Delta \partial_t n_\gamma(t, x) = 0, \\ \partial_t J_\gamma(t, x) &= \Delta \partial_t J_\gamma(t, x) = 0, \\ (K \partial_t V_\gamma)(t, x) &= 0. \end{aligned}$$

The solution persists as long as  $n_\gamma(t, \cdot)$  is positive and the norms  $\|n_\gamma(t, \cdot)\|_{L^\infty(\Omega)}$ ,  $\|\nabla n_\gamma(t, \cdot)\|_{L^\infty(\Omega)}$ ,  $\|J_\gamma(t, \cdot)\|_{L^\infty(\Omega)}$  remain bounded.

Although the machinery of proof is quite standard, there is a difficulty: due to the Zaremba type boundary conditions for  $V_\gamma$ , even a smooth boundary  $\Gamma$  and smooth data  $g_D, g_N \in C^\infty$ ,  $n_\gamma - C \in C^\infty$  do not guarantee that  $V_\gamma \in H^2(\Omega)$ , see [3]. Therefore, we should keep track of the regularity of the coefficients of parabolic systems.

Preparing the proof of Proposition A.1, we start with some known results on the scalar linear fourth order parabolic problem ( $\gamma > 0$ )

$$\begin{cases} \partial_t u(t, x) + \gamma \Delta^2 u(t, x) = f(t, x) \\ u(0, x) = u_0(x), \\ (Bu)(t, x) = 0, \quad (t, x) \in (0, t_*) \times \Gamma, \end{cases} \quad (\text{A.1})$$

where  $B$  describes a boundary operator either of Dirichlet type, or of Neumann type:

$$B = B_D = \begin{pmatrix} 1 \\ \Delta \end{pmatrix}, \quad B = B_N = \begin{pmatrix} \partial_\nu \\ \partial_\nu \Delta \end{pmatrix}.$$

Write  $B_1 = 1$  in the Dirichlet case, and  $B_1 = \partial_\nu$  in the Neumann case.

Recalling the boundary regularity  $\Gamma \in C^4$ , we define an operator  $A = \gamma \Delta^2$  with domain  $D(A) = \{u \in H^4(\Omega) : Bu = 0 \text{ on } \Gamma\}$ .

**Lemma A.2.** *The problem (A.1) has a unique solution  $u \in L^2((0, t_*), D(A))$  with  $\partial_t u \in L^2(Q_*)$  if and only if  $f \in L^2(Q_*)$  and  $u_0 \in \{u \in H^2(\Omega) : B_1 u = 0 \text{ on } \Gamma\}$ .*

*Then this solution  $u$  satisfies the a priori estimates*

$$\|\Delta u\|_{L^\infty((0, t_*), L^2(\Omega))}^2 + \gamma \|\Delta^2 u\|_{L^2(Q_*)}^2 \leq \|\Delta u_0\|_{L^2(\Omega)}^2 + \frac{1}{\gamma} \|f\|_{L^2(Q_*)}^2, \quad (\text{A.2})$$

$$\|\partial_t u\|_{L^2(Q_*)}^2 \leq 2\gamma \|\Delta u_0\|_{L^2(\Omega)}^2 + 4 \|f\|_{L^2(Q_*)}^2, \quad (\text{A.3})$$

$$\|u(t, \cdot)\|_{L^2(\Omega)}^2 \leq 2 \|u_0\|_{L^2(\Omega)}^2 + 2t \|f\|_{L^2((0, t) \times \Omega)}^2, \quad (\text{A.4})$$

$$\|u\|_{L^2(Q_*)}^2 \leq 2t_* \|u_0\|_{L^2(\Omega)}^2 + t_*^2 \|f\|_{L^2(Q_*)}^2, \quad (\text{A.5})$$

$$\|u\|_{L^\infty((0, t_*), H^1(\Omega))}^2 \leq C_0 \left( \|u_0\|_{H^2(\Omega)}^2 + t_*^{1/2} \|f\|_{L^2(Q_*)}^2 \right), \quad (\text{A.6})$$

$$\|u\|_{L^2((0, t_*), H^3(\Omega))}^2 \leq C_0 \left( t_*^{1/4} \|u_0\|_{H^2(\Omega)}^2 + t_*^{1/2} \|f\|_{L^2(Q_*)}^2 \right), \quad (\text{A.7})$$

$$\|u(t', \cdot) - u(t'', \cdot)\|_{H^{2-\alpha}(\Omega)} \leq C_0 |t' - t''|^{\alpha/4} \left( \|u_0\|_{H^2(\Omega)} + \|f\|_{L^2(Q_*)} \right), \quad (\text{A.8})$$

where  $C_0 = C_0(\Omega, \gamma)$ ,  $0 \leq t \leq t_*$ , and  $0 \leq \alpha \leq 2$ ,  $0 \leq t', t'' \leq t_*$ .

*Proof.* The existence of a solution with the claimed regularity follows from the maximal regularity for the operator  $\partial_t + A$ , compare, for instance, [11, Theorem 3.1]. The eigenfunctions of  $A$  form an orthonormal basis of  $L^2(\Omega)$ . Expanding  $u_0$ ,  $f$ , and  $u$  into series of these eigenfunctions and elementary calculations yield (A.2). A direct consequence then is (A.3). Taking the  $L^2(\Omega)$  scalar product of  $\partial_t u + Au = f$  with  $u$  gives  $\partial_t \|u(t, \cdot)\|_{L^2(\Omega)} \leq \|f(t, \cdot)\|_{L^2(\Omega)}$ , from which we quickly deduce (A.4) and (A.5). We interpolate (A.2) with (A.4), obtaining (A.6); and (A.2) with (A.5), obtaining (A.7). Finally, (A.8) is proved as follows:

$$\begin{aligned} \|u(t', \cdot) - u(t'', \cdot)\|_{H^{2-\alpha}(\Omega)} &\leq C \|u(t', \cdot) - u(t'', \cdot)\|_{L^2(\Omega)}^{\alpha/2} \|u(t', \cdot) - u(t'', \cdot)\|_{H^2(\Omega)}^{(2-\alpha)/2} \\ &\leq C \|\partial_t u\|_{L^2((t', t''), L^2(\Omega))}^{\alpha/2} \|u\|_{L^\infty((0, t_*), H^2(\Omega))}^{(2-\alpha)/2} \\ &\leq C |t' - t''|^{\alpha/4} \|\partial_t u\|_{L^2(Q_*)}^{\alpha/2} \|u\|_{L^\infty((0, t_*), H^2(\Omega))}^{(2-\alpha)/2}. \end{aligned}$$

□

Next, we consider linear fourth-order systems for the unknown  $u = (u_1, \dots, u_N)$  with decoupled principal part:

$$\left\{ \begin{aligned} \partial_t u_j(t, x) + \gamma \Delta^2 u_j(t, x) + \sum_{i=1}^N \sum_{|\alpha| \leq 3} a_{i,\alpha}(t, x) \partial_x^\alpha u_i(t, x) &= f_j(t, x), \\ u_j(0, x) &= u_{0,j}(x), \\ (Bu_j)(t, x) &= 0, \quad (t, x) \in (0, t_*) \times \Gamma, \end{aligned} \right. \quad (\text{A.9})$$

where  $j = 1, \dots, N$ , and the boundary operator  $B$  is either  $B_D$  or  $B_N$ . The choice of the boundary operator may depend on  $j$ .

**Lemma A.3.** *Suppose  $f_j \in L^2(Q_*)$  and  $u_{0,j} \in H^2(\Omega)$  with  $B_1 u_{0,j} = 0$  on  $\Gamma$ . Assume  $d \leq 3$  and the regularity properties of the coefficients:*

$$a_{i,\alpha} \in \begin{cases} L^\infty(Q_*) & : |\alpha| = 3, \\ L^2((0, t_*), L^\infty(\Omega)) & : |\alpha| = 2, \\ L^\infty((0, t_*), L^2(\Omega)) & : |\alpha| = 1, \\ L^2(Q_*) & : \alpha = 0. \end{cases}$$

Then there is a number  $\beta_0 > 0$  (depending only on  $\Omega$  and  $\gamma$ ) with the following property: if

$$\sum_{j=1}^N \sum_{|\alpha|=2} \|a_{j,\alpha}\|_{L^2((0,t_*),L^\infty(\Omega))} + \|a_0\|_{L^2(Q_*)} \leq \beta_0, \quad (\text{A.10})$$

and if  $t_* > 0$  is small, then any solution  $u$  to (A.9) with  $u \in L^2((0, t_*), H^4(\Omega))$  and  $\partial_t u \in L^2(Q_*)$  satisfies the a priori estimates (A.6), (A.7), and (A.8). The constant  $C_0$  only depends on  $\Omega$ ,  $\gamma$ , and the norms of the coefficients  $a_{j,\alpha}$ .

The existence and uniqueness of the solution  $u$  is standard.

*Proof.* Put  $\tilde{f}_j = f_j - \sum_{i,\alpha} a_{i,\alpha} \partial_x^\alpha u_i$ . Then we can bring (A.2) into play:

$$\begin{aligned} \|u\|_{L^\infty((0,t_*),H^2(\Omega))}^2 + \|u\|_{L^2((0,t_*),H^4(\Omega))}^2 &\leq C \|\tilde{f}\|_{L^2(Q_*)}^2 + C \|u_0\|_{H^2(\Omega)}^2 \\ &\leq C \|f\|_{L^2(Q_*)}^2 + C \|u\|_{L^2((0,t_*),H^3(\Omega))}^2 \left( \sum_i \sum_{|\alpha|=3} \|a_{i,\alpha}\|_{L^\infty(Q_*)}^2 \right) \\ &\quad + C \|u\|_{L^\infty((0,t_*),H^2(\Omega))}^2 \left( \sum_i \sum_{|\alpha|=2} \|a_{i,\alpha}\|_{L^2((0,t_*),L^\infty(\Omega))}^2 \right) \\ &\quad + C \|u\|_{L^2((0,t_*),H^3(\Omega))}^2 \left( \sum_i \sum_{|\alpha|=1} \|a_{i,\alpha}\|_{L^\infty((0,t_*),L^2(\Omega))}^2 \right) \\ &\quad + C \|u\|_{L^\infty((0,t_*),H^2(\Omega))}^2 \|a_0\|_{L^2(Q_*)}^2 + C \|u_0\|_{H^2(\Omega)}^2, \end{aligned}$$

where we exploited the embeddings  $H^2(\Omega) \subset L^\infty(\Omega)$  and  $H^3(\Omega) \subset C^1(\bar{\Omega})$ . Making use of (A.10) and choosing  $\beta_0$  small, we have

$$\begin{aligned} \|u\|_{L^\infty((0,t_*),H^2(\Omega))}^2 + \|u\|_{L^2((0,t_*),H^4(\Omega))}^2 & \\ &\leq C \|f\|_{L^2(Q_*)}^2 + C \|u\|_{L^2((0,t_*),H^3(\Omega))}^2 + C \|u_0\|_{H^2(\Omega)}^2. \end{aligned} \quad (\text{A.11})$$

Next, we employ (A.7) and (A.6) for the system  $\partial_t + \gamma \Delta^2 u = \tilde{f}$ , and suppose that  $t_*$  is small:

$$\begin{aligned} \|u\|_{L^2((0,t_*),H^3(\Omega))}^2 &\leq C \left( t_*^{1/4} \|u_0\|_{H^2(\Omega)}^2 + t_*^{1/2} \|\tilde{f}\|_{L^2(Q_*)}^2 \right) \\ &\leq C t_*^{1/4} \|u_0\|_{H^2(\Omega)}^2 + C t_*^{1/2} \left( \|f\|_{L^2(Q_*)}^2 + \|u\|_{L^2((0,t_*),H^3(\Omega))}^2 + \|u_0\|_{H^2(\Omega)}^2 \right), \end{aligned}$$

$$\begin{aligned}
\|u\|_{L^2((0,t_*),H^3(\Omega))}^2 &\leq Ct_*^{1/4} \|u_0\|_{H^2(\Omega)}^2 + Ct_*^{1/2} \|f\|_{L^2(Q_*)}^2, \\
\|u\|_{L^\infty((0,t_*),H^1(\Omega))}^2 &\leq C \left( \|u_0\|_{H^2(\Omega)}^2 + t_*^{1/2} \|\tilde{f}\|_{L^2(Q_*)}^2 \right) \\
&\leq C \|u_0\|_{H^2(\Omega)}^2 + Ct_*^{1/2} \left( \|f\|_{L^2(Q_*)}^2 + \|u\|_{L^2((0,t_*),H^3(\Omega))}^2 + \beta_0 \|u\|_{L^\infty((0,t_*),H^2(\Omega))}^2 \right) \\
&\leq C \|u_0\|_{H^2(\Omega)}^2 + Ct_*^{1/2} \|f\|_{L^2(Q_*)}^2.
\end{aligned}$$

These are the estimates (A.7) and (A.6) for the system (A.9). Then we obtain, from (A.11),

$$\|u\|_{L^2((0,t_*),H^2(\Omega))}^2 + \|u\|_{L^2((0,t_*),H^4(\Omega))}^2 \leq C \left( \|u_0\|_{H^2(\Omega)}^2 + \|f\|_{L^2(Q_*)}^2 \right),$$

which is the analogue to (A.2) for the system (A.9). From the system (A.9) we then find an estimate of  $\|\partial_t u\|_{L^2(Q_*)}$ . Interpolating as before yields the desired estimate (A.8) for the system (A.9).  $\square$

*Proof of Proposition A.1.* For simplicity of notation, we drop the subscript  $\gamma$ .

We apply the *a priori* estimates from Lemma A.3 to the linear part of (3.1), which is a special case of (A.9):

$$\left\{ \begin{array}{l} \partial_t n + \gamma \Delta^2 n - \nu_0 \Delta n - \operatorname{div}(J - J_D) = \operatorname{div} J_D, \\ \partial_t(J - J_D) + \gamma \Delta^2(J - J_D) - \nu_0 \Delta(J - J_D) \\ \quad + \frac{\varepsilon^2}{4} \nabla \Delta n - T \nabla n + \frac{1}{\tau}(J - J_D) = f, \\ \quad (n, J - J_D)(0, x) = (n_0, J_0 - J_D)(x), \\ \quad (B_N n, B_D(J - J_D))(t, x) = 0, \quad (t, x) \in (0, t_*) \times \Gamma, \end{array} \right. \quad (\text{A.12})$$

Write this system in the form  $\partial_t U + \gamma \Delta^2 U + A_{\text{lower}} U = F$ , where  $U = (n, J - J_D)$ . The coefficients  $a_{j,\alpha}$  are independent of  $(t, x)$ ; and condition (A.10) can be fulfilled by restricting the time interval. Then we have shown:

There are positive constants  $t_{\max}$  and  $C_0$ , depending only on  $\Omega$ ,  $\gamma$ ,  $\nu_0$ ,  $\varepsilon$  and  $T$  such that, for every  $0 < t_* \leq t_{\max}$ , the solution  $U$  satisfies the following *a priori* estimates:

$$\|U\|_{L^\infty((0,t_*),H^2(\Omega))}^2 + \|U\|_{L^2((0,t_*),H^4(\Omega))}^2 \leq C_0 \left( \|F\|_{L^2(Q_*)}^2 + \|U_0\|_{H^2(\Omega)}^2 \right), \quad (\text{A.13})$$

$$\|\partial_t U\|_{L^2(Q_*)}^2 \leq C_0 \left( \|F\|_{L^2(Q_*)}^2 + \|U_0\|_{H^2(\Omega)}^2 \right), \quad (\text{A.14})$$

$$\|U\|_{L^\infty((0,t_*),H^1(\Omega))}^2 \leq C_0 \left( t_*^{1/2} \|F\|_{L^2(Q_*)}^2 + \|U_0\|_{H^2(\Omega)}^2 \right), \quad (\text{A.15})$$

$$\|U\|_{L^2((0,t_*),H^3(\Omega))}^2 \leq C_0 \left( t_*^{1/2} \|F\|_{L^2(Q_*)}^2 + t_*^{1/4} \|U_0\|_{H^2(\Omega)}^2 \right), \quad (\text{A.16})$$

$$\|U(t', \cdot) - U(t'', \cdot)\|_{H^{2-\alpha}(\Omega)} \leq C_0 |t' - t''|^{\alpha/4} \left( \|F\|_{L^2(Q_*)} + \|U_0\|_{H^2(\Omega)} \right). \quad (\text{A.17})$$

We prove the existence of local solutions to (3.1) by an iteration scheme: set

$$n^0(t, x) = n_0(x), \quad J^0(t, x) = J_0(x), \quad (t, x) \in [0, t_{\max}] \times \Omega,$$



and define  $U^k = (n^k, J^k - J_D)$  as the solution to the linear system

$$\begin{cases} \partial_t U^k + \gamma \Delta^2 U^k + A_{\text{lower}} U^k = F^{k-1}, \\ U^k(0, x) = U_0(x), \\ (B_N n^k, B_D(J^k - J_D))(t, x) = 0, \end{cases} \quad (t, x) \in (0, t_*) \times \Gamma,$$

where  $F^{k-1} = (\text{div } J_D, f^{k-1})^T$  and

$$\begin{aligned} f^{k-1} &= -\frac{1}{\tau} J_D - G(n^{k-1}, J^{k-1}, V^{k-1}) \\ &= -\frac{1}{\tau} J_D + \text{div} \left( \frac{J^{k-1} \otimes J^{k-1}}{n^{k-1}} \right) - n^{k-1} \nabla V^{k-1} \\ &\quad + \varepsilon^2 \text{div} \left( (\nabla \sqrt{n^{k-1}}) \otimes (\nabla \sqrt{n^{k-1}}) \right), \end{aligned}$$

$$\lambda^2 \Delta V^{k-1} = n^{k-1} - C,$$

$$(KV^{k-1})(t, x) = g(x), \quad (t, x) \in (0, t_*) \times \Gamma.$$

As usual, we can assume  $n^k(t, x) \geq \delta_0$  and  $\|n^k(t, \cdot)\|_{L^\infty(\Omega)} \leq \delta_0^{-1}$ ,  $\|J^k(t, \cdot)\|_{L^\infty(\Omega)} \leq \delta_0^{-1}$ , at least for small  $t$ , due to (A.17) and the embedding  $H^{2-\alpha}(\Omega) \subset L^\infty(\Omega)$  for  $0 < \alpha < 1/2$ . Then we have  $\|V^{k-1}(t, \cdot)\|_{H^1(\Omega)} \leq C$ , for such  $t$ .

Suppose that  $\|F^{k-2}\|_{L^2(Q_*)} \leq 1$ . For  $k = 2$ , this is possible by choosing  $t_*$  sufficiently small. Then we get estimates for  $U^{k-1}$  via (A.13)–(A.17). Moreover, we can assume  $t_* \leq 1$ . In the following,  $C$  will denote a constant independent of  $k$ , but dependent on  $\delta_0$ :

$$\begin{aligned} \|F^{k-1}(t, \cdot)\|_{L^2(\Omega)} &\leq C \left( 1 + \|J^{k-1}\|_{H^1(\Omega)} + \|n^{k-1}\|_{H^1(\Omega)} + \|V^{k-1}\|_{H^1(\Omega)} \right) \\ &\quad + C \left( \|\nabla n^{k-1}\|_{L^6(\Omega)}^3 + \|\nabla n^{k-1}\|_{L^\infty(\Omega)} \|\nabla^2 n^{k-1}\|_{L^2(\Omega)} \right), \\ \|F^{k-1}(t, \cdot)\|_{L^2(\Omega)}^2 &\leq C \left( \|U^{k-1}\|_{L^\infty((0,t), H^1(\Omega))}^2 + 1 \right) \\ &\quad + C \left( \|U^{k-1}\|_{L^\infty((0,t), H^2(\Omega))}^6 + \|U^{k-1}(t, \cdot)\|_{H^3(\Omega)}^2 \|U^{k-1}\|_{L^\infty((0,t), H^2(\Omega))}^2 \right) \\ &\leq C \left( C_0(1 + \|U_0\|_{H^2(\Omega)}^2) + 1 + (C_0(1 + \|U_0\|_{H^2(\Omega)}^2))^3 \right) \\ &\quad + C \|U^{k-1}(t, \cdot)\|_{H^3(\Omega)}^2 C_0(1 + \|U_0\|_{H^2(\Omega)}^2), \\ \|F^{k-1}\|_{L^2(Q_*)}^2 &\leq Ct_* \left( C_0(1 + \|U_0\|_{H^2(\Omega)}^2) + 1 + (C_0(1 + \|U_0\|_{H^2(\Omega)}^2))^3 \right) \\ &\quad + C \|U^{k-1}\|_{L^2((0,t_*), H^3(\Omega))}^2 C_0(1 + \|U_0\|_{H^2(\Omega)}^2) \\ &\leq Ct_* (\dots) + Ct_*^{1/4} (C_0(1 + \|U_0\|_{H^2(\Omega)}^2))^2. \end{aligned}$$

From this estimate, we learn how to fix a positive number  $t_*$  with the property that the estimate  $\|F^{k-2}\|_{L^2((0,t_*) \times \Omega)} \leq 1$  implies  $\|F^{k-1}\|_{L^2((0,t_*) \times \Omega)} \leq 1$ . Then we gain a sequence  $(U^k)_k \subset$

$L^2((0, t_*), H^4(\Omega))$ , with uniform in  $k$  estimates as in (A.13)–(A.17). By a similar reasoning, we can show

$$\left\| F^k - F^{k-1} \right\|_{L^2(Q_*)}^2 \leq C t_*^{1/4} \left\| F^{k-1} - F^{k-2} \right\|_{L^2(Q_*)}^2,$$

from which the convergence of the sequence  $(U^k)_k$  to a limit  $U$ ,

$$U^k \rightarrow U \quad \text{in} \quad L^\infty((0, t_*), H^2(\Omega)) \cap L^2((0, t_*), H^4(\Omega))$$

can be deduced provided that  $t_*$  has been chosen sufficiently small. We also have the convergence  $\partial_t U_k \rightarrow U$  in  $L^2(Q_*)$ .

The uniqueness of the solution  $U$  can be shown similarly as the contraction.

Having found this solution  $U$ , we next show better regularity properties by an investigation of the time derivative  $U_t$ . Taking a formal derivative of (A.12), we obtain a system of the form

$$\partial_t U_t + \gamma \Delta^2 U_t + A_{\text{lower}} U_t = \sum_{|\alpha| \leq 2} F_\alpha(J, \nabla J, n, \nabla n, \nabla^2 n, \nabla V) \partial_x^\alpha U_t - \begin{pmatrix} 0 \\ n \nabla V_t \end{pmatrix}.$$

We know that the initial values  $n'_0(x) := (\partial_t n)(0, x)$ ,  $J'_0(x) := (\partial_t J)(0, x)$ , computed via (A.12), satisfy  $\partial_\nu n'_0(x) = 0$ ,  $J'_0(x) = 0$  on  $\Gamma$ . Then we consider the initial-boundary value problem

$$\left\{ \begin{array}{l} \partial_t W + \gamma \Delta^2 W + A_{\text{lower}} W = \sum_{|\alpha| \leq 2} F_\alpha(J, \nabla J, n, \nabla n, \nabla^2 n, \nabla V) \partial_x^\alpha W - \begin{pmatrix} 0 \\ n \Phi(W_1) \end{pmatrix}, \\ (B_N W_1, B_D W_{2, \dots, d}) = 0, \\ W(0, x) = (n'_0, J'_0)(x), \end{array} \right.$$

where  $\Phi$  denotes the mapping  $n_t \mapsto \nabla V_t$  via the Poisson equation  $\lambda^2 \Delta V_t = n_t$  with homogeneous Zaremba boundary conditions. By a variation of Lemma A.3, we see that this problem has a unique solution  $W \in L^2((0, t_*), H^4(\Omega)) \cap L^\infty((0, t_*), H^2(\Omega))$ , and standard arguments show  $U(t, x) = U_0(x) + \int_0^t W(t', x) dt'$ . As a consequence, we have

$$U \in W_2^1((0, t_*), H^4(\Omega)) \cap W_\infty^1((0, t_*), H^2(\Omega)).$$

The persistence of the solution  $U$  is shown by standard Moser type estimates. □

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