Methoden zur Planarisierung von Graphen

Annegret Liebers

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Betreut von Prof. Dr. Dorothea Wagner

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Fakultät für Mathematik und Informatik
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Annegret Liebers

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Advisor: Prof. Dr. Dorothea Wagner

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Universität Konstanz, Germany

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Zusammenfassung

Wir betrachten Verfahren, die einen gegebenen endlichen, ungerichteten, einfachen Graphen in einen planaren Graphen überführen. Wir geben einen Überblick über diese Verfahren sowie über Kennzahlen von Graphen, die beschreiben, wie weit der Graph von der Planarität entfernt ist. Der Schwerpunkt liegt dabei auf dem Löschen von Kanten, auf dem Aufspalten von Knoten und auf dem Parameter Schichtzahl.

Abstract

Given a finite, undirected, simple graph G, we are concerned with operations on G that transform it into a planar graph. We give a survey of results about such operations and related graph parameters. The emphasis is on edge deletion, vertex splitting, and thickness.

Contents

Chapter 1

Introduction

Many problems in discrete mathematics and combinatorial optimization can be viewed as graph problems. Graphs immediately come to mind for modelling networks of all kinds, but also seemingly unrelated problems from areas like transportation or warehousing can turn out to be, e.g. network flow problems, and their solution involves algorithms on graphs [AMO93].

Graphs that can be drawn without edge crossings (i.e. *planar* graphs) have a natural advantage for visualization, but also other graph problems can be easier to solve when restricted to this special class of graphs. "Easier" might mean the following: An intricate algorithm for general graphs may become less complicated (and thus its implementation more feasible), a special algorithm for planar graphs may have a better asymptotic time complexity than the best known algorithm for general graphs, or an intractable problem may become tractable if restricted to planar graphs. The latter case, however, seems to be relatively rare. Johnson [Joh85] lists only two problems (CLIQUE and MAX-CUT) that are NP-complete for general graphs but become polynomial when restricted to planar graphs.

When visualizing nonplanar graphs, a natural approach is to draw the graph in a way as close to planarity as possible (for example with as few edge crossings as possible). This is one of the problems of graph drawing, a field that has grown tremendously within the last decade [DETT94].

When solving a combinatorial optimization problem involving a nonplanar graph, a feasible approach might be to first transform the nonplanar graph into a planar graph, find a solution using the planar graph, and use it to construct a (possibly suboptimal) solution for the original problem.

In any case there is great interest in the question of how far from being planar a given graph is. We survey ways of transforming a nonplanar graph into a planar graph and discuss measures for the nonplanarity of a graph. One approach is to look for the largest induced planar subgraph of a nonplanar graph. Finding an induced subgraph is equivalent to deleting vertices from a graph and will be discussed in Chapter 2. It does not seem to be a very common approach, and there is relatively little literature about it.

Another approach is to look for the largest planar subgraph (without the restriction to induced subgraphs). Considering that instead of deleting a vertex together with all its incident edges it might suffice to just delete some of the edges, it is perhaps not surprising that finding a planar subgraph of a nonplanar graph (i.e. deleting edges) has been studied much more intensively. There is a vast amount of literature about finding a planar subgraph, with an emphasis on algorithmic results. They are the subject of Chapter 3.

Another technique for planarizing a graph is vertex splitting. There are relatively few algorithmic results about vertex splitting, but it turns out that there are many different structural results involving vertex splitting. These are discussed in Chapter 4.

Vertex deletion, edge deletion, and vertex splitting are operations performed on single vertices or edges of the graph in question, i.e. they are local operations. Chapter 5 discusses partitioning the whole graph into several planar layers, hence following a global approach. The greater the number of layers needed, the further away from planarity the graph is. There seem to be almost no algorithmic results about finding this *thickness* of a graph, but there are many structural results about thickness within topological graph theory.

Finally Chapter 6 briefly discusses the problem of drawing a graph so that there are as few edge crossings as possible in the drawing.

We do not study hierarchical graph models such as presented in [Len89, FCE95], nor do we discuss hypergraphs [Ber73, Ber89] or infinite graphs [Die91].

The remainder of this chapter gives definitions and terminology concerning graphs in Section 1.1, and then gives a brief introduction to planar graphs in Section 1.2. For an introduction to algorithms and the definition and use of $O(\cdots)$ and $\Omega(\cdots)$ for asymptotic bounds, the reader is referred to [CLR94]. The complexity classes P and NP and the concept of NP-completeness are also discussed in [CLR94], but a more thorough treatment can be found in [GJ79] and [Pap94].

1.1 Graphs

There are many textbooks on graph theory. Some of the standard ones are [Har69, BM76, Tut84][1]. A more recent text is [TS92]. We recommend it not only because it is new and can therefore include more recent results than the older texts, but also because it covers thoroughly the algorithmic aspects of graph theory. We also find it very well written from a didactic point of view. A very recent text is also [Wes95][2].

Two other texts [Ber73, Ber91], treatments of graph theory from a very algebraic point of view, are listed here for the translation of the English terms into German and French in the index of the first edition [Ber73].

We will now give the definitions and notation concerning graphs that are used in the following chapters.

A *finite, undirected, simple graph* G, denoted $G = (V, E)$, consists of a finite vertex set V and an edge set $E \subseteq \{\{u, v\} \mid u \in V,\ v \in V,\ u \neq v\}$.

[1] Note that [BM76] is out of print. We list it because it is often referred to in the literature for definitions and notation.

[2] [Wes95] was not available to the author at the time of completion of this thesis. The Table of Contents and the Preface are available electronically at http://www.math.uiuc.edu/~west/igtpref.ps .

For an edge $e = \{u, v\} \in E$, u and v are called the *end vertices* of e, and e is said to *connect* (or *join*) u and v. u and v are said to be *adjacent*. u is said to be a *neighbor* of v and vice versa. Furthermore, u and v are said to be *incident* to e (and vice versa). For brevity we often write uv instead of $\{u, v\}$.

A graph is usually visualized by representing each vertex through a point in the plane, and by representing each edge through a curve in the plane, connecting the points corresponding to the end vertices of the edge. We usually do not distinguish between a vertex and the point representing it, or between an edge and the curve representing it. Such a representation is called a *drawing* of the graph if no two vertices are represented by the same point, if the curve representing an edge does not include any point representing a vertex (except that the endpoints of the curve are the points representing the end vertices of the edge), and if two distict edges have at most one point in comon.

We do not consider graphs with infinite vertex or edge sets (*infinite graphs*). We also disregard *directed graphs*, i.e. graphs where each edge is an ordered pair instead of a set with two elements. Furthermore, we do not consider graphs where two vertices may be connected by more than one edge (those edges would be called *multiple edges*), or where the end vertices of an edge are allowed to be identical (such an edge would be called a *loop*). Graphs without multiple edges and loops are called *simple*.

From now on, when we speak of a graph, we always mean a finite, undirected, simple graph.

The number of edges incident to a vertex u is called the *vertex degree* (or simply *degree*) of u. The *minimum (maximum) degree* of a graph G is the minimum (maximum) degree of all vertices of G. If all vertices of a graph have the same degree d, the graph is called *d-regular* (or just *regular*). A 3-regular graph is also called *cubic*.

Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are said to be *isomorphic* if there is a bijection $\nu : V_1 \to V_2$ such that two vertices $u_1 \in V_1$ and $v_1 \in V_1$ are adjacent in G_1 if and only if the vertices $u_2 = \nu(u_1)$ and $v_2 = \nu(v_1)$ are adjacent in G_2. For an illustration, see Figure 4.1. It displays 19 graphs many of which are isomorphic. We usually consider graphs up to isomorphism, i.e. we do not distinguish between two isomorphic graphs.

Given a graph $G = (V, E)$, a graph $G' = (V', E')$ is called a *subgraph* of G if $V' \subseteq V$ and $E' \subseteq E$. If furthermore $V' = V$ then G' is said to be a *spanning* subgraph of G. If $V'' \subset V$ or $E' \subset E$ (or both) then G' is said to be a *proper* subgraph of G. A graph $G'' = (V'', E'')$ is called a *vertex induced* (or simply *induced*) subgraph of G if $V'' \subseteq V$ and $E'' = \{uv \mid u \in V'' \text{ and } v \in V'' \text{ and } uv \in E\}$. In that case we call G'' the *subgraph of G induced by V''*. In Figure 4.1, all the graphs depicted are subgraphs of the graph numbered 18. The graph G and the graphs 1 through 17 are proper subgraphs of graph 18. Graphs 1 through 18 are spanning subgraphs of graph 18, but G is not. In Figure 2.1, the graphs G_1 and G_2 are induced subgraphs of the graph G.

If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are two (not necessarily distinct) subgaphs of a graph $G = (V, E)$, then the subgraph $G' = (V_1 \cup V_2, E_1 \cup E_2)$ of G is called the *union* of G_1 and G_2.[3]

Given a graph $G = (V, E)$, a sequence $v_0 e_1 v_1 e_2 v_2 \ldots e_k v_k$ is called a *path* in G if the

[3]Note that the term *union* is sometimes defined differently (see for example [Har69, p. 21]).

$k+1$ vertices $v_0 \dots v_k$ are elements of V, if they are pairwise distinct except possibly v_0 and v_k, and if v_{i-1} and v_i are the end vertices of e_i for $1 \leq i \leq k$. k is called the *length* of the path. v_0 and v_k are called the *end vertices* of the path, and the path is said to *contain* the vertices $v_0 \dots v_k$. We also say that the path *connects* the vertices v_0 and v_k. If additionally $v_0 = v_k$, the path is called a *cycle*. The length of a shortest cycle in G is called the *girth* of G. If G has no cycles, the girth is undefined.

We denote with P_n the graph consisting only of a path of length $n-1$, where the end vertices of the path are not identical. Observe that the graph P_n has n vertices and $n-1$ edges. C_n denotes a graph consisting of a cycle of length n, having n vertices and n edges. P_3 is depicted as graphs 7 and 8, and C_4 is depicted as graphs 16 and 17 in Figure 4.1. If a path in a graph G includes all vertices of G it is called a *Hamilton path*. If additionally this path is a cycle, it is called a *Hamilton cycle*. Observe that in Figure 4.1, graph 13 contains a Hamilton path, but no Hamilton cycle, whereas graph 14 contains both.

If for every pair of vertices u and v of a graph $G = (V, E)$ there is a path in G connecting u and v then G is said to be *connected*. Otherwise G is said to be *disconnected*. If $V' \subseteq V$ is a vertex set such that the subgraph of G induced by V' is connected and such that for every set $V'' \subseteq V$ with $V' \subset V''$ the subgraph of G induced by V'' is disconnected, then we call the subgraph of G induced by V' a *connected component* (or simply *component*) of G. In Figure 4.1, the graph 1 is disconnected. It consists of two components.

Given a graph $G = (V, E)$ and a vertex $v \in V$ we say that the subgraph G' of G induced by $V \setminus \{v\}$ is obtained by *deleting* v from G. If G' has more components than G then v is said to be a *cut vertex* of G. If $V' \subset V$ is a vertex set such that the subgraph of G induced by $V \setminus V'$ has more components than G, then V' is said to be a *vertex cut* of G. If, for a positive integer k, G has no vertex cut of cardinality less than k, and if G itself is connected, then G is said to be *k-connected* (i.e. at least k vertices have to be deleted from G before the resulting graph is disconnected). Observe that a graph is 1-connected if and only if it is connected, and that a connected graph without cut vertices is 2-connected. In Figure 4.1, graph 6 has one cut vertex, graph 7 has two cut vertices, but graph 16 has none. So graph 16 is 2-connected, but it is not 3-connected because it contains a vertex cut of cardinality 2. Graph 18, however, is 3-connected. Figure 4.12 also depicts 3-connected graphs.

Analogous definitions exist for edges: Given a graph $G = (V, E)$ and an edge $e \in E$ we say that the subgraph $G' = (V, E \setminus \{e\})$ of G is obtained by *deleting* e from G. If G' has more components than G then e is said to be a *cut edge* of G. If $E' \subset E$ is an edge set such that the subgraph $G' = (V, E \setminus E')$ has more components than G, then E' is said to be an *edge cut* of G. If, for a positive integer k, G has no edge cut of cardinality less than k, and if G itself is connected, then G is said to be *k-edge-connected* (i.e. at least k edges have to be deleted from G before the resulting graph is disconnected).

If for a graph $G = (V, E)$, $V' \subseteq V$ is a vertex set such that the subgraph of G induced by V' is 2-connected and such that for every set $V'' \subseteq V$ with $V' \subset V''$ the subgraph of G induced by V'' is not 2-connected, then we call the subgraph of G induced by V' a *block* of G.

If an edge $e = uv$ of a graph $G = (V, E)$ is replaced by a path $ue'v_e e''v$ introducing a new vertex $v_e \neq V$ then we say that the graph $G' = (V \cup \{v_e\}, (E \setminus \{e\}) \cup \{e', e''\})$ is obtained from G by *subdividing* the edge e. If a graph G'' is obtained from G by any number of (possibly zero) subdivisions of edges then G'' is called a *subdivision* of G. It will be clear from the context whether the term subdivision refers to the operation of subdividing an edge or to the resulting graph. For an illustration of subdivisions, see Figure 4.10.

For a graph $G = (V, E)$ and an edge $e = uv \in E$, the graph G' obtained from G by deleting e, identifying u and v and by removing all possibly created loops or multiple edges, is said to have been obtained from G by *contracting* the edge e. A graph obtained from a subgraph of G by any number of (including zero) edge contractions is said to be a *minor* of G. A subgraph of G is always a minor of G, but not vice versa. In Figure 4.1, the graph G is a minor of graphs 1 through 6 and 9 through 18, but it is not a minor of graphs 7 and 8. For another illustration of graph minors, see Figure 5.3.

Besides the paths P_n and the cycles C_n defined above, the following special graphs appear throughout the text:

For $n \geq 2$, the *complete graph*, denoted K_n, consists of n vertices together with all possible $\binom{n}{2}$ edges. So in K_n every vertex is adjacent to every other vertex. We define K_1 to be the graph consisting of a single vertex. K_2 is a single edge with its two end vertices. In Figure 4.1, K_3 is depicted as graph G and K_4 is depicted as graph 18. K_5 is depicted in Figure 4.3 a).

The *complete bipartite graph*, denoted K_{n_1,n_2}, consists of two disjoint vertex sets $V = \{v_1, \ldots v_{n_1}\}$ and $W = \{w_1, \ldots w_{n_2}\}$ and the edge set $E = \{v_i w_j \mid 1 \leq i \leq n_1$ and $1 \leq j \leq n_2\}$ of all edges between vertices in V and vertices in W. Note that $K_{n_1,n_2} = K_{n_2,n_1}$. Observe that $K_{1,1} = K_2$, that $K_{1,2} = P_2$, and that $K_{2,2} = C_4$.

The *hypercube of dimension* n, denoted Q_n, is the graph with 2^n vertices where each vertex has a label consisting of an n-digit binary number between $0 \ldots 0$ and $1 \ldots 1$ and with an edge connecting two vertices if and only if the labels of the vertices differ in a single digit. Observe that Q_n has $n \cdot 2^{n-1}$ edges, that $Q_1 = K_2$ and that $Q_2 = C_4$.

A *tree* is a connected, acyclic graph. A tree with n vertices has $n - 1$ edges.

1.2 Planar Graphs

The class of planar graphs has been widely studied, and most of the textbooks mentioned above contain chapters about planar graphs [Har69, BM76, Tut84, TS92, Wes95]. A wealth of literature studies properties of planar graphs, algorithms for solving problems on planar graphs, and how close other graphs are to planarity. The latter topic results in algorithms that transform a given graph into a planar graph. These results are briefly summarized in Section 4.2 of the annotated graph drawing bibliography by Di Battista et al. [DETT94].

The book by Nishizeki and Chiba [NC88] is a thorough treatment of planar graphs, with an emphasis on algorithms. [Nis90] can be seen as an update of [NC88]. Johnson [Joh85] surveys the algorithmic complexity of problems on graphs, including problems on planar graphs.

A graph G is said to be *planar* if it admits a drawing such that no two edges contain a common point except possibly a common end vertex. Such a drawing of a planar graph is called a *planar embedding* (or simply an *embedding*) of G. Fáry [Fár48] showed that every planar graph has an embedding in which the edges are straight line segments. Each connected subset of the plane that is delimited by a closed curve consisting of vertices and edges of G is called a *face* of the embedding. A face is said to be *incident* to the vertices and edges it is delimited by (and vice versa). All faces except one are bounded subsets of the plane. The unbounded face is called the *outer* face.

Figure 2.1 shows the nonplanar graph G as well as two planar graphs G_1 and G_2. The drawing for G_1 is not an embedding, but the drawing for G_2 is. In Figure 3.1, the graphs G_1, G_2, and G_3 are planar, and the drawing given for each of them is an embedding. The embedding for G_1 contains three faces, one incident to four vertices, another incident to five vertices, and a third one (the outer face) incident to seven vertices.

A planar graph together with an embedding is also called a *plane* graph. For a connected plane graph G with n vertices, m edges and f faces, Euler found the following formula:

$$n - m + f = 2 \qquad \text{(Euler 1750)} \tag{1.1}$$

This can be shown by an induction over m (see for example [NC88]). Note that if a planar graph with $n \geq 3$ vertices has as many edges as possible, then each face is incident to exactly three vertices (for otherwise an additional edge could be added, deviding a face that is incident to more than three vertices into two faces without violating planarity). Euler's formula together with this observation yields the following well known corollary:

$$m \leq 3n - 6 \qquad \text{(for } n \geq 3\text{)} \tag{1.2}$$

We have seen a few properties of planar graphs, but we do not know yet how to decide whether a given graph is planar. We first note that we can restrict our attention to 2-connected graphs as stated by Kelmans [Kel93]: A graph is planar if and only if each of its components is planar. Furthermore, a connected graph is planar if and only if each of its blocks is planar. [Kel93] goes on to show that we may even restrict ourselves to 3-connected graphs.

Now we will give some of the known characterizations of planar graphs. We start with Steinitz's Theorem, relating planar graphs to 3-dimensional convex polytopes. Given a 3-dimensional polytope P, its *edge graph* $G_P = (V_P, E_P)$ is formed as follows. Let V_P be the set of 0-dimensional faces of P (i.e. the so-called vertices of P) and let E_P be the set of 1-dimensional faces of P (the so-called edges of P). Recalling that a polytope is convex by definition and that all graphs considered here are simple, Steinitz's Theorem [SR34] can be stated as follows [Whi84, p. 53],[RZ95]:

Theorem 1.3 (Steinitz 1922) *A graph G is the edge graph of a 3-dimensional polytope if and only if G is planar and 3-connected.*

The most well known characterization of planar graphs is probably the one by Kuratowski [Kur30]:

Theorem 1.4 (Kuratowski [Kur30]) *A graph G is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$ as a subgraph.*

The graphs K_5 and $K_{3,3}$ are the complete graph on 5 vertices and the complete bipartite graph on two times three vertices as defined above. K_5 and $K_{3,3}$ are called the *Kuratowski graphs*. They are depicted in Figure 4.3 a) and 4.10 a), respectively. A proof of Kuratowski's Theorem can be found in [NC88], for example.

Theorem 1.4 was strengthened by Wagner [Wag37b], and, independently, by Hall [Hal43]. Kelmans [Kel93] states the stronger version as follows:

Theorem 1.5 *A 3-connected graph G distinct from K_5 is planar if and only if it does not contain a subdivision of $K_{3,3}$ as a subgraph.*

Wagner [Wag37a], and, independently, Harary and Tutte [HT65] give another characterization:

Theorem 1.6 (Wagner [Wag37a]) *A graph G is planar if and only if it does not contain K_5 or $K_{3,3}$ as a minor.*

For further characterizations of planar graphs see for example [Whi33b, Mac37] and [NC88, Kel93].

An algorithm for determining whether a given graph is planar was first developed by Lempel, Even, and Cederbaum [LEC67]. It was improved by Booth and Lueker [BL76] to run in linear time, using a data structure called PQ-trees. The algorithm can be modified to not only test whether a given graph is planar, but to also yield an embedding for the graph if it is planar. See Kant [Kan93] for a discussion of this algorithm. Another linear time planarity testing algorithm was developed by Hopcroft and Tarjan [HT74]. It follows an edge addition approach, and was modified by Mutzel et al. [Mut92, MMN94] to also yield an embedding if the given graph is planar.

Chapter 2

Vertex Deletion

Given a graph $G = (V, E)$, we can transform it into a planar graph $G' = (V', E')$ in a trivial way by deleting all but four vertices of V from G together with all their incident edges. G' is then a tetrahedron (K_4) or a subgraph thereof, and hence planar. But we would hope to retain more than four vertices of the original graph and still obtain a planar subgraph. This section investigates the question of deleting as few vertices as possible (together with their incident edges) from a given graph G to make it planar. It seems that deleting vertices is too drastic an operation on a given graph to be useful in practice. The author is only aware of few results investigating vertex deletion for planarization.

Definition 2.1 (maximum induced planar subgraph) *If a graph $G' = (V', E')$ is an induced planar subgraph of a graph $G = (V, E)$ such that there is no induced planar subgraph $G'' = (V'', E'')$ of G with $|V''| > |V'|$, then G' is called a* maximum induced planar subgraph *of G.*

So the problem of deleting as few vertices as possible from a graph so that the resulting graph is planar means to find, for a given graph G, a maximum induced planar subgraph of G.

Problem 2.2 (Maximum Induced Planar Subgraph [GJ79, Problem GT21]) *Given a graph $G = (V, E)$ and a positive integer $K \leq |V|$, is there a subset $V' \subseteq V$ with $|V'| \geq K$ such that the subgraph of G induced by V' is planar?*

Yannakakis showed that this problem is NP-complete:

Theorem 2.3 [Yan78] *Maximum Induced Planar Subgraph is NP-complete.*

Actually, Yannakakis showed a far more general result:

Theorem 2.4 [Yan78] *If Π is a graph property satisfying the following four conditions*

1. *Π holds for the graph K_1 consisting of a single vertex.*

2. *There is a graph for which Π does not hold.*

3. *For each positive integer k, there is a graph $G = (V, E)$ with $|V| \geq k$ for which Π holds.*

4. *If Π holds for a graph G and if G' is an induced subgraph of G, then Π holds for G'.*

then the following problem is NP-complete: Given a graph $G = (V, E)$ and a positive integer $K \leq |V|$, is there a subset $V' \subseteq V$ with $|V'| \geq K$ such that Π holds for the subgraph of G induced by V'?

Note that the graph property of being planar satisfies conditions 1 through 4.

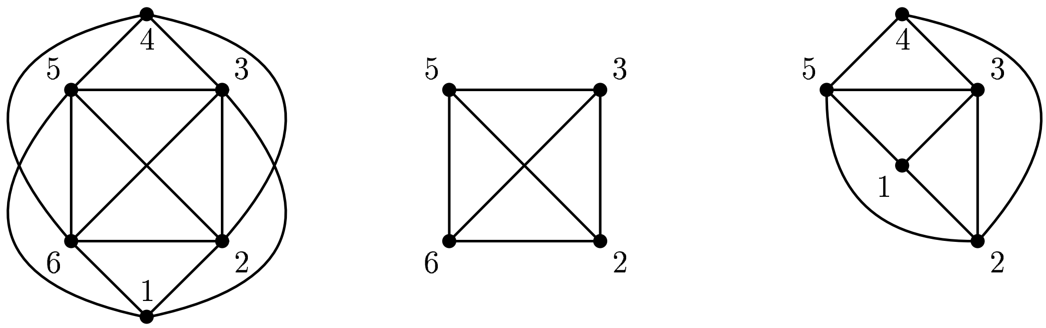
Djidjev and Venkatesan [DV95] show that for a graph G with n vertices and with orientable genus g, there exists a set of $4\sqrt{gn}$ vertices whose removal planarizes G. Furthermore, if G has nonorientable genus g (and n vertices as above), then the deletion of $2\sqrt{2gn}$ vertices planarizes G, and these bounds are tight up to a constant factor. This improves results in [HM87]. Recall that the orientable (nonorientable) genus g of a graph G is the smallest g so that G can be embedded in an orientable (nonorientable) surface of genus g [WB78], [Whi84, Chapters 5 and 6]. But note that it is NP-hard to determine the genus of a given graph [Tho89]. For further results (nonconstructive as well as algorithmic) about deleting vertices from a graph with known genus, see [Hut89, DV95].

Since Maximum Induced Planar Subgraph is an NP-complete problem, we also consider an easier problem:

Definition 2.5 (maximal induced planar subgraph) *If a graph $G' = (V', E')$ is an induced planar subgraph of a graph $G = (V, E)$ such that every subgraph of G induced by a vertex set $V'' = V' \cup \{v\}$ with $v \in V \setminus V'$ is nonplanar, then G' is called a* maximal induced planar subgraph *of G.*

For a given graph G we want to find a maximal induced planar subgraph. Note that every maximum induced planar subgraph is also a maximal induced planar subgraph, but not vice versa. Observe that a maximal induced planar subgraph is maximal with respect to inclusion of its vertex set, whereas a maximum induced planar subgraph is maximal with respect to the cardinality of its vertex set. Analogous definitions concerning the edge set will be used in Chapter 3. Figure 2.1 illustrates maximal and maximum induced planar subgraphs.

A straightforward way of finding, for a given graph G with n vertices and m edges, a maximal induced planar subgraph is the greedy algorithm shown in Figure 2.2. The while loop in lines 2 through 8 is performed n times. The planarity test in line 4 can be done in linear time, and the graph whose planarity is tested has in the worst case the same input size as G, i.e. $n + m$. Since each edge of G is added to E' at most once in line 5, line 5 needs $O(m)$ time over all iterations of the while loop. Thus the overall time complexity of GREEDY MAXIMAL INDUCED PLANAR SUBGRAPH is in $O(n(n+m)+m) = O(n \cdot m)$ (assuming that G is connected, so that $m \in \Omega(n)$).



G $\qquad$ G_1 $\qquad$ G_2

Figure 2.1: G is a nonplanar graph (note that G contains K_5 as a subgraph). G_1 is a maximal induced planar sugraph of G. G_2 is another maximal induced planar subgraph of G, and G_2 is also a maximum induced planar subgraph of G.

```
GREEDY MAXIMAL INDUCED PLANAR SUBGRAPH
INPUT: a graph G = (V, E) with |V| = n and |E| = m
OUTPUT: a maximal induced planar subgraph G' = (V', E') of G
set V' = ∅
while V is nonempty
     choose a vertex v ∈ V
     if the subgraph of G induced by V' ∪ {v} is planar, then
          set E' = E' ∪ {uv | u ∈ V' and uv ∈ E}
          set V' = V' ∪ {v}
          (The graph (V', E') is now the subgraph of the input graph induced by V'.)
     set V = V \ {v}
end of while
```

Figure 2.2: Algorithm GREEDY MAXIMAL INDUCED PLANAR SUBGRAPH

Chapter 3

Edge Deletion and Skewness

If a graph $G = (V, E)$ with an edge $e \in E$ is transformed into a graph $G' = (V, E \setminus \{e\})$ then we say that G' was obtained from G by *edge deletion*. By repeatedly deleting edges from a given nonplanar graph G, G can be transformed into a planar graph G'. We are interested in planarizing G by deleting as few edges as possible.

Deleting edges from a given graph G in order to transform G into a graph G' with a particular property is a common approach (see for example [SC89, Sen90]). We will only discuss edge deletion with the purpose of planarization, a topic that has been studied intensively.

Definition 3.1 (maximum planar subgraph, skewness) *If a graph $G' = (V, E')$ is a planar subgraph of a graph $G = (V, E)$ such that there is no planar subgraph $G'' = (V, E'')$ of G with $|E''| > |E'|$, then G' is called a* maximum planar subgraph *of G, and the number of deleted edges, $|E| - |E'|$, is called the* skewness *of G.*

So the skewness of a graph G is 0 if and only if G is planar. The problem of finding, for a given graph G, a maximum planar subgraph is NP-hard [LG79]. It will be discussed in Section 3.1. For some graph classes, the skewness is known: The skewness of the complete graph K_n is $(n-3)(n-4)/2$ for $n \geq 3$, and the skewness of the complete bipartite graph K_{n_1,n_2} is $n_1 n_2 - 2(n_1 + n_2) + 4$ for $n_1 \geq 2$ and $n_2 \geq 2$. Both results were first published by Kotzig [Kot55]. The skewness of the hypercube of dimension n, Q_n, is $2^n(n-2) - n \cdot 2^{n-1} + 4$ [Cim92].

Definition 3.2 (maximal planar subgraph) *If a graph $G' = (V, E')$ is a planar subgraph of a graph $G = (V, E)$ such that every graph $G'' \in \{(V, E' \cup \{e\}) \mid e \in E \setminus E'\}$ is nonplanar, then G' is called a* maximal planar subgraph *of G.*

In other words a maximal planar subgraph is maximal with respect to inclusion of its edge set, whereas a maximum planar subgraph is maximal with respect to the cardinality of its edge set. Note the analogy with Definitions 2.5 and 2.1 concerning the vertex set of a graph. A maximal planar subgraph can be found in polynomial time, as will be seen in Section 3.2.

Note that every maximum planar subgraph is also a maximal planar subgraph, but not vice versa. Figure 3.1 illustrates maximal and maximum planar subgraphs.

Finally, Section 3.3 discusses approximative and heuristic approaches for finding a large planar subgraph.

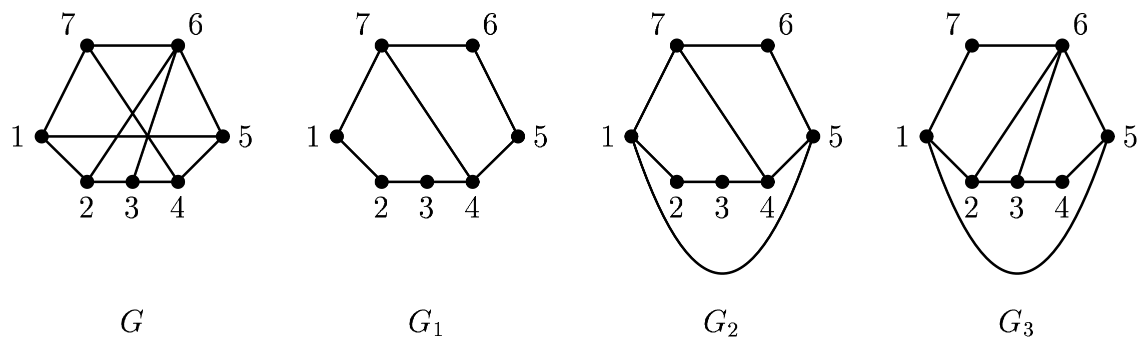


Figure 3.1: G is a nonplanar graph. Note that G contains $K_{3,3}$ as a minor (contract edge $\{2,3\}$). G_1 is a planar subgraph of G, but it is not a maximal planar subgraph: Edge $\{1,5\}$ can be added to G_1 without destroying planarity. The result is G_2. Another maximal planar subgraph of G is G_3. G_3 is also a maximum planar subgraph.

3.1 Finding a Maximum Planar Subgraph

In this section, we study the following problem:

Problem 3.3 (Maximum Planar Subgraph [GJ79, Problem GT27]) *Given a graph $G = (V, E)$ and a positive integer $K \leq |E|$, is there a subset $E' \subseteq E$ with $|E'| \geq K$ such that the graph $G' = (V, E')$ is planar?*

Liu and Geldmacher [LG79], and, independently, Yannakakis [Yan78] showed that this problem is NP-complete:

Theorem 3.4 [LG79, Yan78] *Maximum Planar Subgraph is NP-complete.*

The proof of Liu and Geldmacher is a two step reduction using the following problems:

Problem 3.5 (Vertex Cover [GJ79, Problem GT1])[1] *Given a graph $G = (V, E)$ and a positive integer $K \leq |V|$, is there a vertex cover of size K or less for G, i.e. is there a subset $V' \subseteq V$ of vertices with $|V'| \leq K$ such that for each edge $uv \in E$ at least one of its endpoints u and v belongs to V'?*

Problem 3.6 (Hamilton Path in Graphs Without Triangles) *Given a graph $G = (V, E)$ that does not contain a cycle of length 3, and given two vertices $u \in V$ and $v \in V$, does G contain a Hamilton path from u to v?*

Karp [Kar72] shows Vertex Cover to be NP-complete. [LG79] first reduces Vertex Cover to Hamilton Path in Graphs Without Triangles, and then reduces this problem to Maximum Planar Subgraph.

Since Maximum Planar Subgraph is NP-complete, we cannot expect to find a polynomial time algorithm for this problem. But Jünger and Mutzel [JM93a, JM93b][2]

[1]Liu and Geldmacher call this problem Vertex Edge-Cover.

[2]See [Jün95] for an overview of their work, and [Mut94] for a comprehensive treatment. [JM95] deals with a related problem.

successfully use a branch and cut approach based on polyhedral combinatorics to solve a more general version of Maximum Planar Subgraph for many "real world instances":

Problem 3.7 (Weighted Maximum Planar Subgraph) *Given a graph $G = (V, E)$ with a nonnegative edge weight $w(e)$ for each edge e, and a positive number K, is there a subset $E' \subseteq E$ with $\sum_{e \in E'} w(e) \geq K$ such that the graph $G' = (V, E')$ is planar?*

Djidjev and Venkatesan [DV95] show that for a graph G with m edges, maximum vertex degree d, and orientable genus g, there exists a set of $4\sqrt{dgm}$ edges whose removal planarizes G. Furthermore, if G has nonorientable genus g (and m and d as above), then the deletion of $2\sqrt{2dgm}$ edges planarizes G, and these bounds are tight up to a constant factor. This improves results in [Dji84]. For the corresponding results concerning planarizing vertex sets see Chapter 2.

3.2 Finding a Maximal Planar Subgraph

The problem of finding a maximal planar subgraph for a given graph G with n vertices and m edges is solvable in polynomial time. A straightforward way of finding a maximal planar subgraph is the greedy algorithm shown in Figure 3.2. The while loop in lines 2 through 6 is performed m times in the worst case, and the planarity test in line 4 can be done in linear time. The graph whose planarity is tested has n vertices and at most $3n - 6$ edges (see Equation 1.2), so the planarity test requires $O(n)$ time. Thus the worst case time complexity of GREEDY MAXIMAL PLANAR SUBGRAPH is in $O(m \cdot n)$.

GREEDY MAXIMAL PLANAR SUBGRAPH
INPUT: a graph $G = (V, E)$ with $|V| = n$ and $|E| = m$
OUTPUT: a maximal planar subgraph $G' = (V, E')$ of G
1 set $E' = \emptyset$
2 while E is nonempty and $|E'| < 3n - 6$
(recall that a planar subgraph cannot have more than $3n - 6$ edges)
3 choose an edge $e \in E$
4 if the graph $(V, E' \cup \{e\})$ is planar, set $E' = E' \cup \{e\}$
5 set $E = E \setminus \{e\}$
6 end of while

Figure 3.2: Algorithm GREEDY MAXIMAL PLANAR SUBGRAPH

The standard algorithms for planarity testing [HT74, BL76] are rather complicated to implement. Therefore, algorithms for finding a maximal planar subgraph are sought that not only have a better worst case time complexity than GREEDY MAXIMAL PLANAR SUBGRAPH, but that are also less complicated. One series of results about better polynomial time algorithms for finding a maximal planar subgraph starts with [OT81] which claims to give an $O(n \cdot m)$ algorithm based on the planarity testing algorithm [LEC67, BL76] using PQ-trees. The algorithm starts with one vertex as the initial planar subgraph and then adds one vertex (together with as many of its incident edges as possible) at a time. But [TJS86] points out that the subgraph generated by this algorithm is not always a maximal planar subgraph, and that it is not even always a spanning subgraph. [JTS89, JST89] claims to amend the problem and gives two $O(n^2)$ algorithms, one to find a spanning planar subgraph of a 2-connected graph G, and one to find a maximal planar subgraph by augmenting the previously found spanning planar subgraph. The latter algorithm is shown to be incorrect by [Kan92, Kan93] which also claims to show how to correct the algorithm. But according to [Jün95], the result in [Kan92] is not correct either [Lei, Lei95]. See the discussion in [Jün95, Chapter 3.2.2].

Di Battista and Tamassia [DT89, DT90] define and use $SPQR$-trees to describe the recursive decomposition of a 2-connected planar graph into its 3-connected components. [DT89] obtains an $O(m \log n)$ time algorithm for finding a maximal planar subgraph as a byproduct of an algorithm for incremental planarity testing. An incremental (or *dynamic*dynamic planarity testing) planarity testing algorithm maintains a data structure representing a planar graph $G = (V, E)$ and can handle requests of the following types: a) For two vertices v_1 and v_2 in G with $v_1 v_2 \notin E$, determine whether G stays planar if the edge $v_1 v_2$ is added to G. b) If $v_1 \in V$, $v_2 \in V$, $v_1 v_2 \in E$, add the edge $v_1 v_2$ to G (assuming the corresponding request of type a) yields a positive answer). c) Add a new vertex to G.

Independently, Cai, Han, and Tarjan [CHT93] developed an $O(m \log n)$ algorithm to find a maximal planar subgraph of a graph G with n vertices and m edges. Their algorithm is based on a new version of the Hopcroft and Tarjan planarity testing algorithm [HT74].

La Poutré [La 94] presents algorithms for incremental planarity testing that yield an $O(n + m \cdot \alpha(m, n))$ time algorithm for the maximal planar subgraph problem. This result was recently improved to linear time complexity by Djidjev [Dji95], and, independently, by Hsu [Hsu95].

Given a graph $G = (V, E)$, Djidjev [Dji95] first computes a depth first search tree of G. This spanning tree of G is the initial planar subgraph $G' = (V, E')$ of G. Then for each edge $e \in E \setminus E'$ it is determined whether the graph $(V, E' \cup \{e\})$ is still planar. If so, e is added to E'. The order in which the edges in $E \setminus E'$ are considered is chosen in a clever way so that, with $O(1)$ amortized time per test and insert operation for each edge $e \in E \setminus E'$, the overall time complexity is linear. Many intricate data structures are needed to achieve the $O(1)$ amortized time per test and insert operation. Two of them are BC-trees to describe the decomposition of a connected planar graph into its 2-connected components[3] and $SPQR$-trees to describe the decomposition of a 2-connected planar graph into its 3-connected components [DT89].

[Hsu95] also starts with a depth first search tree of the given graph $G = (V, E)$, and then determines a postordering of the vertices of G. The postordering is a labeling $l : V \to \{1, \ldots, n\}$ so that if u is an ancestor of v in the depth first search tree, then

[3] In [Har69] these trees are called block-cutpoint trees.

$l(u) > l(v)$. The initial planar subgraph G' of G is empty, and the vertices are added in ascending order of their label. So in step i of the algorithm, the vertex with label i (and the edges incident to it) are added to G'. Note that G' is not necessarily connected. According to [Hsu95], the way in which the vertices are added and in which for each edge it is decided whether the edge can be added to G' without destroying planarity, ensures the construction of a maximal planar subgraph in linear time.

3.3 Approximations and Heuristics

Since the Maximum Planar Subgraph problem is NP-complete, and therefore the Weighted Maximum Planar Subgraph problem is too, we cannot expect to find an efficient algorithm to determine a planar subgraph $G' = (V, E')$ of an input graph $G = (V, E)$ so that the sum of the edge weights of edges in E' is maximum. Thus, approximation and heuristic approaches are called for.

First consider a trivial approximation for finding a maximum planar subgraph by observing that for a given graph G with n vertices, any spanning tree of G is a planar subgraph with $n-1$ edges (assume that G is connected), and that a spanning tree can be found in linear time. Furthermore, a planar subgraph of G cannot have more than $3n-6$ edges (see Equation 1.2). So if E' is the edge set of a spanning tree for a given graph G, and if E^* is the edge set of a maximum planar subgraph of G, then the ratio $\frac{|E'|}{|E^*|}$ is bounded (see also [Cim92]):

$$\frac{|E'|}{|E^*|} = \frac{n-1}{|E^*|} \geq \frac{n-1}{3n-6} > \frac{1}{3}$$

In a very recent result this bound was improved for the first time [CFFK96]: Călinescu et al. provide approximations with a lower bound of $\frac{2}{5}$. But note that these bounds do not say anything about the sum of the edge weights of the edges in $|E'|$.

GREEDY MAXIMAL PLANAR SUBGRAPH (Figure 3.2) finds a maximal planar subgraph, which will be at least as good as just taking a spanning tree. The greedy heuristic GREEDY WEIGHTED PLANAR SUBGRAPH [KH78, DFF85, FGG85] for the Weighted Maximum Planar Subgraph problem is the same algorithm as GREEDY MAXIMAL PLANAR SUBGRAPH, except that it considers the edges in the order of nonincreasing weight. So replace line 3 in Figure 3.2 with the following line to obtain GREEDY WEIGHTED PLANAR SUBGRAPH:

```
3        choose an edge e ∈ E with largest edge weight
```

The greedy heuristic does involve repeated planarity testing, and even though planarity testing can be done in linear time, the algorithms are rather complicated. The following heuristics build up a subgraph that is planar by construction, and thus avoid planarity testing.

The Deltahedron Heuristic [FR78, FGG85] starts with a tetrahedron (K_4) as the initial planar subgraph and then adds one vertex at a time, placing each new vertex in one of the faces of the current planar subgraph (see part a) of Figure 3.5 on page 20 for an illustration). The sequence in which the vertices are added is determined by a vertex weight that can be defined in various ways, as discussed below. Figure 3.3 shows the algorithm of the Deltahedron Heuristic, DELTAHEDRON WEIGHTED PLANAR SUBGRAPH.

DELTAHEDRON WEIGHTED PLANAR SUBGRAPH
INPUT: a graph $G = (V, E)$, nonnegative real edge weights $w(e)$ for $e \in E$
OUTPUT: a planar subgraph $G' = (V, E')$ of G
set $EE = \emptyset$
(EE is a set of "extra" edges)
for each $u \in V$, $v \in V$ with $u \neq v$ and $uv \notin E$
    set $E = E \cup \{uv\}$ with $w(uv) = 0$
    set $EE = EE \cup \{uv\}$
(G is now a complete graph.)
assign a vertex weight $W(v)$ to each $v \in V$
sort the vertices by vertex weight in nonincreasing order and
let L be the list of sorted vertices
name the first four vertices of L a, b, c, and d
set $E' = \{ab, ac, ad, bc, bd, cd\}$
(G' is now the tetrahedron on a, b, c, d.)
set $T = \{abc, acd, abd, bcd\}$
(T is the set of faces of G'. By construction, each face of G' is a triangle.)
remove a, b, c, and d from L
while L is nonempty
    let v be the first vertex in L
    choose a face $xyz \in T$ such that
    $w(xv) + w(yv) + w(zv)$ is as large as possible
    set $E' = E' \cup \{xv, yv, zv\}$
    set $T = T \setminus \{xyz\}$
    set $T = T \cup \{xyv, yzv, zxv\}$
    take v out of L
end of while
(G' is now a triangulated graph with n vertices and $3n - 6$ edges.)
if E' contains "extra" edges from EE, eliminate them from E'

Figure 3.3: Algorithm DELTAHEDRON WEIGHTED PLANAR SUBGRAPH

[FGG85] assigns the vertex weights as the sum of the weights of incident edges: $W(v) = W_{sum}(v) = \sum_{u \in V} w(uv)$. [DFF85] suggests to use the maximum of the vertex weights instead of the sum: $W(v) = W_{max}(v) = \max_{u \in V}\{w(uv)\}$, and also provides a worst case analysis for the performance of GREEDY WEIGHTED PLANAR SUBGRAPH (GREEDY for short) and the two versions of DELTAHEDRON WEIGHTED PLANAR SUBGRAPH (DELTAHEDRON for short). To measure the performance of any algorithm A that finds a planar subgraph $G' = (V, E')$ for an input graph

$G = (V, E)$, let P be an instance of the Weighted Maximum Planar Subgraph Problem with a graph $G = (V, E)$ and positive edge weights $w(e)$ for $e \in E$. Let $E^* \subseteq E$ be an optimal edge set, i.e. $G^* = (V, E^*)$ is planar, and $w(E^*) = \sum_{e \in E^*} w(e)$ is as large as possible. Define the worst case ratio ρ_A to be

$$\rho_A = \inf_P \frac{w(E')}{w(E^*)}$$

Clearly $\rho_A \leq 1$ for any algorithm A. The closer ρ_A is to 1, the better A performs (in the worst case). [DFF85] shows that DELTAHEDRON with vertex weights W_{sum} can be arbitrarily bad in the general case but has a performance guarantee if the edge weights are restricted to 0 and 1. DELTAHEDRON with vertex weights W_{max} and GREEDY both have performance guarantees in the general case. Figure 3.4 lists the results presented in [DFF85]. They show that GREEDY is the best algorithm as far as worst case analysis is concerned. Furthermore, computational results [FGG85] on graphs with 10, 20, 30, and 40 vertices suggest that GREEDY does perform well in general. But it also requires a lot of processing resources. [FGG85] suggests improving the result of DELTAHEDRON by edge replacement or vertex relocation operations in a postprocessing phase. Furthermore [FGG85] discusses a Wheel Expansion approach first suggested in [EFG82]. The computational results do not show a good performance of this heuristic, however. They suggest that the improved version of DELTAHEDRON with vertex weights W_{sum} and GREEDY give the best results. (There are no computational results for DELTAHEDRON with vertex weights W_{max}.) Finally, Figure 3.4 also lists the time complexities of the above algorithms as given in [FGG85].

Besides the worst case analysis mentioned above, [DFF85] also analyses a simplification of DELTAHEDRON on random instances of the Weighted Planar Subgraph Problem, and shows it to be asymptotically arbitrarily good under certain assumptions on the distributions of the edge weights.

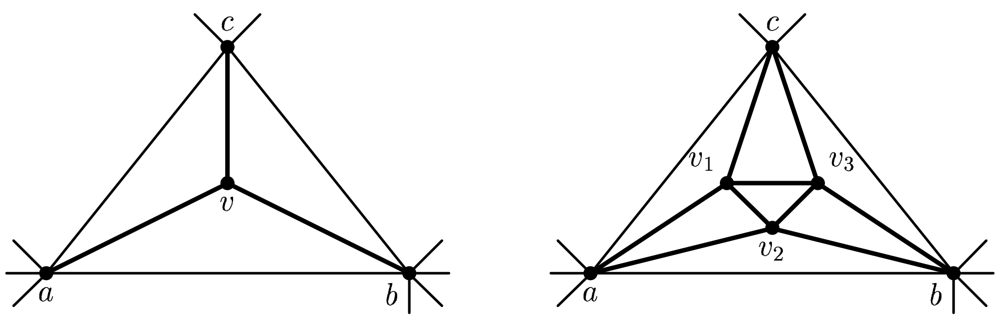
Leung [Leu92] generalizes DELTAHEDRON WEIGHTED PLANAR SUBGRAPH. Starting with a tetrahedron (K_4), a planar subgraph is built such that in each step, the current planar subgraph has only triangular faces. In each step, a single vertex and 3 incident edges (as in DELTAHEDRON) or a set of 3 vertices and 9 incident edges are placed in one of the faces of the current planar subgraph as illustrated in Figure 3.5. Unlike in DELTAHEDRON, the vertices to be inserted are not chosen in any predetermined ordering, but in each step the vertex or the set of 3 vertices, and the face into which to insert them, is determined so that the gain in edge weights per inserted vertex in this step is best possible. The worst case time complexity of this improved approach is $O(n^4 \log n)$, but computational results suggest that the results are better than the ones achieved by the DELTAHEDRON approach discussed in [FR78, FGG85].

A completely different approach is taken by Jünger and Mutzel, who use a heuristic based on linear programming within a branch and cut framework [Jün95, p. 34 f.], [JM93a, JM93b, Mut94]. Computational results on graphs with up to 200 vertices yield that the results obtained are superior to those obtained by other algorithms. But they also show that the running time needed is usually significantly longer than the running time of other algorithms.

For the unweighted case (i.e. all edge weights are 1) there are still other approaches.

A	ρ_A	time complexity for an input graph with n vertices
GREEDY	$\frac{1}{3}$	$O(n^3)$
DELTAHEDRON with vertex weights W_{sum}	0	
DELTAHEDRON with vertex weights W_{sum} when edge weights are restricted to 0 and 1	$\frac{1}{6} \leq \rho_A \leq \frac{2}{9}$	$O(n^2)$
DELTAHEDRON with vertex weights W_{max}	$\frac{1}{6}$	
postprocessing phase for improving results of DELTAHEDRON	—	Computational results suggest an average time complexity of $O(n^3)$.

Figure 3.4: The results of [DFF85] show the worst case performance of three algorithms for finding a planar subgraph with a large sum of edge weights. The time complexity of the algorithms is given in [FGG85].



a) *b)*

Figure 3.5: A step in DELTAHEDRON WEIGHTED PLANAR SUBGRAPH [FR78, FGG85, DFF85] (operation a), see also Figure 3.3), or in its generalization [Leu92] (operation a) or b)). In a), vertex v and 3 incident edges are inserted into face abc. In b), vertices v_1, v_2, v_3 and 9 incident edges are inserted into face abc.

A first heuristic by T. Chiba, Nishioka, and Shirakawa [CNS79] was based on the planarity testing algorithm [HT74].

Cimikowski [Cim92] suggests an approach based on spanning trees. Suppose a graph G with n vertices and m edges is 2-connected and has two edge disjoint spanning trees whose union is planar. Then this union forms a planar subgraph and has $2n-2$ edges. If the graph does not have two such spanning trees, some heuristic edge manipulations are performed, so that the output is still a spanning planar subgraph, but without a guaranteed number of edges. If two spanning trees exist, they can be found in $O(m^2)$ [RT85].

Cimikowski also presents results comparing different heuristics [Cim95].

Takefuji and Lee [TL89, TLC91] and Goldschmidt and Takvorian [GT94] each propose a two-phase heuristic for finding a planar subgraph with as many edges as possible. In the first phase, a linear ordering of the vertices is determined. The vertices are placed on a line according to that ordering, and in the second phase, edges are placed above or below the line. The resulting planar subgraph is thus embedded in a book with 2 pages. The techniques used for each phase are very different in [TL89] and [GT94]. [TL89] places the vertices in a random order in the first phase and uses a neural network technique for the second phase.

[GT94] argues that it is useful to attempt to order the vertices of the input graph $G=(V,E)$ according to a Hamiltonian cycle. Given an ordering of the vertices on a line, three edge sets $A \subseteq E$, $B \subseteq E$, and $C \subseteq E$ must be determined so that $|A|+|B|$ is as large as possible, and so that no two edges of A (B) intersect if all edges of A (B) are placed above (below) the line of vertices. The edges in C are not part of the planar subgraph. If we imagine the vertices of G to lie on the real line, then each edge $e \in E$ can be regarded as an interval defined by its two end vertices. Let $H=(E,F)$ be a graph such that each edge of G is a vertex of H. Let e_1, e_2 be two edges of G and thus two vertices of H, and let i_1 and i_2 be the intervals corresponding to the edges e_1 and e_2 in G. e_1 and e_2 are connected by an edge in H if and only if the intervals i_1 and i_2 intersect but none is contained in the other. Thus H is an overlap graph (also called circle graph). Finding the sets A, B and C as described above is now equivalent to finding a maximum bipartite subgraph of the overlap graph H. Finding a maximum bipartite subgraph of an overlap graph is NP-complete [SL89].

[GT94] now uses the following greedy algorithm to construct a maximal bipartite subgraph of an overlap graph: Find a maximum independent vertex set in H (the vertices of this set are then the edges in A), delete it from H, and find a maximum independent set in the remaining graph (the vertices of this set are then the edges in B). Since the maximum independent set of an overlap graph can be found in polynomial time [Gav73] , this algorithm runs in polynomial time also. [GT94] shows that the number of vertices in the maximal bipartite subgraph found is at least $\frac{3}{4}$ the number of vertices of a maximum bipartite subgraph.

Computational results for the two phase heuristic of Goldschmidt and Takvorian yield that its results are better than those of [TL89]. They were carried out on 18 graphs with up to 100 vertices and on 3 larger graphs with up to 1000 vertices.

The approach of [GT94] is further refined by Resende and Ribeiro [RR95]. They apply a greedy randomized adaptive search procedure (GRASP), a metaheuristic for combinatorial optimization [FR95], to the problem of planarizing a graph through edge deletion. Extensive experimental results indicate that the GRASP compares favorably with the results of [GT94], and that the results are on most instances at least as good as those obtained by [JM93a, JM93b, Mut94]. But note that [RR95] focusses on finding a good solution to a particular given problem, regardless of the time it takes, whereas [JM93a, JM93b, Mut94] impose a time limit and return the best solution found until then.

Chapter 4

Vertex Splitting and Splitting Number

The vertex splitting operation on a graph is the reversal of identifying two vertices:

Definition 4.1 (vertex splitting) *If $G' = (V', E')$ is a graph with two vertices $v_1 \in V'$ and $v_2 \in V'$, and if $G = (V, E)$ is the graph obtained from G' with*

$$\begin{aligned} V &= (V' \setminus \{v_1, v_2\}) \cup \{v\} \quad \textit{and} \\ E &= (E' \setminus \{uv_i \mid u \in V' \textit{ and } i \in \{1,2\} \textit{ and } uv_i \in E'\}) \\ &\quad \cup \{uv \mid u \in V \setminus \{v\} \textit{ and } (uv_1 \in E' \textit{ or } uv_2 \in E')\} \end{aligned}$$

then we say that G' was obtained from G by splitting *the vertex v.*

If a graph G' has been obtained from a graph G by a (possibly empty) sequence of vertex splitting operations, we call G' a *splitting* of G. Note that even if there is a vertex $x \in V'$ such that $xv_1 \in E'$ and $xv_2 \in E'$, no multiple edges are formed in G by the vertex identification operation. Likewise, no loops are formed in G, even if $v_1 v_2 \in E'$.

The vertex identification of given vertices v_1 and v_2 in a given graph G' yields a unique graph G. But the opposite is not true: Given a graph G and one of its vertices, v, there are many ways to split this vertex. Given the graph $G = K_3$, for example, and one of its vertices, v, there are six ways to perform a vertex splitting at v such that the resulting graphs are pairwise non-isomorphic (see Figure 4.1).

One might want to define a vertex splitting in a more general way as the reversal of identifying k vertices of a graph at once, where $k \geq 2$. So a splitting of a vertex v would result in vertices $v_1 \dots v_k$ so that the adjacencies of $v_1 \dots v_k$ cover the adjacencies of v in the original graph. But since splitting a vertex k ways can always be regarded as $(k-1)$ successive vertex splitting operations where each vertex splitting is only a 2-way-splitting, we restrict our definition of vertex splitting to splitting a vertex v into exactly two vertices v_1 and v_2.

The vertex splitting operation has been used in several different ways: Section 4.1 first describes the work of Nora Hartsfield, Brad Jackson and Gerhard Ringel in the 1980s about splitting vertices of complete graphs K_n and of complete bipartite graphs

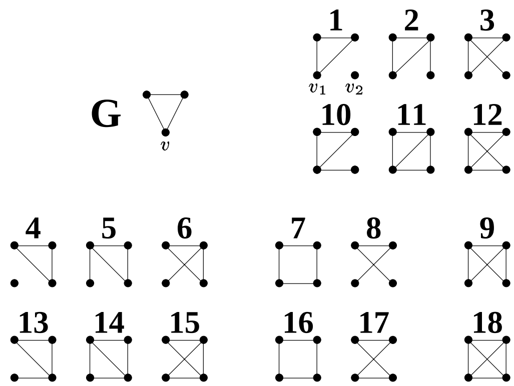


Figure 4.1: Eighteen possible ways to split $G = K_3$ at v. Many of the resulting graphs are isomorphic. Essentially, there are only six ways to split v in G: The graphs numbered 1 and 4 are isomorphic. The graphs numbered 2, 3, 5, 6, 10, and 13 are isomorphic. The graphs numbered 7 and 8 are isomorphic. The graphs 9, 11, 12, 14, and 15 are isomorphic. The graphs 16 and 17 are isomorphic.

K_{n_1,n_2} so that the resulting graph is embeddable in a given surface [JR85, JR84a, JR84b, HJR85, Har86, Har87]. We will be mostly concerned with the case that this surface is the plane. Section 4.1 then describes the work of Eades and Mendonça on determining the complexity status of finding, for a given graph G, the minimum number of vertex splitting operations necessary to yield a planar graph [EM93, Men94]. This remains an open problem.

Section 4.2 gives a brief summery of other work involving vertex splitting operations.

4.1 Planarizing Graphs Through Vertex Splitting

Given a graph G, we want to know the smallest number k, so that G can be obtained from a planar graph G' by k vertex identifications. We call this number the *splitting number* of G, denoted $\sigma(G)$. Clearly $\sigma(G) = 0$ if and only if G is planar. If a planar graph G' was obtained from a graph G by vertex splitting operations, we call G' a *planar* splitting of G. If additionally, G' was obtained by $\sigma(G)$ vertex splitting operations, we call G' an *optimal* planar splitting of G.

For a general surface S, $\sigma(G, S)$ denotes the smallest number k, so that G can be obtained from a graph G' by k vertex identifications, where G' is embeddable on S.

Now consider the different ways of vertex splittings as illustrated in Figure 4.1. The graphs numbered 1 and 7 (and all graphs isomorphic to them) have the same number of edges as the original graph K_3. The other graphs have more edges than K_3: In the graphs numbered 10 through 18, v_1v_2 is an additional edge, and in graphs such as the ones numbered 2 or 18, some vertex u that was adjacent to v in K_3 is now adjacent to both v_1 and v_2. For each u that was adjacent to v and is now adjacent to both v_1 and v_2, we call one of the edges uv_1 and uv_2 *superfluous.* Likewise, we call an edge v_1v_2 superfluous. We say a vertex splitting is *proper* if it does not create superfluous edges, and if none of the resulting vertices v_1 and v_2 is isolated. Otherwise we call it *improper*.

Now observe that when splitting vertices of a graph G with the goal of planarizing it, we can restrict our attention to proper vertex splittings. For assume we obtain a planar graph G' from G by using improper vertex splittings. Now perform the same sequence of vertex splittings on G again, but in each vertex splitting, leave out all superfluous edges. Also, skip all the vertex splittings that create an isolated vertex. This yields a graph G''. Since G'' is a subgraph of G' and since G' is planar, G'' is also planar.

The splitting number is known for complete bipartite graphs (Section 4.1.2) and for complete graphs (Section 4.1.3) as well as for the cartesian product of an m-cycle C_m and an n-cycle C_n. The latter result allows the construction of a graph with genus g and splitting number σ, for any integers $\sigma \geq g \geq 1$ [Sch86].

4.1.1 Lower Bounds for the Splitting Number

The upper bound for the number of edges for planar graphs from Equation 1.2 immediately yields a lower bound for the splitting number:

Let G be a graph with n vertices and m edges, and let $\sigma(G)$ be the splitting number of G. Let G' be a graph obtained from G by $\sigma(G)$ vertex splitting operations so that G' is planar. Then G' has $n' = n + \sigma(G)$ vertices, and by the above argument about superfluous edges, we can construct G' in such a way that $m' = m$. Since G' is planar, Equation 1.2 says that it has at most $m' \leq 3n' - 6$ edges (for $n' \geq 3$). Since $m = m'$, this implies

$$n' \geq \frac{1}{3} \cdot m + 2 \quad \text{for } n' \geq 3$$

Every graph on $n \leq 4$ vertices is planar, so for $n \leq 4$ we have $n' = n$. For $n \geq 5$, we have $n' \geq n$. Therefore, the condition $n' \geq 3$ is equivalent to the condition $n \geq 3$, and we obtain the lower bound

$$\sigma(G) = n' - n \geq \left\lceil \frac{1}{3} \cdot m - n + 2 \right\rceil \quad \text{for } n \geq 3 \tag{4.2}$$

If we know the girth g of a graph G with n vertices and m edges, a better bound for $\sigma(G)$ can be derived [JR85]:

Let again G' be a graph obtained from G by $\sigma(G)$ vertex splitting operations so that G' is planar. Let n' and m' be the number of vertices and edges of G', respectively. Let f' be the number of faces of G' in a given planar embedding. Euler's formula for planar graphs (Equation 1.1) says

$$n' - m' + f' = 2$$

If g' is the girth of G', then every face of G' is incident to at least g' edges. Furthermore, each edge is incident to exactly two faces. Therefore,

$$f' \cdot g' \leq 2 \cdot m'$$

Combining this inequality with Euler's formula, we have

$$2 + m' - n' = f' \leq \frac{2m'}{g'}$$

Since $n' = n + \sigma(G)$, $m' = m$, and $g' \geq g$, we have

$$2 + m - n - \sigma(G) \leq \frac{2m}{g'} \leq \frac{2m}{g}$$

Since $\sigma(G)$ is an integer, we can conclude

$$\sigma(G) \geq \left\lceil m - \frac{2m}{g} - n + 2 \right\rceil \tag{4.3}$$

Note that if a graph G has cycles, but its girth is not known, combining $g \geq 3$ with Equation 4.3 yields Equation 4.2. This is not surprising, since the formula $m' \leq 3n' - 6$ follows from the formula $2 + m' - n' = f'$ with the observation that each of the f' faces is incident to at least 3 edges.

Here we have only considered the number of vertex splitting operations needed to transform a graph G into a graph G' embeddable in the plane. Jackson and Ringel [JR85] consider the general case of, given any surface S, transforming a graph G into a graph G' embeddable in S. For a given graph G and a given surface S, they derive a lower bound for $\sigma(G, S)$ involving the Euler characteristic $E(S)$ of the surface S. For background material on surfaces and their Euler characteristic see Arthur White's "Introduction to Surface Topology" [Whi84, Chapter 5].

4.1.2 The Splitting Number of Complete Bipartite Graphs

First note that the complete bipartite graph K_{n_1,n_2} is planar if and only if $n_1 \in \{1, 2\}$ or $n_2 \in \{1, 2\}$. The girth of K_{n_1,n_2} is 4 (for $n_1, n_2 \geq 2$), so the lower bound 4.3 yields

$$\begin{aligned}\sigma(K_{n_1,n_2}) &\geq \left\lceil n_1 \cdot n_2 - \frac{2n_1 n_2}{4} - n_1 - n_2 + 2 \right\rceil \\ &= \left\lceil \frac{(n_1 - 2)(n_2 - 2)}{2} \right\rceil \end{aligned} \tag{4.4}$$

In [JR85] and[JR84b], Jackson and Ringel show that this lower bound is also an upper bound. Again, they consider the general case of transforming $G = K_{n_1,n_2}$ into a graph G' that is embeddable in a surface S with Euler characteristic $E(S)$. They show that if S is a closed orientable or nonorientable surface, then $\sigma(K_{n_1,n_2}) = \max\left(\left\lceil \frac{(n_1-2)(n_2-2)}{2} \right\rceil - 2 + E(S), 0\right)$. We restrict ourselves to the case where S is the plane. Note that the plane is commonly referred to as S_0, and that $E(S_0) = 2$ [Whi84, Chapter 5].

Theorem 4.5 (Jackson, Ringel [JR85, JR84b]) *The splitting number of the complete bipartite graph K_{n_1,n_2} is*

$$\sigma(K_{n_1,n_2}) = \left\lceil \frac{(n_1 - 2)(n_2 - 2)}{2} \right\rceil \qquad \text{for } n_1, n_2 \geq 2$$

Proof A constructive proof is given in [JR84b]: If $n_1 = 2$ or $n_2 = 2$, then K_{n_1,n_2} is planar, and $\sigma(K_{n_1,n_2})$ as given in Theorem 4.5 is indeed 0. So we can assume $n_2 > 2$. Given $G = K_{n_1,n_2}$, construct a planar graph G' as follows:

Case 1: n_1 or n_2 is an even number. Without loss of generality assume that $n_1 = 2h$ is even. Construct G' with $2h$ white vertices and $n_2 + (h-1)(n_2-2)$ black vertices. Label the white vertices 1^*, -1^*, 2^*, -2^*, ..., h^*, $-h^*$ and place them on a vertical line at the points $(0, \pm 1)$, $(0, \pm 2)$, ..., $(0, \pm h)$. Place the black vertices on a horizontal line at the points $(-1, 0)$, $(0, 0)$, $(1, 0)$, ..., $(n_2 + (h-1)(n_2-2) - 2, 0)$.

In order to label the black vertices we use the following abbreviations for sequences of consecutive labels:

$$\begin{aligned} P_i &= 3, 4, 5, \ldots, n_2 - 2, n_2 - 1 \quad \text{for odd } i \\ P_i &= n_2 - 1, n_2 - 2, \ldots, 5, 4, 3 \quad \text{for even } i \end{aligned}$$

Note that if $n_2 = 3$, the P_i are empty. Label the black vertices in the following order from left to right:

$$\begin{array}{ll} 1, 2, P_1, n_2, P_2, 2, P_3, n_2, P_4, 2, \ldots, P_h, n_2 & \text{if } h \text{ is odd} \\ 1, 2, P_1, n_2, P_2, 2, P_3, n_2, P_4, 2, \ldots, P_h, 2 & \text{if } h \text{ is even} \end{array}$$

Now construct the edges of G'. G' will be a bipartite graph with edges connecting white vertices with black vertices. First join the vertices labeled 1^* and -1^* to the leftmost n_2 black vertices. This creates $2 \cdot n_2$ edges. Then join the white vertices 2^* and -2^* to the black vertices labeled by P_2, to the vertex left of P_2, to the vertex right of P_2, and to the vertex 1. This creates another $2 \cdot n_2$ edges. Similarly, for $3 \le i \le h$, join the vertices i^* and $-i^*$ to the vertices labeled by P_i, to the vertex left and to the vertex right of P_i, and to the vertex 1. The result is a planar, bipartite graph G', in which every face is incident to exactly 4 edges. Figure 4.2 shows the graph G' constructed for $G = K_{6,5}$.

Observe that every vertex i^* and every vertex $-i^*$, for $1 \le i \le h$, is adjacent to n_2 vertices labeled $1, 2, \ldots, n_2$. So if in G' we identify, for each j, $2 \le j \le n_2$, all vertices labeled j, we obtain the original graph $G = K_{n_1,n_2}$. The number of vertices labeled j is

$$k_j = \begin{cases} 1 + \frac{h-1}{2} & \text{for } j = 2 \text{ or } j = n_2 \text{ and if } h \text{ is odd} \\ 1 + \frac{h}{2} & \text{for } j = 2 \text{ and if } h \text{ is even} \\ \frac{h}{2} & \text{for } j = n_2 \text{ and if } h \text{ is even} \\ h & \text{for } 3 \le j \le n_2 - 1 \end{cases} \tag{4.6}$$

The number of vertex identifications involving vertices labeled j is in each case $k_j - 1$. The total number of vertex identifications is $\frac{h-1}{2} + \frac{h-1}{2} + (h-1)(n_2-3) = (h-1)(n_2-2)$ if h is odd, and $\frac{h}{2} + \frac{h}{2} - 1 + (h-1)(n_2-3) = (h-1)(n_2-2)$ if h is even. So in both cases (h odd and h even) $G = K_{n_1,n_2}$ can be constructed from G' by

$$\sigma = (h-1)(n_2-2) = \left(\frac{n_1}{2} - 1\right)(n_2 - 2) = \frac{(n_1-2)(n_2-2)}{2} = \left\lceil \frac{(n_1-2)(n_2-2)}{2} \right\rceil$$

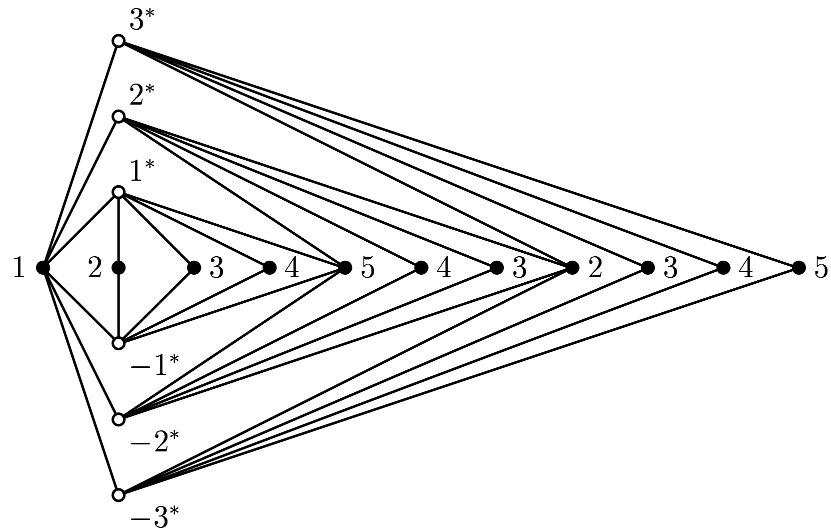


Figure 4.2: An optimal planar splitting of $K_{6,5}$, as suggested in the proof of Theorem 4.5. The sequences P_i are $P_1 = P_3 = 3, 4$ and $P_2 = 4, 3$.

vertex identifications (note that n_1 is even). Therefore, $\sigma(K_{n_1,n_2}) \leq \sigma$. Since σ was also shown to be a lower bound for $\sigma(K_{n_1,n_2})$, we can conclude that $\sigma(K_{n_1,n_2}) = \left\lceil \frac{(n_1-2)(n_2-2)}{2} \right\rceil$ for the case that n_1 or n_2 is an even number.

Case 2: n_1 and n_2 are both odd. Let $n_1 = 2h+1$. Construct the graph G' with $2h$ white vertices and $n_2 + (h-1)(n_2-2)$ black vertices as described in Case 1. G' is now an optimal planar splitting of K_{n_1-1,n_2}. Consider the (n_2-1) quadrilateral faces that contain both the white vertex 1^* and the white vertex -1^*. For $j = 1, 3, 5, \ldots, n_2-2$, place an additional white vertex labeled 0^* inside each of the quadrilaterals $(j, 1^*, j+1, -1^*)$, and join this vertex to the vertices labeled j and $j+1$ by an edge. Place another white vertex labeled 0^* inside the quadrilateral $(n_2-1, 1^*, n_2, -1^*)$ and join it to the vertex labeled n_2. For each j, $1 \leq j \leq n_2$, there is now a vertex labeled 0^* adjacent to a vertex labeled j. We have again constructed a planar graph G' that can be transformed into the original graph $G = K_{n_1,n_2}$ by identifying all vertices with the same label. The number of vertices labeled j is k_j as in Equation 4.6. The number of vertices labeled 0^* is $\frac{n_2-1}{2}+1$. Therefore, the total number of vertex identifications is $\sigma = (h-1)(n_2-2)+\frac{n_2-1}{2} = \left(\frac{n_1-1}{2}-1\right)(n_2-2)+\frac{n_2-1}{2} = \frac{(n_1-2)(n_2-2)}{2}+\frac{1}{2} = \left\lceil\frac{(n_1-2)(n_2-2)}{2}\right\rceil$ (note that n_1 and n_2 are both odd). As in Case 1, we can conclude that $\sigma(K_{n_1,n_2}) = \left\lceil\frac{(n_1-2)(n_2-2)}{2}\right\rceil$. This completes the proof of Theorem 4.5.

4.1.3 The Splitting Number of Complete Graphs

First recall that for $n \geq 5$, the complete graph K_n is nonplanar (this follows from Theorem 1.4). If less than $(n-4)$ vertex splitting operations are performed on a graph K_n with $n > 5$, the resulting graph G' contains (at least) 5 vertices that were not involved in splitting operations. These 5 vertices induce the nonplanar graph K_5, so

G' cannot be planar. This yields the trivial lower bound

$$\sigma(K_n) \geq n - 4 \tag{4.7}$$

The lower bounds 4.2 and 4.3 (with $g = 3$) both yield

$$\sigma(K_n) \geq \left\lceil \frac{1}{3}\binom{n}{2} - n + 2 \right\rceil = \left\lceil \frac{(n-3)(n-4)}{6} \right\rceil \quad \text{for } n \geq 3 \tag{4.8}$$

The lower bound 4.7 is only interesting for $n = 6$ and $n = 7$. For $n \geq 8$, the bound 4.8 is greater than or equal to the bound 4.7.

Hartsfield, Jackson and Ringel show that except for $n = 6, 7$ or 9 the lower bound 4.8 is also an upper bound. Unlike the result in Section 4.1.2 for K_{n_1,n_2}, this result does not extend to general surfaces. In the conference presentations [JR85] and [JR84a], partial results towards finding the splitting number of K_n are announced. [HJR85] then presents a proof for the following theorem:

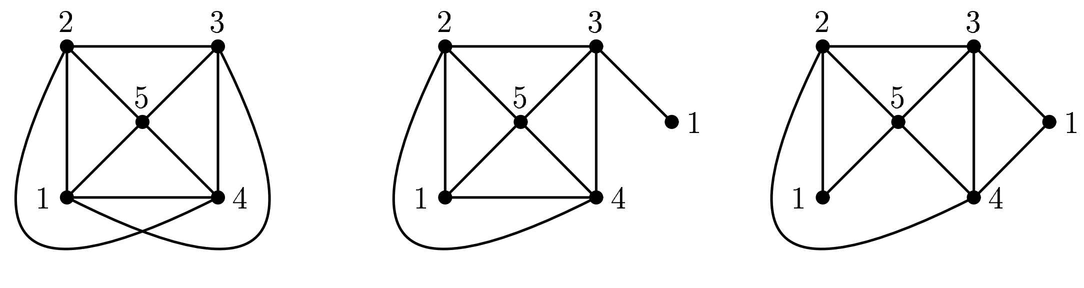
Theorem 4.9 (Hartsfield, Jackson, Ringel [HJR85]) *The splitting number of the complete graph K_n is*

$$\sigma(K_n) = \begin{cases} \left\lceil \frac{(n-3)(n-4)}{6} \right\rceil & \text{for } n \geq 3 \quad \text{and } n \notin \{6,7,9\} \\ \left\lceil \frac{(n-3)(n-4)}{6} \right\rceil + 1 & \text{for } n \in \{6,7,9\} \end{cases}$$

The proof of Theorem 4.9 will not be given here. We will only give optimal planar splittings for small numbers of n and introduce two ideas used in the proof for large n. Note that (4.9) is obviously true for $n = 3$ and $n = 4$. (4.9) yields $\sigma(K_5) = 1$ which is also true as shown in Figure 4.3. Furthermore, (4.9) yields $\sigma(K_6) = 2$ and $\sigma(K_7) = 3$. Figures 4.4 and 4.5 show planar splittings for K_6 and K_7 with 2 and 3 splitting operations, respectively. Since (4.7) yields 2 and 3 as lower bounds for the splitting number of K_6 and K_7, (4.9) is true for $n = 6, 7$. For $n = 8$, (4.9) yields $\sigma(K_8) = 4$, and Figure 4.6 shows an optimal planar splitting of K_8 with 4 splitting operations.

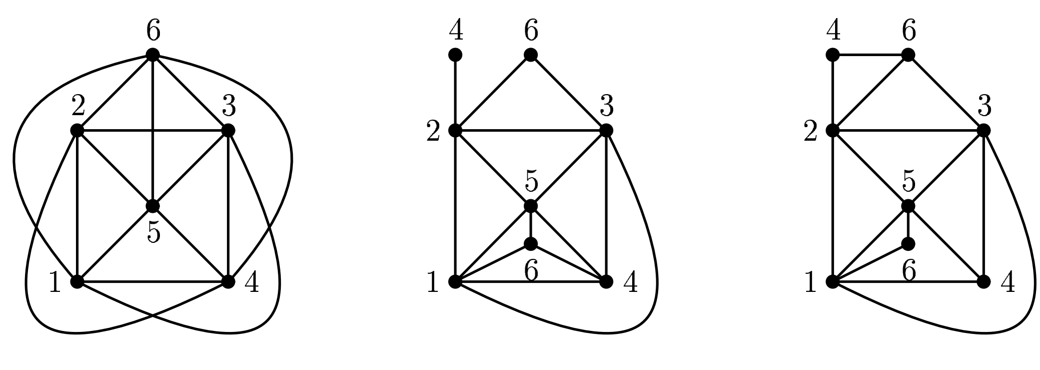
For $n = 9$, Figure 4.7 shows a planar splitting with 6 splitting operations. (4.7) and (4.8) both yield 5 splitting operations as a lower bound for K_9. [HJR85] explains that the proof for $\sigma(K_9) = 6$ involves checking many cases and that Mark Jungerman has verified the proof using a computer. So according to [HJR85], Theorem 4.9 also holds for $n = 9$.

The proof for large n is a meticulous case analysis for the congruence classes of n modulo 12. It makes frequent use of the following observation: Often, a planar splitting of K_n can be constructed from a planar splitting of K_{n-1}. Assume we have a planar splitting of K_{n-1} containing k faces, so that in the union of all vertices incident to these faces, each label i, $1 \leq i \leq (n-1)$, appears at least once. Then a planar splitting of K_n can be obtained from the planar splitting of K_{n-1} by placing a vertex labeled n in each of the k faces and connecting it to every vertex incident to that face. As an example, consider the optimal planar splitting of K_7 shown in Figure 4.5: It results from the optimal planar splitting c) of K_6 shown in Figure 4.4 by placing a vertex labeled 7 in



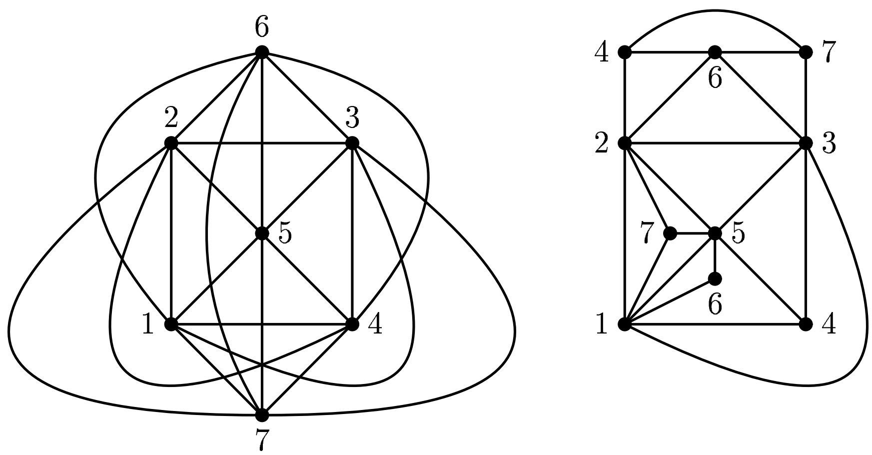
a) b) c)

Figure 4.3: a) K_5. b) and c) The two optimal planar splittings of K_5. All proper planar splittings of K_5 are either isomorphic to b) or to c).



a) b) c)

Figure 4.4: a) K_6. b) and c) Two optimal planar splittings of K_6. Note that $\sigma(K_6) = 2$.



a) b)

Figure 4.5: a) K_7. b) An optimal planar splitting of K_7. Note that $\sigma(K_7) = 3$.

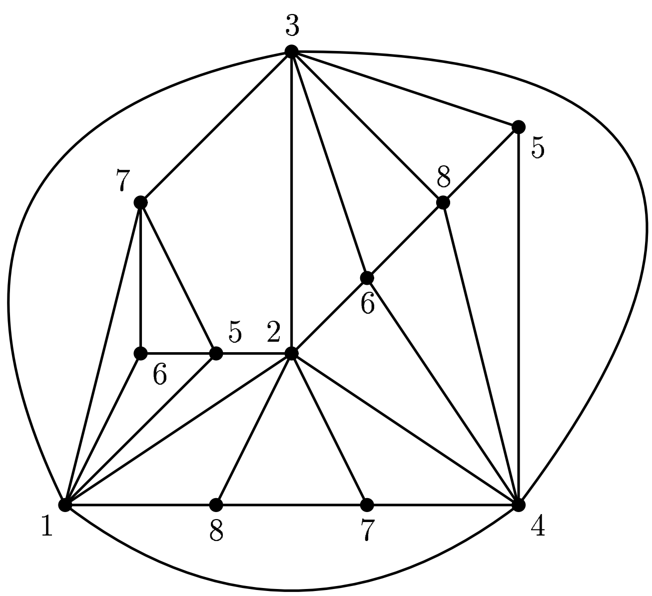


Figure 4.6: An optimal planar splitting of K_8 as given in [HJR85]. Note that $\sigma(K_8) = 4$, and that one of the edges from a vertex labeled 6 to a vertex labeled 1 is superfluous.

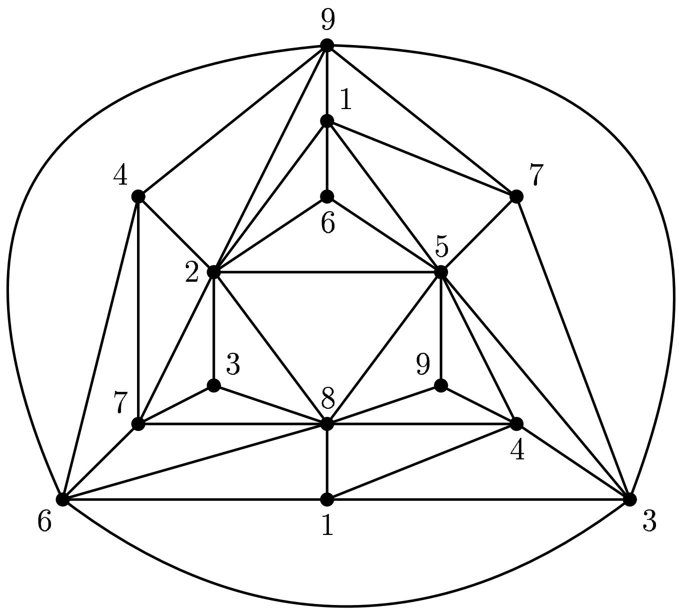


Figure 4.7: An optimal planar splitting of K_9 as exhibited in [HJR85]. $\sigma(K_9) = 6$. Note that there are superfluous edges connecting vertices labeled 1 and 6, 3 and 7, and 9 and 4, respectively.

each of the faces delimited by vertices labeled $1, 2, 5$ and $1, 2, 4, 6, 3$, connecting them to all vertices incident to these two faces, and by then deleting superfluous edges.

Furthermore, the proof of (4.9) is actually carried out in a dual formulation of the problem: A planar splitting of K_n is represented as a map where the countries represent the vertices. Countries that correspond to vertices with the same label belong to a common *empire*. Two empires e_i, e_j are *adjacent* if there exist countries c_i and c_j belonging to the empires e_i and e_j, respectively, that share a common border. Countries whose corresponding vertices are adjacent in the planar splitting share a common border in the map.

Finding an optimal planar splitting of K_n is then equivalent to finding a map with n mutually adjacent empires where the overall number of countries is minimum. Figure 4.8 shows an optimal planar splitting of K_{10}, and Figure 4.9 shows the corresponding map. This map was actually found by Mark Jungerman's program mentioned above [JR84a, HJR85]. Finding maps of mutually adjacent empires is an old problem: In 1890, Heawood [Hea90] found a map of 12 mutually adjacent empires of 2 countries each [JR84a]. Note that indeed $\sigma(K_{12}) = 12$.

Theorem 4.9 extends to two special surfaces: If S is the Klein bottle, or the projective plane, then either $\sigma(K_n, S) = 0$ or $\sigma(K_n, S) = \left\lceil \frac{1}{6}(n-3)(n-4) \right\rceil - 1 + E(S)$ or $\sigma(K_n, S) = \left\lceil \frac{1}{6}(n-3)(n-4) \right\rceil - 2 + E(S)$, where $E(S)$ is again the Euler characteristic of S [HJR85]. Nora Hartsfield determined the splitting number of two more surfaces: In [Har86] Hartsfield shows that the splitting number of the complete graph

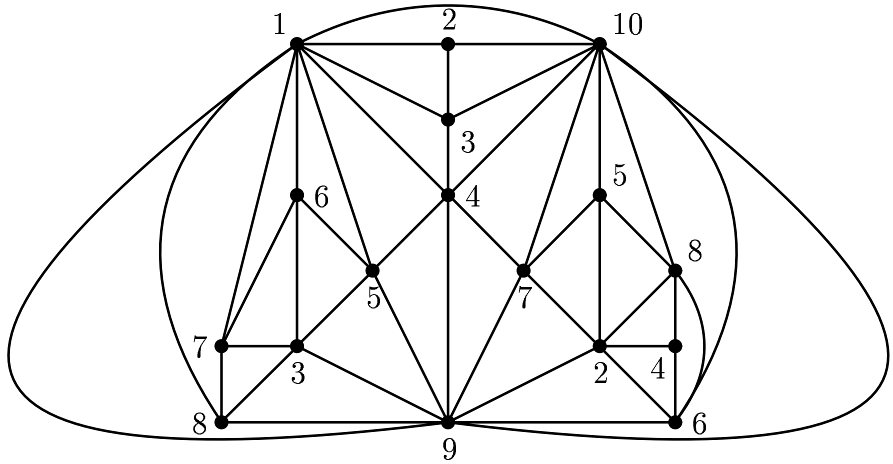


Figure 4.8: An optimal planar splitting of K_{10}. $\sigma(K_{10}) = 7$.

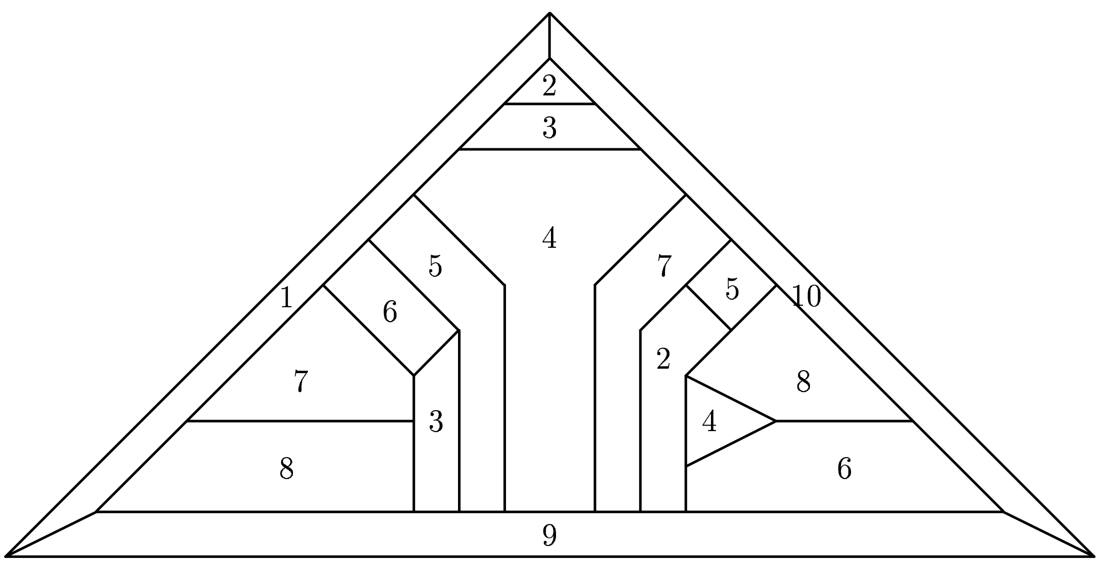


Figure 4.9: The optimal planar splitting of K_{10} of Figure 4.8, represented as a map.

on the torus is $\sigma(K_n, S_1) = 0$ for $n < 8$, and that $\sigma(K_n, S_1) = \left\lceil \frac{1}{6}(n-3)(n-4) \right\rceil - 2$ for $n \geq 8$, where S_1 denotes the torus. [Har87] shows that the splitting number of K_n in the projective plane is $\sigma(K_n, N_1) = 0$ for $n < 7$, $\sigma(K_n, N_1) = 2$ for $n = 7$, and that $\sigma(K_n, N_1) = \left\lceil \frac{1}{6}(n-3)(n-4) \right\rceil - 1$ for $n > 7$.

4.1.4 About the Complexity Status of Determining the Splitting Number

It is not known whether determining the splitting number of a given graph G is an NP-complete problem or not. Mendonça [Men94] defines the following two problems and shows that the first one is NP-complete:

Problem 4.10 (Eligible Set Split Planar Graph [Men94]) *Given a graph $G = (V, E)$, a subset of vertices $S \subseteq V$, and a positive integer $K \leq |S|$*[1]*, can G be transformed into a planar graph G' by K or less vertex splitting operations that involve only vertices in S? The vertices in S are called* eligible *vertices.*

Problem 4.11 (Split Planar Graph [Men94]) *Given a graph $G = (V, E)$ and a positive integer $K < |E|$, can G be transformed into a planar graph G' by K or less vertex splitting operations?*

A reduction from the Maximum Planar Subgraph Problem (see also Section 3.1 shows that Problem 4.10 is NP-complete.

Problem 4.12 (Maximum Planar Subgraph [GJ79, Problem GT27] and [LG79]) *Given a graph $G = (V, E)$ and a positive integer $K \leq |E|$, is there a subset $E' \subseteq E$ with $|E'| \geq K$ such that the graph $G' = (V, E')$ is planar?*

Theorem 4.13 [Men94] *Eligible Set Split Planar Graph (4.10) is NP-complete.*

Proof: Let the graph $G = (V, E)$ and the positive integer $K \leq |E|$ be an arbitrary instance of Maximum Planar Subgraph. Construct an instance of Eligible Set Split-Planar Graph as follows: Replace each edge $e = uv \in E$ with a path $ue'v_e e''v$, i.e. subdivide each edge e once. Call the resulting graph H. Let $K' = |E| - K$ (note that $K' \leq |E| = |S|$), and let $S = \{v_e \mid e \in E\}$ be the set of vertices created through the subdivisions. H, S, and K' define an instance of Eligible Set Split-Planar Graph. G has a planar subgraph with K or more edges if and only if H can be planarized by K' vertex splitting operations on S. For if G has a planar subgraph $G' = (V, E')$ with $|E'| \geq K$ edges, then for each edge $e \in E \setminus E'$, split the vertex v_e in H so that one of the copies of v_e, v_{e1}, is incident to e', and the other one, v_{e2}, is incident to e'', and v_{e1} and v_{e2} are not adjacent. The resulting graph H' is planar and the number of vertex splitting operations was $k' = |E| - |E'| = K' + K - |E'| \leq K'$. On the other hand, if there are $k' \leq K'$ vertex splitting operations on vertices in S that transform H into a planar graph H', then for each vertex $v_e \in S$ that was involved in a vertex splitting, delete the corresponding edge e from G. The resulting graph G' is planar since H' is planar, and the number of deleted edges is $l \leq k' \leq K'$, so G' has $|E'| = |E| - l \geq |E| - K' = K$ edges. Figure 4.10 shows the steps of this reduction for $G = K_{3,3}$.

[1] In [Men94], we actually have "$K < |S|$"

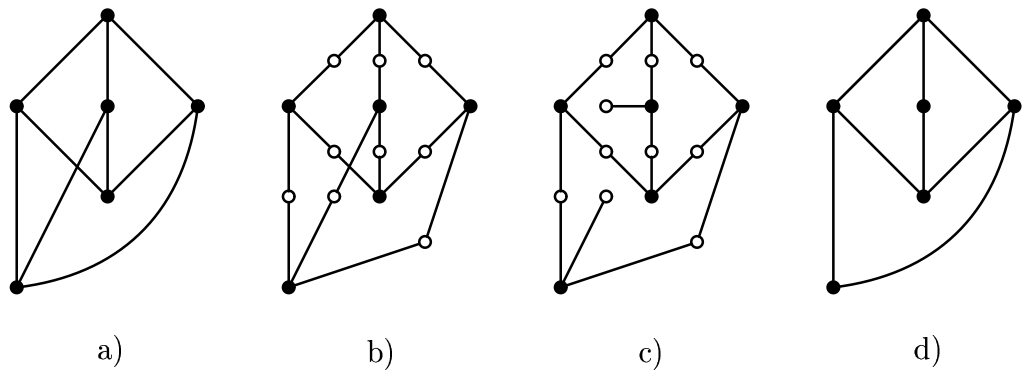


Figure 4.10: Illustration of the reduction for Eligible Set Split Planar Graph from Maximum Planar Subgraph with $K_{3,3}$. a) $K_{3,3}$. b) Every edge is subdivided by a white vertex. c) One of the white vertices needs to be split to planarize the graph in b). d) Alternatively, the deletion of the edge that was subdivided by the vertex split in c) yields a planar subgraph.

If we restrict K in the problem Split Planar Graph (4.11) to $K \leq |V|$ then Split Planar Graph becomes a special case of Eligible Set Split Planar Graph with $S = V$. Neither the restricted version nor the original version of Split Planar Graph has been shown to be NP-complete. A transformation similar to the one for Eligible Set Split Planar Graph does not seem to work for Split Planar Graph [Men94, p. 63].

[Men94] points out that for the class of graphs with vertex degree not greater than 3, Split Planar Graph and Maximum Planar Subgraph are equivalent. If Maximum Planar Subgraph restricted to graphs with vertex degree not greater than 3 were known to be NP-complete, then the following reduction would yield the NP-completeness of Split Planar Graph: A graph $G = (V, E)$ with vertex degrees not greater than 3 has a planar subgraph $G' = (V, E')$ with $|E'| \geq K$ edges if and only if G can be transformed into a planar graph G by less than or equal to $|E| - K$ vertex splitting operations. For assume $E' \subseteq E$ with $|E'| \geq K$ exists so that $G' = (V, E')$ is planar. Then for each edge $e = uv \in E \setminus E'$, perform a proper splitting operation on either u or v in G so that one of the resulting two vertices is only incident to e. The resulting graph is planar, and the number of vertex splitting operations was $|E| - |E'| \leq |E| - K$. On the other hand, assume that G can be planarized by K' vertex splitting operations. Then each (proper) splitting operation yields at least one vertex v with degree 1. Let E'' be the set of edges incident to those vertices. $|E''| \leq K'$. Then $G' = (V, E \setminus E'')$ is a planar graph with $|E| - |E''| \geq |E| - K'$ edges.

Eades and Mendonça have developed and implemented a heuristic for planarizing a graph through vertex splitting [EM93] and [Men94, Sections 4.3.2 and 4.3.3]. It is based on Lempel, Even and Cederbaum's planarity testing algorithm and its implementation using PQ-tree algorithms by Booth and Lueker ([Eve79, Section 8.4], [LEC67], and [BL76]).

4.2 Vertex Splitting in Contexts not Related to Planarity

The vertex splitting operation has appeared in very different contexts. Note, for example, that decomposing a graph into its 2-connected blocks means performing a proper vertex splitting at every cut vertex.

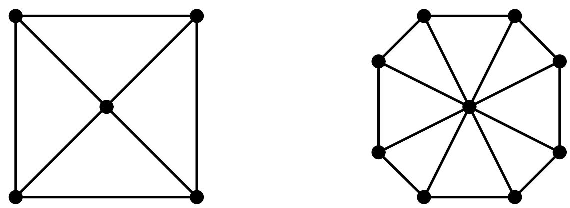
Oxley [Oxl86] studies in detail the class of graphic matroids. The main result is Whitney's 2-Isomorphism Theorem [Whi33a] which characterizes precisely when two graphs have isomorphic polygon matroids:

Theorem 4.14 (Whitney [Whi33a]) *Let G and H be two graphs without isolated vertices. Then their polygon matroids M_G and M_H are isomorphic if and only if H can be obtained from G by a sequence of a) identifying two vertices v_1 and v_2 of G that belong to distinct components of G, b) performing a proper vertex splitting operation at a cut vertex v of G in such a way that the resulting two vertices v_1 and v_2 belong to different components of the resulting graph, and c) twisting G about two of its vertices.*

Note that b) is the inverse operation of a). A graph G can be *twisted* about two of its vertices u and v if $\{u, v\}$ are a vertex cut of G and if u and v are not adjacent in G. Twisting G about u and v means to first separate G into two components G_1 and G_2 by performing a proper vertex splitting operation on u into u_1 and u_2 and on v into v_1 and v_2 so that u_1 and v_1 belong to G_1 and u_2 and v_2 belong to G_2, and by then identifying u_1 with v_2 and u_2 with v_1.

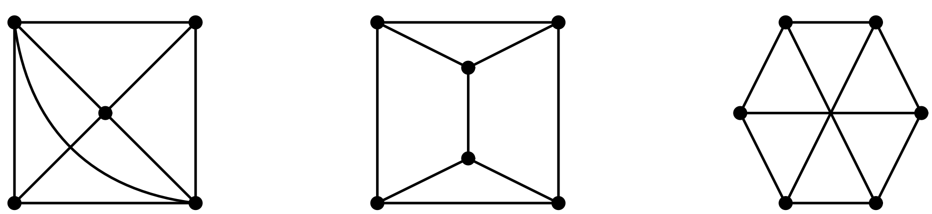
Tutte [Tut61, Tut66] characterizes the class of 3-connected graphs as those obtainable from wheels by edge addition and by the following version of vertex splitting: If a graph H was obtained from a graph G by splitting a vertex v of G with degree at least 4 in such a way that the resulting vertices v_1 and v_2 are adjacent, that H has exactly one more edge than G, and that v_1 and v_2 have both degree at least 3, then H is a *vertex* 3-*splitting* of G. Let W_k be the wheel on k spokes (Figure 4.11 shows W_4 and W_8). Tutte shows that if L_m is the class of 3-connected graphs with m edges then L_{m+1} can be constructed from L_m as follows: If m is odd and $m > 3$, then $W_{\frac{1}{2}(m+1)} \in L_{m+1}$. Furthermore, for every graph $G \in L_m$, every graph H that can be obtained from G by either adding an edge (where the endpoints of the new edge are not adjacent in G) or by 3-splitting a vertex of degree at least 4 in G becomes an element of L_{m+1}. Tutte lists all elements of L_m for $3 \leq m \leq 10$ explicitly. Note that W_4 is the only element of L_8. Figure 4.12 shows the three elements of L_9.

In order to characterize 4-connected graphs, Slater [Sla74] generalizes the notion of vertex 3-splitting: If a vertex v of degree at least $(2k-2)$ in a graph G is split into v_1 and v_2 such that the resulting graph H has exactly one edge more than G does, such that v_1 and v_2 are adjacent in H, and such that both v_1 and v_2 have at least degree k, then h is a vertex k-splitting of G. Slater also considers another version of vertex k-splitting where the original vertex v has degree at least $(2k - 3)$ and where one of the vertices adjacent to v in G is adjacent to both v_1 and v_2 in H (see Figure 4.13). Slater shows that every 4-connected graph can be obtained from the complete graph K_5 by finite sequences of adding edges, applying the two types of vertex 4-splittings described above, and applying two further graph operations [Sla74]. Chen and Kanevsky [CK93] give a different characterization of 4-connected graphs that also involves vertex 4-splitting.



W_4 W_8

Figure 4.11: W_4 and W_8, the wheels on 4 and on 8 spokes, respectively. Note that W_4 is the only element of L_8, the set of all 3-connected graphs with 8 edges [Tut61, Tut66].



$K_{3,3}$

Figure 4.12: L_9, the set of all 3-connected graphs with 9 edges [Tut61, Tut66]. Note that one of these graphs is $K_{3,3}$.

And vertex 3-splitting appears in Gubser's work about characterizing planar graphs that do not have W_6 as a minor [Gub93].

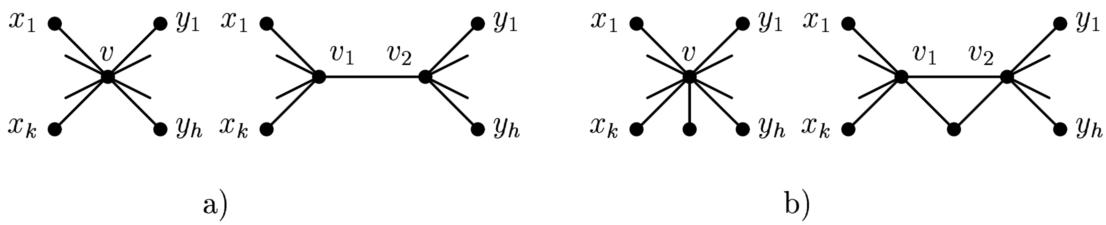


Figure 4.13: a) Vertex k-splitting as used by [Sla74] and [CK93]. We require $h \geq n-1$ and $k \geq n-1$. b) Another version of vertex k-splitting as used in [Sla74]. Here we require $h \geq n-2$ and $k \geq n-2$.

A theorem of Steinitz and Rademacher [SR34] can be restated as follows [Sch91]:

Theorem 4.15 *Every triangulation of the plane can be generated from a planar embedding of K_4 by vertex splitting operations.*

Note that a planar embedding of K_4 is a triangulation of the plane.

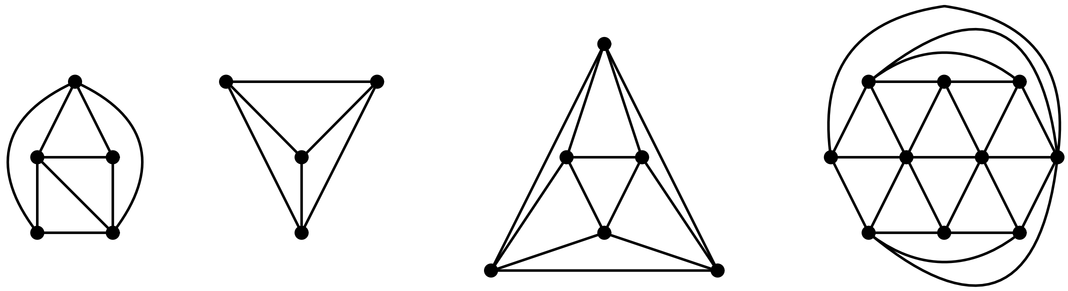
For some other surfaces, similar results have been found ([Bar82] and see the reference list of [Sch91]). Barnette and Edelson [BE88, BE89] showed that for every surface[2] S its triangulations can be generated from a finite set of triangulated graphs by vertex splitting operations. Other related results are [Bar87, Bar90, Bar91, Sch91].

A *locally cyclic* triangulation of a surface S is a triangulation of S such that every cycle of length 3 of the underlying graph constitutes a face of the triangulation. Alternatively, a triangulation is locally cyclic if the induced subgraph on the set of all neighbors of a vertex v is isomorphic to a cycle, for every vertex v of the underlying graph (see Figure 4.14). Malnič and Mohar showed that for each closed, orientable surface, all of its locally cyclic triangulations can be constructed from a finite set of locally cyclic triangulations by vertex splitting operations. For the plane, this finite set consists of K_4 and the octahedron (see Figure 4.14), and for the projective plane, this set was determined in [FMN94].

Given a graph $G = (V, E)$ and a function $f : V \to \mathbf{N}$, Nash-Williams [NW79, NW85a, NW85b, NW87] answers questions of the following type: Can G be transformed into a graph H by a sequence of proper vertex splitting operations such that H has a certain property and such that each $v \in V$ results in $f(v)$ vertices $v_1, v_2, \ldots, v_{f(v)}$ in H? The results are extended to describe situations where not only the number of vertices created from $v \in V$ can be prescribed by f but where also the desired degrees of the vertices $v_1, v_2, \ldots, v_{f(v)}$ in H can be specified. They continue work carried out in [Hil84, HR86].

Given an embedding of a graph G in a surface S, an assignment of k distinct colors $\{1, \ldots, k\}$ to the faces of the embedding such that two faces incident to a common

[2] More precisely: for every 2-manifold



a) b) c) d)

Figure 4.14: Four triangulations of the plane: a) is not locally cyclic, whereas b), c), and d) are all locally cyclic. Note that b) is K_4, also known as the tetrahedron, i.e. the skeleton graph of the regular 3-dimensional convex polytope with 4 faces, and that c) is the octahedron O_3, i.e. the skeleton graph of the regular 3-dimensional convex polytope with 8 faces.

edge of G are assigned different colors is a *k-coloring of the faces*. [Arc84] shows that if G is a graph with minimum vertex degree at least 3, then there exist a surface S, an embedding of G in S, and a 4-coloring of the faces of the embedding if and only if G can be transformed into a cubic graph that has a 3-coloring of the edges by a sequence of the following version of vertex splittings: If a vertex v is split into v_1 and v_2, then v_1 and v_2 are adjacent in the resulting graph, and the number of edges of the resulting graph increases exactly by one. This same version of vertex splitting is used in [Yap81, Yap83], which are about a class of chromatic index critical graphs. [Sel88] reduces the chromatic number of a given graph by proper vertex splitting operations in order to solve a concrete application problem.

Given two rooted trees T_1 and T_2 with n_1 and n_2 vertices, respectively, [Lu84] presents an $O(n_1(n_2)^2)$ algorithm that determines how many operations from a set of 4 allowed vertex splitting and vertex identification operations are necessary to convert one of the trees into the other. Given a graph $G = (V, E)$ and a partition $\{E_1, \ldots, E_k\}$ of E such that each E_i corresponds to a connected subgraph of G, [Nar90] gives an algorithm for finding a shortest sequence of vertex identification and vertex splitting operations that transform G into a graph H in which no cycle contains edges from more than one set E_i. [ME93a, ME93b] solve the following NP-hard problem with genetic algorithms: Given a directed, acyclic graph G and a positive number δ, determine a set X of vertices in G with minimum cardinality so that performing a proper vertex splitting operation on each vertex in X transforms G into a graph where the length of the longest directed path is at most δ.

In [Men94, EM95, EM96] Mendonça et al. address the problem of finding a planar embedding for a graph G with edge weights such that for each edge uv, the Euclidean distance between u and v in the layout is proportional to the weight of the edge uv. In general, finding such a layout for a given graph G with given weights is impossible, but by applying proper vertex splitting operations to G it can be transformed into a graph

H that admits a layout with the desired property. Determining the least number of vertex splitting operations required to achieve this is NP-complete [Men94]. Heuristics are given to solve the problem.

Finally note that [DP93] uses a very special version of vertex splitting, and that [Jac88] uses the term vertex splitting for a different graph operation that is called *lifting* in [Mad78].

Chapter 5

Thickness

In Chapters 2, 3, and 4, we have performed the operations vertex deletion, edge deletion, and vertex splitting on a graph G with the goal of obtaining a new planar graph G'. We now ask for a collection of planar subgraphs of a given graph G, the union of which is G:

Definition 5.1 (thickness) *The* thickness *of a graph G, denoted $\theta(G)$, is the minimum number of planar subgraphs of G whose union is G.*

Clearly the thickness of a graph is 1 if and only if the graph is planar.

As an example, consider the two planar subgraphs of $K_{3,3}$ whose union is $K_{3,3}$ in Figure 5.1, and the three planar subgraphs of K_9 whose union is K_9 in Figure 5.2. Since $K_{3,3}$ is known to be nonplanar (see Theorem 1.4), the exhibition of two planar subgraphs of $K_{3,3}$ whose union forms $K_{3,3}$ shows that $\theta(K_{3,3}) = 2$. The thickness of K_9 is not so easily determined: Figure 5.2 only shows that $\theta(K_9) \leq 3$. [BHK62] shows that indeed $\theta(K_9) = 3$. (Alternative proofs are provided in [Tut63a, Wes86].) See Section 5.3 for further results about the thickness of complete and complete bipartite graphs.

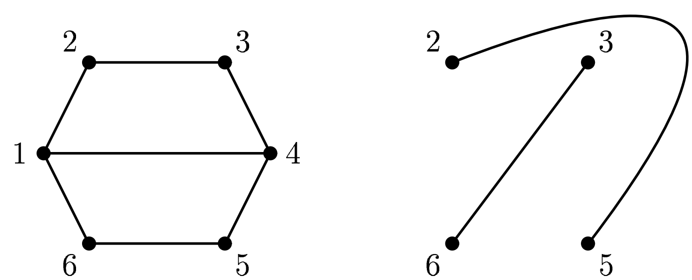


Figure 5.1: Two planar subgraphs of $K_{3,3}$ whose union is $K_{3,3}$.

Since each planar subgraph of a given graph G with n vertices and m edges can have at most $3n - 6$ edges (Equation 1.2), we obtain an immediate lower bound for the thickness of G:

$$\theta(G) \geq \left\lceil \frac{m}{3n-6} \right\rceil$$

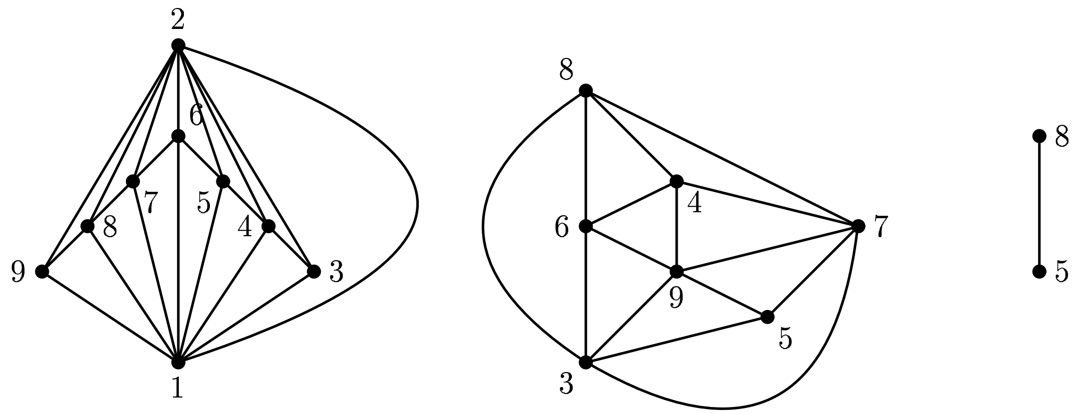


Figure 5.2: Three planar subgraphs G_1, G_2, and G_3 of K_9 whose union is K_9.

Observe that if the graphs in Figure 5.1 were printed onto slides, the two planar subgraphs given could actually be placed on top of each other so that each vertex labeled i in the first subgraph lies exactly on top of the vertex labeled i in the second subgraph. So we not only have two subgraphs whose union is $K_{3,3}$, but we have two *embeddings* of two planar graphs so that the union of the embeddings yields a drawing of $K_{3,3}$. Kainen [Kai73] showed that this observation can be generalized:

Theorem 5.2 [Kai73] *Given a graph G with thickness $\theta(G)$, there exists a drawing of G, and there exist subgraphs $G_1, \ldots G_{\theta(G)}$ whose union is G, such that the drawing of G restricted to G_i is a planar embedding of G_i, for $1 \leq i \leq \theta(G)$.*

Note that the three subgraphs of K_9 in Figure 5.2 are drawn in a way so that the union of their embeddings does not yield K_9.

Knowing the thickness of a given graph can be helpful in some application problems. [AKS91] proposes two new multilayer grid models for VLSI layout and shows for one of them that a graph with n vertices and thickness 2 can be embedded on 2 layers in $O(n^2)$ area. Furthermore, another algorithm embeds a graph with n vertices and thickness t on t layers in $O(n^3)$ area, respecting some additional constraints. [RL92, RL93] give approximate algorithms for scheduling multihop radio networks. They find a schedule whose length is a function of the thickness of the network.

The thickness of graphs has been widely studied as part of topological graph theory, but few algorithmic results for finding the thickness of a graph seem to be available. Early work about thickness and the introduction of the study of thickness into graph theory is described in detail by Hobbs [Hob69]. In particular, Tutte [Tut63b] establishes many results about the thickness of graphs in one of the earliest papers about this topic. A new survey paper about thickness is in preparation [MOS] (see also [Ode94]).

The following sections give a brief summary of the known results about thickness: Section 5.1 describes the result of Mansfield [Man83] that says that determining the thickness of a graph is NP-hard, and mentions heuristic approaches for finding the thickness. Thickness-minimal graphs are discussed in section 5.2, and Section 5.3 lists results about the thickness of graphs belonging to particular classes of graphs. Finally, Section 5.4 mentions two variations of the thickness.

5.1 Finding the Thickness of a Graph

Mansfield [Man83] defines the following problem (that was already mentioned in [GJ79, Problem OPEN3]):

Problem 5.3 (Thickness [Man83]) *Given a graph G and a positive integer K, does the thickness of G satisfy $\theta(G) \leq K$?*

Mansfield shows that this problem is NP-complete for the fixed value $K = 2$, thus establishing the NP-completeness of Thickness. The proof uses a reduction from Planar 3-SAT [GJ79, Problem LO1]. Before we state this problem, recall that given a set $U = \{u_1, \ldots, u_m\}$ of Boolean variables, the set $L = \{u_1, \overline{u_1}, \ldots, u_m, \overline{u_m}\}$ is the set of *literals* over U. A subset of literals $c \subseteq L$ is called a *clause* over U. A clause c is said to be *satisfied* if the disjunction of the literals in c has the Boolean value "true" (for some truth assignment for U). Given U and C, consider the bipartite graph $G_{U,C} = (U \cup C, E)$ with $E = \{uc \mid u \in c \text{ or } \overline{u} \in c\}$.

Problem 5.4 (Planar 3-SAT [GJ79, Problem LO1]) *Given a set U of Boolean variables and a collection C of clauses over U with $|c| \leq 3$ for all $c \in C$, and given that the graph $G_{U,C}$ is planar, is there a truth assignment for U that satisfies all clauses in C simultaneously?*

Lichtenstein [Lic82] showed that Planar 3-SAT is NP-complete. Mansfield first shows that Planar 3-SAT remains NP-complete if each clause contains *exactly* three literals, and then reduces this restricted version of Planar 3-SAT to Thickness with $K = 2$.

So we cannot expect to find a polynomial time algorithm that determines the thickness of a given graph. A heuristic approach for finding an upper bound on the thickness of a graph $G = (V, E)$ is to find a planar subgraph $G' = (V, E')$ of G, to form the difference graph $H = (V, E \setminus E')$, to then find a maximal planar subgraph of H and so on until the difference graph itself is planar. This approach is studied in [Ode94].

5.2 θ-Minimal Graphs

The following facts about thickness and the concept of thickness-minimal graphs (also called θ-minimal graphs) are due to Tutte [Tut63b]: If a graph G has thickness $\theta(G) = t$, then every subgraph of G has thickness at most t. Furthermore, if a subgraph G' of G has exactly one edge less than G or exactly one vertex (and all its incident edges) less than G, then either $\theta(G') = t$ or $\theta(G') = t - 1$. In other words, deleting one edge or deleting one vertex decreases the thickness of a graph by at most one. These facts motivate the following definition:

Definition 5.5 (thickness-minimal graphs)[1] *If a graph G has thickness t and if every proper subgraph of G has thickness less than t, then G is called a* thickness-minimal *(or θ-*minimal*) graph. If G is thickness-minimal with $\theta(G) = t$, we also call G* t-minimal.

The 2-minimal graphs are exactly the subdivisions of K_5 and $K_{3,3}$. Note that if a graph G has thickness $t \geq 2$, then there exists a t-minimal subgraph of G. For $t \geq 2$, every t-minimal graph is 2-connected and has minimum vertex degree at least t and maximum vertex degree at least $2t - 1$. Tutte then establishes the following important theorem:

Theorem 5.6 [Tut63b] *For each integer $t \geq 2$ there exist infinitely many pairwise nonisomorphic t-minimal graphs with maximum vertex degree $2t-1$, and of girth greater than any specified integer N.*

This theorem extablishes the *existence* of infinitely many t-minimal graphs. But given an integer $t \geq 2$, it does not provide an explicit construction of t-minimal graphs. Beineke [Bei67] showed that for any integer $t \geq 2$, the complete bipartite graph $K_{2t-1,4t^2-10t+7}$ is t-minimal. Hobbs and Grossman [HG68a], and, independently, Bouwer and Broere [BB68] showed that $K_{4t-5,4t-5}$ is t-minimal for any integer $t \geq 2$. Hobbs and Grossman [HG68b] also showed that any t-minimal graph is t-edge-connected.

Since $\theta(K_9) = 3$ [BHK62, Tut63a, Wes86], K_9 is a candidate for being 3-minimal. Figure 5.2 displays three subgraphs G_1, G_2, and G_3 of K_9 whose union is K_9, where G_3 consists of a single edge. Thus any proper subgraph of K_9 is the union of a subgraph of G_1 and a subgraph of G_2, and has therefore thickness at most 2. So K_9 is 3-minimal. K_9 appears to be the only θ-minimal complete graph.

Wessel [Wes83, Wes89], and, independently, Širáň and Horák [HŠ87] finally give, for each integer $t \geq 2$, an explicit construction of an infinite number of t-minimal graphs. Širáň and Horák show that the bounds established by [Tut63b] and [HG68b] on connectedness and minimum vertex degree are actually tight: Their graphs are 2-connected, but not 3-connected, they are t-edge-connected, but not $(t + 1)$-edge-connected, and they have minimum vertex degree t.

5.3 Results for Particular Classes of Graphs

There are few classes of graphs for which the thickness is known. For the complete graphs, the thickness was settled in a long process described in detail by White and Beineke [WB78]. It is clear that $\theta(K_1) = \theta(K_2) = \theta(K_3) = \theta(K_4) = 1$, and it is easily seen that $\theta(K_5) = \theta(K_6) = \theta(K_7) = \theta(K_8) = 2$. Figure 5.2 shows that $\theta(K_9) \leq 3$. Battle, Harary, and Kodama [BHK62] were the first to show that indeed $\theta(K_9) = 3$. Alternative proofs were given by Tutte [Tut63a] and Wessel [Wes86]. Beineke and Harary [BH65] showed the formula for $\theta(K_n)$ for most cases, and Alekseev and Goncakov [AG76], and, independently, Vasak [Vas76], completed the result:

$$\theta(K_n) = \left\lfloor \frac{n+7}{6} \right\rfloor \qquad \text{for } n \geq 1,\, n \neq 9,\, n \neq 10$$

[1] [Bei67, Wes83, Wes89] use the term *critical* instead of *minimal.*

$$\theta(K_9) = \theta(K_{10}) = 3$$

For the complete bipartite graph, the thickness is still not settled for all cases. Beineke, Haray, and Moon [BHM64] found the following result:

$$\theta(K_{n_1,n_2}) = \left\lceil \frac{n_1 \cdot n_2}{2(n_1 + n_2 - 2)} \right\rceil$$

except possibly when n_1 and n_2 are both odd, $n_1 \leq n_2$, and there is an integer k such that $n_2 = \left\lfloor \frac{2k(n_1-2)}{n_1-2k} \right\rfloor$. In [Bei67], Beineke gives a more detailed description of the proof than in [BHM64].

The thickness of the hypercube of dimension n, Q_n, is $\theta(Q_n) = \lceil \frac{n+1}{4} \rceil$ [Kle67].

For a graph G and a general surface S, let the *thickness of G on S*, denoted $\theta(G, S)$, be the smallest number of subgraphs of G so that the subgraphs are all embeddable in S and so that their union is G. When S is the torus (also denoted S_1), $\theta(G, S_1)$ is also called the *toroidal thickness* of G. To avoid confusion, the thickness of a graph is sometimes called the *planar thickness*. [WB78] reviews known results about the thickness on other surfaces. [Rin65] and, independently, [Bei69] give the toroidal thickness of the complete graphs. [Bei69] also discusses the complete bipartite graphs, as well as some other surfaces. Further results about the toroidal thickness of graphs are given in [And82b, And82a].

Recently, the thickness of another class of graphs was found. Jünger et al. showed that the thickness of graphs that do not contain K_5 as a minor is at most 2 [JMOS94, Ode94], using a decomposition theorem of Truemper [Tru92]. They were able to extend their result to graphs that do not contain the graph G_{12} as a minor [JMOS95] (G_{12} is depicted in Figure 5.3). Since G_{12} contains K_5 as well as $K_{3,3}$ as a minor, the result for G_{12} implies that the thickness of graphs that do not contain K_5 as a minor and the thickness of graphs that do not contain $K_{3,3}$ as a minor is at most two. For such a graph the thickness can be determined in linear time then, since it can be tested in linear time whether the graph is planar. If it is, its thickness is 1, otherwise it is 2.

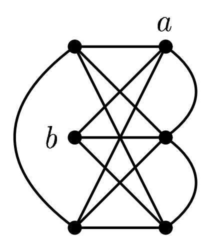


Figure 5.3: The graph G_{12}. G_{12} clearly has $K_{3,3}$ as a subgraph, so it has $K_{3,3}$ as a minor. To see that it also has K_5 as a minor, contract the edge ab.

For some graph classes, bounds on the thickness are known: A graph of orientable genus 1 (i.e. a graph embeddable on the torus) has thickness 2 [Asa87], and a graph of orientable genus 2 has thickness either 2 or 3 [Asa94]. Every graph $G = (V, E)$ has thickness at most $\left\lfloor \sqrt{\frac{|E|}{3}} + \frac{3}{2} \right\rfloor$ [DHS91]. [Wes84] and [Hal91] study the relation betwenn the minimum and maximum vertex degrees of a graph and its thickness.

5.4 Variations of Thickness

Bernhart and Kainen [BK79, Kai90] introduced the *book thickness* of a graph. A *book* B with $n \geq 0$ pages consists of a line L in 3-dimensional space, called the *spine*, together with n distinct half-planes (called the *pages*) with L as their common boundary. A graph G is embeddable in B if the vertices of G can be placed on L and if each edge can be embedded in at most one page of B. The *book thickness* (also called *pagenumber*) of a graph G is the smallest number n so that G can be embedded in a book with n pages. The book thickness has been studied for several classes of graphs, see for example [CLR87, Hea87, MLW88, HI92, Obr93, Mal94, SGB95].

[BS84] showed that any planar graph can be embedded in a 9-page book. [Hea84] lowered this bound to 7 pages and also gave an $O(n^2)$ algorithm to actually find an embedding. Yannkakis [Yan86, Yan89] showed that any planar graph can be embedded in a book with 4 pages. Yannakakis also gives a linear time algorithm to find such an embedding.

If a graph G has a straight line drawing and two subgraphs G_1 and G_2 whose union is G, and if the straight line drawing of G restricted to G_i is a planar embedding of G_i, for $1 \leq i \leq 2$, then G is called *doubly linear*. Clearly any doubly linear graph has thickness at most two. Hutchinson et al. [HSV96] study doubly linear graphs. They show that a doubly linear graph with n vertices has at most $6n - 18$ edges.

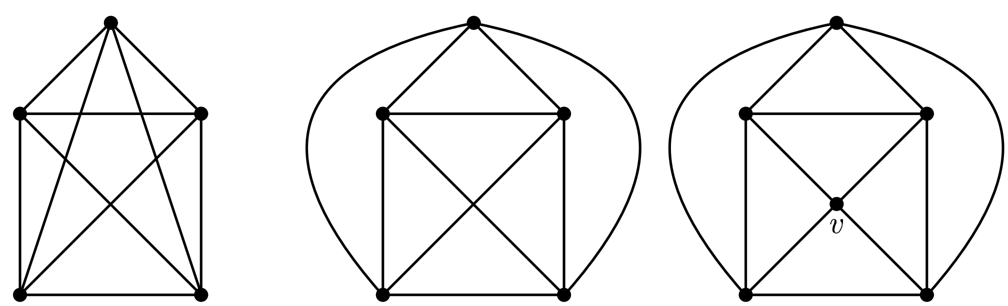
Chapter 6

Crossing Number

In graph drawing, but also in other application areas such as VLSI-Layout, we are interested in a drawing of a given graph with as few edge crossings as possible:

Definition 6.1 (crossing number) *The* crossing number *of a graph G, denoted $\nu(G)$, is the smallest number k so that G can be drawn in the plane with at most k edge crossings.*

Clearly the crossing number of a graph is 0 if and only if the graph is planar, and the crossing number of a graph is bounded from below by the skewnes of the graph. A graph G with crossing number $\nu > 0$ (and a given drawing with ν edge crossings) can be transformed into a planar graph by introducing ν new vertices and placing them at the edge crossings of the drawing. Each edge that was involved in some edge crossing is replaced by at least two new edges. Figure 6.1 illustrates this process.



a) *b)* *c)*

Figure 6.1: a) K_5 in a drawing with 5 edge crossings. b) K_5 with only one edge crossing. c) Vertex v is introduced instead of the edge crossing to planarize K_5.

Problem 6.2 (Crossing Number) *Given a graph G and a positive integer K, is there a drawing of G with K or less edge crossings?*

The complexity status of this problem was mentioned as being open in [GJ79, Problem OPEN3]. Then Garey and Johnson [GJ83] showed that Crossing Number is NP-complete.

There seem to be few results about the exact crossing number of particular graph classes (one is given in [Kle70] for example). A recent survey can be found in [Sch95]. Another recent paper [SSV95] studies the crossing number of a graph with respect to a general surface.

Definition 6.3 (crossing-critical graphs) *If a graph G has crossing number ν and every proper subgraph of G has crossing number less than ν then G is said to be* crossing-critical.

Note the analogy of this definition to thickness-critical graphs discussed in Section 5.2. [Koc87] gives, for any $n \geq 2$, a construction of an infinite family of 3-connected crossing-critical graphs with crossing number n. This improves the result in [Šir84].

List of Figures

Subject Index

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