Universität Konstanz

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Li Chen
Michal Dreher

Konstanzer Schriften in Mathematik und Informatik
Nr. 213, Februar 2006

ISSN 1430-3558

Universität Konstanz
Fach D 188, 78457 Konstanz, Germany
E-Mail: preprints@informatik.uni-konstanz.de
WWW: http://www.informatik.uni-konstanz.de/Schriften/

The Viscous Model of Quantum Hydrodynamics in Several Dimensions

Li Chen* and Michael Dreher†

Abstract

We investigate the viscous model of quantum hydrodynamics in one and higher space dimensions. Exploiting the entropy dissipation method, we prove the exponential stability of the thermal equilibrium state in 1,2, and 3 dimensions, provided that the domain is a box. Further, we show the local in time existence of a solution in the one dimensional case; and in the case of higher dimensions under the assumption of periodic boundary conditions. Finally, we discuss the semiclassical limit.

Keywords: quantum hydrodynamics; exponential decay; entropy dissipation method; local existence of solutions; semiclassical limit.

AMS Mathematics Subject Classification: 35B40, 35Q35, 76Y05

1 Introduction

The flow of charged particles in a semi-conductor can be simulated using different models. Typical examples are the quantum energy transport models (QET), the quantum drift diffusion model (QDD) or the quantum hydrodynamic model (QHD). Derivations of the quantum QET and QDD models can be found in [4]. The quantum hydrodynamic models can be derived directly from the Schrödinger–Poisson system by WKB wave functions [7]; or from the collision Wigner equation by the momentum method and closing the system with the quantum thermal equilibrium distribution [5]; or by the entropy minimization method [9].

In this paper, we study the viscous QHD model, a model which is derived from the Wigner equation with the Fokker–Planck collision operator:

$$\left.\begin{aligned}
&\partial_t n - \operatorname{div} J = \nu_0 \triangle n,\\
&\partial_t J - \operatorname{div}\left(\frac{J\otimes J}{n}\right) - T\nabla n + n\nabla V + \frac{\varepsilon^2}{2} n \nabla\left(\frac{\triangle\sqrt{n}}{\sqrt{n}}\right) = \nu_0 \triangle J - \frac{J}{\tau},\\
&\lambda^2 \triangle V = n - C(x),\\
&n(0,x) = n^0(x), \qquad J(0,x) = J^0(x),
\end{aligned}\right\} \tag{1}$$

where $(t,x) \in (0,\infty)\times\Omega$ and Ω is a domain in $\mathbb{R}^d$.

*Address: Department of Mathematical Sciences, Tsinghua University, Beijing, 100084, P.R. China, lchen@math.tsinghua.edu.cn

†Address: Fachbereich Mathematik und Statistik, P.O.Box D187, Universität Konstanz, 78457 Konstanz, Germany, michael.dreher@uni-konstanz.de

Our boundary conditions are either

$$\left.\begin{aligned} \partial_\nu n(t,x) &= 0, \quad (t,x) \in (0,\infty)\times\partial\Omega, \\ J(t,x) &= 0, \quad (t,x) \in (0,\infty)\times\partial\Omega, \\ \partial_\nu V(t,x) &= 0, \quad (t,x) \in (0,\infty)\times\partial\Omega, \end{aligned}\right\} \tag{2}$$

where ∂_ν denotes the normal derivative, or we assume periodic boundary conditions, i.e.,

$$\Omega = \mathbb{T}^d \qquad \text{is a } d\text{–dimensional torus.} \tag{3}$$

Moreover, we suppose

$$\inf_{x\in\Omega} n^0(x) > 0, \tag{4}$$

$$\int_\Omega \left(n^0(x) - C(x)\right)\,\mathrm{d}x = 0. \tag{5}$$

The last condition is necessary; the Poisson equation for V would not be solvable otherwise.

The unknown functions in this system are the particle density $n = n(t,x)\colon [0,\infty)\times\Omega \to \mathbb{R}_+$, the current density $J = J(t,x)\colon [0,\infty)\times\Omega \to \mathbb{R}^d$, and the electrostatic potential $V = V(t,x)\colon [0,\infty)\times\Omega \to \mathbb{R}$. The given function $C = C(x)\colon \Omega \to \mathbb{R}$ is the doping profile of background charges.

The scaled physical constants are a viscosity constant ν_0 describing the strength of the collisions, a temperature constant T, the Planck constant ε, the momentum relaxation time τ, and the Debye length λ. All these constants are assumed to be positive.

For quantum macroscopic models, some results on local or global existence or long time asymptotics are known. For the QDD model, the existence of weak solutions was shown in [2, 3, 10]; and the semiclassical limit and the long time behaviour were studied in [2, 3]. Concerning the QHD model without viscous terms, the existence of smooth solutions and their long time asymptotics for small initial data were investigated in [8, 13].

It seems that there are less mathematical results for the system (1). The authors are only aware of [6], where the exponential stability of a constant steady state to (1) in a one-dimensional setting with a certain boundary condition was proved, based on the entropy dissipation method [1]. Most of the difficulties arise from the Bohm potential term

$$B(n) = \frac{\triangle\sqrt{n}}{\sqrt{n}},$$

which introduces a third order perturbation to a system which could otherwise be interpreted as a parabolic system coupled to an elliptic equation. In this paper, we follow the approach of [6] and generalize those results to domains of dimension two and three, see Theorem 2.1. Our key ingredient is an estimate on a certain term containing the Bohm potential, Proposition A.1.

Additionally, we are able to prove the local in time existence of sufficiently smooth solutions to (1). The proof relies on the observation that the third-order perturbation term has a good sign, which makes standard energy estimates for parabolic systems possible after having introduced a fourth order viscous regularization.

Physically spoken, the periodic boundary conditions are of restricted interest; however, they enable us to prove the local in time existence of solutions in a one-dimensional setting with boundary conditions (2) immediately, see Theorem 2.5.

In the course of proving the local existence results of the Theorems 2.4 and 2.5, we will obtain a certain *a priori* estimate of the solution, from which we then will be investigating the semiclassical limit $\varepsilon \to +0$, compare Theorem 2.7.

2 Main Results

Our notations are standard: L^p denote the usual Lebesgue spaces, and $H^k(\Omega) := W_2^k(\Omega)$ are the L^2–based Sobolev spaces, for $k \in \mathbb{N}_0$. The brackets $\langle \cdot, \cdot \rangle$ stand for the scalar product in $\mathbb{R}^d$, and $J \otimes J$ is a $d \times d$ matrix with entry $J_k J_l$ at position (k, l).

We list our results:

Theorem 2.1 (Exponential stability). *Let $d = 1, 2, 3$ and $\Omega = \prod_{j=1}^d (a_j, b_j)$ be a box. Let the triple (n, J, V) be a solution to (1) under the boundary conditions (2), and suppose that*

$$
\begin{aligned}
&n \in H^1((0,t^*), H^1(\Omega)) \bigcap L^2((0,t^*), H^3(\Omega)),\\
&J \in H^1((0,t^*), L^2(\Omega)) \bigcap L^2((0,t^*), H^2(\Omega)),\\
&V \in H^1((0,t^*), H^2(\Omega)).
\end{aligned}
$$

We assume that $C = C(x) \equiv C_0 > 0$ in Ω and

$$\inf_{(t,x)\in(0,t^*)\times\Omega} n(t,x) > 0.$$

Define an energy as follows:

$$E(t) = \int_\Omega \left(\frac{\varepsilon^2}{2} (\nabla\sqrt{n})^2 + T\left(n \left(\ln \frac{n}{C_0} - 1 \right) + C_0 \right) + \frac{\lambda^2}{2} (\nabla V)^2 + \frac{|J|^2}{2n} \right) \mathrm{d}x. \tag{6}$$

Let μ_1 denote the first positive eigenvalue of $-\triangle$ on Ω with Neumann boundary conditions, and fix

$$\sigma := \min\left\{ \frac{8T\nu_0}{\varepsilon^2}, \frac{\mu_1 C_0}{\mu_1 \lambda^2 T + C_0}, \frac{2}{\tau} \right\}.$$

Then this energy satisfies the inequality

$$
\begin{aligned}
&\partial_t E(t) \le -\sigma E(t)\\
&\quad - \frac{\varepsilon^2}{2}\nu_0 \left(c_{1,d} \int_\Omega (\triangle\sqrt{n})^2 \,\mathrm{d}x + c_{2,d} \int_\Omega \frac{|\nabla n|^4}{n^3} \,\mathrm{d}x \right) - \nu_0 \sum_l \int_\Omega n \left(\nabla \left(\frac{J_l}{n} \right) \right)^2 \mathrm{d}x,
\end{aligned}
$$

where the numbers $c_{1,d}$ and $c_{2,d}$ are given in the following table:

d	1	2	3
$c_{1,d}$	2	$\frac{1}{3}$	$\frac{1}{9}$
$c_{2,d}$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{7}{144}$

Remark 2.2. *Physically spoken, the four terms in the energy (6) are the quantum energy, the thermodynamic entropy, the electric energy and the kinetic energy. We note that the energy of the steady state $(n, J, V) = (C_0, 0, 0)$ is zero, and that E can not become negative.*

From the above differential inequality of the physical energy we then easily derive decay estimates:

Corollary 2.3. *Assume that the solution mentioned in Theorem 2.1 exists up to* $t = \infty$*. Then the function* $(n, J, \nabla V)$ *decays to the steady state* $(C_0, 0, 0)$ *as follows:*

$$\frac{\varepsilon^2}{2}\left\|\nabla\left(\sqrt{n}(t,\cdot)-\sqrt{C_0}\right)\right\|^2_{L^2(\Omega)} \le e^{-\sigma t}E(0), \tag{7}$$

$$T\left\|\sqrt{n}(t,\cdot)-\sqrt{C_0}\right\|^2_{L^2(\Omega)} \le e^{-\sigma t}E(0), \tag{8}$$

$$\left\|\sqrt{n}(t,\cdot)-\sqrt{C_0}\right\|^2_{L^p(\Omega)} \le C^2_{p,d,\Omega}\left(\frac{2}{\varepsilon^2}+\frac{1}{T}\right)e^{-\sigma t}E(0), \tag{9}$$

$$\frac{\varepsilon^2}{2}\nu_0 c_{1,d}\int_{t=0}^{\infty}\left\|\triangle\left(\sqrt{n}(t,\cdot)-\sqrt{C_0}\right)\right\|^2_{L^2(\Omega)}\,\mathrm{d}t \le E(0), \tag{10}$$

$$\frac{1}{2}\left\|\frac{J(t,\cdot)}{\sqrt{n(t,\cdot)}}\right\|^2_{L^2(\Omega)} \le e^{-\sigma t}E(0), \tag{11}$$

$$\frac{\lambda^2}{2}\left\|\nabla V(t,\cdot)\right\|^2_{L^2(\Omega)} \le e^{-\sigma t}E(0), \tag{12}$$

where $C_{p,d,\Omega}$ *is the norm of the embedding* $H^1(\Omega) \subset L^p(\Omega)$*, with* $1 \le p \le \infty$ *for* $d = 1$*,* $1 \le p < \infty$ *for* $d = 2$*, and* $1 \le p \le 6$ *for* $d = 3$*.*

Theorem 2.4 (Local existence and uniqueness on a torus). *Let* d *be a positive integer and* $\Omega = \mathbb{T}^d$ *be a torus. Let* b *be the smallest integer greater than* $\frac{1}{2}d$*, and* $s \ge b$ *be an integer. Suppose*

$$n^0 \in H^{s+1}(\Omega), \qquad J^0 \in H^s(\Omega), \qquad C \in H^{s-1}(\Omega)$$

and (4), (5)*. Then the problem* (1) *has a solution* (n, J, V)*, local in a time interval* $[0, t^*]$*, with the regularity properties*

$$\left.\begin{aligned} n &\in H^1((0,t^*),H^s(\Omega))\bigcap L^2((0,t^*),H^{s+2}(\Omega)),\\ J &\in H^1((0,t^*),H^{s-1}(\Omega))\bigcap L^2((0,t^*),H^{s+1}(\Omega)),\\ V &\in H^1((0,t^*),H^{s+2}(\Omega))\bigcap L^2((0,t^*),H^{s+4}(\Omega)),\\ (n,\nabla n,J) &\in C([0,t^*]\times\Omega). \end{aligned}\right\} \tag{13}$$

The solution is unique and persists as long as $n(t,\cdot)$ *and* $J(t,\cdot)$ *stay in* $L^\infty(\Omega)$ *and* n *remains positive. The life span* t^* *does not depend on the Sobolev regularity* s*.*

Having shown the local existence in the periodic case, we can take advantage from geometric arguments and consider the case $\Omega \subset \mathbb{R}^1$ effortlessly:

Theorem 2.5 (Local existence and uniqueness in one dimension). *Let* $\Omega \subset \mathbb{R}^1$ *be an open and bounded interval. Suppose*

$$n^0 \in H^2(\Omega), \qquad J^0 \in H^1(\Omega), \qquad C \in L^2(\Omega)$$

and (4), (5)*. The initial functions are assumed to satisfy the compatibility conditions*

$$\partial_\nu n^0(x) = 0, \qquad J^0(x) = 0, \qquad x \in \partial\Omega.$$

Then the problem (1) *with the boundary conditions* (2) *has a local solution* (n, J, V)*, with*

$$\begin{aligned} n &\in H^1((0,t^*),H^1(\Omega))\bigcap L^2((0,t^*),H^3(\Omega)),\\ J &\in H^1((0,t^*),L^2(\Omega))\bigcap L^2((0,t^*),H^2(\Omega)),\\ V &\in H^1((0,t^*),H^3(\Omega))\bigcap L^2((0,t^*),H^5(\Omega)),\\ (n,\nabla n,J) &\in C([0,t^*]\times\overline{\Omega}). \end{aligned}$$

This solution is unique and persists as long as $n(t,\cdot)$ and $J(t,\cdot)$ stay in $L^\infty(\Omega)$ and n remains positive.

Remark 2.6. *The local in time existence in the one-dimensional case can also be proved for solutions of higher regularity provided that the usual compatibility conditions on the initial data are satisfied.*

Theorem 2.7 (Semiclassical limit). *Let Ω be either an open and bounded interval in $\mathbb{R}^1$ or a d-dimensional torus. Suppose that the given data (n^0, J^0, C) of (1) satisfy*

$$n^0 \in H^{b+1}(\Omega), \qquad J^0 \in H^b(\Omega), \qquad C \in H^{b-1}(\Omega)$$

and (4), (5). Let $(n_\varepsilon, J_\varepsilon, V_\varepsilon)$, $\varepsilon > 0$, denote the solutions to (1) with boundary conditions (2) or (3), existing on a domain $[0,t^*]\times\Omega$.

As ε tends to $+0$, a sub-sequence $(n_\varepsilon, J_\varepsilon, V_\varepsilon)_\varepsilon$ then converges to a limit (n, J, V),

$$\begin{aligned}
(n_\varepsilon, J_\varepsilon, \nabla V_\varepsilon) &\longrightarrow (n, J, \nabla V) && \text{in } C([0,t^*]\times\Omega),\\
n_\varepsilon &\rightharpoonup n && \text{in } L^2((0,t^*), H^{b+2}(\Omega)),\\
J_\varepsilon &\rightharpoonup J && \text{in } L^2((0,t^*), H^{b+1}(\Omega)),\\
(n_\varepsilon, J_\varepsilon) &\rightharpoonup^* (n, J) && \text{in } L^\infty((0,t^*), H^b(\Omega)),
\end{aligned}$$

which is a solution to the initial value problem

$$\begin{cases}
\partial_t n - \operatorname{div} J = \nu_0 \triangle n,\\
\partial_t J - \operatorname{div}\left(\dfrac{J\otimes J}{n}\right) - T\nabla n + n\nabla V = \nu_0 \triangle J - \dfrac{J}{\tau},\\
\lambda^2 \triangle V = n - C(x),\\
n(0,x) = n^0(x), \qquad J(0,x) = J^0(x)
\end{cases}$$

with boundary conditions (2) or (3), respectively. The regularity of (n, J, V) is given by

$$\begin{aligned}
&n \in H^1((0,t^*), H^b(\Omega)) \bigcap L^2((0,t^*), H^{b+2}(\Omega)),\\
&J \in H^1((0,t^*), H^{b-1}(\Omega)) \bigcap L^2((0,t^*), H^{b+1}(\Omega)),\\
&V \in H^1((0,t^*), H^{b+2}(\Omega)) \bigcap L^2((0,t^*), H^{b+1}(\Omega)).
\end{aligned}$$

3 Exponential Stability

In this section, we prove Theorem 2.1, following an approach of [6]. Rewrite the energy from (6) as follows:

$$\begin{aligned}
E(t) &= \int_\Omega \left(\frac{\varepsilon^2}{2}(\nabla\sqrt{n})^2 + T\left(n\left(\ln\frac{n}{C_0} - 1\right) + C_0\right) + \frac{\lambda^2}{2}(\nabla V)^2 + \frac{|J|^2}{2n}\right) \mathrm{d}x\\
&= E_1 + E_2 + E_3 + E_4.
\end{aligned}$$

We compute the time derivatives. In the sequel, the zeroes denote boundary integrals that vanish due to the boundary conditions:

$$\begin{aligned}
\partial_t E_1 &= \varepsilon^2 \int_{\partial\Omega} (\partial_\nu \sqrt{n})(\partial_t\sqrt{n})\,\mathrm{d}\sigma - \varepsilon^2 \int_\Omega (\triangle\sqrt{n})(\partial_t\sqrt{n})\,\mathrm{d}x\\
&= 0 - \frac{\varepsilon^2}{2}\int_\Omega B(n)\partial_t n\,\mathrm{d}x = -\frac{\varepsilon^2}{2}\int_\Omega B(n)\left(\operatorname{div} J + \nu_0 \triangle n\right)\,\mathrm{d}x.
\end{aligned}$$

Next, we have

$$
\begin{aligned}
\partial_t E_2 &= T \int_\Omega n_t \ln \frac{n}{C_0} \, \mathrm{d}x = T \int_\Omega (\ln n) \left(\operatorname{div} J + \nu_0 \triangle n\right) \, \mathrm{d}x \\
&= 0 - T \int_\Omega \frac{1}{n} \langle \nabla n, J \rangle \, \mathrm{d}x - T\nu_0 \int_\Omega \frac{|\nabla n|^2}{n} \, \mathrm{d}x \\
&= -T \int_\Omega \frac{1}{n} \langle \nabla n, J \rangle \, \mathrm{d}x - \frac{8T\nu_0}{\varepsilon^2} E_1.
\end{aligned}
$$

Further, we get

$$
\begin{aligned}
\partial_t E_3 &= \lambda^2 \int_\Omega (\nabla V)(\nabla V_t) \, \mathrm{d}x = \lambda^2 \int_{\partial\Omega} V(\partial_\nu V_t) \, \mathrm{d}\sigma - \lambda^2 \int_\Omega V(\triangle V_t) \, \mathrm{d}x \\
&= 0 - \int_\Omega V n_t \, \mathrm{d}x = - \int_\Omega V(\operatorname{div} J + \nu_0 \triangle n) \, \mathrm{d}x \\
&= 0 + \int_\Omega \langle \nabla V, J \rangle \, \mathrm{d}x + \nu_0 \int_\Omega \langle \nabla V, \nabla n \rangle \, \mathrm{d}x.
\end{aligned}
$$

Finally, the identity

$$
\begin{aligned}
\partial_t \left(\frac{1}{2} n^{-1} |J|^2 \right) &= \nu_0 \left(\frac{1}{n} \langle J, \triangle J \rangle - \frac{1}{2n^2} (\triangle n) |J|^2 \right) + \operatorname{div} \left(\frac{1}{2n^2} J |J|^2 \right) \\
&\quad - \frac{1}{n\tau} |J|^2 + T \frac{1}{n} \langle \nabla n, J \rangle - \langle \nabla V, J \rangle - \frac{\varepsilon^2}{2} \left(\operatorname{div}(BJ) - B \operatorname{div} J \right)
\end{aligned}
$$

implies

$$
\begin{aligned}
\partial_t E_4 &= \nu_0 \int_\Omega \left(\frac{1}{n} \langle J, \triangle J \rangle - \frac{1}{2n^2} (\triangle n) |J|^2 \right) \, \mathrm{d}x + \int_{\partial\Omega} \frac{1}{2n^2} J |J|^2 \, \mathrm{d}\vec{\sigma} - \frac{2}{\tau} E_4 \\
&\quad + T \int_\Omega \frac{1}{n} \langle \nabla n, J \rangle \, \mathrm{d}x - \int_\Omega \langle \nabla V, J \rangle \, \mathrm{d}x - \frac{\varepsilon^2}{2} \int_{\partial\Omega} BJ \, \mathrm{d}\vec{\sigma} + \frac{\varepsilon^2}{2} \int_\Omega B \operatorname{div} J \, \mathrm{d}x.
\end{aligned}
$$

The two boundary integrals vanish, due to $J = 0$ on $\partial\Omega$. Concerning the first integral, we can deduce, after partial integration, that

$$
\sum_l \int_\Omega \left(\frac{1}{n} J_l \triangle J_l - \frac{1}{2n^2} J_l^2 \triangle n \right) \, \mathrm{d}x = - \sum_l \int_\Omega n \left(\nabla \left(\frac{J_l}{n} \right) \right)^2 \, \mathrm{d}x.
$$

Summing up, we then find

$$
\begin{aligned}
\partial_t E &= -\frac{\varepsilon^2}{2} \nu_0 \int_\Omega (\triangle n) B(n) \, \mathrm{d}x - \frac{8T\nu_0}{\varepsilon^2} E_1 + \nu_0 \int_\Omega \langle \nabla V, \nabla n \rangle \, \mathrm{d}x \\
&\quad - \nu_0 \sum_l \int_\Omega n \left(\nabla \left(\frac{J_l}{n} \right) \right)^2 \, \mathrm{d}x - \frac{2}{\tau} E_4.
\end{aligned}
$$

For the third term, we bring the constance of the doping profile into play:

$$
\begin{aligned}
\int_\Omega \langle \nabla V, \nabla n \rangle \, \mathrm{d}x &= \int_\Omega \langle \nabla V, \nabla (n - C_0) \rangle \, \mathrm{d}x \\
&= \int_{\partial\Omega} (\partial_\nu V)(n - C_0) \, \mathrm{d}\sigma - \int_\Omega (\triangle V)(n - C_0) \, \mathrm{d}x = 0 - \frac{1}{\lambda^2} \int_\Omega (n - C_0)^2 \, \mathrm{d}x, \\
\int_\Omega \langle \nabla V, \nabla n \rangle \, \mathrm{d}x &= -\lambda^2 \int_\Omega (\triangle V)^2 \, \mathrm{d}x \\
&= -\alpha_0 \lambda^2 \int_\Omega (\triangle V)^2 \, \mathrm{d}x - \frac{1 - \alpha_0}{\lambda^2} \int_\Omega (n - C_0)^2 \, \mathrm{d}x.
\end{aligned}
$$

One easily checks that $x(\ln x - 1) + 1 \leq (x-1)^2$, for $x > 0$, which implies

$$C_0^{-1}(n - C_0)^2 \geq n\left(\ln \frac{n}{C_0} - 1\right) + C_0,$$
$$-\frac{1-\alpha_0}{\lambda^2}\int_\Omega (n - C_0)^2 \,\mathrm{d}x \leq -\frac{(1-\alpha_0)C_0}{\lambda^2 T} E_2, \qquad 0 < \alpha_0 < 1.$$

If μ_1 denotes the first positive eigenvalue of $-\triangle$ on Ω with Neumann boundary conditions, then

$$\|\nabla V\|_{L^2}^2 \leq \frac{1}{\mu_1}\|\triangle V\|_{L^2}^2, \qquad \partial_\nu V = 0 \text{ on } \partial\Omega.$$

As a consequence,

$$-\alpha_0\lambda^2 \int_\Omega (\triangle V)^2 \,\mathrm{d}x \leq -\alpha_0\lambda^2\mu_1 \|\nabla V\|_{L^2}^2 = -\alpha_0\mu_1 E_3.$$

Exploiting Proposition A.1, we then find

$$\partial_t E \leq -\frac{8T\nu_0}{\varepsilon^2}E_1 - \frac{(1-\alpha_0)C_0}{\lambda^2 T}E_2 - \alpha_0\mu_1 E_3 - \frac{2}{\tau}E_4$$
$$-\frac{\varepsilon^2}{2}\nu_0 c_{1,d}\int_\Omega (\triangle\sqrt{n})^2\,\mathrm{d}x - \frac{\varepsilon^2}{2}\nu_0 c_{2,d}\int_\Omega \frac{|\nabla n|^4}{n^3}\,\mathrm{d}x - \sum_l \int_\Omega n\left(\nabla\left(\frac{J_l}{n}\right)\right)^2 \mathrm{d}x,$$

where the numbers $c_{1,d}$ and $c_{2,d}$ are as in the theorem.
We choose $\alpha_0 = \frac{C_0}{\mu_1\lambda^2 T + C_0}$, and obtain

$$\partial_t E \leq -\frac{8T\nu_0}{\varepsilon^2}E_1 - \frac{\mu_1 C_0}{\mu_1\lambda^2 T + C_0}(E_2 + E_3) - \frac{2}{\tau}E_4$$
$$-\frac{\varepsilon^2}{2}\nu_0 c_{1,d}\int_\Omega (\triangle\sqrt{n})^2\,\mathrm{d}x - \frac{\varepsilon^2}{2}\nu_0 c_{2,d}\int_\Omega \frac{|\nabla n|^4}{n^3}\,\mathrm{d}x - \sum_l \int_\Omega n\left(\nabla\left(\frac{J_l}{n}\right)\right)^2 \mathrm{d}x.$$

This completes the proof of Theorem 2.1.

Proof of Corollary 2.3. We directly have $E(t) \leq \exp(-\sigma t)E(0)$, which gives us (7) and (12) immediately. Next, it is easy to check that

$$(y-1)^2 \leq y^2(\ln y^2 - 1) + 1, \qquad 0 < y < \infty,$$

which then yields

$$(\sqrt{n(t,x)} - \sqrt{C_0})^2 \leq n(t,x)\left(\ln\frac{n(t,x)}{C_0} - 1\right) + C_0,$$

and (8) follows quickly, as well as (9). The remaining estimates are proved similarly. □

4 Existence on the Torus

The purpose of this section is to prove Theorem 2.4.

To this end, we choose numbers γ with $0 < \gamma < 1$, and consider a family of parabolic initial value problems

$$\begin{cases} \partial_t n - \operatorname{div} J = \nu_0 \triangle n - \gamma \triangle^2 n, \\ \partial_t J - \operatorname{div} \left(\dfrac{J \otimes J}{n} \right) - T\nabla n + n\nabla V + \dfrac{\varepsilon^2}{2} n \nabla \left(\dfrac{\triangle \sqrt{n}}{\sqrt{n}} \right) = \nu_0 \triangle J - \gamma \triangle^2 J - \dfrac{J}{\tau}, \\ \lambda^2 \triangle V = n - C_\gamma(x), \\ n(0,x) = n_\gamma^0(x), \qquad J(0,x) = J_\gamma^0(x), \end{cases} \tag{14}$$

where $(t,x) \in \mathbb{R} \times \Omega$. We assume that the functions C_γ, n_γ^0, and J_γ^0 belong to $C^\infty(\Omega)$, satisfy the compatibility condition

$$\int_\Omega (n_\gamma^0(x) - C_\gamma(x)) \, \mathrm{d}x = 0,$$

and converge to C, n^0, J^0 in Sobolev norms as follows, for γ tends to zero:

$$\begin{aligned} C_\gamma &\longrightarrow C \text{ in } H^{s-1}(\Omega), \\ n_\gamma^0 &\longrightarrow n^0 \text{ in } H^{s+1}(\Omega), \\ J_\gamma^0 &\longrightarrow J^0 \text{ in } H^s(\Omega). \end{aligned}$$

The system (14) is a fourth order nonlinear parabolic system with third order lower terms. It is standard to show that this problem has a unique solution

$$(n_\gamma, J_\gamma, V_\gamma) \in C^\infty([0,t_\gamma) \times \Omega) \times C^\infty([0,t_\gamma) \times \Omega) \times C^\infty([0,t_\gamma) \times \Omega),$$

for some $t_\gamma > 0$. The solution persists as long as n_γ stays positive and $(n_\gamma, J_\gamma, V_\gamma)$ remain bounded. A proof will be sketched in Lemma B.2.

Our approach is as follows:

- shrink the interval $[0, t_\gamma)$ to guarantee some boundedness assumptions on $(n_\gamma, J_\gamma, V_\gamma)$;
- derive uniform in γ *a priori* estimates of the solutions $(n_\gamma, J_\gamma, V_\gamma)$;
- show that t_γ can not go to zero for $\gamma \to 0$;
- prove convergence of a sub-sequence of $(n_\gamma, J_\gamma, V_\gamma)_\gamma$ for $\gamma \to 0$;
- study the limit of that sub-sequence.

Fix a number δ_0 by the conditions

$$0 < \delta_0 < \min_{x \in \Omega} n^0(x), \qquad \max_{x \in \Omega} n^0(x) < \delta_0^{-1}.$$

In the following computations, we always assume that

$$\delta_0 \leq n_\gamma(t,x) \leq \delta_0^{-1}.$$

For a multi-index $\alpha \in \mathbb{N}_0^d$, define

$$n_{\gamma,\alpha} := \partial_x^\alpha n_\gamma, \qquad J_{\gamma,\alpha} := \partial_x^\alpha J_\gamma, \qquad V_{\gamma,\alpha} := \partial_x^\alpha V_\gamma.$$

Then we obtain

$$\begin{aligned}&\partial_t J_{\gamma,\alpha} - \nu_0 \triangle J_{\gamma,\alpha} + \gamma \triangle^2 J_{\gamma,\alpha} + \frac{1}{\tau} J_{\gamma,\alpha} \\ &\quad = \operatorname{div} \partial_x^\alpha \left(\frac{J_\gamma \otimes J_\gamma}{n_\gamma} \right) + T \nabla n_{\gamma,\alpha} - \partial_x^\alpha (n_\gamma \nabla V_\gamma) - \frac{\varepsilon^2}{2} \partial_x^\alpha \left(n_\gamma \nabla B(n_\gamma) \right).\end{aligned}$$

Multiplying this equation with $J_{\gamma,\alpha}$, integrating over Ω, performing partial integration, and taking advantage from the periodic boundary conditions, we find

$$\begin{aligned}&\frac{1}{2} \partial_t \|J_{\gamma,\alpha}\|_{L^2}^2 + \nu_0 \|\nabla J_{\gamma,\alpha}\|_{L^2}^2 + \gamma \|\triangle J_{\gamma,\alpha}\|_{L^2}^2 + \frac{1}{\tau} \|J_{\gamma,\alpha}\|_{L^2}^2 \\ &\quad = \int_\Omega J_{\gamma,\alpha} \operatorname{div} \partial_x^\alpha \left(\frac{J_\gamma \otimes J_\gamma}{n_\gamma} \right) \mathrm{d}x - T \int_\Omega (\operatorname{div} J_{\gamma,\alpha}) n_{\gamma,\alpha} \, \mathrm{d}x \\ &\qquad - \int_\Omega J_{\gamma,\alpha} \partial_x^\alpha \left(n_\gamma \nabla V_\gamma \right) \mathrm{d}x - \frac{\varepsilon^2}{2} \int_\Omega J_{\gamma,\alpha} \partial_x^\alpha \left(n_\gamma \nabla B(n_\gamma) \right) \mathrm{d}x.\end{aligned}$$

The integrals on the right-hand side are treated as follows:

$$\begin{aligned}&\int_\Omega J_{\gamma,\alpha} \operatorname{div} \partial_x^\alpha \left(\frac{J_\gamma \otimes J_\gamma}{n_\gamma} \right) \mathrm{d}x = -\sum_{k,l} \int_\Omega \left(\partial_k J_{\gamma,\alpha,l} \right) \partial_x^\alpha \left(\frac{J_{\gamma,l} J_{\gamma,k}}{n_\gamma} \right) \mathrm{d}x, \\ &-T \int_\Omega (\operatorname{div} J_{\gamma,\alpha}) n_{\gamma,\alpha} \, \mathrm{d}x = -T \int_\Omega \left(\partial_t n_{\gamma,\alpha} - \nu_0 \triangle n_{\gamma,\alpha} + \gamma \triangle^2 n_{\gamma,\alpha} \right) n_{\gamma,\alpha} \, \mathrm{d}x \\ &\quad = -\frac{T}{2} \partial_t \|n_{\gamma,\alpha}\|_{L^2}^2 - T\nu_0 \|\nabla n_{\gamma,\alpha}\|_{L^2}^2 - T\gamma \|\triangle n_{\gamma,\alpha}\|_{L^2}^2, \\ &n_\gamma \nabla B(n_\gamma) = \frac{1}{2} \nabla \triangle n_\gamma - \frac{1}{2} \sum_l \partial_l \left(\frac{(\partial_l n_\gamma) \nabla n_\gamma}{n_\gamma} \right), \\ &-\frac{\varepsilon^2}{2} \int_\Omega J_{\gamma,\alpha} \partial_x^\alpha \left(n_\gamma \nabla B(n_\gamma) \right) \mathrm{d}x \\ &\quad = \frac{\varepsilon^2}{4} \int_\Omega (\operatorname{div} J_{\gamma,\alpha}) \triangle n_{\gamma,\alpha} \, \mathrm{d}x - \frac{\varepsilon^2}{4} \sum_{k,l} \int_\Omega (\partial_l J_{\gamma,\alpha,k}) \partial_x^\alpha \left(\frac{(\partial_l n_\gamma)(\partial_k n_\gamma)}{n_\gamma} \right) \mathrm{d}x \\ &\quad = \frac{\varepsilon^2}{4} \int_\Omega (\partial_t n_{\gamma,\alpha} - \nu_0 \triangle n_{\gamma,\alpha} + \gamma \triangle^2 n_{\gamma,\alpha}) \triangle n_{\gamma,\alpha} \, \mathrm{d}x - \frac{\varepsilon^2}{4} \sum_{k,l} \int_\Omega \dots \, \mathrm{d}x \\ &\quad = -\frac{\varepsilon^2}{8} \partial_t \|\nabla n_{\gamma,\alpha}\|_{L^2}^2 - \frac{\varepsilon^2}{4} \nu_0 \|\triangle n_{\gamma,\alpha}\|_{L^2}^2 - \frac{\varepsilon^2}{4} \gamma \|\nabla \triangle n_{\gamma,\alpha}\|_{L^2}^2 - \frac{\varepsilon^2}{4} \sum_{k,l} \int_\Omega \dots \, \mathrm{d}x.\end{aligned}$$

Then it follows that

$$\begin{aligned}&\frac{T}{2} \partial_t \|n_{\gamma,\alpha}\|_{L^2}^2 + T\nu_0 \|\nabla n_{\gamma,\alpha}\|_{L^2}^2 + T\gamma \|\triangle n_{\gamma,\alpha}\|_{L^2}^2 \\ &\quad + \frac{\varepsilon^2}{8} \partial_t \|\nabla n_{\gamma,\alpha}\|_{L^2}^2 + \frac{\varepsilon^2}{4} \nu_0 \|\triangle n_{\gamma,\alpha}\|_{L^2}^2 + \frac{\varepsilon^2}{4} \gamma \|\nabla \triangle n_{\gamma,\alpha}\|_{L^2}^2 \\ &\quad + \frac{1}{2} \partial_t \|J_{\gamma,\alpha}\|_{L^2}^2 + \nu_0 \|\nabla J_{\gamma,\alpha}\|_{L^2}^2 + \gamma \|\triangle J_{\gamma,\alpha}\|_{L^2}^2 + \frac{1}{\tau} \|J_{\gamma,\alpha}\|_{L^2}^2 \\ &\quad = -\sum_{k,l} \int_\Omega \left(\partial_k J_{\gamma,\alpha,l} \right) \partial_x^\alpha \left(\frac{J_{\gamma,l} J_{\gamma,k}}{n_\gamma} \right) \mathrm{d}x - \int_\Omega J_{\gamma,\alpha} \partial_x^\alpha \left(n_\gamma \nabla V_\gamma \right) \mathrm{d}x \\ &\qquad - \frac{\varepsilon^2}{4} \sum_{k,l} \int_\Omega (\partial_l J_{\gamma,\alpha,k}) \partial_x^\alpha \left(\frac{(\partial_l n_\gamma)(\partial_k n_\gamma)}{n_\gamma} \right) \mathrm{d}x \\ &\quad = I_{1,\alpha} + I_{2,\alpha} + \frac{\varepsilon^2}{4} I_{3,\alpha}.\end{aligned} \tag{15}$$

We define an energy:

$$\mathcal{E}_k(t) = \sum_{|\alpha|=k} \left(\frac{T}{2} \|n_{\gamma,\alpha}\|_{L^2}^2 + \frac{\varepsilon^2}{8} \|\nabla n_{\gamma,\alpha}\|_{L^2}^2 + \frac{1}{2} \|J_{\gamma,\alpha}\|_{L^2}^2 \right), \qquad k \geq 0, \tag{16}$$

$$\mathcal{E}_{0,\dots,k} := \sum_{l=0}^{k} \mathcal{E}_l. \tag{17}$$

This energy is related to Sobolev space norms via

$$\|n_\gamma\|_{L^2}^2 \leq \frac{2}{T}\mathcal{E}_0,$$
$$\sum_{|\alpha|=k} \|n_{\gamma,\alpha}\|_{L^2}^2 \leq \frac{C}{\varepsilon^2}\mathcal{E}_{k-1}, \qquad k \geq 1,$$
$$\|n_\gamma\|_{H^k}^2 \leq C \left(\frac{1}{T} + \frac{1}{\varepsilon^2} \right) \mathcal{E}_{0,\dots,k-1}.$$

The identity (15) then yields

$$\partial_t \mathcal{E}_k + \sum_{|\alpha|=k} \left(T\nu_0 \|\nabla n_{\gamma,\alpha}\|_{L^2}^2 + \frac{\varepsilon^2}{4}\nu_0 \|\triangle n_{\gamma,\alpha}\|_{L^2}^2 + \nu_0 \|\nabla J_{\gamma,\alpha}\|_{L^2}^2 + \frac{1}{\tau} \|J_{\gamma,\alpha}\|_{L^2}^2 \right)$$
$$\leq \sum_{|\alpha|=k} \left(|I_{1,\alpha}| + |I_{2,\alpha}| + \frac{\varepsilon^2}{4}|I_{3,\alpha}| \right).$$

Next we estimate the integrals $I_{1,\alpha}$, $I_{2,\alpha}$, $I_{3,\alpha}$ in terms of $\mathcal{E}_{0,\dots,k}$. The constants C in the following computations may change from one line to another, and can depend on the order of differentiation $k = |\alpha|$, the space dimension d and the lower bound δ_0 of n^0, but are independent of ν_0, γ, ε, τ, and λ. Recall the embedding $H^b(\Omega) \subset L^\infty(\Omega)$. We will make free use of the estimates

$$\|fg\|_{H^k} \leq C \left(\|f\|_{L^\infty} \|g\|_{H^k} + \|f\|_{H^k} \|f\|_{L^\infty} \right), \qquad k \geq 0,$$
$$\|f(u(\cdot))\|_{H^k} \leq C(\|u\|_{L^\infty}) \|u\|_{H^k}, \qquad k \geq 0, \qquad f(0) = 0.$$

Then we can conclude that

$$\begin{aligned}
|I_{1,\alpha}| &\leq \sum_{l,m} \|\partial_m J_{\gamma,\alpha,l}\|_{L^2} \left\|n_\gamma^{-1} J_{\gamma,l} J_{\gamma,m}\right\|_{H^{|\alpha|}} \\
&\leq C \|\nabla J_{\gamma,\alpha}\|_{L^2} \left(\left\|n_\gamma^{-1}\right\|_{H^{|\alpha|}} \|J_\gamma\|_{L^\infty}^2 + \left\|n_\gamma^{-1}\right\|_{L^\infty} \|J_\gamma\|_{H^{|\alpha|}} \|J_\gamma\|_{L^\infty} \right) \\
&\leq C \|\nabla J_{\gamma,\alpha}\|_{L^2} \left(\|n_\gamma\|_{H^{|\alpha|}} \|J_\gamma\|_{L^\infty}^2 + \|J_\gamma\|_{H^{|\alpha|}} \|J_\gamma\|_{L^\infty} \right) \\
&\leq C \|\nabla J_{\gamma,\alpha}\|_{L^2} \left(\left(1 + T^{-1}\right) \mathcal{E}_{0,\dots,|\alpha|} \right)^{\frac{1}{2}} \left(\mathcal{E}_{0,\dots,b} + \sqrt{\mathcal{E}_{0,\dots,b}} \right) \\
&\leq \frac{\nu_0}{4} \|\nabla J_{\gamma,\alpha}\|_{L^2}^2 + \frac{C}{\nu_0} \left(\mathcal{E}_{0,\dots,b}^2 + \mathcal{E}_{0,\dots,b} \right) \left(1 + T^{-1}\right) \mathcal{E}_{0,\dots,|\alpha|}.
\end{aligned}$$

Concerning the second integral, we have

$$\begin{aligned}
|I_{2,\alpha}| &\leq \|J_{\gamma,\alpha}\|_{L^2} \|n_\gamma \nabla V_\gamma\|_{H^{|\alpha|}} \\
&\leq C \|J_{\gamma,\alpha}\|_{L^2} \left(\|n_\gamma\|_{L^\infty} \|\nabla V_\gamma\|_{H^{|\alpha|}} + \|n_\gamma\|_{H^{|\alpha|}} \|\nabla V_\gamma\|_{L^\infty} \right) \\
&\leq C \|J_{\gamma,\alpha}\|_{L^2} \left(\|V_\gamma\|_{H^{|\alpha|+1}} + \|n_\gamma\|_{H^{|\alpha|}} \|V_\gamma\|_{H^{b+1}} \right) \\
&\leq \frac{C}{\lambda^2} \|J_{\gamma,\alpha}\|_{L^2} \left(\|n_\gamma - C_\gamma\|_{H^{|\alpha|-1}} + \|n_\gamma\|_{H^{|\alpha|}} \|n_\gamma - C_\gamma\|_{H^{b-1}} \right)
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{C}{\lambda^2} \left\| J_{\gamma,\alpha} \right\|_{L^2} \left(\left\| n_\gamma \right\|_{H^{|\alpha|}} + \left\| C_\gamma \right\|_{H^{|\alpha|-1}} + \left\| n_\gamma \right\|_{H^{|\alpha|}} \left(\left\| n_\gamma \right\|_{H^{b-1}} + \left\| C_\gamma \right\|_{H^{b-1}} \right) \right) \\
&\leq \frac{C}{\lambda^2} \left(\left\| J_{\gamma,\alpha} \right\|_{L^2}^2 + \left\| n_\gamma \right\|_{H^{|\alpha|}}^2 \left(1 + \left\| C_\gamma \right\|_{H^{b-1}}^2 + \left\| n_\gamma \right\|_{H^{b-1}}^2 \right) + \left\| C_\gamma \right\|_{H^{|\alpha|-1}}^2 \right) \\
&\leq \frac{C}{\lambda^2} \left(\mathcal{E}_{0,\dots,|\alpha|} + T^{-1} \mathcal{E}_{0,\dots,|\alpha|} \left(1 + \left\| C_\gamma \right\|_{H^{b-1}}^2 + T^{-1} \mathcal{E}_{0,\dots,b-1} \right) + \left\| C_\gamma \right\|_{H^{|\alpha|-1}}^2 \right) \\
&= \frac{C}{\lambda^2} \left(1 + T^{-1} + T^{-1} \left\| C_\gamma \right\|_{H^{b-1}}^2 + T^{-2} \mathcal{E}_{0,\dots,b-1} \right) \mathcal{E}_{0,\dots,|\alpha|} + \frac{C}{\lambda^2} \left\| C_\gamma \right\|_{H^{|\alpha|-1}}^2 .
\end{aligned}
$$

Concerning the last integral, we have

$$
\begin{aligned}
|I_{3,\alpha}| &\leq \sum_{k,l} \left\| \partial_l J_{\gamma,\alpha,k} \right\|_{L^2} \left\| n_\gamma^{-1} (\partial_l n_\gamma)(\partial_k n_\gamma) \right\|_{H^{|\alpha|}} \\
&\leq \sum_{k,l} \left\| \partial_l J_{\gamma,\alpha,k} \right\|_{L^2} \left\| (\partial_l \ln n_\gamma)(\partial_k n_\gamma) \right\|_{H^{|\alpha|}} \\
&\leq C \left\| \nabla J_{\gamma,\alpha} \right\|_{L^2} \left(\left\| \ln n_\gamma \right\|_{H^{|\alpha|+1}} \left\| \nabla n_\gamma \right\|_{L^\infty} + \left\| \nabla \ln n_\gamma \right\|_{L^\infty} \left\| n_\gamma \right\|_{H^{|\alpha|+1}} \right) \\
&\leq C \left\| \nabla J_{\gamma,\alpha} \right\|_{L^2} \left\| n_\gamma \right\|_{H^{|\alpha|+1}} \left\| \nabla n_\gamma \right\|_{L^\infty} .
\end{aligned}
$$

Fix a number θ_d with $\theta_d \in (1/2, 1)$ for even d and $\theta_d \in (2/3, 1)$ for odd d. Then we have

$$
\left\| \nabla n_\gamma \right\|_{L^\infty} \leq C \left\| n_\gamma \right\|_{H^{b+1}}^{\theta_d} \left\| n_\gamma \right\|_{L^\infty}^{1-\theta_d} ,
$$

by Lemma B.3. Together with $\left\| n_\gamma \right\|_{L^\infty} \leq \delta_0^{-1}$, we then get the estimate

$$
|I_{3,\alpha}| \leq C \left\| \nabla J_{\gamma,\alpha} \right\|_{L^2} \left\| n_\gamma \right\|_{H^{|\alpha|+1}} \left\| n_\gamma \right\|_{H^{b+1}}^{\theta_d} .
$$

Now we distinguish two cases.

Case 1: $b + 1 \leq |\alpha| + 2$. Then we have the interpolation inequalities

$$
\begin{aligned}
\left\| n_\gamma \right\|_{H^{b+1}} &\leq C \left\| n_\gamma \right\|_{H^{|\alpha|+2}}^{\frac{1}{|\alpha|+2-b}} \left\| n_\gamma \right\|_{H^b}^{\frac{|\alpha|+1-b}{|\alpha|+2-b}} , \\
\left\| n_\gamma \right\|_{H^{|\alpha|+1}} &\leq C \left\| n_\gamma \right\|_{H^{|\alpha|+2}}^{\frac{|\alpha|+1-b}{|\alpha|+2-b}} \left\| n_\gamma \right\|_{H^b}^{\frac{1}{|\alpha|+2-b}} ,
\end{aligned}
$$

which imply

$$
\left\| n_\gamma \right\|_{H^{|\alpha|+1}} \left\| n_\gamma \right\|_{H^{b+1}}^{\theta_d} \leq C \left\| n_\gamma \right\|_{H^{|\alpha|+2}}^{1-\varrho} \left\| n_\gamma \right\|_{H^b}^{\theta_d+\varrho} ,
$$

where we have set $\varrho = \frac{1-\theta_d}{|\alpha|+2-b} \in (0,1)$. Altogether, we get

$$
\begin{aligned}
|I_{3,\alpha}| &\leq C \left\| \nabla J_{\gamma,\alpha} \right\|_{L^2} \left\| n_\gamma \right\|_{H^{|\alpha|+2}}^{1-\varrho} \left\| n_\gamma \right\|_{H^b}^{\theta_d+\varrho} \\
&\leq C \left\| \nabla J_{\gamma,\alpha} \right\|_{L^2} \left(\left\| \triangle n_\gamma \right\|_{H^{|\alpha|}}^2 + \left\| n_\gamma \right\|_{H^{|\alpha|}}^2 \right)^{\frac{1-\varrho}{2}} \left\| n_\gamma \right\|_{H^b}^{\theta_d+\varrho} .
\end{aligned}
$$

If we apply Young's inequality with the exponents 2, $\frac{2}{1-\varrho}$ and $\frac{2}{\varrho}$ to the right-hand side, we deduce that (putting $k = |\alpha|$)

$$\begin{aligned}
\frac{\varepsilon^2}{4}|I_{3,\alpha}| &\le \frac{\sqrt{\nu_0}}{2}\left\|\nabla J_{\gamma,\alpha}\right\|_{L^2}\cdot\left(\frac{\varepsilon^2\nu_0}{8M(d)}(\|\triangle n_\gamma\|^2_{H^k}+\|n_\gamma\|^2_{H^k})\right)^{\frac{1-\varrho}{2}}\times\\
&\qquad\times C\nu_0^{-1+\frac{\varrho}{2}}\varepsilon^{1+\varrho}\left\|n_\gamma\right\|^{\theta_d+\varrho}_{H^b}\\
&\le \frac{\sqrt{\nu_0}}{2}\left\|\nabla J_{\gamma,\alpha}\right\|_{L^2}\cdot\left(\frac{\varepsilon^2\nu_0}{8M(d)}(\|\triangle n_\gamma\|^2_{H^k}+\|n_\gamma\|^2_{H^k})\right)^{\frac{1-\varrho}{2}}\times\\
&\qquad\times C\nu_0^{-1+\frac{\varrho}{2}}\varepsilon^{1-\theta_d}\left(\varepsilon\left\|n_\gamma\right\|_{H^b}\right)^{\theta_d+\varrho}\\
&\le \frac{\nu_0}{8}\left\|\nabla J_{\gamma,\alpha}\right\|^2_{L^2}+\frac{\varepsilon^2\nu_0}{8M(d)}\left(\|\triangle n_\gamma\|^2_{H^k}+\|n_\gamma\|^2_{H^k}\right)\\
&\quad + C\nu_0^{1-\frac{2}{\varrho}}\varepsilon^{\frac{2(1-\theta_d)}{\varrho}}\left(\varepsilon^2\left\|n_\gamma\right\|^2_{H^b}\right)^{\frac{\theta_d+\varrho}{\varrho}}.
\end{aligned}$$

Recalling that $\varrho = \varrho(k) = \frac{1-\theta_d}{k+2-b}$, we see that

$$\begin{aligned}
\frac{\varepsilon^2}{4}|I_{3,\alpha}| &\le \frac{\nu_0}{8}\left\|\nabla J_{\gamma,\alpha}\right\|^2_{L^2}+\frac{\varepsilon^2\nu_0}{8M(d)}\left\|\triangle n_\gamma\right\|^2_{H^k}+C\nu_0\left(1+\varepsilon^2T^{-1}\right)\mathcal{E}_{0,\dots,k-1}\\
&\quad + C\nu_0^{1-\frac{2}{\varrho}}\varepsilon^{2(k+2-b)}\left(\left(1+\varepsilon^2T^{-1}\right)\mathcal{E}_{0,\dots,b-1}\right)^{1+\frac{\theta_d}{\varrho}}.
\end{aligned}$$

Case 2: $b+1 > |\alpha|+2$**.** In this case, we have $\|n_\gamma\|_{H^{|\alpha|+1}} \le \|n_\gamma\|_{H^{b-1}}$. Applying Young's inequality with the exponents 2, $\frac{2}{\theta_d}$ and $\frac{2}{1-\theta_d}$ to the estimate

$$|I_{3,\alpha}| \le C\left\|\nabla J_{\gamma,\alpha}\right\|_{L^2}\left\|n_\gamma\right\|^{\theta_d}_{H^{b+1}}\left\|n_\gamma\right\|_{H^{b-1}},$$

we get

$$\begin{aligned}
\frac{\varepsilon^2}{4}|I_{3,\alpha}| &\le \frac{\sqrt{\nu_0}}{2}\left\|\nabla J_{\gamma,\alpha}\right\|_{L^2}\cdot\left(\varepsilon^2\left\|n_\gamma\right\|^2_{H^{b+1}}\right)^{\frac{\theta_d}{2}}\cdot C\nu_0^{-\frac{1}{2}}\varepsilon^{2-\theta_d}\left\|n_\gamma\right\|_{H^{b-1}}\\
&\le \frac{\nu_0}{8}\left\|\nabla J_{\gamma,\alpha}\right\|^2_{L^2}+C\left(1+\varepsilon^2T^{-1}\right)\mathcal{E}_{0,\dots,b}\\
&\quad + C\left(\nu_0^{-\frac{1}{2}}\varepsilon^{1-\theta_d}\right)^{\frac{2}{1-\theta_d}}\left(\varepsilon^2\left\|n_\gamma\right\|^2_{H^{b-1}}\right)^{\frac{1}{1-\theta_d}}\\
&\le \frac{\nu_0}{8}\left\|\nabla J_{\gamma,\alpha}\right\|^2_{L^2}+C\left(1+\varepsilon^2T^{-1}\right)\mathcal{E}_{0,\dots,b}\\
&\quad + C\nu_0^{-\frac{1}{1-\theta_d}}\varepsilon^2\left(\left(1+\varepsilon^2T^{-1}\right)\mathcal{E}_{0,\dots,b-2}\right)^{\frac{1}{1-\theta_d}}.
\end{aligned}$$

Having now the estimates in both cases, we choose the number $M(d)$ sufficiently large. Then we can conclude that

$$\begin{aligned}
&\partial_t\mathcal{E}_k+\sum_{|\alpha|=k}\left(T\nu_0\left\|\nabla n_{\gamma,\alpha}\right\|^2_{L^2}+\frac{\varepsilon^2}{8}\nu_0\left\|\triangle n_{\gamma,\alpha}\right\|^2_{L^2}+\frac{\nu_0}{2}\left\|\nabla J_{\gamma,\alpha}\right\|^2_{L^2}+\frac{1}{\tau}\left\|J_{\gamma,\alpha}\right\|^2_{L^2}\right)\\
&\le \frac{C}{\nu_0}\left(\mathcal{E}^2_{0,\dots,b}+\mathcal{E}_{0,\dots,b}\right)\left(1+T^{-1}\right)\mathcal{E}_{0,\dots,k}\\
&\quad + \frac{C}{\lambda^2}\left(1+T^{-1}+T^{-1}\left\|C_\gamma\right\|^2_{H^{b-1}}+T^{-2}\mathcal{E}_{0,\dots,b-1}\right)\mathcal{E}_{0,\dots,k}+\frac{C}{\lambda^2}\left\|C_\gamma\right\|^2_{H^{k-1}}\\
&\quad + CR,
\end{aligned}$$

where the remainder term R equals

$$\nu_0 \left(1+\varepsilon^2 T^{-1}\right) \mathcal{E}_{0,\dots,k-1} + \nu_0^{1-\frac{2}{\varrho(k)}} \varepsilon^{2(k+2-b)} \left(\left(1+\varepsilon^2 T^{-1}\right) \mathcal{E}_{0,\dots,b-1}\right)^{1+\frac{\theta_d}{\varrho(k)}}$$

in case of $k \geq b-1$; and for $k < b-1$ we have

$$R = \left(1+\varepsilon^2 T^{-1}\right) \mathcal{E}_{0,\dots,b} + \nu_0^{-\frac{1}{1-\theta_d}} \varepsilon^2 \left(\left(1+\varepsilon^2 T^{-1}\right) \mathcal{E}_{0,\dots,b-2}\right)^{\frac{1}{1-\theta_d}} .$$

For simplicity, we only discuss the case $s = b$. The other cases $s > b$ run similarly. Summing up for $k = 0, \dots, b$, we find the energy estimate

$$\begin{aligned}
&\partial_t \mathcal{E}_{0,\dots,b} + T\nu_0 \left\|\nabla n_\gamma\right\|_{H^b}^2 + \frac{\varepsilon^2}{8}\nu_0 \left\|\triangle n_\gamma\right\|_{H^b}^2 + \frac{\nu_0}{2}\left\|\nabla J_\gamma\right\|_{H^b}^2 + \frac{1}{\tau}\left\|J_\gamma\right\|_{H^b}^2 \\
&\quad \leq C\left(\left(\frac{1}{\nu_0} + \frac{1}{\nu_0 T}\right)\left(\mathcal{E}_{0,\dots,b}^2 + \mathcal{E}_{0,\dots,b}\right) + (1+\nu_0)\left(1+\varepsilon^2 T^{-1}\right)\right. \\
&\qquad\qquad \left. + \frac{1}{\lambda^2} + \frac{1}{\lambda^2 T} + \frac{1}{\lambda^2 T}\left\|C_\gamma\right\|_{H^{b-1}}^2 + \frac{1}{\lambda^2 T^2}\mathcal{E}_{0,\dots,b-1}\right)\mathcal{E}_{0,\dots,b} \\
&\qquad + \frac{C}{\lambda^2}\left\|C_\gamma\right\|_{H^{b-1}}^2 + C\nu_0^{-\frac{1}{1-\theta_d}}\varepsilon^2\left(\left(1+\varepsilon^2 T^{-1}\right)\mathcal{E}_{0,\dots,b-2}\right)^{\frac{1}{1-\theta_d}} \\
&\qquad + C\nu_0^{-\frac{1+\theta_d}{1-\theta_d}}\varepsilon^2\left(\left(1+\varepsilon^2 T^{-1}\right)\mathcal{E}_{0,\dots,b-1}\right)^{\frac{1}{1-\theta_d}} \\
&\qquad + C\nu_0^{-\frac{3+\theta_d}{1-\theta_d}}\varepsilon^4\left(\left(1+\varepsilon^2 T^{-1}\right)\mathcal{E}_{0,\dots,b-1}\right)^{\frac{1+\theta_d}{1-\theta_d}} . \qquad (18)
\end{aligned}$$

Observe that the right-hand side does not depend on γ.

It exists a number $t^* > 0$ such that $\mathcal{E}_{0,\dots,b} \in C^1([0,t^*])$ and $\mathcal{E}_{0,\dots,b}(t) \leq 2\mathcal{E}_{0,\dots,b}(0)$ for $0 \leq t \leq t^*$.

On the left-hand side, we have a term $\left\|\nabla J_\gamma\right\|_{H^b}^2$. Shrinking the interval $[0,t^*]$ if necessary, we can arrange that $t^* \leq 1$ and

$$\left\|\operatorname{div} J_\gamma\right\|_{L^2((0,t^*),H^{b-1})} \leq 1.$$

By Lemma B.1, we can show that

$$\begin{aligned}
&\left\|n_\gamma(t_2,\cdot) - n_\gamma(t_1,\cdot)\right\|_{L^\infty} \\
&\quad \leq C(\Omega,\alpha,\nu_0)|t_2-t_1|^{\frac{\alpha}{2}}\left(\left\|\operatorname{div} J_\gamma\right\|_{L^2([0,t_\gamma],H^{b-1})} + \left\|n_\gamma^0\right\|_{H^{b+1}}\right) \\
&\quad \leq C|t_2-t_1|^{\frac{\alpha}{2}}, \qquad 0 \leq t_1, t_2 \leq t_\gamma,
\end{aligned}$$

where $0 < \alpha < \frac{1}{2}$. Note that the right-hand side does not depend on γ, $0 < \gamma < 1$. From this we can find a lower bound of the time the solution n_γ needs to touch the boundary of the interval $[\delta_0, \delta_0^{-1}]$.

We list the results obtained so far:

We have determined a number $t^* > 0$ with the property that, for each γ with $0 < \gamma < 1$, we have a solution

$$(n_\gamma, J_\gamma, V_\gamma) \in C^\infty\left([0,t^*]\times\Omega\right) \times C^\infty\left([0,t^*]\times\Omega\right) \times C^\infty\left([0,t^*]\times\Omega\right),$$

satisfying $\delta_0 \leq n_\gamma(t,x) \leq \delta_0^{-1}$ for all $(t,x) \in [0,t^*]\times\Omega$.

These functions satisfy the *a priori* estimates

$$\begin{aligned}
\left\|n_\gamma\right\|_{L^\infty((0,t^*),H^{b+1})} &\leq C, \\
\left\|J_\gamma\right\|_{L^\infty((0,t^*),H^b)} &\leq C, \\
\left\|\triangle n_\gamma\right\|_{L^2((0,t^*),H^b)} &\leq C, \\
\left\|\nabla J_\gamma\right\|_{L^2((0,t^*),H^b)} &\leq C.
\end{aligned}$$

These constants C may depend on T, ν_0, ε, λ, and δ_0, but not on γ.

Because the function C_γ belongs to $H^{b-1}(\Omega)$, we have

$$\|\nabla V_\gamma\|_{L^\infty((0,t^*),H^b(\Omega))} \leq C.$$

From the differential equations, we then obtain the uniform in γ estimates on the time derivatives:

$$\|\partial_t n_\gamma\|_{L^\infty((0,t^*),H^{b-4}(\Omega))} + \|\partial_t J_\gamma\|_{L^\infty((0,t^*),H^{b-4}(\Omega))} \leq C.$$

The embedding $H^{b+1}(\Omega) \subset H^b(\Omega)$ is compact. Therefore, the Aubin Lemma [11] implies that a sub-sequence (which we will not relabel) of $(n_\gamma)_\gamma$ converges in the space $C([0,t^*],H^b(\Omega))$ to a limit function n. The sequence $(n_\gamma)_\gamma$ is bounded in $L^2((0,t^*),H^{b+2}(\Omega))$. By interpolation, we get the strong convergences

$$\begin{aligned} n_\gamma &\longrightarrow n \text{ in } C([0,t^*],H^{b+1-\delta}(\Omega)), \qquad \delta > 0, \\ n_\gamma &\longrightarrow n \text{ in } L^2((0,t^*),H^{b+2-\delta}(\Omega)), \qquad \delta > 0. \end{aligned}$$

And we have the weak convergences

$$n_\gamma \rightharpoonup n \text{ in } L^2((0,t^*),H^{b+2}(\Omega)), \qquad n_\gamma \rightharpoonup^* n \text{ in } L^\infty((0,t^*),H^{b+1}(\Omega)).$$

By a similar reasoning, we can show

$$\begin{aligned} J_\gamma &\longrightarrow J && \text{in } C([0,t^*],H^{b-\delta}(\Omega)), \qquad \delta > 0, \\ J_\gamma &\longrightarrow J && \text{in } L^2((0,t^*),H^{b+1-\delta}(\Omega)), \qquad \delta > 0, \\ J_\gamma &\rightharpoonup J && \text{in } L^2((0,t^*),H^{b+1}(\Omega)), \\ J_\gamma &\rightharpoonup^* J && \text{in } L^\infty((0,t^*),H^b(\Omega)). \end{aligned}$$

Especially, we have the uniform convergences

$$(n_\gamma, \nabla n_\gamma, J_\gamma) \longrightarrow (n, \nabla n, J) \text{ in } C(\overline{Q^*}),$$

where we have put $Q^* := (0,t^*) \times \Omega$. In particular, n and J satisfy the initial conditions $n(0,x) = n^0(x)$ and $J(0,x) = J^0(x)$.

The convergence of $(n_\gamma)_\gamma$ yields the convergence of $(\nabla V_\gamma)_\gamma$, too:

$$\nabla V_\gamma \to V \text{ in } C([0,t^*],H^{b+2-\delta}), \qquad \delta > 0.$$

Finally, we show that (n,J,V) is a solution to (1). By the above reasoning, the identity

$$\lambda^2 \triangle V(t,x) = n(t,x) - C(x), \qquad (t,x) \in \overline{Q^*},$$

is obvious.

Take a function $\varphi \in C_0^\infty(Q^*)$. By the usual arguments, we find

$$\iint_{Q^*} \left(-\varphi_t n_\gamma + \nu_0 (\nabla n_\gamma)(\nabla\varphi) + \gamma n_\gamma \triangle^2 \varphi\right) \mathrm{d}x \,\mathrm{d}t = -\iint_{Q^*} J_\gamma \nabla\varphi \,\mathrm{d}x \,\mathrm{d}t.$$

We send γ to zero and find

$$\begin{aligned} &\iint_{Q^*} (-\varphi_t n + \nu_0 (\nabla n)(\nabla\varphi)) \,\mathrm{d}x \,\mathrm{d}t = -\iint_{Q^*} J\nabla\varphi \,\mathrm{d}x \,\mathrm{d}t, \\ &\iint_{Q^*} (-\varphi_t n + \varphi(-\nu_0 \triangle n - \operatorname{div} J)) \,\mathrm{d}x \,\mathrm{d}t = 0. \end{aligned}$$

We conclude that the function n has distributional time derivative $\partial_t n = \nu_0 \triangle n + \operatorname{div} J$.

We study the terms of the J–equation:

$$\begin{aligned}
\operatorname{div}\left(\frac{J_\gamma \otimes J_\gamma}{n_\gamma}\right) &\longrightarrow \operatorname{div}\left(\frac{J \otimes J}{n}\right) && \text{in } L^2((0,t^*), L^2(\Omega)),\\
T\nabla n_\gamma &\longrightarrow T\nabla n && \text{in } C(\overline{Q^*}),\\
n_\gamma \nabla V_\gamma &\longrightarrow n\nabla V && \text{in } C(\overline{Q^*}),\\
n_\gamma \nabla B(n_\gamma) &\rightharpoonup n\nabla B(n) && \text{in } L^2((0,t^*), L^2(\Omega)),\\
\triangle J_\gamma &\rightharpoonup J && \text{in } L^2((0,t^*), L^2(\Omega)).
\end{aligned}$$

Similarly as for n, we can compute the distributional time derivative of J, and we will see that (n, J, V) solve (1).

To complete the proof of Theorem 2.4, we have to check the uniqueness of the solution: let (n^1, J^1, V^1) and (n^2, J^2, V^2) be two solutions with regularity as in (13). Put

$$n_\Delta = n^1 - n^2, \qquad J_\Delta = J^1 - J^2, \qquad V_\Delta = V^1 - V^2.$$

Then we obtain the system

$$\begin{aligned}
&\partial_t n_\Delta - \nu_0 \triangle n_\Delta = \operatorname{div} J_\Delta,\\
&\partial_t J_\Delta - \nu_0 \triangle J_\Delta + \frac{1}{\tau} J_\Delta - T\nabla n_\Delta + \frac{\varepsilon^2}{2}\nabla \triangle n_\Delta = R_1 - R_2,\\
&\qquad R_j = \operatorname{div}\left(\frac{J^j \otimes J^j}{n^j}\right) - n^j \nabla V^j + \frac{1}{2}\sum_l \partial_l \left(\frac{(\partial_l n^j)(\nabla n^j)}{n^j}\right), \qquad j = 1, 2,\\
&\lambda^2 \triangle V_\Delta = n_\Delta,
\end{aligned}$$

with vanishing initial values for n_Δ and J_Δ. Multiplying the second equation with J_Δ, integrating over Ω, and performing partial integration gives

$$\begin{aligned}
&\frac{1}{2}\partial_t \|J_\Delta\|_{L^2}^2 + \nu_0 \|\nabla J_\Delta\|_{L^2}^2 + \frac{1}{\tau}\|J_\Delta\|_{L^2}^2\\
&\qquad + T\int_\Omega n_\Delta \operatorname{div} J_\Delta \,\mathrm{d}x - \frac{\varepsilon^2}{2}\int_\Omega (\triangle n_\Delta) \operatorname{div} J_\Delta \,\mathrm{d}x\\
&\quad = \int_\Omega J_\Delta (R_1 - R_2)\,\mathrm{d}x,\\
&\frac{1}{2}\partial_t \|J_\Delta\|_{L^2}^2 + \nu_0 \|\nabla J_\Delta\|_{L^2}^2 + \frac{1}{\tau}\|J_\Delta\|_{L^2}^2 + \frac{T}{2}\partial_t \|n_\Delta\|_{L^2}^2 + T\nu_0 \|\nabla n_\Delta\|_{L^2}^2\\
&\qquad + \frac{\varepsilon^2}{2}\partial_t \|\nabla n_\Delta\|_{L^2}^2 + \frac{\varepsilon^2}{4}\nu_0 \|\triangle n_\Delta\|_{L^2}^2\\
&\quad = \int_\Omega J_\Delta (R_1 - R_2)\,\mathrm{d}x.
\end{aligned}$$

Now it is standard to estimate

$$\begin{aligned}
\left|\int_\Omega J_\Delta (R_1 - R_2)\,\mathrm{d}x\right| &\le C\left(\left\|J^j\right\|_{L^\infty}, \left\|n^j\right\|_{L^\infty}, \left\|\nabla n^j\right\|_{L^\infty}, \left\|\nabla V^j\right\|_{L^\infty}\right) \times\\
&\times \left(\|\nabla J_\Delta\|_{L^2}\|J_\Delta\|_{L^2} + \|J_\Delta\|_{L^2}\left(\|n_\Delta\|_{L^2} + \|\nabla V_\Delta\|_{L^2}\right) + \|\nabla J_\Delta\|_{L^2}\|n_\Delta\|_{H^1}\right).
\end{aligned}$$

We apply Young's inequality and find

$$\frac{T}{2}\partial_t \|n_\Delta\|_{L^2}^2 + \frac{\varepsilon^2}{2}\partial_t \|\nabla n_\Delta\|_{L^2}^2 + \frac{1}{2}\partial_t \|J_\Delta\|_{L^2}^2 \le C\left(\|J_\Delta\|_{L^2}^2 + \|n_\Delta\|_{H^1}^2\right).$$

An application of Gronwall's lemma then yields $n_\Delta \equiv 0$, $J_\Delta \equiv 0$, which concludes the proof of Theorem 2.4.

5 Existence in One Dimension

In this section, we prove Theorem 2.5.

We consider (1) and its viscous regularization (14). Put $\Omega = (0, L)$. Our goal is to follow the proof of Theorem 2.4 with $s = b = 1$. Therefore, we choose the approximations C_γ, n_γ^0 and J_γ^0 from the proof of Theorem 2.4 in such a way that

$$\begin{aligned} C_\gamma &\longrightarrow C \text{ in } L^2(\Omega), \\ n_\gamma^0 &\longrightarrow n^0 \text{ in } H^2(\Omega), \\ J_\gamma^0 &\longrightarrow J^0 \text{ in } H^1(\Omega), \end{aligned}$$

subject to the boundary conditions

$$\begin{aligned} &\partial_x^j C_\gamma(x) = 0, && x \in \partial\Omega, && j \geq 1, \\ &\partial_x^j n_\gamma^0(x) = 0, && x \in \partial\Omega, && j \geq 1, \\ &\partial_x^j J_\gamma^0(x) = 0, && x \in \partial\Omega, && j = 0, \quad j \geq 2. \end{aligned}$$

Next, we extend these functions to the interval $(-L, L)$ via

$$C_\gamma(-x) := C_\gamma(x), \qquad n_\gamma^0(-x) := n_\gamma^0(x), \qquad J_\gamma^0(-x) := -J_\gamma^0(x),$$

for $x \in (0, L)$. Then we observe that C_γ, n_γ^0, and J_γ^0 satisfy periodic boundary conditions on the interval $\Omega' := (-L, L)$. We construe Ω' as a torus, and have C_γ, n_γ^0, $J_\gamma^0 \in C^\infty(\Omega')$.

We obtain a fourth order nonlinear parabolic system with third order lower terms. It is standard to show that this problem has a unique and smooth local in time solution $(n_\gamma, J_\gamma, V_\gamma)$.

The function

$$(\tilde{n}_\gamma(t,x), \tilde{J}_\gamma(t,x), \tilde{V}_\gamma(t,x)) := (n_\gamma(t,-x), -J_\gamma(t,-x), V_\gamma(t,-x)),$$

defined for $(t, x) \in [0, t_\gamma] \times \Omega'$, is a solution, too. This is the step where we use $d = 1$. Then the uniqueness of the solution implies

$$n_\gamma(t,-x) = n_\gamma(t,x), \qquad J_\gamma(t,-x) = -J_\gamma(t,x), \qquad V_\gamma(t,-x) = V_\gamma(t,x),$$

for $(t, x) \in [0, t_\gamma] \times \Omega'$. Following the proof of Theorem 2.4, we send γ to zero, and have the convergence of a sub-sequence of $(n_\gamma, J_\gamma, V_\gamma)_\gamma$ to a solution (n, J, V) of the system (1) on $[0, t^*] \times \Omega'$. Clearly, the functions n and V must be even, and J must be odd, which guarantees the boundary conditions (2). The uniqueness can be shown in the same way as for Theorem 2.4. Now the proof of Theorem 2.5 is complete.

6 Semiclassical Limit

Finally, we show Theorem 2.7.

We go back to the proof of Theorem 2.4. Integrating (18) over $[0, t^*]$ and choosing t^* small enough, we can arrange that the solutions $(n_{\varepsilon,\gamma}, J_{\varepsilon,\gamma}, V_{\varepsilon,\gamma})$ to (14) fulfil the *a priori* estimate

$$\begin{aligned} \sup_{t\in[0,t^*]} \mathcal{E}_{0,\dots,b}(t) + \int_{t=0}^{t^*} \left(T\nu_0 \left\| \nabla n_{\varepsilon,\gamma} \right\|_{H^b}^2 + \frac{\varepsilon^2}{8} \nu_0 \left\| \triangle\, n_{\varepsilon,\gamma} \right\|_{H^b}^2 + \frac{\nu_0}{2} \left\| \nabla J_{\varepsilon,\gamma} \right\|_{H^b}^2 \right) \mathrm{d}t \\ \leq 2\mathcal{E}_{0,\dots,b}(0). \end{aligned}$$

The number t^* only depends on the initial energy $\mathcal{E}_{0,\dots,b}(0)$, a bound of $\|C_\gamma\|_{H^{b-1}}$, and the constants ν_0, λ, T, ε_0, where ε_0 is an upper bound of ε, $0 < \varepsilon \le \varepsilon_0$. The constant t^* is independent of γ and ε itself. This gives us the possibility to first send γ to zero, and then ε.

We start with the uniform in γ and ε estimates

$$\begin{aligned}&\|n_{\varepsilon,\gamma}\|^2_{C([0,t^*],H^b)} + \varepsilon^2 \|n_{\varepsilon,\gamma}\|^2_{L^\infty((0,t^*),H^{b+1})} + \|J_{\varepsilon,\gamma}\|^2_{L^\infty((0,t^*),H^b)} \\ &\quad + \|n_{\varepsilon,\gamma}\|^2_{L^2((0,t^*),H^{b+1})} + \varepsilon^2 \|n_{\varepsilon,\gamma}\|^2_{L^2((0,t^*),H^{b+2})} + \|J_{\varepsilon,\gamma}\|^2_{L^2((0,t^*),H^{b+1})} \le C.\end{aligned}$$

We know that the limit $(n_\varepsilon, J_\varepsilon, V_\varepsilon)$ solves (1) and satisfies the corresponding inequality

$$\begin{aligned}&\|n_\varepsilon\|^2_{C([0,t^*],H^b)} + \varepsilon^2 \|n_\varepsilon\|^2_{L^\infty((0,t^*),H^{b+1})} + \|J_\varepsilon\|^2_{L^\infty((0,t^*),H^b)} \\ &\quad + \|n_\varepsilon\|^2_{L^2((0,t^*),H^{b+1})} + \varepsilon^2 \|n_\varepsilon\|^2_{L^2((0,t^*),H^{b+2})} + \|J_\varepsilon\|^2_{L^2((0,t^*),H^{b+1})} \le C.\end{aligned}$$

Making use of $n^0 \in H^{b+1}(\Omega)$ and the maximal regularity property of the parabolic operator $\partial_t - \nu_0 \triangle$, we even get

$$\|n_\varepsilon\|_{L^2((0,t^*),H^{b+2}(\Omega))} \le C.$$

Moreover, the functions n_ε are bounded from below, $n_\varepsilon(t,x) \ge \delta_0 > 0$ for $(t,x) \in [0,t^*] \times \Omega$.

A careful analysis of the differential equations for n_ε and J_ε reveals the uniform in ε bounds

$$\|\partial_t n_\varepsilon\|_{L^\infty((0,t^*),H^{b-2}(\Omega))} + \|\partial_t J_\varepsilon\|_{L^2((0,t^*),H^{b-2}(\Omega))} \le C.$$

We can apply Aubin's Lemma (Corollary 4 in [11]), and find a converging sub-sequence

$$(n_\varepsilon, J_\varepsilon) \longrightarrow (n, J) \text{ in } C([0,t^*] \times \Omega) \text{ and in } C([0,t^*], H^{b-\delta}(\Omega)), \qquad \delta > 0.$$

A direct consequence then is

$$\nabla V_\varepsilon \to \nabla V \text{ in } C([0,t^*], H^b(\Omega)).$$

Additionally, we have the weak convergences

$$\begin{aligned}&n_\varepsilon \rightharpoonup n \text{ in } L^2((0,t^*), H^{b+2}(\Omega)), \qquad\qquad J_\varepsilon \rightharpoonup J \text{ in } L^2((0,t^*), H^{b+1}(\Omega)), \\ &(n_\varepsilon, J_\varepsilon) \rightharpoonup^* (n, J) \text{ in } L^\infty((0,t^*), H^b(\Omega)).\end{aligned}$$

Now fix $Q^* := (0,t^*) \times \Omega$ and choose a test function $\varphi \in C_0^\infty(Q^*)$. Then we have

$$\iint_{Q^*} (-\varphi_t n_\varepsilon - \nu_0 \varphi \triangle n_\varepsilon - \varphi \operatorname{div} J_\varepsilon) \,\mathrm{d}x\,\mathrm{d}t = 0.$$

Sending ε to $+0$ we get $n_t - \nu_0 \triangle n = \operatorname{div} J$ with distributional derivatives. This equation then gives us

$$\partial_t n \in L^2((0,t^*), H^b(\Omega)).$$

Next, after choosing a $\mathbb{R}^d$-valued test function $\varphi \in C_0^\infty(Q^*)$, we can write

$$\begin{aligned}&\iint_{Q^*} \left(-\varphi_t J_\varepsilon + \frac{J_\varepsilon \otimes J_\varepsilon}{n_\varepsilon} \nabla\varphi + (\operatorname{div}\varphi) T n_\varepsilon + \varphi n_\varepsilon \nabla V_\varepsilon \right) \mathrm{d}x\,\mathrm{d}t \\ &= \iint_\Omega \left(-\frac{\varepsilon^2}{2} (\triangle \varphi)(\nabla n_\varepsilon) + \frac{\varepsilon^2}{2} \sum_l (\partial_l \varphi) \frac{(\partial_l n_\varepsilon)(\nabla n_\varepsilon)}{n_\varepsilon} + \nu_0 \varphi \triangle J_\varepsilon - \frac{1}{\tau} \varphi J_\varepsilon \right) \mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Observing

$$\left| \iint_\Omega \sum_l (\partial_l \varphi) \frac{(\partial_l n_\varepsilon)(\nabla n_\varepsilon)}{n_\varepsilon} \,\mathrm{d}x\,\mathrm{d}t \right| \le \frac{C}{\delta_0} \|\nabla\varphi\|_{L^\infty(Q^*)} \|\nabla n_\varepsilon\|^2_{L^2(Q^*)},$$

we can send ε to $+0$, and it follows that

$$\iint_{Q^*} \left(-\varphi_t J + \varphi \left(-\operatorname{div} \left(\frac{J \otimes J}{n} \right) - T\nabla n + n\nabla V - \nu_0 \triangle J + \frac{1}{\tau} J \right) \right) \mathrm{d}x \, \mathrm{d}t = 0.$$

We then deduce that $(\partial_t - \nu_0 \triangle)J = R$ with some $R \in L^2((0,t^*), H^b(\Omega)) \cap L^\infty((0,t^*), H^{b-1}(\Omega))$. By a maximal regularity argument, we then have

$$\partial_t J \in L^2((0,t^*), H^{b-1}(\Omega)).$$

A An Estimate of the Bohm Potential

The main result of this section is the following estimate.

Proposition A.1. *Let $d = 1, 2$ or 3. Assume that the domain $\Omega \subset \mathbb{R}^d$ is a bounded box, $\Omega = \prod_{j=1}^d (a_j, b_j)$. We assume that a function $n \in H^2(\Omega)$ satisfies the following conditions:*

$$\inf_{x\in\Omega} n(x) > 0, \tag{19}$$

$$\partial_\nu n(x) = 0, \qquad x \in \partial\Omega. \tag{20}$$

Then we have the estimates from above

$$\int_\Omega B(n) \triangle n \, \mathrm{d}x \le \frac{5}{2} \int_\Omega (\triangle \sqrt{n})^2 \, \mathrm{d}x + \frac{1}{8} \int_\Omega \frac{|\nabla n|^4}{n^3} \, \mathrm{d}x$$

in all dimensions, and the estimates from below

$$\int_\Omega B(n) \triangle n \, \mathrm{d}x \ge \begin{cases} \dfrac{1}{9} \displaystyle\int_\Omega (\triangle \sqrt{n})^2 \, \mathrm{d}x + \dfrac{7}{144} \displaystyle\int_\Omega \frac{|\nabla n|^4}{n^3} \, \mathrm{d}x & : d = 3, \\ \dfrac{1}{3} \displaystyle\int_\Omega (\triangle \sqrt{n})^2 \, \mathrm{d}x + \dfrac{1}{24} \displaystyle\int_\Omega \frac{|\nabla n|^4}{n^3} \, \mathrm{d}x & : d = 2. \end{cases} \tag{21}$$

In case of $d = 1$, we have

$$\int_\Omega B(n) \triangle n \, \mathrm{d}x = 2 \int_\Omega (\triangle \sqrt{n})^2 \, \mathrm{d}x + \frac{1}{24} \int_\Omega \frac{|\nabla n|^4}{n^3} \, \mathrm{d}x.$$

Proof. The estimate from above is a direct consequence of Young's inequality and

$$B(n) \triangle n = 2(\triangle \sqrt{n})^2 + \frac{\triangle \sqrt{n}}{\sqrt{n}} \cdot 2|\nabla \sqrt{n}|^2.$$

The statement in case of $d = 1$ follows by partial integration.

We start the proof of (21) with some observations: Due to the embedding $H^2(\Omega) \subset C(\overline{\Omega})$ for $d \le 3$, the condition (19) is meaningful. Every function n from $H^2(\Omega)$ satisfying (19) and (20) can be approximated by a sequence $(n_\gamma)_\gamma$ of functions $n_\gamma \in H^3(\Omega)$ that satisfy $\partial_\nu n_\gamma(x) = 0$ on $\partial\Omega$ and $\inf_{x\in\Omega} n_\gamma(x) \ge \frac{1}{2} \inf_{x\in\Omega} n(x)$. The both sides of (21) are continuous mappings from the positive cone of $H^2(\Omega)$ into $\mathbb{R}$. Therefore, we can additionally assume that $n \in H^3(\Omega)$.

Moreover, we will need the following fact: if $p \in H^3(\Omega)$ with $\partial_\nu p = 0$ on $\partial\Omega$, then also $\partial_\nu |\nabla p|^2 = 0$ on $\partial\Omega$. This is the place where we need the assumption that Ω is a box.

Put $p(x) := n^{1/4}(x)$. The assumptions (19) and (20) guarantee $p \in H^3(\Omega)$ and $\partial_\nu p = 0$ on $\partial\Omega$. Then we have

$$B(n) \triangle n = \frac{\triangle(n^{1/2})}{n^{1/2}} \triangle n = 8 \left(p^2(\triangle p)^2 + 3|\nabla p|^4 + 4p(\triangle p)|\nabla p|^2 \right).$$

The last term is the delicate one, since its sign is unknown.

To obtain the estimates from below, we perform partial integration repeatedly:

$$\begin{aligned}
\int_\Omega p(\triangle p)|\nabla p|^2 \,\mathrm{d}x &= -\int_\Omega |\nabla p|^4 \,\mathrm{d}x + \frac{1}{2}\int_\Omega \left(\triangle(p^2)\right)|\nabla p|^2 \,\mathrm{d}x \\
&= -\int_\Omega |\nabla p|^4 \,\mathrm{d}x + \frac{1}{2}\int_\Omega p^2 \triangle |\nabla p|^2 \,\mathrm{d}x + \frac{1}{2}\int_{\partial\Omega} \left((\partial_\nu p^2)|\nabla p|^2 - p^2 \partial_\nu |\nabla p|^2\right) \mathrm{d}\sigma \\
&= -\int_\Omega |\nabla p|^4 \,\mathrm{d}x + \frac{1}{2}\sum_{j,k}\int_\Omega p^2(\partial_k\partial_k((\partial_j p)^2)) \,\mathrm{d}x + 0 \\
&= -\int_\Omega |\nabla p|^4 \,\mathrm{d}x + \sum_{j,k}\int_\Omega p^2(\partial_k\partial_j p)^2 \,\mathrm{d}x + \frac{1}{3}\int_\Omega \left\langle \nabla p^3, \nabla \triangle p \right\rangle \mathrm{d}x \\
&= -\int_\Omega |\nabla p|^4 \,\mathrm{d}x + \sum_{j,k}\int_\Omega p^2(\partial_k\partial_j p)^2 \,\mathrm{d}x \\
&\quad + \frac{1}{3}\int_{\partial\Omega} (\partial_\nu p^3)(\triangle p) \,\mathrm{d}\sigma - \frac{1}{3}\int_\Omega (\triangle p^3)(\triangle p) \,\mathrm{d}x \\
&= -\int_\Omega |\nabla p|^4 \,\mathrm{d}x + \sum_{j,k}\int_\Omega p^2(\partial_k\partial_j p)^2 \,\mathrm{d}x \\
&\quad + 0 - \int_\Omega p^2(\triangle p)^2 \,\mathrm{d}x - 2\int_\Omega p(\triangle p)|\nabla p|^2 \,\mathrm{d}x.
\end{aligned}$$

The last integral is the same as on the left-hand side. Plugging the resulting expression into the integral $\int B(n) \triangle n \,\mathrm{d}x$, we then find

$$\int_\Omega B(n) \triangle n \,\mathrm{d}x = \frac{8}{3}\int_\Omega \left(4\sum_{j,k} p^2(\partial_j\partial_k p)^2 - p^2(\triangle p)^2 + 5|\nabla p|^4 \right) \mathrm{d}x.$$

Using Young's inequality $2|ab| \le a^2 + b^2$, we get the estimate $4\sum_{j,k}(\partial_j\partial_k p)^2 - (\triangle p)^2 \ge \frac{1}{3}(\triangle p)^2$ in case of $d = 3$, and, consequently,

$$\int_\Omega B(n) \triangle n \,\mathrm{d}x \ge \frac{8}{9}\int_\Omega \left(p^2(\triangle p)^2 + 15|\nabla p|^4 \right) \mathrm{d}x.$$

Now we have

$$(\triangle p^2)^2 = 4p^2(\triangle p)^2 + 4|\nabla p|^4 + 8p(\triangle p)|\nabla p|^2 \le 8p^2(\triangle p)^2 + 8|\nabla p|^4,$$

which gives

$$\begin{aligned}
\int_\Omega B(n) \triangle n \,\mathrm{d}x &\ge \frac{1}{9}\int_\Omega (\triangle p^2)^2 \,\mathrm{d}x + \frac{14 \cdot 8}{9}\int_\Omega |\nabla p|^4 \,\mathrm{d}x \\
&= \frac{1}{9}\int_\Omega (\triangle \sqrt{n})^2 \,\mathrm{d}x + \frac{7}{144}\int_\Omega \frac{|\nabla n|^4}{n^3} \,\mathrm{d}x.
\end{aligned}$$

This proof works also for $d = 2$, and the constants can be improved a bit. □

B Some Technicalities

Let $\Omega = \mathbb{T}^d$ be a torus, and consider the problem

$$\begin{cases} \partial_t u - \nu_0 \triangle u + \gamma \triangle^2 u = f, & (t,x) \in (0,T) \times \Omega, \\ u(0,x) = u_0(x), & x \in \Omega, \end{cases}$$

where $\nu_0 > 0$ and $\gamma \geq 0$.

Lemma B.1. *Any smooth solution to this initial value problem satisfies the estimate*

$$\begin{aligned} &\|u(t_2) - u(t_1)\|_{L^\infty(\Omega)} \\ &\quad \leq C(\Omega, \alpha, \nu_0, T) |t_2 - t_1|^{\frac{\alpha}{2}} \left(\|f\|_{L^2((0,T),H^{b-1}(\Omega))} + \|u_0\|_{H^b(\Omega)} + \gamma^{\frac{1}{2}} \|u_0\|_{H^{b+1}(\Omega)} \right) \end{aligned}$$

uniformly in γ, where $0 \leq t_1 < t_2 \leq T$, $0 < \alpha < \frac{1}{2}$, and the constant C remains bounded for $T \to 0$.

Proof. Apply ∇ to this equation, multiply with ∇u, and integrate over Ω:

$$\begin{aligned} \frac{1}{2} \partial_t \|\nabla u\|^2_{L^2(\Omega)} + \nu_0 \sum_l \int_\Omega |\nabla \partial_l u|^2 \, dx + \gamma \|\nabla \triangle u\|^2_{L^2(\Omega)} &= - \int_\Omega f \triangle u \, dx \\ &\leq \frac{1}{2} \left(\frac{d}{\nu_0} \|f\|^2_{L^2(\Omega)} + \nu_0 \sum_l \int_\Omega (\partial_l^2 u)^2 \, dx \right). \end{aligned}$$

If ∇^2 denotes the matrix of all second derivatives, then we obtain the estimates

$$\begin{aligned} \|\nabla u\|^2_{L^\infty((0,T),L^2(\Omega))} &\leq \|\nabla u_0\|^2_{L^2(\Omega)} + \frac{d}{2\nu_0} \|f\|^2_{L^2((0,T),L^2(\Omega))}, \\ \left\|\nabla^2 u\right\|^2_{L^2((0,T),L^2(\Omega))} &\leq \frac{1}{\nu_0} \|\nabla u_0\|^2_{L^2(\Omega)} + \frac{d}{2\nu_0^2} \|f\|^2_{L^2((0,T),L^2(\Omega))}. \end{aligned}$$

The key information here is that the constants do neither depend on T nor on γ.

Apply $\triangle$ to the equation, multiply with $\triangle u$, integrate over Ω, perform partial integration, apply Young's inequality to the right–hand side, integrate over the time interval:

$$\begin{aligned} &\|\triangle u\|^2_{L^\infty((0,T),L^2(\Omega))} + 2\nu_0 \|\nabla \triangle u\|^2_{L^2((0,T),L^2(\Omega))} + \gamma \left\|\triangle^2 u\right\|^2_{L^2((0,T),L^2(\Omega))} \\ &\quad \leq \|\triangle u_0\|^2_{L^2(\Omega)} + \frac{1}{\gamma} \|f\|^2_{L^2((0,T),L^2(\Omega))}. \end{aligned}$$

Finally, using the equation we get

$$\begin{aligned} &\|\partial_t u\|^2_{L^2((0,T),L^2(\Omega))} \\ &\quad \leq 3\nu_0^2 \|\triangle u\|^2_{L^2((0,T),L^2(\Omega))} + 3 \left\|\gamma \triangle^2 u\right\|^2_{L^2((0,T),L^2(\Omega))} + 3 \|f\|^2_{L^2((0,T),L^2(\Omega))} \\ &\quad \leq C(d) \left(\nu_0 \|\nabla u_0\|^2_{L^2(\Omega)} + \gamma \|\triangle u_0\|^2_{L^2(\Omega)} + \|f\|^2_{L^2((0,T),L^2(\Omega))} \right). \end{aligned}$$

The differential equation and the boundary condition do not change after differentiating the equation with respect to x. Therefore, we are allowed to replace $L^2(\Omega)$ everywhere by $H^m(\Omega)$ in the above four estimates.

Assume $0 \le t_1 < t_2 \le T$. Then we have

$$\begin{aligned}
&\|u(t_2) - u(t_1)\|_{L^\infty(\Omega)} \le C \|u(t_2) - u(t_1)\|_{H^{b-\alpha}(\Omega)} \\
&\quad \le C \|u\|^{1-\alpha}_{L^\infty((0,T),H^b(\Omega))} \|u(t_2) - u(t_1)\|^{\alpha}_{H^{b-1}(\Omega)} \\
&\quad \le C \|u\|^{1-\alpha}_{L^\infty((0,T),H^b(\Omega))} \left(\int_{t=t_1}^{t_2} \|\partial_t u\|_{H^{b-1}(\Omega)} \,\mathrm{d}t \right)^{\alpha} \\
&\quad \le C|t_2 - t_1|^{\frac{\alpha}{2}} \|u\|^{1-\alpha}_{L^\infty((0,T),H^b(\Omega))} \|\partial_t u\|^{\alpha}_{L^2((0,T),H^{b-1}(\Omega))} \\
&\quad \le C(\Omega, \alpha, \nu_0, T)|t_2 - t_1|^{\frac{\alpha}{2}} \left(\|f\|_{L^2((0,T),H^{b-1}(\Omega))} + \|u_0\|_{H^b(\Omega)} + \gamma^{\frac{1}{2}} \|u_0\|_{H^{b+1}(\Omega)} \right).
\end{aligned}$$

Here, we have made use of the standard estimate

$$\|u\|^2_{L^\infty((0,T),L^2(\Omega))} \le 2 \left(\|u_0\|^2_{L^2(\Omega)} + T \|f\|^2_{L^2((0,T),L^2(\Omega))} \right).$$

□

Next, we consider fourth-order parabolic systems with nonlocal nonlinear lower order terms

$$\begin{cases} \partial_t Y(t,x) + \gamma \triangle^2 Y(t,x) = F(\{Y\}), & (t,x) \in (0,T) \times \Omega, \\ Y(0,x) = Y_0(x), & x \in \Omega, \end{cases} \tag{22}$$

where Ω is a smooth d-dimensional manifold without boundary and F comprises (local or nonlocal) nonlinear terms of at most third order.

Lemma B.2. *Assume that $Y_0 \in C^\infty(\Omega)$, and that F is defined for functions $Y \in C^\infty(\Omega)$ taking values in a tubular neighbourhood of the graph of Y_0, and satisfies estimates*

$$\begin{aligned}
&\|F(\{Y\})\|_{H^k} \le C_{F,k}(\|Y\|_{L^\infty})(\|Y\|_{H^{k+3}} + \|Y\|_{L^\infty}), \qquad k \ge 0, \\
&\|F(\{Y\}) - F(\{Z\})\|_{L^2} \le C_{F,0} \left(\|Y\|_{L^\infty} + \|Y\|_{H^3} + \|Y\|_{L^\infty} + \|Z\|_{H^3}\right) \|Y - Z\|_{H^3},
\end{aligned}$$

with some continuous and increasing functions $C_{F,k}$.

Then there is a constant $T_* > 0$ such that (22) has a unique solution $Y \in C^\infty([0,T_*] \times \Omega)$.

Proof. Consider first the linear case

$$\partial_t Y(t,x) + \gamma \triangle^2 Y(t,x) = G(t,x), \qquad Y(0,x) = Y_0(x). \tag{23}$$

Multiplying (23) with Y and integrating the resulting equation over $[0,T] \times \Omega$ we quickly get

$$\|Y\|^2_{L^\infty((0,T),L^2(\Omega))} \le 2 \left(\|Y_0\|^2_{L^2(\Omega)} + T \|G\|^2_{L^2((0,T),L^2(\Omega))} \right), \tag{24}$$

$$\|Y\|^2_{L^2((0,T),L^2(\Omega))} \le 2 \left(T \|Y_0\|^2_{L^2(\Omega)} + T^2 \|G\|^2_{L^2((0,T),L^2(\Omega))} \right). \tag{25}$$

Applying $\triangle$ to (23), multiplying with $\triangle Y$ and integrating over Ω, we see

$$\frac{1}{2}\partial_t \|\triangle Y\|^2_{L^2} + \gamma \left\|\triangle^2 Y\right\|^2_{L^2} = \int_\Omega (\triangle G)(\triangle Y) \,\mathrm{d}x \le \frac{\gamma}{2} \left\|\triangle^2 Y\right\|^2_{L^2} + \frac{1}{2\gamma} \|G\|^2_{L^2},$$

$$\|\triangle Y\|^2_{L^\infty((0,T),L^2(\Omega))} + \gamma \left\|\triangle^2 Y\right\|^2_{L^2((0,T),L^2(\Omega))} \le \|\triangle Y_0\|^2_{L^2(\Omega)} + \frac{1}{\gamma} \|G\|^2_{L^2((0,T),L^2(\Omega))}. \tag{26}$$

We interpolate (26) with (24) and with (25), and conclude that

$$\|Y\|^2_{L^\infty((0,T),H^1(\Omega))} + \|Y\|^2_{L^2((0,T),H^3(\Omega))} \le C_0 \left(\|Y_0\|^2_{H^2(\Omega)} + T^{1/2} \|G\|_{L^2((0,T),L^2(\Omega))} \right),$$

with $C_0 = C_0(\gamma, \Omega, T_0)$, $0 < T \leq T_0$, which can be lifted to

$$\|Y\|^2_{L^\infty((0,T),H^{k+1}(\Omega))} + \|Y\|^2_{L^2((0,T),H^{k+3}(\Omega))} \\ \leq C_k \left(\|Y_0\|^2_{H^{k+2}(\Omega)} + T^{1/2} \|G\|_{L^2((0,T),H^k(\Omega))} \right).$$

For b being the smallest integer greater than $d/2$, we choose $k = \max(b-1, 2)$ and find an estimate for Y in the spaces $L^\infty((0,T) \times \Omega)$ and $L^\infty((0,T), H^3(\Omega))$.

Now let us be given the composition operator F. For a moment we assume F to be defined everywhere. Now it is standard to construct an iteration scheme of Picard-Lindelöf type, to exploit the above estimates of solutions to (23), and to show that this iteration scheme converges in $L^2((0,T_*), H^3(\Omega))$ to a solution Y provided that T_* is chosen small enough. The maximal regularity of the system (23) as expressed in (26) then shows $Y \in C^\infty([0,T_*] \times \Omega)$. We can also obtain an estimate of $\partial_t Y$ in $L^\infty((0,T_*) \times \Omega)$.

Next, let us be given the composition operator F, defined in a tubular neighbourhood of the graph of Y_0. We can extend F outside this neighbourhood and then follow the above proof. The estimate on $\partial_t Y$ guarantees that the found solution indeed solves the problem (22) provided that the time interval is short. □

For completeness, we give a proof of an interpolation estimate exploited in the proof of Theorem 2.4.

Lemma B.3. *Let $d \in \mathbb{N}_+$ and b be the smallest integer greater than $\frac{1}{2}d$, and Ω be a bounded domain in $\mathbb{R}^d$ with smooth boundary or a bounded d-dimensional manifold. Put*

$$\theta_d = \left(b + 1 - \frac{d}{2}\right)^{-1} + \kappa = \kappa + \begin{cases} \frac{1}{2} & : d \text{ even}, \\ \frac{2}{3} & : d \text{ odd} \end{cases}$$

with $\kappa > 0$ and $\theta_d < 1$. Then the following interpolation inequality holds:

$$\|\nabla u\|_{L^\infty(\Omega)} \leq C \|u\|^{\theta_d}_{H^{b+1}(\Omega)} \|u\|^{1-\theta_d}_{L^\infty(\Omega)}.$$

Proof. Let $r \gg 1$ be huge. We have $H^{b+1}(\Omega) = F^{b+1}_{2,2}(\Omega)$ and $L^r(\Omega) = F^0_{r,2}(\Omega)$ as well as the complex interpolation

$$\left[F^{s_1}_{p_1,2}, F^{s_2}_{p_2,2}\right]_\theta = F^{s_*}_{p_*,2}, \qquad s_* = \theta s_1 + (1-\theta)s_2, \qquad \frac{1}{p_*} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$$

provided that $1 < p_1, p_2 < \infty$ and $0 \leq \theta \leq 1$. Details on the Triebel-Lizorkin spaces $F^s_{p,q}$ can be found in [12]. In our case, $s_* = \theta_d s_1 + (1-\theta_d)s_2 = (b+1)\theta_d$ and $1/p_* = \theta_d/2 + (1-\theta_d)/r$. It is easy to check that $(b+1)\theta_d - 1 > d\theta_d/2$, hence $s_* - 1 > d/p_*$ for large r, which gives us the embedding $F^{s_*}_{p_*,2} \subset C^1(\overline{\Omega})$. Since Ω is bounded, we also have $L^r(\Omega) \subset L^\infty(\Omega)$, which completes the proof. □

Acknowledgements

Both authors have been supported by the Deutsche Forschungsgemeinschaft (DFG), grant number 446 CHV 113/170, and the National Natural Science Foundation of China (NSFC), grant number 10401019. We express our gratitude to Tsinghua University of Beijing and Konstanz University for hospitality; and thank Robert Denk for helpful discussions.

References

[1] J. Carrillo, A. Jüngel, P. Markowich, G. Toscani, and A. Unterreiter. Entropy dissipation methods for degenerate parabolic problems and generalized Sobolev inequalities. *Monatsh. Math.*, 133(1):1–82, 2001.

[2] L. Chen, X. Chen, and H. Jian. Isentropic quantum drift–diffusion model. preprint, 2006.

[3] L. Chen and Q. Ju. Existence of weak solution and semiclassical limit for quantum drift–diffusion model. to appear in ZAMP, 2006.

[4] P. Degond, F. Méhats, and C. Ringhofer. Quantum energy–transport and drift–diffusion models. *J. Stat. Phys.*, 118:625–667, 2005.

[5] C. L. Gardner. The quantum hydrodynamic model for semiconductor devices. *SIAM J. Appl. Math.*, 54(2):409–427, 1994.

[6] M. Gualdani, A. Jüngel, and G. Toscani. Exponential decay in time of solutions of the viscous quantum hydrodynamic equations. *Appl. Math. Lett.*, 16(8):1273–1278, 2003.

[7] A. Jüngel. *Quasi-hydrodynamic semiconductor equations.* Progress in Nonlinear Differential Equations and their Applications. 41. Basel: Birkhäuser., 2001.

[8] A. Jüngel and H.-L. Li. Quantum Euler–Poisson systems: global existence and exponential decay. *Quart. Appl. Math.*, 62:596–600, 2004.

[9] A. Jüngel and D. Matthes. A derivation of the isothermal quantum hydrodynamic equations using entropy minimization. *Z. Angew. Math. Mech.*, 85:806–814, 2005.

[10] A. Jüngel and R. Pinnau. A positivity-preserving numerical scheme for a nonlinear fourth order parabolic system. *SIAM J. Num. Anal.*, 39:385–406, 2001.

[11] J. Simon. Compact sets in the space $L^p(0,T;B)$. *Ann. Mat. Pura Appl., IV. Ser.*, 146:65–96, 1987.

[12] H. Triebel. *Theory of function spaces II.* Monographs in Mathematics. 84. Basel etc.: Birkhäuser Verlag., 1992.

[13] B. Zhang and J. Jerome. On a steady-state quantum hydrodynamic model for semiconductors. *Nonlin. Anal.*, 26:845–856, 1996.