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Strongly NIP almost real closed fields

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The following conjecture is due to Shelah–Hasson: Any infinite strongly NIP field is either real closed, algebraically closed, or admits a non-trivial definable henselian valuation, in the language of rings. We specialise this conjecture to ordered fields in the language of ordered rings, which leads towards a systematic study of the class of strongly NIP almost real closed fields. As a result, we obtain a complete characterisation of this class.

1 Introduction

The study of tame ordered algebraic structures has received a considerable amount of attention since the notion of o-minimality was introduced in [23]. Frequently, the goal is to give a complete characterisation of such model-theoretically well-behaved structures in terms of their algebraic properties. For instance, a (totally) ordered group is o-minimal if and only if it is abelian and divisible (cf. [23, Proposition 1.4, Theorem 2.1]) and an ordered field is o-minimal if and only if it is real closed (cf. [23, Proposition 1.4, Theorem 2.3]). In fact, these characterisations hold true under the more general tameness condition of weak o-minimality (cf. [4, p. 117] and [21, Theorem 5.1, Theorem 5.3]). While o-minimality and weak o-minimality are comparatively strong tameness conditions, the property NIP ('not the independence property'), introduced in [26], is at the other end of the spectrum; indeed, any o-minimal structure is also NIP (cf. [23, Corollary 3.10] and [24]).

A strategy to examine the class of NIP ordered algebraic structures is to first consider refinements of the property NIP. In this regard, we are mainly concerned with dp-minimal as well as strongly NIP[1] ordered groups and fields. Since any weakly o-minimal theory is dp-minimal (cf. [5, Corollary 4.3]), we obtain the following hierarchy:

$$\text{o-minimal} \;\rightarrow\; \text{weakly o-minimal} \;\rightarrow\; \text{dp-minimal} \;\rightarrow\; \text{strongly NIP} \;\rightarrow\; \text{NIP}.$$

In particular, any divisible ordered abelian group and any real closed field are strongly NIP. A full algebraic characterisation of dp-minimal ordered fields follows from [13, Theorem 6.2] (cf. Proposition 4.4), but so far there has not been a systematic study of the strongly NIP ordered field case.

With this paper, we contribute to the analysis of strongly NIP ordered fields, in light of a conjecture suggested by Shelah in [28, Conjecture 5.34(c)]. This conjecture was verified by Johnson for dp-minimal fields in [15, Theorem 1.6] and more recently for dp-finite[2] fields in [16, Theorem 1.2]. Shelah's conjecture was reformulated as follows in [7, p. 820], [10, p. 2214], [11, p. 720] and [12, p. 183].

Shelah–Hasson Conjecture. *Let K be an infinite strongly NIP field. Then K is either real closed, or algebraically closed, or admits a non-trivial $\mathcal{L}_{\mathrm{r}}$-definable[3] henselian valuation.*

The Shelah–Hasson Conjecture specialised to ordered fields reads as follows.

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[1] A strongly NIP theory is usually said to be strongly dependent.
[2] Dp-finiteness can be classed between dp-minimality and strong NIP in the picture above.
[3] Throughout this work 'definable' always means 'definable with parameters'.

Conjecture 1.1 *Let $(K, <)$ be a strongly NIP ordered field. Then K is either real closed or admits a non-trivial $\mathcal{L}_{\mathrm{or}}$-definable henselian valuation.*

The valuation and model theory of almost real closed fields (cf. [3] and also Definition 3.1) is well-understood. A main achievement of our paper is to show that Conjecture 1.1 is equivalent to the following.

Conjecture 1.2 *Any strongly NIP ordered field $(K, <)$ is almost real closed.*

We highlight this result in the following theorem.[4]

Theorem 1.3 *Conjecture 1.1 and Conjecture 1.2 are equivalent.*

Dp-minimal and more generally strongly NIP ordered abelian groups have already been fully classified (cf. [13, Proposition 5.1] and [11, Theorem 1]). Moreover, Conjecture 1.1 has already been verified for dp-minimal ordered fields in [13, Corollary 6.6]. By a careful analysis of the results of [13], we deduce in Proposition 4.4 that also Conjecture 1.2 holds for dp-minimal ordered fields. Actually, we prove that an ordered field is dp-minimal if and only if it is almost real closed *with respect to some dp-minimal ordered abelian group G*. This latter result raises the question whether the analogous classification holds for strongly NIP ordered fields. We therefore address the following:

Question 1.4 Is it true that an ordered field $(K, <)$ is strongly NIP if and only if it is almost real closed with respect to some strongly NIP ordered abelian group G?

In § 4, we prove our other main result:[5]

Theorem 1.5 *Let $(K, <)$ be an almost real closed field with respect to some ordered abelian group G. Then $(K, <)$ is strongly NIP if and only if G is strongly NIP.*

This answers positively the backward direction of Question 1.4 about the classification of strongly NIP ordered fields. Thus, only the following question remains open.

Question 1.6 Is every strongly NIP ordered field $(K, <)$ almost real closed with respect to some strongly NIP ordered abelian group G?

Finally, we note that a positive answer to Question 1.6 would verify Conjecture 1.2. Conversely, if Conjecture 1.2 is verified, then by Theorem 1.5 the answer to Question 1.6 is positive.

We conclude in § 6 by stating some further open questions motivated by this work.[6]

2 General preliminaries

The set of natural numbers with 0 is denoted by $\mathbb{N}_0$, the set of natural numbers without 0 by $\mathbb{N}$. Let $\mathcal{L}_{\mathrm{r}} = \{+, -, \cdot, 0, 1\}$ be the language of rings, $\mathcal{L}_{\mathrm{or}} = \mathcal{L}_{\mathrm{r}} \cup \{<\}$ the language of ordered rings and $\mathcal{L}_{\mathrm{og}} = \{+, 0, <\}$ the language of ordered groups. Throughout this work, we abbreviate the $\mathcal{L}_{\mathrm{r}}$-structure of a field $(K, +, -, \cdot, 0, 1)$ simply by K, the $\mathcal{L}_{\mathrm{or}}$-structure of an ordered field $(K, +, -, \cdot, 0, 1, <)$ by $(K, <)$ and the $\mathcal{L}_{\mathrm{og}}$-structure of an ordered group $(G, +, 0, <)$ by G.

All notions on valued fields can be found in [8, 20]. Let K be a field and v a valuation on K. We denote the *valuation ring* of v in K by $\mathcal{O}_v$, the *valuation ideal*, i.e., the maximal ideal of $\mathcal{O}_v$, by $\mathcal{M}_v$, the *ordered value group* by vK and the *residue field* $\mathcal{O}_v/\mathcal{M}_v$ by Kv. For $a \in \mathcal{O}_v$ we also denote $a + \mathcal{M}_v$ by $\overline{a}$. For an ordered field $(K, <)$ a valuation is called *convex* (in $(K, <)$) if the valuation ring $\mathcal{O}_v$ is a convex subset of K. In this case, the relation $\overline{a} < \overline{b} :\Longleftrightarrow \overline{a} \neq \overline{b} \wedge a < b$ defines an order relation on Kv making it an ordered field. Note that in ordered fields, henselian valuations are always convex:

Fact 2.1 ([17, Lemma 2.1]) *Let $(K, <)$ be an ordered field and let v be a henselian valuation on K. Then v is convex on $(K, <)$.*

[4] This will be restated as Theorem 5.4.
[5] This will be restated as Theorem 4.12.
[6] A preliminary version of this work is contained in our arXiv preprint [18], which contains also a systematic study of $\mathcal{L}_{\mathrm{or}}$-definable henselian valuations in ordered fields as well as of the class of ordered fields which are dense in their real closure. This systematic study, of independent interest, will be the subject of a separate publication [19].

Let $\mathcal{L}_{\mathrm{vf}} = \mathcal{L}_{\mathrm{r}} \cup \{\mathcal{O}_v\}$ be the *language of valued fields*, where $\mathcal{O}_v$ stands for a unary predicate. Let $(K, \mathcal{O}_v)$ be a valued field. An atomic formula of the form $v(t_1) \geq v(t_2)$, where t_1 and t_2 are $\mathcal{L}_{\mathrm{r}}$-terms, stands for the $\mathcal{L}_{\mathrm{vf}}$-formula $t_1 = t_2 = 0 \vee (t_2 \neq 0 \wedge \mathcal{O}_v(t_1/t_2))$. Thus, by abuse of notation, we also denote the $\mathcal{L}_{\mathrm{vf}}$-structure $(K, \mathcal{O}_v)$ by (K, v). Similarly, we also call $(K, <, v)$ an ordered valued field. We say that a valuation v is $\mathcal{L}$-definable for some language $\mathcal{L} \in \{\mathcal{L}_{\mathrm{r}}, \mathcal{L}_{\mathrm{or}}\}$ if its valuation ring is an $\mathcal{L}$-definable subset of K.

For any ordered abelian groups G_1 and G_2, we denote the *lexicographic sum* of G_1 and G_2 by $G_1 \oplus G_2$. This is the abelian group $G_1 \times G_2$ with the lexicographic ordering $(a, b) < (c, d)$ if $a < c$, or $a = c$ and $b < d$.

Let K be a field and let v and w be valuations on K. We write $v \leq w$ if and only if $\mathcal{O}_v \supseteq \mathcal{O}_w$. In this case we say that w is *finer* than v and v is *coarser* than v. Note that $\leq$ defines an order relation on the set of convex valuations of an ordered field. We call two elements $a, b \in K$ *archimedean equivalent* (in symbols $a \sim b$) if there is some $n \in \mathbb{N}$ such that $|a| < n|b|$ and $|b| < n|a|$. Let $G = \{[a] \mid a \in K^\times\}$, the set of archimedean equivalence classes of $K^\times$. Equipped with addition $[a] + [b] = [ab]$ and the ordering $[a] < [b]$ defined by $a \not\sim b \wedge |b| < |a|$, the set G becomes an ordered abelian group. Then $K^\times \to G, a \mapsto [a]$ defines a convex valuation on K. This is called the *natural valuation* on K and denoted by v_{nat}.[7]

Let $(k, <)$ be an ordered field and let G be an ordered abelian group. We denote the *Hahn field* with coefficients in k and exponents in G by $k((G))$. The underlying set of $k((G))$ consists of all elements in the group product $\prod_{g \in G} k$ with well-ordered support, where the *support* of an element s is given by $\mathrm{supp} s = \{g \in G \mid s(g) \neq 0\}$. We denote an element $s \in k((G))$ by $s = \sum_{g \in G} s_g t^g$, where $s_g = s(g)$ and t^g is the characteristic function on G mapping g to 1 and everything else to 0. The ordering on $k((G))$ is given by $s > 0 :\Longleftrightarrow s(\min \mathrm{supp} s) > 0$. Let $v_{\min}$ be the valuation on $k((G))$ given by $v_{\min}(s) = \min \mathrm{supp} s$ for $s \neq 0$. Note that $v_{\min}$ is convex and henselian. Note further that if k is archimedean, then $v_{\min}$ coincides with v_{nat}.

We repeatedly use the Ax–Kochen–Ershov Principle for ordered fields. This follows from [9, Corollary 4.2(iii)], where all appearing levels in the premise equal 1 (cf. [9, p. 916]).

Fact 2.2 (Ax–Kochen–Ershov Principle) *Let $(K, <, v)$ and $(L, <, w)$ be two ordered henselian valued fields. Then $(Kv, <) \equiv (Lw, <)$ and $vK \equiv wL$ if and only if $(K, <, v) \equiv (L, <, w)$.*

Since we do not use explicitly the definitions of the independence property (IP), 'not the independence property' (NIP), strong NIP and dp-minimality, we refer the reader to [29] for all definitions in this regard. For a structure $\mathcal{N}$, we say that $\mathcal{N}$ is NIP (respectively, strongly NIP and dp-minimal) if its complete theory $\mathrm{Th}(\mathcal{N})$ is NIP (respectively, strongly NIP and dp-minimal). A well-known example of an IP theory is the complete theory of the $\mathcal{L}_{\mathrm{r}}$-structure $(\mathbb{Z}, +, -, \cdot, 0, 1)$ (cf. [29, Example 2.4]). Since $\mathbb{Z}$ is parameter-free definable in the $\mathcal{L}_{\mathrm{r}}$-structure $\mathbb{Q}$ (cf. [25, Theorem 3.1]), also the complete $\mathcal{L}_{\mathrm{r}}$-theory of $\mathbb{Q}$ has IP. Any reduct of a strongly NIP structure is strongly NIP (cf. [27, Claim 3.14 (3)]) and any reduct of a dp-minimal structure is dp-minimal (cf. [22, Observation 3.7]).

3 Almost real closed fields

Algebraic and model theoretic properties of the class of almost real closed fields in the language $\mathcal{L}_{\mathrm{r}}$ have been studied in [3]; in particular, [3, Theorem 4.4] gives a complete characterisation of $\mathcal{L}_{\mathrm{r}}$-definable henselian valuations. In the following, we prove some useful properties of almost real closed fields in the language $\mathcal{L}_{\mathrm{or}}$.

Definition 3.1 Let $(K, <)$ be an ordered field, G an ordered abelian group and v a henselian valuation on K. We call K an *almost real closed field (with respect to v and G)* if Kv is real closed and $vK = G$.

Depending on the context, we may simply say that $(K, <)$ is an almost real closed field without specifying the henselian valuation v or the ordered abelian group $G = vK$.

Remark 3.2 In [3], almost real closed fields are defined as pure fields which admit a henselian valuation with real closed residue field. However, any such field admits an ordering, which is due to the Baer–Krull Representation Theorem (cf. [8, p. 37f.]). We consider almost real closed fields as ordered fields with a fixed order.

Due to Fact 2.1 and the following fact, we do not need to make a distinction between convex and henselian valuations in almost real closed fields.

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[7] Note that the ordered residue field $(Kv_{\mathrm{nat}}, <)$ is always archimedean. Note further that v_{nat} is trivial if and only if $(K, <)$ is archimedean.

Fact 3.3 ([3, Proposition 2.9]) *Let $(K, <)$ be an almost real closed field. Then any convex valuation on $(K, <)$ is henselian.*

[3, Proposition 2.8] implies that the class of almost real closed fields in the language $\mathcal{L}_{\mathrm{r}}$ is closed under elementary equivalence. We can easily deduce that this also holds in the language $\mathcal{L}_{\mathrm{or}}$.

Proposition 3.4 *Let $(K, <)$ be an almost real closed field and let $(L, <) \equiv (K, <)$. Then $(L, <)$ is an almost real closed field.*

P r o o f . Since $L \equiv K$, we obtain by [3, Proposition 2.8] that L admits a henselian valuation v such that Lv is real closed. Hence, $(L, <)$ is almost real closed. □

Corollary 3.5 *Let $(K, <)$ be an ordered field. Then $(K, <)$ is almost real closed if and only if $(K, <) \equiv (\mathbb{R}((G)), <)$ for some ordered abelian group G.*

P r o o f . The forward direction follows from Fact 2.2. The backward direction is a consequence of Proposition 3.4. □

Corollary 3.6 *Let $(K, <)$ be an almost real closed field and let G be an ordered abelian group. Then $(K((G)), <)$ is almost real closed.*

P r o o f . Let v be a henselian valuation on K such that K is almost real closed with respect to v. Since $v_{\min}$ is henselian on $K((G))$, we can compose the two henselian valuations $v_{\min}$ and v in order to obtain a henselian valuation on $K((G))$ with real closed residue field (cf. [8, Corollary 4.1.4]). □

4 Strongly NIP ordered fields

In this section we study the class of strongly NIP ordered fields in light of Conjectures 1.1 & 1.2. A special class of strongly NIP ordered fields are dp-minimal ordered fields. These are fully classified in [13]. In Proposition 4.4 below we show that our query (cf. Question 1.4) holds for dp-minimal ordered fields. An ordered group G is called *non-singular* if G/pG is finite for all prime numbers p.

Fact 4.1 ([13, Proposition 5.1]) *An ordered abelian group G is dp-minimal if and only if it is non-singular.*[8]

Fact 4.2 ([13, Theorem 6.2]) *An ordered field $(K, <)$ is dp-minimal if and only if there exists a non-singular ordered abelian group G such that $(K, <) \equiv (\mathbb{R}((G)), <)$.*

Lemma 4.3 *Let $(K, <)$ be a dp-minimal almost real closed field with respect to some henselian valuation v. Then vK is dp-minimal.*

P r o o f . Since Kv is real closed, it is not separably closed. Thus, by [14, Theorem A], v is definable in the Shelah expansion $(K, <)^{\mathrm{Sh}}$ (cf. [14, § 2]) of $(K, <)$. By [22, Observation 3.8], also $(K, <)^{\mathrm{Sh}}$ is dp-minimal, whence the reduct (K, v) is dp-minimal. By [28, Observation 1.4(2)][9], any structure which is first-order interpretable in (K, v) is dp-minimal (cf. also [2, 13]). Hence, also vK is dp-minimal. □

Proposition 4.4 *Let $(K, <)$ be an ordered field. Then $(K, <)$ is dp-minimal if and only if it is almost real closed with respect to a dp-minimal ordered abelian group.*

P r o o f . Suppose that $(K, <)$ is almost real closed with respect to a dp-minimal ordered abelian group G. By Fact 4.1, G is non-singular. By Fact 2.2, we have $(K, <) \equiv (\mathbb{R}((G)), <)$, which is dp-minimal by Fact 4.2. Hence, $(K, <)$ is dp-minimal.

Conversely, suppose that $(K, <)$ is dp-minimal. By Fact 4.2, we have $(K, <) \equiv (\mathbb{R}((G)), <)$ for some non-singular ordered abelian group G. Since $(\mathbb{R}((G)), <)$ is almost real closed, by Proposition 3.4 also $(K, <)$ is almost real closed with respect to some henselian valuation v. By Lemma 4.3, also vK is dp-minimal, as required. □

As a result, we obtain a characterisation of dp-minimal archimedean ordered fields.

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[8] The saturation condition in [13] can be dropped, as non-singularity of groups transfers via elementary equivalence.

[9] We thank Yatir Halevi for pointing out this reference to us.

Corollary 4.5 *Let $(K, <)$ be a dp-minimal archimedean ordered field. Then K is real closed.*

Proof. The only archimedean almost real closed fields are the archimedean real closed fields. This is due to the fact that any henselian valuation w on an archimedean field L is convex and thus trivial, whence the residue field of Lw is equal to L. Thus, by Proposition 4.4, any archimedean dp-minimal ordered field is real closed. □

We now turn to strongly NIP almost real closed fields, aiming for a characterisation of these (cf. Theorem 4.12). We have seen in Proposition 4.4 that every almost real closed field with respect to a dp-minimal ordered abelian group is dp-minimal. We obtain a similar result for almost real closed fields with respect to a strongly NIP ordered abelian group. The following two results will be exploited.

Fact 4.6 ([11, Theorem 1]) *Let G be an ordered abelian group. Then the following are equivalent:*

(1) G is strongly NIP.

(2) G is elementarily equivalent to a lexicographic sum of ordered abelian groups $\bigoplus_{i\in I} G_i$, where for every prime p, we have $|\{i \in I \mid pG_i \neq G_i\}| < \infty$, and for any $i \in I$, we have $|\{p \text{ prime} \mid [G_i : pG_i] = \infty\}| < \infty$.

Details on angular component maps are given in [6, § 5.4f.]. Recall from § 2 that any henselian valuation on an ordered field is convex (cf. Fact 2.1) and thus naturally induces an ordering on the residue field given by $\overline{a} < \overline{b} :\Longleftrightarrow \overline{a} \neq \overline{b} \wedge a < b$.

Observation 4.7 *Let $(K, <, v)$ be an ordered henselian valued field and let $\mathrm{ac} : K^\times \to Kv^\times$ be an angular component map. Suppose that the induced ordering of K on Kv is $\mathcal{L}_\mathrm{r}$-definable. Then the ordering $<$ is definable in (K, v, ac).*

Proof. Let $\varphi(x)$ be an $\mathcal{L}_\mathrm{r}$-formula such that for any $a \in Kv$ we have $a \geq 0$ if and only if $Kv \models \varphi(a)$. Then the formula $x \neq 0 \to \varphi(\mathrm{ac}(x))$ defines the positive cone of the ordering $<$ on K. □

Lemma 4.8 *Let G be a strongly NIP ordered abelian group. Then the ordered Hahn field $(\mathbb{R}((G)), <)$ is strongly NIP.*

Proof. If $K = \mathbb{R}((G))$ is real closed, then we are done.
Otherwise let $v = v_{\min}$. Then (K, v) is ac-valued with angular component map $\mathrm{ac} : K \to \mathbb{R}$ given by $\mathrm{ac}(s) = s(v(s))$ for $s \neq 0$ and $\mathrm{ac}(0) = 0$. Following a similar argument as [12, p. 188], we obtain that (K, v, ac) is a strongly NIP ac-valued field; more precisely, (K, v, ac) eliminates field quantifiers in the generalised Denef–Pas language (cf. [6, § 5.6], noting that both K and Kv have characteristic 0), whence by [12, Fact 3.5] we obtain that (K, v, ac) is strongly NIP. Since $\mathbb{R}$ is closed under square roots for positive elements, for any $a \in K$ we have $a \geq 0$ if and only if the following holds in K:

$$\exists y\, y^2 = \mathrm{ac}(a)$$

(cf. Observation 4.7). Hence, the order relation $<$ is definable in (K, v, ac). We obtain that $(K, <)$ is strongly NIP. □

Proposition 4.9 *Let $(K, <)$ be an almost real closed field with respect to a strongly NIP ordered abelian group and let G be a strongly NIP ordered abelian group. Then $(K((G)), <)$ is a strongly NIP ordered field.*

Proof. Let H be a strongly NIP ordered abelian group such that $(K, <)$ is almost real closed with respect to H and let w be a henselian valution on K with $wK = H$. As in the proof of Corollary 3.6, we can compose the valuation $v_{\min}$ on $K((G))$ with w on K to obtain a henselian valuation on $K((G))$ with real closed residue field and value group isomorphic to $G \oplus H$. Hence, $(K((G)), <) \equiv (\mathbb{R}((G \oplus H)), <)$. Since G and H are strongly NIP, also $G \oplus H$ is strongly NIP by Fact 4.6. Hence, by Lemma 4.8, also $(K((G)), <)$ is strongly NIP. □

Corollary 4.10 *Let $(K, <)$ be an almost real closed with respect to a henselian valuation v such that vK is strongly NIP. Then $(K, <)$ is strongly NIP.*

Proof. This follows immediately from Proposition 4.9 by setting $G = \{0\}$ and $H = vK$. □

For the proof of Theorem 4.12, we need one further result on general strongly NIP ordered fields, which will also be used for the proof of Theorem 5.4.

Proposition 4.11 *Let $(K, <)$ be a strongly NIP ordered field and let v be a henselian valuation on K. Then also $(Kv, <)$ and vK are strongly NIP.*

Proof. Arguing as in the proof of Lemma 4.3, we obtain that v is definable in $(K, <)^{\mathrm{Sh}}$. Now $(K, <)^{\mathrm{Sh}}$ is also strongly NIP (cf. [22, Observation 3.8]), whence $(K, <, v)$ is strongly NIP. By [28, Observation 1.4(2)], both $(Kv, <)$ and vK are strongly NIP, as they are first-order interpretable in $(K, <, v)$. □

We obtain from Corollary 4.10 and Proposition 4.11 the following characterisation of strongly NIP almost real closed fields.

Theorem 4.12 *Let $(K, <)$ be an almost real closed field with respect to some ordered abelian group G. Then $(K, <)$ is strongly NIP if and only if G is strongly NIP.*

Remark 4.13 Fact 4.6 and Theorem 4.12 give us the following complete characterisation of strongly NIP almost real closed fields: An almost real closed field $(K, <)$ is strongly NIP if and only if it is elementarily equivalent to some ordered Hahn field $(\mathbb{R}((G)), <)$ where G is a lexicographic sum as in Fact 4.6(2).

5 Equivalence of conjectures

Recall our two main conjectures.

Conjecture 1.1 *Let $(K, <)$ be a strongly NIP ordered field. Then K is either real closed or admits a non-trivial $\mathcal{L}_{\mathrm{or}}$-definable henselian valuation.*

Conjecture 1.2 *Any strongly NIP ordered field is almost real closed.*

In this section, we show that Conjectures 1.1 & 1.2 are equivalent (cf. Theorem 5.4).

Remark 5.1 (1) An ordered field is real closed if and only if it is o-minimal (cf. [23, Proposition 1.4, Theorem 2.3]). Hence, for any real closed field K, if $\mathcal{O} \subseteq K$ is a definable convex ring, its endpoints must lie in $K \cup \{\pm\infty\}$. This implies that any definable convex valuation ring must already contain K, i.e., is trivial. Thus, the two cases in the consequence of Conjecture 1.1 are exclusive.

(2) Recall from § 2 that the field $\mathbb{Q}$ is not NIP. By [1], the henselian valuation $v_{\min}$ is $\mathcal{L}_{\mathrm{r}}$-definable in $\mathbb{Q}((\mathbb{Z}))$. Hence, Proposition 4.11 yields that $(\mathbb{Q}((\mathbb{Z})), <)$ is an example of an ordered field which is not real closed, admits a non-trivial $\mathcal{L}_{\mathrm{or}}$-definable henselian valuation but is not strongly NIP.

Lemmas 5.2 & 5.3 below are used in the proof of Theorem 5.4. For the first result, we adapt the proof of [12, Lemma 3.7] to the context of ordered fields.

Lemma 5.2 *Assume that any strongly NIP ordered field is either real closed or admits a non-trivial henselian valuation*[10]. *Let $(K, <)$ be a strongly NIP ordered field. Then $(K, <)$ is almost real closed with respect to the canonical valuation, i.e., the finest henselian valuation on K.*

Proof. Let $(K, <)$ be a strongly NIP ordered field. If K is real closed, then we can take the natural valuation. Otherwise, by assumption, the set of non-trivial henselian valuations on K is non-empty. Let v be the canonical valuation on K. By Proposition 4.11, we have that $(Kv, <)$ is strongly NIP. Note that Kv cannot admit a non-trivial henselian valuation, as otherwise this would induce a non-trivial henselian valuation on K finer than v. Hence, by assumption, Kv must be real closed. □

The next result is obtained from an application of [11, Proposition 5.5].

Lemma 5.3 *Let $(K, <)$ be a strongly NIP ordered field which is not real closed but is almost real closed with respect to a henselian valuation v. Then there exists a non-trivial $\mathcal{L}_{\mathrm{r}}$-definable henselian coarsening of v.*

Proof. By Proposition 4.11, we have that $vK = G$ is strongly NIP. Since K is not real closed, G is non-divisible (cf. [8, Theorem 4.3.7]). By [11, Proposition 5.5], any henselian valuation with non-divisible value group on a strongly NIP field has a non-trivial $\mathcal{L}_{\mathrm{r}}$-definable henselian coarsening. Hence, there is a non-trivial $\mathcal{L}_{\mathrm{r}}$-definable henselian coarsening u of v. □

[10] Note that this valuation does not necessarily have to be $\mathcal{L}_{\mathrm{or}}$-definable.

Theorem 5.4 *Conjecture 1.1 and Conjecture 1.2 are equivalent.*

Proof. Assume Conjecture 1.2, and let $(K, <)$ be a strongly NIP ordered field which is not real closed. Then $(K, <)$ admits a non-trivial henselian valuation v. By Lemma 5.3, it also admits a non-trivial $\mathcal{L}_r$-definable henselian valuation. Now assume Conjecture 1.1. Let $(K, <)$ be a strongly NIP ordered field. By Lemma 5.2, we obtain that K is almost real closed with respect to the canonical valuation v. □

As a final observation, we give two further equivalent formulations of Conjecture 1.2 which follow from results throughout this work.

Observation 5.5 *The following are equivalent:*

(1) Any strongly NIP ordered field $(K, <)$ is almost real closed.

(2) For any strongly NIP ordered field $(K, <)$, the natural valuation v_{nat} on K is henselian.

(3) For any strongly NIP ordered valued field $(K, <, v)$, whenever v is convex, it is already henselian.

Proof. (1) implies (3) by Fact 3.3. Suppose that (3) holds and let $(K, <)$ be strongly NIP. Now v_{nat} is definable in the Shelah expansion $(K, <)^{\mathrm{Sh}}$, as it is the convex closure of $\mathbb{Z}$ in K. Hence, $(K, <, v_{\mathrm{nat}})$ is a strongly NIP ordered valued field. By assumption, v_{nat} is henselian on K, which implies (2). Finally, suppose that (2) holds. Let $(K, <)$ be a strongly NIP ordered field and $(K_1, <)$ an $\aleph_1$-saturated elementary extension of $(K, <)$. Then $K_1 v_{\mathrm{nat}} = \mathbb{R}$, as any Dedekind cut on the rational numbers in K_1 is realised in K_1.

More precisely, let $a \in \mathbb{R}$ and set $L = \{q \in \mathbb{Q} \mid q < a\}$ and $R = \{q \in \mathbb{Q} \mid a < q\}$. Then any finite subset of the 1-type $p(x) = \{q < x \mid q \in L\} \cup \{x < q \mid q \in R\}$ is realised in $\mathbb{Q}$ and thus also in K_1. As $p(x)$ is countable, the $\aleph_1$-saturation of K_1 implies that $p(x)$ is realised in K_1 by some $\alpha \in K_1$. Since $K_1 v_{\mathrm{nat}}$ is archimedean, it embeds as an ordered field into $\mathbb{R}$, i.e., $(\mathbb{Q}, <) \subseteq (K_1 v_{\mathrm{nat}}, <) \subseteq (\mathbb{R}, <)$. Finally, by application of the residue map, for any $q_1, q_2 \in \mathbb{Q}$ with $q_1 < a < q_2$ we obtain $q_1 \leq \overline{\alpha} \leq q_2$. Hence, $\overline{\alpha} = a$. Since a was chosen arbitrary, we obtain $K_1 v_{\mathrm{nat}} = \mathbb{R}$.

By assumption, v_{nat} is henselian on K_1, whence $(K_1, <)$ is almost real closed. By Proposition 3.4, also $(K, <)$ is almost real closed. □

6 Open questions

We conclude with open questions connected to results throughout this work. Conjecture 1.2 for archimedean fields states that any strongly NIP archimedean ordered field is real closed, as the only archimedean almost real closed fields are the real closed ones. Corollary 4.5 shows that any dp-minimal archimedean ordered field is real closed. We can ask whether the same holds for all strongly NIP ordered fields.

Question 6.1 Let $(K, <)$ be a strongly NIP archimedean ordered field. Is K necessarily real closed?

It is shown in [19] that any almost real closed field which is not real closed cannot be dense in its real closure. Thus, any dp-minimal ordered field which is dense in its real closure is real closed. Moreover, if Conjecture 1.2 is true, then, in particular, a strongly NIP ordered field which is not real closed cannot be dense in its real closure.

Question 6.2 Let $(K, <)$ be a strongly NIP ordered field which is dense in its real closure. Is $(K, <)$ real closed?

Note that Question 6.2 is more general than Question 6.1, as a positive answer to Question 6.2 would automatically tell us that any archimedean ordered field is real closed (since every archimedean field is dense in its real closure).

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References

[1] J. Ax, On the undecidability of power series fields, Proc. Amer. Math. Soc. **16**, 846 (1965).
[2] A. Chernikov and P. Simon, Henselian valued fields and inp-minimality, J. Symb. Log. **84**, 1510–1526 (2019).
[3] F. Delon and R. Farré, Some model theory for almost real closed fields, J. Symb. Log. **61**, 1121–1152 (1996).
[4] M. A. Dickmann, Elimination of quantifiers for ordered valuation rings, J. Symb. Log. **52**, 116–128 (1987).
[5] A. Dolich, J. Goodrick, and D. Lippel, Dp-minimality: basic facts and examples, Notre Dame J. Form. Log. **52**, 267–288 (2011).
[6] L. van den Dries, Lectures on the model theory of valued fields, in: Model Theory in Algebra, Analysis and Arithmetic, edited by D. Macpherson and C. Toffalori, Lecture Notes in Mathematics Vol. 2111. (Springer, Heidelberg, 2014), pp. 55–157.
[7] K. Dupont, A. Hasson, and S. Kuhlmann, Definable valuations induced by multiplicative subgroups and NIP fields, Arch. Math. Log. **58**, 819–839 (2019).
[8] A. J. Engler and A. Prestel, Valued Fields, Springer Monographs in Mathematics. (Springer, Berlin, 2005).
[9] R. Farré, A transfer theorem for henselian valued and ordered fields, J. Symb. Log. **28**, 915–930 (1993).
[10] Y. Halevi and A. Hasson, Eliminating field quantifiers in strongly dependent henselian fields, Proc. Amer. Math. Soc. **147**, 2213–2230 (2019).
[11] Y. Halevi and A. Hasson, Strongly dependent ordered abelian groups and henselian fields, Israel J. Math. **232**, 719–758 (2019).
[12] Y. Halevi, A. Hasson, and F. Jahnke, A conjectural classification of strongly dependent fields, Bull. Symb. Log. **25**, 182–195 (2019).
[13] F. Jahnke, P. Simon, and E. Walsberg, Dp-minimal valued fields, J. Symb. Log. **82**, 151–165 (2017).
[14] F. Jahnke, When does NIP transfer from fields to henselian expansions?, preprint (2019), arXiv:1607.02953v3.
[15] W. Johnson, The canonical topology on dp-minimal fields, J. Math. Log. **18**, 1850007 (2018).
[16] W. Johnson, Dp-finite fields VI: the dp-finite Shelah conjecture, preprint (2020), arXiv:2005.13989v1.
[17] M. Knebusch and M. J. Wright, Bewertungen mit reeller Henselisierung, J. Reine Angew. Math. **286/287**, 314–321 (1976).
[18] L. S. Krapp, S. Kuhlmann, and G. Lehéricy, On strongly NIP ordered fields and definable convex valuations, preprint (2019), arXiv:1810.10377v4.
[19] L. S. Krapp, S. Kuhlmann, and G. Lehéricy, Ordered fields dense in their real closure and definable convex valuations, Forum Math. (2021), https://doi.org/10.1515/forum-2020-0030.
[20] S. Kuhlmann, Ordered Exponential Fields, Fields Institute Monographs Vol. 12. (American Mathematical Society, Providence, RI, 2000).
[21] D. Macpherson, D. Marker, and C. Steinhorn, Weakly o-minimal structures and real closed fields, Trans. Amer. Math. Soc. **352**, 5435–5483 (2000).
[22] A. Onshuus and A. Usvyatsov, On dp-minimality, strong dependence and weight, J. Symb. Log. **76**, 737–758 (2011).
[23] A. Pillay and C. Steinhorn, Definable sets in ordered structures, I, Trans. Amer. Math. Soc. **295**, 565–592 (1986).
[24] A. Pillay and C. Steinhorn, Definable sets in ordered structures, III, Trans. Amer. Math. Soc. **309**, 469–476 (1988).
[25] J. Robinson, Definability and decision problems in arithmetic, J. Symb. Log. **14**, 98–114 (1949).
[26] S. Shelah, Stability, the f.c.p., and superstability; model theoretic properties of formulas in first order theory, Ann. Math. Log. **3**, 271–362 (1971).
[27] S. Shelah, Dependent first order theories, continued, Israel J. Math. **173**, 1–60 (2009).
[28] S. Shelah, Strongly dependent theories, Israel J. Math. **204**, 1–83 (2014).
[29] P. Simon, A Guide to NIP Theories, Lecture Notes in Logic Vol. 44 (Cambridge University Press, 2015).

