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Pillay's conjecture for groups definable in weakly o-minimal non-valuational structures

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Abstract

Let G be a group definable in a weakly o-minimal non-valuational structure $\mathcal{M}$. Then G/G^{00}, equipped with the logic topology, is a compact Lie group, and if G has finitely satisfiable generics, then $\dim(G/G^{00}) = \dim(G)$. Our main technical result is that G is a dense subgroup of a group definable in the canonical o-minimal extension of $\mathcal{M}$.

1. *Introduction*

Definable groups have been at the core of model theory for at least a period of three decades, largely because of their prominent role in important applications of the subject, such as Hrushovski's proof of the function field Mordel-l-Lang conjecture in all characteristics [**17**]. Examples include algebraic groups (which are definable in algebraically closed fields) and compact Lie groups (which are definable in o-minimal structures). Groups definable in o-minimal structures are well-understood. The starting point was Pillay's theorem in [40] that every such group admits a definable manifold topology that makes it into a topological group, and the most influential work in the area has arguably been the solution of Pillay's Conjecture over a field [**18**], which brought to light new tools in theories with NIP (not the independence property). While substantial work on NIP groups has since been done (for example, in [**4**]), a full description of definable groups is still missing in many broad NIP settings. In this paper, we provide such a description for groups definable in weakly o-minimal non-valuational structures, and establish Pillay's Conjecture in this setting.

We recall that a structure $\mathcal{M} = \langle M, <, \dots \rangle$ is *weakly o-minimal* if $<$ is a dense linear order and every definable subset of M is a finite union of convex sets. Weakly o-minimal structures were introduced by Cherlin–Dickmann [**3**] in order to study the model theoretic properties of real closed rings. They were later also used in Wilkie's proof of the o-minimality of real exponential field [**35**], as well as in van den Dries' study of Hausdorff limits [**8**]. Macpherson–Marker–Steinhorn [**23**], followed-up by Wencel [**32, 34**], began a systematic study of weakly o-minimal groups and fields, revealing many similarities with the o-minimal setting.

An important dichotomy between valuational structures — those admitting a definable proper non-trivial convex subgroup — and non-valuational ones arose, supported by good evidence that the latter structures resemble o-minimal structures more closely than what the former ones do. For example, strong monotonicity and strong cell decomposition theorems were proved for non-valuational structures in [**32**]. In the same reference, the *canonical o-minimal extension* $\mathcal{N}$ of $\mathcal{M}$ was introduced, which is an o-minimal structure whose domain is the Dedekind completion N of M, and whose induced structure on M is precisely $\mathcal{M}$.

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A further description of $\mathcal{N}$ was recently provided in [**1**]. As far as definable groups are concerned, the existence of a definable group manifold topology was proved in [**33**], extending the aforementioned theorem by Pillay [**27**] for o-minimal structures, whereas in the special case when $\mathcal{M}$ is the trace of a real closed field on a dense elementary subfield, Baro and Martin-Pizarro [**2**] showed that every definable group admits, locally, an algebraic group law.

The main technical result of this paper is that every group G definable in $\mathcal{M}$ is a dense subgroup of a group K definable in $\mathcal{N}$, which is canonical in the sense that it is the smallest such group (Theorems 1.1). The existence of group topology from [**33**] and the local algebraic result from [**2**] are straightforward consequences. Note, however, that the current setting is much richer than that of o-minimal traces (by [**14**]), and that the result herein is global. Moreover, a far reaching application is obtained: we give a sharp description of the smallest type-definable subgroup G^{00} of G of bounded index, and establish Pillay's Conjecture in this setting (Theorem 1.2). Finally, all our results apply yet to another category of definable groups, namely *small groups* in certain dense pairs $\langle \mathcal{N}, P\rangle$ (see Remark 4.14 below).

For the rest of this paper, $\mathcal{M}$ denotes a weakly o-minimal non-valuational structure expanding an ordered group, and $\mathcal{N}$ its canonical o-minimal extension by Wencel [**32**] (some details are given in Section 3). Unless stated otherwise, by 'definable' we mean 'definable in $\mathcal{M}$ with parameters', and by '$\mathcal{N}$-definable', we mean 'definable in $\mathcal{N}$ with parameters'. If $X \subseteq N^n$, $\mathrm{cl}(X)$ denotes the closure of X in the usual order topology. If K is a group definable in $\mathcal{N}$, and $X \subseteq K$, we denote by $\mathrm{cl}_K(X)$ the closure of X in the group topology (given by [**27**]). For every set $X \subseteq N^n$, we define the *dimension of* X as the maximum k such that some projection onto k coordinates contains an open subset of M^k (where $M^0 = \{0\}$). We define $\dim(\emptyset) = -\infty$. This definition of dimension is natural in our setting, since, if $X \subseteq M^n$ is definable, then $\dim X$ equals the usual dimension in the ordered structure $\mathcal{M}$, and if $X \subseteq N^n$ is $\mathcal{N}$-definable, then $\dim X$ equals the usual o-minimal dimension (easily, by [**32**] — see also Section 3).

THEOREM 1.1. *Let $\mathcal{M}$ be a weakly o-minimal non-valuational expansion of an ordered group, and G a definable group. Let $\mathcal{N}$ be the canonical o-minimal extension of $\mathcal{M}$. Then there is an $\mathcal{N}$-definable group K that contains G as a subgroup, and such that for every other $\mathcal{N}$-definable group H that contains G as a subgroup, K $\mathcal{N}$-definably embeds in H. Moreover, $\mathrm{cl}_K(G) = K$ and $\dim G = \dim K$.*

Pillay's Conjecture was stated in [**28**] for groups definable in o-minimal structures, and was established, by splitting different cases and building on previous work, in [**10, 16, 18, 25**]. A version of it in the p-adic setting was further proved in [**24**]. Before stating Pillay's Conjecture, let us fix some terminology. Let $\mathcal{R}$ be any sufficiently saturated structure. By a *small* set or a set of *bounded cardinality*, we mean a set of cardinality smaller than $|R|$. By a type-definable set, we mean an intersection of a bounded collection of definable sets. Let G be a group definable in $\mathcal{R}$, H a normal type-definable subgroup of G, and $\pi : G \to G/H$ the canonical group homomorphism. We say that $A \subseteq G/H$ is closed in the *logic topology* if and only if $\pi^{-1}(A)$ is type-definable.

PILLAY'S CONJECTURE. Let G be a group definable in a sufficiently saturated o-minimal structure. Then:

(1) G has a smallest type-definable subgroup G^{00} of bounded index;
(2) G/G^{00}, equipped with the logic topology, is a compact real Lie group;
(3) if G is definable compact, then $\dim_{Lie} G/G^{00} = \dim G$.

Notes: (a) G^{00} is necessarily normal, (b) see [**26**] for the notion of definable compactness.

If $\mathcal{R}$ is a sufficiently saturated NIP structure, it is also known that every definable group G contains a smallest type-definable subgroup G^{00} of bounded index (Shelah [**29**]), and the notion of definable compactness was generalized to the notion of having *finitely satisfiable generics* (fsg) in [**18**] (see Definition 4.5 below for an alternative definition). As an application of Theorem 1.1, we extend Pillay's Conjecture to the weakly o-minimal setting. To our knowledge, this is the first time that the conjecture is being considered in structures that properly expand an o-minimal one. Recall that $\mathcal{M}$ is our fixed weakly o-minimal non-valuational expansion of an ordered group. It is known that $\mathcal{M}$ is NIP [**30**, A.1.3]. As remarked in Section 4 below, if $\mathcal{M}$ is sufficiently saturated, then so is $\mathcal{N}$ in an appropriate signature.

THEOREM 1.2. *Assume $\mathcal{M}$ is sufficiently saturated. Let G be a definable group, and K as in Theorem 1.1. Then:*

(1) $G^{00} = G \cap K^{00}$;
(2) $G/G^{00} \cong K/K^{00}$, *and hence is a compact real Lie group;*
(3) *if G has fsg, then so does K, and hence* $\dim_{Lie} G/G^{00} = \dim G$.

Outline of the strategy. Let G be a definable group. To show Theorem 1.1, we first prove in Section 2 a group chunk theorem for o-minimal structures, recasting [**5**] with regard to dimension properties instead of topological ones. In Section 3, we establish the existence of an o-minimal group chunk in which G is dense, from which Theorem 1.1 follows. In Section 4, we prove Theorem 1.2.

Terminology/notation. Let $\mathcal{R}$ be a structure, such as our fixed $\mathcal{M}$ or $\mathcal{N}$. Given a set $Z \subseteq R^{2n}$ and $x \in R^n$, we denote

$$Z_x = \{b \in R^n \ : \ (x, b) \in Z\}$$

and

$$Z^x = \{a \in R^n \ : \ (a, x) \in Z\}.$$

By $\pi_1, \pi_2 : R^{2n} \to R^n$, we denote the projections onto the first and second set of n coordinates, respectively. An *embedding* $\sigma : G \to H$ between two groups is simply an injective group homomorphism.

2. *A group chunk theorem*

The goal of this section is to show a group chunk theorem in the spirit of [**5**], but with respect to dimension properties instead of topological ones. It is possible that our Theorem 2.2 below for $\mathcal{N}$ reduces to that reference; however, such a reduction appears to be non-trivial and hence we present a full proof. Moreover, the present account goes through in an abstract dimension-theoretic setting, which we fix next.

For the rest of this section, $\mathcal{R}$ denotes a structure that eliminates imaginaries, and 'definable' means 'definable in $\mathcal{R}$ with parameters'. We assume that there is a map dim from the class of all definable sets to $\{-\infty\} \cup \mathbb{N}$ that satisfies the following properties: for all definable $X, Y \subseteq R^n$, and $a \in R$:

(D1) $\dim\{a\} = 0$, $\dim R = 1$, and $\dim X = -\infty$ if and only if $X = \emptyset$;
(D2) $\dim(X \cup Y) = \max\{\dim X, \dim Y\}$;
(D3) if $\{X_t\}_{t\in I}$ is a definable family of sets, then:
  (a) for $d \in \{-\infty\} \cup \mathbb{N}$, the set $I_d = \{t \in I : \dim X_t = d\}$ is definable; and
  (b) if every X_t has dimension k, and the family is disjoint, then
$$\dim \bigcup_{t\in I} X_t = \dim I + k;$$
(D4) if $f : X \to Y$ is a definable bijection, then $\dim X = \dim Y$.

For example, $\mathcal{N}$ with the usual o-minimal dimension satisfies the above properties [7]. Below we will be using the above properties without specific mentioning. Given definable sets $V \subseteq X \subseteq R^n$, we call V *large* in X if $\dim(X \setminus V) < \dim X$.

Definition 2.1. A *definable group chunk* is a triple (X, i, F), where $X \subseteq R^n$ is a definable set, and $i : X \to R^n$, $F : Z \subseteq X^2 \to R^n$ are definable maps, such that:

(1) i is injective on a large subset of X;
(2) for every $x \in X$, $F(x, -) : Z_x \to F(x, Z_x)$ and $F(-, x) : Z^x \to F(Z^x, x)$ are bijections between large subsets of X;
(3) for every $(x, y) \in Z$, there is a large $S_{(x,y)} \subseteq X$, such that for every $z \in S_{(x,y)}$, the following expressions are defined and are equal:
  (a) $F(F(x, y), z) = F(x, F(y, z))$;
  (b) $F(x, F(i(x), z)) = z = F(F(z, x), i(x))$;
(4) for every $x \in X$, $\pi_1 F^{-1}(x)$ and $\pi_2 F^{-1}(x)$ are large in X.

We often refer to properties (1)–(4) above as 'Axioms'.

Theorem 2.2. *Let (X, i, F) be a definable group chunk, as above. Then:*

(i) *there is a definable group $K = \langle K, *, \mathbf{1}_K \rangle$ with $X \subseteq K$ large in K, and such that for every $(x, y) \in Z$,*
$$F(x, y) = x * y;$$
(ii) *if $H = \langle H, \oplus, \mathbf{1}_H \rangle$ is a definable group and $\sigma : X \to H$ a definable injective map such that for every $(x, y) \in Z$,*
$$\sigma F(x, y) = \sigma(x) \oplus \sigma(y),$$
then K definably embeds in H;
(iii) *if, moreover, $\sigma(X)$ is large in H, then H and K are definably isomorphic.*

Proof. Note that by Axiom (3), for every $a, b \in X$, the set of elements $z \in X$ for which $F(a, F(b, z))$ is defined is large in X. We will be using this fact without mentioning. Also, we will sometimes write ab for $F(a, b)$ to lighten the notation. Finally, let $k = \dim X$.

(i) It is enough to find a definable group $K = \langle K, *, \mathbf{1}_K \rangle$ and a definable injective map $h : X \to K$, with $h(X)$ large in K, and such that for every $(x, y) \in Z$,
$$h(F(x, y)) = h(x) * h(y).$$
Indeed, then, consider the bijection that maps x to $h^{-1}(x)$, if $x \in h(X)$, and is the identity on $K \setminus h(X)$. The induced group structure on $X \cup (K \setminus h(X)$ has the desired properties.

For $a \in X$, denote by F_a the map $F(a, -) : X \to X$. Define a relation $\sim$ on X^2, as follows:
$$(a, b) \sim (c, d) \ \Leftrightarrow \ F_a \circ F_b \text{ agrees with } F_c \circ F_d \text{ on a large subset of } X.$$
By elimination of imaginaries, $\sim$ is a definable equivalence relation. Let $K \subseteq X^2$ be a definable set of representatives for $\sim$. We denote by $[(a, b)]$ the equivalence class of (a, b), and by $[a, b]$ its representative in K. We aim to equip K with a definable group structure $\langle K, *, \mathbf{1}_K \rangle$.

Claim 1. *Let $x \in X$. Then $F_x \circ F_{i(x)}$ agrees with the identity map on a large subset of X.*

Proof of Claim 1. By Axiom (3)(b). □

We may thus denote $\mathbf{1}_K = [x, i(x)] \in K$, for any $x \in X$.

CLAIM 2. *For every $a, b \in X$, if*

$$T = [(a, b)] = \{(x, y) \in X^2 : (x, y) \sim (a, b)\},$$

then $\pi_1(T)$ and $\pi_2(T)$ are large in X.

Proof of Claim 2. Pick $z \in X$ such that $F(a, F(b, z))$ is defined. Let $x = F(a, F(b, z))$. Now the set

$$C := F(Z^z, z) \cap \pi_2 F^{-1}(x) \subseteq X$$

is large, by Axioms (2) and (4). Pick any $e \in C$. Let $c = F(-, e)^{-1}(x)$. Then $F(c, e) = x$ and there is $d \in Z^z$ such that $e = F(d, z)$. That is,

$$F(c, F(d, z)) = F(a, F(b, z)). \tag{1}$$

Clearly, for $e' \neq e$ in C, the corresponding c', d' satisfy $c' \neq c$, by injectivity of F in the second coordinate, and $d' \neq d$, by injectivity of F in the first coordinate. Hence, we obtain large many such elements c and large many such elements d. We will be done if we prove that for every two such c, d,

$$F_c \circ F_d \text{ agrees with } F_a \circ F_b \text{ on a large subset of } X. \tag{2}$$

Fix any two $c, d \in X$ such that (1) holds. Consider the set A of all those $t \in X$, such that all expressions below are defined and are equal:

$$c(d(zt)) = c((dz)t) = (c(dz))t = (a(bz))t = a((bz)t) = a(b(zt)).$$

By Axiom (3)(a), the set A is large in X. Hence, using again injectivity of F in the first coordinate, equality (2) holds for large many functions zt, as required. □

CLAIM 3. (1) *For every $a, b, c, d \in X$, there are $e, x, y, f \in X$, such that $(a, b) \sim (e, x)$, $(c, d) \sim (y, f)$ and $y = i(x)$.*

(2) *For every $a, b, c, d, s, t \in X$, there are $e, x, y, z, w, f \in X$, such that $(a, b) \sim (e, x)$, $(c, d) \sim (y, z)$, $(s, t) \sim (w, f)$, $y = i(x)$ and $w = i(z)$.*

Proof of Claim 3. We only prove (1), as the proof of (2) is similar. Consider the sets

$$S = [(a, b)] = \{(e, x) \in X^2 : (a, b) \sim (e, x)\}$$

and

$$T = [(c, d)] = \{(y, f) \in X^2 : (c, d) \sim (y, f)\}.$$

By Claim (2), the projections $\pi_1(T)$ and $\pi_2(S)$ are large in X. By Axiom (1), $\dim i(\pi_2(S)) = k$, so the set

$$i(\pi_2(S)) \cap \pi_1(T)$$

is non-empty. Take any y in this set and let $x \in \pi_2(S)$ with $y = i(x)$. Then $y \in \pi_1(T)$ and $x \in \pi_2(S)$, so there are $e, f \in X$ with $(a, b) \sim (e, x)$ and $(c, d) \sim (y, f)$, as needed. □

Now, for every $a, b, c, d, e, f \in X$, define the relation

$$R(a, b, c, d, e, f) \Leftrightarrow \text{there are } x, y \in X \text{ such that } (a, b) \sim (e, x),\ (c, d) \sim (y, f)$$
$$\text{and } y = i(x).$$

Clearly, R is a definable relation. By Claim 3(1), for every $a, b, c, d \in X$, there are $e, f \in X$ such that $R(a, b, c, d, e, f)$. Moreover, if $R(a, b, c, d, e, f)$, $R(a', b', c', d', e', f')$, $(a, b) \sim (a', b')$

and $(c, d) \sim (c', d')$, then $(e, f) \sim (e', f')$. Indeed, let x, y, x', y' witnessing the first two relations. Then, by Axiom (3), the set of $z \in X$ for which the following hold

$$e(fz) = e(x(y(fz))) = a(b(c(dz))) = a'(b'(c'(d'z))) = e'(x'(y'(f'z))) = e'(f'z).$$

is large in X. Hence, $(e, f) \sim (e', f')$. We can thus define the following definable operation on K:

$$[a, b] * [c, d] = [e, f] \;\Leftrightarrow\; R(a, b, c, d, e, f).$$

Claim 4. *$K = \langle K, *, \mathbf{1}_K \rangle$ is a definable group.*

Proof of Claim 4. We already saw that the set K and map $*$ are definable. We prove associativity of $*$. Let $a, b, c, d, s, t \in X$. Take e, x, y, z, w, f as in Claim 3(2). Then, by tracing the definitions,

$$([a, b] * [c, d]) * [s, t] = [e, z] * [w, f] = [e, f] = [e, x] * [y, f]$$
$$= [a, b] * ([c, d] * [s, t]).$$

It is also easy to check that $\mathbf{1}_K$ is the identity element, using Claim 3(1), and that $[i(b), i(a)]$ is the inverse of $[a, b]$. □

Consider the map $h : X \to K$ defined as follows. Let $x \in X$. By Axiom (4), there is in particular $(a, b) \in Z$ such that $F(a, b) = x$. By definition of $\sim$, any two such (a, b) are $\sim$-equivalent. We let

$$h(x) = [a, b].$$

Again by definition of $\sim$, it is clear that h is injective.

Claim 5. *The set $h(X)$ is large in K.*

Proof of Claim 5. If not, there must be a set $S \subseteq K \setminus h(X)$ of dimension k $(= \dim X)$. By Claim 2, each $\sim$-class has dimension k. Therefore, the union

$$T = \bigcup_{[a,b] \in S} [(a, b)]$$

has dimension $2k$. Since Z is large in X^2, it must intersect T. But for $(c, d) \in Z \cap T$, we have $F(c, d) \in X$, and hence $[c, d] = h(F(c, d)) \in h(X)$, contradicting the fact that $[c, d] \in S$. □

Finally, we check that for every $(s, t) \in Z$,

$$h(F(s, t)) = h(s) * h(t). \tag{3}$$

Let $a, b, c, d \in X$ such that $s = F(a, b)$ and $t = F(c, d)$. As in the proof of Claim 3(1), we can find $e, x, y, f \in X$ with $F(e, x) = s$ and $F(y, f) = t$ and $y = i(x)$. It follows in particular that

$$h(s) * h(t) = [a, b] * [c, d] = [e, f].$$

Moreover, by Axiom (3), we obtain that for large many $z \in X$,

$$(st)z = ((ex)(yf))z = (ex)((yf)z) = (ex)(y(fz)) = e(x(y(fz))) = e(fz) = (ef)z.$$

Therefore, using Axiom (2), we obtain $st = ef$. Hence

$$h(F(s, t)) = h(F(e, f)) = [e, f],$$

as required.

This ends the proof of (i). For (ii) and (iii), in order to lighten the notation and without loss of generality, we may assume that σ is the identity map.

(ii) We have, for every $(x, y) \in Z$,

$$x * y = F(x, y) = x \oplus y.$$

It follows that for every $a, b, c, d \in X$,

$$a * b = c * d \;\Leftrightarrow\; a \oplus b = c \oplus d.$$

Indeed, take $z \in X$, such that all pairs $(b, z), (a, b * z), (a, b \oplus z)$ are in Z, and hence

$$a * (b * z) = a \oplus (b \oplus z).$$

Therefore

$$a * b * z = c * d * z \;\Leftrightarrow\; a \oplus b \oplus z = c \oplus d \oplus z,$$

as needed.

Denote now by ${}^{-1}$ the inverse map of K. We observe that for every $x \in K$, there are $a, b \in X$, such that $a * b = x$ (indeed, since X is large in K, we can choose $a \in X \cap x * X^{-1}$ and $b = a^{-1} * x$). We can thus define the injective map

$$\tau : K \to H \text{ given by } a * b \mapsto a \oplus b, \text{ where } a, b \in X.$$

It remains to see that τ is a group homomorphism. The proof is similar to the one of property (3) of h above and is left to the reader.

(iii) Since X is large in H, for every $a \in H$, there is $z \in X$, such that both $a \ominus z$ and z are in X, implying that $H = X \oplus X$. Hence, in this case, the homomorphism τ constructed above is also onto. □

3. *Weakly o-minimal structures*

Here we return to the setting of the introduction, where $\mathcal{M} = \langle M, <, +, \dots \rangle$ is a weakly o-minimal non-valuational expansion of an ordered group, and $\mathcal{N} = \langle N, <, +, \dots \rangle$ its canonical o-minimal extension, by Wencel [**32**]. We recall that $\mathcal{N}$ is an o-minimal structure whose domain N is the Dedekind completion of M, and whose induced structure on M is precisely $\mathcal{M}$. The precise construction of $\mathcal{N}$ from [**32**] will not play any particular role here. We will only need the following facts, which follow easily from the strong cell decomposition theorem proved in [**32**].

FACT 3.1 [**32**]. Let X be a definable set. Then:

1. $\mathrm{cl}(X)$ is $\mathcal{N}$-definable;
2. suppose $f : X \subseteq M^n \to M$ is a definable map. Then there is an $\mathcal{N}$-definable map $F : N^n \to N$ that extends f (namely, $F_{\restriction X} = f$);
3. $\dim X = \dim \mathrm{cl}(X)$.

We need some additional terminology: if $X, Z \subseteq N^n$, we call X *dense in* Z if $Z \subseteq \mathrm{cl}(X \cap Z)$. Equivalently, X intersects every relatively open subset of Z. We write $X \triangle Z = (X \setminus Z) \cup (Z \setminus X)$. By a k-*cell*, we mean a cell in $\mathcal{N}$ of dimension k.

Our goal is to establish Proposition 3.4, from which Theorem 1.1 will follow. We first need to ensure that injective definable maps always extend to injective $\mathcal{N}$-definable maps. We prove this uniformly in parameters. In the next lemma, $\pi_1 : M^{k+n} \to M^k$ denotes the projection onto the first k coordinates.

Lemma 3.2. *Let $f : A \subseteq M^{k+n} \to M^m$ be a definable map, and $F : N^{k+n} \to N^m$ an $\mathcal{N}$-definable extension. Assume that for every $t \in \pi_1(A)$, $f_t : A_t \to M^m$ is injective. Then there is an $\mathcal{N}$-definable set $A \subseteq Y \subseteq N^{k+n}$, such that for every $t \in \pi_1(Y)$, $F_t : Y_t \to N^m$ is injective.*

Proof. By Fact 3.1(2), there is an $\mathcal{N}$-definable family of functions $G_t : N^n \to N$, $t \in N^k$, such that for every $t \in \pi_1(A)$, G_t extends f_t^{-1}. Let

$$Y_t = \{x \in N^n : G_t \circ F_t(x) = x\}.$$

Clearly, $A_t \subseteq Y_t$. Moreover, for every $x, y \in Y_t$, if $F_t(x) = F_t(y)$, then

$$x = G_t \circ F_t(x) = G_t \circ F_t(y) = y,$$

and hence $F_{t\restriction Y_t}$ is injective. □

For the rest of this section, we fix a definable group $G = \langle G, \cdot, \mathbf{1}_G\rangle$ with $G \subseteq M^n$. Denote by $^{-1}$ its inverse.

Corollary 3.3. *There are:*

1. *an $\mathcal{N}$-definable $i : N^n \to N^n$ that extends $^{-1}$;*
2. *an $\mathcal{N}$-definable $F : N^{2n} \to N^n$ that extends $\cdot$;*
3. *an $\mathcal{N}$-definable set $X \subseteq N^n$ containing G, such that $i_{\restriction X}$ is injective; and*
4. *an $\mathcal{N}$-definable set $Z \subseteq N^{2n}$ containing G^2, such that*
   - *for every $x \in \pi_1(Z)$, the map $F(x, -) : Z_x \to F(x, Z_x)$ is injective;*
   - *for every $x \in \pi_2(Z)$, the map $F(-, x) : Z^x \to F(Z^x, x)$ is injective.*

Proof. By Fact 3.1(2), there are i, F as in (1) and (2). By Lemma 3.2, (3) and (4) follow. □

We can now achieve the connection to Section 2.

Proposition 3.4. *There is an $\mathcal{N}$-definable group chunk $\langle X, i_{\restriction X}, F_{\restriction Z}\rangle$, such that $G^2 \subseteq Z$, $G \subseteq X \subseteq \mathrm{cl}(G)$, $F_{\restriction G^2} = \cdot$, and $i_{\restriction G} = {}^{-1}$.*

Proof. Let i, F, X, Z be as in Corollary 3.3. Let $X_1 = \mathrm{cl}(G) \cap X$, and denote $Z_1 = Z$.

For every $x \in G$, the set of $z \in X_1$ for which Axiom (3)(b) holds contains G, and hence, since $X_1 \subseteq \mathrm{cl}(G)$, it is large in X_1. Let X_2 be the set of all $x \in X_1$, for which the set of $z \in X_1$ such that Axiom 3(b) holds is large in X_1. Hence $G \subseteq X_2$.

Similarly, for every $x \in G$, $\pi_1 F^{-1}(x)$ and $\pi_2 F^{-1}(x)$ contain G, and hence are large in X_1. Let

$$X_3 = \{x \in X_1 : \pi_1 F^{-1}(x) \text{ and } \pi_2 F^{-1}(x) \text{ are large in } X_1\}.$$

So $G \subseteq X_3$. Therefore, for $X = X_2 \cap X_3 \cap \mathrm{cl}(G)$, we have $G \subseteq X \subseteq \mathrm{cl}(G)$.

Finally, for every $(x, y) \in G^2$, the set of $z \in X$ for which Axiom (3)(a) holds contains G and hence is large in X. Let Z be the set of tuples $(x, y) \in Z_1 \cap X^2$ for which the set of $z \in X$ such that Axiom (3)(a) holds is large in X. Hence $G^2 \subseteq Z$.

It is then straightforward to check that $X, i_{\restriction X}, F_{\restriction Z}$ are as required. □

We are now ready to prove the first result of this paper.

Proof of Theorem 1.1. Let X, Z, F, i be as in Proposition 3.4, and $\langle K, *\rangle$ the $\mathcal{N}$-definable group from Theorem 2.2. Then $G \leqslant K$. Indeed, for every $(x, y) \in G^2 \subseteq Z$, $x \cdot y = F(x, y)$

$= x * y$. We show that for any $\mathcal{N}$-definable group H that contains G as a subgroup, K $\mathcal{N}$-definably embeds in H. Consider $Z' = H^2 \cap Z$ and $X' = \pi_1(Z') \cap \pi_2(Z')$. Then Proposition 3.4 also holds with X', Z', F, i. Apply Theorem 2.2(i) to get a group K'. Observe moreover that X' is large in K. Now, on the one hand, by Theorem 2.2(iii) applied to K' and K, we obtain that the two are $\mathcal{N}$-definably isomorphic. On the other hand, applying Theorem 2.2(ii) to K' and H, we obtain that K' $\mathcal{N}$-definably embeds in H. Therefore K $\mathcal{N}$-definably embeds in H.

For the 'moreover' clause, take a K-open subset U of K. Then $\dim U = \dim K$. By Theorem 2.2(i), X is large in K, and hence $\dim(U \cap X) = \dim X$. By Proposition 3.4, G is dense in X, and hence $U \cap X \cap G \neq \emptyset$, showing $\mathrm{cl}_K(G) = K$. We also have $\dim G = \dim \mathrm{cl}(G) = \dim X = \dim K$. □

Remark 3.5. It follows from Theorem 1.1 that G admits a group topology and contains a large subset on which the group topology coincides with the subspace one. Indeed, the restriction of the K-topology to G is a group topology, and if V is a large subset of K on which the K-topology coincides with the subspace one, then $V \cap G$ also has the same properties with regard to G, as can easily be shown. As mentioned in the introduction, these results already follow from [**33**, Theorem 4.10].

Remark 3.6. Since the subspace and the group topologies coincide on a large subset V of K, it is easy to see that, for every $X \subseteq K$,

$$\dim(\mathrm{cl}_K(X) \triangle \mathrm{cl}(X)) < \dim K.$$

Indeed, since V is large, we may assume $X \subseteq V$. But then the relative closure of X in V in either topology is the same, whereas the remaining parts of the closures have dimension $< \dim K$.

Remark 3.7. It is also worth noting that K is the smallest $\mathcal{N}$-definable group containing G, even up to definable isomorphism. Namely, for every definable group isomorphism $\rho : G \to \rho(G) \subseteq H$ and $\mathcal{N}$-definable group H, there is an $\mathcal{N}$-definable embedding $f : K \to H$. Indeed, by Lemma 3.2, ρ extends to an injective $\mathcal{L}$-definable map $\sigma : X \to \sigma(X)$, with $G \subseteq X \subseteq K$. We can pullback the multiplication and inverse of K (restricted to $\sigma(X)^2$ and $\sigma(X)$) to X^2 and X, respectively. Call those pullbacks F and i. It is then not hard to see that $X, Z = X^2, F, i$ satisfy the assumptions of Theorem 2.2, and, moreover, those of (ii) therein. Hence K $\mathcal{L}$-definably embeds in H, as needed.

We finish this section with some open questions. By [**32**], every definable set is the trace of an $\mathcal{N}$-definable set; that is, of the form $Y \cap M^n$, for some $\mathcal{N}$-definable set $Y \subseteq N^n$. It is therefore natural to ask the following question.

Question 3.8. In Theorem 1.1, can K be chosen so that, moreover, $K \cap M^n = G$?

In general, one can ask for what structures $\mathcal{R}_1 \preccurlyeq \mathcal{R}_2$ the following statement is true: given a group $G = R_2^n \cap X$, for some $\mathcal{R}_2$-definable set X, is G a subgroup of an $\mathcal{R}_2$-definable group K, such that: (a) K is the smallest such group, (b) $G = R_2^n \cap K$?

Finally, the following question was asked by Elias Baro in private communication.

Question 3.9. Assume $\mathcal{M}$ is the trace of a real closed field $\mathcal{N}$ on a dense $\mathcal{R} \preccurlyeq \mathcal{N}$. Is every definable group $\langle \mathcal{N}, \mathcal{R} \rangle$-definably isomorphic to a group definable in $\mathcal{R}$?

4. *Pillay's Conjecture*

In Section 4.2, we will assume further that $\mathcal{M}$ is sufficiently saturated, and prove Theorem 1.2 by means of Claims 4.9, 4.10 and 4.12. It is important for us that the canonical extension $\mathcal{N}$ is presented in some signature so that it also becomes sufficiently saturated (and hence Pillay's Conjecture for $\mathcal{N}$-definable groups holds). Towards that end, we adopt the following definition from [**1**]. Denote by $\mathcal{L}$ the language of $\mathcal{M}$.

Definition 4.1. $\mathcal{M}^*$ is the $\mathcal{L}$-structure whose domain is the Dedekind completion N of M, and for every n-ary $R \in \mathcal{L}$, we interpret $R^{\mathcal{M}^*} = \mathrm{cl}(R^{\mathcal{M}})$.

Fact 4.2. We have:

(1) $\mathcal{M}^*$ and $\mathcal{N}$ have the same definable sets;
(2) $\mathcal{M}$ and $\mathcal{M}^*$ have the same degree of saturation.

Proof. (1) Clearly, $\mathcal{M}^*$ is a reduct of $\mathcal{N}$. For the opposite direction, note that in [**1**], $\mathcal{M}^*$ is denoted by $\mathcal{M}^*_\emptyset$, and $\mathcal{N}$ by $\overline{\mathcal{M}}_M$. The conclusion then follows from [**1**, Theorem 2.10, Proposition 2.7 and Proposition 3.4(2)].

(2) By [**1**, Proposition 3.4(2)]. □

In view of Fact 4.2(1), everything proven for $\mathcal{N}$ in the previous sections still holds for $\mathcal{M}^*$. We may thus, from now on, let $\mathcal{N}$ denote $\mathcal{M}^*$.

4.1. *Preliminaries*

In this subsection, we fix a sufficiently saturated structure $\mathcal{R}$, and definability is taken with respect to $\mathcal{R}$. For a definable group G, the notion of having *finitely satisfiable generics* (fsg) was introduced in [**18**], generalizing the notion of definable compactness from o-minimal structures. By now, several other statements involving Keisler measures have been shown to be equivalent to fsg. Below, we adopt as a definition of fsg a statement that involves *frequency interpretation measures*, first introduced in [**22**]. Our account uses [**30, 31**].

For the next three definitions, G denotes a definable group with $G \subseteq R^n$.

Definition 4.3. A *Keisler measure* μ on G is a finitely additive probability measure on the class $\mathrm{Def}(G)$ of all definable subsets of G; that is, a map $\mu : \mathrm{Def}(G) \to [0,1]$ such that $\mu(\emptyset) = 0$, $\mu(G) = 1$, and for $Y, Z \in \mathrm{Def}(G)$,

$$\mu(Y \cup Z) = \mu(Y) + \mu(Z) - \mu(Y \cap Z).$$

Given a definable set $X \subseteq R^n$ and elements $a_1, \ldots, a_k \in R^n$, we denote

$$\mathrm{Av}(a_1, \ldots, a_k; X)| = \frac{1}{k}|\{i : a_i \in X\}|.$$

Definition 4.4. A Keisler measure μ on G is called a *frequency interpretation measure* (fim) if for every formula $\varphi(x; y)$ and $\varepsilon > 0$, there are $a_1, \ldots, a_k \in R^n$, such that for every $c \in R^m$, and for $X = \varphi(R; c)$, we have

$$|\mu(X) - \mathrm{Av}(a_1, \ldots, a_k; X)| < \varepsilon.$$

Definition 4.5. We say that G has *finitely satisfiable generics* (fsg) if it admits a left-invariant fim Keisler measure.

REMARK 4.6. Our definition of fsg is equivalent to the original definition given in [18]. Indeed, by [31, Theorem 3.32], μ has fim if and only if it is 'generically stable', and by [30, Proposition 8.33], G has fsg in the sense of [18] if and only if G admits a left-invariant generically stable Keisler measure.

Recall from the introduction that by a *small* set or a set of *bounded cardinality*, we mean a set of cardinality smaller than $|R|$. By a type-definable set X, we mean an intersection of a bounded collection of definable sets X_i. We write $X = \bigcap_i X_i$ without specifying the index set.

For the next fact, we assume that $\mathcal{R}$ is our fixed $\mathcal{M}$ or $\mathcal{N}$. For any $X \subseteq R^n$, $\dim X$ is the maximum k such that some projection onto k coordinates contains an open set. This notion is the same with the one defined in the introduction.

FACT 4.7. Let H be a type-definable subgroup of a definable group G. If H has bounded index, then $\dim H = \dim G$.

Proof. Easy, by compactness. □

4.2. *The proof of Pillay's Conjecture*

From now on, we assume that $\mathcal{M}$ is sufficiently saturated. By Fact 4.2, $\mathcal{N}$ is also sufficiently saturated. We also fix a definable group G and the $\mathcal{N}$-definable group K from Theorem 1.1. We write ab for the multiplication in K and G. We let $k = \dim G = \dim K$. By 'type-definable', we mean 'type-definable in $\mathcal{M}$', and by '$\mathcal{N}$-type-definable', we mean 'type-definable in $\mathcal{N}$'. As mentioned in the introduction, since $\mathcal{M}$ has NIP, G^{00} exists. In Claim 4.9, however, we provide a more precise description of G^{00} which further enables us to prove Claim 4.10. We start with a basic lemma.

LEMMA 4.8. *Let $L \leqslant K$ be an $\mathcal{N}$-type-definable subgroup of K with $\dim L = \dim K$. Then $GL = K$.*

Proof. Let $x \in K$. Since $\mathrm{cl}_K(G) = K$ and $\dim xL = \dim L = \dim K$, there must be $y \in xL \cap G$. Thus $x \in yL \subseteq GL$, as needed. □

CLAIM 4.9. *G has a smallest type-definable subgroup G^{00} of bounded index. More precisely, $G^{00} = G \cap K^{00}$.*

Proof. We show that $G \cap K^{00}$ is the smallest type-definable subgroup of G of bounded index. It is certainly a type-definable subgroup of bounded index, since $[G : G \cap K^{00}] \leqslant [K : K^{00}]$. Let H be another such subgroup. We prove that H must contain $G \cap K^{00}$. Let $L = \mathrm{cl}_K(H)$. We claim that L is an $\mathcal{N}$-type-definable subgroup of K of bounded index, and hence it must contain K^{00}. It is certainly a subgroup, since it is the closure of a subgroup of K.

To see that L is $\mathcal{N}$-type-definable, let $H = \bigcap_i H_i$, with H_i definable. We may assume that the family $\{H_i\}$ is closed under finite intersection. Then $\mathrm{cl}_K(H) = \bigcap_i \mathrm{cl}_K(H_i)$. Indeed, for $\subseteq$, the right-hand side is a closed set containing H and hence also $\mathrm{cl}_K(H)$. For $\supseteq$, let $a \in \bigcap_i \mathrm{cl}_K(H_i)$. If $a \notin \mathrm{cl}_K(H)$, then there is an open box B containing a, with $B \cap \bigcap_i H_i = \emptyset$. By compactness, there is i, such that $B \cap H_i = \emptyset$, contradicting $a \in \mathrm{cl}_K(H_i)$.

To see that L has bounded index in K, note that since H has bounded index in G, by Fact 4.7 we have $\dim H = k$. Hence $\dim \mathrm{cl}(H) = k$. By Remark 3.6, $\dim \mathrm{cl}_K(H) = k = \dim K$. By Lemma 4.8, $L = K$, and by the second isomorphism theorem, $[G : G \cap L] = [K : L]$. Hence, since $[G : G \cap L] \leqslant [G : H]$ is bounded, so is $[K : L]$.

We have shown that $K^{00} \subseteq L$. Hence, to prove that $G \cap K^{00} \subseteq H$, it suffices to show $H = G \cap L$. Clearly, H is a subgroup of $G \cap L$. Assume, towards a contradiction, that H is properly contained in $G \cap L$. Then there must be a coset aH of H in $G \cap L$, such that $aH \subseteq (G \cap L) \setminus H$. Since $\dim aH = \dim H = \dim L$, and $\mathrm{cl}_K(H) = L$, it follows that $aH \cap H \neq \emptyset$, a contradiction. □

By Fact 4.7, G^{00} has non-empty interior in the G-topology, and hence it follows that it is open in the G-topology.

Claim 4.10. *The function $f : G/G^{00} \to K/K^{00}$ given by*

$$xG^{00} \mapsto xK^{00}.$$

is an isomorphism of topological groups with respect to the logic topology on both groups. In particular, G/G^{00} is a compact Lie group.

Proof. By Claim 4.9, $G^{00} = G \cap K^{00}$. By Fact 4.7, $\dim K^{00} = \dim K$, and hence, by Lemma 4.8, $GK^{00} = K$. Hence, by the second isomorphism theorem, we obtain that f is a group isomorphism.

It remains to show that f is a homeomorphism, with respect to the logic topology on both G/G^{00} and K/K^{00}. By [**28**, Lemma 2.5], both quotients are compact Hausdorff spaces, and hence, since f is a bijection, it suffices to show that it is continuous. Write $\pi : G \to G/G^{00}$ and $\sigma : K \to K/K^{00}$ for the canonical group homomorphisms. We need to show that for every $X \subseteq G/G^{00}$,

$$\sigma^{-1}(f(X)) \text{ is } \mathcal{N}\text{-type-definable } \Rightarrow \pi^{-1}(X) \text{ is type-definable.}$$

Let

$$X = \{xG^{00} : x \in S\},$$

for some $S \subseteq G$. Then $\pi^{-1}(X) = SG^{00}$, $f(X) = \{xK^{00} : x \in S\}$, and $\sigma^{-1}(f(X)) = SK^{00}$. Hence we need to prove that

$$SK^{00} \text{ is } \mathcal{N}\text{-type-definable } \Rightarrow SG^{00} \text{ is type-definable.} \tag{4}$$

We have

$$SG^{00} = S(K^{00} \cap G) = (SK^{00}) \cap G,$$

since $S \subseteq G$. This implies (4). □

We now turn to the property fsg.

Lemma 4.11. *Assume G has fsg. Then so does K.*

Proof. Since G has fsg, there is a left-invariant fim Keisler measure μ on G. We define $\nu : \mathrm{Def}(K) \to [0, 1]$ by setting $\nu(X) = \mu(X \cap G)$, and show that ν is a left-invariant fim Keisler measure. It is straightforward to see that it is a Keisler measure, since μ is. In what follows, all topological notions for subsets of K are taken with respect to its group topology.

Claim 1. *If $X \subseteq K$ is definable, then $\nu(X) = \nu(\mathrm{int}(X))$, where $\mathrm{int}(X)$ is the interior of X.*

Proof of Claim 1. It suffices to show that if X has empty interior, then $\nu(X) = 0$. For that we will show that if $S \subseteq G$ is definable with empty interior, then $\mu(S) = 0$. This will be enough because $\mathrm{cl}_K(G) = K$. The assumption that S has empty interior implies that $\dim(S) < \dim(G)$, so S is non-generic. Since G has fsg, $\mu(S) = 0$ by [**21**, Fact 3.1(iv)]. □

CLAIM 2. *ν is left-invariant.*

Proof of Claim 2. Let $X \subseteq K$ be a definable set and $k \in K$. We need to show that $\nu(kX) = \nu(X)$. By Claim 1, we may assume that X is open. Let $S = X \cap G$ and $S_k = kX \cap G$. Assume towards a contradiction that $\mu(S) > \mu(S_k)$ (the other inequality is treated similarly). Let $\varepsilon = \mu(S) - \mu(S_k)$. We aim to find $g \in G$, such that $\mu(S) - \mu(gS_k) < \varepsilon$. By applying fim of μ to the union of the families $\{gS : g \in G\}$ and $\{gS_k : g \in G\}$, we obtain $a_1, \ldots, a_r \in M^n$, such that for every $g \in G$,

$$|\mu(gS) - \mathrm{Av}(a_1, \ldots, a_r; gS)| < \frac{\varepsilon}{2}$$

and

$$|\mu(gS_k) - \mathrm{Av}(a_1, \ldots, a_r; gS_k)| < \frac{\varepsilon}{2}.$$

Assume, for simplicity of notation, that $a_i \in S$ if and only if $i \leqslant j$ (some j). Note that $gS_k = gkX \cap G$ for $g \in G$. Continuity of the group operation, and the assumption that X is open assure that $a_1, \ldots, a_j \in hX$ for all $h \in K$ close enough to e. So for $g \in G$ close enough to k^{-1}, we get that $a_1, \ldots a_j \in gkX \cap G = gS_k$. The choice of $a_1, \ldots, a_r$ implies that $\mu(S) < \frac{j}{r} + \frac{\varepsilon}{2}$ and $\mu(gS_k) > \frac{j}{r} - \frac{\varepsilon}{2}$. Hence

$$\mu(S) - \mu(gS_k) < \varepsilon.$$

Since μ is left-invariant, we obtain $\mu(S) - \mu(S_k) < \varepsilon$, a contradiction. □

CLAIM 3. *ν is fim.*

Proof of Claim 3. Let $\varphi(x; y)$ be a formula. Then for every $b \in N^m$ the set $\varphi(M, b)$ is definable, say by $\psi(x, c)$ for some $c \in M^m$. Of course, if b and b' have the same type in the pair $\langle \mathcal{N}, \mathcal{M} \rangle$, then there exists $c' \in M^m$ such that $\varphi(M, b') = \psi(M, c')$. So by compactness there is a formula $\theta(x, z)$, such that for all $b \in N^m$, there is $c \in M^l$ with $\varphi(M, b) = \theta(M, c)$. Now, since μ is fim, for every $\varepsilon > 0$, there are $a_1, \ldots, a_k \in M^m$, such that for every $c \in M^l$, and for $X = \varphi(M^n, c)$, we have

$$|\mu(X) - \mathrm{Av}(a_1, \ldots, a_k; X)| < \varepsilon.$$

But since for every $b \in N^m$, there is $c \in M^l$ with

$$\nu(\varphi(N, b)) = \mu(\varphi(M, b) = \mu(\theta(M, c)),$$

the result follows. □

This ends the proof of the lemma. □

We can now conclude the proof of Theorem 1.2.

CLAIM 4.12. *Suppose G has fsg. Then $\dim G = \dim_{Lie} G/G^{00}$.*

Proof. By Lemma 4.11, K has fsg. By Pillay's Conjecture for o-minimal structures, $\dim_{Lie} K/K^{00} = \dim K = k$. By Claim 4.10, we are done. □

We conclude with some remarks.

Remark 4.13. It is worth mentioning that in [**18**], following the proof of Pillay's Conjecture, the Compact Domination Conjecture was introduced, and was proved by splitting different cases in [**11, 15, 19, 20**]. Compact Domination for an NIP group G turned out to be a crucial property, as it corresponds to the existence of an invariant smooth Keisler measure on G [**30**, Theorem 8.37]. By now it is known that every fsg group definable in a 'distal' NIP structure (which includes weakly o-minimal structures) is compactly dominated [**30**, Chapters 8 & 9]. The present account can actually yield a short proof of compact domination for fsg groups definable in $\mathcal{M}$, which we, however, omit.

Remark 4.14. The results of this paper also apply to another category of definable groups, namely *small groups* in certain dense pairs $\langle \mathcal{N}, P\rangle$. Those pairs include expansions of a real closed field by a dense elementary substructure P or a dense multiplicative divisible subgroup with the Mann property. Following [**6**], a definable set $X \subseteq N^n$ is called *small* if there is an $\mathcal{N}$-definable map $f : N^{mk} \to N^n$ such that $X \subseteq f(P^k)$. By [**12**], every small set is in definable bijection with a set definable in the induced structure on P, which is known to be weakly o-minimal and non-valuational [**6, 9**]. Therefore, we have established Pillay's Conjecture for small groups in dense pairs.

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