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Parameter-Dependent Stochastic Optimal Control in Finite Discrete Time

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Abstract
We prove a general existence result in stochastic optimal control in discrete time, where controls, taking values in conditional metric spaces, depend on the current information and past decisions. The general form of the problem lies beyond the scope of standard techniques in stochastic control theory, the main novelty is a formalization in conditional metric space and the use of conditional analysis. We illustrate the existence result by several examples such as wealth-dependent utility maximization under risk constraints and utility maximization with a conditional dimension. We also provide a discussion as to how our methods compare to techniques based on random sets.

Keywords Conditional analysis · Stochastic optimal control · Conditional metric spaces

Mathematics Subject Classification 93E20 · 28B20 · 03E40

1 Introduction

The present work investigates parameter-dependent stochastic optimization in finite discrete time with the tools of conditional analysis. We consider a forward process

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$(x_t)_{t=0}^{\mathrm{T}}$, for which $x_{t+1} = v_t(x_t, z_t)$ depends on x_t as a function of earlier decisions and an immediate decision z_t chosen recursively in a state-dependent control set $\Theta_t(x_t)$.

Given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^{\mathrm{T}}, \mathbb{P})$, we assume that the forward process x_t and the control process z_t assume values in $\mathcal{F}_t$-conditional metric spaces X_t and Z_t, respectively. An $\mathcal{F}_t$-conditional metric space is a non-empty set X endowed with a vector-valued metric $d : X \times X \to L^0_+(\Omega, \mathcal{F}_t, \mathbb{P})$, satisfying a concatenation property, which encodes information at time t. An example is the space of strongly $\mathcal{F}_t$-measurable[1] functions with values in a metric space with almost everywhere evaluation of the metric. Intuitively, one may imagine a conditional metric space to be a collection of classical metric spaces $X(\omega)$ parameterized by the points $\omega \in \Omega$ of a probability space and glued together in a measurable fashion. When the underlying probability space is not a standard Borel space, or satisfies a similar regularity assumption, and the "fibers" $X(\omega)$ are not separable metric spaces, then this *pointwise* perspective runs into unsolvable measurability problems, usually related to the lack of Borel (or analytic) measurable selectors, or the problem of handling an uncountable collection of null sets. Therefore, instead of trying to model the problem fiberwise, we directly work in the conditional metric space X and build instead on arguments in conditional analysis. Important is that conditional analysis works when we quotient out consequently all null sets. Then, we can rely on the Dedekind completeness of the space $L^0(\mathcal{F})$ and the existence of concatenations of sequences along countable measurable partitions of Ω, which allow for the conditional argumentation. For a methodological discussion, we refer the interested reader to Sect. 5.

We focus on stochastic control problems, which by the Bellman principle can be reduced to a finite number of one-period conditional optimization problems. Our main result shows that the global maximizer is attained. By backward induction, we show that the optimal value function is upper semi-continuous on the conditional metric space X_t. For this, we assume that the control sets $\Theta_t(x_t)$ are conditionally sequentially compact; for a discussion of the notion of conditional compactness, we refer to [1, Sections 3 and 4] and [2, Sections 3.4 and 4]. Then, the existence of the one-period maximizers follows from a conditional version of the fact that a semi-continuous function on a compact space attains its extrema. Moreover, under a regularity condition on the control set—a conditional version of outer semi-continuity in set convergence (see, e.g., [3, Chapter 5, Section B] for the classical definition)—the value function is upper semi-continuous. Under stronger assumptions on the generators, the assumption of conditional compactness on the control set is relaxed in Proposition 4.1 by modifying arguments in [4].

In Sect. 3, we provide sufficient conditions for conditional compactness and conditional outer semi-continuity of the control set. We focus on conditionally finite dimensional control sets. The results are illustrated with applications in mathematical finance. In Example 3.2, we study an optimal consumption problem with local risk constraints on the wealth process. Example 3.3 indicates the importance of conditional Euclidean space with conditional dimension to model control processes with state-dependent dimension. As an application of Proposition 4.1, we derive optimal

[1] That is a Borel measurable function with an essentially separable range.

portfolios w.r.t. dynamic risk measures, for which the risk aversion coefficient is influenced by the current wealth.

Normal integrands, random sets and measurable selection techniques are common tools in the study of parameterized stochastic optimization; see, e.g., [5–8]. In Sect. 5, we discuss the connection of conditional analysis to random sets and normal integrands. In particular, we discuss a one-to-one correspondence between the set of measurable selections of Effros measurable and closed-valued mappings and stable and sequentially closed sets. This indicates that control problems formulated in the language of normal integrands and random sets can equally be formulated in the language of conditional analysis. For a formalization with normal integrands and random sets, measurable selection lemmas provide the main tool to secure measurability. The use of measurable selection arguments is enforced by a pointwise application of standard results in classical analysis, and relies on topological assumptions such as separability and standard Borel spaces. In this regard, conditional analysis provides a measure-theoretic alternative, which does not rely on any topological assumptions, and works as soon as a formalization within its language is reached, which is demonstrated in this article in discrete time stochastic control theory. Conditional analysis approaches measurable functions directly by working with a conditional version of results in classical analysis. The application of conditional versions of classical theorems preserves measurability, see for example the proofs below, in which a conditional version of the Bolzano–Weierstraß theorem, the maximum theorem and the Heine–Borel theorem are employed.

The remainder of this article is organized as follows. In Sect. 2, we introduce the notion of conditional metric spaces and prove the main existence result. In Sects. 3 and 4, we discuss extensions of the main result and provide several examples. The link between conditional analysis and random set theory is established in Sect. 5.

2 Main Result

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Throughout, we identify two sets in $\mathcal{F}$ whenever their symmetric difference is a null set, and identify two functions on Ω if they coincide a.s. (almost surely). Let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. Denote by $\Pi_{\mathcal{G}}$ the set of partitions (A_k) of Ω where $A_k \in \mathcal{G}$ for all k. Let $L^0_{\mathcal{G}}$, $L^0_{\mathcal{G}}(\mathbb{N})$, $L^0_{\mathcal{G},+}$, $L^0_{\mathcal{G},++}$, $\underline{\mathrm{L}}^0_{\mathcal{G}}$, and $\bar{L}^0_{\mathcal{G}}$ denote the spaces of $\mathcal{G}$-measurable random variables with values in $\mathbb{R}$, $\mathbb{N}$, $[0, +\infty[$, $]0, +\infty[$, $\mathbb{R} \cup \{-\infty\}$, and $\mathbb{R} \cup \{\pm\infty\}$, respectively. Recall that $L^0_{\mathcal{G}}$ with the pointwise a.s. order is a Dedekind complete lattice-ordered ring. The essential supremum and the essential infimum are denoted by sup and inf, respectively. Inequalities between random variables with values in an ordered set are always understood in the pointwise a.s. sense.

Definition 2.1 A $\mathcal{G}$-*conditional metric* on a non-empty set X is a function $d : X \times X \to L^0_{\mathcal{G},+}$, such that the following conditions hold:

(i) $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, z) \leq d(x, y) + d(y, z)$,

(iv) for every sequence (x_k) in X and $(A_k) \in \Pi_{\mathcal{G}}$, there exists exactly one element $x \in X$ such that $1_{A_k} d(x, x_k) = 0$ for all $k \in \mathbb{N}$.

The pair (X, d) is called a *$\mathcal{G}$-conditional metric space*.

In the following, we call the unique element in (iv) the *concatenation* of the sequence (x_k) along the partition (A_k), and denote it by $\sum_k 1_{A_k} x_k$. For a sequence (x_n) in a conditional metric space (X, d), we write $x_n \to x$ whenever $d(x, x_n) \to 0$ a.s.. Further, a *conditional subsequence* (x_{n_k}) of (x_n) is of the form $x_{n_k} := \sum_{j \in \mathbb{N}} 1_{\{n_k = j\}} x_j$, where (n_k) is a sequence in $L^0_{\mathcal{G}}(\mathbb{N})$ such that $n_k < n_{k+1}$ for all $k \in \mathbb{N}$.

Definition 2.2 Let (X, d_X) and (Z, d_Z) be *$\mathcal{G}$-conditional metric spaces*, and H and G subsets of X and Z, respectively.

The set H is called *$\mathcal{G}$-stable* if $H \neq \emptyset$ and $\sum_k 1_{A_k} x_k \in H$ for all $(A_k) \in \Pi_{\mathcal{G}}$ and every sequence (x_k) in H, and *sequentially closed* if H contains every $x \in X$, such that there is a sequence (x_k) in H with $x_k \to x$.

A function $f : H \to G$ is said to be *$\mathcal{G}$-stable* if $f\left(\sum_k 1_{A_k} x_k\right) = \sum_k 1_{A_k} f(x_k)$ for all $(A_k) \in \Pi_{\mathcal{G}}$ and every sequence (x_k) in H, where H and G are assumed to be $\mathcal{G}$-stable.

Remark 2.1 1. If (X, d) is a $\mathcal{G}$-conditional metric space, then the metric d is $\mathcal{G}$-stable, i.e., $d\left(\sum_k 1_{A_k} x_k, \sum_k 1_{A_k} y_k\right) = \sum_k 1_{A_k} d(x_k, y_k)$ for all sequences (x_k) and (y_k) in X and $(A_k) \in \Pi_{\mathcal{G}}$. Indeed, denoting by $x = \sum_k 1_{A_k} x_k$ and $y = \sum_k 1_{A_k} y_k$ the respective concatenations, it follows from the triangular inequality that

$$\begin{aligned} 1_{A_k} d(x, y) &\leq 1_{A_k} d(x, x_k) + 1_{A_k} d(x_k, y_k) + 1_{A_k} d(y_k, y) = 1_{A_k} d(x_k, y_k) \\ &\leq 1_{A_k} d(x_k, x) + 1_{A_k} d(x, y) + 1_{A_k} d(y_k, y) = 1_{A_k} d(x, y), \end{aligned}$$

which shows that $1_{A_k} d(x, y) = 1_{A_k} d(\sum_k 1_{A_k} x_k, \sum_k 1_{A_k} y_k) = 1_{A_k} d(x_k, y_k)$ for all $k \in \mathbb{N}$. Summing up over all k yields the desired $\mathcal{G}$-stability.

2. Let (X, d_X) and (Y, d_Y) be two $\mathcal{G}$-conditional metric spaces. Then, its product $X \times Y$ endowed with the $\mathcal{G}$-conditional metric

$$d_{X \times Y}\left((x, y), (x', y')\right) := \max\{d_X(x, x'), d_Y(y, y')\}$$

is a $\mathcal{G}$-conditional metric space.

3. Let (X, d_X) be a $\mathcal{G}$-conditional metric space. Then, the set $\mathbf{X}$ of all pairs $(x, A) \in X \times \mathcal{G}$, where (x, A) and (y, B) are identified if $A = B$ and $1_A d(x, y) = 0$, is a *conditional set* as introduced in [2]; in [2, Section 4] conditional metric spaces are defined.

We next introduce the parameter-dependent stochastic optimal control problem for conditional metric spaces. For a fixed finite time horizon $T \in \mathbb{N}$, we consider a filtration $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_T = \mathcal{F}$. For simplicity, we often abbreviate the index $\mathcal{F}_t$ by t, and write for instance L^0_t for $L^0_{\mathcal{F}_t}$. For each $t = 0, \ldots, T$, let (X_t, d_{X_t}) and (Z_t, d_{Y_t}) be $\mathcal{F}_t$-conditional metric spaces. Our aim is to study control problems, for which the control set Θ_t depends on $\mathcal{F}_t$, but also on a state parameter $x \in X_t$. For every $t = 0, \ldots, T-1$, we assume that the *state-dependent control set* Θ_t satisfies

(c1) $\emptyset \neq \Theta_t(x) \subset Z_t$ for all $x \in X_t$,
(c2) Θ_t is $\mathcal{F}_t$-stable, i.e.,

$$\Theta_t\left(\sum_k 1_{A_k} x_k\right) = \sum_k 1_{A_k} \Theta_t(x_k) := \left\{\sum_k 1_{A_k} z_k : z_k \in \Theta_t(x_k) \text{ for all } k\right\}$$

for all $(A_k) \in \Pi_t$ and every sequence (x_k) in X_t,
(c3) for every $x \in X_t$, the set $\Theta_t(x)$ is *conditionally sequentially compact*, i.e., for every sequence (z_n) in $\Theta_t(x)$, there exists a conditional subsequence $n_1 < n_2 < \cdots$ with $n_k \in L^0_t(\mathbb{N})$ such that $z_{n_k} \to z \in \Theta_t(x)$,
(c4) for every sequence (x_n) in X_t such that $x_n \to x \in X_t$ and every sequence (z_n) in $\Theta_t(x_n)$, there exists a conditional subsequence $n_1 < n_2 < \cdots$ with $n_k \in L^0_t(\mathbb{N})$ and a sequence (z'_k) in $\Theta_t(x)$ such that $d_{Z_t}(z_{n_k}, z'_k) \to 0$ a.s..

Note that $\mathcal{F}_t$-stability of Θ_t implies that $\Theta_t(x)$ is $\mathcal{F}_t$-stable for all $x \in X_t$.

We consider *forward generators*

$$v_t : X_t \times Z_t \to X_{t+1}, \quad t = 0, \ldots, T-1,$$

which are

(v1) $\mathcal{F}_t$-stable, i.e., $v_t\left(\sum_k 1_{A_k} x_k, \sum_k 1_{A_k} z_k\right) = \sum_k 1_{A_k} v_t(x_k, z_k)$ for every partition $(A_k) \in \Pi_t$, and all sequences (x_k) in X_t and (z_k) in Z_t,
(v2) sequentially continuous, i.e., $v_t(x_n, z_n) \to v_t(x, z)$ whenever $x_n \to x$ in X_t and $z_n \to z$ in Z_t.

For every $x_t \in X_t$, we consider the set

$$C_t(x_t) := \Big\{((x_s)_{s=t+1}^{\mathrm{T}}, (z_s)_{s=t}^{T-1}) : x_{s+1} = v_s(x_s, z_s), z_s \in \Theta_s(x_s) \\ \text{for all } s = t, \ldots, T-1\}$$

of all parameter processes $(x_s)_{s=t}^{\mathrm{T}}$, which can be realized by the state-dependent controls $z_s \in \Theta_t(x_s)$ for $s = t, \ldots, T-1$.

As for the objective function, we consider *backward generators*

$$u_t : X_t \times \underline{\mathrm{L}}^0_{t+1} \times Z_t \to \underline{\mathrm{L}}^0_t, \quad t = 0, 1, \ldots, T-1,$$

which are

(u1) $\mathcal{F}_t$-stable, i.e., $u_t\left(\sum_k 1_{A_k} x_k, \sum_k 1_{A_k} y_k, \sum_k 1_{A_k} z_k\right) = \sum_k 1_{A_k} u_t(x_k, y_k, z_k)$ for all $(A_k) \in \Pi_t$, and sequences (x_k) in X_t, (y_k) in Y_t, and (z_k) in Z_t,
(u2) increasing in the second component, i.e., $u_t(x, y, z) \leq u_t(x, y', z)$ whenever $y \leq y'$,
(u3) sequentially upper semi-continuous, i.e.,

$$\limsup_{n \to \infty} u_t(x_n, y_n, z_n) \leq u_t(x, y, z),$$

whenever $x_n \to x$ in X_t, $y_n \to y$ in $\underline{L}^0_{t+1}$, and $z_n \to z$ in Z_t.

We assume that $u_T : X_T \to \underline{L}^0_T$ is $\mathcal{F}_T$-stable and sequentially upper semi-continuous.[2]

Given such a family $(u_t)_{t=0}^{T}$ of backward generators, our goal is to maximize

$$y_t(x_t) := \sup_{((x_s)_{s=t+1}^{T},(z_s)_{s=t}^{T-1})\in C_t(x_t)} u_t(x_t,\cdot,z_t)\circ\cdots\circ u_{T-1}(x_{T-1},\cdot,z_{T-1})\circ u_T(x_T), \tag{1}$$

over all realizable state processes initialized at $x_t \in X_t$. In (1) we consider the composition of the functions $u_T, u_{T-1}(x_{T-1},\cdot,z_{T-1}),\ldots,u_t(x_t,\cdot,z_t)$, where $u_s(x_s,\cdot,z_s)$ denotes the function $\underline{L}^0_{s+1} \to \underline{L}^0_s$, $y \mapsto u_s(x_s,y,z_s)$.

Remark 2.2 The objective function in the stochastic control problem (1) is recursively defined. Its generators are functions between conditional metric spaces which are not necessarily (conditional) expected utilities. In case of (conditional) expected utility, the generators are closely related with dynamic and conditional risk measures; see [9–14]. The preferences which underly conditional expected utility functionals were studied in [15].

In decision theory, there is an extensive literature on recursive utilities starting with the seminal work [16,17]. The preferences therein are defined on sets of temporal lotteries (probability trees), and follow a kind of Bellman recursive structure, which is similar (on a formal level) to the construction above; see [16, Theorem 1]. This was later extended in [18], where non-expected utilities were incorporated as well, and established under the name of Epstein–Zin utilities. See also [19] for a survey on non-expected utility theory. With the techniques of conditional analysis and based on results in BSDE theory, [20] solves a utility maximization problem in continuous time for Epstein–Zin utilities.

The following result shows that the global supremum in (1) is attained and can be reduced to local optimization problems by the following Bellman's principle.

Theorem 2.1 *Suppose that* (c1)–(c4), (v1)–(v2), *and* (u1)–(u3) *are fulfilled. Then, the functions* $y_t\colon X_t \to \underline{L}^0_t$ *are* $\mathcal{F}_t$*-stable and sequentially upper semi-continuous for all* $t = 0,\ldots,T$, *and can be computed by backward recursion*

$$\begin{aligned} y_T(x_T) &= u_T(x_T)\\ y_t(x_t) &= \max_{z_t\in\Theta_t(x_t)} u_t(x_t, y_{t+1}(v_t(x_t,z_t)), z_t), \quad t = 0,\ldots,T-1. \end{aligned}$$

Moreover, for every $x_t \in X_t$ *the process* $((x^*_s)_{s=t}^{T}, (z^*_s)_{s=t}^{T-1})$, *given by* $x^*_t = x_t$, *and the forward recursion* $x^*_{s+1} = v_s(x^*_s, z^*_s)$, *where*

$$z^*_s \in \operatorname*{argmax}_{z_s\in\Theta_s(x^*_s)} u_s\left(x^*_s, y_{s+1}(v_t(x^*_s,z_s)), z_s\right), \quad s = t,\ldots T-1, \tag{2}$$

[2] $u_T : X_T \to \underline{L}^0_T$ is sequentially upper semi-continuous if $\limsup_{n\to\infty} u_T(x_n) \le u_T(x)$ whenever $x_n \to x$ in X_T.

satisfies $((x_s^*)_{s=t+1}^{\mathrm{T}}, (z_s^*)_{s=t}^{T-1}) \in C_t(x_t)$ *and*

$$y_t(x_t) = u_t(x_t, \cdot, z_t^*) \circ \cdots \circ u_{T-1}(x_{T-1}^*, \cdot, z_{T-1}^*) \circ u_T(x_T^*).$$

Proof The proof is by backward induction. For $t = T$, it follows from (1) that $y_T = u_T$, which by assumption is an $\mathcal{F}_t$-stable and sequentially upper semi-continuous function from X_T to $\underline{\mathrm{L}}_T^0$.

As for the induction step, assume that $y_{t+1}\colon X_{t+1} \to \underline{\mathrm{L}}_{t+1}^0$ is $\mathcal{F}_{t+1}$-stable and sequentially upper semi-continuous, and that for each $x_{t+1} \in X_{t+1}$ there exists $((x_s^*)_{s=t+2}^{\mathrm{T}}, (z_s^*)_{s=t+1}^{T-1}) \in C_{t+1}(x_{t+1})$ such that

$$y_{t+1}(x_{t+1}) = u_{t+1}(x_{t+1}, \cdot, z_{t+1}^*) \circ \cdots \circ u_T(x_T^*).$$

By (u1) and (v1), the function

$$X_t \times Z_t \ni (x, z) \mapsto u_t\,(x, y_{t+1}(v_t(x, z)), z)$$

is $\mathcal{F}_t$-stable. Moreover, it is sequentially upper semi-continuous. Indeed, let (x_k, z_k) be a sequence in $X_t \times Z_t$ such that $x_k \to x \in X_t$ and $z_k \to z \in Z_t$. Since $v(x_k, z_k) \to v(x, z)$ by (v2), it follows from the induction hypothesis that

$$\limsup_{k\to\infty} y_{t+1}(v_t(x_k, z_k)) \le y_{t+1}(v(x, z)) < +\infty.$$

Since

$$\begin{aligned}\left\{\sup_{k\ge 1} y_{t+1}(v_t(x_k, z_k)) = +\infty\right\} &= \bigcap_{k\ge 1}\left\{\sup_{k'\ge k} y_{t+1}(v_t(x_{k'}, z_{k'})) = +\infty\right\}\\ &= \left\{\limsup_{k\to\infty} y_{t+1}(v_t(x_k, z_k)) = +\infty\right\},\end{aligned}$$

we have $\sup_{k\ge 1} y_{t+1}(v_t(x_k, z_k)) \in \underline{\mathrm{L}}_{t+1}^0$. Hence, by (u2), (u3) and (v2), we get

$$\begin{aligned}\limsup_{k\to\infty} u_t\,(x_k, y_{t+1}(v_t(x_k, z_k)), z_k) &\le \limsup_{k\to\infty} u_t\left(x_k, \sup_{k'\ge k} y_{t+1}(v_t(x_{k'}, z_{k'})), z_k\right)\\ &\le u_t\left(x, \limsup_{k\to\infty} y_{t+1}(v_t(x_k, z_k)), z\right)\\ &\le u_t\,(x, y_{t+1}(v_t(x, z)), z)\,, \end{aligned} \tag{3}$$

which shows the desired sequential upper semi-continuity. As a consequence, the supremum in

$$f_t(x_t) := \sup_{z\in\Theta_t(x_t)} u_t\,(x_t, y_{t+1}(v_t(x_t, z)), z) \tag{4}$$

is attained for each $x_t \in X_t$. Indeed, since $z \mapsto u_t\,(x, y_{t+1}(v_t(x,z)), z)$ and $\Theta_t(x_t)$ are $\mathcal{F}_t$-stable, it follows from standard properties of the essential supremum that there exists a sequence $z_n \in \Theta_t(x_t)$ such that

$$u_t\,(x_t, y_{t+1}(v_t(x_t, z_n)), z_n) \to f_t(x_t).$$

By (c3), there is a conditional subsequence $n_1 < n_2 < \cdots$ with $n_k \in L^0_t(\mathbb{N})$ such that $z_{n_k} \to z \in \Theta_t(x_t)$ a.s.. Since $z \mapsto u_t\,(x, y_{t+1}(v_t(x,z)), z)$ is sequentially upper semi-continuous and $\mathcal{F}_t$-stable, it follows that

$$u_t\,(x_t, y_{t+1}(v_t(x_t,z)), z) \geq \limsup_{k\to\infty} u_t\left(x_t, y_{t+1}(v_t(x_t, z_{n_k})), z_{n_k}\right) = f_t(x_t),$$

which shows that the supremum in (4) is attained.

We next show that $f_t : X_t \to \underline{\mathrm{L}}^0_{t+1}$ is sequentially upper semi-continuous. By contradiction, suppose that (x_k) is a sequence in X_t such that $x_k \to x \in X_t$ and $f_t(x) < \limsup_{k\to\infty} f_t(x_k)$ on some $A \in \mathcal{F}$ with $\mathbb{P}(A) > 0$. Note that f_t is $\mathcal{F}_t$-stable. Thus, by possibly passing to a conditional subsequence , we can suppose that there exists $r \in L^0_{t,++}$ such that

$$f_t(x) + r < f_t(x_k) \text{ on } A, \quad \text{for all } k \in \mathbb{N}. \tag{5}$$

Denote by $z_k \in \Theta_t(x_k)$ a respective maximizer of $f_t(x_k)$. By (c4), there exists $z'_k \in \Theta_t(x)$ such that $d_{Z_t}(z_k, z'_k) \to 0$ a.s. by possibly passing to a conditional subsequence . By (c3), there exists a conditional subsequence $k_1 < k_2 < \cdots$ with $k_l \in L^0_t(\mathbb{N})$ such that $z'_{k_l} \to z' \in \Theta_t(x)$. Since $d_{Z_t}(z_{k_l}, z'_{k_l}) \to 0$ a.s., by $\mathcal{F}_t$-stability of the conditional metric d_{Z_t}, it follows from the triangular inequality that $z_{k_l} \to z' \in \Theta_t(x)$. By the $\mathcal{F}_t$-stability of f_t and (c2), it follows that z_{k_l} is in $\Theta_t(x_{k_l})$ and maximizes $f_t(x_{k_l})$. Hence, it follows from (3) that

$$\begin{aligned}\limsup_{l\to\infty} f_t(x_{k_l}) &= \limsup_{l\to\infty} u_t\left(x_{k_l}, y_{t+1}(v_t(x_{k_l}, z_{k_l})), z_{k_l}\right)\\ &\leq u_t\left(x, y_{t+1}(v_t(x, z')), z'\right)\\ &\leq \sup_{z\in\Theta_t(x)} u_t\,(x, y_{t+1}(v_t(x,z), z) = f_t(x).\end{aligned}$$

Notice that, due to the $\mathcal{F}_t$-stability of f_t, (5) is satisfied for any conditional subsequence of (x_k). Thus, we have that $f_t(x) + r \leq \limsup_{l\to\infty} f_t(x_{k_l}) \leq f_t(x)$ on A, which is a contradiction. We conclude that f_t is sequentially upper semi-continuous.

Finally, we show that $y_t = f_t$. By induction hypothesis, for every $x_t \in X_t$ and $z_t \in Z_t$, there exists $((x^*_s)^{\mathrm{T}}_{s=t+2}, (z^*_s)^{T-1}_{s=t+1}) \in C_{t+1}(v_t(x_t, z_t))$ such that

$$y_{t+1}(v_t(x_t, z_t)) = u_{t+1}(v_t(x_t, z_t), \cdot, z^*_{t+1}) \circ \cdots \circ u_{T-1}(x^*_{T-1}, \cdot, z^*_{T-1}) \circ u_T(z^*_T).$$

In particular, for $x_t \in X_t$ and $z_t^* \in Z_t$ being a maximizer in (4), it holds

$$\begin{aligned}
f_t(x_t) &= \sup_{z\in\Theta_t(x_t)} u_t\left(x_t, y_{t+1}(v_t(x_t,z)), z\right)\\
&= u_t\left(x_t, y_{t+1}(v_t(x_t,z_t^*)), z_t^*\right)\\
&= u_t\left(x_t, \cdot, z_t^*\right) \circ u_{t+1}(v_t(x_t,z_t^*), \cdot, z_{t+1}^*) \circ \cdots \circ u_{T-1}(x_{T-1}^*, \cdot, z_{T-1}^*) \circ u_T(z_T^*)\\
&= \sup_{(x,z)\in C_{t+1}(v(x_t,z_t^*))} u_t\left(x_t, \cdot, z_t^*\right) \circ u_{t+1}(v_t(x_t,z_t), \cdot, z_{t+1}) \circ \cdots \circ u_T(z_T)\\
&= \sup_{z_t\in\Theta_t(x_t)} \sup_{(x,z)\in C_{t+1}(v(x_t,z_t))} u_t\left(x_t, \cdot, z_t\right) \circ u_{t+1}(v_t(x_t,z_t), \cdot, z_{t+1}) \circ \cdots \circ u_T(z_T)\\
&= \sup_{(x,z)\in C_t(x_t)} u_t\left(x_t, \cdot, z_t\right) \circ u_{t+1}(v_t(x_t,z_t), \cdot, z_{t+1}) \circ \cdots \circ u_T(z_T)\\
&= y_t(x_t).
\end{aligned}$$

This shows that $((x_s^*)_{s=t+1}^{\mathrm{T}}, (z_s^*)_{s=t}^{\mathrm{T}}) \in C_t(x_t)$ is an optimizer of (1) whenever it satisfies the local optimality criterion

$$z_s^* \in \operatorname{argmax}_{z\in\Theta_t(x_t^*)} u_s\left(x_s^*, y_{s+1}(v_t(x_s^*, z_s)), z_s\right) \quad \text{and} \quad x_{s+1}^* = v_s(x_s^*, z_s^*)$$

for all $s = t, \ldots, T$, where $x_t^* = x_t$. In particular, every process which satisfies the forward recursion (2) is an optimizer for (1). □

Example 2.1 As for the illustration, we provide examples of $\mathcal{F}_t$-conditional metric spaces, which are of interest for the control and parameter spaces.

1. Given a nonempty metric space (X, d), denote by $L_t^0(X)$ the set of all strongly $\mathcal{F}_t$-measurable functions $x\colon \Omega \to X$. The metric d extends from X to $L_t^0(X)$ by defining

$$d_{L_t^0(X)}(x, \bar{x})(\omega) := d(x(\omega), \bar{x}(\omega)) \quad \text{for a.a. } \omega \in \Omega \text{ and all } x, \bar{x} \in L_t^0(X).$$

   Then, $(L_t^0(X), d_{L_t^0(X)})$ is a $\mathcal{F}_t$-conditional metric space.
2. The conditional Euclidean space with dimension $n = \sum_k 1_{A_k} n_k \in L_t^0(\mathbb{N})$ is defined as

$$L_t^0(\mathbb{R})^n = \sum_k 1_{A_k} L_t^0(\mathbb{R}^{n_k}) := \left\{ \sum_k 1_{A_k} x_k \colon x_k \in L_t^0(\mathbb{R}^{n_k}) \text{ for all } k \right\}.$$

   The $\mathcal{F}_t$-conditional metric on $L_t^0(\mathbb{R})^n$ is defined by

$$d_{L_t^0(\mathbb{R})^n}(x, \bar{x}) := \sum_k 1_{A_k} d_{L_t^0(\mathbb{R}^{n_k})}(x_k, \bar{x}_k),$$

   where $x = \sum_k 1_{A_k} x_k$ and $\bar{x} = \sum_k 1_{A_k} \bar{x}_k$. Here, $d_{L_t^0(\mathbb{R}^{n_k})}$ denotes the $\mathcal{F}_t$-conditional metric on $L_t^0(\mathbb{R}^{n_k})$. Straightforward verification shows that $(L_t^0(\mathbb{R})^n, d_{L_t^0(\mathbb{R})^n})$ is a $\mathcal{F}_t$-conditional metric space.

3. For $1 \leq p < \infty$, we define the conditional L^p-space

$$L_t^p := \{x \in L_T^0 : \mathbb{E}[|x|^p | \mathcal{F}_t] < +\infty\}$$

with $\mathcal{F}_t$-conditional metric $d_{L_t^p}(x, \bar{x}) := \mathbb{E}[|x - \bar{x}|^p | \mathcal{F}_t]^{1/p}$. Then, $(L_t^p, d_{L_t^p})$ is a $\mathcal{F}_t$-conditional metric space.

3 Compactness Condition for the Control Set

3.1 The Finite Dimensional Case

Suppose that $Z_t = L_t^0(\mathbb{R}^d)$. As shown in Example 2.1, the Euclidean metric of $\mathbb{R}^d$ extends to the $\mathcal{F}_t$-conditional metric $d_{L_t^0(\mathbb{R}^d)} : L_t^0(\mathbb{R}^d) \to L_{t,+}^0$.

Proposition 3.1 *Suppose that for each $t = 0, \ldots, T-1$, the control set Θ_t satisfies (c1), (c2) and the following conditions:*

(i) $\left\{(x, z) \in X_t \times L_t^0(\mathbb{R}^d) : z \in \Theta_t(x)\right\}$ is sequentially closed,

(ii) for every sequence (x_n) in X_t with $x_n \to x \in X_t$ a.s. there exists $M \in L_{t,+}^0$ such that $d_{L_t^0(\mathbb{R}^d)}(z, 0) \leq M$ for all $z \in \bigcup_n \Theta_t(x_n)$.

Then, the control set Θ_t satisfies (c1)–(c4).

Proof Let (x_n) in X_t be a sequence such that $x_n \to x \in X_t$ a.s., and (z_n) a sequence in $\Theta_t(x_n)$. Since by assumption, $d_{L_t^0(\mathbb{R}^d)}(z_n, 0) \leq M$ for some $M \in L_{t,+}^0$, the conditional Bolzano–Weierstrass theorem [1, Theorem 3.8] implies a conditional subsequence $n_1 < n_2 < \cdots$ with $n_k \in L_t^0(\mathbb{N})$ such that $d_{L_t^0(\mathbb{R}^d)}(z_{n_k}, z) \to 0$ a.s. for some $z \in L_t^0(\mathbb{R}^d)$. Since Θ_t satisfies (c2), it holds $z_{n_k} \in \Theta_t(x_{n_k})$, and (i) implies $z \in \Theta_t(x)$. This shows (c4). Further, (c3) follows by considering the constant sequence $x_n = x$ for all $n \in \mathbb{N}$. □

Example 3.1 Let $(S_t)_{t=0}^{\mathrm{T}}$ be a $(\mathcal{F}_t)$-adapted price process with values in $]0, +\infty[^d$. Given an initial investment $x_0 > 0$, we consider the wealth process

$$x_{t+1} = v_t(x_t, z_t) := x_t + \vartheta_t \cdot \Delta S_{t+1},$$

where the control is an investment strategy $\vartheta_t \in L_t^0(\mathbb{R}^d)$. We consider the (wealth-dependent) control set with short-selling constrains

$$\Theta_t(x_t) := \left\{\vartheta_t \in L_t^0(\mathbb{R}^d) \colon x_t = \vartheta_t \cdot S_t,\ 0 \leq \vartheta_t^i,\ i = 1, \ldots, d\right\}.$$

For each t, we consider a bounded measurable function $g_t \colon \Omega \times \mathbb{R}^d \to \mathbb{R}$ such that $g_t(\omega, \cdot)$ is upper semi-continuous for every $\omega \in \Omega$. Next, we show that there exists an optimizer $((x_s^*)_{s=t+1}^{\mathrm{T}}, (\vartheta_s^*)_{s=t}^{T-1}) \in C_t(x_t)$ of the utility maximization problem

$$\sup_{((x_s)_{s=t}^{\mathrm{T}}, (\vartheta_s)_{s=t}^{T-1}) \in C_t(x_t)} \mathbb{E}\left[\sum_{s=t}^{\mathrm{T}} g_s(\cdot, x_s) \mid \mathcal{F}_t\right]. \tag{6}$$

Inspection shows that Θ_t satisfies (c1), (c2) and (i) of Proposition 3.1. As for (ii), for every $\vartheta \in \Theta_t(x)$, it can be checked that $d_{L_t^0(\mathbb{R}^d)}(\vartheta, 0) \leq M(x)$ with $M(x) := \max_{i=1,\ldots,d} \frac{x}{S_t^i}$. Let (x_n) be a sequence in L_t^0 such that $x_n \to x \in L_t^0$. Take $\bar{x} := \sup_n x_n \in L_+^0$. Then, $d_{L^0(\mathbb{R}^d)}(\vartheta, 0) \leq M(\bar{x})$ for all $z \in \bigcup_n \Theta_t(x_n)$, which shows (ii). Finally, for fixed $\alpha > 0$, define $u_T(x) := g_T(\cdot, x)$ and for $t = 0, 1, \ldots, T-1$,

$$u_t(x, y, z) := \mathbb{E}\left[g_t(\cdot, x) + \max\{-\alpha, \min\{\alpha, y\}\} \,|\mathcal{F}_t\right].$$

Then, inspection shows that the backward process $(u_t)_{t=0}^{\mathrm{T}}$ satisfies conditions (u1) and (u2), and due to Fatou's lemma, it also satisfies (u3). Since g_t is bounded, for $\alpha > 0$ large enough, it can be checked by backward induction that

$$u_t(x_t, \cdot, z_t) \circ \cdots \circ u_{T-1}(x_{T-1}, \cdot, z_{T-1}) \circ u_T(x_T) = \mathbb{E}\left[\sum_{s=t}^{\mathrm{T}} g_s(\cdot, x_s) \mid \mathcal{F}_t\right]$$

for every $((x_s)_{s=t+1}^{\mathrm{T}}, (\vartheta_s)_{s=t}^{T-1}) \in C_t(x_t)$. By Proposition 3.1 and Theorem 2.1 it follows that (6) has a global optimizer $((x_s^*)_{s=t+1}^{\mathrm{T}}, (\vartheta_s^*)_{s=t}^{T-1}) \in C_t(x_t)$.

Example 3.2 Let $(S_t)_{t=0}^{\mathrm{T}}$ be a d-dimensional $(\mathcal{F}_t)$-adapted price process. Given an initial investment $x_0 > 0$, we consider the wealth process

$$x_{t+1} = v_t(x_t, z_t) := x_t + \vartheta_t \cdot \Delta S_{t+1} - c_t,$$

where the control $z_t = (\vartheta_t, c_t) \in L_t^0(\mathbb{R}^d) \times L_+^0$ consists of an investment strategy $\vartheta_t \in L_t^0(\mathbb{R})$ and a consumption $c_t \in L_{t,+}^0$. Further, the forward generator $v_t : L_t^0 \times L_t^0(\mathbb{R}^d \times \mathbb{R}_+) \to L_{t+1}^0$ satisfies (v1) and (v2). We assume that the wealth process satisfies the regulatory restriction

$$\rho_t(x_{t+1}) \leq 0, \quad t = 0, \ldots, T-1. \tag{7}$$

In other words, x_{t+1} is acceptable w.r.t. a $\mathcal{F}_t$-conditional convex risk measure $\rho_t : L_{t+1}^0 \to \bar{L}_t^0$ for all $t = 0, \ldots, T-1$. Recall that a $\mathcal{F}_t$-conditional convex risk measure is

- normalized, i.e., $\rho_t(0) = 0$,
- monotone, i.e., $\rho_t(x) \leq \rho_t(y)$ for all $x, y \in L_{t+1}^0$ with $x \geq y$,
- $\mathcal{F}_t$-translation invariant, i.e., $\rho_t(x + m) = \rho(x) - m$ for all $x \in L_{t+1}^0$ and $m \in L_t^0$,
- $\mathcal{F}_t$-convex, i.e., $\rho_t(\lambda x + (1-\lambda)y) \leq \lambda\rho_t(x) + (1-\lambda)\rho_t(y)$ for all $x, y \in L_{t+1}^0$ and $\lambda \in L_t^0$ with $0 \leq \lambda \leq 1$.

By $\mathcal{F}_t$-translation invariance, it follows that (7) is equivalent to

$$\rho_t(\vartheta_t \cdot \Delta S_{t+1}) \leq x_t - c_t.$$

Moreover, ρ_t is $\mathcal{F}_t$-stable as it is $\mathcal{F}_t$-convex; see [1, Lemma 4.3]. Hence, we consider the (wealth-dependent) $\mathcal{F}_t$-stable control set

$$\Theta_t(x_t) := \Big\{ z_t = (\vartheta_t, c_t) \in L^0_t(\mathbb{R}^d \times \mathbb{R}_+) \colon \rho_t(\vartheta_t \cdot \Delta S_{t+1}) \le x_t - c_t \text{ and } 0 \le c_t \le x_t \Big\}.$$

Suppose that for every $\vartheta \in L^0_t(\mathbb{R}^d)$ it holds $\mathbb{P}(\vartheta \cdot \Delta S_{t+1} < 0 \mid \mathcal{F}_t) > 0$ on $\{\vartheta \neq 0\}$, and therefore $\mathbb{P}(\vartheta \cdot \Delta S_{t+1} > 0 \mid \mathcal{F}_t) > 0$ on $\{\vartheta \neq 0\}$. Moreover, we assume that $\rho_t(\vartheta \cdot \Delta S_{t+1}) \in L^0_t$ for all $\vartheta \in L^0_t(\mathbb{R}^d)$, and ρ_t is $\mathcal{F}_t$-*sensitive to large losses*, i.e., $\lim_{m\to\infty} \rho_t(my) = +\infty$ on $\{\mathbb{P}(y < 0 \mid \mathcal{F}_t) > 0\}$. Then, the control set Θ_t satisfies (i) and (ii) of Proposition 3.1. Indeed, consider the function $f_t \colon L^0_t(\mathbb{R}^d \times \mathbb{R}_+) \times L^0_t \to L^0_t$ defined as $f_t(\vartheta, c, x) := \rho_t(\vartheta \cdot \Delta S_{t+1}) + c - x$, which is $\mathcal{F}_t$-convex and therefore sequentially continuous by [1, Theorem 7.2]. Hence, it follows that

$$\Big\{(x, \vartheta, c) \in L^0_t \times L^0_t(\mathbb{R}^d \times \mathbb{R}_+) : (\vartheta, c) \in \Theta_t(x)\Big\} = \Big\{(x, \vartheta, c) \in L^0_t \times L^0_t(\mathbb{R}^d \times \mathbb{R}_+) : f_t(\vartheta, c, x) \le 0\Big\}$$

is $\mathcal{F}_t$-convex and sequentially closed, which shows (i). As for (ii), let (x_n) be a sequence in L^0_t such that $x_n \to x \in L^0_t$. For $\bar{x} := \sup_n x_n \in L^0_t$ one has

$$\Theta_t(x_n) \subset \Theta_t(\bar{x})$$

for all $n \in \mathbb{N}$. Hence, it remains to show that $\Theta_t(\bar{x})$ is $\mathcal{F}_t$-bounded, i.e., there is $M \in L^0_{t,+}$ such that $d_{L^0_t(\mathbb{R}^d)}(\vartheta, 0) + c \le M$ for all $(\vartheta, c) \in \Theta_t(\bar{x})$. Since $\Theta_t(\bar{x})$ contains $(0, 0) \in L^0_t(\mathbb{R}^d \times \mathbb{R}_+)$, by [1, Theorem 3.13] it is enough to show that for each $(\vartheta, c) \in \Theta_t(\bar{x})$ with $(\vartheta, c) \neq (0, 0)$, there exists $k \in \mathbb{N}$ such that $k(\vartheta, c) \notin \Theta_t(\bar{x})$. If $c \neq 0$, this is obvious. Otherwise, it holds $\mathbb{P}(\vartheta \neq 0) > 0$, in which case $\lim_{m\to\infty} \rho_t(m\vartheta \cdot \Delta S_{t+1}) = +\infty$ on $\{\vartheta \neq 0\}$.

By Proposition 3.1 and Theorem 2.1, it follows that for each $x_0 > 0$ and every recursive utility function with backward generators $(u_t)_{t=0}^{\mathrm{T}}$, which satisfy (u1)–(u3), there exists a global optimizer $((x^*_s)_{s=1}^{\mathrm{T}}, (\vartheta^*_s, c^*_s)_{s=0}^{T-1}) \in C_0(x_0)$ of the utility maximization problem (1) such that the local criterion (2) holds.

3.2 Conditional Dimension

Suppose that Z_t is the conditional Euclidean space $L^0_t(\mathbb{R})^{d_t}$ with dimension $d_t = d_t(x) \in L^0_t(\mathbb{N})$, which depends on the parameter $x \in X_t$; see Example 2.1. Let $d_t \colon X_t \to L^0_t(\mathbb{N})$ be an $\mathcal{F}_t$-stable and sequentially continuous, where $L^0_t(\mathbb{N})$ is endowed with the $\mathcal{F}_t$-conditional metric which extends the discrete metric on $\mathbb{N}$. The control set Θ_t is chosen such that the following conditions hold:

(c1) $\emptyset \neq \Theta_t(x) \subset L^0_t(\mathbb{R})^{d_t(x)}$ for all $x \in X_t$, (c2) Θ_t is $\mathcal{F}_t$-stable, i.e.,

$$\Theta_t\left(\sum_k 1_{A_k} x_k\right) = \sum_k 1_{A_k} \Theta_t(x_k) \subset L_t^0(\mathbb{R})^{d_t(\sum_k 1_{A_k} x_k)}$$

for all $(A_k) \in \Pi_t$ and every sequence (x_k) in X_t.

Remark 3.1 Since $Z_t = L_t^0(\mathbb{R})^{d_t(x)}$ depends on the state $x \in X_t$, we are in a more general setting as in Theorem 2.1. However, since $L_t^0(\mathbb{N})$ is endowed with the conditional discrete metric, for every sequence (x_n) in X_t such that $x_n \to x \in X_t$, there exists $n_0 \in L_t^0(\mathbb{N})$ such that $d_t(x_n) = d_t(x)$ for all $n \geq n_0$. In particular, $L_t^0(\mathbb{R})^{d_t(x_n)} = L_t^0(\mathbb{R})^{d_t(x)}$ for all $n \geq n_0$, and Theorem 2.1 still holds true by exploring the arguments on $z_n \in \Theta_t(x_n)$ for sequences $x_n \to x$ in the conditional space $L_t^0(\mathbb{R})^{d_t(x)}$.

A variant of Proposition 3.1 for control sets with conditional dimension can be formulated as follows.

Proposition 3.2 *Suppose that for each $t = 0, \ldots, T-1$, the control set Θ_t satisfies (c1), (c2) and the following conditions:*

(i) $\left\{(x, z) \in X_t \times L_t^0(\mathbb{R})^{d_t(x)} : z \in \Theta_t(x)\right\}$ *is sequentially closed.*
(ii) For every sequence (x_n) in X_t with $x_n \to x \in X_t$, there is $M \in L^0_{t,+}$ and a conditional subsequence $n_1 < n_2 < \cdots$ in $L_t^0(\mathbb{N})$, which satisfy $d_{L_t^0(\mathbb{R}^{d_t(x)})}(z, 0) \leq M$ for all $z \in \bigcup_{k \geq k_0} \Theta_t(x_{n_k})$ for some $k_0 \in L_t^0(\mathbb{N})$, such that $\Theta_t(x_{n_k}) \subset Z_t(d_t(x))$ for all $k \geq k_0$.

Then, the control set Θ_t satisfies (c1)–(c4).

Proof Let (x_n) in X_t be a sequence such that $x_n \to x \in X_t$, and $z_n \in \Theta_t(x_n)$. By Remark 3.1 there exists a conditional subsequence $n_1 < n_2 < \cdots$ with $n_k \in L_t^0(\mathbb{N})$ such that $z_{n_k} \in \Theta_t(x_{n_k}) \subset L_t^0(\mathbb{R})^{d_t(x)}$ for all k. Hence, we can argue similar as in the proof of Proposition 3.1. □

Example 3.3 Consider a portfolio maximization problem, where the number of traded assets depends on past decision. More precisely, given a portfolio $x_t = z_{t-1} = (\vartheta_{t-1}, d_{t-1}) \in L_{t-1}^0(\mathbb{R})^{d_{t-1}} \times L_{t-1}^0(\mathbb{N})$ chosen at time $t-1$ (with initial value $x_{-1} = (\vartheta_{-1}, d_{-1}) \in \mathbb{R}^{d_{-1}} \times \mathbb{N}$), the investor can rebalance the portfolio at time t to

$$x_{t+1} = z_t = (\vartheta_t, d_t) \in \Theta_t(x_t) \subset L_t^0(\mathbb{R})^{d_{t-1}} \times L_t^0(\mathbb{N}).$$

Here, the state spaces and the control spaces $X_{t+1} = Z_t = L_t^0(\mathbb{R})^{d_{t-1}} \times L_t^0(\mathbb{N})$ both depend on the past decision d_{t-1}. In line with Remark 3.1, the convergence $x_t^n = (\vartheta_{t-1}^n, d_{t-1}^n) \to x_t = (\vartheta_{t-1}, d_{t-1})$ is understood as $\vartheta_{t-1}^n \to \vartheta_{t-1}$ in the conditional metric space $L_t^0(\mathbb{R})^{d_{t-1}}$, since $d_{t-1}^n = d_{t-1}$ for all $n \geq n_0$ for some $n_0 \in L_{t-1}^0(\mathbb{N})$. Suppose that the control set Θ_t satisfies (c1), (c2) as well as (i) and (ii) of Proposition 3.2. Then, along the same argumentation as in Proposition 3.2, it follows that Θ_t satisfies (c1)–(c4). Since $v_t(x_t, z_t) := z_t$ satisfies (v1) and (v2), Theorem 2.1 is applicable whenever the backward generators $(u_t)_{t=0}^{\mathrm{T}}$ satisfy (u1)–(u3).

The conditional dimension depending on past decisions allows for instance to add new assets at time t ($d_t > d_{t-1}$) which are traded at $t+1$. Notice that $\Theta_t(\vartheta_{t-1}, d_{t-1})$ denotes the set of all attainable portfolios at time t. For instance, let $S_t \in L^0_{t,++}(\mathbb{R}^d)$ be a price process with fixed $d \in \mathbb{N}$. Without frictions and short-selling constraints one has

$$\Theta_t(\vartheta_{t-1}) := \{\vartheta_t \in L^0_{t,+}(\mathbb{R}^d) : \vartheta_t \cdot S_t = \vartheta_{t-1} \cdot S_t\},$$

which satisfies (c1)–(c4). Transaction costs can be included into the model by considering $\Theta_t(\vartheta_{t-1}) := \{\vartheta_t \in L^0_{t,+}(\mathbb{R}^d) : \vartheta_t - \vartheta_{t-1} \in C_t\}$ for a solvency region $C_t \subset L^0_t(\mathbb{R}^d)$; see, e.g., [21] for a discussion of different market models. Also, the solvency regions can be modeled state-dependently with conditional dimension $d_t \in L^0_t(\mathbb{N})$.

4 Unbounded Control Sets

In this section, we consider unbounded control sets $\Theta_t \equiv L^0_t(\mathbb{R}^d)$ and do not assume constraints on the controls, but derive (c3) and (c4) for upper-level sets of y_t as a result of stronger assumptions on the forward and backward generators. In particular, we additionally need that the backward generators are $\mathcal{F}_t$-sensitive to large losses and increasing in the first argument. Suppose that the forward generators

$$v_t : L^0_t \times L^0_t(\mathbb{R}^d) \to L^0_{t+1}, \quad t = 0, 1, \ldots, T-1,$$

satisfy (v1), (v2) and

- (v3) v_t is increasing in the first component,
- (v4) $v_t(x, \lambda z + (1-\lambda)z') \geq \lambda v_t(x, z) + (1-\lambda)v_t(x, z')$ for all $x \in L^0_t$, $z, z' \in L^0_t(\mathbb{R}^d)$, and $\lambda \in L^0_t$ with $0 \leq \lambda \leq 1$,
- (v5) $\mathbb{P}(v_t(x, z) < x \mid \mathcal{F}_t) > 0$ on $\{z \neq 0\}$ for all $x \in L^0_t$ and $z \in L^0_t(\mathbb{R}^d)$,
- (v6) $v_t(x, 0) = x$ for all $x \in L^0_t$.

As for the backward generators, let $u_T : L^0_T \to L^0_T$ be the identity mapping, and

$$u_t : L^0_t \times \underline{L}^0_{t+1} \times L^0_t(\mathbb{R}^d) \to \underline{L}^0_t, \quad t = 0, \ldots, T-1,$$

satisfy (u1) and (u3) as well as

- (u2′) u_t is increasing in the first and second component,
- (u4) $u_t(x, \lambda y + (1-\lambda)y', \lambda z + (1-\lambda)z') \geq \min\{u_t(x, y, z), u_t(x, y', z')\}$ for all $x \in L^0_t$, $y, y' \in \underline{L}^0_{t+1}$, $z, z' \in L^0_t(\mathbb{R}^d)$, and $\lambda \in L^0_t$ with $0 \leq \lambda \leq 1$,
- (u5) $u_t(x, y + c, z) = u_t(x, y, z) + c$ for all $x \in L^0_t$, $y \in \underline{L}^0_{t+1}$, $z \in L^0_t(\mathbb{R}^d)$ and $c \in L^0_t$,

(u6) $\lim_{m\to\infty} u_t(x, my, mz) = -\infty$ on $\{\mathbb{P}(y < 0 \mid \mathcal{F}_t) > 0\}$ for all $z \in L_t^0(\mathbb{R}^d)$ and $y \in \underline{\mathrm{L}}_{t+1}^0$,

(u7) $u_t(x, 0, 0) = 0$ for all $x \in L_t^0$.

Let $y_t \colon L_t^0 \to \underline{\mathrm{L}}_t^0$ be given as in (1), where

$$C_t(x_t) := \left\{ \left((x_s)_{s=t+1}^{\mathrm{T}}, (z_s)_{s=t}^{T-1} \right) : x_{s+1} = v_s(x_s, z_s), z_s \in L_t^0(\mathbb{R}^d) \right.$$
$$\left. \text{for all } s = t, \dots, T-1 \right\}.$$

Then, the following variant of Theorem 2.1 holds.

Proposition 4.1 *Suppose that (v1)–(v6) and (u1),* $(u2')$, *(u3)–(u7) are fulfilled, and there exists a constant* $K > 0$ *such that*

$$\sup_{z \in L_t^0(\mathbb{R}^d)} u_t(x, v_t(x, z), z) - x \le K$$

for all $t = 0, \dots, T-1$, *and* $x \in L_t^0$. *Then, the functions* $y_t \colon L_t^0 \to \underline{\mathrm{L}}_t^0$ *are* $\mathcal{F}_t$*-stable, increasing and sequentially upper semi-continuous for all* $t = 0, \dots, T$, *and can be computed by backward recursion*

$$y_T(x_T) = u_T(x_T) = x_T$$
$$y_t(x_t) = \max_{z_t \in L_t^0(\mathbb{R}^d)} u_t(x_t, y_{t+1}(v_t(x_t, z_t)), z_t), \quad t = 0, \dots, T-1.$$

Moreover, for every $x_t \in L_t^0$ *the process* $((x_s^*)_{s=t}^{\mathrm{T}}, (z_s^*)_{s=t}^{T-1})$ *given by* $x_t^* = x_t$, *and forward recursion* $x_{s+1}^* = v_s(x_s^*, z_s^*)$, *where*

$$z_s^* \in argmax_{z_s \in L_t^0(\mathbb{R}^d)} u_s\left(x_s^*, y_{s+1}(v_t(x_s^*, z_s)), z_s\right), \quad s = t, \dots T-1, \tag{8}$$

satisfies $((x_s^*)_{s=t+1}^{\mathrm{T}}, (z_s^*)_{s=t}^{T-1}) \in C_t(x_t)$ *and*

$$y_t(x_t) = u_t(x_t, \cdot, z_t^*) \circ \cdots \circ u_{T-1}(x_{T-1}^*, \cdot, z_{T-1}^*) \circ u_T(x_T^*).$$

Proof The proof is similar to Theorem 2.1. However, since the control set is not compact we have to argue differently to show the existence of (4), i.e., that the supremum in

$$y_t(x_t) := \sup_{z \in L_t^0(\mathbb{R}^d)} u_t\left(x_t, y_{t+1}(v_t(x_t, z)), z\right), \quad x_t \in L_t^0,$$

is attained. To do so, we first show that

$$0 \le y_t(x) - x \le K_t \quad \text{for all } x \in L_t^0, \tag{9}$$

where $K_t := (T-t)K$ for all $t = 0, 1, \ldots, T$. For $t = T$, one has $y_T(x) - x = 0$. By induction, suppose that $y_{t+1}(x) - x \leq (T-t)K_{t+1}$. Then, by (u2') and (u5) for every $z \in L^0_t(\mathbb{R}^d)$, it holds

$$\begin{aligned} &u_t\left(x, y_{t+1}(v_t(x,z)), z\right) - x = u_t\left(x, y_{t+1}(v_t(x,z)) - v_t(x,z) + v_t(x,z), z\right) - x \\ &\leq u_t(x, v_t(x,z), z) - x + K_{t+1} \leq K + K_{t+1} = K_t, \end{aligned}$$

so that $y_t(x) - x \leq K_t$. As for the lower bound, suppose by induction that $x \leq y_{t+1}(x)$. By (v6), (u2$'$), (u5) and (u7) it follows that

$$\begin{aligned} y_t(x) &\geq u_t\left(x, y_{t+1}(v_t(x,0)), 0\right) \geq u_t\left(x, y_{t+1}(x), 0\right) \\ &\geq u_t\left(x, x, 0\right) = u_t\left(x, 0, 0\right) + x = x. \end{aligned}$$

Fix $x \in L^0_t$. For $z \in L^0_t(\mathbb{R}^d)$ with $u_t(x, y_{t+1}(v_t(x,0)), 0) \leq u_t(x, y_{t+1}(v_t(x,z)), z)$, it follows from (9) (u5), (u7) and (v6) that

$$\begin{aligned} x &= u_t(x,x,0) \leq u_t(x, y_{t+1}(x), 0) \leq u_t\left(x, y_{t+1}(v_t(x,0)), 0\right) \\ &\leq u_t\left(x, y_{t+1}(v_t(x,z)), z\right) \leq u_t\left(x, v_t(x,z), z\right) + K_{t+1}. \end{aligned}$$

This shows that

$$y_t(x) = \sup_{z \in \Theta_t(x)} u_t\left(x, y_{t+1}(v_t(x,z)), z\right),$$

for the $\mathcal{F}_t$-stable set

$$\Theta_t(x) := \left\{ z \in L^0_t(\mathbb{R}^d) \colon u_t(x, v_t(x,z), z) \geq x - K_{t+1} \right\}.$$

It remains to show that Θ_t satisfies (c1)–(c4). To that end, we verify (i) and (ii) of Proposition 3.1. By (u3) and (v2), it follows that the set

$$\left\{ (x,z) \in L^0_t \times L^0_t(\mathbb{R}^d) : z \in \Theta_t(x) \right\}$$

is sequentially closed, which shows (i) of Proposition 3.1. As for (ii) of Proposition 3.1 let (x_n) be a sequence in L^0_t such that $x_n \to x \in L^0_t$. Defining $\underline{x} := \inf_n x_n \in L^0_t$ as well as $\bar{x} := \sup_n x_n \in L^0_t$, it follows from (u2') and (v3) that

$$\Theta_t(x_n) \subset \left\{ z \in L^0_t(\mathbb{R}^d) \colon u_t(\bar{x}, v_t(\bar{x}, z), z) \geq \underline{x} - K_{t+1} \right\} =: \Theta_t(\underline{x}, \bar{x})$$

for all $n \in \mathbb{N}$. Moreover, by (u4) and (v4), the set $\Theta_t(\underline{x}, \bar{x})$ is $\mathcal{F}_t$-convex. It remains to show that there exists $M \in L^0_t$ such that $d_{L^0_t(\mathbb{R}^d)}(z, 0) \leq M$ for all $z \in \Theta_t(\underline{x}, \bar{x})$. This L^0_t-boundedness of $\Theta_t(\underline{x}, \bar{x})$ would follow from [1, Theorem 3.13], if for all

$z \in \Theta_t(\underline{x}, \bar{x})$ with $z \neq 0$, there exists $A \in \mathcal{F}_t$ with $\mathbb{P}(A) > 0$ such that

$$\lim_{m\to\infty} u_t(\bar{x}, v_t(\bar{x}, mz), mz) = -\infty \quad \text{on } A. \tag{10}$$

Indeed, since by (v5) one has $\mathbb{P}(v_t(\bar{x}, z) < \bar{x} \mid \mathcal{F}_t) > 0$ on $\{z \neq 0\}$, there exists $l \in \mathbb{N}$ such that $A := \{\mathbb{P}\left(|\bar{x}| + l(v_t(\bar{x}, z) - \bar{x}) < 0 \mid \mathcal{F}_t\right) > 0\} \in \mathcal{F}_t$ satisfies $\mathbb{P}(A) > 0$. By (v4), it follows that

$$v_t(\bar{x}, z) \geq \frac{1}{m} v_t(\bar{x}, mz) + \frac{m-1}{m} v_t(\bar{x}, 0),$$

which by (v6) implies $m\,(v_t(\bar{x}, z) - \bar{x}) \geq v_t(\bar{x}, mz) - \bar{x}$ for all $m \in \mathbb{N}$. This shows that

$$\begin{aligned} u_t(\bar{x}, v_t(\bar{x}, mz), mz) &\leq u_t(\bar{x}, |\bar{x}| + v_t(\bar{x}, mz) - \bar{x}, mz) \\ &\leq u_t\left(\bar{x}, \frac{m}{l}\left(|\bar{x}| + l(v_t(\bar{x}, z) - \bar{x})\right), mz\right) \end{aligned}$$

for all $m \in \mathbb{N}$ large enough. Hence, the condition (u6) implies (10). □

Example 4.1 Let $(S_t)_{t=0}^{\mathrm{T}}$ be a $\mathbb{R}^d$-valued adapted stochastic process modeling the discounted stock prices of a financial market model. Given a trading strategy $\vartheta_t \in L_t^0(\mathbb{R}^d)$, $t = 0, \ldots, T-1$, and an initial investment $x_0 \in L_0^0$, we define recursively the wealth process

$$x_{t+1} = v_t(x_t, \vartheta_t) := x_t + \vartheta_t \cdot \Delta S_{t+1}, \quad t = 0, \ldots, T-1,$$

where $\Delta S_{t+1} := S_{t+1} - S_t$ denotes the stock price increment. We assume the no-arbitrage condition:

$$\vartheta \cdot \Delta S_{t+1} \geq 0 \text{ for } \vartheta \in L_t^0(\mathbb{R}^d) \quad \text{implies} \quad \vartheta = 0$$

for all $t = 0, \ldots, T-1$. Then, the forward generator $v_t : L_t^0 \times L_t^0(\mathbb{R}^d) \to L_{t+1}^0$ satisfies (v1)–(v6). As for the backward generators, let $u_T : L_T^0 \to L_T^0$ be the identity and

$$u_t : L_t^0 \times \underline{\mathrm{L}}_{t+1}^0 \to \underline{\mathrm{L}}_t^0, \quad u_t(x, y) := \frac{1}{\gamma_t(x)} g_t(\gamma_t(x) y), \quad t = 0, \ldots, T-1,$$

where $g_t : \underline{\mathrm{L}}_{t+1}^0 \to \underline{\mathrm{L}}_t^0$ is increasing, $\mathcal{F}_t$-concave, $\mathcal{F}_t$-translation invariant, sequentially upper semi-continuous, $g_t(0) = 0$ and $\lim_{r\to\infty} g_t(ry) = -\infty$ on $\{\mathbb{P}(y < 0 \mid \mathcal{F}_t) > 0\}$. The function $\gamma_t : L_t^0 \to L_{t,++}^0$ is $\mathcal{F}_t$-stable, decreasing and sequentially continuous and models the risk aversion depending on the wealth x_t at time t. Then, u_t satisfies the conditions (u1), (u2′), (u3)–(u7). We only verify (u2′) and (u3). To prove (u2′) take

$x_1 \leq x_2$ and $y_1 \leq y_2$. Let $\beta_i := 1/\gamma_t(x_i)$, for $i = 1, 2$. By using the monotonicity and $\mathcal{F}_t$-concavity of g_t, we have

$$g_t\left(\frac{y_2}{\beta_2}\right) \geq g_t\left(\frac{y_1}{\beta_2}\right) \geq \frac{\beta_1}{\beta_2} g_t(\frac{y_1}{\beta_1}) + \frac{\beta_2 - \beta_1}{\beta_2} g_t(0) = \frac{\beta_1}{\beta_2} g_t\left(\frac{y_1}{\beta_1}\right).$$

Multiplying by β_2, we obtain $u_t(x_2, y_2) \geq u_t(x_1, y_1)$. Due to the monotonicity of u_t, it suffices to verify (u3) for decreasing sequences. Indeed, suppose that $x_k \searrow x$ and $y_k \searrow y$. Then, by the monotonicity of u_t we have

$$g_t(\gamma_t(x_k)y_k) \geq \frac{\gamma_t(x_k)}{\gamma_t(x)} g_t(\gamma_t(x)y) \quad \text{for all } k.$$

Thus, by using that g_t is sequentially upper semi-continuous and γ is sequentially continuous we obtain

$$\begin{aligned} g_t(\gamma_t(x)y) &\geq \limsup_{k\to\infty} g_t(\gamma_t(x_k)y_k) \geq \liminf_{k\to\infty} g_t(\gamma_t(x_k)y_k) \\ &\geq \lim_{k\to\infty} \frac{\gamma_t(x_k)}{\gamma_t(x)} g_t(\gamma_t(x)y) = g_t(\gamma_t(x)y). \end{aligned}$$

This shows that $g_t(\gamma_t(x_k)y_k) \to g_t(\gamma_t(x)y)$, and therefore $u_t(x_k, y_k) \to u_t(x, y)$. Given the wealth process $(x_t)_{t=0}^{\mathrm{T}}$, define the backward process

$$y_t(x_t) = \sup_{((x_s)_{s=t}^{\mathrm{T}}, (\vartheta_s)_{s=t}^{T-1}) \in C_t(x_t)} u_t(x_t, \cdot) \circ \cdots \circ u_{T-1}(x_{T-1}, \cdot) \circ u_T(x_T),$$

for $t = 0, \ldots, T-1$, where $C_t(x_t)$ consists of all $((x_s)_{s=t}^{\mathrm{T}}, (\vartheta_s)_{s=t}^{T-1})$ such that $x_{s+1} = x_s + \vartheta_{s+1} \cdot \Delta S_{s+1}$ for all $s = t, \ldots, T-1$. By induction, one can verify that $y_t(x+c) = y_t(x) + c$ for every $c \in L^0_{t-1}$ with $t = 1, \ldots, T$. Suppose there exists $K > 0$ such that

$$u_t\left(x, v_t(x, \vartheta), \vartheta\right) - x \leq \frac{1}{\gamma_t(x)} g_t\left(\gamma_t(x)\vartheta \Delta S_{t+1}\right) \leq K \tag{11}$$

for all $t = 0, \ldots, T-1$, $\vartheta \in L^0_t(\mathbb{R}^d)$, and $x \in L^0_t$. Then, it follows from Proposition 4.1 that

$$y_0(x_0) = \sup_{((x_t)_{t=0}^{\mathrm{T}}, (\vartheta_t)_{t=0}^{T-1}) \in C_0(x_0)} u_0(x_0, \cdot) \circ \ldots \circ u_{T-1}(x_{T-1}, \cdot) \circ u_T(x_T)$$

is attained for all $x_0 \in L^0_0$. For instance one could think of the dynamic entropic preference functional with generators $-\frac{1}{\gamma_t} \log\left(\mathbb{E}[\exp(-\gamma_t y) \mid \mathcal{F}_t]\right)$, where the local risk aversion coefficient $\gamma_t = \gamma_t(x_t)$ depends on the current wealth x_t. Notice that $\lim_{m\to\infty} -\log(\mathbb{E}[\exp(-my) \mid \mathcal{F}_t]) = -\infty$ on $\{\mathbb{E}[y < 0 \mid \mathcal{F}_t] > 0\}$.

5 Connection to Random Set Theory

The aim of this section is to discuss methodological similarities and differences of conditional analysis and random set theory. Random sets are established on the basis of results in classical analysis. They seek a formalization of a randomized problem such that classical theorems can be applied in each state or point of the underlying probability space. Measurable selection lemmas then help to extract a measurable object. To guarantee that such selections exist usually topological countability assumptions such as separability on the involved spaces are necessary. On the other hand, conditional analysis relies on a conditional version of classical results which are directly applied to sets of measurable functions where we consequently identify functions which are equal almost surely. Therefore, the formalization mainly consists in describing those sets for which a conditional version of classical results can be proved [2]. Measurability is then systematically preserved by the application of a conditional version of classical results by construction. A conditional version of classical theorems exists under measure-theoretic countability assumptions such as σ-finiteness and by working on the measure algebra by quotienting out the ideal of null sets from the underlying probability space, while the topological restrictions of random set techniques can be relaxed.

In the following, we show that conditional analysis extends measurable selections and random set theory. More precisely, we establish a correspondence between basic objects in random set theory and their analogues in conditional analysis under the hypothesis of separability. We fix a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a Polish space E. Recall that a closed-valued map $S\colon \Omega \rightrightarrows E$ (i.e., $S(\omega) \subset E$ is a closed set for all $\omega \in \Omega$) is Effros measurable[3] whenever $S^{-1}(O) := \{\omega \in \Omega\colon S(\omega) \cap O \neq \emptyset\} \in \mathcal{F}$ for all open sets O in E. Throughout, we assume that $S(\omega) \neq \emptyset$ for a.a. $\omega \in \Omega$, and we identify S_1 and S_2 whenever $S_1(\omega) = S_2(\omega)$ for a.a. $\omega \in \Omega$. We say that $x \in L^0(E)$ is an a.s. measurable selection of S if $x(\omega) \in S(\omega)$ for a.a. $\omega \in \Omega$. We denote by X_S the set of all a.s. measurable selections of S.

The following theorem is due to Kabanov and Safarian [22, Proposition 5.4.3] in the particular case $E = \mathbb{R}^d$; see also [23, Theorem 2.1.6] and [24, Theorem 2.3][4] and the Remark 5.1 for further details on the relations of these results. Due to limitation of space, we drop our alternative proof of Theorem 5.1 which uses conditional analysis techniques, which can be found in the `arXiv`-version.

Theorem 5.1 *Let $S\colon \Omega \rightrightarrows E$ be a closed-valued and Effros measurable mapping, and let $X \subset L^0(E)$ be decomposable and sequentially closed.*[5] *Then, there exist closed-valued and Effros measurable mappings $S_X\colon \Omega \rightrightarrows E$ and $S_{X_S}\colon \Omega \rightrightarrows E$ satisfying the reciprocality relations $S = S_{X_S}$ and $X = X_{S_X}$.*

[3] There are other measurability concepts besides Effros measurability; see, e.g., [23, Section 1.2]. One of them is graph-measurability, i.e., $\{(\omega, x) \in \Omega \times E\colon x \in S(\omega)\}$ is product-measurable. For closed-valued mappings, graph measurability is equivalent to Effros measurability whenever the underlying measurable space is complete; cf., e.g., [23, Theorem 2.3].

[4] We are indebted to an anonymous referee for these references.

[5] It is easy to see that a decomposable and sequentially closed set is necessarily countably decomposable or stable under countable concatenations in our sense.

A frequently employed concept in stochastic optimal control is a normal integrand; see, e.g., [25,26], that is a function $f\colon \Omega \times E \to \mathbb{R}$ whose epigraphical mapping $S_f\colon \Omega \rightrightarrows E \times \mathbb{R}$, $S_f(\omega) := \{(x,r) \in E \times \mathbb{R}\colon f(\omega,x) \le r\}$, is closed-valued and Effros measurable. A consequence of normality of an integrand is that $f(\omega, x(\omega))$ is measurable in ω whenever $x\colon \Omega \to E$ is a measurable function. Moreover, a normal integrand $f(\omega,x)$ is measurable in ω for fixed x and lower semi-continuous in x for fixed ω; cf. [3, Proposition 14.28]. We obtain the following functional version of Theorem 5.1, where two normal integrands $f\colon \Omega \times E \to \mathbb{R}$ and $g\colon \Omega \times E \to \mathbb{R}$ are identified if their epigraphical mappings coincide a.s..

Corollary 5.1 *Let $u\colon L^0(E) \to L^0$ be stable and sequentially lower semi-continuous and let $f\colon \Omega \times E \to \mathbb{R}$ be a normal integrand. Then, there exist a stable and sequentially lower semi-continuous function $u_f\colon L^0(E) \to L^0$ and a normal integrand $f_u\colon \Omega \times E \to \mathbb{R}$ such that $u_{f_u} = u$ and $f_{u_f} = f$.*

Proof Due to normality, $u_f\colon L^0(E) \to L^0$ given by $x \mapsto (\omega \mapsto f(\omega, x(\omega)))$ is well defined. Direct inspection shows that u_f is stable and sequentially lower semi-continuous. Conversely, put $X := \{(x,r) \in L^0(E \times \mathbb{R})\colon u(x) \le r\}$. By assumption, X is a stable and sequentially closed subset of $L^0(E \times \mathbb{R})$. By Theorem 5.1, there exist an Effros measurable and closed-valued map $S_X\colon \Omega \rightrightarrows E \times \mathbb{R}$ corresponding to X. Thus $f_u\colon \Omega \times E \to \mathbb{R}$ defined by $f(\omega,x) := \inf S(\omega)_x$ a.s. is a normal integrand where $S(\omega)_x$ denotes the x-section of $S(\omega)$. It follows from the reciprocality relations in Proposition 5.1 that $u_{f_u} = u$ and $f_{u_f} = f$. □

We compare the assumptions which underly conditional analysis and random set theory. Conditional analysis is applicable under the following two purely measure-theoretic hypotheses:

- A probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ needs to be fixed a priori in order to identify[6] sets, functions, relations, etc. almost surely. By the almost sure identification, we work with equivalence classes of functions and sets, and thus basically change to a pointfree perspective.
- One consequently works in the context of conditional sets [2]. In particular, all involved sets must satisfy stability under countable concatenations; cf., Definition 2.2.

Conditional analysis does not rely on the following topological assumptions which are prevalent in random set theory:

- standard Borel space,[7] measure completeness, closed-valued mappings and Polish spaces.

The connections provided by Theorem 5.1 and Corollary 5.1 suggest that a stochastic control problem can equally be formalized in the language of conditional set theory.

[6] Actually, one needs to fix a σ-ideal $\mathcal{I}$ of $\mathcal{F}$ such that the quotient $\mathcal{F}/\mathcal{I}$ is a complete Boolean algebra. The ideal of null sets of a σ-finite measure is one such example, see [2] and the references therein for more examples.

[7] For example, for purposes of a dynamical programming principle in finite discrete time stochastic optimal control, more precisely the existence of disintegration of measure, the underlying measure space is additionally assumed to be standard Borel in [31].

In, e.g., [6,25–28] some form of integrability is always assumed, which leads to further technicalities in the proofs; see also [29,30] and the references therein for basic studies on the relations of (conditional) expectations and integrands. The main results in Sect. 2 are established for general utilities which are not necessarily in the form of expected utilities, and no integrability assumptions are required.

Remark 5.1 Let E be a separable Banach space, and let $L^p(E)$ be the Bochner space of all p-integrable functions $x\colon \Omega \to E$ for $p \in [1, \infty]$. For a set-valued mapping $S\colon \Omega \rightrightarrows E$, denote by $X_S^p := X_S \cap L^p(E)$ the set of p-integrable selections of S. Let $X \subset L^p(E)$ be norm-closed. The result of Kabanov and Safarian [22, Theorem 5.4.3] states that $X = X_S^p$ for an Effros measurable closed-valued mapping $S\colon \Omega \rightrightarrows E$ if and only if X is finitely decomposable and L^p-closed, see also [23, Theorem 2.1.6], while [24, Theorem 2.3] by Molchanov and Lépinette replace L^p-closed by closed with respect to convergence in probability.

6 Conclusions

Conditional analysis allows for a treatment of stochastic control problems, which offers a viable alternative to classical measurable selection arguments. In particular, as controls and states are modeled in general conditional metric spaces, the mathematical restrictions of random set techniques such as finite dimension, separability, and completeness can be relaxed. The existence result Theorem 2.1 covers a wide variety of situations such as wealth-dependent utility maximization under risk constraints, or utility maximization, where the number of traded assets depends on past decisions. We applied the novel notion of conditional compactness [2], which works in finite and infinite dimensional settings thanks to a conditional version of the Heine–Borel theorem [2, Theorem 4.6]. Conditional compactness extends the notion of compact-valued and Effros measurable mappings; see [32]. The control sets work in any conditional metric space. This involves many examples, which are out of reach of the existing technology, for example conditional L^p-spaces on general probability spaces, $L^0(\mathbb{R})^n$ with a conditional dimension, and $L^0(X)$, where X is a non-separable metric space. Another example are conditional weak topologies, which are not included in this article for which conditional analysis offers extensive tools as well. We plan to explore this direction in future work.

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