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THE SPIN-COATING PROCESS: ANALYSIS OF THE FREE BOUNDARY VALUE PROBLEM

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Abstract. In this paper, an accurate model for the spin-coating process is presented and investigated from the analytical point of view. More precisely, the spin-coatong process is being described as a one-phase free boundary value problem for Newtonian fluids in the rotational setting. The method presented is based on a transformation of the free boundary value problem to a quasilinear evolution equation on a fixed domain. The keypoint for solving the latter equation will be so-called maximal regularity approach. In order to pursue this one needs to determine the precise regularity classes for the associated inhomogenous linearized equations. This is being achieved by applying the Newton polygon method to the boundary sumbol.

1. Introduction

The spin-coating process may be roughly speaking described as follows: it is a method of placing a small drop of coating material, in liquid form, on the center of a disc, which is then spun rapidly about its axis. The drop is then driven by two competing forces: centrifugal forces cause the liquid to be thrown radially outwards, whereas surface tension forces will work against this spreading. For large centrifugal forces, the coating material film thins.

Of particular interest is the situation where the coating material is a polymer dissolved in a solvent. As the film thins, the solvent evaporates and the solution viscosity increases, reducing the radial flow. Eventually, the viscosity becomes so large that relative motion virtually ceases and the process is completed by evaporating the residual solvent.

Spin-coating has many applications. The process is used, for example, in manufacturing micro-electronic devices or magnetic storage discs. In all cases a uniform layer is required and essential.

It has to be stressed that complete mathematical models describing all the above effects do not seem to exist.

In order to develop an accurate model and to investigate it rigorously from an analytical point of view, we describe the spin-coating process as a one-phase free boundary value problem for a Newtonian fluid subject to surface tension and rotational effects.

More precisely, let $\Gamma_0 \subset \Omega(0)$ be a surface which bounds a region $\Omega(0)$ filled with a viscous, incompressible fluid. Denoting by $\Gamma(t)$ the position of the boundary at time t, $\Gamma(t)$ is then the interface seperating the fluid occupying the region $\Omega(t)$ and its complement. In the following, the normal on $\Gamma(t)$ is denoted by $\nu(t,\cdot)$ and $V(t,\cdot)$ and $\kappa(t,\cdot)$ denote the normal velocity and mean curvature of $\Gamma(t)$, respectively. Assume that the free surface may be described as the graph of a height function h. Thus, the region $\Omega(t)$ describing the fluid is occupying may be represented as $\Omega(t) := \{(x,y); x \in \mathbb{R}^2, y \in (0,h)\}$, where $h = h(t,x)$ is the height function. The boundary of $\Omega(t)$ splits into the free surface on the top part $\Gamma^+(t) = \{(x, h(t,x)) : x \in \mathbb{R}^2)$ and the bottom part $\Gamma^-(t) = \{(x,0) : x \in \mathbb{R}^2\}$, the interface of the fluid with a solid phase. In the situation of spin-coating it is natural to consider the case where $\Gamma^+(0)$ is close to a plane, i.e. $\Gamma(0)$ is a graph over $\mathbb{R}^2$ given by a function h_0. Then the motion of the fluid is

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governed for $u = (v, w)^T$ with $v = (u_1, u_2)^T$ by the following set of equations:

$$
\tag{1.1}
\begin{cases}
\rho(\partial_t u + (u \cdot \nabla)u) & = \nu\Delta u - \nabla q - \rho[2\omega \times u + \omega \times (\omega \times (x, y))] & \text{in } \Omega(t) \\
\operatorname{div} u & = 0 & \text{in } \Omega(t) \\
-T\nu & = \sigma H \nu & \text{on } \Gamma^+(t) \\
V & = u \cdot \nu & \text{on } \Gamma^+(t) \\
v & = ch^\alpha \partial_3 v, & \text{on } \Gamma^-(t) \\
w & = 0 & \text{on } \Gamma^-(t) \\
u(0) & = u_0 & \text{in } \Omega(0) \\
h(0) & = h_0 + \delta & \text{in } \mathbb{R}^2.
\end{cases}
$$

The first equation represents the equation for momentum subject to Coriolis and centrifugal forces given by $2\omega \times u$ and $\omega \times (\omega \times (x, y))$, respectively. Here ρ, ν and ω denote the density and viscosity of the fluid and ω the speed of rotation. The second equation is the condition that the fluid is incompressible. The third equation says that there is a jump of the stress tensor

$$T = \mu(\nabla u + (\nabla u)^T) - qI$$

in normal direction at the interface Γ which is determined by its mean curvature κ and by the surface tension σ. Further, V denotes the velocity of the free surface Γ in normal direction. The fifth and sixth equations above describe wetting phenomena at the bottom part $\Gamma^-(t)$ of $\Omega(t)$. Note that the classical Dirichlet condition holds only for the third component w of the fluid velocity u. In the case, where a contact line exists and the liquid on a solid substrate spreads and displaces the surrounding fluid, say gas, it is well known that the classical homogeneous Dirichlet condition for u leads to a nonintegrable singularity at the contact line, see [HS71] and [DD74]. This singularities can be relieved by allowing relative motion, i.e. slip, between the liquid and the solid near the contact line. This means that the condition of no penetration is retained and tangential relative motion is allowed. The *Navier slip condition* on Γ^- demands that the velocity at the interface to be proportional to its normal derivative:

$$v = k(h)\partial_3 v.$$

The function $k(\cdot)$ describes the slip parameter and depends on the height h. In the fifth equation above we assume that k is of the form $k(h) = ch^\alpha$, where c and α are positive constants.

On the top part, our problem differs from known one- or two-phase flow models through *Coriolis* and *centrifugal force*. Wellposedness results in the non-rotating setting for one-phase flows with surface tension are due to Solonnikov [Sol87] [Sol99], [Sol04] and Shibata and Shimuzu [SS07]. In the setting of spin-coating it is natural to consider infinite layer-like domains. Note that the results cited do not cover this situation. An additional difficulty arising in infinite layers is the localization of the pressure term q. Our approach to circumvent this difficulty is a localization technique for the *reduced* Stokes system on two half spaces. Estimates for the solution of Laplace's equation subject to various boundary conditions in Sobolev spaces of negative order will play a key role.

The case of an ocean of infinite extend bounded below by a solid surface and bounded above by a free surface was treated by Beale [Bea84], Tani [Tan96], and Tani and Tanaka [TT95]. The two-phase problem without rotational effects was investigated by Denisova in [Den91] and [Den94], by Tanaka in [Tan95] and by Prüss and Simonett in [PS09].

Wellposedness results for the spin-coating system on the other hand seem not to be known and are the objectives of this paper. Keypoint of our approach are optimal regularity properties of the boundary symbol. They will be achieved by the so-called Newton polygon technique.

Our main result says that the above set of equations admits a unique, local, strong solution (u, p, h) in the space of maximal parabolic regularity provided the intitial data u_0 and h_0 belong to certain function spaces and are small. The precise regularity assertions will be given in Theorem 2.1 in the following section.

Some comments on notation and function spaces are in order. Let $s \in \mathbb{R}$, $1 < p < \infty$, Ω be a domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega = \Gamma$ and X be a Banach space. Then $H^s_p(\Omega, X)$ denotes the Bessel potential space of order s. The Slobodeckij space $W^s_p(\Omega, X)$ is defined as $W^s_p(\Omega, X) := B^s_{pp}(\Omega, X)$, where

$B^s_{pp}(\Omega, X)$ denotes the corresponding Besov space. Moreover, for $T \in (0,\infty)$ set $J := (0,T)$. Then the space ${}_0W^s_p(J,X)$ is defined for $s \geq 0$ with $s - 1/p \notin \mathbb{N}_0$, as

$$ {}_0W^s_p(J,X) := \begin{cases} \{u \in W^s_p(J,X) : u(0) = \ldots = u^{(k)}(0) = 0\}, \text{ if } s \in (k + \frac{1}{p}, k+1+\frac{1}{p}) \text{ for } k \in \mathbb{N}_0, \\ W^s_p(J,X), \quad \text{if} \quad 0 \leq s < \frac{1}{p}. \end{cases} $$

The spaces ${}_0H^s_p(J,X)$ are defined analogously. The homogeneous versions of the above spaces will be denoted by $\widehat{H}^s_p(\Omega,X)$ and $\widehat{W}^s_p(\Omega,X)$. Moreover, we set ${}^0\widehat{H}^1_p(\Omega,\Gamma) := \{\varphi \in \widehat{H}^1_p(\Omega) : \gamma\varphi = 0 \text{ on } \Gamma\}$ and

$$ {}_0\widehat{H}^{-1}_p(\Omega,\Gamma) := \left({}^0\widehat{H}^1_{p'}(\Omega,\Gamma)\right)'. $$

Here γ denotes the trace operator $u \mapsto u|_\Gamma$. The trace operator γ depends of course on Ω, Γ and the smoothness of the underlying space. However, in order to simplify our notation, we always denote the trace operator by γ whenever no misunderstanding may occur.

For more information about the Navier-Stokes equations in fixed domains, we refer e.g. to [Gal94], [Ama00] and [FKS05] and in the rotational setting e.g. to [CT07], [GHH06] and [HS09].

2. Main Result

In this section, we prove that the free boundary value problem describing the spin-coating process, i.e. on the free surface, the Navier-Stokes equations with surface tension in the rotational setting and the Navier-Stokes equations with Navier's condition on the fixed bottom boundary are locally well posed.

Theorem 2.1. *Let $p > 5$. Then there exist $\varepsilon > 0$ and $T > 0$ such that for all $u_0 \in W^{2-2/p}_p(\Omega(0))$ and all $h_0 \in W^{3-2/p}_p(\mathbb{R}^2)$ with $\operatorname{div} u_0 = 0$ on $\Omega(0)$ and satisfying*

$$ \|(u_0,h_0)\|_{W^{2-2/p}_p(\Omega(0)) \times W^{3-2/p}_p(\mathbb{R}^2)} < \varepsilon, $$

there exists a unique solution (u,q,h) of equation (1.1) within the regularity classes

$$ \begin{aligned} u &\in H^1_p(J, L_p(\Omega(t))^3) \cap L_p(J, H^2_p(\Omega(t))^3), \\ q &\in \{p \in L_p(J, \widehat{H}^1_p(\Omega(t))) : \gamma q \in W^{1/2-1/2p}_p(J, L_p(\Gamma^+(t))) \cap L_p(J, W^{1-1/p}_p(\Gamma^+(t)))\}, \\ h &\in W^{2-1/2p}_p(J, L_p(\Gamma^+(t))) \cap H^1_p(J, W^{2-1/p}_p(\Gamma^+(t))) \cap L_p(J, W^{3-1/p}_p(\Gamma^+(t))). \end{aligned} $$

Note that due to our asumption $p > 5$ we have

$$ h \in C(J, BUC^2(\Gamma(t))) \quad \text{and } \partial_t h \in C(J, BUC^1(\Gamma(t))). $$

This implies that the normal of $\Omega(t)$, the normal velocity V of $\Gamma(t)$ and its mean curvature are well defined and continuous. In particular, the equations on the free boundaries given in (1.1) can be understood pointwise. For u we have

$$ u(t) \in BUC^1(\Omega(t)) \quad \text{and } \nabla u(t) \in BUC(\Omega(t)), \quad t \in J. $$

Some comments of our approach how to prove the above result are in order: first, by applying the Hanzawa transform to problem (1.1), we obtain a set of equations on a fixed layer-like domain $D := \mathbb{R}^2 \times (0,\delta)$ of fixed height δ with top and bottom boundaries $\Gamma^+ = \mathbb{R}^2 \times \{\delta\}$ and $\Gamma^- = \mathbb{R}^2 \times \{0\}$. In section 3 we will verify that the set of equations for $\rho = \nu = 1$ on the fixed domain D are of the form

$$ \left\{ \begin{aligned} \partial_t u - \Delta u + \nabla q &= \chi_R \Lambda \times (\Lambda x) - 2\omega \times u + F(u,q,h) && \text{in } J \times D, \\ \operatorname{div} u &= F_d(u,h) && \text{in } J \times D, \\ -\gamma T(u,q)\nu_D - \sigma \Delta_x h \nu_D &= G_+(u,q,h) && \text{on } J \times \Gamma^+, \\ \partial_t h - \gamma w &= H(u,h) && \text{on } J \times \Gamma^+, \\ \gamma v - \gamma c \delta^\alpha \partial_y v &= G_-(u,h) && \text{on } J \times \Gamma^-, \\ \gamma w &= 0 && \text{on } J \times \Gamma^-, \\ u|_{t=0} &= u_0 && \text{in } D, \\ h|_{t=0} &= h_0 && \text{in } \mathbb{R}^2. \end{aligned} \right. \tag{2.1} $$

for certain functions F, F_2, G_+, G_- and H. Splitting the normal stress into its tangential and normal component, the linearization of (2.1) leads to the following linear inhomogeneous problem

$$
\left\{
\begin{array}{rll}
\partial_t u - \Delta u + \nabla q & = f_1 & \text{in } J \times D, \\
\operatorname{div} u & = f_d & \text{in } J \times D, \\
\gamma \partial_y v + \gamma \nabla_x w & = g_v & \text{on } J \times \Gamma^+, \\
\gamma 2 \partial_y w - \gamma q - \sigma \Delta_x h & = g_w & \text{on } J \times \Gamma^+, \\
\partial_t h - \gamma w & = f_h & \text{on } J \times \Gamma^+, \\
\gamma v - \gamma c \delta^\alpha \partial_y v & = g_- & \text{on } J \times \Gamma^-, \\
\gamma w & = 0 & \text{on } J \times \Gamma^-, \\
u|_{t=0} & = u_0 & \text{in } D, \\
h|_{t=0} & = h_0 & \text{on } \mathbb{R}^2
\end{array}
\right. \tag{2.2}
$$

Note that in the first line above the Coriolis term $2\omega \times u$ is neglected. It will be included lateron, however, in the definition of F_1.

Secondly, we show maximal L_p-regularity for the linearized problem (2.2) in Section 4. To this end, we split the original problem into two model problems defined on half spaces, use the equivalence of the Stokes problem and the reduced Stokes problem explained in Appendix B as well as the Newton polygon technique explained in Appendix C. Finally, a fixed-point argument yields the existence of a unique solution (u, p, h) to equation (1.1) belonging to the regularity class described in Theorem 2.1.

A second comment about the regularity class seems also to be in order: assume that the linear problem (2.2) admits a solution (u, p, h) satisfying

$$
u \in H_p^1(J, L_p(D)^3) \cap L_p(J, H_p^2(D)^3), \qquad q \in L_p(J, \widehat{H}_p^1(D)),
$$

then the right hand sides f_1 and f_d need to satisfy $f_1 \in L_p(J \times \Omega)$ and

$$
f_d \in H_p^1(J, {}_0\widehat{H}_p^{-1}(D)) \cap L_p(J, H_p^1(D)),
$$

since the operator div maps L_p into H_p^{-1}. By trace theory, u_0 necessarily belongs to $W_p^{2-2/p}(D)$. Moreover, the trace of u belongs to the class

$$
Y_0 := W_p^{1/2-1/2p}(J, L_p(\Gamma^+)) \cap L_p(J, W_p^{2-1/p}(\Gamma^+)),
$$

and that of ∇u to

$$
Y_1 := W_p^{1/2-1/2p}(J, L_p(\Gamma^+)^3) \cap L_p(J, W_p^{1-1/p}(\Gamma^+)^3).
$$

Thus $g_v \in Y_1$ and if in addition $q \in W_p^{1/2-1/2p}(J, L_p(\Gamma^+)) \cap L_p(J, W_p^{1-1/p}(\Gamma^+))$, then also $g_w \in Y_1$. The equation for h is defined on Y_0; hence h should naturally belong to

$$
W_p^{2-1/2p}(J, L_p(\Gamma^+)) \cap H_p^1(J, W_p^{2-1/p}(\Gamma^+)).
$$

The fourth equation above is defined in Y_1 and contains the term Δ_x. Thus h should also belong to $L_p(J, W_p^{3-1/p}(\Gamma^+))$ and the natural space for h is

$$
W_p^{2-1/2p}(J, L_p(\Gamma^+)) \cap H_p^1(J, W_p^{2-1/p}(\Gamma^+)) \cap L_p(J, W_p^{3-1/p}(\Gamma^+)).
$$

This also implies $h_0 \in W_p^{3-2/p}(\mathbb{R}^2)$.

Having thus observed that the above regularity for h is necessary for u and q solving (2.2) and belonging to the above regularity classes, our main result shows that under these assumptions the nonlinear problem admits a unique solution in the above regularity classes.

3. Hanzawa transformation

In this section we transform the problem (1.1) to a problem on the fixed domain $D = \mathbb{R}^2 \times (0, \delta)$ for some $\delta > 0$. The top and bottom boundary of D are given by $\Gamma^+ = \mathbb{R}^2 \times \{\delta\}$ and $\Gamma^- = \mathbb{R}^2 \times \{0\}$, respectively. To this end, we define for $J := (0, T)$

$$
\Theta : J \times \mathbb{R}^2 \times (0, \delta) \to \bigcup_{t \in J} \{t\} \times \Omega(t), \qquad \Theta(t, x, y) := (t, x, \frac{y h(t, x)}{\delta})
$$

as well as $\theta(t,x,y) := (x, yh(t,x)/\delta)$. Thus $\Theta(t,x,y) = (t, \theta(t,x,y))$ for all $t \in J$, $x \in \mathbb{R}^2$ and $y \in (0,\delta)$. We then define the transformed variables by

$$\begin{aligned}
(\Theta^* u)(t,x,y) &:= u(\Theta(t,x,y)),\\
v(t,x,y) &:= \begin{bmatrix} (\Theta^* u)_1(t,x,y) \\ (\Theta^* u)_2(t,x,y) \end{bmatrix}\\
w(t,x,y) &:= (\Theta^* u_3)(t,x,y) := u_3(\Theta(t,x,y)),\\
\pi(t,x,y) &:= (\Theta^* q)(t,x,y) := q(\Theta(t,x,y)).
\end{aligned}$$

Then, the Jacobian of Θ and its inverse are of the form

$$D\Theta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ y\partial_t h/\delta & y\partial_1 h/\delta & y\partial_2 h/\delta & h/\delta \end{pmatrix} \text{ and } D\Theta^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{y\delta}{h^2}\partial_t h & -\frac{y\delta}{h^2}\partial_1 h & -\frac{y\delta}{h^2}\partial_2 h & \delta/h \end{pmatrix}.$$

By means of this coordinate transformation we obtain the assertions:

$$\begin{aligned}
\Theta^* \partial_t u &= \partial_t v - \frac{y}{h}(\partial_y v)\partial_t h,\\
\Theta^* \partial_j u &= \partial_j v - \frac{y}{h}(\partial_y u)\partial_j h, \quad j = 1,2,\\
\Theta^* \partial_j^2 u &= \partial_j^2 v - 2\frac{y}{h}(\partial_j \partial_y v)\partial_j h + \frac{y^2}{h^2}(\partial_y^2 v)(\partial_j h)^2 - y(\partial_y v)\left(\frac{h\partial_j^2 h - 2(\partial_j h)^2}{h^2}\right), \quad j = 1,2,\\
\Theta^* \partial_y u &= \frac{\delta}{h}\partial_y v,\\
\Theta^* \partial_y^2 u &= \frac{\delta^2}{h^2}\partial_y^2 v,\\
\Theta^* \Delta u &= [\Delta_x + \frac{\delta^2}{h^2}\partial_y^2]v - 2\frac{y}{h}\nabla_x \partial_y v \nabla_x h + \frac{y^2}{h^2}|\nabla_x h|^2 \partial_y^2 v - \frac{y}{h}(\partial_y v)\Delta h + 2\frac{y}{h^2}(\partial_y v)|\nabla_x h|^2,\\
\Theta^*(u\cdot\nabla)u &= ((\Theta^* u)\cdot(\nabla_x, \frac{\delta}{h})\partial_y)(\Theta^* u) - \frac{y}{h}(\partial_y(\Theta^* u))(v\cdot\nabla_x)h.\\
\Theta^* \nabla q &= (\nabla_x, \frac{\delta}{h}\partial_y)\pi - \frac{y}{h}(\partial_y \pi)(\nabla_x, 0)^T.
\end{aligned}$$

The fourth equation of (1.1) is transformed via the outer normal ν given by

$$\nu := \frac{1}{\sqrt{1+|\nabla_x h|^2}}(-\partial_1 h, -\partial_2 h, 1)^T$$

into

$$\partial_t h = \Theta^* V_\nu \sqrt{1+|\nabla_x h|^2} = \Theta^* \nu \cdot w\sqrt{1+|\nabla_x h|^2} = -\nabla_x h\cdot v + w.$$

In order to compute the transformed stress tensor on Γ^+, we note first that the outer normal ν at the free surface and the outer normal $\nu_D = (0,0,1)^T$ at Γ^+ are related through

$$\nu = \frac{1}{\sqrt{1+|\nabla_x h|^2}}(I+K)^T \nu_D, \quad \text{with } K := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\partial_1 h & -\partial_2 h & 0 \end{pmatrix}.$$

Employing this representation, we compute the transformed stress tensor on the upper boundary to be equal to

$$\Theta^* T(u,q) = \nabla(\Theta^* u)D\theta^{-1} + D\theta^{-T}\nabla(\Theta^* u)^T - Iq.$$

Writing $D\theta^{-1}$ as $D\theta^{-1} = I_{\delta/h}(I+K)$ with $I_{\delta/h} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \delta/h \end{pmatrix}$, we obtain

$$\begin{aligned} \Theta^*T(u,q)\nu &= \frac{1}{\sqrt{1+|\nabla_x h|^2}}\left[\nabla(\Theta^*u)(I_{\delta/h}(I+K)) + (I_{\delta/h}(I+K))^T\nabla(\Theta^*u)^T - I\pi\right](I+K)^T\nu_D \\ &= \frac{1}{\sqrt{1+|\nabla_x h|^2}}\left[(\nabla_x, \frac{\delta}{h}\partial_3)(\Theta^*u) + ((\nabla_x, \frac{\delta}{h}\partial_3)(\Theta^*u))^T - I\pi\right]\nu_D \\ &\quad + \frac{1}{\sqrt{1+|\nabla_x h|^2}}\big[-\nabla_x(\Theta^*u)\nabla_x h + \frac{\delta}{h}|\nabla_x h|^2\partial_3(\Theta^*u) - ((\nabla_x, \frac{\delta}{h}\partial_3)v)^T\nabla_x h \\ &\quad -\frac{\delta}{h}\partial_3 w(\nabla_x h, 0)^T + \frac{\delta}{h}(\nabla_x h, \partial_3 v)(\nabla_x h, 0)^T + \pi(\nabla_x h, 0)^T\big]. \end{aligned}$$

The mean curvature κ is given by

$$\kappa = -\nabla_x\cdot\left(\frac{\nabla_x h}{\sqrt{1+|\nabla_x h|^2}}\right) = -\frac{\Delta_x h}{\sqrt{1+|\nabla_x h|^2}} + \sum_{j,k=1}^{2}\frac{\partial_j h\partial_k h}{(1+|\nabla_x h|^2)^{\frac{3}{2}}}\partial_j\partial_k h.$$

The transformed Navier-slip condition on the lower boundary reads as

$$v = \Theta^*w' = ch^\alpha\Theta^*\partial_y u = ch^\alpha\frac{\delta}{h}\partial_y(\Theta^*u) = c\delta h^{\alpha-1}\partial_y(\Theta^*u).$$

Summarizing, the equation (1.1) reads in transformed coordinates as

$$(3.1)\qquad \left\{\begin{array}{rll} \partial_t u - \Delta u + \nabla\pi &= \chi_R\omega\times(\omega\times(x,y)) + F_1(u,\pi,h) & \text{in } J\times D, \\ \operatorname{div} u &= F_d(u,h) & \text{in } J\times D, \\ \gamma T(u,\pi)\nu_D - \sigma\Delta_x h\nu_D &= G_+(u,\pi,h) & \text{on } J\times\Gamma^+, \\ \partial_t h - \gamma w &= H(u,h) & \text{on } J\times\Gamma^+, \\ \gamma v - \gamma c\delta^\alpha\partial_y v &= G_-(u,h) & \text{on } J\times\Gamma^-, \\ \gamma w &= 0 & \text{on } J\times\Gamma^-, \\ u|_{t=0} &= u_0 & \text{in } D, \\ h|_{t=0} &= h_0+\delta & \text{in } \mathbb{R}^2, \end{array}\right.$$

where the functions on the right hand side above are given by

$$\begin{aligned} F_1(u,p,h) &:= \frac{y}{h}(\partial_y u)\partial_t h + (\delta^2/h^2-1)\partial_y^2 u - 2\frac{y}{h}\nabla_x\partial_y u\nabla_x h + \frac{y^2}{h^2}|\nabla_x h|^2\partial_y^2 u - \frac{y}{h}(\partial_y u)\Delta_x h \\ &\quad + 2\frac{y}{h^2}(\partial_y u)|\nabla_x h|^2 + \frac{y}{h}(\partial_y\pi)(\nabla_x h, 0)^T - (u\cdot(\nabla_x, \frac{\delta}{h}\partial_y))u + \frac{y}{h}(\partial_y u)v\cdot\nabla_x h \\ &\quad + (1-\frac{\delta}{h})(0,0,\partial_y\pi)^T - 2\omega\times u \\ F_d(u,h) &:= (1-\frac{\delta}{h})\partial_y u + \frac{y}{h}\partial_y v\cdot\nabla_x h \\ G^+(u,p,h) &:= (1-\delta/h)\partial_y(v,2w) + \sigma(\frac{1}{\sqrt{1+|\nabla_x h|^2}}-1)\Delta_x h\cdot\nu_D - \sigma\sum_{j,k=1}^{2}\frac{\partial_j h\partial_k h}{(1+|\nabla_x h|^2)^{3/2}}\partial_j\partial_k h\cdot\nu_D \\ &\quad - \frac{\sigma}{\sqrt{1+|\nabla_x h|^2}}(\Delta_x h - \sum_{j,k=1}^{2}\frac{\partial_j h\partial_k h}{1+|\nabla_x h|^2}\partial_j\partial_k h)(\nabla_x h, 0)^T - [-\nabla_x u\nabla_x h + \frac{\delta}{h}|\nabla_x h|^2\partial_y u \\ &\quad - ((\nabla_x, \frac{\delta}{h}\partial_y)v)^T\nabla_x h - \frac{\delta}{h}\partial_y w(\nabla_x h, 0)^T + \frac{\delta}{h}(\nabla_x h, \partial_y v)(\nabla_x h, 0)^T + \pi(\nabla_x h, 0)^T] \\ H(u,h) &:= -\nabla_x h\cdot v \\ G^-(u,h) &:= c\delta(h^{\alpha-1} - \delta^{\alpha-1})\partial_y u. \end{aligned}$$

4. Maximal regularity for the linearized problem

It is the aim of this section to prove maximal regularity estimates for the linearized problem (2.2). To this end, we introduce the function space $\mathbb{F}$ associated with the right hand side of (2.2) as

$$\mathbb{F}(J,D) := \mathbb{F}_1(J,D) \times \mathbb{F}_d(J,D,\Gamma_+) \times \mathbb{G}_+(J,\Gamma^+) \times \mathbb{H}(J,\Gamma^+) \times \mathbb{G}_-(J,\Gamma^-) \times \mathbb{I}_1(D) \times \mathbb{I}_2(\Gamma^+),$$

where

$$\begin{array}{rcl} \mathbb{F}_1(J,D) & := & L_p(J,L_p(D)^3), \\ \mathbb{F}_d(J,D,\Gamma^+) & := & H_p^1(J,{}_0\widehat{H}_p^{-1}(D,\Gamma^+)) \cap H_p^{1/2}(J,L_p(D)) \cap L_p(J,H_p^1(D)), \\ \mathbb{G}_+(J,\Gamma^+) & := & W_p^{1/2-1/2p}(J,L_p(\Gamma^+)^3) \cap L_p(J,W_p^{1-1/p}(\Gamma^+)^3), \\ \mathbb{H}(J,\Gamma^+) & := & W_p^{1-1/2p}(J,L_p(\Gamma^+)) \cap L_p(J,W_p^{2-1/p}(\Gamma^+)), \\ \mathbb{G}_-(J,\Gamma^-) & := & W_p^{1/2-1/2p}(J,L_p((\Gamma^-)^2) \cap L_p(J,W_p^{1-1/p}(\Gamma^-)^2), \\ \mathbb{I}_1(D) & := & W_p^{2-2/p}(D)^3, \\ \mathbb{I}_2(\Gamma^+) & := & W_p^{3-2/p}(\Gamma^+), \end{array}$$

as well as the solution space

$$\mathbb{E}(J,D) := \mathbb{E}_1(J,D) \times \mathbb{E}_2(J,D,\Gamma^+) \times \mathbb{E}_3(J,\Gamma^+)$$

with

$$\begin{array}{rcl} \mathbb{E}_1(J,D) & := & H_p^1(J,L_p(D)^3) \cap L_p(J,H_p^2(D)^3), \\ \mathbb{E}_2(J,D,\Gamma^+) & := & \{\pi \in L_p(J,\widehat{H}_p^1(D)) : \pi \in W_p^{1/2-1/2p}(J,L_p(\Gamma^+)) \cap L_p(J,W_p^{1-1/p}(\Gamma^+))\}, \\ \mathbb{E}_3(J,\Gamma^+) & := & W_p^{2-1/2p}(J,L_p(\Gamma^+)) \cap H_p^1(J,W_p^{2-1/p}(\Gamma^+)) \cap L_p(J,W_p^{3-1/p}(\Gamma^+)). \end{array}$$

Theorem 4.1. *Let $T>0$, $J:=(0,T)$, $p\in(1,\infty)$ with $p\neq 3/2,3$. Then there exists a unique solution $(u,\pi,h)\in\mathbb{E}(J,D)$ of the problem*

$$(4.1)\qquad \left\{\begin{array}{rcll} \partial_t u - \Delta u + \nabla\pi & = & f_1 & \text{in } J\times D, \\ \operatorname{div} u & = & f_d & \text{in } J\times D, \\ \gamma\partial_y v + \gamma\nabla_x w & = & g_v & \text{on } J\times\Gamma^+, \\ \gamma\partial_y w - \gamma p - \sigma\Delta_x h & = & g_w & \text{on } J\times\Gamma^+, \\ \partial_t h - \gamma w & = & f_h & \text{on } J\times\Gamma^+, \\ \gamma v - c\delta^\alpha\gamma\partial_y v & = & g_- & \text{on } J\times\Gamma^-, \\ \gamma w & = & 0 & \text{on } J\times\Gamma^-, \\ u(0) & = & u_0 & \text{in } D, \\ h(0) & = & h_0 & \text{on } \Gamma^+ \end{array}\right.$$

if and only if $(f_1, f_d, g=(g_v,g_w), f_h, g_-, u_0, h_0)$ belong to $\mathbb{F}(J,D)$ and satisfy the compatibility conditions

$$g_v(0) = \gamma\partial_y v_0 + \gamma\nabla_x w_0 \ \text{ on } \Gamma^+ \quad \text{and} \quad g_-(0) = \gamma v_0 - c\delta^\alpha\gamma\partial_y v_0 \ \text{ on } \Gamma^-$$

in the case $p>3$ and

$$\gamma w_0 = 0 \ \text{ on } \Gamma^- \quad \text{and} \quad f_d(0) = \operatorname{div} u_0$$

in the case $p>3/2$.

Note that we omitted δ in the initial value because $(u,p,h+\delta)$ solves the linearized equations with the initial value $h_0+\delta$ if and only if (u,p,h) solves these equations with the initial value h_0.

In order to prove Theorem 4.1, we consider first the problem

$$(4.2)\qquad \begin{cases} \partial_t u - \Delta u + \nabla \pi &= f_1 \quad \text{in } J \times D, \\ \operatorname{div} u &= f_d \quad \text{in } J \times D, \\ \gamma \partial_y v + \gamma \nabla_x w &= 0 \quad \text{on } J \times \Gamma^+, \\ 2\gamma \partial_y w - \gamma \pi - \gamma \sigma \Delta_x h &= 0 \quad \text{on } J \times \Gamma^+, \\ \partial_t h - \gamma w &= 0 \quad \text{on } J \times \Gamma^+, \\ \gamma v - c\delta^\alpha \gamma \partial_y v &= 0 \quad \text{on } J \times \Gamma^-, \\ \gamma w &= 0 \quad \text{on } J \times \Gamma^-, \\ u|_{t=0} &= 0 \quad \text{in } D, \\ h|_{t=0} &= 0 \quad \text{on } \Gamma^+ \end{cases}$$

Then the following result holds.

Lemma 4.2. *Let $p \in (1,\infty)$, $p \neq 3/2, 3$ and let $f_1 \in \mathbb{F}_1(J,D)$ and $f_d \in \mathbb{F}_d(J,D,\Gamma^+)$ satisfying $f_d|_{t=0} = 0$ if $p > 3/2$. Then there exist $J_\tau := [0,\tau]$ for some $\tau > 0$ and a solution $(u,\pi,h) \in \mathbb{E}(J_\tau, D)$ of (4.2) in J_τ satisfying*

$$\|(u,\pi,h)\|_{\mathbb{E}(J_\tau,D)} \leq C \left(\|f_1\|_{\mathbb{F}_1(J_\tau,D)} + \|f_d\|_{\mathbb{F}_d(J_\tau,D,\Gamma^+)} \right).$$

In order to prove Lemma 4.2 it suffices, thanks to Proposition B.1, to consider the *reduced* Stokes problem defined by

$$(4.3)\qquad \begin{cases} \partial_t u - \Delta u + \nabla T_2(u,h) &= f_1 \quad \text{in } J \times D, \\ \gamma \partial_y v + \gamma \nabla_x w &= 0 \quad \text{on } J \times \Gamma^+, \\ \gamma \operatorname{div} u &= g_r \quad \text{in } J \times \Gamma^+, \\ \partial_t h - \gamma w &= 0 \quad \text{on } J \times \Gamma^+, \\ \gamma v - c\delta^\alpha \gamma \partial_y v &= 0 \quad \text{on } J \times \Gamma^-, \\ \gamma w &= 0 \quad \text{on } J \times \Gamma^-, \\ u|_{t=0} &= 0 \quad \text{in } D, \\ h|_{t=0} &= 0 \quad \text{on } \mathbb{R}^2, \end{cases}$$

where $f_1 \in \mathbb{F}_1(J,D)$, $g_r \in G^+(J,\Gamma^+)$ satisfying $g_r(0) = 0$ if $p > 3$. Here

$$G^+(J,\Gamma^+) := W_p^{1/2-1/2p}(J, L_p(\Gamma^+)) \cap L_p(J, W_p^{1-1/p}(\Gamma^+)),$$

and $T_2(u,h)$ is defined by $T_2(u,h) := p_2$, where $p_2 \in H_p^1(D)$ is the unique solution of

$$(4.4)\qquad \begin{cases} \Delta p = 0 & \text{in } D, \\ \gamma \partial_y p = \gamma_\nu (\Delta u - \nabla \operatorname{div} u) & \text{on } \Gamma^-, \\ \gamma p = 2\gamma \partial_y w - \sigma \Delta_x h & \text{on } \Gamma^+, \end{cases}$$

which is guaranteed by Proposition A.5 provided $(u,h) \in \mathbb{E}_1 \times \mathbb{E}_3$. Here $\gamma_\nu u := \gamma \nu \cdot u$.

In order to construct a solution for equation (4.3), we employ a localization procedure to transfer the reduced Stokes problem to two problems in an half-space. More precisely, consider in $D^- := \mathbb{R}^3_+$ the set of equations

$$(4.5)\qquad \begin{cases} \partial_t u - \Delta u + \nabla T_2^-(u) &= f_- \quad \text{in } J \times D^-, \\ \gamma v - c\delta^\alpha \gamma \partial_y v &= 0 \quad \text{on } J \times \Gamma^-, \\ \gamma w &= 0 \quad \text{on } J \times \Gamma^-, \\ u(0) &= 0 \quad \text{in } D^-, \end{cases}$$

where $T_2^-(u)$ is defined by $T_2^-(u) := p_2^-$, where p_2^- is the unique solution of the equation

$$(4.6)\qquad \begin{cases} \Delta p &= 0 \quad \text{in } D^-, \\ \gamma \partial_y p &= \gamma \partial_\nu (\Delta u - \nabla \operatorname{div} u) \quad \text{on } \Gamma^-. \end{cases}$$

Note that by Proposition A.1 and Propostion A.4, T_2^- is well-defined due to the assumption on u.

The following lemma is a consequence of Theorem 5.1 and Proposition B.2(2).

Lemma 4.3. *Let $p \in (1,\infty)$, $p \neq 3/2, 3$. Then there exists $C > 0$ such that for $f_- \in \mathbb{F}_1(J, D^-)$ there exists a unique solution $u_- \in \mathbb{E}_1(J, D^-)$ of equation* (4.5) *satisfying*

$$\|u_-\|_{\mathbb{E}_1(J,D^-)} \leq C\|f_-\|_{\mathbb{F}_1(J,D^-)}.$$

Similary as above, we consider the reduced system also in $D^+ := \mathbb{R}^2 \times (-\infty, \delta)$. It reads as

$$\left\{\begin{array}{rcll} \partial_t u - \Delta u + \nabla T_2^+(u,h) &=& f_+ & \text{in } J \times D^+, \\ \gamma\partial_y v + \gamma\nabla_x w &=& 0 & \text{on } \ J \times \Gamma^+, \\ \gamma \mathrm{div}\, u &=& g_r & \text{on } \ J \times \Gamma^+, \\ \partial_t h - \gamma w &=& 0 & \text{in } J \times \Gamma^+, \\ u(0) &=& 0 & \text{in } D^+, \\ h(0) &=& 0 & \text{in } \Gamma^+, \end{array}\right. \tag{4.7}$$

where $T_2^+(u,h)$ is defined as $T_2^+(u,h) := p_2^+$, where p_2^+ is the unique solution of

$$\left\{\begin{array}{rll} \Delta p &= 0 & \text{in } D^+, \\ \gamma\partial_y p &= 2\gamma\partial_y w - \sigma\Delta_x h & \text{on } \Gamma^+. \end{array}\right. \tag{4.8}$$

By Proposition A.1 and Propostion A.4, T_2^+ is well-defined. Finally, Theorem 5.2 and Proposition B.1 imply the following result.

Lemma 4.4. *Let $p \in (1,\infty)$, $p \neq 3/2, 3$. Then there exists a constant $C > 0$ such that for $f_+ \in \mathbb{F}_1(J, D^+)$ and $g_r \in G^+(J, \Gamma^+)$ satisfying $g(0) = 0$ in the case $p > 3$, there exists a unique solution $(u_+, h) \in \mathbb{E}_1(J, D^+) \times \mathbb{E}_3(J, \Gamma^+)$ of equation* (4.7) *satisfying*

$$\|u_+\|_{\mathbb{E}_1(J,D^+)} + \|h\|_{\mathbb{E}_3(J,\Gamma^+)} \leq C\left(\|f_+\|_{\mathbb{F}_1(J,D^+)} + \|g_r\|_{G^+(J,\Gamma^+)}\right).$$

Proof of Lemma 4.2. Let $\chi_- \in C^\infty(\mathbb{R})$ be a cut-off function satisfying $0 \leq \chi_- \leq 1$ such that

$$\chi_-(y) = \left\{\begin{array}{ll} 1 & \text{for} \quad y \leq \delta/3, \\ 0 & \text{for} \quad y \geq 2\delta/3, \end{array}\right.$$

holds. Moreover, set $\chi_+ := 1 - \chi_-$. It follows from Lemma 4.4 and Lemma 4.3 that for $f \in \mathbb{F}_1(J, D)$ and $g \in G^+(J, \Gamma^+)$ there exist unique solutions (u_+, h) of equation (4.7) and u_- of (4.5), respectively. Inserting $u = \chi_+ u_+ + \chi_- u_-$ in (4.3), we obtain

$$\left\{\begin{array}{rcll} \partial_t u - \Delta u + \nabla T_2(u,h) &=& f + S(f,g) & \text{in } J \times D, \\ \gamma\partial_y v + \gamma\nabla_x w &=& 0 & \text{on } J \times \Gamma^+, \\ \partial_t h - \gamma w &=& 0 & \text{in } J \times \Gamma^+, \\ \gamma \mathrm{div}\, u &=& g & \text{on } J \times \Gamma^+, \\ \gamma v - c\delta^\alpha \gamma\partial_y v &=& 0 & \text{on } J \times \Gamma^-, \\ \gamma w &=& 0 & \text{on } J \times \Gamma^-, \\ u(0) &=& 0 & \text{in } D, \\ h(0) &=& 0 & \text{in } \mathbb{R}^2, \end{array}\right.$$

where $S(f,g)$ is given by

$$\begin{aligned} S(f,g) :=& \nabla T_2(u,h) - \chi_+ \nabla T_2^+(u_+,h) - \chi_- \nabla T_2^-(u_-) - (\Delta\chi_+)u_+ - (\Delta\chi_-)u_- \\ &- 2(\nabla\chi_-)(\nabla u_-) - 2(\nabla\chi_+)(\nabla u_+) \\ =& \big[\nabla\left(T_2(u,h) - \chi_+ T_2^+(u_+,h) - \chi_- T_2^-(u_-)\right)\big] \\ &+ \big[- (\Delta\chi_+)u_+ - (\Delta\chi_-)u_- - 2(\nabla\chi_-)(\nabla u_-) - 2(\nabla\chi_+)(\nabla u_+)\big] \\ &+ \big[(\nabla\chi_+)T_2^+(u_+,h) + (\nabla\chi_-)T_2^-(u_-)\big] \\ =:& S_1(f,g) + S_2(f,g) + S_3(f,g). \end{aligned}$$

By Hölder's inequality and Sobolev's embedding theorem, for $T_0 > 0$ there exist $\beta > 0$ and $C > 0$, independent of $T < T_0$, such that

$$\begin{aligned}\|S_2(f,g)\|_{\mathbb{F}_1(0,T,D)} &\leq C\left(\|u_-\|_{L_p(0,T;H^1_p(D))} + \|u_+\|_{L_p(0,T;H^1_p(D))}\right)\\ &\leq CT^\beta\left(\|u_+\|_{\mathbb{E}_1(0,T,D)} + \|u_-\|_{\mathbb{E}_1(0,T,D)}\right).\end{aligned}$$

Hence, by Propositon 4.4 and 4.3, we obtain

$$\|S_2(f,g)\|_{\mathbb{F}_1(0,T,D)} \leq CT^\beta\left(\|f\|_{\mathbb{F}_1(0,T,D)} + \|g\|_{G^+(0,T,\Gamma^+))}\right).$$

In order to estimate $S_1(f,g)$, note that φ defined by $\varphi := T_2(u,h) - \chi_+T_2^+(u_+,h) - \chi_-T_2^-(u_-)$ solves the equation

$$\left\{\begin{array}{rcll}\Delta\varphi &=& -\mathrm{div}\,\nabla(\chi_+T_2(u_+,h) + \chi_-T_2^-(u_-)) & \text{in } D,\\ \gamma\partial_y\varphi &=& 0 & \text{on } \Gamma^-,\\ \gamma\varphi &=& 0 & \text{on } \Gamma^+.\end{array}\right.$$

It follows from Proposition A.5 that

$$\|\varphi\|_{L_p(0,T;H^1_p(D))} \leq C\| - \mathrm{div}\,\nabla(\chi_+T_2(u_+,h) + \chi_-T_2^-(u_-))\|_{L_p(0,T;{}_0H_p^{-1}(D,\Gamma^+))}.$$

For $\varphi \in^0 H^1_p(D,\Gamma^+)$ we estimate the above right hand side further as

$$\begin{aligned}&|\langle \mathrm{div}\,\nabla\left(\chi_+T_2(u_+,h) + \chi_-T_2(u_-)\right),\varphi\rangle|\\ &= |\langle(\Delta\chi_+)T_2(u_+,h) + 2(\nabla\chi_+)(\nabla T_2(u_+,h)),\varphi\rangle + \langle(\Delta\chi_-)T_2(u_-) + 2(\nabla\chi_-)(\nabla T_2(u_-),\varphi\rangle|\\ &\leq C\left(\|T_2^+(u_+,h)\|_{L_p(0,T;H^1_p(\mathbb{R}^2\times(\delta/3,2\delta/3)))} + \|T_2^-(u_-)\|_{L_p(0,T;H^1_p(\mathbb{R}^2\times(\delta/3,2\delta/3)))}\right)\|\varphi\|_{L_p(D)}.\end{aligned}$$

Consider the solutions of (4.6) and (4.8). Their Fourier transform are given by

$$\begin{aligned}\hat{p}^-(\xi,y) &= (i\frac{\xi}{|\xi|}\partial_y\hat{v}_-(\xi,0) + \|\xi\|\hat{w}_-(\xi,0))\mathrm{e}^{-|\xi|y},\\ \hat{p}^+(\xi,y) &= \left(2\partial_y\hat{w}_+(\xi,\delta) + \sigma|\xi|^2\hat{h}(\xi)\right)\mathrm{e}^{-|\xi|(\delta-y)}\frac{1}{\|\xi\|}.\end{aligned}$$

By Mikhlin's multiplier theorem there exists a constant $C > 0$, independent of u_+, u_- and h such that

$$\begin{aligned}\|T_2^-(u_-)\|_{L_p(0,T;H^1_p(\mathbb{R}^2\times(1/3,2/3)))} &\leq C\left(\|\gamma\partial_yu_-\|_{L_p(0,T;L_p(\mathbb{R}^2))} + \|\gamma u_-\|_{L_p(0,T;L_p(\mathbb{R}^2))}\right),\\ \|T_2^+(u_+,h)\|_{L_p(0,T;H^1_p(\mathbb{R}^2\times(1/3,2/3)))} &\leq C\left(\|\gamma\partial_yu_+\|_{L_p(0,T;L_p(\partial D^+))} + \|\Delta_xh\|_{L_p(0,T;L_p(\mathbb{R}^2))}\right).\end{aligned}$$

Hence,

$$\begin{aligned}\|w\|_{L_p(0,T;H^1_p(D))} \leq& C\big(\|\gamma\partial_yu_+\|_{L_p(0,T;L_p(\partial D^+))} + \|\gamma u_-\|_{L_p(0,T;L_p(\mathbb{R}^2))}\\ &+ \|\gamma\partial_yu_-\|_{L_p(0,T;L_p(\mathbb{R}^2))} + \|\Delta_xh\|_{L_p(0,T;L_p(\mathbb{R}^2))}\big).\end{aligned}$$

By similar arguments as above we obtain

$$\begin{array}{rcl}\|S_1(f,g)\|_{\mathbb{F}_1(0,T;D)} &\leq& CT^\beta\left(\|f\|_{\mathbb{F}_1(0,T,D)} + \|g\|_{G^+(0,T,\Gamma^+)}\right),\\ \|S_3(f,g)\|_{\mathbb{F}_1(0,T,D)} &\leq& CT^\beta\left(\|f\|_{\mathbb{F}_1(0,T,D)} + \|g\|_{G^+(0,T,\Gamma^+)}.\right).\end{array}$$

Summarizing, it follows that

$$\|S(f,g)\|_{\mathbb{F}_1(0,T,D)} \leq \frac{1}{2}\|f\|_{\mathbb{F}_1(0,T,D)} + \|g\|_{G^+(0,T,\Gamma^+)})$$

provided T is small enough. The assertion thus follows by a Neumann series argument. □

Proof of Theorem 4.1. Suppose that the data $f_1, f_d, g_v, g_w, f_h, g_-, u_0, h_0$ satisfies the assumptions given in Theorem 4.1. In a first step we show that the existence of a triple $(\tilde{u}, \tilde{\pi}, \tilde{h}) \in \mathbb{E}(J, D)$ satisfying the last seven equations of (4.1). Keeping the notation introduced in the proof of Lemma 4.2, we define further cut-off functions $\chi_1, \chi_2 \in C^\infty(\mathbb{R})$ satisfying

$$\chi_1(y) = \begin{cases} 1 & \text{for} \quad y \leq \frac{2}{3}\delta, \\ 0 & \text{for} \quad y \geq \frac{5}{6}\delta, \end{cases} \qquad \text{and} \qquad \chi_2(y) = \begin{cases} 1 & \text{for} \quad y \geq \frac{1}{3}\delta, \\ 0 & \text{for} \quad y \leq \frac{1}{6}\delta. \end{cases}$$

By [DPZ08, Theorem 2.1] (see also [DPZ08, Example 3.7]), there exists $(w_+, \tilde{h}) \in \mathbb{E}_1 \times \mathbb{E}_3$ satisfying

$$\left\{ \begin{array}{rcll} \partial_t w_+ - \Delta w_+ & = & 0 & \text{in } J \times D^+, \\ \partial_t \tilde{h} - w_+ & = & f_h & \text{on } J \times \Gamma^+, \\ 2\gamma\partial_y w_+ - \sigma\Delta_x \tilde{h} & = & \mathrm{e}^{\Delta_x \cdot}(\gamma\partial_y w_0 - \sigma\Delta_x \tilde{h}_0) & \text{on } J \times \Gamma^+, \\ w_+(0) & = & \chi_1 w_0 & \text{in } D^+, \\ \tilde{h}(0) & = & h_0 & \text{on } \Gamma^+. \end{array} \right.$$

Here $e^{\Delta_x \cdot}$ refers to the time variable t. We further set

$$\pi_+(t, x, y) = \mathrm{e}^{-\sqrt{-\Delta_x}(\delta - y)} \left(g_w - \mathrm{e}^{\Delta_x \cdot}(\gamma\partial_y w_0 - \sigma\Delta_x \tilde{h}_0) \right)(t, x).$$

Then $(w_+, \pi_+, \tilde{h})$ is a solution of the equation

$$\left\{ \begin{array}{rcll} \partial_t \tilde{h} - w_+ & = & f_h & \text{on } \ J \times \Gamma^+, \\ 2\gamma\partial_y w_+ - \gamma\pi_+ - \sigma\Delta_x \tilde{h} & = & g_w & \text{on } \ J \times \Gamma^+, \\ w_+(0) & = & \chi_1 w_0 & \text{in } \ D^+, \\ \tilde{h}(0) & = & h_0 & \text{on } \ \Gamma^+. \end{array} \right.$$

Furthermore, denote by v_+ the solution of the equation

$$\begin{array}{rcll} \partial_t v_+ - \Delta v_+ & = & 0 & \text{in } \ J \times D^+, \\ \gamma\partial_y v_+ & = & g_v - \gamma\nabla_x w_+ & \text{on } \ J \times \Gamma^+, \\ v_+(0) & = & \chi_1 v_0 & \text{in } \ D^+. \end{array}$$

Hence, by construction the triple $(u_+ = (v_+, w_+), \pi_+, \tilde{h})$ satisfies the equation

$$\begin{array}{rcll} \partial_t \tilde{h} - w_+ & = & f_h & \text{on } \ J \times \Gamma^+, \\ \gamma\partial_y v_+ + \gamma\nabla_x w_+ & = & g_v & \text{on } \ J \times \Gamma^+, \\ 2\gamma\partial_y w_+ - \gamma\pi^+ - \sigma\Delta_x \tilde{h} & = & g_w & \text{on } \ J \times \Gamma^+, \\ u_+(0) & = & \chi_1 u_0 & \text{in } \ D^+, \\ \tilde{h}(0) & = & \tilde{h}_0 & \text{on } \ \Gamma^+. \end{array}$$

Furthermore, by [DHP07, Theorem 2.1] there exists a function u_- satisfying

$$\begin{array}{rcll} \partial_t u_- - \Delta u_- & = & 0 & \text{in } \ J \times D^-, \\ \gamma v_- - c\delta^\alpha \gamma\partial_y v_- & = & g_- & \text{on } \ J \times \Gamma^-, \\ \gamma w_- & = & 0 & \text{on } \ J \times \Gamma^-, \\ u_-(0) & = & \chi_2 u_0 & \text{in } \ D^-, \end{array}$$

provided $\gamma w_0 = 0$ on Γ^- if $p > 3/2$ and $\gamma v_0 - c\delta^\alpha \gamma\partial_y v_0 = g_-(0)$ on Γ^- if $p > 3$.

Setting $\tilde{u} := \chi_+ u_+ + \chi_- u_-$ and $\tilde{p} := \chi_+ p_+$ and chossing $\tilde{h}$ as above, we see that the triple $(\tilde{u}, \tilde{p}, \tilde{h}) \in \mathbb{E}(J, D^+)$ satisfies the last seven equations of (4.1). Moreover, given f_1, f_d as in Theorem 4.1, it follows from Lemma 4.2 that there exists a solution of (4.2) in some small time intervall $J_\tau = (0, \tau)$. By repeating these arguments, we obtain a solution of (4.1) on an arbitrary time intervall J.

Finally, the uniqueness of the solution to (4.1) follows by the following duality argument. To this end, assume that (u,h,p) satisfies

$$(4.9)\qquad \left\{\begin{array}{rcll} \partial_t u - \Delta u + \nabla\pi &=& 0 & \text{in } J\times D,\\ \operatorname{div} u &=& 0 & \text{in } J\times D,\\ \gamma\partial_y v + \gamma\nabla_x w &=& 0 & \text{on } J\times\Gamma^+,\\ 2\gamma\partial_y w - \gamma\pi - \sigma\Delta_x h &=& 0 & \text{on } J\times\Gamma^+,\\ \partial_t h - \gamma w &=& 0 & \text{on } J\times\Gamma^+,\\ \gamma v - c\delta^\alpha\gamma\partial_y v &=& 0 & \text{on } J\times\Gamma^-,\\ \gamma w &=& 0 & \text{on } J\times\Gamma^-,\\ u(0) &=& 0 & \text{in } D,\\ h(0) &=& 0 & \text{on } \Gamma^+ \end{array}\right.$$

and for $\tilde f \in {\mathbb{F}_1}'$ let $(\tilde u,\tilde h,\tilde p)$ be the solution of the *dual backward problem*

$$(4.10)\qquad \left\{\begin{array}{rcll} -\partial_t \tilde u - \Delta \tilde u + \nabla\tilde\pi &=& \tilde f & \text{in } J\times D,\\ \operatorname{div} \tilde u &=& 0 & \text{in } J\times D,\\ \gamma\partial_y \tilde v + \gamma\nabla_x \tilde w &=& 0 & \text{on } J\times\Gamma^+,\\ 2\gamma\partial_y \tilde w - \gamma\tilde\pi - \sigma\Delta_x \tilde h &=& 0 & \text{on } J\times\Gamma^+,\\ -\partial_t \tilde h - \gamma \tilde w &=& 0 & \text{on } J\times\Gamma^+,\\ \gamma \tilde v - c\delta^\alpha\gamma\partial_y \tilde v &=& 0 & \text{on } J\times\Gamma^-,\\ \gamma \tilde w &=& 0 & \text{on } J\times\Gamma^-,\\ \tilde u(T) &=& 0 & \text{in } D,\\ \tilde h(T) &=& 0 & \text{on } \Gamma^+ \end{array}\right.$$

Integration by parts yields

$$\begin{aligned} 0 =&\langle \partial_t u - \Delta u + \nabla\pi, \tilde u\rangle_{J\times D} = \langle u, -\partial_t\tilde u - \Delta\tilde u\rangle_{J\times D} + \langle u(s),\tilde u(s)\rangle_D\big|_0^T - \langle\gamma\partial_y u,\gamma\tilde u\rangle_{J\times\Gamma^+}\\ &- \langle\gamma u,\gamma\tilde u\rangle_{J\times\Gamma^-} + \langle\gamma u,\gamma\partial_y\tilde u\rangle_{J\times\Gamma^+} + \langle\gamma u,\gamma\partial_y\tilde u\rangle_{J\times\Gamma^-} + \langle\gamma\pi,\gamma\tilde w\rangle_{J\times\Gamma^+} + \langle\gamma\pi,\gamma\tilde w\rangle_{J\times\Gamma^-}\\ =&: I_1 + \ldots + I_8. \end{aligned}$$

Since $u(0) = \tilde u(T) = 0$, we have $I_2 = 0$. The two equations on Γ^- imply $I_4 + I_6 + I_8 = 0$. Moreover, the equations describing the tangential stresses on Γ^+, integration by parts, and the divergence free conditions yield

$$\begin{aligned} I_5 + I_3 =&\langle\gamma v,\gamma\partial_y\tilde v\rangle_{J\times\Gamma^+} - \langle\gamma\partial_y v,\gamma\tilde v\rangle_{J\times\Gamma^+} + \langle\gamma w,\gamma\partial_y\tilde w\rangle_{J\times\Gamma^+} - \langle\gamma\partial_y w,\gamma\tilde w\rangle_{J\times\Gamma^+}\\ =&\langle\gamma v,-\gamma\nabla_x\tilde w\rangle_{J\times\Gamma^+} + \langle\gamma\nabla_x w,\gamma\tilde v\rangle_{J\times\Gamma^+} + \langle\gamma w,\gamma\partial_y\tilde w\rangle_{J\times\Gamma^+} - \langle\gamma\partial_y w,\gamma\tilde w\rangle_{J\times\Gamma^+}\\ =&\langle\gamma\nabla_x v,\gamma\tilde w\rangle_{J\times\Gamma^+} - \langle\gamma w,\gamma\nabla_x\tilde v\rangle_{J\times\Gamma^+} + \langle\gamma w,\gamma\partial_y\tilde w\rangle_{J\times\Gamma^+} - \langle\gamma\partial_y w,\gamma\tilde w\rangle_{J\times\Gamma^+}\\ =&-\langle\gamma\partial_y w,\gamma\tilde w\rangle_{J\times\Gamma^+} + \langle\gamma w,\gamma\partial_y\tilde w\rangle_{J\times\Gamma^+} + \langle\gamma w,\gamma\partial_y\tilde w\rangle_{J\times\Gamma^+} - \langle\gamma\partial_y w,\gamma\tilde w\rangle_{J\times\Gamma^+}\\ =&\langle\gamma w,2\gamma\partial_y\tilde w\rangle_{J\times\Gamma^+} - \langle 2\gamma\partial_y w,\gamma\tilde w\rangle_{J\times\Gamma^+}. \end{aligned}$$

The equations for the normal stress on Γ^+, the equations for the normal velocity of Γ^+ as well as integration by parts yield

$$\begin{aligned} I_5 + I_3 =&\langle\gamma w,\gamma\tilde\pi + \sigma\Delta'\tilde h\rangle_{J\times\Gamma^+} - \langle\gamma\pi + \sigma\Delta' h,\gamma\tilde w\rangle_{J\times\Gamma^+}\\ =&\langle\gamma w,\gamma\tilde\pi\rangle_{J\times\Gamma^+} - \langle\gamma\pi,\gamma\tilde w\rangle_{J\times\Gamma^+} + \langle\partial_t h,\sigma\Delta_x\tilde h\rangle_{J\times\Gamma^+} + \langle\sigma\Delta_x h,\partial_t\tilde h\rangle_{J\times\Gamma^+}\\ =&\langle\gamma w,\gamma\tilde\pi\rangle_{J\times\Gamma^+} - \langle\gamma\pi,\gamma\tilde w\rangle_{J\times\Gamma^+} - \langle\partial_t\nabla_x h,\sigma\nabla_x\tilde h\rangle_{J\times\Gamma^+} + \langle\sigma\nabla_x\partial_t h,\nabla\tilde h\rangle_{J\times\Gamma^+}\\ =&\langle\gamma w,\gamma\tilde\pi\rangle_{J\times\Gamma^+} - \langle\gamma\pi,\gamma\tilde w\rangle_{J\times\Gamma^+}. \end{aligned}$$

Summarizing, we obtain

$$0 =\langle u, -\partial_t\tilde u - \Delta\tilde u\rangle_{J\times D} + \langle\gamma w,\gamma\tilde\pi\rangle_{J\times\Gamma^+} = \langle u,-\partial_t\tilde u - \Delta\tilde u + \nabla\tilde\pi\rangle_{J\times D} = \langle u,\tilde f\rangle_{J\times D},\quad \tilde f\in{\mathbb{F}_1}',$$

which implies that $u\equiv 0$. Now, it is not difficult to show that $h = 0$ and $\pi = 0$. □

5. Analysis of model problems in the half-space

We start this section with the model problem related to the bottom boundary. Similarly as above, by Proposition B.2, the reduced Stokes problem (4.5) is uniquely solvable if and only if the problem

$$\left\{\begin{array}{rcll} \partial_t u - \Delta u + \nabla\pi &=& f_1 & \text{in } J\times D^-,\\ \operatorname{div} u &=& f_d & \text{in } J\times D^-,\\ \gamma v - c\delta^\alpha\gamma\partial_y v &=& 0 & \text{on } J\times\Gamma^-,\\ \gamma w &=& 0 & \text{on } J\times\Gamma^-,\\ u|_{t=0} &=& 0 & \text{in } D^-. \end{array}\right. \tag{5.1}$$

is uniquely solvable. It was shown in [Shi08, Theorem 5.1] that, given $(f_1,f_3)\in\mathbb{F}_1(J,D^-)\times\mathbb{G}^-(J,\Gamma^-)$ and $\beta>0$, the system

$$\left\{\begin{array}{rcll} \partial_t u - \Delta u + \nabla\pi &=& f_1 & \text{in } J\times D^-,\\ \operatorname{div} u &=& 0 & \text{in } J\times D^-,\\ \beta\gamma v - \gamma\partial_y v &=& f_3 & \text{on } J\times\Gamma^-,\\ \gamma w &=& 0 & \text{on } J\times\Gamma^-,\\ u|_{t=0} &=& 0 & \text{in } D^-. \end{array}\right. \tag{5.2}$$

there exists a unique $(u,\pi)\in\mathbb{E}_1(J,D^-)\times L_p(J,\widehat{H}^1_p(D^-))$ satisfying (5.2) and a constant $C>0$ such that

$$\|u\|_{\mathbb{E}_1}+\|\pi\|_{L^p(\widehat{H}^1_p)}\le C(\|f_1\|_{\mathbb{F}_1}+\|f_3\|_{\mathbb{G}^-}). \tag{5.3}$$

To be precise, in [Shi08, Theorem 5.1] only the case $\beta=0$ was considered. However, by a perturbation argument similar to the one employed in the proof of Proposition 4.2, the result cited above extends to the case $\beta\ge 0$.

Consider now $(f_1,f_d)\in\mathbb{F}_1(J,D^-)\times\mathbb{F}_d(J,D^-,\emptyset)$. It is well known that there exists a unique solution (u_1,p_1) to the half space problem with pure Dirichlet boundary conditions

$$\left\{\begin{array}{rcll} \partial_t u - \Delta u + \nabla\pi &=& f_1 & \text{in } J\times D^-,\\ \operatorname{div} u &=& f_d & \text{in } J\times D^-,\\ \gamma u &=& 0 & \text{on } J\times\Gamma^-,\\ u|_{t=0} &=& 0 & \text{in } D^-, \end{array}\right. \tag{5.4}$$

satisfying estimate (5.3). Furthermore, let (u_2,p_2) be the unique solution to (5.2) with right hand sides $f_1=0$, $f_3=\gamma\partial_y u_1$, and with $\beta=\frac{1}{c\delta^\alpha}$. Then $(u,\pi):=(u_1,\pi_1)+(u_2,\pi_2)$ solves the system (5.1). Summarizing, we have the following result

Lemma 5.1. *For every $(f_1,f_d)\in\mathbb{F}_1(J,D^-)\times\mathbb{F}_d(J,D^-,\emptyset)$ there exist a unique $(u,\pi)\in\mathbb{E}_1(J,D^-)\times L_p(J,\widehat{H}^1_p(D^-))$ satisfying (5.1) and a constant $C>0$ such that*

$$\|u\|_{\mathbb{E}_1}+\|\pi\|_{L_p(\widehat{H}^1_p)}\le C(\|f_1\|_{\mathbb{F}_1}+\|f_d\|_{\mathbb{F}_2}). \tag{5.5}$$

In the following, we consider the model problem related to the top boundary. More precisely, we aim to give a proof of Proposition 4.4. By Proposition B.2, it suffices to show that for every $f_1\in\mathbb{F}_1(J,D^+)$ and $f_d\in\mathbb{F}_d(J,D^+,\Gamma^+)$ satisfying the compatibility condition $f_d(0)=0$ if $p>\frac{3}{2}$, there exists a unique solution $(u,p,h)\in\mathbb{E}_1(J,D^+)\times\mathbb{E}_2(J,D^+,\Gamma^+)\times\mathbb{E}_3(J,\Gamma^+)$ of the equations

$$\left\{\begin{array}{rcll} \partial_t u - \Delta u + \nabla\pi &=& f_1 & \text{in } \mathbb{R}_+\times D^+,\\ \operatorname{div} u &=& f_d & \text{in } \mathbb{R}_+\times D^+,\\ \gamma\partial_y v + \gamma\nabla_x w &=& 0 & \text{on } \mathbb{R}_+\times\Gamma^+,\\ 2\gamma\partial_y w - \gamma\pi - \sigma\Delta_x h &=& 0 & \text{on } \mathbb{R}_+\times\Gamma^+,\\ \partial_t h - \gamma w &=& 0 & \text{on } \mathbb{R}_+\times\Gamma^+,\\ u(0) &=& 0 & \text{in } D^+,\\ h(0) &=& 0 & \text{on } \Gamma^+. \end{array}\right. \tag{5.6}$$

The transformation $y \mapsto \delta - y$, $w \mapsto -w$ yields the following boundary value problem in the half-space $\mathbb{R}^3_+$

$$(5.7)\qquad \begin{cases} \partial_t u - \Delta u + \nabla \pi &= f_1 \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^3_+, \\ \operatorname{div} u &= f_d \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^3_+, \\ \gamma \partial_y v + \gamma \nabla_x w &= 0 \quad \text{on } \mathbb{R}_+ \times \mathbb{R}^2 \\ 2\gamma \partial_y w - \gamma \pi - \sigma \Delta_x h &= 0 \quad \text{on } \mathbb{R}_+ \times \mathbb{R}^2, \\ \partial_t h - \gamma w &= 0 \quad \text{on } \mathbb{R}_+ \times \mathbb{R}^2, \\ u(0) &= 0 \quad \text{in } \mathbb{R}^3_+, \\ h(0) &= 0 \quad \text{on } \mathbb{R}^2. \end{cases}$$

In order to formulate the main result of this section, we introduce the spaces ${}_0\mathbb{F}_1(J,D)$, ${}_0\mathbb{F}_d(J,D,\Gamma^+)$, ${}_0\mathbb{E}_j(J,D)$ for $j = 1,2,3$, ${}_0\mathbb{G}^+(J,\Gamma^+)$, ${}_0\mathbb{G}^-(J,\Gamma^-)$ and ${}_0\mathbb{H}(J,\Gamma^+)$ of zero time traces at the origin and set

$${}_0\mathbb{F}(J,D) :=_0 \mathbb{F}_1(J,D) \times {}_0\mathbb{F}_d(J,D,\Gamma^+) \times {}_0\mathbb{G}^+(J,\Gamma^+) \times {}_0\mathbb{H}(J,\Gamma^+) \times {}_0\mathbb{G}^-(J,\Gamma^-) \times \{0\} \times \{0\}$$

as well as

$${}_0\mathbb{E}(J,D) := {}_0\mathbb{E}_1(J,D) \times \mathbb{E}_2(J,D,\Gamma^+) \times {}_0\mathbb{E}_3(J,\Gamma^+).$$

Then the following result holds true.

Theorem 5.2. *Let $1 < p < \infty$ and $p \neq 3/2, 3$. Then, for every $(f_1, f_d) \in \mathbb{F}_1(J,\mathbb{R}^3_+) \times {}_0\mathbb{F}_d(J,\mathbb{R}^3_+,\mathbb{R}^2)$, equation* (5.7) *has a unique solution $(u,p,h) \in {}_0\mathbb{E}(J,\mathbb{R}^3_+)$ satisfying*

$$\|(u,p,h)\|_{\mathbb{E}(J,\mathbb{R}^3_+)} \leq C\left(\|f_1\|_{\mathbb{F}_1} + \|f_d\|_{\mathbb{F}_d}\right).$$

for some $C > 0$.

By classical results, for every $(f_1, f_d) \in \mathbb{F}_1(J,\mathbb{R}^3_+) \times {}_0\mathbb{F}_d(J,\mathbb{R}^3_+,\mathbb{R}^2)$, there exists a unique solution $(\bar{u},\bar{\pi}) \in \mathbb{E}_1(J,\mathbb{R}^3_+) \times \mathbb{E}_2(J,\mathbb{R}^3_+,\mathbb{R}^2)$ of the classical Stokes problem

$$\begin{aligned} \partial_t u - \Delta u + \nabla p &= f_1 \quad \text{in } J \times \mathbb{R}^3_+, \\ \operatorname{div} u &= f_d \quad \text{in } J \times \mathbb{R}^3_+, \\ \gamma \partial_y v + \gamma \nabla_x w &= 0 \quad \text{on } J \times \mathbb{R}^2, \\ 2\gamma \partial_y w - \gamma p &= 0 \quad \text{on } J \times \mathbb{R}^2, \\ v|_{t=0} &= 0 \quad \text{in } \mathbb{R}^3_+. \end{aligned}$$

Note here that $\gamma w \in {}_0\mathbb{H}(J,\mathbb{R}^2)$. Considering $u - \bar{u}$ and $\pi - \bar{\pi}$, we see that (5.7) is uniquely solvable if and only if

$$(5.8)\qquad \begin{cases} \partial_t u - \Delta u + \nabla \pi = 0 & \text{in } J \times \mathbb{R}^3_+, \\ \operatorname{div} u = 0 & \text{in } J \times \mathbb{R}^3_+, \\ \partial_t h + \gamma w = f_h & \text{on } J \times \mathbb{R}^2, \\ \gamma \partial_y v + \gamma \nabla_x w = 0 & \text{on } J \times \mathbb{R}^2, \\ 2\gamma \partial_y w - \gamma \pi - \sigma \Delta_x h = 0 & \text{on } J \times \mathbb{R}^2, \\ u(0) = 0 & \text{in } \mathbb{R}^3_+, \\ h(0) = 0 & \text{on } \mathbb{R}^2 \end{cases}$$

has a unique solution. Therefore, Theorem 5.2 is a consequence of the following result.

Proposition 5.3. *For every $f_h \in {}_0\mathbb{H}(J,\mathbb{R}^2)$, equation* (5.8) *has a unique solution $(u,p,h) \in {}_0\mathbb{E}(J,\mathbb{R}^3_+)$.*

We subdivide the proof in several steps.
(i) *Symbols of the solution operators.*
Applying Laplace transform in t and partial Fourier transform in $x \in \mathbb{R}^2$, we obtain the following system

of ordinary differential equations in y for the transformed functions $\hat{u}$, $\hat{p}$ and $\hat{h}$.

$$
\begin{aligned}
\omega^2\hat{u} - \partial_y^2\hat{u} + (i\xi, \partial_y)^T\hat{\pi} &= 0, \quad y > 0,\\
i\xi \cdot \hat{v} + \partial_y \hat{w} &= 0, \quad y > 0,\\
\lambda\hat{h} + \gamma\hat{w} &= \hat{f}_h,\\
\gamma\partial_3\hat{v} + i\xi\gamma\hat{w} &= 0,\\
2\gamma\partial_y\hat{w} - \gamma\hat{\pi} + \sigma|\xi|^2\hat{h} &= 0.
\end{aligned}
\tag{5.9}
$$

Here we have set $\omega := \omega(|\xi|, \lambda) := \sqrt{\lambda + |\xi|^2}$. Multiplying the first equation with $(i\xi, \partial_y)$ and combing it with the second one yields $(-|\xi|^2 + \partial_y^2)\hat{\pi} = 0$ for $y > 0$. The only stable solution of this equation is given by

$$\hat{\pi}(\xi, y) = \hat{p}_1(\xi)e^{-|\xi|y}, \quad \xi \in \mathbb{R}^2,\ y > 0.$$

To solve the above system we employ the ansatz

$$\hat{v}(\xi, y) = -\int_0^\infty k_-(\xi, y, s)i\xi\hat{\pi}(\xi, s)ds + \hat{\phi}_v(\xi)e^{-\omega y}, \quad \xi \in \mathbb{R}^2,\ y > 0, \tag{5.10}$$

$$\hat{w}(\xi, y) = -\int_0^\infty k_+(\xi, y, s)\partial_y\hat{\pi}(\xi, s)ds + \hat{\phi}_w(\xi)e^{-\omega y}, \quad \xi \in \mathbb{R}^2,\ y > 0, \tag{5.11}$$

with

$$k_\pm(\xi, y, s) := \frac{1}{2\omega}\left(e^{-\omega|y-s|} \pm e^{-\omega(y+s)}\right).$$

Here the initial values $\hat{p}_1(\xi)$ and $\hat{\phi}(\xi) = (\hat{\phi}_u(\xi), \hat{\phi}_w(\xi))^T$ still have to be determined. It then follows that

$$\gamma\partial_y\hat{w} = -\omega\hat{\phi}_w \ \text{ on } \mathbb{R}^2 \quad \text{ and } \quad i\xi\cdot\gamma\hat{v} \ = i\xi\cdot\hat{\phi}_v(\xi) \quad \text{ on } \quad \mathbb{R}^2. \tag{5.12}$$

Formula (5.11) for $\hat{w}$ implies

$$\hat{w}|_{y=0} = -\int_0^\infty k_+(\xi, 0, s)\partial_y\hat{p}(s)ds + \hat{\phi}_w = \frac{|\xi|}{\omega}\hat{p}_1\int_0^\infty e^{-\omega s}e^{-|\xi|s}ds + \hat{\phi}_w = \frac{|\xi|}{\omega(\omega + |\xi|)}\hat{p}_1 + \hat{\phi}_w. \tag{5.13}$$

In view of $\partial_y k_-(\xi, 0, s) = e^{-\omega s}$ we see that

$$\gamma\partial_y\hat{v}|_{y=0} = -\omega\hat{\phi}_v - \frac{i\xi}{\omega + |\xi|}\hat{p}_1. \tag{5.14}$$

Inserting (5.12)–(5.14) into the last four equations of (5.9), we obtain

$$i\xi\cdot\hat{\phi}_v - \omega\hat{\phi}_w = 0, \tag{5.15}$$

$$\lambda\hat{h} + \frac{|\xi|}{\omega(\omega + |\xi|)}\,\hat{p}_1 + \hat{\phi}_w = \hat{f}_h \tag{5.16}$$

$$-\omega\hat{\phi}_v - i\xi\,\frac{\omega - |\xi|}{\omega(\omega + |\xi|)}\,\hat{p}_1 + i\xi\hat{\phi}_w = 0 \tag{5.17}$$

$$-2\omega\hat{\phi}_w - \hat{p}_1 + \sigma|\xi|^2\hat{h} = 0. \tag{5.18}$$

Multiplying (5.17) by $i\xi$ from the left and inserting the product into (5.15), we obtain

$$(\omega^2 + |\xi|^2)\hat{\phi}_w = \frac{|\xi|(\omega - |\xi|)}{\omega(\omega + |\xi|)}\,\hat{p}_1.$$

Inserting this expression into (5.16) and (5.18) yields

$$
\begin{aligned}
\lambda\hat{h} + c_1(\xi, \lambda)\hat{p}_1 &= \hat{f}_h,\\
\sigma|\xi|^2\hat{h} - c_2(\xi, \lambda)\hat{p}_1 &= 0,
\end{aligned}
$$

where c_1 and c_2 are defined as

$$c_1(\lambda,\xi) := \frac{|\xi|}{\omega(\omega+|\xi|)} + \frac{|\xi|^2(\omega-|\xi|)}{\omega(\omega+|\xi|)(\omega^2+|\xi|^2)} = \frac{|\xi|}{\lambda+2|\xi|^2},$$
$$c_2(\lambda,\xi) := \frac{2|\xi|^2(\omega-|\xi|)}{(\omega+|\xi|)(\omega^2+|\xi|^2)} + 1 = \frac{\omega^3+\lambda|\xi|+3\omega|\xi|^2}{(\omega+|\xi|)(\lambda+2|\xi|^2)}.$$

Hence, we obtain the following representation formulas for $\hat{h}, \hat{p}_1$ and $\hat{\phi}_w$

$$\hat{h} = \frac{c_2(\lambda,\xi)}{\lambda c_2(\lambda,\xi)+\sigma|\xi|^2 c_1(\lambda,\xi)}\hat{f}_h = \frac{\omega^3+\lambda|\xi|+3\omega|\xi|^2}{\lambda(\omega^3+\lambda|\xi|+3\omega|\xi|^2)+\sigma|\xi|^3(\omega+|\xi|)}\hat{f}_h =: \frac{m_1(\lambda,|\xi|)}{m_2(\lambda,|\xi|)}\hat{f}_h, \tag{5.19}$$

$$\hat{p}_1 = \frac{(\omega+|\xi|)(\omega^2+|\xi|^2)\sigma|\xi|^2}{m_1(\lambda,|\xi|)}\hat{h}, \tag{5.20}$$
$$\hat{\phi}_w = \frac{|\xi|^2(\omega-|\xi|)\sigma|\xi|^2}{\omega m_1(\lambda,|\xi|)}\hat{h}.$$

(ii) *Maximal regularity for the related solution operators.*
In the following we show maximal regularity for the solution operators defined above. Our strategy to do this relies on the joint $\mathcal{H}^\infty$-calculus for the Laplacian and the time derivative, see e.g. [KW01]. As the symbols $m_1(\lambda,|\xi|)$ and $m_2(\lambda,|\xi|)$ are not quasi-homogeneous with respect to λ and ξ, we will apply the theory of the Newton polygon which, for the convienince of the reader, is summarized in Appendix C.

First, we collect some basics on the weighted space $\mathbb{F}^r_{p,\rho}(J,X)$, which will be used frequently in the sequel. These spaces are defined for $r\in\mathbb{N}_0$ and $\rho\geq 0$ by

$$\mathcal{F}^r_{p,\rho}(J,X) := \Big\{u\in\mathcal{D}'(J,X) : \Psi_\rho(\frac{d}{dt})^k u \in L^p(J,X) \quad (0\leq k\leq r)\Big\}$$

equipped with its canonical norm $\|u\|_{\mathcal{F}^r_{p,\rho}(J,X)}$ defined by

$$\|u\|^p_{\mathcal{F}^r_{p,\rho}(J,X)} := \sum_{k=0}^r \|\Psi_\rho(\frac{d}{dt})^k u\|^p_{L^p(J,X)}.$$

Here the operator Ψ_ρ is defined by multiplication with $e^{-\rho t}$, that is,

$$\Psi_\rho u(t) := e^{-\rho t}u(t), \quad t\in J. \tag{5.21}$$

For $r\in\mathbb{R}_+\setminus\mathbb{N}$ the spaces $\mathcal{F}^r_{p,\rho}(J,X)$ are defined by complex or real interpolation. More precisely, if $\mathcal{F}=H$ we set

$$\mathcal{F}^r_{p,\rho}(J,X) = H^r_{p,\rho}(J,X) := \Big[H^{[r]}_{p,\rho}(J,X), H^{[r]+1}_{p,\rho}(J,X)\Big]_{r-[r]} \tag{5.22}$$

and, if $\mathcal{F}=W$, we set

$$\mathcal{F}^r_{p,\rho}(J,X) = W^r_{p,\rho}(J,X) := \Big(H^{[r]}_{p,\rho}(J,X), H^{[r]+1}_{p,\rho}(J,X)\Big)_{r-[r],p}, \tag{5.23}$$

where $[r] := \max\{k\in\mathbb{N}_0 : k\leq r\}$. The assertions of the following lemma can be verified easily.

Lemma 5.4. *Let $1<p<\infty$, $r,\rho,\omega\geq 0$, and X be a Banach space. Further, let $T\in(0,\infty)$, $J=(0,T)$, and $\mathcal{F}\in\{H,W\}$. Then the following holds:*

(i) $\|\cdot\|_{L^p_\rho(J,X)} \leq \|\cdot\|_{L^p_\omega(J,X)} \leq e^{(\rho-\omega)T}\|\cdot\|_{L^p_\rho(J,X)}$ $\quad(T>0, 0\leq\omega\leq\rho)$.

(ii) $\Psi_\rho\in Isom(\mathcal{F}^r_{p,\rho}((0,T_0),X),\mathcal{F}^r_p((0,T_0),X))$ *for each $T_0\in(0,\infty]$. Furthermore, the norms*

$$\|\cdot\|_{W^r_{p,\rho}((0,T_0),X)}, \quad \|\Psi_\rho\cdot\|_{W^r_p((0,T_0),X)}, \quad \text{and } \|\cdot\|_{H^{[r]}_{p,\rho}((0,T_0),X)} + \langle\langle\Psi_\rho(d/dt)^{[r]}u\rangle\rangle_{r-[r],p,X}$$

are equivalent, where

$$\langle\langle g\rangle\rangle_{r-[r],p,X} := \sum_{|\alpha|=[r]}\Big(\int_\Omega\int_\Omega \frac{\|\partial^\alpha g(x)-\partial^\alpha g(y)\|^p_X}{|x-y|^{n+(r-[r])p}}\mathrm{d}x\mathrm{d}y\Big)^{1/p}.$$

(iii) $\mathcal{F}^r_{p,\rho}(J,X) = \mathcal{F}^r_p(J,X)$ *for* $T<\infty$ *with equivalent norms.*
(iv) $\mathcal{F}^r_{p,\omega}(\mathbb{R}_+,X) \hookrightarrow \mathcal{F}^r_{p,\rho}(\mathbb{R}_+,X)$ *for* $0\le\omega\le\rho$.
(v) *there exists an extension operator*

$$E:\mathcal{F}^r_{p,\rho}(J,X)\to\mathcal{F}^r_{p,\omega}(\mathbb{R}_+,X).$$

(vi) *The assertions (i) to (v) remain valid if* $\mathcal{F}$ *is replaced by* ${}_0\mathcal{F}$.
(vii) *For reflexive* X *statements (i) to (v) remain valid for* $\mathcal{F}^{-r}_{p,\omega}(\mathbb{R}_+,X) := (\mathcal{F}^r_{p',\omega}(\mathbb{R}_+,X'))'$, $r>0$.

In the following we often use the equivalence stated in (iii) without further notice. We further define related operators on the space ${}_0\mathcal{F}^r_{p,\rho}(\mathcal{K}^s_p) := {}_0\mathcal{F}^r_{p,\rho}(\mathbb{R}_+,\mathcal{K}^s_p(\mathbb{R}^n))$ for $1<p<\infty$ and $r,s\in\mathbb{R}$. Here

$$\mathcal{F},\mathcal{K}\in\{H,W\},$$

i.e., by $\mathcal{K}^s_p$ we mean either the space H^s_p or the space W^s_p. We define

$$Gu := \partial_t u,\quad u\in\mathscr{D}(G) := {}_0\mathcal{F}^{r+1}_{p,\rho}(\mathcal{K}^s_p)$$

and

$$D_n u := (-\Delta)^{1/2}u,\quad u\in\mathscr{D}(D_n) := {}_0\mathcal{F}^r_{p,\rho}(\mathcal{K}^{s+1}_p).$$

We recall from [DSS08, Proposition 2.7] that

$$G, D_n \in \mathcal{RH}^\infty({}_0\mathbb{F}^r_{p,\rho}(\mathcal{K}^s_p)),\quad \phi^{\mathcal{R},\infty}_G = \pi/2,\quad \phi^{\mathcal{R},\infty}_{D_n}=0,$$

i.e., G and D_n admit an $\mathcal{R}$-bounded $\mathcal{H}^\infty$-calculus with $\mathcal{RH}^\infty$-angle $\phi^{\mathcal{R},\infty}_G = \pi/2$ and $\phi^{\mathcal{R},\infty}_{D_n}=0$, respectively.

The following lemma is crucial for the proof of Proposition 5.3. It establishes closedness and invertibility of the operators that correspond to the symbols ω, m_1, m_2 on their natural domains.

Lemma 5.5. *Let* $1<p<\infty$ *and* $r,s\ge 0$. *There exists a* $\rho_0\ge 0$ *such that for all* $\rho\ge\rho_0$ *we have that*

(i) $\omega(D_n,G)\in Isom\left(\mathscr{D}(\omega(D_n,G)),\, {}_0\mathcal{F}^r_{p,\rho}(\mathcal{K}^s_p)\right)$, *where*

$$\mathscr{D}(\omega(D_n,G)) = {}_0\mathcal{F}^{r+1/2}_{p,\rho}(\mathcal{K}^s_p)\cap{}_0\mathcal{F}^r_{p,\rho}(\mathcal{K}^{s+1}_p),$$

(ii) $m_1(D_n,G)\in Isom\left(\mathscr{D}(m_1(D_n,G)),\, {}_0\mathcal{F}^r_{p,\rho}(\mathcal{K}^s_p))\right)$, *where*

$$\mathscr{D}(\omega(D_n,G)) = {}_0\mathcal{F}^{r+3/2}_{p,\rho}(\mathcal{K}^s_p)\cap{}_0\mathcal{F}^r_{p,\rho}(\mathcal{K}^{s+3}_p),$$

(iii) $m_2(D_n,G)\in Isom\left(\mathscr{D}(m_2(D_n,G)),\, {}_0\mathcal{F}^r_{p,\rho}(\mathcal{K}^s_p))\right)$, *where*

$$\mathscr{D}(\omega(D_n,G)) = {}_0\mathcal{F}^{r+5/2}_{p,\rho}(\mathcal{K}^s_p)\cap{}_0\mathcal{F}^{r+1}_{p,\rho}(\mathcal{K}^{s+3}_p)\cap{}_0\mathcal{F}^r_{p,\rho}(\mathcal{K}^{s+4}_p).$$

Proof. We intend to apply Corollary C.3. To this end, let $\varphi_0\in(\pi/3,\pi/2)$, $\varphi\in(0,\pi)$, and set

$$\Sigma_{\varphi,\varphi_0} := \left(\overline{\Sigma}_\varphi\setminus\{0\}\right)\times\left(\overline{\Sigma}_{\pi-\varphi_0}\setminus\{0\}\right).$$

Adopting the notation of Appendix C, we need to show that

$$P^j_R(z,\lambda)\neq 0\qquad ((z,\lambda)\in\Sigma_{\varphi,\varphi_0}, R>0, j=1,2,3)\,, \tag{5.24}$$

where $P^j_R(z,\lambda) = \lim_{\sigma\to\infty}\sigma^{-d_R(P^j)}P^j(\sigma^R\lambda,\sigma z)$ and where $P^1 := \omega$, $P^2 := m_1$, $P^3 := m_2$, and $d_R(P^j)$ is defined as in (C.2).

(i) In this case $I=\{(0,0,1)\}$, i.e., $N(P^1) = \operatorname{conv}\{(0,0),(0,1/2),(1,0)\}$. This implies that

$$d_R(P^1) = \begin{cases} 1, & 0<R\le 2,\\ R/2, & R>2.\end{cases}$$

From this we easily calculate that

$$P^1_R(z,\lambda) = \begin{cases} z, & 0<R<2,\\ \sqrt{\lambda+z^2}, & R=2,\\ \sqrt{\lambda}, & R>2.\end{cases}$$

It is also obvious that for $\varphi \in (0, \varphi_0/2)$ we deduce

$$\sqrt{\lambda + z^2} \neq 0 \qquad ((z,\lambda) \in \Sigma_{\varphi,\varphi_0}) .$$

Thus, condition (5.24) is satisfied for P^1. Corollary C.3 then implies (i).
(ii) Also here the related Newton polygon is still a triangle. We have $I = \{(2,0,1),(1,1,0),(0,0,3)\}$, i.e.,

$$N(P^2) = \operatorname{conv}\{(0,0),(2,1/2),(3,0),(1,1),(0,3/2)\} = \operatorname{conv}\{(0,0),(0,3/2),(3,0)\},$$

since $(2,1/2)$ and $(1,1)$ lie on the line connecting $(0,3/2)$ and $(3,0)$. This gives us

$$P_R^2(z,\lambda) = \begin{cases} z^3, & 0 < R < 2, \\ m_1(z,\lambda), & R = 2, \\ \lambda^{3/2}, & R > 2. \end{cases}$$

Obviously condition (5.24) is fulfilled for P_R^2 and $R \neq 2$. We denote the three summands of P_2^2 by $P_2^{2,k}$, $k = 1,2,3$, and pick $(z,\lambda) \in \Sigma_{\varphi,\varphi_0}$ such that $\arg\lambda \geq 0$. Then we get that

$$-\varphi \leq \arg\sqrt{\lambda + z^2} \leq \frac{\pi - \varphi_0}{2} < \frac{\pi}{3},$$

hence that

$$-3\varphi \leq \arg P_2^{2,k}(z,\lambda) \leq \frac{3(\pi - \varphi_0)}{2} < \pi \quad (k = 1,2,3).$$

Choosing φ sufficiently small, say $\varphi < \frac{3\varphi_0 - \pi}{6}$, yields that

$$-3\varphi \leq \arg P_2^2(z,\lambda) \leq \frac{3(\pi - \varphi_0)}{2} < \pi$$

as well. This implies $P_2^2(z,\lambda) \neq 0$ for all $(z,\lambda) \in \Sigma_{\varphi,\varphi_0}$ satisfying $\arg\lambda \geq 0$. If $(z,\lambda) \in \Sigma_{\varphi,\varphi_0}$ such that $\arg\lambda \leq 0$, we obtain completely analogous that

$$3\varphi \geq \arg P_2^2(z,\lambda) \geq -\frac{3(\pi - \varphi_0)}{2} > -\pi,$$

which yields $P_2^2(z,\lambda) \neq 0$ also in this case. Thus, condition (5.24) is satisfied for P_R^2 and Corollary C.3 yields (ii).

At this point we remark that ω and m_1 are homogeneous symbols, i.e. the Newton polygon is a triangle. Therefore the proof of assertions (i) and (ii) can also be based on a compactness argument. This is no longer possible for m_2, since there the related Newton polygon has four (real) vertices.
(iii) Similar geometric observations as above show that here

$$N(P^3) = \operatorname{conv}\{(0,0),(0,5/2),(3,1),(4,0)\},$$

since $(2,1/2)$ and $(1,1)$ lie on the line connecting $(0,3/2)$ and $(3,0)$. This gives us

$$P_R^3(z,\lambda) = \begin{cases} 2\sigma z^4, & 0 < R < 1, \\ 3\lambda z^3 + 2\sigma z^4, & R = 1, \\ 3\lambda z^3, & 1 < R < 2, \\ \lambda m_2(z,\lambda), & R = 2, \\ \lambda^{5/2}, & R > 2. \end{cases}$$

If φ is chosen as in (ii), we already know that

$$P_R^3(z,\lambda) \neq 0 \qquad ((z,\lambda) \in \Sigma_{\varphi,\varphi_0})$$

for $R = 2$. Observe that all other cases are obvious except the case $R = 1$. For $R = 1$ again pick $(z,\lambda) \in \Sigma_{\varphi,\varphi_0}$ such that $\arg\lambda \geq 0$. Then,

$$-4\varphi \leq \arg(3\lambda z^3 + 2\sigma z^4) \leq \pi - \varphi_0 + 3\varphi < \pi.$$

Consequently, choosing $\varphi < \min\{(3\varphi_0 - \pi)/6,\ \varphi_0/7\}$ we obtain $P_1^3(z,\lambda) \neq 0$. Argueing in the same way for $(z,\lambda) \in \Sigma_{\varphi,\varphi_0}$ satisfying $\arg\lambda \leq 0$ finally results in

$$P_1^3(z,\lambda) \neq 0 \qquad ((z,\lambda) \in \Sigma_{\varphi,\varphi_0}) .$$

Thus, the assertion follows from Corollary C.3.

□

The mapping properties derived in Lemma 5.5 allow us to finish the proof of Proposition 5.3. For the remaining proof we denote by ${}_0\mathbb{E}_{j,\rho}$ and ${}_0\mathbb{F}_{j,\rho}$ the weighted versions of the spaces of solutions and right hand sides, respectively.

Proof of Proposition 5.3. Let ρ_0 as in Lemma 5.5 and choose $\rho > \rho_0$. Applying Lemma 5.5(iii) for $n = 2$, we obtain

$$\begin{aligned} m_2(D_n, G)^{-1} f_h \quad &\in \quad {}_0W^{7/2-1/2p}_{p,\rho}(\mathbb{R}_+, L_p(\mathbb{R}^n)) \cap {}_0W^{2-1/2p}_{p,\rho}(\mathbb{R}_+, H^3_p(\mathbb{R}^n)) \cap {}_0W^{1-1/p}_{p,\rho}(\mathbb{R}_+, H^4_p(\mathbb{R}^n)) \\ &\cap \quad {}_0H^{5/2}_{p,\rho}(\mathbb{R}_+, W^{2-1/p}_p(\mathbb{R}^n)) \cap {}_0H^1_{p,\rho}(\mathbb{R}_+, W^{5-1/p}_p(\mathbb{R}^n)) \cap L_{p,\rho}(\mathbb{R}_+, W^{6-1/p}_p(\mathbb{R}^n)). \end{aligned} \tag{5.25}$$

Next, Lemma 4.3 of [DSS08] implies

$${}_0W^{5/2}_{p,\rho}(\mathbb{R}_+, W^{2-1/p}_p(\mathbb{R}^n)) \cap L_{p,\rho}(\mathbb{R}_+, W^{6-1/p}_p(\mathbb{R}^n)) \hookrightarrow {}_0H^2_{p,\rho}(\mathbb{R}_+, W^{2-1/p}_p(\mathbb{R}^n)) \cap {}_0H^{3/2}_{p,\rho}(\mathbb{R}_+, W^{3-1/p}(\mathbb{R}^n)).$$

By virtue of Lemma 5.5(ii) these two embeddings yield

$$m_1(D_n, G) m_2(D_n, G)^{-1} f_h \in {}_0\mathbb{E}_{3,\rho}(\mathbb{R}_+, \mathbb{R}^2).$$

By representation (5.19) we see that $h \in {}_0\mathbb{E}_{3,\rho}(\mathbb{R}_+, \mathbb{R}^2)$. Next, by virtue of (5.19) and (5.20) we obtain

$$p_1 \quad = \quad \sigma\left(\omega(D_n, G) + D_n\right)\left(\omega(D_n, G)^2 + D_n^2\right) D_n^2 m_1(D_n, G)^{-1} g_1.$$

Analogously, from (5.25) and by the fact that (see again [DSS08, Lemma 4.3])

$${}_0H^1_{p,\rho}(\mathbb{R}_+, W^{2-1/p}_p(\mathbb{R}^n)) \cap L_{p,\rho}(\mathbb{R}_+, W^{3-1/p}_p(\mathbb{R}^n)) \hookrightarrow {}_0W^{1-1/p}_{p,\rho}(\mathbb{R}_+, H^2_p(\mathbb{R}^n)) \cap {}_0W^{1/2-1/2p}_{p,\rho}(\mathbb{R}_+, H^2(\mathbb{R}^n))$$

it follows that

$$p_1 \in \mathbb{G}^+_\rho(\mathbb{R}_+, \mathbb{R}^2).$$

Given $f_h, \partial_t h \in {}_0\mathbb{H}_\rho(\mathbb{R}_+, \mathbb{R}^2)$, let $(u,p) \in {}_0\mathbb{E}_{1,\rho}(\mathbb{R}_+, \mathbb{R}^3_+) \times {}_0\mathbb{E}_{2,\rho}(\mathbb{R}_+, \mathbb{R}^3_+, \mathbb{R}^2)$ the solution of the Stokes equations with Neumann boundary conditions

$$\left\{ \begin{aligned} \partial_t u - \Delta u + \nabla p = 0 \quad & \text{in } \mathbb{R}_+ \times \mathbb{R}^3_+, \\ \nabla \cdot u = 0 \quad & \text{in } \mathbb{R}_+ \times \mathbb{R}^3_+, \\ \gamma w = f_h - \partial_t h \quad & \text{on } \mathbb{R}_+ \times \mathbb{R}^2, \\ \gamma_0 \partial_y v + \gamma \nabla \bar{w} = 0 \quad & \text{on } \mathbb{R}_+ \times \mathbb{R}^2, \\ u(0) = 0 \quad & \text{in } \mathbb{R}^3_+. \end{aligned} \right. \tag{5.26}$$

For a finite time interval $J = (0,T)$ and $f_h \in {}_0\mathbb{H}_\rho(J, \mathbb{R}^2)$ we know in view of Lemma 5.4(v) and (vi) that there exists an extension $\widetilde{f_h} \in {}_0\mathbb{H}_\rho(\mathbb{R}_+, \mathbb{R}^2)$. Let (u,p,h) denote the restriction of the solution $(\widetilde{u}, \widetilde{p}, \widetilde{h})$ to $\widetilde{f_h}$ to the interval J. Then, we obviously have $(u,p,h) \in {}_0\mathbb{E}_\rho(J, \mathbb{R}^3_+)$ and (u,p,h) solves (5.7) on J. Moreover, (u,p,h) depends continuously on f_h, since the extension operator given in Lemma 5.4(v) is continuous. This proves the result for $\rho > \rho_0$. On the other hand, by Lemma 5.4(iii),(vi) the norms on the weighted spaces ${}_0\mathbb{E}_{j,\rho}(J,D)$, $j = 1,2,3$, and ${}_0\mathbb{F}_{j,\rho}(J,D)$, $j = 1,\ldots,5$, are equivalent for different weights $e^{-p\rho t}$, $\rho \geq 0$. Thus, the assertion of Proposition 5.3 is true for all $\rho \geq 0$, hence in particular for $\rho = 0$.

Finally, the uniqueness assertion can be proved by similar arguments as in the proof of Theorem 4.1.

□

6. The nonlinear problem

In this section we construct a unique solution to the spincoating system described in (3.1) by an application of the contraction mapping principle. To this end, we rewrite this system as

$$L\Phi = ((N(\Phi) + f), 0, \Phi_0) \tag{6.1}$$

where $\Phi = (u, p, h)$, $\Phi_0 = (u_0, h_0)$, $f = (f_1, 0, 0, 0, 0)$ with $f_1 \in \mathbb{F}_1(J, D)$ and L is the linear operator represented by the left hand side of (3.1). The nonlinear operator N is given by and

$$N(\Phi) = (F_1(\Phi), F_d(\Phi), G^+(\Phi), H(\Phi), G^-(\Phi)),$$

where the functions F_1, F_d, G^+, H and G^- are defined as in (3.1). Recall from Theorem 4.1 that

$$L : \mathbb{E}(J, D) \to \mathbb{F}(J, D) \times \{0\} \times \mathbb{I}$$

is an isomorphism, where $\mathbb{I}$ is defined as $I := \mathbb{I}_1(D) \times \mathbb{I}_2(\Gamma^+)$. Thus, in order to solve equation (6.1) by the contraction mapping principle, we need to show first that $N(\Phi) \in \mathbb{F}(J, D)$, whenever $\Phi \in \mathbb{E}(J, D)$. The following embedding properties of the related function spaces will be useful in the sequel.

Lemma 6.1. *Let $p > 5$ and $J = (0, T)$ with $T \in (0, \infty)$. Then the following assertions hold.*

(i) $\mathbb{E}_1(J, D) \hookrightarrow H_p^{1/2}(J, H_p^1(D))$.

(ii) $\mathbb{E}_1(J, D) \hookrightarrow \mathrm{BUC}(J, W_p^{2-2/p}(D)) \hookrightarrow \mathrm{BUC}(J, \mathrm{BUC}^1(D))$.

(iii) $\mathbb{E}_3(J, \Gamma^+) \hookrightarrow \mathrm{BUC}(J, W_p^{3-2/p}(\mathbb{R}^2))$.

(iv) $\mathbb{H}(J, \Gamma^+) \hookrightarrow \mathrm{BUC}(J, W_p^{2-3/p}(\mathbb{R}^2))$.

(v) $\mathbb{G}^+(J, \Gamma^+) \hookrightarrow \mathrm{BUC}(J, W_p^{1-3/p}(\mathbb{R}^2)) \hookrightarrow \mathrm{BUC}(J \times \mathbb{R}^2)$.

(vi) *The space $\mathbb{G} := W_p^{1-1/2p}(J, L_p(\mathbb{R}^2)) \cap L_p(J, W_p^{1-1/p}(\mathbb{R}^2))$ is an algebra with respect to the scalar multiplication of functions. In particular,*

$$\left\| \frac{g}{\delta + h} \right\|_{\mathbb{G}} \leq C\,(\|g\|_{\mathbb{G}} + \|h\|_{\mathbb{G}})$$

for any $g \in \mathbb{G}$ and any $h \in \mathbb{G}$ satisfying $\|h\|_{L^\infty(J\times\mathbb{R}^2)} \leq \delta/2$.

(vii) *The assertions of (v) also hold for the space $\mathbb{H}(J, \Gamma^+)$.*

(viii) *The space $H_p^{1/2}(J, L_p(D)) \cap L_p(J, H_p^1(D))$ is an algebra with respect to the multiplication of scalar functions.*

Proof. The embedding (i) follows by an application of the mixed derivative theorem (see e.g. [DHP03]), whereas embedding (ii) follows from [Ama95, Theorem III.4.10.2] and Sobolev's embedding due to our assumption $p > 5$. The assertions (iii) and (iv) and the first emdedding in (v) follow from a general trace results, see e.g. [DSS08, Lemma 4.4]). The second embedding in (v) is then a consequence of the Sobolev embedding and our assumption $p > 5$. Relation (vi) is proved in [DSS08, Lemma 6.6]. Using the fact that $\mathbb{H}(J, \Gamma^+) \hookrightarrow \mathbb{G} \hookrightarrow \mathrm{BUC}(J \times \mathbb{R}^2)$, we may copy the proof of (vi) given in [DSS08, Lemma 6.6] verbatim to obtain (vii). Assertion (viii) follows by the same argument as (vii). □

Next, we establish the desired mapping properties of the nonlinearity.

Lemma 6.2. *Let $T > 0, J = (0, T), p > 5$ and $\mathbb{B}_\varrho(J, D) := \{\Phi \in \mathbb{E}(J, D) : \|\Phi\|_{\mathbb{E}(J,D)} \leq \varrho\}$. Then there exists $r > 0$ such that*

$$N(\mathbb{B}_r(J, D)) \hookrightarrow \mathbb{F}(J, D).$$

Moreover, $N \in C^1(\mathbb{B}_r(J, D), \mathbb{F}(J, D))$ and

$$N(0) = 0, \qquad DN(0) = 0,$$

where $DN : \mathbb{B}_r(J, D) \to \mathscr{L}(\mathbb{E}(J, D), \mathbb{F}(J, D))$ denotes the Fréchet derivative of N.

Proof. Observe first that by Lemma 6.1(iii) and by Sobolev's embedding we obtain $\|h\|_\infty \leq C\|h\|_{\mathbb{E}_3} \leq Cr$. Choosing $r > 0$ such that $Cr \leq \delta/2$ yields

$$\|h + \delta\|_\infty \geq |\delta - \|h\|_\infty| \geq \delta/2. \tag{6.2}$$

In a first step we show that $F_1(\mathbb{B}_r(J, D)) \hookrightarrow \mathbb{F}_1(J, D)$. To this end, by (6.2) and Lemma 6.1(ii), the first term of F_1 can be estimated as

$$\|\frac{y}{h+\delta}(\partial_y u)\partial_t h\|_{L_p(J,L_p(D))} \leq C\|u\|_{L^\infty(J,W^{1,\infty}(\mathbb{R}^n))}\|h\|_{\mathbb{E}_3} \leq C\|\Phi\|_{\mathbb{E}}^2.$$

Similarly, by writing $\frac{\delta^2}{(\delta+h)^2} - 1 = -\frac{h^2+2\delta h}{(\delta+h)^2}$, we obtain for the second term of F_1

$$\|(\frac{\delta^2}{(\delta+h)^2} - 1)\partial_y^2 u\|_{L_p(J,L_p(D))} \leq C\|u\|_{\mathbb{E}_1}\left(\|h\|_\infty^2 + \|h\|_\infty\right) \leq C\|\Phi\|_{\mathbb{E}}^2,$$

where we assumed that $r \leq 1$. All the other terms of F_1 may be estimated in a similar way, which proves that $F_1(\mathbb{B}_r(J,D)) \hookrightarrow \mathbb{F}_1(J,D)$.

In order to show that $H(\mathbb{B}_r(J,D)) \hookrightarrow \mathbb{H}(J,\Gamma^+)$ note that

$$\|v\cdot\nabla_x h\|_{\mathbb{H}} \leq C\|\nabla_x h\|_{\mathbb{H}}\|v\|_{\mathbb{H}} \leq C\|h\|_{\mathbb{E}_3}\|u\|_{\mathbb{E}_1} \leq C\|\Phi\|_{\mathbb{E}}^2$$

follows by the fact that $\mathbb{H}(J,\Gamma^+)$ is an algebra thanks to Lemma 6.1(vii).

Next, we consider the embedding $G^+(\mathbb{B}_r(J,D)) \hookrightarrow \mathbb{G}^+(J,\Gamma^+)$. Choosing r sufficiently small and due to Lemma 6.1(iii) and the fact that $W_p^{3-2/p}(\mathbb{R}^n) \hookrightarrow \mathrm{BUC}(\mathbb{R}^n)$, we may assume that $\|h\|_\infty \leq \delta/2$. Writing the first term of G^+ as

$$(1 - \frac{\delta}{\delta+h})\partial_y(v,2w)^T = \frac{h/\delta}{1+h/\delta}\partial_y(v,2w)^T$$

we obtain by Lemma 6.1(vi)

$$\|(1-\frac{\delta}{\delta+h})\partial_y(v,2w)^T\|_{\mathbb{G}^+} \leq C\|\frac{h}{1+h/\delta}\|_{\mathbb{G}}\|\partial_y u\|_{\mathbb{G}^+} \leq C\|h\|_{\mathbb{G}}\|\partial_y u\|_{\mathbb{G}^+} \leq C\|\Phi\|_{\mathbb{E}}^2.$$

In order to estimate the terms of G^+ involving the square root $\sqrt{1+|\nabla_x h|^2}$, we employ the Taylor expansion $\sqrt{1+s} = 1 + g(s)$ with $g(s) = \sum_{k=1}^\infty \binom{1/2}{k} s^k$. The above series converges absolutely for $|s| < 1$. Hence,

$$\begin{aligned}\|g(|\nabla_x h|^2)\|_{\mathbb{G}} &\leq \sum_{k=1}^\infty \|\nabla_x h\|_{\mathbb{G}}^{2k} \leq \frac{\|\nabla_x h\|_{\mathbb{G}}^2}{1-r^2}\\ \|g(|\nabla_x h|^2)\|_\infty &\leq \|g(|\nabla_x h|^2)\|_{\mathbb{G}} \leq C\frac{r^2}{1-r^2} \leq \frac{1}{2}\end{aligned}$$

provided r is small enough. Writing

$$\frac{1}{\sqrt{1+|\nabla_x h|^2}} - 1 = \frac{1}{1+g} - 1 = \frac{-g}{1+g},$$

we obtain for the second term of G^+ by an iterative application of Lemma 6.1(vi) that

$$\|\sigma\frac{g}{1+g}\Delta_x h\nu_D\|_{\mathbb{G}^+} \leq C\|\frac{g}{1+g}\|_{\mathbb{G}}\|\Delta_x h\|_{\mathbb{G}} \leq C\|g\|_{\mathbb{G}}\|h\|_{\mathbb{E}_3} \leq C\|\Phi\|_{\mathbb{E}}^3.$$

Analogously, we estimate the third term as

$$\begin{aligned}\left\|\sigma\sum_{j,k=1}^2 \frac{\partial_j h\partial_k h}{(1+|\nabla_x h|^2)^{3/2}}\partial_j\partial_k h\nu_D\right\|_{\mathbb{G}^+} &\leq C\sum_{j,k=1}^2 \left\|\frac{\partial_j h}{\sqrt{1+|\nabla_x h|^2}}\right\|_{\mathbb{G}}\left\|\frac{\partial_k h}{1+|\nabla_x h|^2}\right\|_{\mathbb{G}}\|\partial_j\partial_k h\|_{\mathbb{G}}\\ &\leq C\sum_{j,k=1}^2 \left(\|\partial_j h\|_{\mathbb{G}} + \|g\|_{\mathbb{G}}\right)\left(\|\partial_k h\|_{\mathbb{G}} + \|\nabla_x h\|_{\mathbb{G}}^2\right)\|h\|_{\mathbb{E}_3}\\ &\leq C\|\Phi\|_{\mathbb{E}}^3.\end{aligned}$$

The remaining terms of G^+ may be estimated in a similar way, where for the last term we use the fact that $p \in \mathbb{E}_2(J,D\Gamma^+)$ implies that $p|_{\Gamma^+} \in \mathbb{G}^+(J,D)$.

Recalling that $\|h\|_{\mathbb{G}^-} \leq \|h\|_{\mathbb{E}_3} \leq r \leq \delta/2$ we may write

$$(h+\delta)^{\alpha-1} - \delta^{\alpha-1} = \delta^{\alpha-1}\sum_{k=1}^\infty \binom{\alpha-1}{k}\left(\frac{h}{\delta}\right)^k.$$

This yields

$$\|c[(h+\delta)^{\alpha-1}-\delta^{\alpha-1}]\partial_y u\|_{\mathbb{G}^-} \leq C\sum_{k=1}^{\infty}\|\frac{h}{\delta}\|_{\mathbb{G}^-}^k\|\partial_y u\|_{\mathbb{G}^-} \leq C\|h\|_{\mathbb{G}^-(J)}\|u\|_{\mathbb{E}_1} \leq C\|\Phi\|_{\mathbb{E}}^2.$$

In the following step, we consider the corresponding embedding relations for the function F_d, i.e. $F_d(\mathbb{B}_r(J,D)) \hookrightarrow \mathbb{F}_d(J,D,\Gamma^+)$. Consider first the space $H^1(J,{}_0H_p^{-1}(\mathbb{R}_+^{n+1}))$. In view of ${}^0H_{p'}^1 \overset{d}{\hookrightarrow} L_{p'}$ we have $L_p \hookrightarrow {}_0H_p^{-1}$. This implies

$$F_d(u,h) \in L_p(J,L_p) \hookrightarrow L_p(J,{}_0H_p^{-1}),$$

by using, for instance, that $1-\delta/(\delta+h)$, $\frac{y}{h}\nabla_x h \in L^\infty(J\times\mathbb{R}^2)$. In order to treat the time derivative of the first term of F_d, we write

$$\partial_t\left(1-\frac{\delta}{\delta+h}\right)\partial_y u = \frac{\partial_t h}{\delta+h}\partial_y w + \frac{h}{\delta+h}\partial_y\partial_t w - h\partial_y w\frac{\partial_t h}{(\delta+h)^2}. \tag{6.3}$$

By similar arguments as above we see that the first and the third term of the right hand side of (6.3) belong to $L_p(J,L_p)$, and hence to $L_p(J,{}_0H_p^{-1})$. For the second term in (6.3), pick $\varphi \in {}^0H_{p'}^1$. Integration by parts in y yields

$$\int_D\left(\frac{h}{\delta+h}\partial_y\partial_t w\right)\varphi \mathrm{d}x = -\int_D \frac{h}{\delta+h}\partial_t w\partial_y\varphi \mathrm{d}x,$$

where we used the fact that $w|_{y=0}=0$. This yields

$$\begin{aligned}\|\frac{h(t)}{\delta+h(t)}\partial_y\partial_t w(t)\|_{{}_0H_p^{-1}} &= \sup_{\|\varphi\|_{{}^0H_{p'}^1}=1}\left|\int_D\frac{h(t)}{\delta+h(t)}\partial_t w(t)\partial_y\varphi \mathrm{d}x\right| \\ &\leq C\left\|\frac{h(t)}{\delta+h(t)}\right\|_\infty\|\partial_t u(t)\|_{L_p},\end{aligned}$$

hence $\|\frac{h}{\delta+h}\partial_y\partial_t w\|_{L_p(J,{}_0H_p^{-1})} \leq C\|\Phi\|_{\mathbb{E}}^2$. The $H_p^1(J,{}_0H_p^{-1})$-norm of the second term of F_d can be estimated similarly. In fact, the first and the second term of the time derivative

$$\partial_t\left(\frac{y}{h}\partial_y v\cdot\nabla_x h\right) = \frac{y}{h}\partial_y v\cdot\nabla_x\partial_t h - \frac{y}{h^2}\partial_t h\partial_y v\cdot\nabla_x h + \frac{y}{h}(\partial_y\partial_t v)\cdot\nabla_x h$$

belong to $L_p(J,L_p)$ and hence to $L_p(J,{}_0H_p^{-1})$. For the third term we again employ integration by parts, which yields

$$\int_D\frac{y}{h}(\partial_y\partial_t v)\cdot\nabla_x h\varphi \mathrm{d}x = -\int_D\frac{\partial_t v}{h}\cdot\nabla_x h(\varphi+y\partial_y\varphi)\mathrm{d}x.$$

The fact that $\partial_t v \in L_p(J,L_p)$, $\nabla_x h/h \in L^\infty(J\times\mathbb{R}^n)$, and $\varphi+y\partial_y\varphi \in L_{p'}(J,L_{p'})$ then results in

$$\|\frac{y}{h}(\partial_y\partial_t v)\cdot\nabla_x h\|_{L_p(J,{}_0H_p^{-1})} \leq C\|\Phi\|_{\mathbb{E}}^2.$$

Summarizing, we obtain

$$\|F_d(\Phi)\|_{L_p(J,{}_0H_p^{-1})} \leq C\|\Phi\|_{\mathbb{E}}^2.$$

In order to obtain the same estimate for the $E = H_p^{1/2}(J,L_p(\mathbb{R}_+^3)) \cap L_p(J,H_p^1(\mathbb{R}_+^3))$-norm, note that by Lemma 6.1(viii) also this space is an algebra. Furthermore, since $h/(\delta+h)$ does not depend on y, we deduce for this term that

$$\left\|\frac{h}{\delta+h}\right\|_E \leq C\left\|\frac{h/\delta}{1+h/\delta}\right\|_{\mathbb{H}} \leq C\|h\|_{\mathbb{E}_3} \leq C\|\Phi\|_{\mathbb{E}}$$

in virtue of Lemma 6.1(vii). This implies for the first term of F_d

$$\|(1-\frac{\delta}{h})\partial_y u\|_E \leq C\left\|\frac{h}{\delta+h}\right\|_E\|\partial_y u\|_E \leq C\|\Phi\|_{\mathbb{E}}^2.$$

Here we used Lemma 6.1(viii) and the fact that $\nabla\mathbb{E}_1(J,D) \hookrightarrow E$ which holds by Lemma 6.1(i). Similarly $\|y\nabla_x h/h\|_E \le C\|\Phi\|_{\mathbb{E}}$ and thus also the second term of F_d belongs to E. Summarizing, we arrive at

$$F_d(\mathbb{B}_r(J,D)) \hookrightarrow \mathbb{F}_d(J,D,\Gamma^+).$$

Finally, the additional assertions for N follow immediately from the structure of N. In particular, $DN(0)=0$ follows from the fact that N contains only nonlinear terms of second or higher order. The proof is complete. □

Lemma 6.2 puts us into the position to prove the existence and uniqueness of a strong solution to system (3.1) for small initial data. For $t>0$ and $(x,y)\in D$, we set

$$f_1(t,(x,y)) := \chi_R\,\omega\times(\omega\times(x,y)).$$

Theorem 6.3. *Let $p>5$, $T\in(0,\infty)$ and $J=(0,T)$. Then there exists an $\varepsilon>0$ such that for each $(f_1,u_0,h_0)\in\mathbb{F}_1(J,D)\times\mathbb{I}_1\times\mathbb{I}_2$ satisfying*

$$\|(f_1,u_0,h_0)\|_{\mathbb{F}_1\times\mathbb{I}_1\times\mathbb{I}_2} < \varepsilon,$$

there exists a unique solution $(u,p,h)\in\mathbb{E}(J,D)$ of system (3.1).

Proof. By Lemma 6.2, the system (3.1) is equivalent to the fixed point problem

$$\Phi = L^{-1}(N(\Phi)+f,0,\Phi_0) =: K(\Phi), \quad \Phi\in\mathbb{B}_r(J,D).$$

We set $\mathbb{X}_r := \{\Phi\in\mathbb{B}_r(J,D):\ \Phi(0)=\Phi_0\}$ and choose $r>0$ such that

$$\sup_{v\in\mathbb{B}_r(J,D)} \|DN[v]\|_{\mathscr{L}(\mathbb{E},\mathbb{F})} \le \frac{1}{2\|L^{-1}\|},$$

which is possible since $DN\in C(\mathbb{B}_r(J,D),\mathscr{L}(\mathbb{E}(J,D),\mathbb{F}(J,D)))$ and $DN(0)=0$. Setting

$$\|L^{-1}\| := \|L^{-1}\|_{\mathscr{L}(\mathbb{F}(J,D)\times\{0\}\times\mathbb{I},\mathbb{E}(J,D))},$$

which is finite due to Theorem 4.1, Lemma 6.2 and the mean value theorem imply the estimate

$$\begin{aligned}
\|K(\Phi)\|_{\mathbb{E}} &\le \|L^{-1}\|\Big(\|N(\Phi)-N(0)\|_{\mathbb{F}} + \|(f_1,\Phi_0)\|_{\mathbb{F}_1\times\mathbb{I}}\Big)\\
&\le \|L^{-1}\|\left(\sup_{v\in\mathbb{B}_r(J,D)}\|DN[v]\|_{\mathscr{L}(\mathbb{E},\mathbb{F})}\|\Phi\|_{\mathbb{E}} + \|(f_1,\Phi_0)\|_{\mathbb{F}_1\times\mathbb{I}}\right)\\
&\le \frac{r}{2} + \|L^{-1}\|\varepsilon, \qquad \Phi\in\mathbb{X}_r.
\end{aligned}$$

By choosing $\varepsilon\le r/2\|L^{-1}\|$ we conclude that $K(\mathbb{X}_r)\subset\mathbb{X}_r$. To see that K is contractive, we observe that

$$\begin{aligned}
\|K(\Phi_1)-K(\Phi_2)\|_{\mathbb{E}} &\le \|L^{-1}\|\|N(\Phi_1)-N(\Phi_2)\|_{\mathbb{F}}\\
&\le \|L^{-1}\|\sup_{\Psi\in\mathbb{B}_r(J,D)}\|DN[\Psi]\|_{\mathscr{L}(\mathbb{E},\mathbb{F})}\|\Phi_1-\Phi_2\|_{\mathbb{E}}\\
&\le \frac{1}{2}\|\Phi_1-\Phi_2\|_{\mathbb{E}}, \qquad \Phi_1,\Phi_2\in\mathbb{X}_r.
\end{aligned}$$

Consequently, K is a contraction on $\mathbb{X}_r$ and the assertion follows by the contraction mapping principle. □

Appendix A. The Laplace-equation and the heat-equation in negative Sobolev spaces

In this first section of the appendix we present regularity estimates for the Laplace and the heat equation in Sobolev spaces of negative order. They will be useful in the treatment of the reduced Stokes equation in Appendix B. Many of the results given below are well-known in the elliptic case. For the parabolic case, this is less true. Hence, for the convenience of the reader, we give a sketch of the corresponding proofs.

Proposition A.1. *Let* $\Omega = \mathbb{R}^3_+, D^+,$ *or* D^-.

(a) *For* $f \in \widehat{H}^{-1}_p(\Omega)$ *there exists a unique solution* $u \in \widehat{H}^1_{p,0}(\Omega)$ *of*

$$\begin{array}{rcll} -\Delta u &=& f & \text{in } \Omega, \\ u &=& 0 & \text{on } \partial\Omega, \end{array} \tag{A.1}$$

i.e. $\langle \nabla u, \nabla \varphi\rangle_\Omega = \langle f, \varphi\rangle_\Omega$, $\varphi \in \widehat{H}^1_{p',0}(\Omega)$. *Moreover, there exists* $C > 0$, *independent of* f, *such that*

$$\|u\|_{\widehat{H}^1_p(\Omega)} \le C\|f\|_{\widehat{H}^{-1}_p(\Omega)}.$$

(b) *For* $f \in \widehat{H}^{-1}_{p,0}(\Omega) := (\widehat{H}^1_{p'}(\Omega))'$ *there exists a unique solution* $u \in \widehat{H}^1_p(\Omega)$ *of*

$$\begin{array}{rcll} -\Delta u &=& f & \text{in } \Omega, \\ \partial_\nu u &=& 0 & \text{on } \partial\Omega, \end{array} \tag{A.2}$$

i.e. $\langle \nabla u, \nabla \varphi\rangle_\Omega = \langle f, \varphi\rangle_\Omega$, $\varphi \in \widehat{H}^1_{p'}(\Omega)$. *Moreover, there exists* $C > 0$, *independent of* f, *such that*

$$\|u\|_{\widehat{H}^1_p(\Omega)} \le C\|f\|_{\widehat{H}^{-1}_{p,0}(\Omega)}.$$

(c) *For* $f \in L_p(\Omega)^3$ *there exists a unique solution* $u \in \widehat{H}^1_p(\Omega)$ *of*

$$\begin{array}{rcll} -\Delta u &=& \operatorname{div} f & \text{in } \Omega, \\ \partial_\nu u &=& f\cdot\nu & \text{on } \partial\Omega, \end{array} \tag{A.3}$$

i.e. $\langle \nabla u, \nabla \varphi\rangle_\Omega = \langle f, \nabla\varphi\rangle_\Omega$, $\varphi \in \widehat{H}^1_{p'}(\Omega)$. *Moreover, there exists* $C > 0$, *independent of* f, *such that*

$$\|u\|_{\widehat{H}^1_p(\Omega)} \le C\|f\|_{L_p(\Omega)^3}.$$

Proof. Without loss of generality we may assume $\Omega = \mathbb{R}^3_+$. By the theory of singular integrals, it is easy to see that for $\tilde f \in \widehat{H}^{-1}_p(\mathbb{R}^3)$ there exists a solution $\tilde u \in \widehat{H}^1_p(\mathbb{R}^3)$ of

$$-\Delta \tilde u = \tilde f \qquad \text{in } \mathbb{R}^3.$$

Since $C^\infty_{c,0}(\Omega) = \{g \in C^\infty_c(\Omega) : \int_\Omega g = 0\}$ is dense in $\widehat{H}^{-1}_p(\Omega)$ and $\widehat{H}^{-1}_{p,0}(\Omega)$, it suffices to consider $f \in C^\infty_{c,0}(\Omega)$. In that case, the existence of a solution of (A.1) or (A.2) follows by extending f odd to $\mathbb{R}^3$ or even to $\mathbb{R}^3$, respectively. The uniqueness follows from a standard argument for harmonic functions. Finally, (c) follows easily from (b) since $\langle f, \nabla\varphi\rangle_\Omega \le \|f\|_{L_p(\Omega)^3}\|\varphi\|_{\widehat{H}^1_{p'}(\Omega)}$. □

The crucial part in the proof of the proposition above is that the odd extension to $\mathbb{R}^3$ of some $f \in \widehat{H}^{-1}_p(\mathbb{R}^3_+)$ is always contained in $\widehat{H}^{-1}_p(\mathbb{R}^3)$. Note, however, that the even extension of some $f \in \widehat{H}^{-1}_p(\mathbb{R}^3_+)$ is not contained in $\widehat{H}^{-1}_p(\mathbb{R}^3)$, in general, whereas the even extension of some $f \in \widehat{H}^{-1}_{p,0}(\mathbb{R}^3_+)$ is contained in $\widehat{H}^{-1}_p(\mathbb{R}^3)$. By the equation $\langle \nabla u, \nabla \varphi\rangle_\Omega = \langle f, \varphi\rangle_\Omega$, $\varphi \in \widehat{H}^1_{p'}(\Omega)$, we also see that $f \in \widehat{H}^{-1}_{p,0}(\mathbb{R}^3_+)$ is a necessary condition in the latter case.

Some remarks about the trace in (A.3) are in order. Multiplying (A.3) by $\varphi \in \widehat{H}^1_{p'}(\Omega)$ and integrating by parts, we end up with

$$\langle \nabla u, \nabla \varphi\rangle_\Omega - \langle f, \nabla\varphi\rangle_\Omega = \langle \partial_\nu u - f\cdot\nu, \varphi\rangle_{\partial\Omega}.$$

If u is a solution of (A.3), we thus have

$$\langle \partial_\nu u - f\cdot\nu, \varphi\rangle_{\partial\Omega} = 0, \quad \varphi \in \widehat{H}^1_{p'}(\Omega).$$

Hence the trace in (A.3) is well-defined. However, note that this procedue does neither allow to define $\partial_\nu u$ nor $f\cdot\nu$ but $\partial_\nu u - f\cdot\nu$ only.

In the following $L_{p,\rho}$, $H^1_{0,\rho}$, etc., denote the weighted Sobolev spaces as introduced in Section 5.

Proposition A.2. *Let $\rho > 0$ and $\Omega = \mathbb{R}^3_+, D^+,$ or D^-.*

(a) *For $f \in L_{p,\rho}(\mathbb{R}_+, \widehat{H}^{-1}_p(\Omega))$ there exists a unique solution $u \in L_{p,\rho}(\mathbb{R}_+, H^1_{p,0}(\Omega)) \cap H^1_{p,\rho}(\mathbb{R}_+, \widehat{H}^{-1}_p(\Omega))$ of*

$$\text{(A.4)} \qquad \begin{aligned} \partial_t u - \Delta u &= f && \text{in } \mathbb{R}_+\times\Omega,\\ u &= 0 && \text{on } \mathbb{R}_+\times\partial\Omega,\\ u(0) &= 0 && \text{in } \Omega, \end{aligned}$$

i.e. $\langle \partial_t u(t), \varphi\rangle_\Omega + \langle \nabla u(t), \nabla\varphi\rangle_\Omega = \langle f(t), \varphi\rangle_\Omega$, $t>0$, $\varphi \in \widehat{H}^1_{p',0}(\Omega)$.

(b) *For $f \in L_{p,\rho}(\mathbb{R}_+, \widehat{H}^{-1}_{p,0}(\Omega))$ there exists a unique solution $u \in L_{p,\rho}(\mathbb{R}_+, H^1_p(\Omega)) \cap H^1_{p,\rho}(\mathbb{R}_+, \widehat{H}^{-1}_{p,0}(\Omega))$ of*

$$\text{(A.5)} \qquad \begin{aligned} \partial_t u - \Delta u &= f && \text{in } \mathbb{R}_+\times\Omega,\\ \partial_\nu u &= 0 && \text{on } \mathbb{R}_+\times\partial\Omega,\\ u(0) &= 0 && \text{in } \Omega, \end{aligned}$$

i.e. $\langle \partial_t u(t), \varphi\rangle_\Omega + \langle \nabla u(t), \nabla\varphi\rangle_\Omega = \langle f(t), \varphi\rangle_\Omega$, $t>0$, $\varphi \in \widehat{H}^1_{p'}(\Omega)$.

(c) *For $f \in L_{p,\rho}(\mathbb{R}_+, L_p(\Omega)^3)$ there exists a unique solution $u \in L_{p,\rho}(\mathbb{R}_+, H^1_p(\Omega)) \cap H^1_{p,\rho}(\mathbb{R}_+, \widehat{H}^{-1}_{p,0}(\Omega))$ of*

$$\text{(A.6)} \qquad \begin{aligned} \partial_t u - \Delta u &= \operatorname{div} f && \text{in } \mathbb{R}_+\times\Omega,\\ \partial_\nu u &= f\cdot\nu && \text{on } \mathbb{R}_+\times\partial\Omega,\\ u(0) &= 0 && \text{in } \Omega, \end{aligned}$$

i.e. $\langle \partial_t u(t), \varphi\rangle_\Omega + \langle \nabla u(t), \nabla\varphi\rangle_\Omega = \langle f(t), \varphi\rangle_\Omega$, $t>0$, $\varphi \in \widehat{H}^1_{p'}(\Omega)$.

Proof. Again, it suffices to consider the case where $\Omega = \mathbb{R}^3_+$. We start with the problem

$$\text{(A.7)} \qquad \begin{aligned} \partial_t \tilde{u} - \Delta\tilde{u} &= \tilde{f} && \text{in } \mathbb{R}_+\times\mathbb{R}^3,\\ \tilde{u}(0) &= 0 && \text{in } \mathbb{R}^3, \end{aligned}$$

where $\tilde{f} \in C^\infty_c(\mathbb{R}_+, C^\infty_{c,0}(\Omega))$. By the joint H^∞-calculus of ∂_t and Δ, it follows that $\tilde{u} := (\partial_t - \Delta)^{-1}\tilde{f}$ satisfies $\|\tilde{u}\|_{L_{p,\rho}(\mathbb{R}_+, L_p(\mathbb{R}^3))} \le C\|\tilde{f}\|_{L_{p,\rho}(\mathbb{R}_+, \widehat{H}^{-1}_p(\mathbb{R}^3))}$ and

$$\begin{aligned} \|\tilde{u}\|_{L_{p,\rho}(\mathbb{R}_+, \widehat{H}^1_p(\mathbb{R}^3))} &\le C\|(-\Delta)^{\frac12}(\partial_t-\Delta)^{-1}\tilde{f}\|_{L_{p,\rho}(\mathbb{R}_+, L_p(\mathbb{R}^3))}\\ &= C\|(-\Delta)^{\frac12}(\partial_t-\Delta)^{-1}(-\Delta)^{\frac12}(-\Delta)^{-\frac12}\tilde{f}\|_{L_{p,\rho}(\mathbb{R}_+, L_p(\mathbb{R}^3))}\\ &\le C\|(-\Delta)^{-\frac12}\tilde{f}\|_{L_{p,\rho}(\mathbb{R}_+, L_p(\mathbb{R}^3))} \le C\|\tilde{f}\|_{L_{p,\rho}(\mathbb{R}_+, \widehat{H}^{-1}_p(\mathbb{R}^3))}. \end{aligned}$$

Now, integration by parts yields

$$\langle \partial_t\tilde{u}(t), \varphi\rangle_{\mathbb{R}^3} = \langle \tilde{f}(t), \varphi\rangle_{\mathbb{R}^3} - \langle \nabla\tilde{u}(t), \nabla\varphi\rangle_{\mathbb{R}^3}, \quad \varphi \in \widehat{H}^1_{p'}(\mathbb{R}^3),\ t>0.$$

Hence, $\partial_t\tilde{u} \in L_{p,\rho}(\mathbb{R}_+, \widehat{H}^{-1}_p(\mathbb{R}^3))$. Therefore, extending f in the same way as in the previous proposition to $\mathbb{R}^3$, the assertion of the proposition follows.

□

Proposition A.3. *Let $\rho > 0$ and $\Omega = \mathbb{R}^3, \mathbb{R}^3_+, D^+$ or D^-. Then the following embeddings hold true.*

(a) $L_{p,\rho}(\mathbb{R}_+, H^1_{p,0}(\Omega)) \cap H^1_{p,\rho}(\mathbb{R}_+, \widehat{H}^{-1}_p(\Omega)) \hookrightarrow H^{\frac12}_{p,\rho}(\mathbb{R}_+, L_p(\Omega))$.

(b) $L_{p,\rho}(\mathbb{R}_+, H^1_p(\Omega)) \cap H^1_{p,\rho}(\mathbb{R}_+, \widehat{H}^{-1}_{p,0}(\Omega)) \hookrightarrow H^{\frac12}_{p,\rho}(\mathbb{R}_+, L_p(\Omega))$.

Proof. We start with the case $\mathbb{R}^3$. By the joint H^∞-calculus of ∂_t and Δ, we obtain

$$\begin{aligned}\|\partial_t^{\frac12}\tilde u\|_{L_{p,\rho}(\mathbb{R}_+,L_p(\mathbb{R}^3))} &= \|\partial_t^{\frac12}(-\Delta)^{\frac12}(\partial_t-\Delta)^{-1}(\partial_t-\Delta)(-\Delta)^{-\frac12}\tilde u\|_{L_{p,\rho}(\mathbb{R}_+,L_p(\mathbb{R}^3))}\\ &\le C\left(\|\partial_t(-\Delta)^{-\frac12}\tilde u\|_{L_{p,\rho}(\mathbb{R}_+,L_p(\mathbb{R}^3))}+\|(-\Delta)^{\frac12}\tilde u\|_{L_{p,\rho}(\mathbb{R}_+,L_p(\mathbb{R}^3))}\right)\\ &\le C\|\tilde u\|_{L_{p,\rho}(\mathbb{R}_+,H^1_p(\mathbb{R}^3))\cap H^1_{p,\rho}(\mathbb{R}_+,\widehat H^{-1}_p(\mathbb{R}^3))}.\end{aligned}$$

Now, the proposition follows by extending u to $\mathbb{R}^3$ in the same way as before. □

Proposition A.4. *Let $\Omega=\mathbb{R}^3_+, D_+$ or D_-.*

(a) *For $g\in W_p^{1-\frac1p}(\partial\Omega)$ there exists a unique solution $u\in\widehat H^1_p(\Omega)$ of*

$$\begin{array}{rcll}-\Delta u&=&0&\text{in }\Omega,\\ u&=&g&\text{on }\partial\Omega.\end{array}\tag{A.8}$$

(b) *Let $\rho>0$. For $g\in W^{\frac12-\frac1{2p}}_{p,\rho}(\mathbb{R}_+,L_p(\mathbb{R}^2))\cap L_{p,\rho}(\mathbb{R}_+,W_p^{1-\frac1p}(\mathbb{R}^2))$ satisfying the compatibility condition $g|_{t=0}=0$ if $p>3$ there exists a unique solution $u\in L_{p,\rho}(\mathbb{R}_+,H^1_p(\Omega))\cap H^1_{p,\rho}(\mathbb{R}_+,\widehat H^{-1}_p(\Omega))\cap H^{\frac12}_{p,\rho}(\mathbb{R}_+,L_p(\Omega))$ of*

$$\begin{array}{rcll}\partial_t u-\Delta u&=&0&\text{in }\mathbb{R}_+\times\Omega,\\ u&=&g&\text{on }\mathbb{R}_+\times\partial\Omega,\\ u(0)&=&0&\text{in }\Omega.\end{array}\tag{A.9}$$

Proof. Again, we consider the case $\mathbb{R}^3_+$ only. (a) The solution of (A.8) is given by $u=\mathrm{e}^{-\sqrt{-\Delta_{\mathbb{R}^2}}\cdot}g$. Hence, it follows from interpolation theory for analytic semigroups that

$$\|u\|_{\widehat H^1_p(\mathbb{R}^3_+)}\le C\|\sqrt{-\Delta_{\mathbb{R}^2}}u\|_{L_p(\mathbb{R}^3_+)}\le C\|g\|_{(D(\sqrt{-\Delta_{\mathbb{R}^2}}),L_p(\mathbb{R}^2))_{1-\frac1p,p}}\le C\|g\|_{W_p^{1-\frac1p}(\mathbb{R}^2)}.$$

(b) The joint H^∞-calculus of ∂_t and $\Delta_{\mathbb{R}^2}$ yields that

$$\sqrt{\partial_t-\Delta_{\mathbb{R}^2}}:D(\sqrt{\partial_t-\Delta_{\mathbb{R}^2}})\to L_{p,\rho(\mathbb{R}^2_+,L_p(\mathbb{R}^2))},$$
$$D(\sqrt{\partial_t-\Delta_{\mathbb{R}^2}}):=_0 H^{\frac12}_{p,\rho}(\mathbb{R}_+,L_p(\mathbb{R}^2))\cap L_{p,\rho}(\mathbb{R}_+,H^1_p(\mathbb{R}^2))$$

generates an analytic semigroup on $L_{p,\rho}(\mathbb{R}_+,L_p(\mathbb{R}^2))$. Since the solution of (A.9) is given by $u=\mathrm{e}^{-\sqrt{\partial_t-\Delta_{\mathbb{R}^2}}\cdot}g$, it follows in a similar way as above that

$$\begin{aligned}\|u\|_{H^{\frac12}_{p,\rho}(\mathbb{R}_+,L_p(\mathbb{R}^3_+))\cap L_{p,\rho}(\mathbb{R}_+,H^1_p(\mathbb{R}^3_+))}&\le C\|\sqrt{\partial_t-\Delta_{\mathbb{R}^2}}u\|_{L_p(\mathbb{R}^3_+)}\le C\|g\|_{(D(\sqrt{\partial_t-\Delta_{\mathbb{R}^2}}),L_p(\mathbb{R}^2)_{1-\frac1p,p}}\\ &\le C\|g\|_{W^{\frac12-\frac1{2p}}_{p,\rho}(\mathbb{R}_+,L_p(\mathbb{R}^2))\cap L_{p,\rho}(\mathbb{R}_+,W_p^{1-\frac1p}(\mathbb{R}^2))}.\end{aligned}$$

Now, integrating by parts equation (A.9), we obtain $u\in H^1_{p,\rho}(\mathbb{R}_+,\widehat H^{-1}_p(\mathbb{R}^3))$. □

Proposition A.5. *For $g\in W_p^{1-\frac1p}(\Gamma^+)$, $f_1\in {}_0H^{-1}_p(D)$ and $f_2\in L_p(D)^3$ there exists a unique solution $u\in H^1_p(D)$ of*

$$\begin{array}{rcll}-\Delta u&=&f_1+\operatorname{div}f_2&\text{in }D,\\ \gamma u&=&g&\text{on }\Gamma^+.\\ \gamma\partial_y u&=&\gamma_\nu f_2&\text{on }\Gamma^-.\end{array}\tag{A.10}$$

For a proof we refer to [Abe06].

Proposition A.6. *Let $1 < p < \infty$, $p \neq 3/2, 3$, $f \in L_p(J, {}_0H_p^{-1}(D))$ and $g \in \mathbb{G}^+(J, \Gamma^+)$ satisfy the compatibility condition $g|_{t=0} = 0$ if $p > 3$. Then there exists a unique solution $w \in \mathbb{F}_2(J, D, \Gamma_+)$ of the initial and boundary value problem*

$$(\text{A.11}) \qquad \begin{aligned} (\partial_t - \Delta)w &= f && \text{in } J \times D, \\ \gamma w &= g && \text{on } J \times \Gamma^+, \\ \gamma \partial_y w &= 0 && \text{on } J \times \Gamma^-, \\ w|_{t=0} &= 0 && \text{in } D, \end{aligned}$$

i.e. $\langle \partial_t w(t), \varphi \rangle_D + \langle \nabla w(t), \nabla \varphi \rangle_D = -\langle f, \nabla \varphi \rangle_D$, $t > 0$, $\varphi \in {}^0H_p^1(D)$ *and* $\gamma w = g$ *on* Γ^+.

Proof. Let $\rho > 0$, $f \in C_c^\infty(\mathbb{R}_+, C_{c,0}^\infty(D))$ and $g \in \mathbb{G}^+(J, \Gamma^+)$. Without loss of generality let $\delta = 1$. We define $\tilde{f}$ by

$$\tilde{f}(x,y) := \begin{cases} f(x,y), & 0 < y < 1, \ x \in \mathbb{R}^2, \\ -f(x, 2-y), & 1 < y < 2, \ x \in \mathbb{R}^2, \\ 0, & 2 < y, \ x \in \mathbb{R}^2. \end{cases} .$$

Then, $\tilde{f} \in L_{p,\rho}(\mathbb{R}_+, H_{p,0}^{-1}(\mathbb{R}_+^3))$ and by Proposition A.2 there exists a unique solution $\tilde{u} \in L_{p,\rho}(\mathbb{R}_+, H_p^1(\Omega)) \cap H_{p,\rho}^1(\mathbb{R}_+, \widehat{H}_{p,0}^{-1}(\Omega))$ of (A.5), where the right-hand side f is replaced by $\tilde{f}$. Thus a solution u of (A.11) is given by $u = w + \tilde{u}$, where w solves

$$(\text{A.12}) \qquad \begin{aligned} \partial_t w - \Delta w &= 0 && \text{in } \mathbb{R}_+ \times D, \\ \gamma w &= g - \gamma \tilde{u} && \text{on } \mathbb{R}_+ \times \Gamma^+, \\ \gamma \partial_y w &= 0 && \text{on } \mathbb{R}_+ \times \Gamma^-, \\ w|_{t=0} &= 0 && \text{in } \mathbb{R}_+^3. \end{aligned}$$

In the following, we construct a solution w to (A.12) by semigroup theory. More precisely,

$$\begin{aligned} A &:= \sqrt{\partial_t - \Delta_{\mathbb{R}^2}}, \\ D(A) &:= {}_0H_{p,\rho}^{\frac{1}{2}}(\mathbb{R}_+, L_p(\mathbb{R}^2)) \cap L_{p,\rho}(\mathbb{R}_+, H_p^1(\mathbb{R}^2)), \end{aligned}$$

is the generator of an analytic semigroup $(\mathrm{e}^{-Ay})_{y \geq 0}$ on $X := L_{p,\rho}(\mathbb{R}_+, L_p(\mathbb{R}^2))$, see the proof of Proposition A.3(b). Since the growth bound $\omega(\mathrm{e}^{-Ay})$ of $(\mathrm{e}^{-Ay})_{y \geq 0}$ is strictly negative, we may define $\tilde{g} = (1 + \mathrm{e}^{-2A})^{-1}(g - \gamma \tilde{u})$. Then, w defined by

$$w(t,x,y) := \left((\mathrm{e}^{-Ay} + \mathrm{e}^{-A(2-y)}) \tilde{g} \right)(t,x)$$

is a solution to (A.12). Hence, it follows by interpolation that

$$w \in L_{p,\rho}(\mathbb{R}_+, H_p^1(D)) \cap H_{p,\rho}^{\frac{1}{2}}(\mathbb{R}_+, L_p(D)) = L_{p,\rho}(\mathbb{R}_+, D(A)) \cap H_{p,\rho}^1(\mathbb{R}_+, L_p(D))$$

if and only if $\tilde{g} \in (X, D(A))_{1-\frac{1}{p},p}$. Since $(1 + \mathrm{e}^{-2A})^{-1}$ is an isomorphism from $(X, D(A))_{1-\frac{1}{p},p}$ onto itself, we obtain

$$\tilde{g} \in (X, D(A))_{1-\frac{1}{p},p} \Longleftrightarrow g \in (X, D(A))_{1-\frac{1}{p},p} = L_p(J, W_p^{1-1/p}(\Gamma^+)) \cap W_{p,\rho}^{\frac{1}{2}-\frac{1}{2p}}(J, L_p(\Gamma^+)).$$

Integrating by parts yields $w \in H_{p,\rho}^1(\mathbb{R}_+, {}_0H_p^{-1}(D))$. Finally, the uniquess of w follows from by a duality argument. ☐

Appendix B. The reduced Stokes problem

In this section, we rewrite (4.2) as a reduced Stokes problem which is equivalent to (4.2). Applying divergence to the first equation in (4.2) we obtain

$$(\text{B.1}) \qquad \Delta p = \operatorname{div} f_1 - (\partial_t - \Delta) f_d$$

in the sense of distributions. Noting $\gamma_\nu \partial_t u = \partial_t \gamma w = 0$ on Γ^-, we apply γ_ν to the first equation in (4.2) and obtain $\gamma \partial_y p = \gamma_\nu f_1 + \gamma_\nu \Delta u$ on Γ^-. It is now advantageous to insert the term $\gamma \partial_y (f_d - \operatorname{div} u) = 0$ into the boundary condition on Γ^-. On the upper boundary Γ^+, we use the condition written in (4.2). Then,

$$\begin{aligned} \gamma p &= 2\gamma \partial_y w - \sigma \Delta_x h && \text{on } \Gamma^+, \\ \gamma \partial_y p &= \gamma_\nu f_1 + \gamma_\nu (\Delta u - \nabla \operatorname{div} u) + \gamma \partial_y f_d && \text{on } \Gamma^-. \end{aligned} \tag{B.2}$$

To solve (B.1)–(B.2), we split $p = p_1 + p_2$ where p_1 satisfies

$$\begin{aligned} \Delta p_1 &= \operatorname{div} f_1 - (\partial_t - \Delta) f_d && \text{in } D, \\ \gamma p_1 &= 0 && \text{on } \Gamma^+, \\ \gamma \partial_y p_1 &= \gamma_\nu f_1 + \gamma \partial_y f_d && \text{on } \Gamma^-, \end{aligned} \tag{B.3}$$

and p_2 satisfies (4.4). Note that p_1 and p_2 are well-defined by Proposition A.5 and the fact that $\operatorname{div}(\Delta u - \nabla \operatorname{div} u) = 0$. Defining the solution operators $T_1(f_1, f_d) := p_1$ and $T_2(u, h) := p_2$ for the boundary value problems (B.3) and (4.4), respectively, we may formulate the following proposition.

Proposition B.1. *Let $p \in (1, \infty)$, $p \neq 3/2, 3$. Then the following statements are equivalent:*

(1) *For every $f_1 \in \mathbb{F}_1(J, D)$ and $f_d \in \mathbb{F}_d(J, D, \Gamma^+)$ satisfying $f_d|_{t=0} = 0$ if $p > 3/2$ there exists a unique solution $(u, \pi, h) \in \mathbb{E}_1(J, D) \times \mathbb{E}_2(J, D, \Gamma^+) \times \mathbb{E}_3(J, \Gamma^+)$ of (4.2).*

(2) *For every $f_1 \in \mathbb{F}_1(J, D)$ and $g_r \in \mathbb{G}^+(J, \Gamma^+)$ satisfying $g_r|_{t=0} = 0$ if $p > 3$ there exists a unique solution $(u, h) \in \mathbb{E}_1(J, D) \times \mathbb{E}_3(J, \Gamma^+)$ of (4.3).*

Proof. (2)$\Longrightarrow$(1). We set $f_1 := f$ and choose f_d as the solution of the problem

$$\begin{aligned} (\partial_t - \Delta) f_d &= \operatorname{div} f && \text{in } J \times D, \\ \gamma f_d &= g && \text{on } \Gamma^+, \\ \gamma \partial_y f_d &= -\gamma_\nu f && \text{on } \Gamma^-, \\ f_d|_{t=0} &= 0 && \text{on } D. \end{aligned}$$

Here the unique solvability is guaranteed by Proposition A.6. Solving (4.2) with $\tilde{f}_1$, $\tilde{f}_d$, we see that p satisfies $\Delta p = 0$ with boundary conditions

$$\begin{aligned} \gamma p &= 2\gamma \partial_y w - \sigma \Delta_x h && \text{on } \Gamma^+, \\ \gamma \partial_y p &= \gamma_\nu (\Delta u - \nabla \operatorname{div} u) && \text{on } \Gamma^-. \end{aligned}$$

Thus, $p = p_2 = T_2(u, h)$ by definition of T_2 and Proposition A.5. Moreover, we have

$$\begin{aligned} \partial_t u - \Delta u + \nabla T_2(u, h) &= f_1 = \tilde{f}_1 && \text{in } D, \\ \gamma \operatorname{div} u &= \gamma f_d = && \text{on } \Gamma^+. \end{aligned}$$

Therefore, (u, h) is a solution of (4.3).

(1)$\Longrightarrow$(2). Thanks to $f_d|_{t=0} = 0$ we have $g|_{t=0} = 0$ for $p > 3$. Therefore, there exists a solution (u, h) of (4.3) with $f := f_1 - \nabla T_1(f_1, f_d)$ and $g = \gamma f_d$ on Γ^+. Setting $p := T_1(f_1, f_d) + T_2(u, h)$, we see that $(u, p, h) \in \mathbb{E}(J, D)$ solves all equations of (4.2) except the second line by construction. To show that also the second equality in (4.2) holds, we note that $\operatorname{div} u$ satisfies

$$(\partial_t - \Delta) \operatorname{div} u = \operatorname{div} \tilde{f}_1 - \operatorname{div} \nabla T_2(u, h) = \operatorname{div} f_1 - \Delta T_1(f_1, f_d) - \Delta T_2(u, h) = (\partial_t - \Delta) f_d$$

with boundary conditions

$$\begin{aligned} \gamma \operatorname{div} u &= \gamma f_d && \text{on } \Gamma^+, \\ \gamma \partial_y \operatorname{div} u &= \gamma_\nu \nabla \operatorname{div} u = \gamma_\nu f_1 + \gamma \partial_y f_d + \gamma_\nu \Delta u - \gamma \partial_y p = \gamma \partial_y f_d && \text{on } \Gamma^+. \end{aligned}$$

The unique solvability of this boundary value problem, see Proposition A.6, implies $\operatorname{div} u = f_d$. □

The following proposition can be proved in a similar way as Proposition B.1.

Proposition B.2. *The following statements are equivalent:*

(1) *For every* $f_1 \in \mathbb{F}_1(J, D^+)$ *and* $f_d \in \mathbb{F}_d(J, D^+, \Gamma^+)$ *satisfying* $f_d|_{t=0} = 0$ *if* $p > 3/2$*, there exists a unique solution* $(u, \pi, h) \in \mathbb{E}_1(J, D^+) \times \mathbb{E}_2(J, D^+, \Gamma^+) \times \mathbb{E}_3(J, D^+)$ *of* (5.6).
(2) *For every* $f_+ \in \mathbb{F}_1(J, D^+)$ *and* $g \in \mathbb{G}^+(J, \Gamma^+)$ *satisfying* $g|_{t=0} = 0$ *if* $p > 3$ *there exists a unique solution* $(u_+, h) \in \mathbb{E}_1(J, D^+) \times \mathbb{E}_3(J, D^+)$ *of* (4.7).

Moreover, the following statements are also equivalent:

(1) *For every* $f_1 \in \mathbb{F}_1(J, D^-)$ *and* $f_d \in \mathbb{F}_d(J, D^-, \emptyset)$ *there exists a unique solution* $(u, p) \in \mathbb{E}_1(J, D^-) \times \mathbb{E}_2(J, D^-, \emptyset)$ *of* (5.1).
(2) *For every* $f_- \in \mathbb{F}_1(J, D^-)$ *there exists a unique solution* $u_- \in \mathbb{E}_1(J, D^-)$ *of* (4.5).

Appendix C. Estimates for inhomogeneous symbols

For fixed $\epsilon \in (0, \frac{\pi}{2})$ and $\theta \in (0, \pi)$ we will consider polynomial symbols $P \colon \overline{\Sigma}_\epsilon \times \overline{\Sigma}_\theta \to \mathbb{C}$ of the form

(C.1)
$$P(z, \lambda) = \sum_{(\alpha,\beta,\gamma)\in I} a_{\alpha\beta\gamma} z^\alpha \lambda^\beta \omega(z, \lambda)^\gamma \quad \big((z, \lambda) \in \overline{\Sigma}_\epsilon \times \overline{\Sigma}_\theta\big)$$

with $a_{\alpha\beta\gamma} \in \mathbb{C} \setminus \{0\}$, $\omega(z, \lambda) := \sqrt{\lambda + z^2}$, and $I \subset \mathbb{N}_0^3$ being a finite set of exponents. To analyze this symbol, we will follow the Newton polygon approach described in [GV92] and [DMV98].

For this, we define the Newton polygon $N(P) \subset [0, \infty)^2$ as the convex hull of all points $(\alpha + \gamma, \beta + \frac{\gamma}{2})$ with $(\alpha, \beta, \gamma) \in I$, their projections onto the coordinate axes, and the origin. Denote the vertices of $N(P)$ by $v_0 := (0, 0), v_1, \ldots, v_{J+1}$, numbered in clockwise direction. Then for $v_j = (p_j, q_j)$ the vector $\frac{1}{\sqrt{1+r_j^2}}(1, r_j)$ with

$$r_j := -\frac{p_{j+1} - p_j}{q_{j+1} - q_j} \quad (j = 1, \ldots, J)$$

is an exterior normal to the edge $[v_j v_{j+1}]$ connecting v_j and v_{j+1}.

For simplicity, we assume that $N(P)$ has no edge parallel to the coordinate axes but not lying on the axis. More precisely, we assume

$$r_1 > r_2 > \cdots > r_J > 0.$$

In this case $N(P) = \operatorname{conv}(\tilde{I})$ with

$$\tilde{I} := \{(0, 0)\} \cup \{(\alpha + \gamma, \beta + \gamma/2), (\alpha, \beta, \gamma) \in I\}.$$

The main idea of the Newton polygon approach is to deal with different inhomogeneities by assigning a weight $r > 0$ to the co-variable λ with respect to z, i.e. to set $|\lambda| \approx |z|^r$. In a natural way, for $r > 0$ the r-degree $d_r(P)$ is defined as

(C.2)
$$d_r(P) := \max\{\alpha + r\beta + \gamma \max\{1, r/2\}, (\alpha, \beta, \gamma) \in I\}.$$

Note that in the same way for $\omega(z, \lambda) = \sqrt{\lambda + z^2}$ the r-degree is given by

$$d_r(\omega) = \begin{cases} 1, & r \le 2, \\ r/2, & r \ge 2. \end{cases}$$

The r-principal part of P is given by

$$P_r(z, \lambda) := \lim_{\rho\to\infty} \rho^{-d_r(P)} P(\rho z, \rho^r \lambda) \quad \big((z, \lambda) \in \Sigma_\epsilon \times \Sigma_\theta\big).$$

Obviously the "leading exponents" for weight r are given by

$$I_r := \{(\alpha, \beta, \gamma) \in I, \alpha + r\beta + \gamma \max\{1, r/2\} = d_r(P)\},$$

and we get

(C.3)
$$P_r(z, \lambda) = \sum_{(\alpha,\beta,\gamma)\in I_r} a_{\alpha\beta\gamma} z^\alpha \lambda^\beta \omega_r(z, \lambda)^\gamma$$

with

$$\omega_r(z,\lambda) = \begin{cases} \sqrt{\lambda}, & r > 2, \\ \sqrt{\lambda + z^2}, & r = 2, \\ z, & r < 2. \end{cases}$$

Geometric observations show that

$$I_r = \begin{cases} \{(\alpha,\beta,\gamma) \in I, (\alpha+\gamma, \beta+\frac{\gamma}{2}) \in [v_j v_{j+1}]\} & \text{if } r = r_j \quad (j = 1,\dots,J), \\ \{(\alpha,\beta,\gamma) \in I, (\alpha+\gamma, \beta+\frac{\gamma}{2}) = v_j\} & \text{if } r_{j-1} < r < r_j \quad (j = 1,\dots,J+1). \end{cases}$$

Here we formally have set $r_0 := \infty$ and $r_{J+1} := 0$.

In the next result we show how symbols satisfying a non-degeneracy condition (see (C.4)) give rise to isomorphic operators on their natural domain arising from the vertices of the Newton polygon. To this end, let $r, s \geq 0$ and

$$\mathbb{F}, \mathcal{K} \in \{H, W\}.$$

Then by $\mathcal{K}_p^s$ we either mean the space H_p^s or the space W_p^s. For notational convenience we set

$${}_0\mathbb{F}_p^s(\mathcal{K}_p^r) := {}_0\mathbb{F}_p^s(\mathbb{R}_+, \mathcal{K}_p^r(\mathbb{R}^n)).$$

In [DSS08, Theorem 3.2] the following result is proved.

Theorem C.1. *Let $1 < p < \infty$, $\rho, r, s \geq 0$, and let A, B be resolvent commuting operators such that for each $\gamma, \sigma \geq 0$,*

(i) *$\mathscr{D}(A) = {}_0\mathbb{F}_{p,\rho}^\sigma(\mathcal{K}_p^{\gamma+1})$ and $\mathscr{D}(B) = {}_0\mathbb{F}_{p,\rho}^{\sigma+1}(\mathcal{K}_p^\gamma)$,*

(ii) *$A, B \in \mathcal{H}^\infty({}_0\mathbb{F}_{p,\rho}^\sigma(\mathcal{K}_p^\gamma))$ with $\phi_A^\infty = 0$ and $\phi_B^\infty < \pi$.*

Furthermore, let P be a symbol as defined in (C.1) and let $v_j = (\alpha_j, \beta_j)$, $j = 0,\dots,J+1$ be the vertices of the Newton polygon corresponding to P. Suppose that P satisfies

$$P_r(z,\lambda) \neq 0 \quad (z \in \overline{\Sigma}_\epsilon \setminus \{0\}, \lambda \in \overline{\Sigma}_\theta \setminus \{0\}, r > 0). \tag{C.4}$$

for some $\epsilon > 0$ and $\theta > \phi_B^\infty$. Then there exists a $\lambda_0 > 0$ such that

$$P(A, B+\lambda_0) : \mathscr{D}(P(A, B+\lambda_0)) \to {}_0\mathbb{F}_{p,\rho}^s(\mathbb{R}_+, \mathcal{K}_p^r(\mathbb{R}^n))$$

is invertible, where

$$\mathscr{D}(P(A, B+\lambda_0)) = \bigcap_{j=1}^{J+1} {}_0\mathbb{F}_{p,\rho}^{s+\beta_j}(\mathcal{K}_p^{r+\alpha_j}).$$

In our applications, the operator B will always be the time derivative. For this reason we recall the following well-known fact.

Lemma C.2. *Let $1 < p < \infty$, $r, \rho \geq 0$, $\mathbb{F} \in \{H, W\}$ and X be a UMD space. Let G be the operator defined in the space ${}_0\mathbb{F}_{p,\rho}^r(\mathbb{R}_+, X)$ by*

$$Gu = \frac{d}{dt}u, \quad u \in D(G) := {}_0\mathbb{F}_{p,\rho}^{r+1}(\mathbb{R}_+, X). \tag{C.5}$$

Then $G \in \mathcal{H}^\infty({}_0\mathbb{F}_{p,\rho}^r(\mathbb{R}_+, X))$, i.e., G admits a bounded H^∞-calculus on ${}_0\mathbb{F}_{p,\rho}^r(\mathbb{R}_+, X)$ with H^∞-angle $\phi_G^\infty = \pi/2$.

By employing the shift $\mathrm{e}^{-\lambda_0 t}$ and Lemma 5.4 we immediately obtain the following result.

Corollary C.3. *Let $r, s, \rho \geq 0$ and $1 < p < \infty$. Let G be the time derivative operator as defined in (C.5) and A an operator such that for each $\gamma, \sigma \geq 0$,*

(i) *$\mathscr{D}(A) = {}_0\mathbb{F}_{p,\rho}^\sigma(\mathbb{R}_+, \mathcal{K}_p^{\gamma+1})$,*

(ii) *$A \in \mathcal{H}^\infty({}_0\mathbb{F}_{p,\rho}^\sigma(\mathbb{R}_+, \mathcal{K}_p^\gamma))$ with $\phi_A^\infty = 0$.*

Furthermore, let P be a symbol satisfying the assumptions of Theorem (C.1). Then, if λ_0 is the constant obtained in Theorem C.1, for $\omega \geq \lambda_0$ the operator $P(A,G) : \mathscr{D}(P(A,G)) \to {}_0\mathbb{F}^s_{p,\omega+\rho}\mathcal{K}^r_p(\mathbb{R}^n))$ is invertible, where

$$\mathscr{D}(P(A,G)) := \bigcap_{j=1}^{J+1} {}_0\mathbb{F}^{s+\beta_j}_{p,\omega+\rho}(J,\mathcal{K}^{r+\alpha_j}_p).$$

Proof. Denote by G_ρ the time derivative operator in the space ${}_0\mathbb{F}^s_{p,\rho}(J,\mathcal{K}^r_p(\mathbb{R}^n))$. Then

$$(\lambda - G_\omega)\Psi_\omega^{-1}u = \Psi_\omega^{-1}(\lambda-\rho-G_\rho)u \qquad (u \in D(G_\rho)),$$

which implies that

$$(\lambda - G_\omega)^{-1} = \Psi_\omega^{-1}(\lambda-(G_\rho+\omega))^{-1}\Psi_\omega.$$

By the Cauchy integral representation for the bounded holomorphic function $\lambda \mapsto P(A,\lambda)^{-1}$ this implies

$$P(A,G_\omega)^{-1} = \Psi_\omega^{-1}P(A,G_\rho+\omega)^{-1}\Psi_\omega.$$

By the assumption $\omega \geq \lambda_0$, the result then follows from Theorem C.1 with $B = G_\rho$, Theorem C.2, and from the fact that

$$\Psi_\omega \in Isom(\mathbb{F}^r_{p,\rho+\omega}(\mathbb{R}_+,X),\mathbb{F}^r_{p,\rho}(\mathbb{R}_+,X)),$$

which is an obvious consequence of Lemma 5.4(ii). □

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