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Abstract: We consider a hyperbolicly perturbed Navier-Stokes initial value problem in $\mathbb{R}^n$, $n = 2,3$, arising from using a Cattaneo type relation instead of a Fourier type one in the constitutive equations. The resulting system is a hyperbolic one with quasilinear nonlinearities. The global existence of smooth solutions for small data is proved, and relations to the classical Navier-Stokes systems are discussed.

1 Introduction

The classical Navier-Stokes equations in the whole space $\mathbb{R}^n$, $n = 2,3$,

$$u_t - \mu\Delta u + ((u\cdot\nabla)u) + \nabla p = 0 \quad \text{in } (0,\infty)\times\mathbb{R}^n, \tag{1.1}$$

$$\operatorname{div} u = 0 \quad \text{in } (0,\infty)\times\mathbb{R}^n, \tag{1.2}$$

$$u(0,\cdot) = u_0, \qquad u_t(0,\cdot) = u_1 \quad \text{in } \mathbb{R}^n, \tag{1.3}$$

with $\mu > 0$ being the viscosity, for the velocity vector $u = u(t,x) : (0,\infty)\times\mathbb{R}^n \to \mathbb{R}^n$ of a fluid, and $p = p(t,x) : (0,\infty)\times\mathbb{R}^n \to \mathbb{R}$ the related pressure, arise from the transport law

$$u_t + (u\cdot\nabla)u + \nabla p = \operatorname{div} S \tag{1.4}$$

and the constitutive law for the tensor S,

$$S = \frac{\mu}{2}(\nabla u + (\nabla u)'), \tag{1.5}$$

together with the incompressibility (zero divergence) condition (1.2) and initial conditions (1.3). We replace the Fourier type relation (1.5) by the Cattaneo type relation

$$\tau S_t + S = \frac{\mu}{2}(\nabla u + (\nabla u)') \tag{1.6}$$

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cf. the Ortroyd model in (1.11) below. The Fourier type constitutive assumption (1.5) — in addition to the pressure contribution — leads to the well-known parabolic type classical Navier-Stokes system (1.1)–(1.3); there, in particular, we have the effect of an infinite propagation speed of signals, as it is well-known as modeling problem/paradox for heat equations, or, more generally, for flux type equations (diffusion problems, ...) where the flux relation is given by the Fourier type. There are applications, however, where it is more reasonable to work with hyperbolic models, cf [23] and the references therein. It has also been observed experimentally that there exist hyperbolic heat waves. First, one is naturally led to models with a delayed flux relation

$$S(t+\tau,\cdot) = \frac{1}{2}(\nabla u + (\nabla u)')(t,\cdot), \tag{1.7}$$

with a small (*small* relatively to other physical constants in the system) relaxation parameter $\tau > 0$. The Cattaneo type constitutive law (1.6) can be interpreted as a *formal* Taylor expansion of order one in t.

Remark: *Formal higher-order Taylor expansions may lead to ill-posed problems, cf. the examples by Dreher, Quintanilla and Racke [6].*

Differentiating the transport equation (1.4) with respect to t, and using the new relation (1.6), we obtain the new hyperbolicly perturbed Navier-Stokes system

$$\begin{aligned} \tau u_{tt} - \mu\Delta u + u_t + \nabla p + \tau\nabla p_t &= -(u\cdot\nabla)u - (\tau u_t\cdot\nabla)u - (\tau u\cdot\nabla)u_t && (1.8)\\ &\qquad\qquad\text{in} \quad (0,\infty)\times\mathbb{R}^n, \\ \operatorname{div} u &= 0 \qquad\qquad\qquad \text{in} \quad (0,\infty)\times\mathbb{R}^n, && (1.9)\\ u(0,\cdot) = u_0, \qquad & u_t(0,\cdot) = u_1 \qquad\qquad \text{in} \quad \mathbb{R}^n, && (1.10) \end{aligned}$$

It will turn out that, in this system, at least the vorticity $\nabla\times u$ has finite propagation speed.

The classical Navier-Stokes system (1.1)–(1.3) has been and is widely discussed. The global in time well-posedness is of great interest not only in fluid dynamics and has led over the years to many mathematical contributions, also to still open problems, the most prominent one being the question of global existence of smooth solutions to any (possibly large, smooth) data ("million dollar problem"). Under minimal assumptions on the data u_0, the existence of a weak solution is guaranteed by the results of Leray [14] and Hopf [11]. The uniqueness of (u,p) up to an additive constant for the pressure p is known in two space dimensions, hence also the global existence of large strong solutions, but the uniqueness in three space dimensions remains open in general. The global existence of *small* strong solutions has been proved, see e.g. the books of Ladyzhenskaya [13], Constantin [5], Temam [26], von Wahl [28], and the references therein. For an elementary approach to the classical Navier-Stokes equations we also refer to the monograph [9].

For *large* data, strong global solutions are know to exist only under very restrictive additional assumptions on the data: see Ladyzhenskaya [13] for rotational symmetry, Ukhovskii and Iudovich [27] for axial symmetry, Mahalov, Titi and Leibovich [16] for helical symmetry, or for approximately symmetric data, see Ponce, Racke, Sideris and Titi [21], for large initial data with uniformly large vorticity see [3], [15], for highly oscillating nondecaying large initial data cf. [10]. For further discussions on global solvability we also refer to [4].

Here, we consider the hyperbolic version (1.8)–(1.10). There it not only the hyperbolic character of a wave equation for u, that complicates things by less regularization properties, but we notice the nonlinearities which are — in contrast to the classical case — of highest order, see the term $(\tau u \cdot \nabla)u_t$. In view of known results on global existence for small data or blow-up even for small data, respectively, for wave equations or heat type equations, these quadratic nonlinearities touch the critical borderline. We recall that for nonlinear heat equations — the linearized version of which will show the same decay rates for solutions as solutions to damped wave equations, see below — it is known from Fujita [8], see also Ponce [19], that for quadratic perturbations the nonlinear heat equation in three dimensions has global small solutions. However, this is in general not the case in dimension two. Nevertheless, we will be able to prove the global existence for small data also in two space dimensions since the appearing nonlinearities also have derivatives, and derivatives of solutions to *damped* wave equations have a better decay rate compared to the solution itself — this is the same property as known for heat equations.

Remark: *The equation (1.8) can be regarded as a* damped *wave equation only for* small *values of* u*, since the term* $(\tau u_t \cdot \nabla)u$ *might disturb the positive damping term* u_t *for large* u*. — This is another hint to think about a possible blow-up situation for large data.*

Therefore, we will combine and apply techniques known for nonlinear heat equations, where additional trouble will arise through the Helmholtz projections, see below.

Quadratic nonlinearities are worse in two than in three space dimensions, and a blow-up of solutions cannot yet be excluded. In fact, we have the *conjecture* that strong solutions to the hyperbolic Navier-Stokes system (1.8)–(1.10) in two space dimensions blow up in finite time if the data are sufficiently large.

This conjecture might turn out to be wrong if someone is able to prove the global large well-posedness of large strong solutions — as it is the case for the (simpler) classical Navier-Stokes system. But *if* the conjecture is true, it would have two important consequences:

(i) The comparison to the classical Navier-Stokes system in two dimensions would demonstrate that the modeling of fluid dynamics by the classical system is sensitive versus small changes — predicting global solutions in one model and a blow-up in the other one —, and hence the big question of global large solutions in three dimensions might also be a more mathematical one in the sense that the model might be not appropriate.

(ii) It would be the first *nonlinear* example where a change from the Fourier type law to a Cattaneo type law gives opposite information (global existence versus blow-up). This was known up to now only for linearized equations from the recent study of Timoshenko type systems in the work of Fernández Sare and Racke [7].

On the other hand, the conjecture might be wrong. Then the conclusion is that both models — with Fourier or Cattaneo law, respectively — behave similar (leaving the big question in three dimensions open in both cases). This would fit to observations in different systems in thermoelasticity, where the change from one to the other model does not change the qualitative and even quantitative behavior essentially, see the survey in [23] and the references therein. Nevertheless, the results in [7] show that, a priori, it is not evident that both systems yield the same description, despite the fact that the systems are formally close to each other (τ being small). Our contribution shows on the level of small perturbations of equilibria (small data) that the two systems — classical versus hyperbolic — are comparable.

We remark that our new Navier-Stokes system is related to the Oldroyd model which considers instead of (1.6) the more general model

$$\tau S_t + S = \mu(\mathcal{E} + \nu \mathcal{E}_t). \tag{1.11}$$

where $\mathcal{E} := \frac{1}{2}(\nabla u + (\nabla u)')$, cf. de Araújo, de Menzenes and Marinho [2] and Joseph [12]; in comparison to our model we have $\nu = 0$. If $\nu \neq 0$ then, from the point of derivatives getting involved, S is on a similar level as $\mathcal{E}$, as in the classical case.

In [18], Paicu and Raugel considered a hyperbolic perturbation of the classical Navier-Stokes equations consisting in adding the term τu_{tt} to the equation (1.1). The global well-posedness for mild solutions in two dimensions for sufficiently small τ, and the global existence for small data and sufficiently small τ in three dimensions in analogy to the classical case are proved. Of course, keeping the nonlinearity $(u \cdot \nabla)u$ is essential there and cannot be compared to our situation with the quasilinear additional nonlinearities in(1.8). In [18], a number of justifications for their model is presented, see the references therein.

In order to prove a global existence theorem for small data, we apply the Leray projector P onto solenoidal fields, in order to eliminate the pressure terms. Once knowing u, one can determine the pressure p by solving the linear problem

$$-\Delta p - \Delta p_t = \mathrm{div}\ \{(u \cdot \nabla)u + (\tau u_t \cdot \nabla)u + (\tau u \cdot \nabla)u_t\}. \tag{1.12}$$

P projects L^2-vector fields onto the divergence free fields,

$$P : (L^2(\mathbb{R}^n))^n \longrightarrow L^2_\sigma(\mathbb{R}^n) := \left\{w \in (L^2_\sigma(\mathbb{R}^n))^n : \mathrm{div}\, w = 0\right\}.$$

This leads to the well-known orthogonal decomposition

$$(L^2(\mathbb{R}^n))^n = L^2_\sigma(\mathbb{R}^n) \oplus_\perp G_2(\mathbb{R}^n),$$

where $G_2(\mathbb{R}^n) := \{\nabla v : v \in L^2_{loc}(\mathbb{R}^n),\ \nabla v \in (L^2(\mathbb{R}^n))^n\}$.

Applying the projector P to (1.8) we arrive at the following system involving u only,

$$\tau u_{tt} - \mu \Delta u + u_t = -P((u \cdot \nabla)u) - P((\tau u_t \cdot \nabla)u) - P((\tau u \cdot \nabla)u_t) \quad \text{in} \quad (0,\infty) \times \mathbb{R}^n, \tag{1.13}$$

$$\mathrm{div}\, u = 0 \quad \text{in} \quad (0,\infty) \times \mathbb{R}^n, \tag{1.14}$$

$$u(0,\cdot) = u_0, \qquad u_t(0,\cdot) = u_1 \quad \text{in} \quad \mathbb{R}^n. \tag{1.15}$$

In the following we shall prove the global existence to the problem (1.13)–(1.15); the pressure is then determined by (1.12) (up to constants, as usual).

The Helmholtz projection can be regarded as a continuous operator on any $W^{m,p} \cap L^2$-space, provided $1 < p < \infty$. Since we need to estimate the nonlinearities in particular in Section 4, we have to avoid L^∞- resp. L^1-norms. On the other hand, for the nonlinear problem in question we are — with quadratic nonlinearities involving u and one derivative of u in two space dimensions — at the borderline of possible global existence theorems for small data (cf. the comments above); for these cases, e.g. for nonlinear heat equations, one usually exploits the decay of solutions in L^∞. This has to be avoided here and leads to some additional technical difficulties.

The paper is organized as follows: In Section 2, we will recall the local existence theorem from our paper [24] of ($H^m =$)$W^{m,2}$)-valued solutions. The subsequent remarks on finite propagation speed and on possible blow-up phenomena in Section 3 precede the first a priori estimate in Section 4, where a priori energy estimates for the local solution will be proved in H^m-norms. In Section 5, the decay known for linearized damped wave equations will be exploited to prove additional weighted a priori estimates in L^q-norms ($q \neq \infty$). These, together with the energy estimates from Section 5 will lead in Section 6 to the global existence theorem for small data given in Theorem 6.1. In an Appendix, Moser-type inequalities for composite functions are collected.

We use the standard Sobolev spaces $W^{m,p} = W^{m,p}(\mathbb{R}^n)$, $1 \leq p \leq \infty$, and $L^p = W^{0,p}$, with norms $\|\cdot\|_{m,p}$ and $\|\cdot\|_p$, respectively, cp. [1]. Occasionally, we shall omit the number of the copies of the space needed for vector fields. $\langle \cdot, \cdot \rangle$ will denote the inner product in L^2.

2 Local existence

In [24] we obtained the following local existence theorem:

Theorem 2.1 *Let $n \geq 2$ and $m > n/2$. For each*

$$(u_0, u_1) \in \left(W^{m+2,2}(\mathbb{R}^n) \cap L^2_\sigma(\mathbb{R}^n)\right) \times \left(W^{m+1,2}(\mathbb{R}^n) \cap L^2_\sigma(\mathbb{R}^n)\right)$$

there exists a time $T > 0$ and a unique solution (u, p) to the equations (1.8)–(1.10) satisfying

$$u \in C^2([0,T], W^{m,2}(\mathbb{R}^n)) \cap C^1([0,T], W^{m+1,2}(\mathbb{R}^n)) \cap C^0([0,T], W^{m+2,2}(\mathbb{R}^n) \cap L^2_\sigma(\mathbb{R}^n)),$$

$$\nabla(p + \tau p_t) \in C^0([0,T], W^{m,2}(\mathbb{R}^n)).$$

The existence time T can be estimated from below as

$$T > \frac{1}{1 + C(\|u_0\|_{m+2,2} + \|u_1\|_{m+1,2})}$$

with a constant $C > 0$ depending only on m and the dimension n.

3 Remarks on finite propagation speed and on blow-up phenomena

For the local solution provided in the previous section, we can prove the finite propagation speed for the vorticity $v := \nabla \times u$. Namely, v satisfies the differential equation

$$\tau v_{tt} - \Delta v + v_t + (\tau u \cdot \nabla) v_t + \Big\{ (u \cdot \nabla) v + (\tau u_t \cdot \nabla) v + (2-n)\,(1 + \tau \partial_t) J(\nabla u) v) \Big\} = 0, \quad (3.1)$$

where $J(\nabla u)$ denotes the Jacobi matrix of the first derivatives of u. The part in brackets $\{\dots\}$ involves at most first-order derivatives of v. Therefore, the general energy estimates for hyperbolic equations of second order — after transformation to a first-order symmetric-hyperbolic system — apply as described in [22], and give the finite propagation speed. We remark that this can still not be expected for u due to the pressure terms.

Below, we shall prove the global existence for small data (for u). It will remain open if there is a blow-up to be expected for large data. An ansatz could be to look at the hyperbolic system (3.1) for the vorticity v (also involving u in the coefficients, of course), and to try to apply methods known for large data blow-up situations as in the work of Sideris [25].

The consequences for the classical Navier-Stokes equations as well as for the relation between Fourier type and Cattaneo type models have been mentioned in the introduction.

4 High energy estimates

In order to be able to continue a local solution to a global one — for small data —, we shall prove in this section and in the next one suitable a priori estimates. We start with an estimate for the higher-order energy term $E_m(t)$ defined below.

Let $u \in C^0([0,T], W^{m+2,2}) \cap C^1([0,T], W^{m+1,2}) \cap C^2([0,T], W^{m,2})$ be the local solution to

$$\begin{aligned} \tau u_{tt} - \mu \Delta u + u_t &= -P((u \cdot \nabla)u) - P((\tau u_t \cdot \nabla)u) - P((\tau u \cdot \nabla)u_t) \quad (4.1)\\ &\qquad\qquad \text{in} \quad (0,T) \times \mathbb{R}^n, \\ &\equiv N_1 + N_2 + N_3 \\ \operatorname{div} u &= 0 \qquad\qquad \text{in} \quad (0,T) \times \mathbb{R}^n, \quad (4.2)\\ u(0,\cdot) = u_0, \qquad & u_t(0,\cdot) = u_1 \qquad \text{in} \quad \mathbb{R}^n, \quad (4.3) \end{aligned}$$

$m > n/2, n = 2, 3$, for data $(u_0, u_1) \in W^{m+2,2} \times W^{m+1,2}$ according to Theorem 2.1.

Let, for $0 \le t \le T$,

$$E_m(t) := \frac{1}{2} \sum_{|\alpha| \le m+1} (\|\nabla^\alpha u_t\|_2^2 + \|\nabla^\alpha \nabla u\|_2^2 + \varepsilon_2 \|\nabla^\alpha u\|_2^2)(t), \quad (4.4)$$

where $0 < \varepsilon_2$ will be fixed in the proof of the following Theorem, for which we assume $m \in \mathbb{N}$, $m > n/2$. Then we have for any t, with constants c_1, c_2 being independent of t,

$$c_1 E_m(t) \le \|(u(t), u_t(t))\|_{W^{m+2,2} \times W^{m+1,2}} \le c_2 E_m(t). \quad (4.5)$$

Theorem 4.1 *: There is $C > 0$, being independent of T and of the data (u_0, u_1) such that for $0 \leq t \leq T$,*

$$E_m(t) \leq C E_m(0) e^{C \int_0^t (\|u\|_\infty^2 + \|u_t\|_{1,\infty} + \|\nabla u\|_\infty)(r) dr} \tag{4.6}$$

Remark: *The quadratic character of the term $\|u\|_\infty^2$ is essential for the subsequent sections.*

Proof: We use multiplicative techniques exploiting the derivative character of some nonlinear terms.

Let $|\alpha| \leq m+1$. Applying ∇^α to the differential equation (4.1), then multiplying by $\nabla^\alpha u_t$ in $L^2(\mathbb{R}^n)$ yields

$$\frac{\tau}{2} \frac{d}{dt} \|\nabla^\alpha u_t\|_2^2 + \frac{\mu}{2} \frac{d}{dt} \|\nabla^\alpha \nabla u\|_2^2 + \|\nabla^\alpha u_t\|_2^2 = \sum_{j=1}^{3} \langle \nabla^\alpha N_j, \nabla^\alpha u_t \rangle, \tag{4.7}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2(\mathbb{R}^n)$.

Remark: *In (4.7) we have assumed w.l.o.g. - and will do so in the sequel - that functions are real valued. Otherwise taking real parts would lead to the same conclusions.*

Estimation of $\langle \nabla^\alpha N_1, \nabla^\alpha u_t \rangle$:

Using $\nabla^\alpha P = P \nabla^\alpha$ and well-known Moser-type unequalities that are listed in the Appendix, we obtain

$$\begin{aligned} |\langle \nabla^\alpha N_1, \nabla^\alpha u_t \rangle| &\leq C \left(\|u\|_\infty \| \|\nabla^\alpha \nabla u\|_2 + \|\nabla u\|_\infty \| \|\nabla^\alpha u\|_2 \right) \|\nabla^\alpha u_t\|_2 \\ &\leq C \left(\|u\|_\infty^2 + \|\nabla u\|_\infty \right) E_m + \frac{1}{4} \|\nabla^\alpha u_t\|_2. \end{aligned} \tag{4.8}$$

From now on the letter C will denote positive constants not depending on T or on the data.

Estimation of $\langle \nabla^\alpha N_2, \nabla^\alpha u_t \rangle$:

As before, we get

$$|\langle \nabla^\alpha N_2, \nabla^\alpha u_t \rangle| \leq C (\|u_t\|_\infty + \|\nabla u\|_\infty) E_m. \tag{4.9}$$

Estimation of $\langle \nabla^\alpha N_3, \nabla^\alpha u_t \rangle$:

$$\begin{aligned} \langle \nabla^\alpha N_3, \nabla^\alpha u_t \rangle &= -\tau \langle (u \cdot \nabla \nabla^\alpha) u_t, \nabla^\alpha u_t \rangle \\ &\quad -\tau \langle \nabla^\alpha \big((u \cdot \nabla) u_t \big) - (u \cdot \nabla^\alpha \nabla) u, \nabla^\alpha u_t \rangle \equiv R_1 + R_2. \end{aligned} \tag{4.10}$$

Using $\operatorname{div} u = 0$, we obtain for a typical term

$$\begin{aligned} R_1 &= -\tau \langle u_j \partial_j \partial_k^m \partial_t u_r, \partial_k^m \partial_t u_r \rangle \\ &= \tau \langle u_j \partial_k^m \partial_t u_r, \partial_j \partial_k^m \partial_t u_r \rangle \\ &= -R_1, \end{aligned} \tag{4.11}$$

hence

$$R_1 = 0.$$

$$|R_2| \leq \tau C (\|\nabla u\|_\infty + \|\nabla u_t\|_\infty) E_m. \tag{4.12}$$

Now we apply again ∇^α to (4.1), but muliply with $\varepsilon_2\nabla^\alpha u$ in $L^2(\mathbb{R}^n)$, where $\varepsilon_2 > 0$ will be chosen small enough below. Then we obtain

$$\varepsilon_2\tau\langle\nabla^\alpha u_{tt}, \nabla^\alpha u\rangle + \varepsilon_2\mu\|\nabla^\alpha\nabla u\|_2^2 + \varepsilon_2\langle\nabla^\alpha u_t, \nabla^\alpha u\rangle = \varepsilon_2\sum_{j=1}^{3}\langle\nabla^\alpha N_j, \nabla^\alpha u\rangle,$$

or,

$$\begin{aligned}\varepsilon_2\tau\frac{d}{dt}\langle\nabla^\alpha u_t, \nabla^\alpha u\rangle \quad &- \quad \varepsilon_2\tau\|\nabla^\alpha u_t\|_2^2 + \varepsilon_2\mu\|\nabla^\alpha\nabla u\|_2^2 \\ &+ \quad \varepsilon_2\frac{d}{dt}\frac{1}{2}\|\nabla^\alpha u\|_2^2 = \varepsilon_2\sum_{j=1}^{3}\langle\nabla^\alpha N_j, \nabla^\alpha u\rangle\end{aligned} \tag{4.13}$$

Consider the Lyapunov functional

$$\widetilde{E}_m := E_m + \varepsilon_2\tau\sum_{|\alpha|\le m}\langle\nabla^\alpha u_t, \nabla^\alpha u\rangle. \tag{4.14}$$

If

$$0 < \varepsilon_2\tau < \frac{1}{2}$$

then

$$\exists C_1, C_2 > 0 \ \ \forall t \ : \ C_1 E_m(t) \le \widetilde{E}_m(t) \le C_2 E_m(t). \tag{4.15}$$

The term $-\varepsilon_2\tau\|\nabla^\alpha u_t\|_2^2$ in (4.13) will be dominated by the term $\|\nabla^\alpha u_t\|_2^2$ in (4.7) if $\varepsilon_2\tau$ is small enough ($\varepsilon_2\tau < \frac{1}{2}$ e.g.). This fixes ε_2.

Estimation of $\langle\nabla^\alpha N_1, \nabla^\alpha u\rangle$:

$$\begin{aligned}|\langle\nabla^\alpha N_1, \nabla^\alpha u_\rangle| \quad &\le \quad C\left(\|u\|_\infty\|\|\nabla^\alpha\nabla u\|_2 + \|\nabla u\|_\infty\|\|\nabla^\alpha u\|_2\right)\|\nabla^\alpha u\|_2 && (4.16)\\ &\le \quad C\left(\|u\|_\infty^2 + \|\nabla u\|_\infty\right)E_m. && (4.17)\end{aligned}$$

Estimation of $\langle\nabla^\alpha N_2, \nabla u\rangle$:

$$|\langle\nabla^\alpha N_2, \nabla^\alpha u\rangle| \quad \le \quad C\left(\|u_t\|_\infty + \|\nabla u\|_\infty\right)E_m. \tag{4.18}$$

Estimation of $\langle\nabla^\alpha N_3, \nabla u\rangle$:

For $\alpha = 0$ we have

$$|\langle N_3, u\rangle| \le C\|\nabla u_t\|_\infty E_m. \tag{4.19}$$

For $|\alpha| > 0$ we get

$$\begin{aligned}|\langle\nabla^\alpha N_3, \nabla^\alpha u\rangle| \quad &= \quad \tau|\langle\nabla^{\alpha-1}\big((u\cdot\nabla)u_t\big), \nabla^\alpha\nabla u\rangle| \\ &\le \quad C\left(\|u\|_\infty\|\nabla^\alpha u_t\|_2\|\nabla^\alpha\nabla u\|_2 + \|\nabla^{\alpha-1}u\|_2\|\nabla u_t\|_\infty\|\nabla^\alpha\nabla u\|_2\right) \\ &\le \quad C\left(\|u\|_\infty^2 + \|\nabla u_t\|_\infty\right)E_m + \frac{1}{4}\|\nabla^\alpha u_t\|_2^2.\end{aligned} \tag{4.20}$$

The last term in (4.20) can be dominated by $\|\nabla^\alpha u_t\|_2$ in (4.7) — as well as $\frac{1}{4}\|\nabla^\alpha u_t\|_2^2$ in (4.8). Summing up for $0 \le |\alpha| \le m$ all estimates for the (left and) right-hand sides of (4.7) and of (4.13), we obtain, after integration in time from 0 to t,

$$\widetilde{E}_m(t) \le CE_m(0) + C\int_0^t (\|u\|_\infty^2 + \|\nabla u\|_\infty + \|u_t\|_{1,\infty})(r)E_m(r)dr$$

which, by the equivalence of $\widetilde{E}_m$ and E_m given in (4.15) and Gronwalls inequality, yields the assertion (4.10).

□

5 Weighted a priori estimates

Let $u \in C^0([0,T], W^{m+2,2}) \cap C^1([0,T], W^{m+1,2}) \cap C^2([0,T], W^{m,2})$ be again the local solution to

$$\begin{aligned} \tau u_{tt} - \mu\Delta u + u_t &= -P((u\cdot\nabla)u) - P((\tau u_t\cdot\nabla)u) - P((\tau u\cdot\nabla)u_t) \\ &\qquad\qquad \text{in} \quad (0,T)\times\mathbb{R}^n, \\ &\equiv N_1 + N_2 + N_3 \end{aligned} \tag{5.1}$$

$$\operatorname{div} u = 0 \qquad \text{in} \quad (0,T)\times\mathbb{R}^n, \tag{5.2}$$

$$u(0,\cdot) = u_0, \qquad u_t(0,\cdot) = u_1 \qquad \text{in} \quad \mathbb{R}^n, \tag{5.3}$$

$m > n/2, n = 2, 3$, according to Theorem 2.1.

In order to prove a weighted a priori estimate to exploit the expected decay in time, we shall use the following decay estimates for solutions to the corresponding linearized equations.

Lemma 5.1 *Let v be the solution to*

$$\tau v_{tt} - \mu\Delta v + v_t = 0 \qquad \text{in} \;\; (0,\infty)\times\mathbb{R}^n, \tag{5.4}$$

$$v(0,\cdot) = v_0, \quad v_t(0,\cdot) = v_1 \qquad \text{in} \;\; \mathbb{R}^n. \tag{5.5}$$

Then we have for $\alpha \in \mathbb{N}_0^n$, $j \in \mathbb{N}_0$ and $1 \le p \le 2 \le q \le \infty$ with $1/q + 1/p = 1$

$$\|\nabla^\alpha \partial_t^j v(t,\cdot)\|_2 \le C_{|\alpha|,j}(1+t)^{-(\frac{|\alpha|}{2}+j)}\|(v_0,v_1)\|_{X_2}, \tag{5.6}$$

where

$$X_2 := \begin{cases} L^2 \times L^2 & if \quad |\alpha| + j = 0, \\ W^{|\alpha|+j,2} \times W^{|\alpha|+j-1,2} & if \quad |\alpha| + j \ge 1, \end{cases}$$

$$\|\nabla^\alpha \partial_t^j v(t,\cdot)\|_q \le C_{q,|\alpha|,j}(1+t)^{-\left\{\frac{n}{2}(1-\frac{2}{q})+\frac{|\alpha|}{2}+j\right\}}\|(v_0,v_1)\|_{Y_q}, \tag{5.7}$$

where

$$Y_q := W^{m_q,p} \times W^{m_q-1,p}$$

with

$$m_q := \big[(1-\frac{2}{q})([\frac{n}{2}]+4)\big] \le 5 =: m_0, \tag{5.8}$$

$$\|\nabla^\alpha \partial_t^j v(t,\cdot)\|_2 \le C_{|\alpha|,j}(1+t)^{-(\frac{n}{4}+\frac{|\alpha|}{2}+j)}\|(v_0,v_1)\|_{Z_2}, \tag{5.9}$$

where

$$Z_2 := \begin{cases} (L^2\times L^2)\cap(L^1\times L^1) & if \quad |\alpha|+j=0, \\ (W^{|\alpha|+j,z}\times W^{|\alpha|+j-1,2}\cap(L^1\times L^1) & if \quad |\alpha|+j=1. \end{cases}$$

Proof: The assertions follow from Lemma 1 in [17], taking $m=2$ resp. $m=1$ there, and interpolation.

□

Denoting by $w(t)g$ the solution v to

$$Lv \equiv \tau v_{tt} - \mu\Delta v + v_t = 0 \qquad \text{in } (0,\infty)\times\mathbb{R}^n, \tag{5.10}$$

$$v(0,\cdot)=0,\ v_t(0,\cdot)=g \qquad \text{in } \mathbb{R}^n, \tag{5.11}$$

we have the following representation of the local solution u to (5.1)–(5.3).

Lemma 5.2 *:*

$$u(t) = w(t)(u_1+\frac{1}{\tau}u_0) + \partial_t w(t)u_0 + \frac{1}{\tau}\int_0^t w(t-r)\underbrace{\sum_{j=1}^3 N_j(r)}_{=:f(r)}\,dr \tag{5.12}$$

Proof: Writing $u(t)\equiv v_1(t)+v_2(t)+v_3(t)$ according to (5.12), we get

$$Lv_1 = 0, \quad Lv_2 = 0, \tag{5.13}$$

moreover,

$$\partial_t v_3(t) = \underbrace{\frac{1}{\tau}w(0)f(t)}_{=0} + \frac{1}{\tau}\int_0^t \partial_t w(t-r)f(r)dr, \tag{5.14}$$

$$\begin{aligned} \tau\partial_t^2 v_3(t) &= \partial_t w(t-r)f(r)|_{r=t} + \frac{1}{\tau}\int_0^t \tau\partial_t^2 w(t-r)f(r)dr \\ &= f(t) + \int_0^t \mu\Delta w(t-r)f(r)dr - \int_0^t \partial_t w(t-r)f(r)dr \\ &= f(t) + \mu\Delta v_3(t) - \partial_t v_3(t), \end{aligned}$$

hence

$$Lv_3 = f$$

which gives, using (5.13)

$$Lu = f.$$

Moreover, using (5.14),

$$u(0,\cdot) = u_0,$$

$$\begin{aligned} u_t(0,\cdot) &= u_1 + \frac{1}{\tau}u_0 + \partial_t^2 w(t)u_{0|t=0} + \int_0^t \partial_t w(t-r)\frac{1}{\tau}f(r)dr|_{t=0} \\ &= u_1 + \frac{1}{\tau}u_0 + \frac{1}{\tau}\big(\mu\Delta w(t)u_0 - \partial_t w(t)u_0\big)_{|t=0} \\ &= u_1 + \frac{1}{\tau}u_0 + \frac{1}{\tau}(-u_0) \\ &= u_1. \end{aligned}$$

□

The main weighted a priori estimate for the local solution will be a bound on the following quantity. Let $0 \le T_1 \le T$, and define

$$M(T) \equiv M_{m,m_1,q}(T) := \sup_{0\le t\le T} \Big\{(1+t)^{1-\frac{2}{q}}\|u(t)\|_{m_1,q} + (1+t)^{\frac{3}{2}-\frac{2}{q}}(\|u_t(t)\|_{m_1,q} + \|\nabla u(t)\|_{m_1,q}) + (1+t)^{\frac{1}{2}}\|u(t)\|_{m,2} + (1+t)(\|u_t(t)\|_{m,2} + \|\nabla u(t)\|_{m,2})\Big\}, \quad (5.15)$$

where $\infty > q > 4$ will be arbitrary, but fixed, and $m, m_1 \in \mathbb{N}$ have to satisfy $m_1 \ge 3, m \ge m_1+9$, and are also fixed.

With quadratic nonlinearities in two space dimensions we are at the borderline concerning the possibility to prove suitable a priori estimates to finally get a global small solution. Usually, for nonlinear heat equations, for example, one uses L^∞-estimates and L^1-spaces. This seems to be not possible here because of the fact that the Helmholtz projector P is not continuous in these spaces, cf. the introduction. This explains why we try to use L^q-norms, $4 < q < \infty$ in addition to the also not avoidable L^2-spaces. We follow the approach by Ponce in [19] with the modification of avoiding L^∞- estimates. Since the approach works the better, the better the decay rates of solutions to the linearized equations are, the situation in two dimensions is more difficult (actually borderline case) than in three dimensions. For the simplicity of the presentation we have therefore chosen time weights that are suitable for the two-dimensional case, but, the more, work in three dimensions. The decay rates finally obtained for the global solution could be improved for the three dimensional case.

Let $1 < p \le 2$ satisfy $1/q + 1/p = 1$.

Theorem 5.3 *: There is $\delta > 0$ such that if the initial data (u_0, u_1) satisfy*

$$\|u_0\|_{m+2,2} + \|u_1\|_{m+1,2} + \|u_0\|_1 + \|u_1\|_1 + \|u_0\|_{m_1+6,p} + \|u_1\|_{m_1+5,p} < \delta, \quad (5.16)$$

then there is $M_0 > 0$, independent of T, such that

$$M(T) \le M_0. \quad (5.17)$$

Proof: We use the representation (5.12) and write u as

$$u(t) = v_1(t) + v_2(t) + v_3(t)$$

with

$$v_1(t) = w(t)(u_1 + \frac{1}{\tau}u_0), \quad v_2(t) = \partial_t w(t)u_0, \quad v_3(t) = \frac{1}{\tau}\int_0^t w(t-r)f(r)dr$$

where

$$f(r) = \sum_{j=1}^{3} N_j(r)$$

with N_j given in (5.1). Since div $u = 0$, we may also write

$$N_j = \nabla \cdot P\widetilde{N}_j \tag{5.18}$$

where

$$\widetilde{N}_1 := u \otimes u,\ \widetilde{N}_2 := \tau u_t \otimes u,\ \widetilde{N}_3 := \tau u \otimes u_t, \tag{5.19}$$

e.g.,

$$N_1 = -P((u\cdot\nabla)u) = \sum_{k=1}^{3} \partial_k P(u_k u).$$

Writing the nonlinearities N_j as derivatives as in (5.18) is crucial for the convergence of some integrals below (in the estimate of $(1+t)^{1-2/q}\|u(t)\|_{m_1,q}$), while for others the previous representation with the derivative ∇ in front of u resp. u_t is more appropriate (as in the estimate of $(1+t)^{3/2-2/q}(\|u_t(t)\|_{m_1,q} + \|\nabla u(t)\|_{m_1,q})$, for example).

We start with the

I. Estimate for $\|u(t)\|_{m_1,q}$.

Using Lemma 5.1 frequently in the sequel, we obtain – using from now on the latter C to denote positive constants that do not depend on T or on the data –,

$$\|v_1(t)\|_{m_1,q} \le C(1+t)^{-(1-\frac{2}{q})}(\|u_0\|_{m_0-1+m_1,p} + \|u_1\|_{m_0-1+m_1,p}) \le C\delta(1+t)^{-(1-\frac{2}{q})}, \tag{5.20}$$

$$\|v_2(t)\|_{m_1,q} \le C(1+t)^{-(1-\frac{2}{q})}\|u_0\|_{m_0+m_1,p} \le C\delta(1+t)^{-(1-\frac{2}{q})}, \tag{5.21}$$

$$\begin{aligned}\|v_3(t)\|_{m_1,q} &\le C\int_0^t (1+t-r)^{-((1-\frac{2}{q})+\frac{1}{2})}\sum_{j=1}^{3}\|P\widetilde{N}_j(r)\|_{m_0+m_1,p}\,dr,\\ &\le C\int_0^t (1+t-r)^{-((1-\frac{2}{q})+\frac{1}{2})}\sum_{j=1}^{3}\|\widetilde{N}_j(r)\|_{m_0+m_1,p}\,dr,\end{aligned} \tag{5.22}$$

where we used the continuity of the Helmholtz projection P in $W^{m,\varrho}$ if $1 < \varrho < \infty$.

Remark: *Since P is not continuous in $W^{m,1}$ we have to modify the arguments from [19, 20] where standard $L^\infty - L^1$-estimates could be used, while we have to circumvent this using*

L^q-*estimates with data in* $W^{m,p}$ *spaces for* $q < \infty$ *and* $p > 1$.
The nonlinearities can be estimated as follows (cf. standard inequalities in [20], [22]),

$$\|\widetilde{N}_1(r)\|_{m_0+m_1,p} \leq C\|u(r)\|_{m_0+m_1,2}\|u(r)\|_{m_0+m_1,p_1} \tag{5.23}$$

where

$$\frac{1}{p} = \frac{1}{2} + \frac{1}{p_1}, \quad i.e. \quad p_1 = \frac{2p}{2-p} > 2,$$

$$\|\widetilde{N}_2(r)\|_{m_0+m_1,p} + \|\widetilde{N}_3(r)\|_{m_0+m_1,p} \leq C\|u_t(r)\|_{m_0+m_1,2}\|u(r)\|_{m_0+m_1,p_1} \tag{5.24}$$

Using

$$W^{m,\varrho} \hookrightarrow L^\mu \text{ for } \varrho \leq \mu \text{ and } m\varrho > n \tag{5.25}$$

we conclude

$$\|\widetilde{N}_1(r)\|_{m_0+m_1,p} \leq C\|u(r)\|_{m_0+m_1,2}\|u(r)\|_{m_0+2+m_1,2}, \tag{5.26}$$

$$\|\widetilde{N}_2(r)\|_{m_0+m_1,p} + \|\widetilde{N}_3(r)\|_{m_0+m_1,p} \leq C\|u_t(r)\|_{m_0+m_1,2}\|u(r)\|_{m_0+2+m_1,2}, \tag{5.27}$$

hence

$$(1+r)\|\widetilde{N}_1(r)\|_{m_0+m_1,p} \leq C\Big((1+r)^{\frac{1}{2}}\|u(r)\|_{m_0+2+m_1,2}\Big)^2 \leq C(M(T))^2, \tag{5.28}$$

$$\begin{aligned}(1+r)(\|\widetilde{N}_2(r)\|_{m_0+m_1,p} &+ \|\widetilde{N}_3(r)\|_{m_0+m_1,p}) \\ &\leq C\Big(\big((1+r)^{\frac{1}{2}}\|u_t(t)\|_{m_0+m_1,2}\big)\big((1+r)^{\frac{1}{2}}\|u(r)\|_{m_0+2+m_1,2}\big)\Big) \\ &\leq C(M(T))^2.\end{aligned} \tag{5.29}$$

Combining (5.22), (5.28), (5.29) we get

$$\begin{aligned}(1+t)^{1-\frac{2}{q}}\|v_3(t)\|_{m_1,q} &\leq C(M(T))^2\int_0^t (1+t-r)^{\left((1-\frac{2}{q})+\frac{1}{2}\right)}(1+r)^{-1}(1+t)^{1-\frac{2}{q}}dr \\ &\leq C(M(T))^2\end{aligned} \tag{5.30}$$

where we used the following well-known Lemma (cf. [22]).

Lemma 5.4 *: Let* $\alpha, \beta, \gamma \geq 0$*. Then*

$$\sup_{t\geq 0}\int_0^t (1+t-r)^{-\alpha}(1+r)^{-\beta}(1+t)^{\gamma}dr < \infty$$

if and only if the following conditions (i)–(iii) are satisfied:

(i) $\alpha + \beta - \gamma \geq 1$,

(ii) $\alpha \geq \gamma$ *and* $\beta \geq \gamma$,

(iii) (*if* $\beta = 1$ *then* $\alpha > \gamma$) *and* (*if* $\alpha = 1$ *then* $\beta > \gamma$).

To obtain (5.30) we take $\alpha = 1 - 2/q + \frac{1}{2},\ \beta = 1,\ \gamma = 1 - 2/q$ and conclude that the conditions (i)–(iii) of Lemma 5.4 are satisfied.

Combining (5.20), (5.21), and (5.30) we get the first one of the desired estimates, i.e. for $\|u(t)\|_{m_1,q}$,

$$\sup_{0\le t\le T}(1+t)^{1-2/q}\|u(t)\|_{m_1,q} \le C\delta + C(M(T))^2. \tag{5.31}$$

II. Estimate for $\|u_t(t)\|_{m_1,q} + \|\nabla u(t)\|_{m_1,q}$.

We have the representations (cp. (5.12), (5.14))

$$\begin{aligned} u_t(t) &= \partial_t w(t)(u_1 + \frac{1}{\tau}u_0) + \partial_t^2 w(t)u_0 + \frac{1}{\tau}\int_0^t \partial_t w(t-r)f(r)dr \\ &= \partial_t w(t)(u_1 + \frac{1}{\tau}u_0) + w(t)(\frac{\mu}{\tau}\Delta u_0) - \frac{1}{\tau}\partial_t w(t)u_0 + \frac{1}{\tau}\int_0^t \partial_t w(t-r)f(r)dr \\ &= w(t)(\frac{\mu}{\tau}\Delta u_0) + \partial_t w(t)u_1 + \frac{1}{\tau}\int_0^t \partial_t w(t-r)f(r)dr, \end{aligned} \tag{5.32}$$

and

$$\nabla u(t) = \nabla w(t)(u_1 + \frac{1}{\tau}u_0) + \nabla\partial_t w(t)u_1 + \int_0^t \nabla w(t-r)f(r)dr. \tag{5.33}$$

In analogy to the estimates for $\|u(t)\|_{m_1,q}$ above, we may conclude the following series of estimates, with the essential difference that now $f(r)$ keeps the derivative ∇ in front of u resp. u_t. This will allow to justify the convergence of integrals as in (5.30) (that otherwise would be divergent).

Writing $\nabla u = v_1 + v_2 + v_3$ according to (5.33) we have

$$\|v_1(t)\|_{m_1,q} \le C\delta(1+t)^{-((1-\frac{2}{q})+\frac{1}{2})}, \tag{5.34}$$

$$\|v_2(t)\|_{m_1,q} \le C\delta(1+t)^{-((1-\frac{2}{q})+\frac{1}{2})}, \tag{5.35}$$

Writing $N_j = P\hat{N}_j, j = 1,2,3$, we obtain

$$\|v_3(t)\|_{m_1,q} \le C\int_0^t (1+t-r)^{-((1-\frac{2}{q})+\frac{1}{2})}\sum_{j=1}^3 \|\hat{N}_j(r)\|_{m_0+m_1,p}\, dr, \tag{5.36}$$

with $p_1 = 2p/(2-p)$ as before, we get

$$\begin{aligned} \|\hat{N}_1(r)\|_{m_0+m_1,p} &\le C\|u(r)\|_{m_0+m_1,2}\|\nabla u(r)\|_{m_0+m_1,p_1} \\ &\le C\|u(r)\|_{m_0+m_1,2}\|\nabla u(r)\|_{m_0+m_1+2,2}, \end{aligned} \tag{5.37}$$

$$\begin{aligned} \|\hat{N}_2(r)\| &\le C\|u_t(r)\|_{m_0+m_1,2}\|\nabla u(r)\|_{m_0+m_1,p_1} \\ &\le C\|u_t(r)\|_{m_0+m_1,2}\|\nabla u(r)\|_{m_0+m_1,2}, \end{aligned} \tag{5.38}$$

$$\begin{aligned}\|\hat{N}_3(r)\| &\le C\|u(r)\|_{m_0+m_1,2}\|\nabla u_t(r)\|_{m_0+m_1,p_1}\\ &\le C\|u(r)\|_{m_0+m_1,2}\|u_t(r)\|_{m_0+m_1+3,2}. \end{aligned} \tag{5.39}$$

Hence, we get

$$\begin{aligned}(1+r)^{\frac{3}{2}}\|\hat{N}_1(r)\|_{m_0+m_1,p} &\le C\big((1+r)^{\frac{1}{2}}\|u(r)\|_{m_0+m_1,2}\big)\cdot\big((1+r)\|\nabla u(r)\|_{m_0+m_1+2,2}\big)\\ &\le C(M(T))^2, \end{aligned} \tag{5.40}$$

similarly

$$(1+r)^{\frac{3}{2}}(\|\hat{N}_2(r)\|_{m_0+m_1,p}+\|\hat{N}_3(r)\|_{m_0+m_1,p})\le C(M(T))^2. \tag{5.41}$$

Combining (5.36), (5.40), and (5.41) we obtain

$$\begin{aligned}(1+t)^{(1-\frac{2}{q})+\frac{1}{2}}\|v_3(t)\|_{m_1,q} &\le C(M(T))^2\int_0^t(1+t-r)^{-((1-\frac{2}{q})+\frac{1}{2})}(1+r)^{-\frac{3}{2}}(1+t)^{1-\frac{2}{q}+\frac{1}{2}}dr\\ &\le C(M(T))^2 \end{aligned} \tag{5.42}$$

by Lemma 5.4. The estimates (5.34), (5.35), (5.42) yield

$$\sup_{0\le t\le T}(1+t)^{\frac{3}{2}-\frac{2}{q}}\|\nabla u(t)\|_{m_1,q}\le C\delta+C(M(T))^2. \tag{5.43}$$

Analogously we get the estimate

$$\sup_{0\le t\le T}(1+t)^{\frac{3}{2}-\frac{2}{q}}\|u_t(t)\|_{m_1,q}\le C\delta+C(M(T))^2. \tag{5.44}$$

III. Estimate for $\|u(t)\|_{m,2}$.

Writing $u=v_1+v_2+v_3$ according to (5.12) we have, in particular using (5.9) from Lemma 5.1,

$$\begin{aligned}\|v_1(t)\|_{m,2} &\le C(1+t)^{-\frac{1}{2}}(\|u_0\|_{m-1,2}+\|u_1\|_{m-1,2}+\|u_0\|_1+\|u_1\|_1)\\ &\le C\delta(1+t)^{-\frac{1}{2}}, \end{aligned} \tag{5.45}$$

$$\begin{aligned}\|v_2(t)\|_{m,2} &\le C(1+t)^{-\frac{1}{2}}(\|u_0\|_{m,2}+\|u_0\|_1)\\ &\le C\delta(1+t)^{-\frac{1}{2}}, \end{aligned} \tag{5.46}$$

We observe that the L^1-norms appear for the initial data, but will not appear for the nonlinearities (where it would cause trouble due to the Helmholtz projection).

$$\|v_3(t)\|_{m,2}\le C\int_0^t(1+t-r)^{-\frac{1}{2}}\sum_{j=1}^3\|\widetilde{N}_j(r)\|_{m+1,2}\,dr. \tag{5.47}$$

The nonlinearities $\widetilde{N}_j$ are now estimated as follows.

$$\begin{aligned}\|\widetilde{N}_j(r)\|_{m+1,2} &\le C(\|u(r)\|_\infty+\|u_t(r)\|_\infty)(\|u(r)\|_{m,2}+\|\nabla^{m+1}u(r)\|_2)\\ &\le C(\|u(r)\|_{m_1,q}+\|u_t(r)\|_{m_1,q})(\|u(r)\|_{m,2}+\|\nabla^{m+1}u(r)\|_2). \end{aligned} \tag{5.48}$$

We have from (5.12)

$$\|\nabla^{m+1}u(r)\|_2 \leq C\delta(1+r)^{-\frac{m+1}{2}} + C\int_0^r (1+r-\lambda)^{-(\frac{m+1}{2}+\frac{1}{2})} \sum_{j=1}^{3} \|\widetilde{N}_j(r)\|_{m+1,2}\, d\lambda. \tag{5.49}$$

Moreover, using the energy estimate from Theorem 4.1,

$$\begin{aligned} \|\widetilde{N}_j(\lambda)\|_{m+1,2} &\leq C(\|u(\lambda)\|_\infty + \|u_t(\lambda)\|_\infty)\|u(\lambda)\|_{m+1,2} \\ &\leq C\delta(\|u(\lambda)\|_{2,q} + \|u_t(\lambda)\|_{2,q}) \cdot e^{C\int_0^\lambda (\|u(\varrho)\|_{2,q}^2 + \|u_t(\varrho)\|_{3,q} + \|\nabla u(\varrho)\|_{2,q}) d\varrho}. \end{aligned} \tag{5.50}$$

The estimates (5.49), (5.50) imply

$$\begin{aligned} (1+r)^{1-\frac{2}{q}}\|\nabla^{m+1}u(r)\|_2 &\leq C\delta(1+r)^{-(\frac{m}{2}-(1-\frac{2}{q}))} + C\delta\int_0^r (1+r-\lambda)^{-(\frac{m}{2}+\frac{1}{2})} \cdot \\ &\cdot\Big\{(1+\lambda)^{-(1-\frac{2}{q})}(1+r)^{1-\frac{2}{q}}\big[(1+\lambda)^{1-\frac{2}{q}}\|u(\lambda)\|_{m_1,q}\big] \\ &+(1+\lambda)^{-(\frac{3}{2}-\frac{2}{q})}(1+r)^{1-\frac{2}{q}}\big[(1+\lambda)^{\frac{3}{2}-\frac{2}{q}}\|u_t(\lambda)\|_{m_1,q}\big]\Big\} \cdot \\ &\cdot e^{C\int_0^\lambda (1+\varrho)^{-2(1-\frac{2}{q})}\left[(1+\varrho)^{2(1-2/q)}\|u(\varrho)\|_{m_1,q}^2\right] + (1+\varrho)^{-(3/2-2/q)} \cdot \left[(1+\varrho)^{(3/2-2/q)}\left(\|u_t(\varrho)\|_{m_1,q} + \|\nabla u(\varrho)\|_{m_1,q}\right)\right] d\varrho} d\lambda \\ &\leq C\delta\big(1 + M(T)e^{C((M(T))^2+M(T))}\big) \end{aligned} \tag{5.51}$$

since

$$\sup_{r\geq 0}\int_0^r (1+r-\lambda)^{-(\frac{m+1}{2}+\frac{1}{2})}(1+\lambda)^{-(1-\frac{2}{q})}(1+r)^{1-\frac{2}{q}}d\lambda +$$

$$\sup_{r\geq 0}\int_0^r (1+r-\lambda)^{-(\frac{m}{2}+\frac{1}{2})}(1+\lambda)^{-(\frac{3}{2}-\frac{2}{q})}(1+r)^{\frac{3}{2}-\frac{2}{q}}dr +$$

$$\sup_{\lambda\geq 0}\int_0^\lambda (1+\varrho)^{-2(1-\frac{2}{q})} + (1+\varrho)^{-(\frac{3}{2}-\frac{2}{q})}d\varrho < \infty$$

by Lemma 5.4, using $q > 4$. Combining (5.48) and (5.51) we obtain

$$\begin{aligned} (1+r)^{1-\frac{2}{q}+\frac{1}{2}}\|\widetilde{N}_j(r)\|_{m+1,2} &\leq C(1+r)^{1-\frac{2}{q}}\big(\|u(r)\|_{m_1,q} + \|u_t(r)\|_{m_1,q}\big) \cdot \\ &\cdot\big((1+r)^{\frac{1}{2}}\|u(r)\|_{m,2} + (1+r)^{\frac{1}{2}}\|\nabla^{m+1}u(r)\|_2\big) \\ &\leq C(M(T))^2 + C\delta M(T)\Big(1 + M(T)e^{C((M(T))^2+M(T))}\Big). \end{aligned} \tag{5.52}$$

By (5.45), (5.46),(5.47), and (5.52) we conclude

$$\begin{aligned}(1+t)^{\frac{1}{2}}\|u(t)\|_{m,2} &\leq C\delta + C\int_0^t (1+t-r)^{-\frac{1}{2}}(1+r)^{-(1-\frac{2}{q}+\frac{1}{2})}(1+t)^{1/2}\cdot \\ &\quad \cdot(1+r)^{1-2/q+1/2}\sum_{j=1}^{3}\|\widetilde{N}_j(r)\|_{m+1,2}dr \qquad (5.53)\\ &\leq C\delta + C(M(T))^2 + C\delta M(T)\Big(1+M(T))e^{C((M(T))^2+M(T))}\Big),\end{aligned}$$

since

$$\sup_{t\geq 0}\int_0^t (1+t-r)^{-\frac{1}{2}}(1+r)^{-(1-\frac{2}{q}+\frac{1}{2})}(1+t)^{\frac{1}{2}}dr < \infty. \qquad (5.54)$$

IV. Estimates for $\|u_t(t)\|_{m,2} + \|\nabla u(t)\|_{m,2}$.

Using the representations (5.32) for u_t and (5.33) for ∇u, respectively, we obtain the analogous series of estimates as for $\|u(t)\|_{m,2}$ in part III above, in particular: (5.48) turns into

$$\|\widetilde{N}_j(r)\|_{m+2,2} \leq C(\|u(r)\|_{m_1,q} + \|u_t(r)\|_{m_1,q})(\|u(r)\|_{m,2} + \|\nabla^{m+2}u(r)\|_{m,2}), \qquad (5.55)$$

and (5.53) turns into

$$\begin{aligned}(1+t)(\|u_t(t)\|_{m,2} + \|\nabla u(t)\|_{m,2}) &\leq C\delta + C(M(T))^2 \\ &\quad + C\delta M(T)\Big(1+M(T)e^{C(M(T))^2+(M(T))}\Big), \qquad (5.56)\end{aligned}$$

since

$$\sup_{t\geq 0}\int_0^t (1+t-r)^{-1}(1+r)^{-(1-\frac{2}{q}+\frac{1}{2})}(1+t)dr < \infty. \qquad (5.57)$$

Summarizing (5.31), (5.43), (5.44), (5.53), and (5.56), we have

$$M(T) \leq C\delta + C(M(T))^2 + C\delta M(T)\big(1+M(T)e^{C((M(T))^2+M(T))}\big). \qquad (5.58)$$

In (5.58) we may replace T by any T_1 with $0 \leq T_1 \leq T$, and we conclude by standard arguments (cf. [22], [19]) that for sufficiently small $\delta > 0$ we have

$$M(T) \leq M_0 \qquad (5.59)$$

where M_0 is the first zero of the function h with

$$h(x) := Cx + Cx^2 + C\delta x(1 + xe^{C(x^2+x)}) - x.$$

This proves Theorem 5.3.

□

6 Global existence

The a priori estimates in Theorem 4.1 (high energy estimates) and in Theorem 5.3 (weighted a priori estimates) allow us in a standard way (cf.[22]) to prove the global existence theorem

Theorem 6.1 *: Let $m_1 \geq 3, m \geq m_1 + 9, \infty > q > 4, 1/q + 1/p = 1$.*
There exists $\varepsilon > 0$ such that if

$$\|u_0\|_{m+2,2} + \|u_0\|_{m+1,2} + \|u_0\|_1 + \|u_1\|_1 + \|u_0\|_{m_1+6,p} + \|u_0\|_{m_1+5,p} < \varepsilon,$$

then there exists a unique global solution (u, p) to the hyperbolic Navier-Stokes equations (1.8)–(1.10), satisfying

$$u \in C^2([0,\infty), W^{m,2}) \cap C^1([0,\infty), W^{m+1,2}) \cap C^0([0,\infty), W^{m+2,2}),$$

$$\nabla(p + \tau p_t) \ \in \ C^0([0,\infty), W^{m,2}(\mathbb{R}^n)).$$

Remark: *Since $\sup_{t\geq 0} M(t) \leq M_0$, we have the corresponding decay rates for $u(t), u_t(t)$ and $\nabla u(t)$ in $\|\cdot\|_{m_1,q}$ and in $\|\cdot\|_{m,2}$, respectively.*

Proof: Let u be the local solution to (5.1), (5.2), (5.3) according to Theorem 2.1. We obtain using Theorem 4.1 and Theorem 5.3

$$\|E_m(t)\| \leq CE_m(0)e^{C(M_0^2+M_0)} \leq CE_m(0)$$

where C is independent of T and of the data. Using (4.5), we get

$$\|(u(t), u_t(t))\|_{W^{m+2,2}\times W^{m+1,2}} \leq \hat{C}\|(u_0, u_1)\|_{W^{m+2,2}\times W^{m+1,2}},$$

where

$$\hat{C} := \frac{c_2 C}{c_1}.$$

Choosing

$$O < \varepsilon < \frac{\delta}{\hat{C}}$$

we conclude

$$E_m(T) < \delta$$

and are thus able to continue a local solution to a global one (as usual, cf. [22, p. 91]).
□

7 Appendix

The following inequalities have been frequently used in the preceding sections and are often quoted as "Moser-type inequalities".

Lemma 7.1 *Let $m \in \mathbb{N}$. There there is a constant $c = c(m,n) > 0$ such that for all $f, g \in W^{m,2}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}_0^n$, $|\alpha| \leq m$, the following inequalities hold:*

$$\|\nabla^\alpha(fg)\|_2 \leq c(\|f\|_\infty \|\nabla^m g\|_2 + \|g\|_\infty \|\nabla^m f\|_2), \tag{7.1}$$

$$\|\nabla^\alpha(fg) - f \cdot \nabla^\alpha g\|_2 \leq c(\|\nabla f\|_\infty \|\nabla^{m-1} g\|_2 + \|g\|_\infty \|\nabla^m f\|_2). \tag{7.2}$$

For a **Proof** see [22, Lemma 4.9].

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