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Stability for a transmission problem in thermoelasticity with second sound*

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Abstract: We consider a semilinear transmission problem for a coupling of an elastic and a thermoelastic material. The heat conduction is modeled by Cattaneo's law removing the physical paradox of infinite propagation speed of signals. The damped, totally hyperbolic system is shown to be exponentially stable.

1 Introduction

Systems consisting of a purely elastic part and another thermoelastic part with a transmission taking place at the boundary between the two parts naturally rise the question whether the dissipation being present through heat conduction in the thermoelastic part is sufficient to (exponentially) stabilize the whole system.

In contrast, if the elastic system is augmented by interior friction or friction type boundary conditions, then this dissipation is strong enough to yield exponential stability, cp. [3, 4, 6, 7, 8].

For the coupling of an elastic part, say with reference configuration $\Omega_1 := (L_1, L_2) \subset \mathbb{R}$, to a thermoelastic part $\Omega := (0, L_1) \cup (L_2, L_3)$, with $0 < L_1 < L_2 < L_3$, Marzocchi, Muñoz Rivera and Naso [5] proved the exponential stability modeling the vibrations in Ω_1 by a wave equation, and modeling the vibrations and the thermal behavior by classical thermoelasticity. The latter means that the classical Fourier law is used for the relation between the heat flux q and the temperature gradient θ_x, leading to the known paradox of infinite propagation speed of signals in the system. Their system corresponds to the case $\tau = 0$ (and $f_2 = 0$) in the following system where Fourier's law is replaced by Cattaneo's law ($\tau > 0$).

Thus we study the following transmission problem for the displacement $u = u(t, x)$ in Ω, the displacement $v = v(t, x)$ in Ω_1, the temperature difference (relative to a fixed reference temperature) $\theta = \theta(t, x)$, and the heat flux $q = q(t, x)$, the latter two both in Ω:

$$u_{tt} - \alpha u_{xx} + \beta\theta_x + f_1(u) = 0 \quad \text{in } (0, \infty) \times \Omega \tag{1.1}$$

$$\theta_t + \gamma q_x + \delta u_{tx} + f_2(\theta) = 0 \quad \text{in } (0, \infty) \times \Omega \tag{1.2}$$

$$\tau q_t + q + \kappa\theta_x = 0 \quad \text{in } (0, \infty) \times \Omega \tag{1.3}$$

$$v_{tt} - b v_{xx} = 0 \quad \text{in } (0, \infty) \times \Omega_1 \tag{1.4}$$

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with initial conditions

$$u(0,\cdot)=u_0,\quad u_t(0,\cdot)=u_1,\quad \theta(0,\cdot)=\theta_0,\quad q(0,\cdot)=q_0 \quad \text{in } \Omega \tag{1.5}$$

$$v(0,\cdot)=v_0,\quad v_t(0,\cdot)=v_1 \quad \text{in } \Omega_1 \tag{1.6}$$

and boundary conditions (transmission conditions) for $t\in(0,\infty),\ j=1,2,$

$$u(t,0)=u(t,L_3)=\theta(t,0)=\theta(t,L_3) = 0 \tag{1.7}$$

$$q(t,L_1)=q(t,L_2) = 0 \tag{1.8}$$

$$u(t,L_j)=v(t,L_j),\quad \alpha u_x(t,L_j)-\beta\theta(t,L_j) = bv_x(t,L_j). \tag{1.9}$$

Here $\alpha,\beta,\gamma,\delta,\tau,\kappa,b$ are positive constants, and the smooth nonlinearities f_1,f_2 are assumed to satisfy for $s\in\mathbb{R}$:

$$sf_1(s)\geq 0,\quad |f_j(s)|\leq \mu_j|s|,\quad j=1,2 \tag{1.10}$$

with constants $\mu_1,\mu_2>0$.
The case $\tau=0, f_2=0$ corresponds to the system in [5]. The right-hand sides considered there are here assumed to be zero just for simplicity.
On the level of pure heat conduction Fourier's law leads to the standard parabolic equation for the temperature,

$$\theta_t-\gamma\kappa\theta_{xx}=0$$

while Cattaneo's law leads to a damped wave equation

$$\tau\theta_{tt}+\theta_t-\gamma\kappa\theta_{xx}=0.$$

In both cases one has exponential stability. Also for classical thermoelastic boundary value problems, both Fourier's and Cattaneo's law yield exponential stability, cp. [2, 9]. But the conclusion that this equivalence should always happen is wrong; recent investigations in [1] show Timoshenko type systems where a coupling to heat conduction is modeled by Fourier's law gives exponential stability, while a coupling via Cattaneo's law does (surprisingly) not. Therefore, it is a priori an open question whether the system (1.1) – (1.9) is exponentially stable, despite the knowledge on the case $\tau=0$ from [5]. We shall give a positive answer to this question here using appropriate energy functionals, also allowing additionally $f_2\neq 0$. Moreover, the limit $\tau\to 0$ is studied comparing the two systems.
The paper is organized as follows. In Section 2 we demonstrate the global well-posedness of a solution to (1.1) – (1.9). Section 3 contains the proof of the main result on exponential stability. In Section 4 the limit $\tau\to 0$ is considered.

2 Global well-posedness

Here, we can follow [5] to prove the unique global existence of a solution to (1.1) - (1.9). The new appearance of the nonlinearity f_2 in (1.2) requires a solution concept of strong solutions.
Let

$$\begin{aligned} H^1_L(\Omega) &:= \Big\{w\in H^1(\Omega)\ \mid\ w(0)=w(L_3)=0\Big\},\\ H^1_R(\Omega) &:= \Big\{w\in H^1(\Omega)\ \mid\ w(L_1)=w(L_2)=0\Big\},\\ V &:= \Big\{(u,v)\in H^1_L(\Omega)\times H^1(\Omega_1)\ \mid\ u(L_j)=v(L_j), j=1,2\Big\}.\end{aligned}$$

Definition 2.1 *Let $T > 0$ and $I := [0, T]$. Then we call (u, v, θ, q) a weak solution to (1.1) – (1.9), for $(u_0, v_0) \in V, u_1, \theta_0, q_0 \in L^2(\Omega)$, if*

$$(u, v) \in L^\infty(I, V), \qquad (u_t, v_t) \in L^\infty(I, L^2(\Omega) \times L^2(\Omega_1)) \tag{2.1}$$

$$\theta \in L^\infty(I, L^2(\Omega)), \qquad q \in L^\infty(I, L^2(\Omega)) \tag{2.2}$$

and one has

$$\int_\Omega^T \int_\Omega \{u\phi_{tt} + \alpha u_x \phi_x - \beta\theta\phi_x + f_1(u)\phi\} dxdt + \int_0^T \int_{\Omega_1} \{vw_{tt} + bv_x w_x\} dxdt \tag{2.3}$$
$$= \int_\Omega \{u_1\phi(0,\cdot) - u_0\phi_t(0,\cdot)\} dx + \int_{\Omega_1} \{v_1 w(0,\cdot) - v_0\phi_t(0,\cdot)\} dx$$

$$\int_\Omega^T \int_\Omega \{-\theta\psi_t + \gamma q\psi_x - \delta u_x\psi_t + f_2(\theta)\psi\} dxdt = \int_\Omega \{\theta_0\psi(0,\cdot) + \delta u_{0,x}\ \psi(0,\cdot)\} dx \tag{2.4}$$

$$\int_\Omega^T \int_\Omega \{-q\chi_t + q\chi + \kappa\theta\chi_x\} dxdt = \int_{\Omega_1} q_0\chi(0,\cdot) dx \tag{2.5}$$

for all $(\phi, w) \in C^2(I, V)$ with $\phi(T,\cdot) = \phi_t(T,\cdot) = 0, w(T,\cdot) = w_t(T,\cdot) = 0$, $\psi \in C^2(I, H_L^1(\Omega))$ with $\psi(T,\cdot) = \psi_t(T,\cdot) = 0$, and all $\chi \in C^2(I, H_R^1(\Omega))$ with $\chi(T,\cdot) = \chi_t(T,\cdot) = 0$.

Definition 2.2 *A weak solution is called a strong solution if*

$$(u, v) \in C^0(I, (H^2(\Omega) \times H^2(\Omega)) \cap V) \cap C^1(I, V) \cap C^2(I, L^2(\Omega)) \tag{2.6}$$

$$\theta \in C^0(I, H_L^1(\Omega)) \cap C^1(I, L^2(\Omega)), \quad q \in C^0(I, (H_R^1(\Omega)) \cap C^1(I, L^2(\Omega)). \tag{2.7}$$

Then we have the following result about existence and uniqueness of solutions

Theorem 2.1 *Let $(u_0, v_0) \in (H^2(\Omega) \times H^2(\Omega)) \cap V, \theta_0 \in H_L^1(\Omega), q_0 \in H_R^1(\Omega)$ satisfying the compatibility condition*

$$\alpha u_{0,x}(L_j) - \beta\theta_0(L_j) = bv_{0,x}(L_j), \quad j = 1, 2.$$

Then there is a unique strong solution (u, v, θ, q) to (1.1) – (1.9).

Proof. We sketch the proof since the Faedo-Galeskin method works as in [5]. Let $\{(\varphi_j, w_j) | j \in \mathbb{N}\}$ be an orthonormal (in L^2) basis (ONB) in $(H^2(\Omega) \times H^2(\Omega)) \cap V$, let $\{\psi_j | j \in \mathbb{N}\}$ be an ONB in $H^2(\Omega) \cap H_L^1(\Omega)$, and let $\{\xi_j | j \in \mathbb{N}\}$ be an ONB in $H^2(\Omega) \cap H_R^1(\Omega)$. In the ansatz

$$u^N(t,x) = \sum_{j=1}^N a_j(t)\varphi_j(x), \quad v^N(t,x) = \sum_{j=1}^N d_j(t)w_j(x)$$

$$\theta^N(t,x) = \sum_{j=1}^N b_j(t)\varphi_j(x), \quad q^N(t,x) = \sum_{j=1}^N p_j(t)\xi_j(x)$$

the set $\{a_j, d_j, b_j, p_j | j = 1, ..., N\}$ is determined by solving the system of nonlinear ordinary differential equations given by requiring, for $j = 1, ..., N$,

$$\int_\Omega u_{tt}^N \varphi_j + \alpha u_x^N \varphi_{j,x} - \beta \theta^N \varphi_{j,x} + f_1(u^N)\varphi_j dx = 0 \tag{2.8}$$

$$\int_{\Omega_1} v_{tt}^N w_j + b v_x^N w_{j,x} dx = 0 \tag{2.9}$$

$$\int_\Omega \theta_t^N \psi_j + \gamma q_x^N \psi_j + \delta u_{tx}^N \psi_j + f_2(\theta^N) dx = 0 \tag{2.10}$$

$$\int_\Omega \tau q_t^N \chi_j + q^N \chi_j + \kappa \theta_x^N \psi_j dx = 0. \tag{2.11}$$

Initial data are given as usual, e.g.

$$a_j(0) = a_0^j \ , \quad \text{where} \quad u_0(x) = \sum_{j=1}^{\infty} a_0^j \varphi_j(x).$$

Then a unique solution $\{a_j, b_j, d_j, p_j | j = 1, ..., N\}$ exists in $[0, T_N]$ for some $T_N \leq T$. The following estimates prove $T_N = T$ (arbitrary). Let

$$F_1(s) := \int_0^s f_1(r) dr \tag{2.12}$$

Multiplying (2.8) by $\kappa\delta a_j'(t)$, (2.9) by $\kappa\delta d_j'(t)$, (2.10) by $\kappa\beta b_j(t)$ and (2.11) by $\beta\gamma p_j(t)$, integration and summation over $j = 1, ..., N$ yields

$$\frac{d}{dt} E^N(t) \ \leq \ -\gamma \int_\Omega |q^N|_{dx}^2 - \int_\Omega f_2(\theta^N)\theta^N dx \ \leq \ \text{const.} E^N(t) \tag{2.13}$$

where

$$\begin{aligned} E^N(t) \ := \ & \frac{1}{2} \int_\Omega \{\kappa\delta |u_t^N|^2 + \alpha\kappa\delta |u_x^N|^2 + 2\kappa\delta \ F_1(u^N) + \kappa\beta |\theta^N|^2 + \tau\gamma\beta |q^N|^2\} dx \\ & + \int_{\Omega_1} \{\kappa\delta |v_t^N|^2 + \beta\kappa\delta |v_x^N|^2\} dx. \end{aligned} \tag{2.14}$$

By (2.13) we get the following boundedness

$$\begin{aligned} (u^N, v^N) \quad & \text{bounded in} \quad L^\infty(I, V) \\ (u_t^N v_t^N) \quad & \text{bounded in} \quad L^\infty(I, L^2(\Omega) \times L^2(\Omega_1)) \\ \theta^N \quad & \text{bounded in} \quad L^\infty(I, L^2(\Omega)) \\ q^N \quad & \text{bounded in} \quad L^\infty(I, L^2(\Omega)). \end{aligned}$$

Weak-$*$ convergence leads to a limit (u, v, θ, q). By the lemma of Aubin ([10, p.97]) we conclude that

$$u_N \to u \quad \text{a.e. in} \quad I \times \Omega$$

and then

$$f_1(u^N) \to f_1(u) \quad \text{weakly in} \quad L^2(I, L^2(\Omega)).$$

If $f_2 = 0$ this would suffice to conclude that (u, v, θ, q) is a weak solution, by letting $N \to \infty$ in (2.8) – (2.11). For $f_2 \neq 0$ we differentiate (2.8) – (2.11) with respect to t and get a priori estimates also for

$$\int_\Omega |\theta_t^N|^2 dx,$$

finally allowing to conclude

$$f_2(\theta^N) \to f_2(\theta)$$

and we recognize that (u, v, θ, q) is a weak, and then strong solution. The uniqueness is proved as follows:

Let $(u^*, v^*, \theta^*, q^*) := (u_1 - u_2, v_1 - v_2, \theta_1 - \theta_2, q_1 - q_2)$ be the difference of two strong solutions. Substracting the differential equations, then multiplying by u_t^*, v_t^*, θ^*, and q^*, respectively, one obtains for

$$P(t) := \frac{1}{2} \int_\Omega \{\kappa\delta |u_t^*|^2 + \alpha\kappa\delta |u_x^*|^2 + \kappa\beta |\theta^*|^2 + \tau\gamma\beta |q^*|^2\} dx + \frac{1}{2} \int_{\Omega_1} \{\kappa\delta |v_t^*|^2 + b\kappa\delta |v_x^*|^2\} dx$$

that

$$\begin{aligned} \frac{d}{dt} P(t) &\leq -\gamma \int_\Omega |q^*|^2 dx - \int_\Omega (f_1(u_1) - f_1(u_2)) u_t^* dx \\ &\quad - \int_\Omega (f_2(\theta_1) - f_2(\theta_2)) \theta^* dx \\ &\leq cP(t) \end{aligned}$$

for some constant $c > 0$, implying $P = 0$, since $P(0) = 0$. This yields $u^* = 0$, $v^* = 0$, $\theta^* = 0$, and $q^* = 0$.

Q.E.D.

3 Exponential stability

For the proof of exponential stability we may assume without loss of generality that

$$\gamma = \kappa, \quad \beta = \delta. \tag{3.1}$$

Otherwise, a multiplication of (1.2) by $\rho_2 := \kappa/\gamma$ and of (1.1) by $\rho_1 := (\delta\kappa)/(\beta\gamma)$ yields the desired equality (3.1), and the additional constructs ρ_2 in front of θ_t, and ρ_1 in front of u_{tt} can be dealt with in the energies below in an obvious manner.
Let (u, v, θ, q) be a strong solution to (1.1) – (1.9). Let

$$\begin{aligned} E_1(t) &:= \frac{1}{2} \int_\Omega \{u_t^2 + \alpha u_x^2 + \theta^2 + \tau q^2 + 2F_1(u)\} dx + \frac{1}{2} \int_{\Omega_1} \{v_t^2 + b v_x^2\} dx \\ &\equiv E_1(u, v, \theta, q) \end{aligned}$$

$$
\begin{aligned}
E_2(t) &:= E_1(u_t, v_t, \theta_t, q_t) - 2\int\limits_\Omega F_1(u_t)dx \\
\mathcal{E}(t) &:= E_1(t) + E_2(t).
\end{aligned}
$$

We have

$$
\begin{aligned}
\frac{d}{dt}E_1(t) &= -\int\limits_\Omega q^2 dx - \int\limits_\Omega f_2(\theta)\theta dx \\
\frac{d}{dt}E_2(t) &= -\int\limits_{\Omega_1} q_t^2 dx - \int\limits_\Omega f_2'(\theta)\theta_t^2 dx - \int\limits_\Omega f_1'(u)u_t u_{tt} dx.
\end{aligned} \tag{3.2}
$$

The technical difficulty in comparison to [5] consists in the fact that θ_x is no longer equivalent to q but only to the highest derivative q_t. This can be overcome as follows.
Multiplying equation (1.2) by u_{xt} we obtain

$$
\begin{aligned}
\frac{d}{dt}\int\limits_\Omega \theta u_{xt} dx - [\theta u_{tt}]_{\partial\Omega} + \int\limits_\Omega \theta_x u_{tt} dx + [\kappa q u_{xt}]_{\partial\Omega} - \kappa\frac{d}{dt}\int\limits_\Omega q u_{xx} dx + \\
+\kappa\int\limits_\Omega q_t u_{xx} + \delta\int\limits_\Omega u_{xt}^2 dx + \int\limits_\Omega f_2(\theta)u_{xt} dx &= 0,
\end{aligned}
$$

implying

$$
\begin{aligned}
\frac{d}{dt}\int\limits_\Omega \Big\{\theta u_{xt} - \kappa q u_{xx}\Big\} dx &= -\delta\int\limits_\Omega u_{xt}^2 dx - \int\limits_\Omega \theta_x u_{tt} dx - \kappa\int\limits_\Omega q_t u_{xx} dx - \int\limits_\Omega f_2(\theta)u_{xt} dx \\
&\quad +\theta(t,L_1)u_{tt}(t,L_1) - \theta(t,L_2)u_{tt}(t,L_2) \\
&\quad +\kappa q(t,0)u_{xt}(t,0) - \kappa q(t,L_3)u_{xt}(t,L_3).
\end{aligned} \tag{3.3}
$$

Muliplying (1.1) by u and (1.2) by v, respectively we get, using (1.10) for f_1,

$$
\begin{aligned}
\frac{d}{dt}\Big\{\int\limits_\Omega u_t u dx + \int\limits_{\Omega_1} v_t v dx\Big\} &\le \int\limits_\Omega u_t^2 dx + \int\limits_{\Omega_1} v_t^2 dx - \alpha\int\limits_\Omega u_x^2 dx - b\int\limits_{\Omega_1} v_x^2 dx - \delta\int\limits_\Omega \theta_x u dx \\
&\quad +\delta\theta(t,L_1)u(t,L_1) - \delta\theta(t,L_2)u(t,L_2).
\end{aligned} \tag{3.4}
$$

Mulitplying (1.2) by θ_t we get

$$
-\kappa\frac{d}{dt}\int\limits_\Omega q\theta_x dx = -\int\limits_\Omega \theta_t^2 dx - \kappa\int\limits_\Omega q_t\theta_x dx - \delta\int\limits_\Omega u_{xt}\theta_t dx - \int\limits_\Omega f_2(\theta)\theta_t dx. \tag{3.5}
$$

Muliplying (1.1) by u_{xx} yields

$$
\begin{aligned}
-\frac{d}{dt}\int\limits_\Omega u_t u_{xx} dx &= -a\int\limits_\Omega u_{xx}^2 dx + \int\limits_\Omega u_{xt}^2 dx + \delta\int\limits_\Omega \theta_x u_{xx} dx + \int\limits_\Omega f_1(u)u_{xx} dx. \\
&\quad -u_t(t,L_1)u_{xt}(t,L_1) + u_t(t,L_2)u_{xt}(t,L_2)
\end{aligned} \tag{3.6}
$$

Let p_1 be a piecewise linear function on Ω being a straight line joining $p_1(0) > 0$ to $p_1(L) < 0$ on $(0, L)$, and a straight line joining $p_1(L_2 > 0)$ to $p_1(L_3) < 0$ on (L_2, L_3).
A differentiation of (1.1) with respect to t and a multiplication by $p_1 u_{xt}$ yields

$$\begin{aligned} \frac{d}{dt}\int_\Omega p_1 u_{tt} u_{xt} dx &= -\frac{1}{2}\int_\Omega p_{1,x}(u_{tt}^2 + \alpha u_{xt}^2)dx + \frac{1}{2}\Big[p_1(u_{tt}^2 + \alpha u_{xt}^2)\Big]_{\partial\Omega} \\ &\quad -\delta\int_\Omega \theta_{xt} p_1 u_{xt} dx - \int_\Omega f_1'(u) u_t p_1 u_{xt} dx. \end{aligned} \tag{3.7}$$

Mulitplying (1.2) by $p_1\theta_{xt}$ gives

$$\begin{aligned} -\kappa\frac{d}{dt}\int_\Omega p_1 q_x \theta_x d_x &= -\frac{1}{2}\int_\Omega p_{1,x}\theta_t^2 + \frac{1}{2}\Big[p_1(\theta_t^2 + \frac{\kappa^2}{\tau}\theta_x^2)\Big]_{\partial\Omega} - \frac{1}{2}\frac{\kappa^2}{\tau}\int_\Omega p_{1,x}\theta_x^2 dx \\ &\quad + \frac{d}{dt}\int_\Omega f_2(\theta) p_1 \theta_x dx - \int_\Omega f_2'(\theta) p_1 \theta_t \theta_x dx. \end{aligned} \tag{3.8}$$

Combining (3.7) and (3.8) we obtain

$$\begin{aligned} \underbrace{\frac{d}{dt}\int_\Omega \{p_1 u_{tt} u_{xt} - \kappa p_1 q_x \theta_x - f_2(\theta) p_1 \theta_x\} dx}_{=:J_1(t)} &= \frac{1}{2}\Big[p_1(u_{tt}^2 + \alpha u_{xt}^2 + \theta_t^2 + \frac{\kappa^2}{\tau}\theta_x^2)\Big]_{\partial\Omega} + \frac{\kappa}{\tau}\int_\Omega p_1 q_x \theta_x dx \\ &\quad - \int_\Omega f_2'(\theta)\theta_t p_1 \theta_x dx - \int_\Omega f_1'(u) u_t p_1 u_{xt} dx. \\ &\quad -\frac{1}{2}\int_\Omega p_{1,x}(u_{tt}^2 + \alpha u_{xt}^2 + \theta_t^2 + \frac{\kappa^2}{\tau}\theta_x^2) dx \end{aligned} \tag{3.9}$$

Analogously, let p_2 be a straight line joining $p_2(L_1) < 0$ to $p_2(L_2) > 0$ on Ω_1. Then we have

$$\underbrace{\frac{d}{dt}\int_{\Omega_1} v_{tt} p_2 v_{xt} dx}_{=:J_2(t)} = \frac{1}{2}\Big[p_2(v_{tt}^2 + b v_{xt}^2)\Big]_{\partial\Omega_1} - \frac{1}{2}\int_{\Omega_1} p_{2,x}(v_{tt}^2 + b v_{xt}^2) dx. \tag{3.10}$$

From (3.9) and (3.10), respectively, we obtain

$$\begin{aligned} \frac{d}{dt}J_1(t) &\leq -d_1\Big[u_{tt}^2(t, L_1) + u_{tt}^2(t, L_2) + u_{xt}^2(t, 0) + u_{xt}^2(t, L_1) + u_{xt}^2(t, L_2) \\ &\quad + u_{xt}^2(t, L_3) + \theta_t^2(t, L_1) + \theta_t^2(t, L_2) + \theta_x^2(t, 0) + \theta_x^2(t, L_2)\Big] \\ &\quad + d_2\int_\Omega (u_{xx}^2 + u_{xt}^2 + \theta_t^2 + \theta_x^2) dx + d_2\int_\Omega |q_x \theta_x| dx, \end{aligned} \tag{3.11}$$

$$\frac{d}{dt}J_2(t) \leq d_3\Big[u_{tt}^2(t, L_2) + u_{xt}^2(t, L_2) + \theta_t^2(t, L_2)\Big] - d_4\int_{\Omega_1} (v_{tt}^2 + v_{xt}^2) dx,$$

implying

$$\frac{d}{dt}\Big\{\frac{d_1}{2d_3}J_2(t)\Big\} \leq \frac{d}{2}\Big[u_{tt}^2(t, L_2) + u_{xt}^2(t, L_2) + \theta_t^2(t, L_2)\Big] - d_5\int_{\Omega_1} (v_{tt}^2 + v_{xt}^2) dx, \tag{3.12}$$

where $d_1, d_2, ...$ (will) denote positive constants (μ_1, μ_2 in (1.10) are assumed to be less than a fixed constant, since they will be chosen small enough later on). The estimates (3.11), (3.12) imply

$$\begin{aligned} \frac{d}{dt}\Big\{J_1(t) + \frac{d_1}{2d_3}J_2(t)\Big\} \quad &\leq \quad -\frac{d_1}{2}\Big[u_{tt}^2(t,L_1) + u_{tt}^2(t,L_2) + u_{xt}^2(t,0) + u_{xt}^2(t,L_1) + u_{xt}^2(t,L_2) \\ &\quad +u_{xt}^2(t,L_3) + \theta_t^2(t,L_1) + \theta_t^2(t,L_2) + \theta_x^2(t,0) + \theta_x^2(t,L_2)\Big] \\ &\quad -d_5\int_{\Omega_1}(v_{tt}^2 + v_{xt}^2)dx + d_6\int_{\Omega}(u_{xx}^2 + u_{xt}^2 + \theta_t^2 + \theta_x^2)dx. \end{aligned} \tag{3.13}$$

We conclude from (3.5)

$$-4\kappa\, d_6\frac{d}{dt}\int_{\Omega} q\theta_x dx \quad \leq \quad -2d_6\int_{\Omega}\theta_t^2 + d_7\int_{\Omega}|q_t\theta_x|dx + d_7\int_{\Omega}u_{xt}^2dx + d_7\int_{\Omega}\theta_x^2dx. \tag{3.14}$$

Combining (3.13) and (3.14), and denoting

$$J_3(t) := J_1(t) + \frac{d_1}{2d_3}\, J_2(t) - 4\kappa\, d_6\int_{\Omega} q\theta_x dx$$

as well as the boundary terms in (3.13) by $B(t)$, we conclude

$$\begin{aligned} \frac{d}{dt}J_3(t) \quad &\leq \quad -\frac{d_1}{2}B(t) - d_5\int_{\Omega_1}(v_{tt}^2 + v_{xt}^2)dx - d_6\int_{\Omega}\theta_t^2 dx \\ &\quad +d_8\int_{\Omega}(u_{xx}^2 + u_{xt}^2 + \theta_x^2 + q_t^2)dx. \end{aligned} \tag{3.15}$$

We get from (3.6) for $\varepsilon_1 > 0$ and some C_{ε_1} denoting a positive constant depending on ε_1,

$$\begin{aligned} -\frac{d}{dt}\int_{\Omega} u_t u_{xx} dx \quad &\leq \quad -\frac{\alpha}{2}\int_{\Omega}u_{xx}^2dx + \frac{\alpha\varepsilon_1}{8d_8}\Big[u_{xt}^2(t,L_1) + u_{xt}^2(t,L_2)\Big] \\ &\quad +C_{\varepsilon_1}\int_{\Omega}(u_{xt}^2 + \theta_x^2)dx + \mu_1\int_{\Omega}u_x^2dx \end{aligned}$$

implying

$$\begin{aligned} \frac{d}{dt}\Big\{\frac{-4d_8}{\alpha}\int_{\Omega}u_t u_{xx}dx\Big\} \quad &\leq \quad -2d_8\int_{\Omega}u_{xx}^2dx + \frac{\varepsilon_1}{2}\Big[u_{xt}^2(t,L_1) + u_{xt}^2(t,L_2)\Big] \\ &\quad +\frac{4\mu_1 d_8}{\alpha}\int_{\Omega}u_x^2dx + C_{\varepsilon_1}\int_{\Omega}(u_{xt}^2 + \theta_x^2)dx. \end{aligned} \tag{3.16}$$

Adding (3.15), (3.16) we get for

$$J_4(t) \;:=\; J_3(t) - \frac{4d_8}{\alpha}\int_{\Omega}u_t(u_{tt} + u\theta_x)dx$$

that

$$\frac{d}{dt}J_4(t) \leq -\frac{1}{2}(d_1-\varepsilon_1)B(t) - d_5\int\limits_{\Omega_1}(v_{tt}^2+v_{xt}^2)dx - d_6\int\limits_{\Omega}\theta_t^2dx - d_8\int\limits_{\Omega}u_{xx}^2dx$$
$$+C_{\varepsilon_1}\int\limits_{\Omega}(u_{xt}^2+\theta_x^2+q_t^2)dx + \mu_1 d_9\int\limits_{\Omega}u_x^2dx. \tag{3.17}$$

Let

$$J_5(t) := \int\limits_{\Omega}u_t u dx + \int\limits_{\Omega_1}v_t v dx.$$

Then we get for $\varepsilon_2 > 0$ from (3.4)

$$\varepsilon_2\frac{d}{dt}J_5(t) \leq \varepsilon_2\int\limits_{\Omega}u_t^2dx + \varepsilon_2\int\limits_{\Omega}v_t^2dx - \frac{\alpha\varepsilon_2}{2}\int\limits_{\Omega}u_x^2dx - \frac{b\varepsilon}{2}\int\limits_{\Omega}v_x^2dx + C_{\varepsilon_2}\int\limits_{\Omega}\theta_x^2dx. \tag{3.18}$$

With

$$J_6(t) := J_4(t) + \varepsilon_2\, J_5(t)$$

we conclude from (3.17), (3.18)

$$\frac{d}{dt}J_6(t) \leq -\frac{1}{2}(d_1-\varepsilon_1)B(t) - d_5\int\limits_{\Omega_1}(v_{tt}^2+v_{xt}^2)dx - d_6\int\limits_{\Omega}\theta_t^2dx - d_8\int\limits_{\Omega}u_{xx}^2dx$$
$$-\left(\frac{\alpha\varepsilon_2}{2}-\mu_1 d_9\right)\int\limits_{\Omega}u_x^2dx - \frac{b\varepsilon_2}{2}\int\limits_{\Omega_1}v_x^2dx + \varepsilon_2\int\limits_{\Omega}u_t^2dx$$
$$+\varepsilon_2\int\limits_{\Omega_1}v_t^2dx + C_{\varepsilon_1}\int\limits_{\Omega}(u_{xt}^2+q_t^2)dx + C_{\varepsilon_1,\varepsilon_2}\int\limits_{\Omega}\theta_x^2dx. \tag{3.19}$$

The equation (3.3) implies for $N > 0$ that

$$N\frac{d}{dt}\int\limits_{\Omega}(\theta u_{xt}-\kappa q u_{xx})dx \leq -N\delta\int\limits_{\Omega}u_{xt}^2dx + \frac{\varepsilon_1}{2}\Big[u_{tt}^2(t,L_1)+u_{tt}^2(t,L_2)+u_{xt}^2(t,0)+u_{xt}^2(t,L_3)\Big]$$
$$+\frac{d_8}{2}\int\limits_{\Omega}u_{xx}^2dx + C_{N,\varepsilon_1}\int\limits_{\Omega}\theta_x^2dx + C_{N,\delta}\Big[q^2(t,0)+q^2(t,L_3)\Big]. \tag{3.20}$$

With

$$J_7(t) := J_6(t) + N\int\limits_{\Omega}(\theta u_{xt}-\kappa q u_{xx})dx$$

a combination of (3.19), (3.20) yields

$$\frac{d}{dt}J_7(t) \leq -\frac{1}{2}(d_1-2\varepsilon_1)B(t) - d_5\int\limits_{\Omega}(v_{tt}^2+v_{xt}^2)dx - d_6\int\limits_{\Omega}\theta_t^2dx - \frac{d_8}{2}\int\limits_{\Omega}u_{xx}^2dx$$
$$-\Big(\frac{\alpha\varepsilon_2}{2}-\mu_1 d_9\Big)\int\limits_{\Omega}u_x^2 - \frac{b\varepsilon_2}{2}\int\limits_{\Omega}v_x^2dx - (N\delta - C_{\varepsilon_1} - \mu_1)\int\limits_{\Omega}u_{xt}^2dx + \varepsilon_2\int\limits_{\Omega}u_t^2dx$$
$$+\varepsilon_2\int\limits_{\Omega}v_t^2dx + C_{\varepsilon_1,\varepsilon_2,N}\int\limits_{\Omega}(\theta_x^2+q_t^2)dx + C_{N,\varepsilon_1}\Big[q^2(t,0)+q^2(t,L_3)\Big]. \tag{3.21}$$

Observing

$$-\frac{d_5}{2}\int_{\Omega_1} v_{xt}^2 dx \;\le\; d_{10}\int_{\Omega} u_{xt}^2 dx - d_{11}\int_{\Omega_1} v_t^2 dx$$

arising from

$$v_t(t,x) \;=\; u_t(t,L_1) + \int_{L_1}^{x} v_{xt}(t,y)dy$$

we conclude from (3.21)

$$\begin{aligned}\frac{d}{dt}J_7(t) \;\le\; & -\frac{1}{2}(d_1-2\varepsilon_1)B(t) - \frac{d_5}{2}\int_{\Omega_1}(v_{tt}^2+v_{xt}^2)dx - (d_{11}-\varepsilon_2)\int_{\Omega_1} v_t^2 dx \\ & -d_6\int_{\Omega}\theta_t^2 dx - \frac{d_8}{2}\int_{\Omega} u_{xx}^2 dx - \Big(\frac{\alpha\varepsilon_2}{2}-\mu_1 d_9\Big)\int_{\Omega} u_x^2 dx \\ & -\frac{b\varepsilon_2}{2}\int_{\Omega_1} v_x^2 dx - (N\delta - d_{10} - C_{\varepsilon_1} - \mu_1)\int_{\Omega} u_{xt}^2 dx \\ & +\varepsilon_2\int_{\Omega} u_t^2 dx + C_{\varepsilon_1,\varepsilon_2,N}\int_{\Omega}(\theta_x^2+q_t^2)dx + C_{\varepsilon_1,N}\Big[q^2(t,0)+q^2(t,L_3)\Big]. \qquad (3.22)\end{aligned}$$

Choosing first ε_1 such that $d_1 - 2\varepsilon_1 > \frac{d_1}{2}$, then N such that $N\delta - d_{10} - C_{\varepsilon_1} - \mu_1 > d_{12}$ for some $d_{12} > 0$, then ε_2 such that $d_{11} - \varepsilon_2 > \frac{d_{11}}{2}$, and assuming $\frac{\alpha\varepsilon_2}{2} - \mu_1 d_9 > \frac{\alpha\varepsilon_2}{4}$, we get from (3.22)

$$\begin{aligned}\frac{d}{dt}J_7(t) \;\le\; & -\frac{d_5}{2}\int_{\Omega_1}(v_{tt}^2+v_{xt}^2)dx - d_6\int_{\Omega}\theta_t^2 dx - \frac{d_8}{2}\int_{\Omega} u_{xx}^2 dx \\ & -\frac{d_{11}}{2}\int_{\Omega_1} v_t^2 dx - \frac{\alpha\varepsilon_2}{4}\int_{\Omega} u_x^2 dx - \frac{b\varepsilon_2}{2}\int_{\Omega_1} v_x^2 dx - d_{12}\int_{\Omega} u_{xt}^2 dx \\ & +\varepsilon_2\int_{\Omega} u_t^2 dx + C_{\varepsilon_1,\varepsilon_2,N}\int_{\Omega}(\theta_x^2+q_t^2)dx + C_{\varepsilon_1,N}\Big[q^2(t,0)+q^2(t,L_3)\Big]. \qquad (3.23)\end{aligned}$$

Observing that for $\varepsilon_3 > 0$

$$C_{\varepsilon_1,N}q^2(t,0) \;=\; -C_{\varepsilon_1,N}\int_0^{L_1}\frac{d}{dx}(q^2)dx \;\le\; \frac{\varepsilon_3}{2}\int_{\Omega} q_x^2 dx + C_{\varepsilon_1,N,\varepsilon_3}\int_{\Omega} q^2 dx$$

(analogously for $C_{\varepsilon_1,N}\, q^2(t,L_3)$) we conclude from (3.23)

$$\begin{aligned}\frac{d}{dt}J_7(t) \;\le\; & -\frac{d_5}{2}\int_{\Omega_1}(v_{tt}^2+v_{xt}^2)dx - d_6\int_{\Omega}\theta_t^2 dx - \frac{d_8}{2}\int_{\Omega} u_{xx}^2 dx \\ & -\frac{d_{11}}{2}\int_{\Omega_1} v_t^2 dx - \frac{\alpha\varepsilon_2}{4}\int_{\Omega} u_x^2 dx - \frac{b\varepsilon_2}{2}\int_{\Omega_1} v_x^2 dx - d_{12}\int_{\Omega} u_{xt}^2 dx \\ & +\varepsilon_2\int_{\Omega} u_t^2 dx + C_{\varepsilon_1,\varepsilon_2,N,\varepsilon_3}\int_{\Omega}(\theta_x^2+q_t^2+q^2)dx + \varepsilon_3\int_{\Omega} q_x^2 dx \qquad (3.24)\end{aligned}$$

Using Poincaré's estimate and equations (1.2), (1.3) we have

$$\int_\Omega u_t^2 dx \ \le\ d_{13} \int_\Omega u_{tx}^2 dx,$$

and

$$\int_\Omega q_x^2 dx \ \le\ d_{14} \int_\Omega (\theta_t^2 + u_{xt}^2) dx + d_{15} \int_\Omega (q^2 + q_t^2) dx.$$

This combined with (3.24) yields for small $\varepsilon_2, \varepsilon_3$ (hence necessarily small μ_1)

$$\begin{aligned}\frac{d}{dt} J_5(t) \ \le\ & -\frac{d_5}{2} \int_{\Omega_1} (v_{tt}^2 + v_{xt}^2) dx - d_{16} \int_\Omega \theta_t^2 dx - \frac{d_8}{2} \int_\Omega u_{xx}^2 dx - \frac{d_{11}}{2} \int_{\Omega_1} v_t^2 dx \\ & -\frac{\alpha \varepsilon_2}{4} \int_\Omega u_x^2 dx - \frac{b\varepsilon_2}{2} \int_{\Omega_1} v_x^2 dx - d_{17} \int_\Omega u_{xt}^2 dx - d_{18} \int_\Omega u_t^2 dx \\ & -d_{19} \int_\Omega q_x^2 dx + C_{\varepsilon_1, \varepsilon_2, N, \varepsilon_3} \int_\Omega (q^2 + q_t^2) dx. \end{aligned} \tag{3.25}$$

For $M > 0$ we define the final Lyapunov functional $\mathcal{L}(t)$ as

$$\mathcal{L}(t) := J_7(t) + M\mathcal{E}(t).$$

Using the equations (1.1) and (1.3) we can produce negative terms $-\int_\Omega \theta_x^2 dx$ and $-\int_\Omega u_{tt}^2 dx$, and we conclude from (3.2) and (3.25), for μ_1, μ_2 sufficiently small, and M large enough,

$$\frac{d}{dt} \mathcal{L}(t) \ \le\ -d_{20} \mathcal{E}(t). \tag{3.26}$$

Since $\mathcal{L}(t)$ is equivalent to $\mathcal{E}(t)$ for M sufficiently large, i.e. there are positive constants C_1, C_2 such that for all t we have

$$C_1 \mathcal{E}(t) \ \le\ \mathcal{L}(t) \ \le\ C_2 \mathcal{E}(t)$$

we conclude from (3.26) as usual

$$\mathcal{E}(t) \ \le\ C_0 e^{-d_0 t} \mathcal{E}(0)$$

for some constants $d_0, C_0 > 0$ being independent of the data. Thus we have proved

Theorem 3.1 *If μ_1, μ_2 (from (1.10)) are sufficently small, the strong solution (u, v, θ, q) to (1.1) – (1.9) given in Theorem 2.1 decays exponentially i.e.*

$$\exists d_0, C_0 > 0 \quad : \quad \mathcal{E}(t) \le C_0 e^{-d_0 t} \mathcal{E}(0) \qquad \forall t \ge 0$$

where C_0 and d_0 are independent of the initial data.

4 The limit $\tau \to 0$

As shown above, the qualitative behavior — exponential stability — is the same for the case $\tau > 0$ as for the case $\tau = 0$. Now we compare the two systems and show that the energy of the

difference is of order $\mathcal{O}(\tau^2)$.
Let $(u^\tau, v^\tau, \theta^\tau, q^\tau)$ denote the solution to (1.1) – (1.9) for $\tau > 0$, and let $(u^0, v^0, \theta^0, q^0)$ denote the solution for $\tau = 0$ with

$$q^0 := -\kappa\theta^0_x.$$

We assume compatible initial data, i.e. the same data $(u_0, u_1, v_0, v_1, \theta_0)$ and the compatibility condition

$$q_0 \;=\; -\kappa\theta_{0,x}. \tag{4.1}$$

Let

$$(\hat{u}, \hat{v}, \hat{\theta}, \hat{q}) := (u^\tau - u^0, v^\tau - v^0, \theta^\tau - \theta^0, q^\tau - q^0)$$

denote the difference of the solutions. Then $(\hat{u}, \hat{v}, \hat{\theta}, \hat{q})$ satisfies

$$\hat{u}_{tt} - \alpha\hat{u}_{xx} + \delta\hat{\theta}_x + f_1(u^\tau) - f_1(u^0) \;=\; 0 \tag{4.2}$$

$$\hat{\theta}_t + \kappa\hat{q}_x + \delta\hat{u}_{xt} + f_2(\theta^\tau) - f_2(\theta^0) \;=\; 0 \tag{4.3}$$

$$\begin{aligned} \tau\hat{q}_t + \hat{q} + \kappa\hat{\theta}_x \;&=\; -\tau q^0_t \\ &=\; \tau\kappa\theta^0_{tx} \end{aligned} \tag{4.4}$$

$$\hat{v}_{tt} - b\hat{v}_{xx} \;=\; 0 \tag{4.5}$$

with zero initial conditions and boundary conditions (1.7) – (1.9). Let, for $0 \le t \le T$, the energy term $F_1(t)$ be defined as (cp. $E_1(t)$)

$$F_1(t) \;:=\; \frac{1}{2}\int_\Omega \hat{u}_t^2 + \alpha\hat{u}_x^2 + \hat{\theta}^2 + \tau\hat{q}^2 dx + \frac{1}{2}\int_{\Omega_1} \hat{v}_t^2 + b\hat{v}_x^2 dx.$$

Then, by (4.1) – (4.5), we have

$$\begin{aligned} \frac{d}{dt}F_1(t) \;&=\; -\int_\Omega \hat{q}^2 dx - \tau\kappa\int_\Omega \theta^0_{tx}\hat{q}dx + \int_\Omega (f_1(\hat{u}^\tau) - f_1(\hat{u}^0))\hat{u}_t dx + \int_\Omega (f_2(\theta^\tau) - f_2(\theta^0))\hat{\theta}dx \\ &\le\; -\frac{1}{2}\int_\Omega \hat{q}^2 dx + \frac{\tau^2\kappa^2}{2}\int_\Omega |\theta^0_{tx}|^2 dx + \mu_1\int_\Omega |u^\tau - u^0||\hat{u}_t|dx + \mu_2\int_\Omega |\theta^\tau - \theta^0||\hat{\theta}|dx \\ &\le\; \frac{\tau^2\kappa^2}{2}\int_\Omega |\theta^0_{tx}|^2 dx + C_T F_1(t) \end{aligned}$$

with a positive contant C_T essentially depending only on T. Hence

$$F_1(t) \;\le\; C_T\int_0^t F_1(r)dr + \frac{\tau^2\kappa^2}{2}\int_0^t\int_\Omega |\theta^0_{tx}|^2 dxdr,$$

implying

$$F_1(t) \;\le\; \tau^2\Big\{\frac{C_T\kappa^2}{2}\int_0^t\int_\Omega |\theta^0_{tx}|^2 dxdr\Big\}e^{C_T t}.$$

Using the exponential stability result from [5] we know

$$c^* := \sup_{t>0} \int_0^t \int_\Omega |\theta_{tx}^0|^2 dx dr < \infty,$$

hence we conclude

Theorem 4.1 *The first-order energy F_1 of the difference of solutions to the Cattaneo system ($\tau > 0$) and the Fourier system ($\tau = 0$) (1.1)– (1.9), with assumed compatibility (4.1), is of order $\mathcal{O}(\tau^2)$, this is:*

$$\forall t \in [0,T] \quad : \quad F_1(t) = \mathcal{O}(\tau^2) \qquad as \quad \tau \to 0.$$

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