Quasi-Ordered Rings: a uniform Study of Orderings and Valuations

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Abstract

The subject of this thesis is a systematic investigation of quasi-ordered rings. A quasi-ordering $\preceq$, sometimes also called preordering, is usually understood to be a binary, reflexive and transitive relation on a set. In his note *Quasi-Ordered Fields* ([19]), S. M. Fakhruddin introduced totally quasi-ordered fields by imposing axioms for the compatibility of $\preceq$ with the field addition and multiplication. His main result states that any quasi-ordered field $(K, \preceq)$ is either already an ordered field or there exists a valuation v on K such that $x \preceq y \Leftrightarrow v(y) \leq v(x)$ holds for all $x, y \in K$. Hence, quasi-ordered fields provide a uniform approach to the classes of ordered and valued fields.

At first we generalise quasi-orderings and the result by Fakhruddin that we just mentioned to possibly non-commutative rings with unity. We then make use of it by stating and proving mathematical theorems simultaneously for ordered and valued rings. Key results are:

(1) We develop a notion of compatibility between quasi-orderings and valuations. Given a quasi-ordered ring $(R, \preceq)$, this yields, among other things, a characterisation of all valuations v on R such that $\preceq$ canonically induces a quasi-ordering on the residue class domain Rv. Moreover, it leads to a uniform definition of the rank of $(R, \preceq)$.

(2) Given a valued ring (R, v), we characterise all v-compatible quasi-orderings on R in terms of the quasi-orderings on Rv by establishing a Baer-Krull theorem for quasi-ordered rings. From this, we also derive such a theorem for ordered rings, respectively valued rings.

(3) We equip the set of all quasi-orderings on a ring R with a coarsening relation $\leq$, which is defined as follows:

$$\preceq_1 \leq \preceq_2 :\Leftrightarrow \forall x, y \in R\colon (0 \preceq_1 x \preceq_1 y \Rightarrow x \preceq_2 y)$$

That way, we unify three different notions at once, namely coarsenings of valuations, inclusions of orderings, and compatibility of orderings and valuations. We prove that the set of all quasi-orderings on R with fixed support forms a rooted tree with respect to $\leq$, i.e. a partially ordered set with a unique maximal element such that for each such quasi-ordering all its coarsenings are linearly ordered.

As an application of this tree structure theorem we deduce that the set of all quasi-orderings on R, partially ordered by a slight modification of the coarsening relation $\leq$, is a spectral set, i.e. order-isomorphic to the spectrum of some commutative ring.

(4) We develop a notion of the quasi-real spectrum of a ring, in analogy to the real spectrum as known from real algebraic geometry. For that purpose, we generalise both the Harrison and the Tychonoff topology to the space of all quasi-orderings on a given ring. We establish the results that this space is a spectral space with respect to the Harrison topology and a Boolean space with respect to the Tychonoff topology.

Furthermore, we introduce partially quasi-ordered rings and show that they provide a uniform approach to strict partial valuations and division closed partial orderings. Last but not least, we give a model-theoretic application of quasi-orderings. We subsume the results that the theories of algebraically closed valued fields and real closed fields adjoined with a convex valuation admit quantifier elimination.

Deutsche Zusammenfassung (German Summary)

Gegenstand dieser Thesis ist das Studium quasiangeordneter Ringe. Gemeinhin versteht man unter einer Quasianordnung $\preceq$, auch Präordnung genannt, eine binäre, reflexive und transitive Relation auf einer Menge. In seinem Paper *Quasi-Ordered Fields* ([19]) führt S. M. Fakhruddin total quasiangeordnete Körper ein, indem er Axiome für die Verträglichkeit von $\preceq$ mit Addition und Multiplikation formuliert. Sein Hauptresultat besagt, dass jeder quasiangeordnete Körper $(K, \preceq)$ entweder schon ein angeordneter Körper ist, oder eine Bewertung v auf K existiert, so dass $x \preceq y \Leftrightarrow v(y) \leq v(x)$ für alle $x, y \in K$ gilt. Somit bieten quasiangeordnete Körper einen einheitlichen Zugang zu angeordneten und bewerteten Körpern.

In dieser Arbeit verallgemeinern wir Quasianordnungen und das eben beschriebene Resultat zunächst auf möglicherweise nicht-kommutative Ringe mit Eins. Dies machen wir uns zu Nutze, um mathematische Sätze simultan für angeordnete und bewertete Ringe zu beweisen. Zentrale Ergebnisse dieser Arbeit sind:

(1) Wir entwickeln einen Begriff von Kompatibilität zwischen Quasianordnungen und Bewertungen. Ist $(R, \preceq)$ ein quasiangeordneter Ring, so führt uns dies unter anderem zu einer Charakterisierung all derjenigen Bewertungen v auf R, für die $\preceq$ kanonisch eine Quasianordnung auf dem Restklassenring Rv induziert. Ferner leiten wir hieraus eine einheitliche Definition für den Rang eines quasiangeordneten Rings $(R, \preceq)$ ab.

(2) Für einen bewerteten Ring (R, v) charakterisieren wir die v-kompatiblen Quasianordnungen auf R durch die Quasianordnungen auf Rv mittels eines Baer-Krull Theorems. Hieraus leiten wir dann entsprechende Theoreme für angeordnete Ringe, beziehungsweise bewertete Ringe, ab.

(3) Wir versehen den Raum aller Quasianordnungen auf einem Ring R mit einer Vergröberungs-Relation $\leq$, welche wie folgt definiert ist:

$$\preceq_1 \leq \preceq_2 :\Leftrightarrow \forall x, y \in R\colon (0 \preceq_1 x \preceq_1 y \Rightarrow x \preceq_2 y)$$

Auf diese Weise vereinheitlichen wir drei unterschiedliche Sachverhalte auf einmal, nämlich Vergröberungen von Bewertungen, Inklusionen von Anordnungen sowie Kompatibilität zwischen Anordnungen und Bewertungen. Wir beweisen, dass die Menge aller Quasianordnungen auf R mit festem Support eine Baumstruktur bezüglich $\leq$ besitzt, d.h. eine partiell angeordnete Menge mit einem Maximum bildet, so dass zu jedem Element die Menge all seiner Vergröberungen linear angeordnet ist.

Als eine Anwendung dieses Baumstruktur-Theorems erhalten wir, dass die Menge aller Quasianordnungen auf R, versehen mit einer leichten Modifizierung der Vergröberungs-Relation $\leq$, eine spektrale Menge bildet, d.h. ordnungsisomorph zum Spektrum eines kommutativen Rings ist.

(4) Wir entwickeln den Begriff des quasi-reellen Spektrums eines Rings, in Analogie zu dem aus der reellen algebraischen Geometrie bekannten reellen Spektrum. Hierzu verallgemeinern wir die Harrison und die Tychonoff Topologie auf den Raum aller Quasianordnungen eines Rings. Wir beweisen, dass dies bezüglich der Harrison Topologie ein spektraler Raum und bezüglich der Tychonoff Topologie ein boolscher Raum ist.

Zudem führen wir partielle Quasianordnungen auf Ringen ein und zeigen, dass diese strikte partielle Bewertungen und divisionsabgeschlossene partielle Anordnungen vereinheitlichen. Schließlich fassen wir mittels Quasianordnungen die Resultate zusammen, dass die Theorien der algebraisch abgeschlossenen bewerteten Körper und der reell abgeschlossenen Körper mit konvexer Bewertung Quantorenelimination besitzen.

Contents

Introduction

Goals. The purpose of the present thesis is to provide a systematic and uniform study of the classes of ordered and valued rings. At this point we cannot put enough emphasis on the word uniformity, since we will treat orderings and valuations on rings with multiplicative identity simultaneously throughout the entire thesis - not only when formulating our results, but also when proving them. This is achieved by the notion of quasi-orderings, which provides an axiomatic subsumption of orderings and valuations.

Quasi-ordered fields were introduced in 1987 by Fakhruddin in his note [19]. Let K be a field and $\preceq$ a binary, reflexive, transitive and total relation on K. Write $x \sim_{\preceq} y$ if $x \preceq y \preceq x$, and otherwise $x \nsim_{\preceq} y$. Then $(K, \preceq)$ is called a quasi-ordered field if the following axioms are satisfied for all $x, y, z \in K$:

(Q1) If $x \sim_{\preceq} 0$, then $x = 0$.
(Q2) If $x \preceq y$ and $0 \preceq z$, then $xz \preceq yz$.
(Q3) If $x \preceq y$ and $z \nsim_{\preceq} y$, then $x + z \preceq y + z$.

Fakhruddin's main theorem states that any quasi-ordered field $(K, \preceq)$ is either an ordered field or else there is a valuation v on K such that $x \preceq y \Leftrightarrow v(y) \leq v(x)$ for all $x, y \in K$ ([19, Theorem 2.1]). Slightly adapting these axioms, we will show that this dichotomy generalises to the class of all possibly non-commutative rings with unity. While Fakhruddin left his investigation on quasi-ordered fields at the said result, we are building upon it and develop the theory of quasi-ordered rings without exploiting their dichotomous nature. In this manner, we are able to treat ordered rings and valued rings simultaneously and uniformly.

The main goal of our research on quasi-ordered rings is to unify the theories of ordered and valued rings as much as possible. To this end, we consider various results from both disciplines, real algebra and valuation theory, and try to formulate and establish them for quasi-ordered rings. That way, we demonstrate that numerous concepts and results in these theories are essentially very similar to one another and can be dealt with simultaneously.

A beautiful example of this is the notion of special valuations, which was introduced in [35] by Knebusch and D. Zhang. A valuation v on a ring R can be proven to be special if and only if for any $x \in R \backslash v^{-1}(\infty)$ there exists some $y \in R$ such that $v(xy) \leq 0$. Evidently, via the relationship $x \preceq y \Leftrightarrow v(y) \leq v(x)$, this characterisation may be literally generalised to quasi-ordered rings by demanding that for any $x \in R$, which is not on the same level as 0, there is some $y \in R$ such that $1 \preceq xy$. Replacing the quasi-ordering with an ordering, this yields precisely Krivine's notion of pseudo-fields ([36]), or equivalently, orderings that are maximal elements in the real spectrum of R. Therefore, special valuations and pseudo-fields turn out to be their respective counterparts in terms of quasi-orderings.

Likewise, if $\preceq_1$ and $\preceq_2$ are quasi-orderings on a ring R, the condition

$$\forall x, y \in R\colon 0 \preceq_1 x \preceq_1 y \Rightarrow x \preceq_2 y$$

even unifies three different notions at once: depending on whether $\preceq_1$ and $\preceq_2$ are orderings or valuations, this means either that the valuation $\preceq_2$ is coarser than the valuation $\preceq_1$, or that the ordering $\preceq_2$ contains the ordering $\preceq_1$, or that the valuation $\preceq_2$ is compatible with the ordering $\preceq_1$.

Throughout this thesis, we will encounter much more such similarities of ordered and valued rings, and develop a whole theory of quasi-ordered rings without making use of their dichotomy. An extensive description of the contents of this thesis is given in the reader's guide below.

Motivation and literary background. There are two main reasons why we do study quasi-ordered rings. First the great importance of both, orderings and valuations, in various mathematical areas, and second the close relationship between these two classes.

According to Roquette's survey on the history of valuation theory ([67]), valuations were first axiomatically defined on fields by Kürschák in 1912 ([42]) as what today is called an absolute value. For the time being, valuations were primarily utilised in algebraic number theory. Kürschák himself introduced them for studying the p-adic fields $\mathbb{Q}_p$ that Hensel had discovered before. Major results of that time are the classification of all valuations on $\mathbb{Q}$ by Ostrowski, the local-global principle by Hasse-Minkowski and Hensel's lemma. In 1932, Krull gave a crucial generalisation of Kürschák's notion of valuation by replacing the value group $\mathbb{R}$ with an arbitrary ordered abelian group ([37]). Thereafter, valuations have found many applications in various other mathematical areas such as algebraic geometry, real algebra, model theory and functional analysis.

In the sixties, Bourbaki ([9]) and Manis ([52]) were the first to introduce valuations on rings. Among other things, Manis established an approximation theorem for this class in the sense of the respective result by Artin and Whaples for absolute values on fields ([5]). Since then, there have been numerous publications on valued rings, for instance the works of Gräter on the approximation theorem ([22],[23]), the development of the valuation spectrum by Huber and Knebusch ([28], [29]), the study of ring valuations in real algebra by Marshall ([53]), Powers ([61]) and others, and the books of Knebusch and D. Zhang ([35]), respectively Knebusch and Kaiser ([32]), on Manis valuations and Prüfer extensions.

The theory of formally real fields and real closed fields was developed in the twenties by the groundbreaking works of Artin and Schreier ([3], [4]), Artin solving Hilbert's 17-th problem on the representation of non-negative polynomials in $\mathbb{R}[X]$ as sum of squares over $\mathbb{R}(X)$. Afterward, ordered fields were considered in model theory, resulting in the famous Tarski-Seidenberg theorem ([71], [69]). Orderings were extensively studied from the seventies onwards, leading to the foundation of modern real algebraic geometry. The latter studies polynomials and semialgebraic sets over real closed fields. Major results in that area, amongst others, are the Positivstellensätze of Schmüdgen ([68]) and Putinar ([66]). In the eighties, the area of o-minimality arose (cf. [15]). The last-mentioned subject deals with certain ordered structures in a model-theoretic way and can in some sense be seen as a generalisation of real algebraic geometry. Orderings also appear in further mathematical areas such as optimisation (e.g. semidefinite programming).

Shortly after and independently of the works of Artin and Schreier, Baer and Krull discovered that there is a strong relationship between orderings and valuations ([6], [37]). They showed that any ordering P on a field K canonically induces a valuation v_P on K, the so-called *natural valuation* associated to P, that describes the ordering in terms of its Archimedean equivalence classes. More precisely, the valuation ring of v_P consists of all elements of K which are not infinitely large with respect to P, and its valuation ideal of all infinitely small elements. From there on, the relationship between orderings and valuations, and the search for a common foundation for these two classes have been a reoccurring theme in the literature.

We believe that our research on quasi-ordered rings makes a valuable contribution to that matter and provides a better understanding of the relationship of orderings and valuations.

Let us give a (presumably incomplete) overview of what has been done before in that regard, where we focus on attempts to unify these objects. In 1966, Harrison studied the class of maximal preprimes of a commutative ring R, where a preprime is a non-empty subset S of R fulfilling the axioms $S + S \subseteq S, SS \subseteq S$ and $-1 \notin S$. Harrison proved that any maximal preprime of R canonically induces a Manis valuation on R and vice versa ([26, Proposition 2.5]). These maximal preprimes, also called Harrison primes, were further investigated in various publications, e.g. by Becker in 1979 ([7], cf. Theorem 5.3).

In 1974, van Maaren studied the class of pseudo-orders on fields, which properly contains the class of field orderings. Van Maaren showed that pseudo-orders which satisfy a certain condition are generated by valuations ([51, Theorem 2.13]).

A different approach was taken by Dress in 1977. He proved the trichotomy that any subset T of a field K satisfying the conditions

(1) $1 \in T$,
(2) $1 - T \subseteq T$,
(3) $a, b \in T \Rightarrow ab \in T$,
(4) $(ab)/(a + b - 1) \in T \Rightarrow a \in T$ or $b \in T$,
(5) $a, b \in T, ab \neq 0 \Rightarrow a/b \in T$ or $b/a \in T$,

is either a valuation ring in K, or a $[0, 1]$-interval with respect to some ordering of K (and therefore uniquely determines this ordering), or the set $\{0, 1\}$ ([14, Corollary to Theorem 2]). Eight years later, Fakhruddin introduced quasi-ordered fields and showed the dichotomy that any quasi-ordering is either an ordering or a valuation ([19, Theorem 2.1]; see also the first part of this introduction).

Unrelated to the work of Fakhruddin, D. Zhang axiomatically introduced quasi-ordered fields and rings in a different way in 2002 ([74]). For him, a quasi-ordering on a field K is a binary, reflexive, transitive and total relation $\leq$ on K such that for all $x, y, z \in K$ the axioms

(1) $x \leq y \Rightarrow xz \leq yz$,
(2) $x \geq 1, y \geq 1 \Rightarrow x + y \geq 1$,
(3) $1 < 0$,

are fulfilled. Zhang deduced that the map $v \mapsto \leq_v$ is a bijection between the set of all quasi-orderings on K and the set of all valuations on K ([74, Proposition 2.3]), where - in analogy to Fakhruddin's quasi-orderings (except for the non-reversing of the ordering) - $x \leq_v y \Leftrightarrow v(x) \leq v(y)$. The special feature of this notion is that if one exchanges $+$ with $\cdot$ and 0 with 1 in these axioms above, this yields a natural axiomatisation of non-reduced ordered fields, i.e. ordered fields except that anti-symmetry may possibly fail.

In 2007, Efrat took a very pragmatic approach by simply defining localities on a field to be either orderings or valuations ([16, Chapter 7]). Hence, localities and Fakhruddin's quasi-orderings refer to the very same objects, the only difference being that localities are not defined axiomatically. Via localities, Efrat was able to state and prove numerous results simultaneously for ordered and valued fields - as a matter of fact we will refer to quite a few of them within this thesis. However, due to the localities' nature, Efrat frequently needs to distinguish whether a given locality is an ordering or a valuation. In fact, our research on quasi-orderings only differs in two aspects from his research on localities, namely our demand for uniformity and that we are rather interested in rings than fields.

Very recently, in 2019, Anscombe, Dittmann and Fehm picked up this notion of Efrat and proved approximation theorems for spaces of localities ([2]).

In the past years, quasi-orderings in the sense of Fakhruddin were reconsidered. Around 2016, Lehéricy derived a notion of quasi-ordered groups from the respective notion for fields. Instead of a strict dichotomy, he obtained that quasi-orderings on abelian groups are a hybrid of orderings and valuations ([47]), behaving like an ordering on an initial segment and like a valuation elsewhere. In a further publication, S. Kuhlmann and Lehéricy managed to establish a Baer-Krull theorem for quasi-ordered groups ([39]). Next, in 2017, S. Kuhlmann, Matusinski and Point derived a notion of rank for quasi-ordered fields ([40]). Also in 2017, the author of this thesis introduced quasi-ordered rings ([60]). They were studied in more detail in further publications by S. Kuhlmann and him ([41], [59]). Their respective results on quasi-ordered rings are all included in this thesis.

Readers' guide. Chapter 1 is devoted to the introduction of quasi-ordered rings and to establishing the fundamental result of this thesis, namely that any quasi-ordered ring $(R, \preceq)$ is either an ordered ring or else a valued ring (R, v) such that $x \preceq y \Leftrightarrow v(y) \leq v(x)$ holds for all $x, y \in R$. We give two different proofs. If the ring under consideration is commutative, we derive it directly from the respective result of Fakhruddin ([19, Theorem 2.1]) by exploiting that any commutative quasi-ordered ring is naturally associated to a quasi-ordered field. For the general case we adapt and slightly improve Fakhruddin's proof. Furthermore, we introduce special and Manis quasi-orderings and show that they yield a uniform approach to special valuations and pseudo-fields, respectively Manis valuations and orderings on skew-fields.

In the second chapter we study the relationship of quasi-orderings and valuations by introducing different compatibility notions. Given a quasi-ordered ring $(R, \preceq)$, we call a valuation v on R for instance strongly $\preceq$-compatible, if

$$\forall x, y \in R \colon 0 \preceq x \preceq y \Rightarrow v(y) \leq v(x).$$

This coincides with the usual notion of compatibility of orderings and valuations if $\preceq$ is an ordering, while it means that v is coarser than w if $\preceq$ is induced by a valuation w on R. The main results of this chapter describe all the valuations on R that are (strongly) $\preceq$-compatible. Among other things, this yields a characterisation of when a quasi-ordering $\preceq$ on a ring R canonically induces a quasi-ordering on the residue class domain Rv induced by a given valuation v on R. Moreover, strong compatibility enables us to introduce the natural valuation of a quasi-ordering, eventually leading to a uniform definition of the rank of a quasi-ordered ring.

Chapter 3 seamlessly continues the investigation of strong $\preceq$-compatibility between quasi-orderings and valuations. Given a valued ring (R, v), we characterise the quasi-orderings on R for which v is strongly $\preceq$-compatible in terms of the quasi-orderings on the residue class domain Rv induced by v. To this end, we establish a so-called *Baer-Krull theorem* for quasi-ordered rings. Moreover, we give various applications of this result and also derive Baer-Krull theorems for ordered rings, respectively valued rings.

In the following two chapters we move from studying quasi-ordered rings $(R, \preceq)$ to studying the set $\mathcal{Q}(R)$ of all quasi-orderings on R. In the fourth chapter, we equip this set with a coarsening relation $\leq$ defined as follows:

$$\preceq_1 \leq \preceq_2 :\Leftrightarrow \forall x, y \in R \colon 0 \preceq_1 x \preceq_1 y \Rightarrow x \preceq_2 y.$$

Note that $\leq$ arises from strong compatibility by replacing the valuation v with a quasi-ordering. The main result of this chapter states that the set of all quasi-orderings on R with fixed support forms a rooted tree with respect to the coarsening relation $\leq$, i.e. a partially ordered set admitting a unique maximal element such that for any member the set of all its coarsenings is linearly ordered. This improves the original result by Efrat ([16, Corollary 7.3.6]) on two levels - first by generalising it from fields to arbitrary rings with unity, and second by providing a uniform definition of the coarsening relation on the one hand, and a uniform proof of the statement itself on the other hand.

As the main application of this tree structure theorem we obtain that $(\mathcal{Q}(R), \leq')$ is a spectral set, where $\leq'$ is a slight modification of $\leq$. In other words, there is a commutative ring S such that $(\mathcal{Q}(R), \leq')$ is order-isomorphic to the set of all prime ideals of S, the so-called spectrum of S, partially ordered by inclusion. We prove that the same result also applies to the set of all special quasi-orderings, respectively Manis quasi-orderings, of a ring. As a further consequence of the tree structure theorem we obtain that dependency is an equivalence relation on the set of all quasi-orderings with fixed support, where two quasi-orderings are called dependent if they admit a non-trivial common coarsening.

In Chapter 5 we consider $\mathcal{Q}(R)$ as a topological space. We equip it with (the analogue of) the Harrison topology as known from the real spectrum of a ring, i.e. with the topology generated by all sets of the form

$$U(a,b) := \{\preceq \in \mathcal{Q}(R) \colon a \prec b\} \quad (a, b \in R)$$

In analogy to the real spectrum, we obtain that $\mathcal{Q}(R)$ is a spectral space with respect to the Harrison topology, and that $\mathcal{Q}(R)$ is a Boolean space with respect to the the corresponding constructible topology, i.e. the topology generated by the sets $U(a,b)$ and

$$V(a,b) := \{\preceq \in \mathcal{Q}(R) \colon a \preceq b\} \quad (a, b \in R)$$

as an open subbasis.

So far we have only dealt with totally quasi-ordered rings. The subject of Chapter 6 is the development of partial quasi-orderings. We axiomatically introduce partially quasi-ordered rings and establish the dichotomy, that any partial quasi-ordering is either a division closed partial ordering (cf. [20, p. 165]) or a strict partial valuation ([73, Definition 1.1.1]). In a further step, we show that any partial quasi-ordering $\preceq$ on a field extends to a quasi-ordering, if all squares are non-negative with respect to $\preceq$.

Chapter 7 is about the model theory of quasi-ordered fields. We uniformly define quasi-real closed fields and the quasi-real closure of a quasi-ordered field $(K, \preceq)$. The former subsumes the classes of real-closed fields and algebraically closed fields, while the latter coincides with the real closure of $(K, \preceq)$ (if $\preceq$ is an ordering), respectively the algebraic closure of K (if $\preceq$ is a valuation). Exploiting Fakhruddin's dichotomy, we then deduce that the theory of quasi-real closed fields adjoined with a non-trivial compatible valuation admits quantifier elimination. Moreover, we obtain that this theory is model complete and the model companion of the theory of quasi-ordered fields adjoined with a non-trivial compatible valuation.

To make this thesis easier to read, most chapters begin with a preliminary section, where relevant basic definitions and results are recalled. Moreover, a symbol index is attached at the very end of this thesis.

1. The Notion of Quasi-Ordered Rings

In the present chapter we introduce quasi-ordered rings and prove that they provide an axiomatic unification of the classes of ordered and valued rings. This establishes the basis for all our further results within this thesis.

Section 1.1 serves the purpose of defining quasi-ordered rings. Moreover, we verify that any ordered ring and any valued ring is a quasi-ordered ring. The Sections 1.2 and 1.3 are devoted to proving the converse, namely that any quasi-ordered ring is either an ordered ring or a valued ring. In the former section it is shown that for commutative rings this result may be derived from the respective result for fields ([19, Theorem 2.1]), while in the latter section we give a proof for arbitrary rings.

We continue by introducing special (Section 1.4) and Manis quasi-orderings (Section 1.5). Special quasi-orderings yield a uniform approach to special valuations on the one hand, and orderings that are maximal elements in the real spectrum of the given ring on the other hand. Manis quasi-orderings provide a unification of Manis valuations and orderings with maximal support, if the ring is commutative. In general, they subsume Manis valuations with support $\{0\}$ and skew-field orderings.

Preliminaries on ordered rings, valued rings and quasi-ordered fields.

Here, we briefly recall the notions of ordered and valued rings (including weak, special and Manis valuations), and give a few examples of these classes. Moreover, we summarise Fakhruddin's work on quasi-ordered fields, which yields an axiomatic subsumption of ordered and valued fields.

Convention Throughout the entire thesis, the term *ring* shall always refer to a possibly non-commutative ring with multiplicative identity $1 \neq 0$. Furthermore, the term *domain* shall always include non-commutative rings without zero-divisors.

Definition 1.1. Let R be a ring and $\leq$ a binary, reflexive, transitive and total relation on R. Then $(R, \leq)$ is called an *ordered ring* if for all $a, b, x, y, z \in R$ the following holds:

(OR1) $0 < 1$

(OR2) If $x \leq y$ and $0 \leq a, b$, then $axb \leq ayb$.

(OR3) If $xy \leq 0$, then $x \leq 0$ or $y \leq 0$.

(OR4) If $x \leq y$, then $x + z \leq y + z$.

Remark 1.2.

(1) The corresponding positive cone $P := P_{\leq} := \{x \in R \colon 0 \leq x\}$ satisfies the axioms $P + P \subseteq P,\ PP \subseteq P,\ P \cup -P = R$, and

$$\mathfrak{q}_P := P \cap -P = \{x \in R \colon 0 \leq x \leq 0\}$$

is a prime ideal of R, the so-called *support* of P. In fact, $\mathfrak{q}_P$ is a two-sided completely prime ideal ([48, Theorem 1.2]), i.e. a proper two-sided ideal such that $xy \in \mathfrak{q}_P$ implies $x \in \mathfrak{q}_P$ or $y \in \mathfrak{q}_P$ for all $x, y \in R$. Therefore, $R/\mathfrak{q}_P$ is a domain.

(2) The orderings on R with support $\mathfrak{q}$ are in natural 1:1 correspondence with the orderings on $R/\mathfrak{q}$ with support $\{0\}$. If R is commutative, then the orderings on $R/\mathfrak{q}$ with support $\{0\}$ are in natural 1:1 correspondence with the orderings on $K := \mathrm{Quot}(R/\mathfrak{q})$ (cf. [54, Proposition 5.1.1], [55, p. 3764]).

(3) If R is a division ring, then the axioms (OR1) and (OR3) in Definition 1.1 may be replaced with the condition that $\leq$ is antisymmetric. For the corresponding positive cone $P_{\leq}$ this means $\mathfrak{q}_P = \{0\}$.

Example 1.3. The ordering in (4) will turn out to be a Manis ordering, whereas the ordering in (6) is special and not Manis (see Section 1.4 and 1.5).

(1) If $(D, \leq)$ is an ordered skew-field and $R \subseteq D$ a subring, then $(R, \leq)$ is an ordered domain with support $\{0\}$.

(2) Let K be a field and G an ordered abelian additive group. Equipped with obvious operations $+$ and $\cdot$, the set of all formal series

$$K((G)) := \left\{ \sum_{g \in G} s(g)t^g \mid s\colon G \to K \text{ such that } \operatorname{supp}(s) \text{ is well-ordered} \right\}$$

is a field ([25]), where $\operatorname{supp}(s) := \{g \in G\colon s(g) \neq 0\}$. It is called a *field of generalised power series*. If $\leq$ is an ordering on K, then

$$s > 0 :\Leftrightarrow s(\min(\operatorname{supp}(s))) > 0$$

defines an ordering on $K((G))$. By (1), this gives rise to an ordering on any subring of $K((G))$.

(3) Consider the polynomial ring $\mathbb{Q}[X]$, and let $r \in \mathbb{R}$. Then $0 \leq f :\Leftrightarrow 0 \leq f(r)$ w.r.t. the ordering on $\mathbb{R}$ defines an ordering on $\mathbb{Q}[X]$.
   (a) If r is algebraic, this is an Archimedean ordering with support (f), where $f = \operatorname{MinPol}(r|\mathbb{Q})$.
   (b) If r is transcendental, this is an Archimedean ordering with support $\{0\}$. The former is because $\mathbb{Q}$ is dense in $\mathbb{R}$.

(4) Consider the commutative ring $R = C([0,1], \mathbb{R})$ of all continuous maps $f\colon [0,1] \to \mathbb{R}$ with pointwise addition and multiplication, and let $x \in [0,1]$. Then $f \leq g :\Leftrightarrow f(x) \leq g(x)$ defines an ordering on R.

(5) (cf. [44, Ex. 17.2]) Any free algebra $K\langle X_i\colon i \in I\rangle$ over a formally real field K can be ordered.

(6) (cf. [44, Ex. 17.3]) Consider the Weyl algebra

$$R := \mathbb{R}\langle X, Y\rangle/(XY - YX - 1).$$

We write x and y for the residue classes of X and Y, respectively. Since $yx = xy - 1$, any $r \in R$ can be written in the canonical form

$$r = r_0(x) + r_1(x)y + \ldots + r_n(x)y^n$$

for some $n \in \mathbb{N}_0$, where $r_i \in \mathbb{R}[X]$ for all $0 \leq i \leq n$. The chain

$$\mathbb{R} < x < x^2 < \ldots < y < xy < x^2y < \ldots < y^2 < xy^2 < x^2y^2 < \ldots$$

defines a non-Archimedean ordering on R.

Definition 1.4. ([9], [56]) Let R be a ring, $(\Gamma, +, \leq)$ a totally ordered monoid and ∞ a symbol such that $\gamma < \infty$ and $\infty = \infty + \infty = \gamma + \infty = \infty + \gamma$ for all $\gamma \in \Gamma$. A surjective map $v\colon R \to \Gamma \cup \{\infty\}$ is called a *weak valuation* on R if for all $x, y \in R$ the following holds:

(VR1) $v(0) = \infty$,

(VR2) $v(1) = 0$,

(VR3) $v(xy) = v(x) + v(y)$,

(VR4) $v(x + y) \geq \min\{v(x), v(y)\}$.

$\Gamma_v := \Gamma$ is called the *value monoid*, and $\mathfrak{q}_v := v^{-1}(\infty) = \{x \in R \colon v(x) = \infty\}$ the *support* of v. A weak valuation v is called *trivial*, if Γ_v is the trivial monoid $\{0\}$.

Definition 1.5. A weak valuation v on a ring R is said to be a

(1) a *valuation*, if $(\Gamma_v, +, \leq)$ is cancellative, i.e. if $a + x + b \leq a + y + b$ implies $x \leq y$ for all $a, b, x, y \in \Gamma_v$.

(2) a *Manis valuation*, if Γ_v is even a group (cf. [52]).

Remark 1.6.

(1) Valuations on commutative rings are usually defined by demanding that v maps into the group generated by $v(R \backslash \mathfrak{q}_v)$. In this case, a valuation is called Manis if it is surjective. Note that these two definitions of valuations are equivalent. Indeed, if R is commutative and if v is a valuation on R in the sense of Definition 1.5, then Γ_v embeds in an ordered group ([31, Corollary 1]). Conversely, $v(R \backslash \mathfrak{q}_v)$ is a monoid for any map v on R that satisfies the above axioms (VR1) - (VR4).

However, as was proven by Mal'cev, Chihata and Vinogradov, there are ordered cancellative monoids which cannot be embedded in any ordered group (cf. [45, Theorem 9.8]). Consequently, whenever we consider possibly non-commutative valued rings, we stick to Definition 1.5. For valuations on commutative rings, we occasionally suppose that v maps into the group Γ_v for the sake of convenience.

(2) If R is a division ring, then the axioms (VR1) and (VR2) may be subsumed by the condition $v(x) = \infty$ if and only if $x = 0$. In this case the notions of weak valuation, valuation and Manis valuation all coincide.

Later on, we also work with special valuations, respectively special quasi-orderings.

Definition 1.7. ([35, Chapter I]) Let R be a commutative ring and v a valuation on R with value group Γ (see Remark 1.6(1)).

(1) The *characteristic subgroup* $c_v(\Gamma)$ of Γ is the smallest convex subgroup of Γ containing all elements $v(x)$ such that $x \in R$ and $v(x) \leq 0$, i.e.

$$c_v(\Gamma) = \{\gamma \in \Gamma \colon v(x) \leq \gamma \leq -v(x) \text{ for some } x \in R \text{ with } v(x) \leq 0\}.$$

(2) v is called *special*, if $c_v(\Gamma) = \Gamma$.

Lemma 1.8. (cf. [35, Proposition I.2.2]) *Let R be a commutative ring and let v be a valuation on R. Then*

(1) $\mathfrak{q}_v \subseteq \{r \in R \colon v(xr) > 0$ *for all* $x \in R\}$

(2) The following are equivalent:

(i) v is special,

(ii) $\mathfrak{q}_v = \{r \in R \colon v(xr) > 0$ *for all* $x \in R\}$.

Proof. The proof of (1) is trivial. For (2) first suppose that v is special, and let $r \in R$ such that $v(xr) = v(x) + v(r) > 0$ for all $x \in R$. This implies $v(r) > -v(x)$ for all $x \in R$. Thus, $v(r) \notin c_v(\Gamma)$. Since v is special, it follows $r \in \mathfrak{q}_v$.

For the converse, suppose that (ii) holds and assume that v is not special. Then we find some $\gamma \in \Gamma \backslash c_v(\Gamma)$, w.l.o.g. $\gamma \in v^{-1}(R \backslash q_v)$. Let $r \in R$ such that $v(r) = \gamma$. Since $\gamma \notin c_v(\Gamma)$, we know that $v(r) > 0$. We claim that $v(xr) > 0$ for all $x \in R$. By (ii), this yields $r \in \mathfrak{q}_v$, the desired contradiction. So let $x \in R$, w.l.o.g. $v(x) < 0 < v(r)$. From $v(r) \notin c_v(\Gamma)$, it follows $-v(x) < v(r)$. Hence, $0 < v(xr)$. □

The previous lemma yields a generalisation of special valuations to possibly non-commutative rings. We further show that we do not have to distinguish between left and right special valuations.

Lemma 1.9. *Let R be a ring, v a valuation on R and $x, y \in R$. Then $v(xy) \leq 0$ if and only if $v(yx) \leq 0$.*

Proof. Let w.l.o.g. $x, y \notin \mathfrak{q}_v$. By cancellation in Γ_v, we obtain

$$v(x) + v(y) \leq 0 \Leftrightarrow v(y) + v(x) + v(y) \leq v(y) \Leftrightarrow v(y) + v(x) \leq 0.$$

□

Definition 1.10. Let R be a ring and let v be a valuation on R. We call v *special*, if $\mathfrak{q}_v = \{r \in R \colon v(xr) > 0 \text{ for all } x \in R\}$.

Remark 1.11. Let R be a ring.

(1) By definition and Lemma 1.9, a valuation v on R is
   (a) special, if and only if for any $x \in R \backslash \mathfrak{q}_v$ we find some $y \in R$ such that $v(xy) \leq 0$, or equivalently, $v(yx) \leq 0$.
   (b) Manis, if and only if for any $x \in R \backslash \mathfrak{q}_v$ we find some $y \in R$ such that $v(xy) = 0$, or equivalently, $v(yx) = 0$.

(2) Evidently, any Manis valuation is a special valuation, any special valuation a valuation, and any valuation a weak valuation.

In Lemma 1.12 and Remark 1.13 we observe first similarities between ordered rings and valued rings.

Lemma 1.12. *Let R be a ring and v a weak valuation on R. Then $\mathfrak{q}_v$ is a two-sided completely prime ideal of R.*

Proof. By (VR1) and (VR2) we obtain $0 \in \mathfrak{q}_v$ and $1 \notin \mathfrak{q}_v$. Now let $x, y \in \mathfrak{q}_v$. Then $v(x + y) \geq \min\{v(x), v(y)\} = \infty$ by (VR4), i.e. $v(x + y) = \infty$. Thus, $x + y \in \mathfrak{q}_v$. Next, let $x \in \mathfrak{q}_v$ and $r \in R$. Then $v(xr) = \infty + v(r) = \infty$ by (VR3), so $xr \in \mathfrak{q}_v$. Analogously, it follows $rx \in \mathfrak{q}_v$. Hence, $\mathfrak{q}_v$ is a proper two-sided ideal. Finally, suppose that $xy \in \mathfrak{q}_v$ and $y \notin \mathfrak{q}_v$. Then $\infty = v(xy) = v(x) + v(y)$ and $v(y) < \infty$. Thus, $v(x) = \infty$, so $x \in \mathfrak{q}_v$. This proves that $\mathfrak{q}_v$ is also completely prime. □

Remark 1.13. By the previous lemma, $R/\mathfrak{q}_v$ is a domain. The valuations on R with support $\mathfrak{q}$ are in natural 1:1 correspondence with the valuations on $R/\mathfrak{q}$ with support $\{0\}$. If R is commutative, then the valuations on $R/\mathfrak{q}$ with support $\{0\}$ are in natural 1:1 correspondence with the valuations on $K := \mathrm{Quot}(R/\mathfrak{q})$ (cf. [61, p. 256]).

Lemma 1.14. *Let R be a ring, v a weak valuation on R, and $x, y \in R$ such that $v(x) \neq v(y)$. Then $v(x + y) = \min\{v(x), v(y)\}$.*

Proof. As in the case of valued fields, see for instance [18, p. 20, (1.3.4)]. □

Example 1.15.

(1) If (D, v) is a valued skew-field and $R \subseteq D$ a subring, then (R, v) is a valued domain with support $\{0\}$.

(2) Consider the field $K((G))$ of generalised power series (see Example 1.3(2)) for some field K and some ordered abelian additive group G. Then the assignment $v_{\min}(f) = \min(\mathrm{supp}(s))$ for $s \neq 0$ defines a valuation on $K((G))$. By (1), $v_{\min}$ yields a valuation on any subring of $K((G))$.

(3) Let R be a ring and $\mathfrak{q}$ a two-sided completely prime ideal of R. Then the *trivial valuation*

$$v_{\mathfrak{q}} \colon R \to \{0, \infty\},\ x \mapsto \left\{ \begin{array}{ll} 0, & x \notin \mathfrak{q} \\ \infty, & x \in \mathfrak{q} \end{array} \right.$$

with support $\mathfrak{q}$ is a Manis valuation.

(4) Let R be a domain. Then the map

$$v\colon R[X,Y] \to \mathbb{Z} \cup \{\infty\},\ 0 \neq \sum_{i,j} a_{ij} X^i Y^j \mapsto \min\{i - j\colon a_{ij} \neq 0\}$$

is a Manis valuation (see Proposition 1.16 below).

(5) Consider some free algebra $R := S\langle X_i\colon i \in I\rangle$ over a domain S. Then the assignment $f \mapsto -\deg(f)$ defines a special valuation on R, which is not Manis. Here, $\deg(f)$ refers to the total degree of f.

(6) $(\mathbb{R}[[X]], v_{\min})$ (see Example (2) above) is a valued ring such that $v_{\min}$ is not special.

(7) Let R be a domain and $-\infty$ a symbol such that $-\infty + n := -\infty =: n + (-\infty)$ for all $n \in \mathbb{N}_0 \cup \{-\infty\}$, and $-\infty + \infty := \infty =: \infty + (-\infty)$, and $-\infty < \mathbb{N}_0$. Set $\Gamma := \mathbb{N}_0 \cup \{-\infty\}$. Then the map

$$v\colon R[X,Y] \to \Gamma \cup \{\infty\},\ 0 \neq \sum_{i,j} a_{ij} X^i Y^j \mapsto \min\{i + j(-\infty)\colon a_{ij} \neq 0\},$$

is a weak valuation. It is not a valuation, because $v(Y) \leq v(Y) + v(Y)$, but $v(0) > v(Y)$, whence the value monoid Γ is not cancellative.

The following result gives rise to a whole class of valued rings and is later used to produce (counter-)examples. Inductively, it generalises to arbitrary many variables.

Proposition 1.16. *Let R be a ring, $u\colon R \to \Gamma_u \cup \{\infty\}$ a valuation, $\Gamma_u \subseteq \Gamma$ an ordered cancellative monoid, and $\gamma \in \Gamma \cup \{\infty\}$. For $f = \sum_{i=0}^n a_i X^i \in R[X]$ define*

$$v(f) = \begin{cases} \infty & \text{if } f = 0, \\ \min\limits_{0 \leq i \leq n} \{u(a_i) + i\gamma\} & \text{otherwise.} \end{cases}$$

Then $v\colon R[X] \to \Gamma_v \cup \{\infty\}$ is a valuation that extends u.

Proof. Completely analogue to the field case, see e.g. [18, Theorem 2.2.1]. □

We conclude our discussion on valued rings by giving a binary description of (weak) valuations, which is due to D. Zhang.

Definition 1.17. Let R be a ring. Two weak valuations v, w on R are said to be *equivalent*, if

$$\forall x, y \in R\colon v(x) \leq v(y) \Leftrightarrow w(x) \leq w(y).$$

Following the convention from the literature, we identify equivalent weak valuations.

Remark 1.18. [74, Definition 2.3, Theorem 2.1] Let R be a commutative ring. D. Zhang has shown that (the equivalence class of) a weak valuation v on R is a binary, reflexive, transitive and total relation $\preceq$ on R such that for all $x, y, z \in R$:

(1) $0 \prec 1$

(2) If $x \preceq y$, then $xz \preceq yz$.

(3) If $x \preceq y$, then $x + y \preceq y$.

More precisely, the map $v \mapsto \preceq_v$ is a natural bijective correspondence between the set of all weak valuations on R and the set of all such relations on R, where $x \preceq_v y :\Leftrightarrow v(y) \leq v(x)$ for all $x, y \in R$.

Adjoining the axiom

(4) If $xz \preceq yz$ and $0 \prec z$, then $x \preceq y$,

we obtain that $v \mapsto \preceq_v$ is a natural bijection between the set of all valuation on R and the set of all relations on R satisfying these conditions. By Remark 1.11, we may further generalise Zhang's result to special valuations and Manis valuations.

Finally, we recall quasi-ordered fields.

Definition 1.19. Let S be a set. A (total) *quasi-ordering* on S is a binary, reflexive, transitive and total relation $\preceq$ on S. If $\preceq$ is a quasi-ordering on S, we call $(S, \preceq)$ *a quasi-ordered set.*

Lemma 1.20. *Let $(S, \preceq)$ be a quasi-ordered set. Then*

$$x \sim_{\preceq} y :\Leftrightarrow x \preceq y \text{ and } y \preceq x$$

defines an equivalence relation on S.

Proof. Symmetry is clear. Reflexivity and transitivity follow from the respective properties of $\preceq$. $\square$

Notation 1.21. If $(S, \preceq)$ is a quasi-ordered set, then $\sim := \sim_{\preceq}$ shall always denote the equivalence relation from Lemma 1.20. For $x \in S$ we denote by

$$E(x) := E_{\preceq}(x) := \{y \in S \colon y \sim x\}$$

the equivalence class of x w.r.t. $\sim$. Furthermore, we define

$$x \prec y :\Leftrightarrow x \preceq y \text{ and } y \not\preceq x,$$

$$x \not\sim y :\Leftrightarrow x \prec y \text{ or } y \prec x.$$

Definition 1.22. ([19]) Let K be a field and $\preceq$ a quasi-ordering on K. Then $(K, \preceq)$ is called a *quasi-ordered field*, if for all $x, y, z \in K$ the following holds:

(Q1) If $x \sim 0$, then $x = 0$.

(Q2) If $x \preceq y$ and $0 \preceq z$, then $xz \preceq yz$.

(Q3) If $x \preceq y$ and $z \not\sim y$, then $x + z \preceq y + z$.

The purpose of quasi-ordered fields is to subsume the classes of ordered fields and valued fields, as displayed in the following theorem:

Theorem 1.23. ([19, Theorem 2.1]) *Any quasi-ordered field $(K, \preceq)$ is either an ordered field or there is a valuation v on K such that $x \preceq y \Leftrightarrow v(y) \leq v(x)$ for all $x, y \in K$.*

Proof sketch: Fakhruddin first shows that $\sim$ is preserved under multiplication. From this, he deduces that $E(x) \neq \{x\}$ if and only if $E(1) \neq \{1\}$ for any $x \in K^{\times}$ ([19, Proposition 3.2]).

The quasi-orderings with $E(1) = \{1\}$ correspond to orderings, unless $\operatorname{char}(K) = 2$, and the proof of this is rather straightforward ([19, Theorem 3.3]). In the case where $E(1) \neq \{1\}$, the valuation v inducing $\preceq$ is constructed as follows: the value group is given by $\Gamma := K^{\times}/E(1)$ and the map v assigns $0 \neq x$ to its equivalence class $E(x)$. It remains to verify that this indeed yields a valuation on K that induces $\preceq$. The details are worked out in the more general setting of rings in Section 1.3. There we take a simpler approach by distinguishing whether $-1 \prec 0$ or $0 \prec -1$.

1.1. **Introduction of Quasi-Ordered Rings.**

We begin this section by giving the central definition of the entire thesis, namely that of a quasi-ordered ring $(R, \preceq)$ (Definition 1.24). In analogy to orderings and valuations, we obtain that the support $\mathfrak{q}_{\preceq} := \{x \in R \colon x \sim 0\}$ of a quasi-ordering $\preceq$ on a ring R is a two-sided completely prime ideal of R (Proposition 1.30). This observation will play a crucial role in Section 1.2. Next, we show that our notion of quasi-ordered rings is consistent with Fakhruddin's notion of quasi-ordered fields (Proposition 1.31). At the end of this section we prove that any ordered ring and any valued ring is a quasi-ordered ring (Proposition 1.32 and Proposition 1.33).

Definition 1.24. Let R be a ring and $\preceq$ a quasi-ordering on R (see Definition 1.19 and also Notation 1.21). Then $(R, \preceq)$ is called a *quasi-ordered ring* if for all $a, b, x, y, z \in R$ the following holds:

(QR1) $0 \prec 1$

(QR2) If $x \preceq y$ and $0 \preceq a, b$, then $axb \preceq ayb$.

(QR3) If $axb \preceq ayb$ and $0 \prec a, b$, then $x \preceq y$.

(QR4) If $x \preceq y$ and $z \not\sim y$, then $x + z \preceq y + z$.

The set $\mathfrak{q}_{\preceq} := \{x \in R \colon x \sim 0\}$, is called the *support* of $\preceq$.

In a first naive attempt of finding the right notion of quasi-ordered rings, we simply took the axioms of an ordered ring (see Definition 1.1) and adjusted the axiom of addition in the sense of Fakhruddin, i.e. by additionally demanding that $z \not\sim y$. In fact, this leads to a slightly different dichotomy involving Marshall's and Y. Zhang's notion of weak valuations ([56, p. 193] or Definition 1.4 above), as we will see in Section 1.3. Therefore, we also define:

Definition 1.25. A *weakly quasi-ordered ring* is a quasi-ordered ring except that the axiom (QR3) is replaced with the axiom

(QR3') If $xy \preceq 0$, then $x \preceq 0$ or $y \preceq 0$.

Weak quasi-orderings will play a minor role in this thesis. They only appear in the present section and in Section 1.3.

Lemma 1.26. *Any quasi-ordered ring is a weakly quasi-ordered ring.*

Proof. Let $xy \preceq 0$. If $x \not\preceq 0$, then the totality of $\preceq$ implies $0 \prec x$. So we may apply (QR3) and obtain $y \preceq 0$. Thus, (QR3') is fulfilled. □

The converse is not true. Consider for instance $R = \mathbb{R}[X, Y]$. We define a relation $\preceq'$ on R as follows: first declare $0 \prec' r$ for all $r \in \mathbb{R}\backslash\{0\}$. For monomials $f = rX^iY^j$ and $g = sX^mY^n$ in $R\backslash\{0\}$ define

$$f \preceq' g :\Leftrightarrow \begin{cases} \text{either } 0 < n, \\ \text{or } j = n = 0 \text{ and } i \leq m. \end{cases}$$

So the relation $\preceq'$ can be described by the chain

$$0 \prec' 1 \prec' X \prec' X^2 \prec' X^3 \prec' \ldots \prec' Y \sim XY \sim X^2Y \sim \ldots \sim Y^2 \ldots$$

and the rule $rf \sim f$ for all monomials f and all $r \in \mathbb{R}\backslash\{0\}$. Extend $\preceq'$ to the whole of R by declaring for $0 \neq f, g \in R$:

$f \preceq g :\Leftrightarrow$ the $\preceq'$ -greatest monomial of f is smaller than or equivalent to the one of g.

Proposition 1.27. $(R, \preceq)$ *as just defined is a weakly quasi-ordered ring, but not a quasi-ordered ring.*

Proof. Clearly, $\preceq$ is reflexive, transitive, total and satisfies (QR1). Moreover, it is evident that $\preceq'$ is preserved under multiplication, whence $\preceq$ fulfils (QR2). For the proof of (QR3') note that R is an integral domain and that $0 \prec x$ for all $x \neq 0$. It follows that $fg \preceq 0$ implies $f = 0$ or $g = 0$. Hence, $f \preceq 0$ or $g \preceq 0$. Last but not least, we verify axiom (QR4). Suppose that $f \preceq g$ and $h \not\sim g$. The condition $h \not\sim g$ particularly ensures, that the greatest $\preceq'$-monomial of h is not the additive inverse of the greatest $\preceq'$-monomial of g. From this it is easy to see that $f + h \preceq g + h$. Hence, $(R, \preceq)$ is a weakly quasi-ordered ring.

However, since $X^2Y \preceq XY$ and $0 \prec Y$, (QR3) implies the contradiction $X \prec X^2$. Therefore, $(R, \preceq)$ is not a quasi-ordered ring. □

Next, we show that the support $\mathfrak{q}_{\preceq}$ of a weak quasi-ordering $\preceq$ on a ring R is a two-sided completely prime ideal of R (see Proposition 1.30). In a retrospect, this will be a first little example of a uniform treatment of ordered and valued rings (see Remark 1.2(1), respectively Lemma 1.12).

Lemma 1.28. *Let $(R, \preceq)$ be a weakly quasi-ordered ring and let $x, y \in R$ such that $x \nsim 0$ and $y \sim 0$. Then $x + y \sim x$. In particular, $y \sim 0$ if and only if $-y \sim 0$.*

Proof. Since $x \nsim 0$ and $y \sim 0$, we obtain $x \nsim y$. So from $y \preceq 0$ we deduce that $x + y \preceq x$, and from $0 \preceq y$ we deduce that $x \preceq x + y$. Thus, $x + y \sim x$.

If $y \sim 0$, then $-y \nsim 0$ implies $0 = (-y) + y \sim -y \nsim 0$, a contradiction. Therefore, $-y \sim 0$. The converse is obviously also true for symmetry reasons. $\square$

Lemma 1.29. *Let $(R, \preceq)$ be a weakly quasi-ordered ring and let $x \in R$ such that $x \prec 0$. Then $0 \prec -x$.*

Proof. By Lemma 1.28 we get $x, -x \nsim 0$. Hence, (QR4) implies $0 \prec -x$. $\square$

Proposition 1.30. *Let $(R, \preceq)$ be a weakly quasi-ordered ring. The support $\mathfrak{q}_{\preceq}$ is a two-sided completely prime ideal of R.*

Proof. Clearly, $0 \in \mathfrak{q}_{\preceq}$. Now let $x, y \in \mathfrak{q}_{\preceq}$, and assume that $x + y \notin \mathfrak{q}_{\preceq}$. So we have $x + y \nsim 0$, but $-y \sim 0$ by Lemma 1.28. The same lemma yields the contradiction

$$0 \sim x = (x + y) - y \sim x + y \nsim 0.$$

Next, suppose that $x \in \mathfrak{q}_{\preceq}$ and $r \in R$. If $0 \preceq r$, then evidently $xr \in \mathfrak{q}_{\preceq}$ by (QR2). If $r \prec 0$, then Lemma 1.29 and Lemma 1.28 imply $0 \prec -r$ and $-x \sim 0$. Therefore, $xr = (-x)(-r) \sim 0$, so again $xr \in \mathfrak{q}_{\preceq}$. Analogously, it follows that $rx \in \mathfrak{q}_{\preceq}$. Thus, $\mathfrak{q}_{\preceq}$ is a two-sided ideal of R.

It remains to show that $\mathfrak{q}_{\preceq}$ is completely prime. Axiom (QR1) states that $1 \notin \mathfrak{q}_{\preceq}$, i.e. $\mathfrak{q}_{\preceq} \neq R$. Finally, assume that $xy \in \mathfrak{q}_{\preceq}$, but $x, y \notin \mathfrak{q}_{\preceq}$. By Lemma 1.28, also $-xy \in \mathfrak{q}_{\preceq}$. So we may w.l.o.g. assume that $0 \prec x, y$ as if for instance $x \prec 0$, then $0 \prec -x$ (Lemma 1.29). But if $0 \prec x, y$, then also $0 \prec xy$ by (QR3'), contradicting the fact that $xy \in \mathfrak{q}_{\preceq}$. Hence, $\mathfrak{q}_{\preceq}$ is completely prime. $\square$

The next result proves that our definition of quasi-ordered rings is consistent with Fakhruddin's definition of quasi-ordered fields (see Definition 1.22).

Proposition 1.31. *If $(R, \preceq)$ is a weakly quasi-ordered ring such that R is a field, then $(R, \preceq)$ is a quasi-ordered field. Conversely, any quasi-ordered field is a commutative quasi-ordered ring.*

Proof. Let $(R, \preceq)$ be a weakly quasi-ordered ring such that R is a field. Proposition 1.30 tells us that $\mathfrak{q}_{\preceq} = \{0\}$, so (Q1) is fulfilled. All the other axioms of a quasi-ordered field are also imposed on weakly quasi-ordered rings.

Now let $(K, \preceq)$ be a quasi-ordered field. We only have to verify (QR1) and (QR3). First assume that $1 \preceq 0$. By (Q1) and (Q3) we obtain $0 \preceq -1$. So (Q2) implies that also $0 \preceq 1$, a contradiction to (Q1). Hence, $0 \prec 1$. Next, let $0 \prec z$. Note that if $z^{-1} \prec 0$, then (Q2) yields the contradiction $1 \preceq 0$. So we have $0 \prec z^{-1}$. Thus, if $xz \preceq yz$ and $0 \prec z$, then we get $x \preceq y$ by (Q2). $\square$

We conclude this section by establishing the two main (and in fact only) examples of quasi-ordered rings, namely that of ordered rings (see Definition 1.1) and that of valued rings (see Definition 1.5).

Proposition 1.32. *Any ordered ring $(R, \leq)$ is a quasi-ordered ring.*

Proof. We only have to verify axiom (QR3) (see Definition 1.1 and Definition 1.24). So let $axb \leq ayb$ and $0 < a, b$. Then $a(x-y)b \leq 0$ by axiom (O2). So from axiom (O3) it follows that $x - y \leq 0$, i.e. that $x \leq y$. Thus, (QR3) holds. □

Proposition 1.33. *Let (R, v) be a (weakly) valued ring. Then $(R, \preceq)$ is a (weakly) quasi-ordered ring with support $\mathfrak{q}_v$, where $x \preceq y :\Leftrightarrow v(y) \leq v(x)$.*

Proof. First let (R, v) be a weakly valued ring. Clearly, $\preceq$ is reflexive, transitive and total, since the ordering $\leq$ on the value monoid of v has these properties. Further note that $v(1) = 0 < \infty = v(0)$, so $0 \prec 1$. This shows that (QR1) is fulfilled. Next, we establish (QR2). From $x \preceq y$ it follows that $v(y) \leq v(x)$. Hence,

$$v(ayb) = v(a) + v(y) + v(b) \leq v(a) + v(x) + v(b) = v(axb),$$

and therefore $axb \preceq ayb$. For the verification of (QR3') suppose that $xy \preceq 0$, i.e. $\infty \leq v(xy)$. Then Lemma 1.12 implies $v(x) = \infty$ or $v(y) = \infty$. Consequently, $x \preceq 0$ or $y \preceq 0$. Axiom (QR4) is shown via proof by cases. So let $x \preceq y$ and $z \not\sim y$. We deal with the case $z \prec y$, the case $y \prec z$ is proven similarly. Then $v(y) < v(z)$. Moreover, $v(y) \leq v(x)$. Applying Lemma 1.14 yields

$$v(y+z) = v(y) \leq \min\{v(x), v(z)\} \leq v(x+z),$$

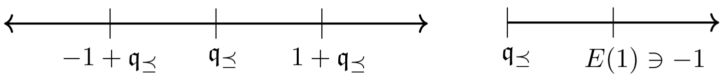
whence $x + z \preceq y + z$. Thus, $(R, \preceq)$ is a weakly quasi-ordered ring.

Now let (R, v) be a valued ring. It remains to verify (QR3). So suppose that $y \prec x$, and let $0 \prec a, b$, i.e. $v(x) < v(y)$ and $v(a), v(b) < \infty$. Then $v(axb) \leq v(ayb)$. Moreover, equality cannot hold, because then cancellation in Γ_v implies $v(x) = v(y)$, a contradiction. Hence, $v(axb) < v(ayb)$, and therefore $ayb \prec axb$. Thus, $(R, \preceq)$ is a quasi-ordered ring.

Finally, note that $x \in \mathfrak{q}_\preceq$ if and only if $v(x) = v(0) = \infty$, i.e. if and only if $x \in \mathfrak{q}_v$. Consequently, $\mathfrak{q}_\preceq = \mathfrak{q}_v$. □

Remark 1.34. Let $(R, \preceq)$ be a weakly quasi-ordered ring. If $\preceq$ is an ordering, then $E(x) = x + \mathfrak{q}_\preceq$ for all $x \in R$. If $\preceq$ is induced by a weak valuation v on R, then $E(x) \supseteq x + \mathfrak{q}_\preceq$ and $-x \in E(x)$ for all $x \in R$, since $v(x) = v(-x)$. Furthermore, $v(x) \leq \infty = v(0)$, and therefore $0 \preceq x$ for any $x \in R$.

Hence, the quasi-orderings obtained in Proposition 1.32 (LHS) and Proposition 1.33 (RHS) can be displayed by the following number rays:



1.2. **The Dichotomy of Quasi-Ordered Rings - Commutative Case.**

Let $(R, \preceq)$ be a commutative quasi-ordered ring. In this section, we first prove that $\preceq$ canonically induces a quasi-ordering $\trianglelefteq$ on the field $K := \mathrm{Quot}(R/\mathfrak{q}_\preceq)$ (Corollary 1.38). By applying Fakhruddin's dichotomy (see Theorem 1.23) to $(K, \trianglelefteq)$, we then deduce that $\preceq$ is either an ordering or a valuation on R (Theorem 1.40). The proof presented here was already published in 2018 by the author of this thesis ([60]).

A more general proof of the dichotomy, which also applies to non-commutative rings and weak quasi-orderings, is given in the following section. We decided to include the present one since it takes a very different approach and, more importantly, its key ingredients (Lemma 1.35, Proposition 1.37, Corollary 1.38) will play a major role later on in this thesis.

Lemma 1.35. *Let $(R, \preceq)$ be a quasi-ordered ring and $I \subseteq \mathfrak{q}_{\preceq}$ a two-sided ideal of R. Then $(R/I, \preceq')$ is also a quasi-ordered ring, where $\overline{x} \preceq' \overline{y} :\Leftrightarrow x \preceq y$.*
If $I = \mathfrak{q}_{\preceq}$, then $(R/I, \preceq')$ is a quasi-ordered domain with support $\{\overline{0}\}$.

Proof. Let $a \in I \subseteq \mathfrak{q}_{\preceq}$ and let $r \in R$. Then Lemma 1.28 (if $r \notin \mathfrak{q}_{\preceq}$), respectively Proposition 1.30 (if $r \in \mathfrak{q}_{\preceq}$), tells us that $a + r \sim r$. Hence, if $x \preceq y$ and $c, d \in I$, these results imply

$$x + c \sim x \preceq y \sim y + d.$$

Thus, $\preceq'$ is well-defined. It is clear that $\preceq'$ is reflexive, transitive and total. Also the axioms (QR1) – (QR4) are easily verified. Finally, if $I = \mathfrak{q}_{\preceq}$, then Proposition 1.30 implies that R/I is a domain. In that case the support of $\preceq'$ is trivial, since

$$\overline{x} \in \mathfrak{q}_{\preceq'} \Leftrightarrow \overline{x} \sim \overline{0} \Leftrightarrow x \sim 0 \Leftrightarrow x \in \mathfrak{q}_{\preceq} \Leftrightarrow \overline{x} = \overline{0}.$$

□

Lemma 1.36. *Let $(R, \preceq)$ be a quasi-ordered ring and let $x \in R$. Then $0 \preceq x^2$. Moreover, if $x \in R \backslash \mathfrak{q}_{\preceq}$, then $0 \prec x^2$.*

Proof. By Lemma 1.29, we know that $0 \preceq x$ or $0 \preceq -x$ for all $x \in R$. Applying (QR2) results in $0 \preceq x^2$. The second assertion follows from Proposition 1.30. □

In the proof of the following proposition we will frequently make use of the previous lemma without mentioning it explicitly all the time.

Proposition 1.37. *Let $(R, \preceq)$ be a commutative quasi-ordered ring and $S \subseteq R$ a multiplicative set such that $S \cap \mathfrak{q}_{\preceq} = \emptyset$. Then $\preceq$ uniquely extends to a quasi-ordering $\trianglelefteq$ on the localisation R_s as follows:*

$$\forall a, x \in R, \forall b, y \in S\colon ab^{-1} \trianglelefteq xy^{-1} :\Leftrightarrow aby^2 \preceq xyb^2.$$

Moreover, $\mathfrak{q}_{\trianglelefteq} = R_S \mathfrak{q}_{\preceq}$.

Proof. In this proof we always assume for any fraction that the nominator lies in R and that the denominator lies in S, the latter being a multiplicative set without zero-divisors according to Proposition 1.30.

We first show that $\trianglelefteq$ is well-defined. So suppose that $ab^{-1} \trianglelefteq pq^{-1}$, and let ab^{-1} and xy^{-1} be equal fractions, i.e. $ay = bx$. Note that $ab^{-1} \trianglelefteq pq^{-1}$ means $abq^2 \preceq pqb^2$. (QR2) implies $abq^2y^2 \preceq pqb^2y^2$. By the equality $ay = bx$, we get $b^2q^2xy \preceq pqb^2y^2$. Since $0 \prec b^2$, applying (QR3) yields $xyq^2 \preceq pqy^2$. Hence, $xy^{-1} \trianglelefteq pq^{-1}$. Analogously, it is proven that pq^{-1} may be replaced with any equal fraction.

Reflexivity and totality of $\trianglelefteq$ are clear, as well as the fact that $\trianglelefteq$ extends $\preceq$. Next, we verify transitivity. So let $ab^{-1} \trianglelefteq xy^{-1}$ and $xy^{-1} \trianglelefteq pq^{-1}$, i.e. $aby^2 \preceq xyb^2$ and $xyq^2 \preceq pqy^2$. This implies $aby^2q^2 \preceq xyb^2q^2$ and $xyb^2q^2 \preceq pqy^2b^2$. Transitivity of $\preceq$ yields $aby^2q^2 \preceq pqy^2b^2$, so (QR3) tells us $abq^2 \preceq pqb^2$. Thus, $ab^{-1} \trianglelefteq pq^{-1}$.

It remains to establish the axioms (QR1) - (QR4). The fact that $\frac{0}{1} \vartriangleleft \frac{1}{1}$ follows immediately from $0 \prec 1$ and the definition of $\trianglelefteq$. Thus, (QR1) is fulfilled.

Next, we prove (QR2). So suppose that $\frac{a}{b} \trianglelefteq \frac{x}{y}$ and $\frac{0}{1} \trianglelefteq \frac{p}{q}$. Then $aby^2 \preceq xyb^2$ and $0 \preceq pq$. The latter yields $0 \preceq pq^3$. Multiplying pq^3 on both sides of the inequality $aby^2 \preceq xyb^2$ implies $aby^2pq^3 \preceq xyb^2pq^3$. Consequently, $\frac{ap}{bq} \trianglelefteq \frac{xp}{yq}$.

For the proof of (QR3) let $\frac{0}{1} \vartriangleleft \frac{x}{y}$ and $\frac{ax}{by} \trianglelefteq \frac{px}{qy}$, i.e. $0 \prec xy$ and $abxy^3q^2 \preceq pqxy^3b^2$. Since $0 \prec xy$ and $0 \prec y^2$, we may cancel xy^3 in the second inequality via (QR3). That way, we obtain $abq^2 \preceq pqb^2$. Hence, $\frac{a}{b} \vartriangleleft \frac{p}{q}$.

Last but not least, we prove that $\trianglelefteq$ fulfils (QR4). Suppose that $\frac{a}{b} \trianglelefteq \frac{x}{y}$ and $\frac{p}{q} \nsim \frac{x}{y}$, i.e. $aby^2 \preceq xyb^2$ and $pqy^2 \nsim xyq^2$. We have to show $\frac{aq+bp}{bq} \trianglelefteq \frac{xq+yp}{yq}$, i.e.

$$(aq + bp)bqy^2q^2 \preceq (xq + yp)yqb^2q^2,$$

or equivalently

$$aq^4by^2 + b^2pq^3y^2 \preceq xq^4yb^2 + py^2q^3b^2.$$

From $aby^2 \preceq xyb^2$, it follows $aby^2q^4 \preceq xyb^2q^4$. If we show $py^2q^3b^2 \not\sim xq^4yb^2$, then we are done by axiom (QR4). But this is an easy consequence of the assumption $pqy^2 \not\sim xyq^2$ and axiom (QR3) applied to $b^2q^2 \succ 0$.

So far we have shown that $(R_S, \trianglelefteq)$ is a quasi-ordered ring. If $\trianglelefteq'$ is another extension of $\preceq$ to R_S, then

$$\frac{a}{b} \trianglelefteq' \frac{x}{y} \Leftrightarrow \frac{aby^2}{1} \trianglelefteq' \frac{xyb^2}{1} \Leftrightarrow aby^2 \preceq xyb^2 \Leftrightarrow \frac{a}{b} \trianglelefteq \frac{x}{y},$$

where in the first equivalence we utilised Lemma 1.36 and (QR2), respectively Lemma 1.36 and (QR3). This shows that $\trianglelefteq$ is the unique extension of $\preceq$ to R_S. Finally,

$$\frac{r}{s} \in \mathfrak{q}_{\trianglelefteq} \Leftrightarrow \frac{r}{s} \sim_{\trianglelefteq} \frac{0}{1} \Leftrightarrow rs \sim_{\preceq} 0 \Leftrightarrow r \in \mathfrak{q}_{\preceq}$$

by Proposition 1.30 and the fact that $S \cap \mathfrak{q}_{\preceq} = \emptyset$. □

Corollary 1.38. *If $(R, \preceq)$ is a commutative quasi-ordered ring, then $(K, \trianglelefteq)$ is a quasi-ordered field, where $K := Quot(R/\mathfrak{q}_{\preceq})$ and $\trianglelefteq$ is given by*

$$\forall a, x \in R,\ \forall b, y \in R\backslash\{0\}\colon ab^{-1} \trianglelefteq xy^{-1} :\Leftrightarrow aby^2 \preceq xyb^2.$$

Proof. By Lemma 1.35, $(R/\mathfrak{q}_{\preceq}, \preceq')$ is a quasi-ordered domain with support $\{\overline{0}\}$. Proposition 1.37 implies that $(\mathrm{Quot}(R/\mathfrak{q}_{\preceq}), \trianglelefteq)$ is a quasi-ordered ring. According to Proposition 1.31, it is a quasi-ordered field. □

Definition 1.39. Let $(R, \preceq)$ be a commutative quasi-ordered ring. We call the pair $(K, \trianglelefteq)$ from Corollary 1.38 the *quasi-ordered field associated* to $(R, \preceq)$.

We are now in a position to generalise Fakhruddin's dichotomy (see Theorem 1.23) to commutative quasi-ordered rings.

Theorem 1.40. *Let R be a commutative ring and let $\preceq$ be a binary relation on R. Then $(R, \preceq)$ is a quasi-ordered ring if and only if it is either an ordered ring or there is a unique valuation v on R such that $x \preceq y \Leftrightarrow v(y) \leq v(x)$ for all $x, y \in R$. Moreover, the support of the quasi-ordering $\preceq$ coincides with the support of the ordering, respectively the support of the valuation.*

Proof. One of these two implications was already shown in Proposition 1.32 and Proposition 1.33. Now let $(R, \preceq)$ be a commutative quasi-ordered ring and $(K, \trianglelefteq)$ its associated quasi-ordered field. By Fakhruddin's dichotomy (Theorem 1.23), we obtain that $\trianglelefteq$ is either an ordering on K or is induced by a valuation on K.

First suppose that there is a valuation $\nu\colon K \to \Gamma \cup \{\infty\}$ such that $\overline{x} \trianglelefteq \overline{y} \Leftrightarrow \nu(\overline{y}) \leq \nu(\overline{x})$ for all $\overline{x}, \overline{y} \in K$. The restriction v' of ν to $R/\mathfrak{q}_{\preceq}$ is obviously a valuation on $R/\mathfrak{q}_{\preceq}$ with support $\{\overline{0}\}$ and the same value group such that $\overline{x} \preceq' \overline{y} \Leftrightarrow v'(\overline{y}) \leq v'(\overline{x})$ for all $\overline{x}, \overline{y} \in R/\mathfrak{q}_{\preceq}$. Define $v\colon R \to \Gamma \cup \{\infty\}$ by $v(x) = v'(\overline{x})$. Then v is easily seen to be a valuation as well. Moreover,

$$x \in \mathfrak{q}_v \Leftrightarrow v(x) = \infty \Leftrightarrow v'(\overline{x}) = \infty \Leftrightarrow \overline{x} = \overline{0} \Leftrightarrow x \in \mathfrak{q}_{\preceq},$$

i.e. the supports of $\preceq$ and v coincide. By Lemma 1.35, it follows:

$$\forall x, y \in R\colon v(y) \leq v(x) \Leftrightarrow v'(\overline{y}) \leq v'(\overline{x}) \Leftrightarrow \overline{x} \preceq' \overline{y} \Leftrightarrow x \preceq y.$$

Moreover, if w is another valuation on R that induces $\preceq$, then

$$v(x) \leq v(y) \Leftrightarrow y \preceq x \Leftrightarrow w(x) \leq w(y)$$

for all $x, y \in R$, so v and w are equivalent, i.e. equal.

If $\trianglelefteq$ is an ordering on K, then the restriction $(R/\mathfrak{q}_{\preceq}, \preceq')$ is an ordered ring with support $\{0\}$. Recall from Lemma 1.35 that $\overline{x} \preceq' \overline{y} \Leftrightarrow x \preceq y$. From this, it is easy to see that $(R, \preceq)$ is an ordered ring with support $\mathfrak{q}_{\preceq}$.

Finally, note that $-1 \prec 0$ if $\preceq$ is an ordering, whereas $0 \prec -1$ if $\preceq$ is induced by some valuation on R. This establishes that $\preceq$ is either an ordering or comes from a valuation on R. $\square$

Remark 1.41. Fakhruddin's proof easily generalises to division rings. Nonetheless, we cannot deduce an analogue of Theorem 1.40 for possibly non-commutative rings from it in the way we did for commutative rings. In fact, not every ordered domain can be embedded in a division ring (cf. [45, Theorem 9.11]), which is why Proposition 1.37 does not apply to arbitrary quasi-ordered rings.

However, the latter result does hold whenever the set S is a *right denominator* set (cf. [45, Definition 10.5]), i.e. both *right permutable* ($rS \cap sR \neq \emptyset$ for any $r \in R$ and any $s \in S$) and *right reversible* (for any $r \in R$, if $s'r = 0$ for some $s' \in S$, then $rs = 0$ for some $s \in S$). By [45, Theorem 10.6], then and only then the right ring of fractions RS^{-1} exists with the following definitions of equality of fractions, addition and multiplication:

(1) $(r, s) \sim (r', s')$ in $R \times S :\Leftrightarrow \exists b, b' \in R\colon sb = s'b' \in S$ and $rb = r'b' \in R$,

(2) $\frac{r_1}{s_1} + \frac{r_2}{s_2} = \frac{r_1 s + r_2 r}{t}$, where $t = s_1 s = s_2 r$ for some $s \in S$ and $r \in R$,

(3) $\frac{r_1}{s_1} \cdot \frac{r_2}{s_2} = \frac{r_1 r}{s_2 s}$, where $s_1 r = r_2 s$ for some $s \in S$ and $r \in R$.

So let $(R, \preceq)$ be a quasi-ordered ring and $S \subseteq R$ a multiplicative right denominator set such that $S \cap \mathfrak{q}_{\preceq} = \emptyset$. For $r_1, r_2 \in R$ and $s_1, s_2 \in S$, w.l.o.g. $0 \prec s_1, s_2$, the quasi-ordering $\preceq$ extends to a quasi-ordering $\trianglelefteq$ on RS^{-1} as follows:

$$r_1 s_1^{-1} \trianglelefteq r_2 s_2^{-1} :\Leftrightarrow r_1 s \preceq r_2 r \text{ and } s_1 s = s_2 r \text{ for some } r \in R \text{ and } s \in S.$$

Indeed, by right permutability of S, we may choose some $r \in R$ and some $s \in S$ such that $s_1 s = s_2 r$, i.e. such that $s_1 = s_2 r s^{-1}$. Consequently,

$$\begin{aligned} r_1 s_1^{-1} \trianglelefteq r_2 s_2^{-1} &\Leftrightarrow r_1 \trianglelefteq r_2 s_2^{-1} s_1 \\ &\Leftrightarrow r_1 \trianglelefteq r_2 r s^{-1} \\ &\Leftrightarrow r_1 s \preceq r_2 r. \end{aligned}$$

Alternatively, we may also choose some $r \in R$ and some $s \in S$ such that $r_2 s = s_1 r$. The same arguing then yields the following definition of $\trianglelefteq$:

$$r_1 s_1^{-1} \trianglelefteq r_2 s_2^{-1} :\Leftrightarrow r_1 r \preceq r_2 s \text{ and } r_2 s = s_1 r \text{ for some } r \in R \text{ and } s \in S.$$

Proving that $\trianglelefteq$ indeed defines a quasi-ordering on RS^{-1}, however, is very tedious. Therefore, we take a whole different approach in the following section in order to establish Theorem 1.40 for possibly non-commutative quasi-ordered rings.

1.3. **The Dichotomy of Quasi-Ordered Rings - General Case.**

In the present section we give a proof of the dichotomy of quasi-ordered rings that applies to possibly non-commutative rings, and also takes weak quasi-orderings into consideration (Theorem 1.50). While we exploited Fakhruddin's dichotomy in the previous section, here we alter and simplify his proof such that it no longer relies on the existence of multiplicative inverses. This is achieved by distinguishing whether $-1 \prec 0$ or $0 \prec -1$, instead of whether $E(1)$ is trivial or not. That way, we also get rid of the pathological case of characteristic 2 (cf. [19, Theorem 3.3]).

Lemma 1.42. *Let $(R, \preceq)$ be a weakly quasi-ordered ring with $-1 \prec 0$. Then $0 \preceq x$ if and only if $-x \preceq 0$ for any $x \in R$. Moreover, if $x \in R \backslash \mathfrak{q}_{\preceq}$, then either $x \prec -x$ or $-x \prec x$.*

Proof. If $x \preceq 0$, then $0 \preceq -x$ by Lemma 1.29 or Proposition 1.30. If $0 \preceq x$, then $-x \preceq 0$ by the assumption $-1 \prec 0$ and axiom (QR2). The moreover statement is obvious, since either $-x \prec 0 \prec x$ or $x \prec 0 \prec -x$. $\square$

Proposition 1.43. *Any weakly quasi-ordered ring $(R, \preceq)$ with $-1 \prec 0$ is an ordered ring.*

Proof. We only have to verify axiom (OR4). So suppose that $x \preceq y$. Then $x - y \preceq 0$. Indeed, either $-y \sim 0$, and then $x - y \sim x \preceq y \preceq 0$ by Lemma 1.28; or $-y \not\sim 0$, and then $-y \not\sim y$ by Lemma 1.42, whence $x - y \preceq 0$ follows from $x \preceq y$ via axiom (QR4). With the same argument we get that $x - y \preceq 0$ implies $x + z \preceq y + z$. $\square$

Remark 1.44. According to Lemma 1.26, Proposition 1.32, and Proposition 1.43 the following are equivalent:

(1) $(R, \preceq)$ is a weakly quasi-ordered ring with $-1 \prec 0$,
(2) $(R, \preceq)$ is a quasi-ordered ring with $-1 \prec 0$,
(3) $(R, \preceq)$ is an ordered ring.

It remains to show that any (weak) quasi-ordering $\preceq$ on a ring R with $0 \prec -1$ is induced by a (weak) valuation on R.

Lemma 1.45. *Let $(R, \preceq)$ be a weakly quasi-ordered ring such that $0 \prec -1$. Then $0 \preceq x$ for any $x \in R$.*

Proof. By Lemma 1.29 we know that $0 \preceq x$ or $0 \preceq -x$ for all $x \in R$. Since $0 \prec -1$, (QR2) implies $0 \preceq x$ for any $x \in R$. $\square$

Corollary 1.46. *Let $(R, \preceq)$ be a weakly quasi-ordered ring such that $0 \prec -1$.*

(1) If $x \preceq y$, then $ax \preceq ay$ for all $a, b, x, y \in R$.
(2) $x \sim -x$ for all $x \in R$.

Proof. (1) is trivial by Lemma 1.45 and (QR2). For (2), note that $-1 \preceq 1$ or $1 \preceq -1$ by the totality of $\preceq$. Applying (QR2) with $0 \prec -1$ yields that the converse also holds, whence $-1 \sim 1$. Since $\sim$ is preserved under multiplication according to (1), this implies $-x \sim x$ for all $x \in R$. $\square$

For the rest of this section let $(R, \preceq)$ be a weakly quasi-ordered ring with $0 \prec -1$. Define $H := (R \backslash \mathfrak{q}_{\preceq})/{\sim}$. We equip H with the addition $[x] + [y] := [xy]$ and the ordering $[x] \leq [y] :\Leftrightarrow y \preceq x$.

Lemma 1.47. *$(H, +, \leq)$ is a totally ordered monoid. Moreover, if $(R, \preceq)$ is a quasi-ordered ring, then $(H, +, \leq)$ is cancellative.*

Proof. We first show that $+$ is well-defined. So suppose that $[x] = [x']$ and $[y] = [y']$, i.e. $x \sim x'$ and $y \sim y'$. From $x \sim x'$ it follows $xy' \sim x'y'$, and from $y \sim y'$ it follows $xy \sim xy'$ by Corollary 1.46(1). Hence, $xy \sim xy' \sim x'y'$, and therefore

$$[x] + [y] = [xy] = [x'y'] = [x'] + [y'].$$

By axiom (QR1), $1 \in R \backslash \mathfrak{q}_{\preceq}$, and $[1] \in H$ is easily seen to be the neutral element of H. It is also clear that $+$ is associative, whence $(H, +)$ is a monoid.

Next, we show that H is totally ordered by $\leq$. We first prove that $\leq$ is well-defined. So let $[x] \leq [y]$, and let $a \in [x]$ and $b \in [y]$, i.e. $a \sim x$ and $b \sim y$. Then

$$[x] \leq [y] \Leftrightarrow y \preceq x \Leftrightarrow b \preceq a \Leftrightarrow [a] \leq [b].$$

It is clear that $\leq$ is reflexive, transitive and total. For anti-symmetry suppose that $[x] \leq [y]$ and $[y] \leq [x]$. Then $x \sim y$, i.e. $[x] = [y]$. Finally, notice that if $y \preceq x$, then $yz \preceq xz$ for any $z \in R$ by Corollary 1.46(1). Consequently, $[xz] \leq [yz]$ follows from $[x] \leq [y]$. Hence, $(H, +, \leq)$ is a totally ordered monoid.

For the moreover statement suppose that $[axb] \leq [ayb]$ in H. Then $ayb \preceq axb$. Furthermore, $0 \prec a, b$ by Corollary 1.46(1), so (QR3) yields $y \preceq x$, i.e. $[x] \leq [y]$. Consequently, $(H, +, \leq)$ is cancellative whenever $(R, \preceq)$ is a quasi-ordered ring. $\square$

We will verify in Proposition 1.49 below that $\preceq$ is induced by the (weak) valuation

$$v\colon R \to H \cup \{\infty\},\ x \mapsto \begin{cases} [x], & x \in R\backslash\mathfrak{q}_{\preceq} \\ \infty, & x \in \mathfrak{q}_{\preceq} \end{cases}$$

Lemma 1.48. *Let $(R, \preceq)$ be a weakly quasi-ordered ring such that $0 \prec -1$. Then $x + y \preceq \max\{x, y\}$ for all $x, y \in R$.*

Proof. Completely analogue to the field case, see ([19, Lemma 4.1]). $\square$

Proposition 1.49. *Let $(R, \preceq)$ be a (weakly) quasi-ordered ring with $0 \prec -1$. Then there is a (weak) valuation v on R such that $x \preceq y$ if and only if $v(y) \leq v(x)$ for all $x, y \in R$. Moreover, v is unique and $\mathfrak{q}_{\preceq} = \mathfrak{q}_v$.*

Proof. Let $(R, \preceq)$ be a weakly quasi-ordered ring. We first show that the map v that we defined above is a weak valuation. By construction, v is surjective. Now let us verify the axioms (VR1) - (VR4). Obviously, $v(0) = \infty$ and $v(1) = [1] = 0_H$ by definition of v, so (VR1) and (VR2) are fulfilled. For (VR3) note that $xy \in \mathfrak{q}_{\preceq}$ if and only if $x \in \mathfrak{q}_{\preceq}$ or $y \in \mathfrak{q}_{\preceq}$ (Proposition 1.30), so let w.l.o.g. $x, y \notin \mathfrak{q}_{\preceq}$. Then

$$v(xy) = [xy] = [x] + [y] = v(x) + v(y),$$

as desired. Axiom (VR4) is immediately implied by Lemma 1.48. Hence, v is a weak valuation.

If $(R, \preceq)$ is a quasi-ordered ring and $v(a) + v(x) + v(b) \leq v(a) + v(y) + v(b) \in \Gamma_v$, then $ayb \preceq axb$ in R with $0 \prec a, b$. So (QR3) yields $y \preceq x$, i.e. $v(x) \leq v(y)$. Thus, $(H, +, \leq)$ is cancellative, whence v is a valuation.

Now let $x, y \in R$, w.l.o.g. $x, y \notin \mathfrak{q}_{\preceq}$. Then

$$v(x) \leq v(y) \Leftrightarrow [x] \leq [y] \Leftrightarrow y \preceq x.$$

If w is another such weak valuation, then

$$w(x) \leq w(y) \Leftrightarrow y \preceq x \Leftrightarrow [x] \leq [y] \Leftrightarrow v(x) \leq v(y),$$

for all $x, y \in R$, so v and w are equivalent, i.e. equal. Finally,

$$x \in \mathfrak{q}_v \Leftrightarrow v(x) = v(0) \Leftrightarrow x \sim 0 \Leftrightarrow x \in \mathfrak{q}_{\preceq},$$

so $\mathfrak{q}_v = \mathfrak{q}_{\preceq}$. This finishes the proof. $\square$

Proposition 1.32, Proposition 1.33, Proposition 1.43 and Proposition 1.49 prove the following generalisation of Theorem 1.40:

Theorem 1.50. *Let R be a ring and let $\preceq$ be a binary relation on R. Then $(R, \preceq)$ is a (weakly) quasi-ordered ring if and only if it is either an ordered ring or there is a unique (weak) valuation v on R such that $x \preceq y \Leftrightarrow v(y) \leq v(x)$ for all $x, y \in R$. Moreover, the support of the (weak) quasi-ordering $\preceq$ coincides with the support of the ordering, respectively the support of the (weak) valuation.*

1.4. **Special Quasi-Orderings.**

In the preliminaries we derived a definition of special valuations on possibly non-commutative rings (see Definition 1.10). Here, we go one step further by introducing special quasi-orderings (Definition 1.54). That way, we obtain a notion of special orderings, which coincides with Krivine's notion of *pseudo-fields* ([36]). Hence, special quasi-orderings unify the classes of special valuations and pseudo-fields. Further characterisations of special quasi-orderings are provided (Proposition 1.57, Proposition 1.58). Special quasi-orderings will reappear in Chapter 4.

In Remark 1.11(1), we pointed out that a valuation v on a ring R is special if and only if for any $x \in R\backslash\mathfrak{q}_v$ we find some $y \in R$ such that $v(xy) \leq 0 = v(1)$, i.e. such that $1 \preceq_v xy$. That is precisely the condition, which we will impose on (right) special quasi-orderings. As in the valued case, left and right special quasi-orderings coincide, as we will see in Proposition 1.53 below.

Lemma 1.51. *Let $(R, \preceq)$ be a quasi-ordered ring and let $x, y, z \in R$. If $x \preceq y$ and $z \preceq 0$, then $yz \preceq xz$ and $zy \preceq zx$.*

Proof. For symmetry reasons, it suffices to show that $yz \preceq xz$. We may assume that $z \notin \mathfrak{q}_{\preceq}$. Moreover, if $y \in \mathfrak{q}_{\preceq}$, then $x, z \preceq 0$, whence $0 \preceq -x, -z$ by Lemma 1.29. Thus, by (QR2), $yz \sim 0 \preceq xz$. So we may further assume that $y \notin \mathfrak{q}_{\preceq}$. Now $x \preceq y$ and $0 \preceq -z$ implies $-xz \preceq -yz$. We claim that $yz \nsim -yz$. Once this is shown, it follows from $-xz \preceq -yz$ that $yz - xz \preceq 0$. The latter implies $yz \preceq xz$. Indeed, either $x \nsim 0$ and therefore $xz \nsim 0$ (Proposition 1.30), so that we can apply (QR4); or $x \sim 0$, and therefore $yz - xz \sim yz \preceq 0 \sim xz$ (Lemma 1.28).

Assume for the sake of a contradiction that $yz \sim -yz$. This yields $0 \preceq yz, -yz$ (Lemma 1.29). As $y \notin \mathfrak{q}_{\preceq}$, either $0 \prec y$ or $0 \prec -y$. So via (QR3) we deduce from either $0 \preceq yz$ (if $0 \prec y$) or $0 \preceq -yz$ (if $0 \prec -y$) that $0 \preceq z$. Hence $z \sim 0$, the desired contradiction. □

Lemma 1.52. *Let $(R, \preceq)$ be a quasi-ordered ring and let $x, y, z \in R$. If $xz \preceq yz$ (respectively $zx \preceq zy$) and $z \prec 0$, then $y \preceq x$.*

Proof. Suppose that $xz \preceq yz$ and $z \prec 0$. Assume that $x \prec y$. Then Lemma 1.51 yields $yz \preceq xz$, whence $xz \sim yz$. Since $0 \prec -z$ (Lemma 1.29), we obtain via (QR3) that $-x \sim -y$. By (QR2) (if $0 \prec -1$), respectively Lemma 1.51 (if $-1 \prec 0$), it follows that $x \sim y$, a contradiction. Thus, $y \preceq x$. □

Proposition 1.53. (see Lemma 1.9) *Let $(R, \preceq)$ be a quasi-ordered ring and let $x, y \in R$. Then $1 \preceq xy$ if and only if $1 \preceq yx$.*

Proof. Let $1 \preceq xy$. If $x \prec 0$, then Lemma 1.51 implies $xyx \preceq x$. Consequently, Lemma 1.52 yields $1 \preceq yx$. If $0 \prec x$, then we just apply axiom (QR2) instead of Lemma 1.51, and axiom (QR3) instead of Lemma 1.52. □

Definition 1.54. Let R be a ring. We call a quasi-ordering $\preceq$ on R *special*, if for any $x \in R\backslash\mathfrak{q}_{\preceq}$ there is some $y \in R$ such that $1 \preceq xy$ (or equivalently, $1 \preceq yx$).

Special valuations were formally introduced in [35] by Knebusch and D. Zhang. These are precisely the valuations that admit no proper primary specialisation in the valuation spectrum of a commutative ring (cf. [29]). Krivine showed that a preordering T on a real ring R is a maximal preordering if and only if R/T is special, respectively a pseudo-field ([36, Theorem 6]). Furthermore, special orderings are the maximal elements in the real spectrum of a commutative ring (cf. [13, Theorem 13.2.8], see also Remark 4.36).

Example 1.55.

(1) In Example 1.3(6) we considered the Weyl algebra R equipped with the ordering $<$ that is uniquely determined by the chain
$$\mathbb{R} < x < x^2 < \ldots < y < xy < x^2y < \ldots < y^2 < xy^2 < x^2y^2 < \ldots$$
This ordering is special, since $1 \leq r(ry)$ for any $r \in R\backslash\{0\}$.

(2) The unique ordering on $R = \mathbb{R}[X]$ such that $0 < X < \mathbb{R}^{>0}$ is not special, since $|Xr| < 1$ for any $r \in R$.

We conclude this section by giving characterisations of special quasi-orderings.

Definition 1.56. Let $(R, \preceq)$ be a quasi-ordered ring and let $M \subseteq R$ be an additive subgroup of R. Then M is *convex* (w.r.t. $\preceq$), if $0 \preceq x \preceq y \in M$ implies $x \in M$ for all $x, y \in R$.

If $\preceq = \preceq_v$ for some valuation v on R, this is the usual notion of v-convexity (see for instance [52], [35, Definition I.1.8]).

Proposition 1.57. (cf. [34, Theorem 4.8] or [13, Proposition 13.2.7])
Let $(R, \preceq)$ be a quasi-ordered ring. The following are equivalent:

(1) $\preceq$ is special,

(2) $\mathfrak{q}_{\preceq}$ is the only proper two-sided $\preceq$-convex ideal of R.

Proof. First suppose that (1) holds, and assume that there is a proper two-sided convex ideal I of R such that $I \neq \mathfrak{q}_{\preceq}$. Then $\mathfrak{q}_{\preceq} \subsetneq I$, since $\mathfrak{q}_{\preceq}$ is obviously the smallest convex subgroup of R. Let $x \in I\backslash\mathfrak{q}_{\preceq}$. By (1), we find some $y \in R$ such that $0 \prec 1 \preceq xy \in I$, so by convexity also $1 \in I$, a contradiction. Hence, $\mathfrak{q}_{\preceq}$ is the only such ideal of R.

Conversely, assume that there is some $x \in R\backslash\mathfrak{q}_{\preceq}$ such that $xy \prec 1$ for all $y \in R$. Consider $I := \{y \in R\colon yr \prec 1 \text{ for all } r \in R\}$. Then $\mathfrak{q}_{\preceq} \subsetneq I \subsetneq R$, since $x \in I\backslash\mathfrak{q}_{\preceq}$ and $1 \in R\backslash I$. We conclude by showing that I is a two-sided convex ideal, which is a contradiction to (2). Clearly, $0 \in I$ and $IR \subseteq I$. Proposition 1.53 tells us that also $RI \subseteq I$. Moreover, $I + I \subseteq I$ follows from Lemma 1.48 if $0 \prec -1$, whereas it is an easy consequence of Proposition 1.43 if $-1 \prec 0$. Thus, I is a two-sided ideal. It is convex, since if $0 \preceq a \preceq b \in I$ and $r \in R$, then either $ar \preceq br \preceq 1$ by (QR2), or $ar \preceq 0 \prec 1$ by Lemma 1.51. $\square$

In Proposition 1.57, we may also state in (2) that $\mathfrak{q}_{\preceq}$ is the only proper convex left, respectively right ideal of R. This is evident from the proof and Proposition 1.53.

Proposition 1.58. (cf. [13, Proposition 13.2.7])
Let $(R, \preceq)$ be a commutative quasi-ordered ring and let $(K, \trianglelefteq)$ be its associated quasi-ordered field. Then the following are equivalent:

(1) $\preceq$ is special,

(2) $\preceq'$ is special,

(3) $R/\mathfrak{q}_{\preceq}$ is cofinal in K.

Proof. The equivalence of (1) and (2) is trivial by Lemma 1.35. Now let $\preceq'$ be special and let $\overline{x}/\overline{y} \in K$, w.l.o.g. $\overline{0} \prec' \overline{x}, \overline{y}$. By (2), we find some $\overline{a} \in R/\mathfrak{q}_{\preceq}$ such that $\overline{1} \preceq' \overline{ya}$. Therefore, $\overline{x} \preceq' \overline{yax}$. Since $\overline{0} \prec \overline{y}$, we obtain by (QR2) that $\overline{x}/\overline{y} \trianglelefteq \overline{ax}$. Hence, (2) implies (3).

Conversely, let $R/\mathfrak{q}_{\preceq}$ be cofinal in K. We show that $\preceq'$ is special. Let $\overline{0} \neq \overline{x} \in R/\mathfrak{q}_{\preceq}$, and assume that $\overline{xy} \prec' \overline{1}$ for all $\overline{y} \in R/\mathfrak{q}_{\preceq}$. Then $\overline{y} \lhd \overline{1}/\overline{x}^2$ for all $\overline{y} \in R/\mathfrak{q}_{\preceq}$, a contradiction to (2). Thus, $\preceq'$ is special. $\square$

1.5. **Manis Quasi-Orderings and the Inverse Property.**

We conclude Chapter 1 by introducing a further class of quasi-orderings, namely Manis quasi-orderings. Just like special quasi-orderings, they arise from the respective notion from valuation theory. In 1969, Manis generalised valuations from fields to commutative rings, transferring various results in the process. He demanded that valuations shall map onto groups ([52]), making his definition stronger than the usual definition of a valuation (see Definition 1.5). Amongst others, Manis established an approximation theorem for Manis valued rings. The key tool for that is the so-called inverse property, which ever since was subject to numerous publications on valued rings (e.g. [1], [17], [22], [23], [32]).

In the present section we will first derive a definition of Manis quasi-ordered rings (Definition 1.63). We then prove the following dichotomy: if R is a ring and $\preceq$ a binary relation on R, then $\preceq$ is a Manis quasi-ordering if and only if $\preceq$ is either a Manis valuation or $R/\mathfrak{q}_\preceq$ is an ordered division ring (Theorem 1.65). Thus, Manis quasi-orderings unify support $\{0\}$ Manis valuations and skew-field orderings (Corollary 1.66). Finally, we introduce the inverse property for families of Manis quasi-orderings (Definition 1.68), and prove that it applies to any family of Manis orderings (Proposition 1.77). Manis quasi-orderings will reappear in Chapter 4.

In Remark 1.11(1), we saw that a valuation v on a ring R is Manis if and only if for any $x \in R\backslash\mathfrak{q}_v$ we find some $y \in R$ such that $v(xy) = v(1)$. Hence, (right) Manis quasi-orderings shall be quasi-orderings such that

$$\forall x \in R\backslash\mathfrak{q}_\preceq,\ \exists y \in R\colon\ xy \sim 1.$$

As for special quasi-orderings, we first show that left Manis quasi-orderings and right Manis quasi-orderings coincide, i.e. that $xy \sim 1$ if and only if $yx \sim 1$ for any quasi-ordered ring $(R, \preceq)$, see Corollary 1.62 below. This relies on the fact that $\sim$ is stable under multiplication, as well as under cancellation.

Lemma 1.59. *Let $(R, \preceq)$ be a quasi-ordered ring and let $x, y \in R$. If $x \sim y$, then $x \sim -y$ or $0 \sim x - y$.*

Proof. If $x, y \sim 0$, then $x \sim -y$, as $\mathfrak{q}_\preceq$ is an ideal. So suppose that $x, y \nsim 0$, and assume that $x \nsim -y$. We show that $0 \sim x - y$. Note that $y \preceq x \nsim -y$. Thus, $0 \preceq x - y$ by (QR4). Likewise, $x \preceq y \sim x \nsim -y$, so $y \nsim -y$, and therewith $x - y \preceq 0$. Hence, $0 \sim x - y$. □

Corollary 1.60. *Let $(R, \preceq)$ be a quasi-ordered ring. Then $\sim$ is preserved under multiplication, i.e. if $x \sim y$, then also $axb \sim ayb$ for all $x, y, a, b \in R$.*

Proof. We show that $x \sim y$ implies $ax \sim ay$, the proof of right multiplication being analogue. The cases $0 \preceq a$ (axiom (QR2)) and x, y in $\mathfrak{q}_\preceq$ ($\mathfrak{q}_\preceq$ is an ideal) are both trivial. So suppose that $0 \npreceq a$ and $x, y \nsim 0$. Then $0 \preceq -a$. Lemma 1.59 gives rise to a proof by cases.

First suppose $0 \sim x - y$. Since $-x \nsim 0$, we have $-x \nsim x - y$. Hence, $0 \sim x - y$ yields $-x \sim -y$ by (QR4). Because of $0 \preceq -a$, axiom (QR2) implies $ax \sim ay$.

Now suppose $0 \nsim x - y$. Then also $0 \nsim y - x$. Thus, Lemma 1.59 implies $x \sim -y$ and $y \sim -x$. Therefore, $-x \sim y \sim x \sim -y$. So we have $-x \sim -y$ and $0 \preceq -a$, whence the claim follows by applying (QR2). □

Corollary 1.61. *Let $(R, \preceq)$ be a quasi-ordered ring. Then $\sim$ is preserved under cancellation, i.e. if $axb \sim ayb$ and $0 \nsim a, b$, then also $x \sim y$ for all $x, y, a, b \in R$.*

Proof. Due to symmetry, we may w.l.o.g. assume that $b = 1$. If $0 \prec a$, the claim follows immediately by (QR3). If $a \prec 0$, then $0 \prec -a$ (Lemma 1.29), so $ax \sim ay$ implies $-x \sim -y$. By Corollary 1.60 we get $x \sim y$. □

Corollary 1.62. *Let $(R, \preceq)$ be a quasi-ordered ring and let $x, y \in R$. Then $xy \sim 1$ if and only if $yx \sim 1$.*

Proof. Due to symmetry, it suffices to show that $xy \sim 1$ implies $yx \sim 1$. So let $xy \sim 1$. Then Corollary 1.60 yields $yxy \sim y$. Since $xy \sim 1$, we know that $y \nsim 0$. Therefore, $yxy \sim y$ implies $yx \sim 1$ (Corollary 1.61). □

Definition 1.63. Let R be a ring. A quasi-ordering $\preceq$ on R is called *Manis*, if for any $x \in R \backslash \mathfrak{q}_{\preceq}$ there is some $y \in R$ such that $1 \sim xy$ (or equivalently, $1 \sim yx$).

Remark 1.64. Alternatively, we may define Manis quasi-orderings on a ring R as quasi-orderings such that

(1) (QR3) is omitted and

(2) For any $x \in R \backslash \mathfrak{q}_{\preceq}$ there is some $y \in R$ such that both, $xy \sim 1$ and $yx \sim 1$.

This relies on the observation that $\sim$ is preserved under multiplication, even if (QR3) fails - in fact, Corollary 1.60 only requires Lemma 1.59 and that $\mathfrak{q}_{\preceq}$ is an ideal. Now suppose that $axb \preceq ayb$ and $0 \prec a, b$. If $\preceq$ satisfies condition (2), then we find some $c, d \in R^{\succ 0}$ such that $ca \sim 1$ and $bd \sim 1$. Corollary 1.60 and (QR2) imply

$$x \sim caxbd \preceq caybd \sim y.$$

Thus, (QR3) follows from (2) and the other axioms of a quasi-ordered ring.

Manis quasi-orderings give rise to the following dichotomies (Theorem 1.65 and Corollary 1.66):

Theorem 1.65. *Let $(R, \preceq)$ be a quasi-ordered ring. Then $\preceq$ is Manis if and only if either $(R/\mathfrak{q}_{\preceq}, \preceq')$ is an ordered division ring or there is a unique Manis valuation v on R such that $x \preceq y \Leftrightarrow v(y) \leq v(x)$ for all $x, y \in R$.*

Proof. By Theorem 1.50, $\preceq$ is either an ordering or induced by a unique valuation v on R. In the latter case, v is obviously a Manis valuation by Remark 1.11(1) and Definition 1.63. So suppose from now on that $\preceq$ is an ordering on R. Hence, for all $x, y \in R$ we have $x \sim y$ if and only if $x = y + c$ for some $c \in \mathfrak{q}_{\preceq}$.

First suppose that $\preceq$ satisfies the Manis property, and let $\overline{0} \neq \overline{x} \in R/\mathfrak{q}_{\preceq}$. Then we find some $y \in R$ such that $xy \sim 1$, i.e. $xy = 1 + c$ for some $c \in \mathfrak{q}_{\preceq}$. Hence, $\overline{xy} = \overline{1}$. Analogously, we obtain $\overline{yx} = \overline{1}$. Therefore, $(R/\mathfrak{q}_{\preceq}, \preceq')$ is an ordered division ring.

Conversely, let $R/\mathfrak{q}_{\preceq}$ be a division ring and let $\overline{0} \neq \overline{x} \in R/\mathfrak{q}_{\preceq}$. Then we find some $y \in R/\mathfrak{q}_{\preceq}$ such that $\overline{xy} = \overline{1}$, whence $xy \sim 1$ by Lemma 1.28. Thus, $\preceq$ has the Manis property. □

Consequently, if R is a commutative ring, a Manis quasi-ordering on R is either a Manis valuation or an ordering whose support is a maximal ideal of R. In the special case where $\mathfrak{q}_{\preceq} = \{0\}$, we obtain:

Corollary 1.66. *Let $(R, \preceq)$ be a quasi-ordered ring with support $\{0\}$. Then $\preceq$ is Manis if and only if either $(R, \preceq)$ is an ordered division ring or there is a unique Manis valuation v on R with support $\{0\}$ such that $x \preceq y \Leftrightarrow v(y) \leq v(x)$ for all $x, y \in R$.*

Proof. This follows immediately from Theorem 1.65. □

Example 1.67.

(1) In Example 1.3(4) we considered the ring $R = C([0,1], \mathbb{R})$, chose some $x \in [0,1]$, and ordered R by $f \leq g :\Leftrightarrow f(x) \leq g(x)$. This is a Manis ordering. Indeed, if $f \in R$ such that $f(x) = r \neq 0$, then the constant map $g(x) = 1/r$ is also in R and $fg \sim 1$.

(2) The ordering on the Weyl algebra R from Example 1.3(6) is not Manis, since for any $r \in R$ either $ry < 0$ or $ry > 1$.

Finally, we generalise the inverse property to families of Manis quasi-orderings.

Definition 1.68. ([52]) Let R be a ring and $\mathcal{F} = (\preceq_i : i \in I)$ a family of Manis quasi-orderings on R. Then $\mathcal{F}$ has the *inverse property*, if for every $x \in R$ there is some $y \in R$ such that $xy \sim_{\preceq_i} 1$ for all $i \in I$ with $x \notin \mathfrak{q}_{\preceq_i}$.

We first establish some very basic results on the inverse property. Thereafter, we prove that it applies to any family of Manis orderings.

The first result yields a reformulation of the inverse property in such a way, that we do not have to restrict to those $i \in I$ for which $x \notin \mathfrak{q}_{\preceq_i}$.

Lemma 1.69. (cf. [32, Chapter II, Remark 3.2]) *A family $\mathcal{F} = (\preceq_i : i \in I)$ of Manis quasi-orderings on a ring R has the inverse property if and only if for every $x \in R$ there is some $y \in R$ such that $x^2 y \sim_{\preceq_i} x$ for all $i \in I$.*

Proof. First suppose that $\mathcal{F}$ has the inverse property, and let $x \in R$. Then there exists some $y \in R$ such that $xy \sim_{\preceq_i} 1$ for all $i \in I$ with $x \notin \mathfrak{q}_{\preceq_i}$. Since $\sim_{\preceq_i}$ is preserved under multiplication (Corollary 1.60), this implies $x^2 y \sim_{\preceq_i} x$ for all $i \in I$ with $x \notin \mathfrak{q}_{\preceq_i}$. Moreover, if $x \in \mathfrak{q}_{\preceq_i}$ for some $i \in I$, then $x^2 y \sim_{\preceq_i} 0 \sim_{\preceq_i} x$.

Conversely, suppose that $x^2 y \sim_{\preceq_i} x$ for all $i \in I$. Then it follows immediately from Corollary 1.61 that $xy \sim_{\preceq_i} 1$ for all $i \in I$ such that $x \notin \mathfrak{q}_{\preceq_i}$. □

Lemma 1.70. (cf. [32, Chapter II, Remark 3.4]) *Let $(\preceq_i : i \in I)$ be a family of Manis quasi-orderings on a ring R, and J a two-sided ideal of R such that $J \subseteq \mathfrak{q}_{\preceq_i}$ for all $i \in I$. The following are equivalent:*

(1) $(\preceq_i : i \in I)$ has the inverse property,

(2) $(\preceq_i' : i \in I)$ has the inverse property.

Proof. This follows immediately from Lemma 1.35 and Lemma 1.69, because we have $x^2 y \preceq_i x \Leftrightarrow \overline{x^2 y} \preceq_i' \overline{x}$ for all $x, y \in R$ and all $i \in I$. □

For the next result, we introduce a coarsening relation $\leq$ on the set of all quasi-orderings on a ring R. It will be the central object of Chapter 4, where we establish a tree structure theorem for the set of all quasi-orderings on R equipped with $\leq$.

Definition 1.71. Let R be a ring, and let $\preceq_1$ and $\preceq_2$ be quasi-orderings on R. We say that $\preceq_2$ is *coarser* than $\preceq_1$ (or $\preceq_1$ *finer* than $\preceq_2$), written $\preceq_1 \leq \preceq_2$, if

$$\forall x, y \in R \colon 0 \preceq_1 x \preceq_1 y \Rightarrow x \preceq_2 y.$$

Remark 1.72. Definition 1.71 unifies three different notions at once:

(1) either the ordering $\preceq_2$ contains the ordering $\preceq_1$,

(2) or the valuation $\preceq_2$ is strongly compatible with the ordering $\preceq_2$,

(3) or the valuation $\preceq_2$ is coarser than the valuation $\preceq_1$.

The details for (2) and (3) are worked out in Section 2.1. Because of (3), Corollary 1.74 below is a generalisation of [52, Proposition 9].

Lemma 1.73. *Let R be a ring and let $\preceq_1 \leq \preceq_2$ be quasi-orderings on R. If $x \sim_{\preceq_1} y$, then also $x \sim_{\preceq_2} y$ for all $x, y \in R$.*

Proof. If $0 \preceq_1 x \sim_{\preceq_1} y$, this is an immediate consequence of Definition 1.71. If $x \prec_1 0$, then we obtain $0 \preceq_1 -x \sim_{\preceq_1} -y$ by Lemma 1.29 and Corollary 1.60. Hence, $-x \sim_{\preceq_2} -y$. Again applying Corollary 1.60 yields $x \sim_{\preceq_2} y$. □

Corollary 1.74. *Let R be a ring and let $(\preceq_i : i \in I)$ and $(\preceq_j : j \in J)$ be families of Manis quasi-orderings on R such that*

(1) $(\preceq_i : i \in I)$ has the inverse property,

(2) For every $j \in J$ there is some $i \in I$ such that $\preceq_i \leq \preceq_j$.

Then $(\preceq_j : j \in J)$ has also the inverse property.

Proof. Let $x \in R$. By Lemma 1.69 we find some $y \in R$ such that $x^2y \sim_{\preceq_i} x$ for all $i \in I$. For $j \in J$ let $i_j \in I$ such that $\preceq_{i_j} \leq \preceq_j$. Then Lemma 1.73 tells us that $x^2y \sim_{\preceq_j} x$. Hence, it follows from Lemma 1.69 that $(\preceq_j : j \in J)$ also has the inverse property. □

Lastly, we consider the inverse property for families of Manis orderings.

Lemma 1.75. *Let R be a ring, $\leq_1$ and $\leq_2$ orderings on R with equal support, say $\mathfrak{q}$, and $x, y \in R$. Then $xy \sim_{\leq_1} 1$ if and only if $xy \sim_{\leq_2} 1$.*

Proof. We obtain

$$\begin{aligned} xy \sim_{\leq_1} 1 &\Leftrightarrow xy + c = 1 \text{ for some } c \in \mathfrak{q} \\ &\Leftrightarrow xy \sim_{\leq_2} 1. \end{aligned}$$

□

Lemma 1.76. *Let R be a ring and let $\leq_1, \leq_2$ be Manis orderings on R such that $\mathfrak{q}_{\leq_1} \neq \mathfrak{q}_{\leq_2}$. Then $\mathfrak{q}_{\leq_1}$ and $\mathfrak{q}_{\leq_2}$ are coprime.*

Proof. Since the supports are different, there exists (w.l.o.g.) some $x \in \mathfrak{q}_{\leq_1} \backslash \mathfrak{q}_{\leq_2}$. Hence, $\overline{x} \neq \overline{0}$ in $R/\mathfrak{q}_{\leq_2}$. By Theorem 1.65, we find some $\overline{y} \in R/\mathfrak{q}_{\leq_2}$ such that $\overline{xy} = \overline{1}$ in $R/\mathfrak{q}_{\leq_2}$. Thus, $xy + c = 1$ for some $c \in \mathfrak{q}_{\leq_2}$. Note that $xy \in \mathfrak{q}_{\leq_1}$, so we have $1 \in \mathfrak{q}_{\leq_1} + \mathfrak{q}_{\leq_2}$. Therefore, $\mathfrak{q}_{\leq_1} + \mathfrak{q}_{\leq_2} = R$, i.e. $\mathfrak{q}_{\leq_1}$ and $\mathfrak{q}_{\leq_2}$ are coprime. □

Proposition 1.77. *Let R be a ring and $\mathcal{F} = (\leq_i : i \in I)$ a family of Manis orderings on R. Then $\mathcal{F}$ has the inverse property.*

Proof. Let $x \in R$. By definition of the inverse property and Lemma 1.75 it suffices to consider the maximal subfamily $\mathcal{G}$ of $\mathcal{F}$ such that the Manis orderings in $\mathcal{G}$ have pairwise distinct supports not containing x, say $\mathcal{G} = (\leq_j : j \in J)$ for some $J \subseteq I$.

By Lemma 1.76, the supports of the elements in $\mathcal{G}$ are pairwise coprime. Moreover, the supports $\mathfrak{q}_{\leq_j}$ are two-sided ideals of R. Hence, the Chinese remainder theorem yields

$$R/\textstyle\bigcap_{j \in J} \mathfrak{q}_{\leq_j} \cong \displaystyle\prod_{j \in J} R/\mathfrak{q}_{\leq_j}.$$

By Theorem 1.65 and the fact that $x \notin \mathfrak{q}_{\leq_j}$ for all $j \in J$, it follows that $(\overline{x}^{\mathfrak{q}_{\leq_j}})_{j \in J}$ is invertible. Thus, $\overline{x}^{\bigcap_{j \in J} \mathfrak{q}_{\leq_j}}$ is also invertible, whence we find some $y \in R$ such that

$$\overline{xy}^{\bigcap_{j \in J} \mathfrak{q}_{\leq_j}} = \overline{1}^{\bigcap_{j \in J} \mathfrak{q}_{\leq_j}}.$$

Consequently, $xy + c = 1$ for some $c \in \bigcap_{j \in J} \mathfrak{q}_{\leq_j}$, which implies $xy \sim_{\preceq_j} 1$ for all $j \in J$. It follows that $\mathcal{G}$, and hence also $\mathcal{F}$, has the inverse property.

□

2. On the Interplay of Quasi-Orderings and Valuations

We have seen in Chapter 1 that a quasi-ordering on a ring is either an ordering or a valuation (Theorem 1.50). In the present chapter we give first explicit examples of how this yields a uniform treatment of ordered and valued rings. In Section 2.1 we introduce two different compatibility notions for quasi-orderings and valuations. Given a quasi-ordered ring $(R, \preceq)$, the first one gives rise to a characterisation of all valuations v on R such that $\preceq$ canonically induces a quasi-ordering on the residue class domain Rv of v. The second one unifies the usual notion of compatibility of orderings and valuations on the one hand (see Preliminaries below), and the coarsening relation of valuations on the other hand. In Section 2.2, we first introduce the natural valuation $v_{\preceq}$ of a quasi-ordering $\preceq$. Following the approach in [40], we then derive a uniform notion for the rank of a quasi-ordered ring.

Many of the results presented here were published by S. Kuhlmann and the author of this thesis in [41].

Preliminaries on the compatibility of orderings and valuations.

Here, we introduce our notations for the valuation ring, valuation ideal and residue class domain of a valued ring (R, v). They are not only used in this chapter, but also for the rest of this thesis. Moreover, given an ordered field $(K, \leq)$, we recall the characterisation of all $\leq$-compatible valuations on K (see Theorem 2.3).

Notation 2.1. Given a valued ring (R, v), we denote by $R_v := \{x \in R \colon v(x) \geq 0\}$ the *valuation ring* of v, and by $I_v := \{x \in R \colon v(x) > 0\}$ the *valuation ideal* of v. Moreover, we define $U_v := R_v \backslash I_v = \{x \in R \colon v(x) = 0\}$. Last but not least, we refer to $Rv := R_v/I_v$ as the *residue class domain* of v. The latter is justified by the following observation:

Lemma 2.2. *I_v is a two-sided completely prime ideal of R_v.*

Proof. The fact that I_v is a proper two-sided ideal of R_v is easily deduced from the axioms of a ring valuation. Furthermore, if $xy \in I_v$, then $0 < v(xy) = v(x) + v(y)$, and therefore $v(x) > 0$ or $v(y) > 0$. Hence, $x \in I_v$ or $y \in I_v$. □

In the theory of real valued fields, there is a well-established notion of compatibility between orderings and valuations - a valuation v on a field K is said to be *compatible* with an ordering $\leq$ on K, if

$$\forall x, y \in K \colon 0 \leq x \leq y \Rightarrow v(y) \leq v(x).$$

This condition can also be expressed in terms of the valuation ring K_v, the valuation ideal I_v, or the residue class field Kv, as the following characterisation shows (cf. e.g. [46, Theorem 2.3, Proposition 2.9], [62, Lemma 7.2], [18, Proposition 2.2.4]):

Theorem 2.3. *Let $(K, \leq)$ be an ordered field, and let v be a valuation on K. Then the following are equivalent:*

(1) v is compatible with $\leq$,

(2) K_v is convex,

(3) I_v is convex,

(4) $I_v < 1$,

(5) $\leq$ canonically induces an ordering $\leq^*$ on Kv via

$$x + I_v \leq^* y + I_v :\Leftrightarrow \exists c_1, c_2 \in I_v \colon x + c_1 \leq y + c_2.$$

We are particularly interested in the push-down of quasi-orderings, i.e. an analogue of condition (5). It will play an essential role in Chapter 3, where we prove a Baer-Krull theorem for quasi-ordered rings.

2.1. **Compatibility of Quasi-Orderings and Valuations.**

The main goal of this section is to establish an analogue of Theorem 2.3 for quasi-ordered rings. To begin with, we derive a compatibility notion for quasi-orderings and valuations on rings by taking the one for orderings and valuations on fields (see Preliminaries), and replacing the ordering $\leq$ with a quasi-ordering $\preceq$ (Definition 2.4). We deduce that if $\preceq$ is a valuation, then $\preceq$ is compatible with v if and only if v is a coarsening of $\preceq$ as valuations (Proposition 2.7).

Unfortunately, Theorem 2.3 utterly fails if we just replace the ordered field $(K, \leq)$ with a quasi-ordered ring $(R, \preceq)$, even if R is commutative (Example 2.9, Remark 2.18). In response, we first alter the definition of compatibility (Definition 2.10). This results in a characterisation of those quasi-orderings, which push down to the residue class domain of the valuation under consideration, i.e. in criteria of when condition (5) of Theorem 2.3 applies to quasi-orderings (Theorem 2.15). Afterward, we prove that the said theorem holds for quasi-ordered rings with the exception of condition (4), if we only consider Manis valuations (Theorem 2.19). Next, we show that for local valuations Theorem 2.3 applies to quasi-ordered rings in full generality (Proposition 2.25). A fortiori, we obtain that we may replace the ordered field $(K, \leq)$ in the said theorem with a quasi-ordered field $(K, \preceq)$ (Theorem 2.27).

Definition 2.4. Let $(R, \preceq)$ be a quasi-ordered ring and let v be a valuation on R. We say that v is *strongly compatible* (abbr. s-compatible) with $\preceq$, if

$$\forall x, y \in R\colon 0 \preceq x \preceq y \Rightarrow v(y) \leq v(x).$$

Remark 2.5. Strong compatibility is a specialisation of our coarsening relation from Definition 1.71, where the valuation v is replaced with a quasi-ordering.

If $\preceq = \preceq_w$ for some valuation w on R, then v is strongly compatible with $\preceq_w$ if and only if $w(x) \leq w(y)$ implies $v(x) \leq v(y)$ for all $x, y \in R$. This leads to the notion of coarsenings of valuations.

Definition 2.6. Let R be a ring, and let v and w be some valuations on R. Then v is said to be *coarser* than w (or w *finer* than v), written $w \leq v$, if there exists an order-preserving homomorphism $\varphi\colon \Gamma_w \to \Gamma_v$ such that $v = \varphi \circ w$.

Note that in the literature $w \leq v$ is often times understood as w *is coarser than* v (cf. [16], [24], [55], [61]), whereas other authors use our notation (cf. [32],[52]).

Proposition 2.7. *Let R be a ring, and let v and w be some valuations on R. The following are equivalent:*

(1) v is strongly $\preceq_w$-compatible,

(2) v is a coarsening of w.

Proof. We first show that (1) implies (2). The homomorphism $\varphi\colon \Gamma_w \to \Gamma_v$ from Definition 2.6 is uniquely determined by the equality $v(x) = \varphi(w(x))$ for all $x \in R$, since w is surjective. By (1), φ is well-defined and order-preserving. Moreover, $\varphi(0) = \varphi(w(1)) = v(1) = 0$ and

$$\begin{aligned}\varphi(x + y) &= \varphi(w(a) + w(b)) = \varphi(w(ab)) = v(ab) = v(a) + v(b) \\ &= \varphi(w(a)) + \varphi(w(b)) = \varphi(x) + \varphi(y),\end{aligned}$$

where $a, b \in R$ are chosen such that $w(a) = x$ and $w(b) = y$. Consequently, φ is an order-preserving homomorphism, whence v is coarser than w.

Conversely, if $w(x) \leq w(y)$, then $v(x) = \varphi(w(x)) \leq \varphi(w(y)) = v(y)$. $\square$

Example 2.8. Let $(R, \leq)$ be an ordered ring. We introduce the *natural valuation* of $\leq$, which we will need later on. For $x, y \in R$ define

$$x \sim^+ y :\Leftrightarrow \exists n \in \mathbb{N} \colon n|x| \geq |y| \text{ and } n|y| \geq |x|,$$

the so-called *Archimedean equivalence relation*, and

$$x <<^+ y :\Leftrightarrow \forall n \in \mathbb{N} \colon n|x| < |y|.$$

We equip $\Gamma := (R \backslash \mathfrak{q}_{\leq})/{\sim^+}$ with the addition $[x] + [y] := [xy]$ and the ordering $[x] < [y] :\Leftrightarrow y <<^+ x$, where $[x]$ denotes the equivalence class of x with respect to $\sim^+$. It is easy to verify that the map $v \colon R \mapsto (\Gamma, +, <)$,

$$x \mapsto \begin{cases} [x], & x \in R \backslash \mathfrak{q}_{\leq} \\ \infty, & \text{else} \end{cases}$$

defines a valuation on R with support $\mathfrak{q}_{\leq}$.

Its valuation ring $R_v = \{x \in R \colon x \sim^+ 1 \text{ or } x <<^+ 1\}$ is the convex hull of $\mathbb{Z}$ in R, and its valuation ideal is given by $I_v = \{x \in R \colon x <<^+ 1\}$.

The valuation v is s-compatible with $\leq$. If $x, y \in R$ with $0 < x \leq y$, then $x <<^+ y$ or $x \sim^+ y$. Hence, $[y] \leq [x]$, and therefore $v(y) \leq v(x)$. It is the finest such valuation on R. Indeed, let w be another strongly $\leq$-compatible valuation on R, and suppose that $v(x) \leq v(y)$ for some $x, y \in R$, w.l.o.g. $0 \leq x, y$. Since $v(x) \leq v(y)$, we find some $n \in \mathbb{N}$ such that $0 \leq y \leq nx$. By s-compatibility of w with $\leq$, this implies $w(x) = w(nx) \leq w(y)$. Therefore, w is a coarsening of v.

Convexity with respect to some quasi-ordering was already defined in Section 1.4 (see Definition 1.56(2)), whence we may consider Theorem 2.3 for quasi-ordered rings. As the following examples show, it fails if we replace the field K with a ring R and the ordering $\leq$ with a quasi-ordering $\preceq$, even if R is commutative.

Example 2.9.

(i) Let $(R, \leq)$ be some ordered ring. We equip the polynomial ring $R[X]$ with the ordering $0 \leq f :\Leftrightarrow 0 \leq f(0)$ and the valuation $v(f) = -\deg(f)$. Then $R_v = R$ and $I_v = \{0\}$, so obviously the conditions (4) and (5) of Theorem 2.3 are fulfilled. However, the inequalities $0 \leq X \leq 0$ imply that neither I_v nor R_v is convex with respect to $\leq$. Moreover, we have $0 \leq X + 1 \leq 1$, but $v(X+1) = -1 < 0 = v(1)$, so (1) is also not satisfied.

(ii) Let p be a prime number and v the p-adic valuation on $\mathbb{Z}$. Moreover, let $\leq$ be the unique ordering on $\mathbb{Z}$. Then $R_v = \mathbb{Z}$ is convex, so (2) holds. However, it is easy to see that all the other conditions of Theorem 2.3 are not satisfied.

(iii) For $0 \neq f = \sum_i a_i X^i \in \mathbb{Z}[X]$ consider the valuations v, w on $\mathbb{Z}[X]$ given by $v(f) = -\deg(f)$, respectively $w(f) = \min\{i \colon a_i \neq 0\}$ (see Proposition 1.16) Then (1) and (2) are false, whereas (3) - (5) are fulfilled.

Hence, Theorem 2.3 fails for (commutative) quasi-ordered rings, no matter if the quasi-ordering is an ordering or a valuation. In fact, none of these conditions are equivalent anymore for quasi-ordered rings. We will give an complete overview in Remark 2.18.

In what follows, we introduce a weaker compatibility notion, which is necessary and sufficient for a quasi-ordering on R to canonically induce a quasi-ordering on the residue class domain Rv of a $\preceq$-compatible valuation v on R. Thereafter, in order to establish an analogue of Theorem 2.3 for quasi-ordered rings, we strengthen the assumption on v by demanding that it is a Manis valuation.

Definition 2.10. Let $(R, \preceq)$ be a quasi-ordered ring, and let v be a valuation on R. Then v is said to be *compatible* with $\preceq$, if I_v is a convex subset of R_v, i.e. if $0 \preceq x \preceq y \in I_v$ implies $x \in I_v$ for all $x, y \in R_v$.

Remark 2.11. If R is a ring and if v and w are valuations on R, then v is $\preceq_w$-compatible if and only if $w(x) \geq w(y)$, and $v(y) > 0$, and $v(x) \geq 0$ all together imply $v(x) > 0$. From this, we may easily derive that $I_v \subseteq I_w$.

Proof. Assume that there is some $y \in I_v \backslash I_w$. Then $w(1) \geq w(y)$, and $v(y) > 0$, and $v(1) \geq 0$. Hence, $v(1) > 0$, a contradiction. Thus, $I_v \subseteq I_w$. □

Example 2.12. Let $(R, \leq)$ be some ordered ring with support $\mathfrak{q}_{\leq} = \{0\}$. We extend $\leq$ to $R[X]$ by declaring $0 \neq \sum_i a_i X^i > 0 :\Leftrightarrow a_k > 0$, where $k := \min\{j \colon a_j \neq 0\}$. Moreover, we equip $R[X]$ with the valuation $v(f) = -\deg(f)$. Then v is not strongly $\leq$-compatible, since $0 \leq X \leq 1$, but $v(X) = -1 < 0 = v(1)$. However, v is $\leq$-compatible, because $0 \leq f \leq g \in I_v = \{0\}$ implies $f = 0 \in I_v$.

Lemma 2.13. *Let $(R, \preceq)$ be a quasi-ordered ring.*

(1) Any strongly $\preceq$-compatible valuation is $\preceq$-compatible. The converse of this statement is false.

(2) If v is a Manis valuation on R, then v is compatible with $\preceq$ if and only if it is strongly compatible with $\preceq$.

Proof. We first prove (1). So let v denote a strongly $\preceq$-compatible valuation on R, and suppose that $0 \preceq x \preceq y \in I_v$. Then $0 < v(y) \leq v(x)$, and therefore also $x \in I_v$. Hence, v is $\preceq$-compatible. In Example 2.12 we have shown that not any $\preceq$-compatible valuation is strongly $\preceq$-compatible.

Now let v be a Manis valuation that is $\preceq$-compatible, and suppose $0 \preceq x \preceq y$. We have to show that $v(y) \leq v(x)$. So we may assume that $x \notin \mathfrak{q}_v$. Since v is Manis and $x \notin \mathfrak{q}_v$, we find some $a \in R$, w.l.o.g. $0 \preceq a$ (Lemma 1.29), such that $v(a) = -v(x)$. Via (QR2), we obtain $0 \preceq ax \preceq ay$ with $ax \in U_v \subseteq R_v$. If $ay \in I_v$, then the $\preceq$-compatibility of v yields $ax \in I_v$, a contradiction. Hence, $ay \notin I_v$, i.e. $v(ay) \leq v(ax)$. By cancellation in Γ_v, this implies $v(y) \leq v(x)$. □

Also note that if $(R, \preceq)$ is a quasi-ordered ring and v a strongly $\preceq$-compatible valuation on R, then $\mathfrak{q}_{\preceq} \subseteq \mathfrak{q}_v$. If v is just $\preceq$-compatible, this is not necessarily true (see Example 2.12).

Our next aim is to give a characterisation of all valuations v on R, such that v is $\preceq$-compatible for some quasi-ordering $\preceq$. This will be exactly these valuations, for which $\preceq$ canonically induces a quasi-ordering on the residue class domain Rv.

Lemma 2.14. *Let $(R, \preceq)$ be a quasi-ordered ring, v a $\preceq$-compatible valuation and $u \in U_v$.*

(1) If $c \in I_v$, then $c \not\sim u$.

(2) If $0 \prec u$, then $0 \prec u + c$ for all $c \in I_v$.

(3) If $u \prec 0$, then $u + c \prec 0$ for all $c \in I_v$.

Proof. For (1), assume that $c \sim u$ for some $c \in I_v$, w.l.o.g. $0 \prec u$. Then we know that $0 \preceq u \preceq c \in I_v$ and $u \in R_v$, whence the $\preceq$-compatibility of v yields $u \in I_v$, a contradiction. Thus, $c \not\sim u$ for all $c \in I_v$.

Next, we show that (2) holds. So let $0 \prec u$, and assume that $u + c \preceq 0$ for some $c \in I_v$. This implies $c \notin q_{\preceq}$, as otherwise $u \sim u+c$ (Lemma 1.28). Hence, we obtain $u \preceq -c$ by (QR4). Therefore, $0 \prec u \preceq -c$. The $\preceq$-compatibility of v implies $u \in I_v$, a contradiction. So we have $0 \prec u + c$.

Finally, we prove (3) by contraposition. Suppose that $0 \preceq u + c$ for some $c \in I_v$. By Lemma 1.14 we know that $u + c \in U_v$, whence (1) yields $0 \prec u + c$. So it follows from (2) that $0 \prec (u + c) - c = u$. □

In the following theorem, the equivalence of (1) and (4) is our main interest. It was previously shown for fields in a non-uniform way by Efrat ([16, Proposition 7.2.1(a)]).

Theorem 2.15. *Let $(R, \preceq)$ be a quasi-ordered ring and let v be a valuation on R. The following are equivalent:*

(1) v is $\preceq$-compatible,

(2) $I_v^{\succeq 0}$ is an initial segment of $R_v^{\succeq 0}$,

(3) $U_v^{\succeq 0}$ is a final segment of R_v,

(4) $\preceq$ canonically induces a quasi-ordering $\preceq^*$ on Rv via

$$x + I_v \preceq^* y + I_v :\Leftrightarrow \exists c_1, c_2 \in I_v \colon x + c_1 \preceq y + c_2.$$

Proof. The equivalence of (1), (2) and (3) is clear by definition of $\preceq$-compatibility, the fact that $R_v^{\succeq 0}$ is the disjoint union of $U_v^{\succeq 0}$ and $I_v^{\succeq 0}$, and Lemma 2.14(1).

Next, we show that (4) implies (1). So let $0 \preceq x \preceq y \in I_v$ with $x \in R_v$. Then $\overline{0} \preceq^* \overline{x} \preceq^* \overline{y} = \overline{0}$. Since the support of $\preceq^*$ is trivial, this yields $\overline{x} = \overline{0}$, and therefore $x \in I_v$.

Finally, we prove that (1) implies (4). First of all we verify that $\preceq^*$ is well-defined. So suppose that $\overline{x} \preceq^* \overline{y}$, and let $\overline{x} = \overline{x_1}$ and $\overline{y} = \overline{y_1}$, say $x = x_1 + c_1$ and $y = y_1 + c_2$ for some $c_1, c_2 \in I_v$. Then there exist some $c_3, c_4 \in I_v$ such that $x + c_3 \preceq y + c_4$. But then also $x_1 + (c_1 + c_3) \preceq y_1 + (c_2 + c_4)$, so we get $\overline{x_1} \preceq^* \overline{y_1}$.

Evidently, $\preceq^*$ is reflexive and total. Next, we show transitivity. So suppose that $\overline{x} \preceq^* \overline{y}$ and $\overline{y} \preceq^* \overline{z}$, i.e. $x + c_1 \preceq y + c_2$ and $y + d_1 \preceq z + d_2$ for some $c_1, c_2, d_1, d_2 \in I_v$. If $y \in U_v$, then $-c_2 + d_1 \not\sim y + c_2$ by Lemma 2.14(1). So from $x + c_1 \preceq y + c_2$ it follows via (QR4) that

$$x + c_1 - c_2 + d_1 \preceq y + c_2 - c_2 + d_1 = y + d_1 \preceq z + d_2,$$

and therefore $\overline{x} \preceq^* \overline{z}$. If $y \in I_v$, then $y + c_2 \in I_v$ and $y + d_1 \in I_v$. By Lemma 2.14, this means that x is either in $U_v^{\prec 0}$ or in I_v, and that z is either in $U_v^{\succ 0}$ or in I_v. We only consider the case where both elements are in I_v, the other ones being trivial. So let $x, z \in I_v$. Then $x + (z - x) \preceq z + 0$, and therefore $\overline{x} \preceq^* \overline{z}$.

Now we prove that $\mathfrak{q}_{\preceq^*} = \{\overline{0}\}$. So let $\overline{x} \sim \overline{0}$. Then there exist $c_1, c_2, d_1, d_2 \in I_v$ such that $x + c_1 \preceq c_2$ and $d_1 \preceq x + d_2$. Assume that $x \in R_v \setminus I_v = U_v$. If $0 \prec x$, then Lemma 2.14 yields $0 \prec x + c_1 - c_2$, whence $c_2 \prec x + c_1$, a contradiction to $x + c_1 \preceq c_2$. Likewise, if $x \prec 0$, then Lemma 2.14 yields $x + d_2 - d_1 \prec 0$, whence $x + d_2 \prec d_1$, a contradiction to $d_1 \preceq x + d_2$. Therefore, $x \in I_v$, i.e. $\overline{x} = \overline{0}$. Thus, $\mathfrak{q}_{\preceq^*} = \{\overline{0}\}$.

It remains to verify the axioms (QR1) - (QR4). Obviously, (QR1) is fulfilled. For (QR2) we only show that $\overline{x} \preceq^* \overline{y}$ and $\overline{0} \preceq^* \overline{z}$ implies $\overline{xz} \preceq^* \overline{yz}$, the proof of left-multiplication being analogue. We may assume that $z \in U_v$. So Lemma 2.14 implies $0 \preceq z$. Moreover, $\overline{x} \preceq^* \overline{y}$ means that $x + d_1 \preceq y + d_2$ for some $d_1, d_2 \in I_v$. So (QR2) yields $xz + d_1 z \preceq yz + d_2 z$, and therefore $\overline{xz} \preceq^* \overline{yz}$.

For (QR3), we only prove right-cancellation, the proof of left-cancellation follows due to symmetry reasons. So let $\overline{xz} \preceq^* \overline{yz}$ and $\overline{0} \prec^* \overline{z}$. We have to show that $\overline{x} \preceq^* \overline{y}$. By Lemma 2.14, we know that $0 \prec z$. So if $xz \preceq yz$, then $x \preceq y$ by (QR3), whence $\overline{x} \preceq^* \overline{y}$. So we may assume that $yz \prec xz$. Further note that $z \in U_v$. Next, we show that we may also assume that $x, y \in U_v$.

First suppose that $\overline{x} = \overline{0}$ and $\overline{y} \prec^* \overline{0}$. Then $\overline{yz} \preceq^* \overline{0}$ by (QR3). But equality cannot hold, because neither $z \in I_v$, nor $y \in I_v$. Thus, $\overline{yz} \prec^* \overline{0} = \overline{xz}$, a contradiction to the assumption $\overline{xz} \preceq^* \overline{yz}$. Likewise, if $\overline{y} = \overline{0}$ and $\overline{0} \prec^* \overline{x}$, then $\overline{yz} = \overline{0} \prec \overline{xz}$, again a contradiction. Hence, suppose from now on that $x, y \in U_v$.

Since $x, y, z \in U_v$, also $xz \in U_v$ and $yz \in U_v$. Hence, it follows from $\overline{xz} \preceq^* \overline{yz}$ that there is some $c \in I_v$ such that $xz \preceq yz + c$. So we have $yz \prec xz \preceq yz + c$. The rest of the proof is done by case distinction.

If $0 \prec -1$, then all elements are non-negative by Lemma 1.45. Since yz is in U_v and $I_v \subseteq R_v$ is convex, this implies $c \prec yz$ (otherwise, $0 \prec yz \preceq c \in I_v$ yields $yz \in I_v$). From Lemma 1.48, it follows that $yz \prec yz+c \preceq \max\{yz+c\} = yz$, a contradiction. Therefore, $\overline{x} \preceq^* \overline{y}$.

Finally suppose that $-1 \prec 0$. Consider the inequalities $yz \prec xz \preceq yz + c$. By Lemma 2.14(2) and (3), yz, $yz + c$, and therefore also xz all have the same sign, which is contrary to the sign of $-yz$. So we may add $-yz$ to these two inequalities and obtain $0 \preceq (x - y)z \preceq c$. Since v is $\preceq$-compatible, this yields $(x - y)z \in I_v$. Thus, the facts that I_v is completely prime and $z \notin I_v$ imply $\overline{x} = \overline{y}$, This finishes the proof of (QR3).

Last but not least, we verify axiom (QR4). So let $\overline{x} \preceq^* \overline{y}$ and $\overline{z} \not\sim \overline{y}$. Then we find some $c_1, c_2 \in I_v$ such that $x + c_1 \preceq y + c_2$. Moreover, $\overline{y} \not\sim \overline{z}$ implies that either $y + e_1 \prec z + e_2$ for all $e_1, e_2 \in I_v$ or $z + e_1 \prec y + e_2$ for all $e_1, e_2 \in I_v$. Either way, $z \not\sim y + c_2$. But then $x + z + c_1 \preceq y + z + c_2$ by (QR4), whence $\overline{x + z} \preceq^* \overline{y + z}$. $\square$

As a supplement to Theorem 2.15(4), we obtain:

Proposition 2.16. *Let $(R, \preceq)$ be a quasi-ordered ring and v a valuation on R such that v is $\preceq$-compatible.*

(1) If $\preceq$ is an ordering, then $\preceq^*$ is an ordering.

(2) If $\preceq$ is a valuation, then $\preceq^*$ is a valuation. More precisely, if $\preceq = \preceq_w$, then $\preceq^*$ is the quotient valuation (cf. e.g. [16, p.56], [18, p.45])

$$w/v\colon Rv \to \Gamma_{w/v},\ a + I_v \mapsto \begin{cases} w(a) & \text{if } a \in U_v, \\ \infty & \text{if } a \in I_v. \end{cases}$$

Proof. First suppose that $\preceq$ is an ordering. It suffices to show that $\overline{x} \preceq^* \overline{y}$ implies $\overline{x + z} \preceq^* \overline{y + z}$. From $\overline{x} \preceq^* \overline{y}$, it follows $x + c_1 \preceq y + c_2$ for some $c_1, c_2 \in I_v$. Since $\preceq$ is an ordering, we get $x + z + c_1 \preceq y + z + c_2$. Thus, $\overline{x + z} \preceq^* \overline{y + z}$.

Now suppose that $\preceq = \preceq_w$ for some valuation w on R. We first verify that w/v is well-defined. So let $a \in U_v$ and $c \in I_v$. We have to prove that $w(a) = w(a + c)$. By Lemma 1.14, it suffices to show that $w(a) < w(c)$. If not, then $w(a) \geq w(c)$ with $c \in I_v$ and $a \in R_v$. So the $\preceq_w$-compatibility of v implies $a \in I_v$, a contradiction. Hence, $w(a) < w(c)$, and therefore $w(a) = w(a + c)$.

It is easy to see that w/v satisfies the axioms (VR1) and (VR2) of Definition 1.4, and also that $\Gamma_{w/v}$ is cancellative, as $\Gamma_w \supseteq \Gamma_{w/v}$ has this property. For (VR3) note that $ab \in I_v$ if and only if $a \in I_v$ or $b \in I_v$, since I_v is a completely prime ideal of R_v (see Lemma 2.2). Thus, $w/v(ab+I_v) = \infty$ if and only if $w/v(a+I_v)+w/v(b+I_v) = \infty$. From this observation (VR3) is easily deduced. The proof of (VR4) is done by a similar case distinction. Therefore, w/v defines a valuation on Rv. Its support is $\{0\}$, since $x \in R_v \cap \mathfrak{q}_w$ implies $0 \preceq_w x \preceq_w 0$ with $x \in R_v$, whence $x \in I_v$ by the $\preceq_w$-compatibility of v. Moreover, for $x, y \in U_v$ (i.e. $\overline{x}, \overline{y} \neq \overline{0}$) we have

$$\begin{aligned} \overline{x} \preceq_w^* \overline{y} &\Leftrightarrow x + c_1 \preceq_w y + c_2 \text{ for some } c_1, c_2 \in I_v \\ &\Leftrightarrow w(y + c_2) \leq w(x + c_1) \text{ for some } c_1, c_2 \in I_v \\ &\Leftrightarrow w(y) \leq w(x) \\ &\Leftrightarrow w/v(\overline{y}) \leq w/v(\overline{x}), \end{aligned}$$

where the third equivalence follows exactly as in the proof of the well-definition of w/v above. $\square$

We can say more about the relationship of quasi-orderings on R and quasi-orderings on Rv. Recall that if $\preceq_1$ and $\preceq_2$ are quasi-orderings on a ring R, then $\preceq_2$ is said to be coarser than $\preceq_1$, written $\preceq_1 \leq \preceq_2$, if $0 \preceq_1 x \preceq_1 y$ implies $x \preceq_2 y$ for all $x, y \in R$ (see Definition 1.71).

Proposition 2.17. (cf. [16, Proposition 7.2.1(b)]) *Let R be a ring, let $\preceq_1, \preceq_2$ be quasi-orderings on R, and let v be a valuation on R which is compatible with $\preceq_1$ and $\preceq_2$. If $\preceq_1 \leq \preceq_2$ in R, then $\preceq_1^* \leq \preceq_2^*$ in Rv.*

Proof. Suppose that $\overline{0} \preceq_1^* \overline{x} \preceq_1^* \overline{y}$ for some $\overline{x}, \overline{y} \in Rv$. Then we find $c_1, c_2, c_3, c_4 \in I_v$ such that $c_1 \preceq_1 x + c_2$ and $x + c_3 \preceq_1 y + c_4$. We have to show that $\overline{x} \preceq_2^* \overline{y}$.

If $x \in U_v$, then $c_1 \preceq_1 x+c_2$ implies $0 \preceq_1 x+(c_2-c_1) \in U_v$ by Lemma 2.14(1). Hence, by Lemma 2.14(2), $0 \preceq_1 x + c$ for all $c \in I_v$. In particular, $0 \preceq_1 x + c_3 \preceq_1 y + c_4$. Since $\preceq_1 \leq \preceq_2$, this implies $x + c_3 \preceq_2 y + c_4$, and therefore $\overline{x} \preceq_2^* \overline{y}$.

If $x \in I_v$, then we may w.l.o.g. assume that $y \in U_v$. So from $x + c_3 \preceq_1 y + c_4$, it follows, like above, that $0 \preceq_1 0 \preceq_1 y + c_4$. Therefore, $\preceq_1 \leq \preceq_2$ implies that $0 \preceq_2 y + c_4$. Hence, $\overline{x} = \overline{0} \preceq_2^* \overline{y}$. ☐

Remark 2.18. In Example 2.9 we have seen that Theorem 2.3 does not hold if we replace the ordered field $(K, \leq)$ with a quasi-ordered ring $(R, \preceq)$ - not even when R is commutative and $\preceq$ is either an ordering or a valuation. In fact, none of the conditions from this theorem are equivalent anymore for a quasi-ordered ring $(R, \preceq)$. What we do get is the following tabular of non-trivial implications, where

(1) v is strongly compatible with $\preceq$,
(2) R_v is a convex subset of R,
(3) I_v is a convex subset of R,
(4) $I_v \prec 1$,
(5) $\preceq$ canonically induces a quasi-ordering $\preceq^*$ on Rv via

$$x + I_v \preceq^* y + I_v :\Leftrightarrow \exists c_1, c_2 \in I_v \colon x + c_1 \preceq y + c_2.$$

$\Rightarrow$	(1)	(2)	(3)	(4)	(5)
(1)		✓	✓	✓	✓
(2)	×		×	×	×
(3)	×	×		✓	✓
(4)	×	×	×		×
(5)	×	×	×	✓	

Proof. We first show that (1) implies (2) and (3). So let $0 \preceq y \preceq z$ with $z \in R_v$. Then (1) yields $0 \leq v(z) \leq v(y)$, whence $y \in R_v$. The same arguments work for I_v instead of R_v. In order to prove that (1) implies (4) and (5), it suffices to show that (3) does so. Condition (3) obviously yields (4). Moreover, it implies that I_v is a convex subset of R_v, i.e. that v is $\preceq$-compatible. Hence, (5) follows from (3) by Theorem 2.15. However, it does not imply (1) and (2), see Example 2.9(iii).

In Example 2.9(ii) we have seen that (2) implies none of the other conditions. The same holds for (4), even if v is Manis, as we will see in Example 2.22 and 2.23. Next, we show that (5) implies (4). So assume that $1 \preceq x$ for some $x \in I_v$. Then $\overline{1} \preceq^* \overline{x} = \overline{0}$, but for quasi-ordered rings (QR1) demands that $0 \prec 1$, a contradiction. Thus, $I_v \prec 1$. The fact that (5) does not imply any of the other conditions follows from Example 2.9(i). ☐

However, with the additional assumption that v is Manis, we get very close to an analogue of Theorem 2.3.

Theorem 2.19. *Let $(R, \preceq)$ be a quasi-ordered ring and let v be a Manis valuation on R.*

(a) The following are equivalent:

(1) v is compatible with $\preceq$,

(2) v is strongly compatible with $\preceq$,

(3) I_v is a convex subset of R,

(4) $\preceq$ canonically induces a quasi-ordering $\preceq^*$ on Rv via

$$x + I_v \preceq^* y + I_v :\Leftrightarrow \exists c_1, c_2 \in I_v \colon x + c_1 \preceq y + c_2.$$

Moreover, any of these conditions implies that R_v is convex subset of R.

(b) If v is non-trivial, then

(5) R_v is a convex subset of R

is equivalent to the conditions $(1) - (4)$*.*

Proof. We have already seen that (1), (2) and (4) are equivalent (Lemma 2.13(2) and Theorem 2.15). Moreover, clearly (3) implies (1). Next, we show that (2) implies (3). So let $0 \preceq x \preceq y \in I_v$. Then $0 < v(y) \leq v(x)$ by (2), and therefore $x \in I_v$. Hence, the assertions (1) - (4) are equivalent. We conclude part (a) by proving that (2) implies that R_v is a convex subset of R. Again, $0 \preceq x \preceq y \in R_v$ implies $0 \leq v(y) \leq v(x)$, and therefore $x \in R_v$.

In order to prove that (b) is fulfilled, it suffices to show that (5) implies (1). So let $0 \preceq x \preceq y \in I_v$ and $x \in R_v$. Assume that $x \in U_v$. We choose some $z \in R^{\succeq 0}$ such that $v(z) = -v(y) < 0$. Then $0 \preceq xz \preceq yz$ with $v(xz) < 0$ and $v(yz) = 0$, i.e. $xz \notin R_v$ and $yz \in R_v$. However, (5) yields $xz \in R_v$, a contradiction. Thus, $x \in R_v \backslash U_v = I_v$. □

Remark 2.20.

(1) The assumption in (b) that v is non-trivial is crucial, no matter which kind of a quasi-ordering $\preceq$ is.

In the ordered case consider $\mathbb{Z}$ with its unique ordering and the trivial valuation v mapping the even integers to ∞ and the odd integers to 0. Then $R_v = \mathbb{Z}$ is a convex subset of $\mathbb{Z}$, whereas $I_v = 2\mathbb{Z}$ is not.

In the valued case consider again $\mathbb{Z}$ with the valuation v as before, and let w be the p-adic valuation on $\mathbb{Z}$ for some prime number $p > 2$. Then $R_v = \mathbb{Z}$ is again obviously a convex subset of itself. However, we obtain $0 = w(2) \leq w(1) = 0$, but $v(1) = 0 < v(2)$, so (2) is not satisfied.

(2) Instead of v being non-trivial we may as well demand that $\mathfrak{q}_v = \mathfrak{q}_\preceq$ in part (b), i.e. that the supports of v and $\preceq$ coincide. In that case if $y \in \mathfrak{q}_v$, then also $x \in \mathfrak{q}_v \subseteq I_v$. Otherwise, if $y \notin \mathfrak{q}_v$, we may argue just as in the proof of Theorem 2.19(b).

While the condition on the supports is consistent with the field case (cf. Theorem 2.3), the assumption that v is non-trivial seems less restraining.

(3) $I_v \prec 1$ (see Theorem 2.3) is an easy consequence of the conditions stated in Theorem 2.19. It follows for instance immediately from (3).

Proposition 2.7 and Theorem 2.19 yield a characterisation of all Manis valuations on a ring, which are coarser than a given valuation.

Proposition 2.21. *Let R be a ring and let v and w be valuations on R such that v is non-trivial and Manis. Then v is a coarsening of w if and only if one of the following equivalent conditions is satisfied for all $x, y \in R$:*

(1) $w(x) \leq w(y) \Rightarrow v(x) \leq v(y)$,

(2) $w(x) \leq w(y) \wedge 0 \leq v(x) \Rightarrow 0 \leq v(y)$,

(3) $w(x) \leq w(y) \wedge 0 < v(x) \Rightarrow 0 < v(y)$,

(4) $w/v\colon Rv \to \Gamma_{w/v} \cup \{\infty\},\ x + I_v \mapsto \begin{cases} \infty & \text{if } x \in I_v \\ w(x) & \text{else} \end{cases}$ *defines a valuation with support* $\{0\}$.

Proof. This follows immediately from Proposition 2.7 and Theorem 2.19(2) - (5) in the case where the quasi-ordering $\preceq$ comes from a valuation w on R. Moreover, Proposition 2.16(2) was used for the explicit representation of w/v in (4). $\square$

The following examples show that $I_v \prec 1$ is not equivalent to all the other conditions of Theorem 2.19, regardless of whether $\preceq$ is a valuation (Example 2.22) or an ordering (Example 2.23), even if v is a non-trivial Manis valuation.

Example 2.22. Let $v_p\colon \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}$ denote the p-adic valuation for some prime number $p \in \mathbb{N}$. Apply Proposition 1.16 with $\gamma = 0$ to extend v_p to a valuation $v\colon \mathbb{Q}[X] \to \mathbb{Z} \cup \{\infty\}$. The valuation v is Manis, since v_p is Manis and $\Gamma_v = \Gamma_{v_p}$. We do the same procedure with w instead of v, except that this time $\gamma = 1$. Then $v = w$ on $\mathbb{Q}$, and $w(f) = v(f) + i$ for some $i \geq 0$ if $f \in \mathbb{Q}[X] \backslash \mathbb{Q}$. This implies $I_v \prec_w 1$. Indeed, $f \in I_v$ means $v(f) > 0$. But then also $w(f) > 0 = w(1)$, and therefore $f \prec_w 1$. However, v is not s-compatible with $\preceq_w$. For instance we have $v(X^2) = v(X)$, but $w(X) < w(X^2)$.

Example 2.23. Let v be the trivial valuation on $\mathbb{Z}$ with support $\{0\}$, i.e. $v(x) = 0$ for all $x \neq 0$. Extend v via Proposition 1.16 to a valuation on $\mathbb{Z}[X, Y]$ with $\gamma = 1$ for X and $\gamma = -1$ for Y. So for any $0 \neq f = \sum_{i,j} a_{ij} X^i Y^j \in \mathbb{Z}[X, Y]$, we have $v(f) = \min\{i - j\colon a_{ij} \neq 0\}$. Then v is a Manis valuation with value group $\mathbb{Z}$, since evidently $v(X^m) = m$ or $v(Y^{-m}) = m$ for any $m \in \mathbb{Z}$. Order $\mathbb{Z}[X, Y]$ by declaring $f \geq 0 :\Leftrightarrow f(0) \geq 0$. Then $v(f) \leq 0$, if $a_{00} \neq 0$. Therefore, $I_v \subseteq \langle X, Y\rangle = \mathfrak{q}_{\leq}$, so $I_v < 1$. However, I_v is not a convex subset of R, since $0 \leq Y \leq 0$, but $Y \notin I_v$.

We conclude this section by imposing a suitable extra condition on v, such that $I_v \prec 1$ becomes equivalent to the conditions (1) - (4) of Theorem 2.19.

Definition 2.24. (cf. [24], [35, Ch. I, Definition 5]) Let R be a commutative ring. A valuation v on R is called *local*, if the pair (R_v, I_v) is local, i.e. if I_v is the unique maximal ideal of R_v.

A characterisation of local valuations is given in [35, Ch. I, Proposition 1.3] and [24, Proposition 5], respectively. Valuations on fields are always local Manis valuations.

Proposition 2.25. *Let $(R, \preceq)$ be a commutative quasi-ordered ring and v a local Manis valuation on R. The following are equivalent:*

(1) *v is compatible with* $\preceq$,

(2) $I_v \prec 1$.

Proof. Clearly, (1) implies (2). Now suppose that (2) holds and let $0 \preceq x \preceq y \in I_v$ with $x \in R_v$. If $x \notin I_v$, then x is a unit in R_v. Moreover, $0 \preceq x$ implies that also $0 \preceq x^{-1}$. Hence, $x \preceq y$ yields $1 \preceq yx^{-1}$ and $v(yx^{-1}) = v(y) > 0$, i.e. $yx^{-1} \in I_v$. This is a contradiction to (2). Consequently, $x \in I_v$, so v is $\preceq$-compatible. $\square$

Corollary 2.26. *Let v, w be valuations on a commutative ring R such that v is Manis and local. Then v is coarser than w if and only if $I_v \subseteq I_w$.*

Proof. This is an immediate consequence of Proposition 2.7 and Proposition 2.25 in the case where $\preceq \, = \, \preceq_w$. □

By what we have shown so far, we obtain that Theorem 2.3 applies in full generality to quasi-ordered fields.

Theorem 2.27. (cf. [40, Theorem 2.2]) *Let $(K, \preceq)$ be a quasi-ordered field and v a valuation on K. Then the following are equivalent:*

(1) v is compatible with $\preceq$,

(2) K_v is convex,

(3) I_v is convex,

(4) $I_v \prec 1$,

(5) $\preceq$ canonically induces a quasi-ordering $\preceq^*$ on Kv via

$$x + I_v \preceq^* y + I_v :\Leftrightarrow \exists c_1, c_2 \in I_v \colon x + c_1 \preceq y + c_2.$$

Proof. This is an immediate consequence of Theorem 2.19, Remark 2.20(2), and Proposition 2.25, since any field valuation is Manis and local. □

2.2. **The Rank of a Quasi-Ordered Ring.**

In this section we give a uniform definition of the rank of a quasi-ordered ring. To this end, we first define the natural valuation $v_\preceq$ of a quasi-ordering $\preceq$ to be the finest valuation of support $\mathfrak{q}_\preceq$ that is strongly $\preceq$-compatible (Definition 2.32). The rank of $(R, \preceq)$ is then defined as the order type of the set of all strict coarsenings of $v_\preceq$ (Definition 2.34). We show that this definition is consistent with the rank of a quasi-ordered field as developed in [40] (Corollary 2.36).

Simply said, the rank serves as a measure for the size of a given structure. The rank of a valued field (K, v) is the order type of all non-trivial valuation rings K_w such that w is coarser than v, i.e. the order type of all non-trivial coarsenings of v. The rank of an ordered field $(K, \leq)$ is the order type of all non-trivial valuation rings K_w such that w is $\leq$-compatible, or equivalently the order type of all non-trivial convex subrings of K with respect to $\leq$. In other words, the rank of $(K, \leq)$ is the rank of (K, v), where v denotes the natural valuation induced by $\leq$.

In [40], S. Kuhlmann, Matusinki and Point established Theorem 2.27 by exploiting Fakhruddin's dichotomy, and derived a definition of the rank of a quasi-ordered field as follows:

Proposition 2.28. *Let $(K, \preceq)$ be a quasi-ordered field. Then the $\preceq$-compatible valuations on K are linearly ordered by the coarsening relation $\leq$.*

Proof. If v and w are $\preceq$-compatible, then Theorem 2.27(2) yields $K_v \subseteq K_w$ or vice versa, whence $v \leq w$ or $w \leq v$. □

Hence, to any quasi-ordering $\preceq$ on K there is a finest valuation $v_\preceq$ on K that is $\preceq$-compatible. If $\preceq = v$ is a valuation, then $v = v_\preceq$. If $\preceq = \leq$ is an ordering, then $v_\preceq$ is the natural valuation of $\leq$. Either way, the rank of $(K, v_\preceq)$ coincides with the rank of (K, v), respectively $(K, \leq)$.

In the following we first prove an analogue of Proposition 2.28 for quasi-ordered rings. Afterward, we generalise the rank to this class.

Notation 2.29. For a ring R we denote by $\mathcal{V}(R)$ the set of all valuations on R. If $\mathfrak{q}$ is a two-sided completely prime ideal of R, we define $\mathcal{V}_{\mathfrak{q}}(R) := \{v \in \mathcal{V}(R) \colon \mathfrak{q}_v = \mathfrak{q}\}$.

Lemma 2.30. *Let $(R, \preceq)$ be a quasi-ordered ring and v the trivial valuation with support $\mathfrak{q}_{\preceq}$. Then v is strongly $\preceq$-compatible.*

Proof. Let $0 \preceq x \preceq y$. If $y \in \mathfrak{q}_{\preceq}$, then so is x, whence $v(x) = v(y) = \infty$. Otherwise, $v(y) = 0 \leq v(x)$. □

Proposition 2.31. *Let $(R, \preceq)$ be a quasi-ordered ring. Then the set*

$$\{v \in \mathcal{V}_{\mathfrak{q}_{\preceq}}(R) \colon v \textit{ is strongly } \preceq \textit{-compatible}\}$$

is linearly ordered by the coarsening relation $\leq$.

Proof. (cf. [56, Lemma 4.2]) Otherwise, we find strongly $\preceq$-compatible valuations $v_1, v_2 \in \mathcal{V}_{\mathfrak{q}_{\preceq}}(R)$ and $a_i, b_i \in R$ such that $v_1(a_1) \leq v_1(a_2)$, but $v_2(a_2) < v_2(a_1)$, and $v_2(b_1) \leq v_2(b_2)$, but $v_1(b_2) < v_1(b_1)$. Consequently, $v_1(a_1b_2) < v_1(a_2b_1)$ and $v_2(a_2b_1) < v_2(a_1b_2)$. By Lemma 1.29 and the fact that $v_i(x) = v_i(-x)$ for all $x \in R$, we may w.l.o.g. assume that $0 \preceq a_i, b_i$, and therefore $0 \preceq a_1b_2$ and $0 \preceq a_2b_1$. But then the contraposition of v_i being strongly $\preceq$-compatible implies that both, $a_2b_1 \prec a_1b_2$ and $a_1b_2 \prec a_2b_1$, a contradiction. Hence, this set is linearly ordered. □

So if $(R, \preceq)$ is a quasi-ordered ring, then Lemma 2.30 and Proposition 2.31 yield a finest valuation $v \in \mathcal{V}_{\mathfrak{q}_{\preceq}}(R)$ that is strongly $\preceq$-compatible, namely the intersection of all strongly $\preceq$-compatible valuations.

Definition 2.32. Let $(R, \preceq)$ be a quasi-ordered ring. We call the finest strongly $\preceq$-compatible valuation $v \in \mathcal{V}_{\mathfrak{q}_{\preceq}}(R)$ the *natural valuation* of $\preceq$, and we denote it by $v_{\preceq}$.

Remark 2.33. Let $(R, \preceq)$ be a quasi-ordered ring.

(1) If $\preceq$ is induced by a valuation v, then $v_{\preceq} = v$.
(2) If $\preceq$ is an ordering, then $v_{\preceq}$ is the natural valuation in the usual sense (see Example 2.8).

Proposition 2.31 particularly implies that for a valued ring (R, v) the coarsenings of v of support $\mathfrak{q}_v$ are linearly ordered. Therefore, we may define the rank of a quasi-ordered ring in terms of valuations as follows:

Definition 2.34. The *rank* of a quasi-ordered ring $(R, \preceq)$, written $\mathrm{rk}(R, \preceq)$, is the order type of the linearly ordered set $\{v \in \mathcal{V}_{\mathfrak{q}_{\preceq}}(R) \colon v_{\preceq} < v\}$, i.e. the order type of the set of all strict coarsenings of $v_{\preceq}$.

Next, we show that the rank of a commutative quasi-ordered ring is equal to the rank of its associated quasi-ordered field.

Given a valuation v on a commutative ring R, we denote by v' the valuation on $R/\mathfrak{q}_v$ defined by $v'(\overline{x}) := v(x)$, and by ν the unique extension of v' to $K := \mathrm{Quot}(R/\mathfrak{q}_v)$.

Lemma 2.35. *Let $(R, \preceq)$ be a commutative quasi-ordered ring and v a valuation on R with support $\mathfrak{q}_{\preceq}$. The following are equivalent:*

(1) v is strongly compatible with $\preceq$,
(2) v' is strongly compatible with $\preceq'$,
(3) ν is (strongly) compatible with $\trianglelefteq$.

Proof. The equivalence of (1) and (2) easily follows from the definition of $\preceq'$ and v', respectively. We conclude by showing that (2) and (3) are equivalent. It is clear that strong compatibility in K implies strong compatibility in $R/\mathfrak{q}_{\preceq}$, as it is a universal statement. Conversely, let $0 \trianglelefteq \frac{x}{y} \trianglelefteq \frac{a}{b}$. Then $0 \preceq' xyb^2 \preceq' aby^2$. By strong $\preceq'$-compatibility of v', we obtain that $v'(aby^2) \leq v'(xyb^2)$, and therefore $v'(ay) \leq v'(xb)$. Thus, $\nu(\frac{a}{b}) \leq \nu(\frac{x}{y})$. $\square$

Corollary 2.36. *The rank of a commutative quasi-ordered ring $(R, \preceq)$ coincides with the rank of its associated quasi-ordered field $(K, \trianglelefteq)$.*

Proof. By Lemma 2.35 we know that $v = v_{\preceq}$ if and only if $\nu = \nu_{\trianglelefteq}$, where $\trianglelefteq$ and ν denote the unique extensions of $\preceq$ and v to $\mathrm{Quot}(R/\mathfrak{q}_{\preceq})$. Moreover, by Lemma 2.35 and Remark 1.13, $v_{\preceq}$ and $\nu_{\trianglelefteq}$ have the same number of strict coarsenings. Hence, $(R, \preceq)$ and $(K, \trianglelefteq)$ have the same rank. $\square$

Corollary 2.37. *Let (R, v) be a commutative valued ring. Then the rank of (R, v) coincides with the rank of the value group Γ_v.*

Proof. This is because the ranks of (R, v) and (K, ν) coincide (Corollary 2.36), the latter being equal to the rank of $\Gamma_\nu = \Gamma_v$. $\square$

Obviously, a valued ring (R, v) has rank 0 if and only if v is a trivial valuation. We conclude this section by considering ordered rings of rank 0.

If $(R, \leq)$ is an ordered ring, we call an element $0 < r \in R$ *infinitely large* if $r > n$ for all $n \in \mathbb{N}$, and *infinitely small* if $nr < 1$ for all $n \in \mathbb{N}$.

Lemma 2.38. ([43, Lemma 17.20]) *Let $(R, \leq)$ be an ordered ring. The following are equivalent:*

(1) For any $a, b > 0$ in R, there exists an integer $n \geq 1$ such that $na > b$.

(2) R admits neither infinitely large nor infinitely small elements.

Proof. We first show that (1) implies (2). So let $0 < r \in R$ be arbitrary. Then $r < n \cdot 1 = n$ for some $n \in \mathbb{N}$, whence r is not infinitely large. Likewise, $mr > 1$ for some $m \in \mathbb{N}$, so r is not infinitely small either.

Now suppose that (2) holds and let $0 < a, b$. Then we find some $m, n \in \mathbb{N}$ such that $na > 1$ and $m > b$, since a is not infinitely small and b is not infinitely large. Therefore, $mna > m > b$. $\square$

Definition 2.39. [43, p. 282] An ordered ring $(R, \leq)$ is called *Archimedean*, if it satisfies one of the equivalent conditions from Lemma 2.38.

Example 2.40. Let $(R, \leq)$ be an ordered ring. Then $(Rv_{\leq}, \leq^*)$ is an Archimedean ordered domain, because $R_{v_{\leq}}$ contains no infinitely large elements and $I_{v_{\leq}}$ consists precisely of all infinitely small elements (see Example 2.8).

Corollary 2.41. (to Lemma *2.38*) *An ordered ring $(R, \leq)$ has rank 0 if and only if it is Archimedean.*

Proof. $(R, \leq)$ has rank 0 if and only if the natural valuation $v_{\leq}$ is trivial (Lemma 2.30). The latter is equivalent to the fact that the Archimedean equivalence relation $\sim^+$ (Example 2.8) on $R \backslash \mathfrak{q}_{\preceq}$ is trivial, which is precisely the statement of Lemma 2.38(1). $\square$

For a further discussion on the rank of quasi-ordered fields we refer the interested reader to [40].

3. Baer-Krull Theorems for Quasi-Ordered Fields and Rings

In Chapter 2 we fixed a quasi-ordered ring $(R, \preceq)$ and characterised the valuations on R, which are (strongly) $\preceq$-compatible. In the present chapter we turn the situation around - given a valued ring (R, v), we characterise the quasi-orderings $\preceq$ on R such that v is strongly $\preceq$-compatible. For that purpose we establish a Baer-Krull theorem for commutative quasi-ordered rings. This theorem describes the said quasi-orderings on R in terms of the quasi-orderings on the residue class domain Rv.

For the sake of convenience, we first prove the Baer-Krull theorem for quasi-ordered fields (Section 3.1). Afterward, in Section 3.2, we generalise it to commutative quasi-ordered rings and give a few applications. We conclude this chapter by deducing and discussing Baer-Krull theorems for commutative ordered rings in Section 3.3, respectively for commutative valued rings in Section 3.4.

Quite a few of the results presented here, including a Baer-Krull theroem for quasi-ordered rings, were already published by S. Kuhlmann and the author of this thesis in [41].

Preliminaries on the Baer-Krull theorem for ordered fields.

In the present chapter, we use the following notation (the notation introduced in (4) will also play an important role in the following chapters):

Notation 3.1. Let (R, v) be a commutative valued ring and $\mathfrak{q}$ a prime ideal of R.

(1) As before, Γ_v denotes the value group (see Remark 1.6(1)), $\mathfrak{q}_v$ the support, R_v the valuation ring, I_v the valuation ideal, and $Rv = R_v/I_v$ the residue class domain of v (see Notation 2.1).

(2) A *quadratic system of representatives of* R *w.r.t.* v (cf. [18, p. 37]) is a family $\{\pi_i : i \in I\} \subseteq R \backslash \mathfrak{q}_v$ such that the elements $\overline{v(\pi_i)}$ form an $\mathbb{F}_2$-basis of $\overline{\Gamma_v} := \Gamma_v/2\Gamma_v$. Such a system exists for any commutative valued ring (R, v), since for any $\gamma \in \Gamma_v$ we have $\gamma \in v(R)$ or $-\gamma \in v(R)$. We set $\gamma_i := v(\pi_i)$.

(3) Given a quasi-ordering $\preceq$ on R such that v is strongly $\preceq$-compatible, we denote by $\preceq^*$ the induced quasi-ordering on Rv (see Theorem 2.19(4)). By $\eta_\preceq$ we denote the map $I \to \{-1, 1\}$ defined by $\eta_\preceq(i) = 1$ if and only if $0 \preceq \pi_i$.

(4) We denote by $\mathcal{Q}(R)$ the set of all quasi-orderings, by $\mathcal{O}(R)$ the set of all orderings, and by $\mathcal{V}(R)$ the set of all valuations on R. We add $\mathfrak{q}$ as subscript to restrict to quasi-orderings (resp. orderings, or valuations) with support $\mathfrak{q}$, and we add v as exponent to restrict to quasi-orderings (resp. orderings, or valuations) which are strongly compatible with v.

So for example $\mathcal{Q}^v_\mathfrak{q}(R)$ denotes the set of all quasi-orderings $\preceq$ on R with support $\mathfrak{q}$ such that v is strongly $\preceq$-compatible.

Our goal is to generalise the following theorem, the so-called *Baer-Krull theorem*, to the class of quasi-ordered rings. Based upon the works of Baer ([6]) and Krull ([37]), we follow its exposition given by Engler and Prestel in [18, Theorem 2.2.5].

Theorem 3.2. *Let* (K, v) *be a valued field and* $\{\pi_i : i \in I\}$ *a quadratic system of representatives of* K *w.r.t.* v*. Then the map*

$$\psi \colon \mathcal{O}^v(K) \to \{-1, 1\}^I \times \mathcal{O}(Kv), \leq \mapsto (\eta_\leq, \leq^*)$$

is a bijective correspondence.

3.1. **A Baer-Krull Theorem for Quasi-Ordered Fields.**

The sole aim of this section is to prove a Baer-Krull theorem for quasi-ordered fields (Theorem 3.7 and Corollary 3.8). Even though we are ultimately interested in quasi-ordered rings, it is more convenient to first establish the said theorem for quasi-ordered fields, and then deduce it for rings by exploiting the fact that strong compatibility of quasi-orderings and valuations is preserved when moving on to the quotient field (see Lemma 2.35). In fact, when we gave a direct proof for quasi-ordered rings in [41], we required that the valuation v is Manis - in Section 3.2, however, we will be able to replace the Manis property with a weaker condition on the valuation v.

Further note that we cannot expect the map ψ from Theorem 3.2 to be a bijection, if we replace $\mathcal{O}^v(K)$ with $\mathcal{Q}^v(K)$, and $\mathcal{O}(Kv)$ with $\mathcal{Q}(Kv)$. Recall that if a quasi-ordering $\preceq$ is induced by a valuation, then $0 \preceq x$ for any x (Lemma 1.45). Hence, in that case $\eta_\preceq$ is the constant map $\eta = 1$. For that reason what we may and what we will get for quasi-ordered fields is that the map

$$\psi\colon \mathcal{Q}^v(K) \to \{-1,1\}^I \times \mathcal{O}(Kv) \sqcup \{1\}^I \times \mathcal{V}(Kv),\ \preceq\ \mapsto (\eta_\preceq, \preceq^*)$$

is a bijection. Thus, there is some non-uniformity, which, however, will only play a minor role in the following exposition.

Let us now establish a Baer-Krull theorem for quasi-ordered fields. For that purpose we fix a valued field (K,v), a quadratic system $\{\pi_i\colon i \in I\}$ of representatives of K w.r.t. v (see Notation 3.1), and a tuple

$$(\eta^\times, \preceq^\times) \in \{-1,1\}^I \times \mathcal{O}(Kv) \sqcup \{1\}^I \times \mathcal{V}(Kv).$$

The main part of the proof of the Baer-Krull theorem is to construct a quasi-ordering on K that is mapped to $(\eta^\times, \preceq^\times)$ under the analogue of the map ψ from Theorem 3.2. To this end, adapting the proof given in [18], we define a binary relation $\preceq$ on K as a function of $\preceq^\times$ and $\eta^\times$ as follows: for $x, y \in K$ such that $x \in K^\times$ or $y \in K^\times$, we consider

$$\gamma := \gamma_{x,y} := \max\{-v(x), -v(y)\} \in \Gamma_v.$$

Then there is a unique representation $\overline{\gamma} = \sum_{i \in I_{x,y}} \overline{\gamma_i}$ of $\overline{\gamma}$ in $\Gamma_v/2\Gamma_v$. Hence,

$$\gamma = \sum_{i \in I_{x,y}} \gamma_i + 2v(a_{x,y}) = v\left(\prod_{i \in I_{x,y}} \pi_i a_{x,y}^2\right)$$

for some unique subset $I_{x,y} \subseteq I$, and some $a_{x,y} \in K^\times$, which is unique up to its value, i.e. instead of $a_{x,y}$ we may have chosen any other element $b_{x,y} \in K$ with $v(b_{x,y}) = v(a_{x,y})$. In what follows, we will just write $\sum_i, \prod_i$ and a instead of $\sum_{i \in I_{x,y}}, \prod_{i \in I_{x,y}}$ and $a_{x,y}$, respectively, whenever x and y are clear from the context.

Lemma 3.3. *Let (K,v) be a valued field and $x, y \in K$. With the above notation, $x\prod_i \pi_i a^2$, $y\prod_i \pi_i a^2 \in K_v$. Moreover, $x\prod_i \pi_i a^2 \in I_v$ if and only if $v(x) > v(y)$.*

Proof. We have

$$\begin{aligned} v\left(x\prod_i \pi_i a^2\right) &= v(x) + \sum_i v(\pi_i) + 2v(a) = v(x) + \gamma \\ &= v(x) + \max\{-v(x), -v(y)\} \geq 0, \end{aligned}$$

and analogously for $y\prod_i \pi_i a^2$, so both are in K_v. Moreover,

$$x\prod_i \pi_i a^2 \in I_v \Leftrightarrow v(x) + \max\{-v(x), -v(y)\} > 0 \Leftrightarrow v(x) > v(y).$$

□

In particular, we can always take residues of both, $x\prod_i \pi_i a^2$ and $y\prod_i \pi_i a^2$. The moreover statement of Lemma 3.3 will be of great importance in the proof of the following result:

Main Lemma 3.4. *Let (K,v) be a valued field. Using the above notation, we define for $x,y \in K$ with $x \in K^\times$ or $y \in K^\times$, that*

$$x \preceq y :\Leftrightarrow \begin{cases} \textit{either} & \overline{x\prod_i \pi_i a^2} \preceq^\times \overline{y\prod_i \pi_i a^2} \quad \textit{and} \quad \prod_i \eta^\times(i) = 1, \\ \textit{or} & \overline{y\prod_i \pi_i a^2} \preceq^\times \overline{x\prod_i \pi_i a^2} \quad \textit{and} \quad \prod_i \eta^\times(i) = -1. \end{cases}$$

Moreover, we declare $0 \preceq 0$. Then $\preceq$ defines a quasi-ordering on K such that v is $\preceq$-compatible.

Proof. For the sake of convenience and uniformity, we treat $\preceq^\times$ and $\eta^\times$ as an arbitrary quasi-ordering on Kv, respectively an arbitrary map from I to $\{-1,1\}$, for as long as possible. The distinction whether $\preceq^\times$ is an ordering or a valuation (in which case $\eta^\times$ is trivial) is only necessary when verifying axiom (Q3).

First of all we show that $\preceq$ is well-defined. Recall that $a \in K^\times$ is only unique up to its value. So let $b \in K^\times$ such that $v(a) = v(b)$, and suppose that

$$\overline{x\prod_i \pi_i a^2} \preceq^\times \overline{y\prod_i \pi_i a^2}.$$

Define $z = ba^{-1}$. Then $v(z) = 0$, so Lemma 1.36 implies $\overline{0} \prec^\times \overline{z}^2$. Applying (Q2) yields

$$\overline{x\prod_i \pi_i b^2} \preceq^\times \overline{y\prod_i \pi_i b^2}.$$

Obviously, $\preceq$ is reflexive and total. Next we prove transitivity. So let $x \preceq y$ and $y \preceq z$, w.l.o.g. $x \in K^\times$ or $z \in K^\times$. The proof is done by distinguishing four cases. First suppose that $v(p) = v(q) \leq v(r)$ with $p,q,r \in \{x,y,z\}$ pairwise distinct. Then $\gamma_{x,y} = \gamma_{x,z} = \gamma_{y,z} \in \Gamma_v$ all coincide, so $I_{x,y} = I_{x,z} = I_{y,z}$ and $a_{x,y} = a_{x,z} = a_{y,z}$. Hence, transitivity of $\preceq$ follows immediately from transitivity of $\preceq^\times$. It remains to verify the cases where there is a unique smallest element among $v(x), v(y)$ and $v(z)$. First suppose that $v(x) < v(y), v(z)$. Then $\gamma_{x,y} = -v(x) = \gamma_{x,z}$, i.e. $I_{x,y} = I_{x,z}$ and $a_{x,y} = a_{x,z}$. We only consider the subcase $\prod_{i \in I_{x,y}} \eta^\times(i) = -1$, since the opposite one is dealt with analogously. From $x \preceq y$ and $v(x) < v(y)$ it follows

$$\overline{y\prod_{i \in I_{x,y}} \pi_i a_{x,y}^2} = \overline{0} \preceq^\times \overline{x\prod_{i \in I_{x,y}} \pi_i a_{x,y}^2}$$

(Lemma 3.3). Now $v(x) < v(z)$ and again Lemma 3.3 imply

$$\overline{z\prod_{i \in I_{x,y}} \pi_i a_{x,y}^2} = \overline{0} \preceq^\times \overline{x\prod_{i \in I_{x,y}} \pi_i a_{x,y}^2}.$$

Thus, $x \preceq z$. Next, suppose that $v(y) < v(x), v(z)$. Then $\gamma_{x,y} = -v(y) = \gamma_{y,z}$, i.e. $I_{x,y} = I_{y,z}$ and $a_{x,y} = a_{y,z}$. Again, we only deal with the case $\prod_{i \in I_{x,y}} \eta^\times(i) = -1$. From $v(y) < v(x)$ and $x \preceq y$ it follows

$$\overline{y\prod_{i \in I_{x,y}} \pi_i a_{x,y}^2} \preceq^\times \overline{x\prod_{i \in I_{x,y}} \pi_i a_{x,y}^2} = \overline{0}.$$

Likewise, $v(y) < v(z)$ and $y \preceq z$ implies

$$\overline{z\prod_{i \in I_{x,y}} \pi_i a_{x,y}^2} = \overline{0} \preceq^\times \overline{y\prod_{i \in I_{x,y}} \pi_i a_{x,y}^2}.$$

Therefore,

$$\overline{y \prod_{i \in I_{x,y}} \pi_i a_{x,y}^2} = \overline{0}.$$

On the other hand, $v(y) < v(x)$ yields via Lemma 3.3 that $y \prod_{i \in I_{x,y}} \pi_i a_{x,y}^2 \notin I_v$, a contradiction. The case $v(z) < v(x), v(y)$ is proven exactly like the case where $v(x)$ is the unique smallest value.

It remains to verify the axioms (Q1) - (Q3) and compatibility of $\preceq$ and v. For the proof of **(Q1)** assume that there is some $x \in K^\times$ such that $x \sim 0$. Then

$$\overline{0} \preceq^\times \overline{x \prod_i \pi_i a^2} \preceq^\times \overline{0},$$

whence $x \prod_i \pi_i a^2 \in I_v$. However, from $v(x) < v(0)$ and Lemma 3.3, it follows that $x \prod_i \pi_i a^2 \notin I_v$, a contradiction. Hence, (Q1) is fulfilled.

Next, we verify **(Q2)**, i.e. we show that $x \preceq y$ and $0 \preceq z$ implies $xz \preceq yz$. We may w.l.o.g. assume that $z \in K^\times$, and that $x \in K^\times$ or $y \in K^\times$. Therefore,

$$\begin{aligned}\gamma_{xz,yz} &= \max\{-v(xz), -v(yz)\} = \max\{-v(z), -v(0)\} + \max\{-v(x), -v(y)\} \\ &= \gamma_{0,z} + \gamma_{x,y} \in \Gamma_v.\end{aligned}$$

Hence, $I_{xz,yz}$ is the (w.l.o.g.) disjoint union of $I_{x,y}$ and $I_{0,z}$, which implies

$$\prod_{i \in I_{xz,yz}} \eta^\times(i) = \prod_{i \in I_{x,y}} \eta^\times(i) \cdot \prod_{i \in I_{0,z}} \eta^\times(i).$$

Moreover, $a_{xz,yz} = a_{x,y} a_{0,z}$.

First consider the case $\prod_{i \in I_{0,z}} \eta^\times(i) = 1$. This implies $\overline{0} \preceq^\times \overline{z \prod_{i \in I_{0,z}} \pi_i a_{0,z}^2}$.
If $\prod_{i \in I_{x,y}} \eta^\times(i) = -1 = \prod_{i \in I_{xz,yz}} \eta^\times(i)$, then

$$\overline{y \prod_{i \in I_{x,y}} \pi_i a_{x,y}^2} \preceq^\times \overline{x \prod_{i \in I_{x,y}} \pi_i a_{x,y}^2}.$$

Applying (Q2) yields

$$\overline{yz \prod_{i \in I_{xz,yz}} \pi_i a_{x,y}^2 a_{0,z}^2} \preceq^\times \overline{xz \prod_{i \in I_{xz,yz}} \pi_i a_{x,y}^2 a_{0,z}^2},$$

whence $xz \preceq yz$. The subcase $\prod_{i \in I_{x,y}} \eta^\times(i) = 1 = \prod_{i \in I_{xz,yz}} \eta^\times(i)$ is proven analogously. The proof of the case $\prod_{i \in I_{0,z}} \eta^\times(i) = -1$ is also almost the same, we just apply Lemma 1.52 instead of axiom (Q2).

The proof of axiom **(Q3)** is divided into five cases. First suppose that $v(x) < v(z)$ or $v(y) < v(z)$. Then $\gamma_{x,y} = \gamma_{x+z,y+z}$. Moreover, $\overline{z \prod_{i \in I_{x,y}} \pi_i a_{x,y}^2} = \overline{0}$ by Lemma 3.3. Thus, clearly $x \preceq y$ implies $x + z \preceq y + z$. Further note that if $\preceq^\times$ is an ordering and $x \prec y$, then we obtain $x + z \prec y + z$, because orderings preserve strict inequalities under addition. We will exploit this fact below.

If $v(z) < v(x)$ and $v(z) < v(y)$, then $\gamma_{x,z} = \gamma_{y,z} = \gamma_{x+z,y+z}$. Lemma 3.3 implies

$$\overline{x \prod_{i \in I_{x,z}} \pi_i a_{x,z}^2} = \overline{0} = \overline{y \prod_{i \in I_{y,z}} \pi_i a_{y,z}^2}.$$

From this, it is easy to see that $x+z \preceq y+z$. Now suppose that $v(x) = v(z) < v(y)$. Then $\gamma_{x,y} = \gamma_{y,z} = \gamma_{x+z,y+z}$. We distinguish two subcases. If $\prod_{i \in I_{x,y}} \eta^\times(i) = 1$, then $x \preceq y$ yields

$$\overline{x \prod_{i \in I_{x,y}} \pi_i a_{x,y}^2} \preceq^\times \overline{y \prod_{i \in I_{x,y}} \pi_i a_{x,y}^2}.$$

As $z \not\sim y$, one may add $\overline{z \prod_{i \in I_{x,y}} \pi_i a_{x,y}^2}$ on either side. If $\prod_{i \in I_{x,y}} \eta^\times(i) = -1$, then $\eta^\times \neq 1$, whence $\preceq^\times$ is an ordering. Thus, one may add $\overline{z \prod_{i \in I_{x,y}} \pi_i a_{x,y}^2}$ on either side as well. This proves the present case. The case $v(y) = v(z) < v(x)$ is proven analogously. Finally, suppose that

$$v(x) = v(y) = v(z) \in \Gamma_v.$$

Then

$$\gamma_{x+z,y+z} = \max\{-v(x+z), -v(y+z)\} \leq -v(z).$$

First suppose that equality holds. Then $\max\{-v(x+z), -v(y+z)\} = -v(z)$, i.e. all $\gamma' s$ coincide. If $\prod_i \eta^\times(i) = 1$, the claim follows immediately by applying axiom (Q3). If $\prod_i \eta^\times(i) = -1$, then $\preceq^\times$ must be an ordering and we may add $\overline{z \prod_i \pi_i a^2}$ to either side anyway.

Last but not least assume that $<$ holds, i.e. $\max\{-v(x+z), -v(y+z)\} < -v(z)$. Then $v(z) < v(x+z), v(y+z)$. By Lemma 3.3, $\overline{p \prod_{i \in I_{x,y}} \pi_i a_{x,y}^2} \neq \overline{0}$ for $p \in \{x, y, z\}$, whereas

$$\overline{(x+z) \prod_{i \in I_{x,y}} \pi_i a_{x,y}^2} = \overline{0} = \overline{(y+z) \prod_{i \in I_{x,y}} \pi_i a_{x,y}^2}.$$

Therefore,

$$\overline{x \prod_{i \in I_{x,y}} \pi_i a_{x,y}^2} = \overline{y \prod_{i \in I_{x,y}} \pi_i a_{x,y}^2} = \overline{-z \prod_{i \in I_{x,y}} \pi_i a_{x,y}^2}.$$

In particular, we may assume that $\preceq^\times$ is an ordering, since for a valuation

$$\overline{y \prod_{i \in I_{x,y}} \pi_i a_{x,y}^2} \sim \overline{-y \prod_{i \in I_{x,y}} \pi_i a_{x,y}^2} = \overline{z \prod_{i \in I_{x,y}} \pi_i a_{x,y}^2},$$

a contradiction to the assumption $y \not\sim z$. We claim that $x + z \sim 0 \sim y + z$, which clearly implies $x + z \preceq y + z$. Assume for the sake of a contradiction that $x + z \not\sim 0$. If $x + z \prec 0$, it follows from the case "$v(x) < v(z)$" above (where $x + z$ plays the role of x, 0 the one of y and $-z$ the one of z; recall that $v(x + z) < v(z)$) and the fact that $\preceq^\times$ is an ordering, that $x \prec -z$. However, this contradicts the fact that $x \sim -z$. Likewise, if $0 \prec x + z$, it follows from the case "$v(y) < v(z)$" that $-z \prec x$, again a contradiction. Therefore $x + z \sim 0$. The same reasoning shows that $y + z \sim 0$ as well. Hence, (Q3) is satisfied.

Finally, we show that v is $\preceq$-compatible. By Theorem 2.27(4) it suffices to show that $I_v \prec 1$. So let $x \in I_v$. Then

$$\gamma_{x,1} = \max\{-v(x), -v(1)\} = 0.$$

Therefore, $I_{x,1} = \emptyset$, $\prod_{i \in I_{x,1}} \eta^\times(i) = 1$, and we may choose $a_{x,1} = 1$. Consequently, we obtain $x \prec 1 \Leftrightarrow \overline{0} = \overline{x} \prec^\times \overline{1}$. Since $\preceq^\times$ is a quasi-ordering, the RHS is fulfilled, whence $x \prec 1$. So v is $\preceq$-compatible. □

Remark 3.5.

(1) Let $\preceq$ denote the quasi-ordering from the Main Lemma 3.4. If $x \in U_v$ and $y \in R_v$, or vice versa, then

$$x \preceq y \Leftrightarrow \overline{x} \preceq^\times \overline{y}.$$

This is because, as we just witnessed at the very end of the previous proof, in this case $\gamma_{x,y} = 0$, whence $I_{x,y} = \emptyset$ and $v(a_{x,y}) = 0$. The former fact yields $\prod_{i \in I_{x,y}} \eta^\times(i) = 1$. Moreover, we may choose $a_{x,y} = 1$, since $a_{x,y}$ is only unique up to its value.

(2) The main lemma can be proven for commutative quasi-ordered rings, if the valuation v is Manis (cf. [41, Main Lemma 4.5]).

For the proof of the Baer-Krull theorem we require one more lemma. It will be used to compare the size of two quasi-orderings on K. For the sake of generality, we prove it for arbitrary quasi-ordered rings.

Recall from Notation 1.21 that

$$E(x) = E_{\preceq}(x) = \{y \in S \colon x \sim y\},$$

where $(S, \preceq)$ is a quasi-ordered set.

Lemma 3.6. *Let $(R, \preceq)$ be a quasi-ordered ring. Then $E(0) + \{x\} \subseteq E(x)$ for any $x \in R$. Moreover, if $E(0) + \{x\} \subsetneq E(x)$, then $E(x) = -E(x)$.*

Proof. The first statement is an easy consequence of Lemma 1.28. For the moreover part, let $z \in E(x)$ be arbitrary and $y \in E(x) \backslash (E(0) + \{x\})$. We will show that $-y \in E(x)$. From $z \sim x \sim y$ and Corollary 1.60 it then follows $-z \sim -y \sim x$. This implies $-z \in E(x)$, and therefore $E(x) = -E(x)$.

So assume for the sake of a contradiction that $-y \notin E(x)$. Recall that $y \in E(x)$. So we have $y \sim x \not\sim -y$. By (QR4), we obtain $x - y \in E(0)$, i.e. $y \in E(0) + \{x\}$, a contradiction. Hence, $-y \in E(x)$, as desired. □

We can finally prove a Baer-Krull theorem for quasi-ordered fields. Its formulation relies on Notation 3.1.

Theorem 3.7. (*Baer-Krull theorem for quasi-ordered fields I*)
Let (K, v) be a valued field and $\{\pi_i \colon i \in I\}$ a quadratic system of representatives of K w.r.t. v. Then

$$\psi \colon \mathcal{Q}^v(K) \to \{-1, 1\}^I \times \mathcal{Q}(Kv), \ \preceq \mapsto \ (\eta_{\preceq}, \preceq^*)$$

is a well-defined map such that the restriction $\psi \restriction \psi^{-1}(\mathcal{A}) \colon \psi^{-1}(\mathcal{A}) \to \mathcal{A}$ is a bijection, where $\mathcal{A} := \{-1, 1\}^I \times \mathcal{O}(Kv) \sqcup \{1\}^I \times \mathcal{V}(Kv)$.

Proof. By Theorem 2.27(5) the map ψ is well-defined. Now let $(\eta^\times, \preceq^\times) \in \mathcal{A}$ be arbitrary. We first prove that ψ maps the quasi-ordering $\preceq$ constructed in the Main Lemma to the tuple $(\eta^\times, \preceq^\times)$. First we verify that $\eta_{\preceq} = \eta^\times$. Note that $\gamma_{\pi_i,0} = -v(\pi_i) = -\gamma_i$, whence $\gamma_{\pi_i,0} = v(\pi_i a^2)$ for some $a \in K^\times$ according to the construction in the beginning of this chapter. Thus, $0 \preceq \pi_i$ if and only if either $\overline{0} \preceq^\times \overline{(\pi_i a)^2}$ and $\eta^\times(i) = 1$, or $\overline{(\pi_i a)^2} \preceq^\times \overline{0}$ and $\eta^\times(i) = -1$. However, the second case cannot occur, since non-zero squares are always strictly positive (Lemma 1.36). Altogether, we obtain

$$\eta_{\preceq}(i) = 1 \Leftrightarrow 0 \preceq \pi_i \Leftrightarrow \eta^\times(i) = 1,$$

and therefore $\eta_{\preceq} = \eta^\times$.

Next, we prove that $\preceq^* = \preceq^\times$. So let $x, y \in K_v$, w.l.o.g. not x and y in I_v. Then also $x + c$ and $y + d$ are not both in I_v for all $c, d \in I_v$, so Remark 3.5(1) tells us that $x + c \preceq y + d \Leftrightarrow \overline{x + c} \preceq^\times \overline{y + d}$. Thus,

$$\begin{aligned} \overline{x} \preceq^* \overline{y} &\Leftrightarrow \exists c_1, c_2 \in I_v \colon x + c_1 \preceq y + c_2 \\ &\Leftrightarrow \exists c_1, c_2 \in I_v \colon \overline{x + c_1} \preceq^\times \overline{y + c_2} \\ &\Leftrightarrow \overline{x} \preceq^\times \overline{y}. \end{aligned}$$

This shows that the map $\psi \restriction \psi^{-1}(\mathcal{A}) \colon \psi^{-1}(\mathcal{A}) \to \mathcal{A}$ is surjective.

We conclude by showing that this map is also injective. So let $\preceq_1 \in \psi^{-1}(\mathcal{A})$ be arbitrary, and denote by $\preceq_2$ the quasi-ordering on K defined by $\eta_{\preceq_1}$ and $\preceq_1^*$ via our Main Lemma. We prove that $\preceq_1 = \preceq_2$. First we show the inclusion $\preceq_1 \subseteq \preceq_2$. So let x and y in K, w.l.o.g. not both equal to 0, and let I, π_i and a be as in the definition of the quasi-ordering $\preceq_2$ with respect to x and y. If $\prod_i \eta_{\preceq_1}(i) = -1$, then $\prod_i \pi_i a^2 \prec_1 0$, so we obtain by Lemma 1.51 and Lemma 1.52 that

$$\begin{aligned} x \preceq_1 y &\Leftrightarrow y \prod_i \pi_i a^2 \preceq_1 x \prod_i \pi_i a^2 \\ &\Rightarrow \overline{y \prod_i \pi_i a^2} \preceq_1^* \overline{x \prod_i \pi_i a^2} \\ &\Leftrightarrow x \preceq_2 y. \end{aligned}$$

Likewise, if $\prod_i \eta_{\preceq_1}(i) = 1$, we may apply (Q2) instead of the Lemmas and get the same result. Thus, $\preceq_1 \subseteq \preceq_2$. For the rest of the proof we distinguish the cases $-1 \nsim_{\preceq_2} 1$ and $-1 \sim_{\preceq_2} 1$.

If $-1 \nsim_{\preceq_2} 1$, then also $-x \nsim_{\preceq_2} x$ for any $x \in K^\times$, since $\sim_{\preceq_2}$ is preserved under multiplication ([19]; or Corollary 1.60). Thus, $E_{\preceq_2}(x) \neq -E_{\preceq_2}(x)$ for all $x \in K^\times$. From Lemma 3.6, it follows $E_{\preceq_2}(x) = \{x\}$ for all $x \in K$. Therefore, no quasi-ordering can be properly contained in $\preceq_2$, whence $\preceq_1 \subseteq \preceq_2$ already implies $\preceq_1 = \preceq_2$. So suppose for the rest of this proof that $-1 \sim_{\preceq_2} 1$. We distinguish the subcases $v(x) \neq v(y)$ and $v(x) = v(y)$.

If $v(x) \neq v(y)$, then Lemma 3.3 states that $x \prod_i \pi_i a^2 \in U_v$ and $y \prod_i \pi_i a^2 \in I_v$, or vice versa. We show $\preceq_1 = \preceq_2$ by proving that the only implication $\Rightarrow$ above is also an equivalence. First suppose that $x \prod_i \pi_i a^2 \in U_v$ and $y \prod_i \pi_i a^2 \in I_v$, and assume for the sake of a contradiction that

$$\overline{0} = \overline{y \prod_i \pi_i a^2} \preceq_1^* \overline{x \prod_i \pi_i a^2}, \text{ but } x \prod_i \pi_i a^2 \prec_1 y \prod_i \pi_i a^2.$$

By definition of $\preceq_1^*$, along with Lemma 2.14(1), we find some $c \in I_v$ such that $c \preceq_1 x \prod_i \pi_i a^2 \prec_1 y \prod_i \pi_i a^2$. This is a contradiction to Lemma 2.14(2) and (3), since either $u \prec_1 I_v$ or $I_v \prec_1 u$ for any $u \in U_v$. Likewise, if $x \prod_i \pi_i a^2 \in I_v$ and $y \prod_i \pi_i a^2 \in U_v$, and if the only implication $\Rightarrow$ above is not an equivalence, then

$$y \prod_i \pi_i a^2 + c_1 \preceq_1 x \prod_i \pi_i a^2 + c_2 \prec_1 y \prod_i \pi_i a^2$$

for some $c_1, c_2 \in I_v$. Taking residues yields $\overline{y \prod_i \pi_i a^2} = \overline{0}$, a contradiction.

So finally suppose that $v(x) = v(y)$, and assume for the sake of a contradiction that $x \sim_{\preceq_2} y$, but $x \prec_1 y$. Choose some $a \in K^\times$ such that $0 \prec_1 a$ (and hence $0 \prec_2 a$) and $v(a) = -v(x)$. By axiom (Q2), $ax \prec_1 ay$ if and only if $x \prec_1 y$, and by Corollary 1.60 $ax \sim_{\preceq_2} ay$ if and only if $x \sim_{\preceq_2} y$. So we may replace x and y with ax and ay. In other words, we may w.l.o.g. assume that $v(x) = v(y) = 0$. So from $y \preceq_2 x$ and Remark 3.5(1) it follows $\overline{y} \preceq_1^* \overline{x}$. Thus, there exist some $c_1, c_2 \in I_v$ such that $y + c_1 \preceq_1 x + c_2$, respectively, $y \preceq_1 x + c$ for $c := c_2 - c_1$ (Lemma 2.14(1)). Since $-1 \sim_{\preceq_2} 1$, also $-1 \sim_{\preceq_1} 1$. Otherwise $-1 \preceq_1 0$, but $-1 \npreceq_2 0$, contradicting the fact that $\preceq_1 \subseteq \preceq_2$. Therefore, Lemma 1.45 tells us that $0 \prec_1 x$ for all $x \in K^\times$. In particular, $c \prec_1 y$, since $0 \preceq_1 y \preceq_1 c$ would imply $y \in I_v$ by Theorem 2.27. So [19, Lemma 4.1], respectively Lemma 1.48, implies the contradiction

$$y \preceq_1 x + c \preceq_1 \max\{x, c\} \prec_1 y.$$

Hence, $\preceq_1 = \preceq_2$. This finishes the proof of the Baer-Krull theorem. $\square$

For the sake of uniformity, we so far did not make use of the dichotomy that every quasi-ordered field is either an ordered or a valued field (see Theorem 1.23). Taking this result into consideration, the Baer-Krull theorem simplifies as follows:

Corollary 3.8. (*Baer-Krull theorem for quasi-ordered fields II*)
Let (K,v) be a valued field and $\{\pi_i : i \in I\}$ a quadratic system of representatives of K w.r.t. v. Then the map

$$\psi \colon \mathcal{Q}^v(K) \to \mathcal{A}, \ \preceq \ \mapsto \ (\eta_{\preceq}, \preceq^*)$$

is a bijection, where $\mathcal{A} := \{-1,1\}^I \times \mathcal{O}(Kv) \sqcup \{1\}^I \times \mathcal{V}(Kv)$.

Proof. Theorem 1.23 (resp. [19, Theorem 2.1]) and Proposition 2.16 imply that $\psi^{-1}(\mathcal{A})$ coincides with the domain of ψ, whence the statement follows immediately from Theorem 3.7. □

3.2. **A Baer-Krull Theorem for Quasi-Ordered Rings.**

In this main section of Chapter 3 we establish two Baer-Krull theorems for commutative quasi-ordered rings (R,v). The first one (Theorem 3.9) is weaker in both, the assumption and the conclusion. It imposes no condition on the valuation v, but characterises the set $\mathcal{Q}^v_{\mathfrak{q}_v}(R)$ in terms of the quasi-orderings on $K\nu$ instead of Rv, where K denotes the quotient field of $R/\mathfrak{q}_v$ and ν the unique extension of v from R to K. The second one (Theorem 3.13) applies only to what we call C-valuations (see Definition 3.11), but characterises $\mathcal{Q}^v_{\mathfrak{q}_v}(R)$ in terms of the quasi-orderings on Rv.

As an application of the Baer-Krull theorem we obtain that if $(R,v) \subseteq (S,w)$ is an immediate extensions of commutative valued rings, then any $\preceq \in \mathcal{Q}^v_{\mathfrak{q}_v}(R)$ uniquely extends to a quasi-ordering in $\mathcal{Q}^w_{\mathfrak{q}_w}(S)$ (Proposition 3.22). We conclude this section by transferring a result from Efrat, stating that the same conclusion holds if the ring extension is integral instead of immediate (Proposition 3.25). As throughout the entire chapter we refer to Notation 3.1.

Theorem 3.9. *(Baer-Krull theorem for commutative quasi-ordered rings I)*
Let (R,v) be a commutative valued ring, $\{\pi_i : i \in I\}$ a quadratic system of representatives of R w.r.t. v, and ν the unique extension of v to $K := Quot(R/\mathfrak{q}_v)$. Then the map

$$\psi \colon \mathcal{Q}^v_{\mathfrak{q}_v}(R) \to \mathcal{A}, \ \preceq \ \mapsto \ (\eta_{\trianglelefteq}, \trianglelefteq^*)$$

is a bijection, where $\mathcal{A} := \{-1,1\}^I \times \mathcal{O}(K\nu) \sqcup \{1\}^I \times \mathcal{V}(K\nu)$.

Proof. We have $\Gamma_v = \Gamma_\nu$, so $\{\pi_i : i \in I\}$ is also a quadratic system of represenatives of K w.r.t. ν. Applying Corollary 3.8 to (K,ν) yields the bijective correspondence

$$\varphi \colon \mathcal{Q}^v(K) \to \mathcal{A}, \ \trianglelefteq \ \mapsto \ (\eta_{\trianglelefteq}, \trianglelefteq^*).$$

From Corollary 1.38 and the uniqueness stated in Proposition 1.37, it follows that $\preceq \ \mapsto \ \trianglelefteq$ defines a bijection between the set $\mathcal{Q}_{\mathfrak{q}_v}(R)$ of all quasi-orderings on R with support $\mathfrak{q}_v$, and the set $\mathcal{Q}(K)$ of all quasi-orderings on K. Moreover, Lemma 2.35 implies that $\preceq$ is strongly v-compatible if and only if $\trianglelefteq$ is strongly ν-compatible. Hence, we know that

$$\lambda \colon \mathcal{Q}^v_{\mathfrak{q}_v}(R) \to \mathcal{Q}^\nu(K), \ \preceq \ \mapsto \ \trianglelefteq \, .$$

is a bijection. Therefore, $\psi = \varphi \circ \lambda$ is also a bijection. □

In a further step we want to replace the residue class field $K\nu$ in the co-domain of the map ψ from Theorem 3.9 with the residue class domain Rv induced by v. Note that if (R,v) is a commutative valued ring, we can first consider the quotient field $K := \mathrm{Quot}(R/\mathfrak{q}_v)$, and then take the residue class domain $K\nu$, where ν denotes the unique extension of v from R to K. However, we can also first take the residue class domain Rv, and then consider its quotient field $L := \mathrm{Quot}(Rv)$.

Lemma 3.10. *Let (R, v) be a commutative valued ring, and let ν denote the unique extension of v to $K = Quot(R/\mathfrak{q}_v)$. Then $L := Quot(Rv)$ is a subfield of $K\nu$.*

Proof. We consider the canonical map

$$\varphi\colon R_v \to K\nu,\ x \mapsto \frac{x + \mathfrak{q}_v}{1 + \mathfrak{q}_v} + I_\nu.$$

Note that

$$x \in \ker(\varphi) \Leftrightarrow \frac{x + \mathfrak{q}_v}{1 + \mathfrak{q}_v} \in I_\nu \Leftrightarrow x + \mathfrak{q}_v \in I_{v'} \Leftrightarrow x \in I_v,$$

where v' denotes the valuation on $R/\mathfrak{q}_v$ defined by $v'(\overline{x}) = v(x)$ (see Lemma 1.35). Therefore, the homomorphism theorem tells us that the field $K\nu$ contains the domain $Rv = R_v/I_v$. Hence, it also contains its quotient field L. □

Definition 3.11. ([72, p. 975]) Let (R, v) be a commutative valued ring, and let ν denote the unique extension of v to $K := \mathrm{Quot}(R/\mathfrak{q}_v)$. We say that v is a *commuting valuation* (abbr. C-valuation) if $\mathrm{Quot}(Rv) = K\nu$.

Remark 3.12. In [72], Valente calls these valuations special. We renamed them here, because they do not coincide with special valuations as introduced in [35] by Knebusch and D. Zhang. *Commuting* refers to the fact that taking the quotient field and taking the residue class domain commutes for such valuations.

Theorem 3.13. *(Baer-Krull theorem for commutative quasi-ordered rings II) Let R be a commutative ring, v a C-valuation on R, and $\{\pi_i\colon i \in I\}$ a quadratic system of representatives of R w.r.t. v. Then the map*

$$\mathcal{Q}^v_{\mathfrak{q}_v}(R) \to \mathcal{A},\ \preceq\ \mapsto\ (\eta_\preceq, \preceq^*)$$

is a bijection, where $\mathcal{A} := \{-1, 1\}^I \times \mathcal{O}_0(Rv) \sqcup \{1\}^I \times \mathcal{V}_0(Rv)$.

Proof. Let $K := \mathrm{Quot}(R/\mathfrak{q}_v)$, and denote by ν the unique extension of v to K. Since v is a C-valuation, restriction $\trianglelefteq^* \mapsto \preceq^*$ is a bijective correspondence between the quasi-orderings on $K\nu = \mathrm{Quot}(Rv)$ and the quasi-orderings on Rv with support $\{0\}$ (see Proposition 1.37). Moreover, $\eta_\trianglelefteq = \eta_\preceq$, since $\pi_i \in R$ for all $i \in I$. Therefore, the claim follows immediately from Theorem 3.9. □

S. Kuhlmann and the author of this thesis already proved the previous theorem in [41, Corollary 4.11], except that they demanded v to be a Manis valuation instead of a C-valuation. The next result implies that Theorem 3.13 is a generalisation of their result.

Proposition 3.14. ([72, Proposition 2.2]) *Let R be a commutative ring. Any Manis valuation on R is a C-valuation. The converse may be false.*

Proof. We first show that any Manis valuation is a C-valuation. So let v be a Manis valuation on R. By Lemma 3.10, it suffices to show that $K\nu$ is a subfield of $\mathrm{Quot}(Rv)$, where $K := \mathrm{Quot}(R/\mathfrak{q}_v)$ and ν the unique extension of v to K. We argue again via the homomorphism theorem. This time, we consider the map

$$\varphi\colon K_\nu \to \mathrm{Quot}(Rv),\ \frac{x + \mathfrak{q}_v}{y + \mathfrak{q}_v} \mapsto \frac{x + I_v}{y + I_v}.$$

Since $\mathfrak{q}_v \subseteq I_v$, the choice of representatives in the domain does not matter. To show that φ is well-defined, we additionally have to prove that φ maps any element $\frac{x+\mathfrak{q}_v}{y+\mathfrak{q}_v} \in K_\nu$ to an element in $\mathrm{Quot}(Rv)$. The fact that this element lies in K_ν implies that $v'(y + \mathfrak{q}_v) \leq v'(x + \mathfrak{q}_v)$, and therewith $v(y) \leq v(x)$. Since v is Manis, we find some $a \in R$ such that $v(a) = -v(y)$, i.e. $v(ay) = 0 \leq v(ax)$. Thus, $ay \in U_v$ and $ax \in R_v$, so $\frac{ax+I_v}{ay+I_v} \in \mathrm{Quot}(Rv)$. Therefore, also $\frac{x+I_v}{y+I_v} \in \mathrm{Quot}(Rv)$, as this fraction equals $\frac{ax+I_v}{ay+I_v}$.

It is easy to see that φ is a ring homomorphism. It remains to show that I_ν is the kernel of φ. We have

$$\frac{x+\mathfrak{q}_v}{y+\mathfrak{q}_v} \in \ker(\varphi) \Leftrightarrow v(x) > v(y) \Leftrightarrow v'(x+\mathfrak{q}_v) > v'(y+\mathfrak{q}_v) \Leftrightarrow \nu\left(\frac{x+\mathfrak{q}_v}{y+\mathfrak{q}_v}\right) > 0,$$

so indeed $\ker(\varphi) = I_\nu$. Applying the homomorphism theorem implies that v is a C-valuation. Consequently, any Manis valuation is a C-valuation.

For the converse consider $R = \mathbb{Z}[X]$ and $v\colon R \to \mathbb{Z} \cup \{\infty\}$, $f \mapsto -\deg(f)$. Then $v(R) = -\mathbb{N}_0$, whence v is not Manis. However, $\mathrm{Quot}(Rv) = \mathbb{Q} = K\nu$. Hence, v is a C-valuation. □

Also any p-adic valuation v_p on $\mathbb{Z}$ is a C-valuation, as $\mathrm{Quot}(Rv_p) = \mathbb{F}_p = K\nu_p$. Valente claims that not any valuation is a C-valuation ([72]).

We continue our investigation by considering our second Baer-Krull theorem for quasi-ordered rings under further assumptions. First suppose that the value group Γ_v of v is 2-divisible. Then $\overline{\Gamma_v} = \Gamma_v/2\Gamma_v = \emptyset$, whence also $I = \emptyset$ (see Notation 3.1(2)). Thus, Theorem 3.13 implies:

Proposition 3.15. *Let R be a commutative ring and v a C-valuation on R such that Γ_v is 2-divisible. Then the map*

$$\mathcal{Q}^v_{\mathfrak{q}_v}(R) \to \mathcal{Q}_0(Rv), \ \preceq \mapsto \preceq^*$$

is a bijection.

Remark 3.16.

(1) The 2-divisibility of the value group Γ_v erases precisely the non-uniform component $\{-1,1\}^I$ of the set $\mathcal{A}$ in Theorem 3.13.
(2) In the same way, 2-divisibility of Γ_v also simplifies all our other versions of the Baer-Krull theorem, i.e. Theorem 3.7, Corollary 3.8, Theorem 3.9, and the ones yet to be derived for commutative ordered rings.

Next, we study the connection between our Baer-Krull theorem and immediate extensions of valued rings.

Definition 3.17. An extension $(R,v) \subseteq (S,w)$ of commutative valued rings is called *immediate* if $v(R) = w(R)$ and $Rv = Sw$, i.e. if their value groups and residue class domains coincide.

Example 3.18. Let R be some commutative ring. Then $R[X] \subseteq R[[X]]$ is an immediate extension w.r.t. the valuation $0 \neq \sum a_i X^i \mapsto \min_i\{a_i \neq 0\}$.

Lemma 3.19. *If $(R,v) \subseteq (S,w)$ is an immediate extension of commutative valued rings such that w is a C-valuation, then v is also a C-valuation.*

Proof. Since the extension is immediate, we have $\mathrm{Quot}(Rv) = \mathrm{Quot}(Sw)$. Moreover, since w is a C-valuation, we also have $\mathrm{Quot}(Sw) = L\omega$, where L denotes the quotient field of $S/\mathfrak{q}_w$, and ω the unique extension of w to L. Correspondingly, let K denote the quotient field of $R/\mathfrak{q}_v$, and ν the unique extension of v to K. Then

$$\mathrm{Quot}(Rv) = \mathrm{Quot}(Sw) = L\omega \supseteq K\nu \supseteq \mathrm{Quot}(Rv),$$

where the first inclusion is deduced by applying the homomorphism theorem to the canonical map $\varphi\colon K_\nu \to L\omega$, and the second one is implied from Lemma 3.10.

Thus, $\mathrm{Quot}(Rv) = K\nu$, so v is a C-valuation. □

Definition 3.20. Let $R \subseteq S$ be a ring extension and $\preceq_1$ a quasi-ordering on R. We say that a quasi-ordering $\preceq_2$ on S is an *extension of* $\preceq_1$, if

$$R^2 \cap \preceq_2 = \preceq_1 .$$

If there exists such a quasi-ordering $\preceq_2$ on S, we also say that $\preceq_1$ *extends* to S.

Obviously, if $\preceq_2$ is an extension of $\preceq_1$, then $\preceq_2$ is an ordering (respectively a valuation) if and only if $\preceq_1$ is an ordering (respectively a valuation).

Example 3.21. If $(R, \preceq)$ is a quasi-ordered ring and $x \in R$, then it is easy to see that $f \preceq' g :\Leftrightarrow f(x) \preceq g(x)$ defines a quasi-ordering on $R[X]$, which extends $\preceq$.

Inductively, we may deduce that any quasi-ordering on a ring R extends to any polynomial ring $R[X_1, \ldots, X_n]$ in finitely many variables.

Proposition 3.22. ([46, Proposition 3.17] or [38, Corollary 10.7] for ordered fields) *Let $(R, v) \subseteq (S, w)$ be an immediate extension of commutative valued rings such that w is a C-valuation. Any $\preceq_R \in \mathcal{Q}^v_{\mathfrak{q}_v}(R)$ admits a unique extension $\preceq_S \in \mathcal{Q}^w_{\mathfrak{q}_w}(S)$.*

$$\begin{tikzcd}
R \arrow[r, "immediate\ ext."] \arrow[d, dash] & S \arrow[d, dash] \\
v \arrow[r, dotted] \arrow[d, dash, "s\text{-}compatible"'] & w \arrow[d, dash, "s\text{-}compatible"] \\
\preceq_R \arrow[r, dotted] & \preceq_S
\end{tikzcd}$$

Proof. Consider the map

$$\rho\colon \mathcal{Q}^w_{\mathfrak{q}_w}(S) \to \mathcal{Q}^v_{\mathfrak{q}_v}(R),\ \preceq \mapsto \preceq \cap R^2.$$

The map ρ is well-defined, because strong compatibility is a universal statement, and therefore preserved under restriction. We conclude our proof by showing that ρ is bijective.

Since w is a C-valuation, Lemma 3.19 implies that v is also a C-valuation. So we may apply Theorem 3.13 to both, (R, v) and (S, w). Thus, we find a bijective map

$$\varphi_R\colon \mathcal{Q}^v_{\mathfrak{q}_v}(R) \to \{-1, 1\}^{I_R} \times \mathcal{O}_0(Rv) \sqcup \{1\}^{I_R} \times \mathcal{V}_0(Rv),\ \preceq \mapsto (\eta_\preceq, \preceq^*),$$

and analogously a bijective map φ_S. Since $(R, v) \subseteq (S, w)$ is immediate, we know that $\Gamma := \Gamma_v = \Gamma_w$. Therefore, when applying Theorem 3.13 to (R, v) and (S, w), we may choose the same quadratic system of representatives of R w.r.t. v, respectively of S w.r.t. w. Thus, we may assume that $I := I_R = I_S$. Moreover, since the extension $(R, v) \subseteq (S, w)$ is immediate, we know that $Rv = Sw$, whence the co-domains of φ_R and φ_S coincide.

We prove that ρ is bijective by showing that the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{Q}^w_{\mathfrak{q}_w}(S) \arrow[r, "\rho"] \arrow[d, "\varphi_S"] & \mathcal{Q}^v_{\mathfrak{q}_v}(R) \arrow[d, "\varphi_R"] \\
\varphi_S(\mathcal{Q}^w_{\mathfrak{q}_w}(S)) \arrow[r, "id"] & \varphi_R(\mathcal{Q}^v_{\mathfrak{q}_v}(R))
\end{tikzcd}$$

Let $\preceq \in \mathcal{Q}^w_{\mathfrak{q}_w}(S)$. Then $id(\varphi_S(\preceq)) = (\eta_\preceq, \preceq^*)$, and $\varphi_R(\rho(\preceq)) = (\eta_{\preceq \cap R^2}, (\preceq \cap R^2)^*)$. Clearly, $\eta_\preceq = \eta_{\preceq \cap R^2}$, since we chose $\pi_i \in R$ for all $i \in I$. So it remains to verify that $\preceq^* = (\preceq \cap R^2)^*$. But this follows immediately from the fact that the residue class domains Sw and Rv coincide. $\square$

In [16, Theorem 20.1.1(a)], Efrat proves a result for localities on algebraic field extensions that is very similar to Proposition 3.22. There, $\preceq_S$ is given and the existence and uniqueness of the valuation w is derived, such that w extends v and $\preceq_S \in \mathcal{Q}^w_{\mathfrak{q}_w}(S)$.

Due to this similarity, we transfer Efrat's result to commutative quasi-ordered rings here, even though it does not rely on the Baer-Krull theorem at all. This is achieved by arguing via the associated quasi-ordered field (see Definition 1.39).

Lemma 3.23. *Let $R \subseteq S$ be an extension of commutative rings, let $\preceq_R$ be a quasi-ordering on R, and $\preceq_S$ a quasi-ordering on S. Then the following are equivalent:*

(1) $\preceq_S$ extends $\preceq_R$ from R to S,

(2) $\preceq'_S$ extends $\preceq'_R$ from $R/\mathfrak{q}_{\preceq_R}$ to $S/\mathfrak{q}_{\preceq_S}$,

(3) $\trianglelefteq_S$ extends $\trianglelefteq_R$ from $K := Quot(R/\mathfrak{q}_{\preceq_R})$ to $L := Quot(S/\mathfrak{q}_{\preceq_S})$.

Proof. In the following, given a quasi-ordering $\preceq$, we write $(x, y) \in \preceq$ for $x \preceq y$.

First, we show that (1) implies (2). By Lemma 1.35 we get:

$$\begin{aligned}(\overline{x}, \overline{y}) \in (R/\mathfrak{q}_{\preceq_R})^2 \cap \preceq'_S &\Leftrightarrow (x, y) \in R^2 \cap \preceq_S \\ &\Leftrightarrow (x, y) \in \preceq_R \\ &\Leftrightarrow (\overline{x}, \overline{y}) \in \preceq'_R .\end{aligned}$$

Likewise, (3) follows from (2) via Proposition 1.37:

$$\begin{aligned}\left(\overline{ab^{-1}}, \overline{xy^{-1}}\right) \in K^2 \cap \trianglelefteq_S &\Leftrightarrow \left(\overline{aby^2}, \overline{xyb^2}\right) \in (R/\mathfrak{q}_{\preceq_R})^2 \cap \preceq'_S \\ &\Leftrightarrow \left(\overline{aby^2}, \overline{xyb^2}\right) \in \preceq'_R \\ &\Leftrightarrow \left(\overline{ab^{-1}}, \overline{xy^{-1}}\right) \in \trianglelefteq_R .\end{aligned}$$

It remains to show that (1) follows from (3). Like above, we obtain:

$$\begin{aligned}(x, y) \in R^2 \cap \preceq_S &\Leftrightarrow (\overline{x/1}, \overline{y/1}) \in K^2 \cap \trianglelefteq_S \\ &\Leftrightarrow (\overline{x/1}, \overline{y/1}) \in \trianglelefteq_R \\ &\Leftrightarrow (x, y) \in \preceq_R .\end{aligned}$$

□

Remark 3.24. For the proof of the next proposition, we also require the following basic facts:

(1) If $R \subseteq S$ is an integral extension of commutative rings, $\mathfrak{p} \in \mathrm{Spec}(R)$ and $\mathfrak{q} \in \mathrm{Spec}(S)$ such that $R \cap \mathfrak{q} = \mathfrak{p}$, then $R/\mathfrak{p} \subseteq S/\mathfrak{q}$ is an integral extension of commutative domains.

(2) If $R \subseteq S$ is an integral extension of commutative domains, then the field extension $\mathrm{Quot}(S)|\mathrm{Quot}(R)$ is algebraic.

Proposition 3.25. *Let $R \subseteq S$ be an integral extension of commutative rings, v a valuation on R, and $\preceq_R \in \mathcal{Q}^v_{\mathfrak{q}_v}(R)$ such that $\preceq_R$ extends to a quasi-ordering $\preceq_S$ on S. Then there is a unique valuation w on S extending v such that $\preceq_S \in \mathcal{Q}^w_{\mathfrak{q}_w}(S)$.*

Proof. Since $\preceq_S$ is an extension of $\preceq_R$, we know that $\mathfrak{q}_{\preceq_R} = R \cap \mathfrak{q}_{\preceq_S}$. Hence, Remark 3.24(1) tells us that $R/\mathfrak{q}_{\preceq_R} \subseteq S/\mathfrak{q}_{\preceq_S}$ is an integral extension of commutative domains. Let K and L denote the quotient fields of $R/\mathfrak{q}_{\preceq_R}$, respectively $S/\mathfrak{q}_{\preceq_S}$. Then $L|K$ is an algebraic field extension by Remark 3.24(2). Moreover, Corollary 1.38 implies that $\preceq_R$ and $\preceq_S$ extend to quasi-orderings $\trianglelefteq_R$ and $\trianglelefteq_L$ on K and L, respectively. Furthermore, Lemma 3.23 tells us that $\trianglelefteq_S$ is an extension of $\trianglelefteq_R$.

Finally, Lemma 2.35 implies $\trianglelefteq_R \in \mathcal{Q}^{\nu}(K)$, where ν denote the unique extension of v from R to K.

Therefore, we may apply [16, Theorem 20.1.1(a)]. Thus, there is a unique valuation ω on L extending ν such that $\trianglelefteq_S \in \mathcal{Q}^{\omega}(L)$. Let w be the restriction from L to S. Then it follows again by Lemma 2.35 and Lemma 3.23 that w is still an extension of v such that $\preceq_S \in \mathcal{Q}^{w}_{\mathfrak{q}_w}(S)$. The valuation w is unique, because any other valuation on S with these properties would extend to L, and therefore contradict the uniqueness established in [16, Theorem 20.1.1(a)]. □

3.3. **A Baer-Krull Theorem for Ordered Rings.**

In this section we derive from Theorem 3.13 a Baer-Krull theorem for commutative ordered rings (Theorem 3.26), and give various applications. Proposition 3.29 and Proposition 3.31 describe the relationship between non-Archimedean orderings on R and real valuations on R, while Proposition 3.32 transfers Proposition 3.22 and Proposition 3.25 from quasi-ordered rings to ordered rings.

Theorem 3.26. (*Baer-Krull theorem for commutative ordered rings*)
Let R be a commutative ring, v a C-valuation on R, and $\{\pi_i \colon i \in I\}$ a quadratic system of representatives of R w.r.t. v. Then the map

$$\mathcal{O}^{v}_{\mathfrak{q}_v}(R) \to \{-1,1\}^I \times \mathcal{O}_0(Rv),\ \leq\ \mapsto\ (\eta_{\leq}, \leq^*)$$

is a bijection.

Proof. We have already seen in Proposition 2.16 that $\preceq$ is an ordering if and only $\preceq^*$ is an ordering, whence the claim follows immediately from Theorem 3.13. □

If R is a field, then this result obviously coincides with Theorem 3.2. Further note that if Γ_v is 2-divisible, then Theorem 3.26 simplifies in the same manner as Theorem 3.13 (see Remark 3.16).

Proposition 3.27. *Let R be a commutative ring and v a C-valuation on R such that Γ_v is 2-divisible. Then the map*

$$\mathcal{O}^{v}_{\mathfrak{q}_v}(R) \to \mathcal{O}_0(Rv),\ \preceq\ \mapsto\ \preceq^*$$

is a bijection.

Proof. This is proven exactly like Proposition 3.15. □

By imposing extra conditions on the residue class domain Rv, we may also eliminate $\mathcal{O}_0(Rv)$ from the co-domain of our bijection in Theorem 3.26.

Proposition 3.28. *Let R be a commutative ring, v a C-valuation on R, and $\{\pi_i \colon i \in I\}$ a quadratic system of representatives of R w.r.t. v. Further suppose that $\overline{0}$ is not a sum of non-zero squares in Rv, and that for each $\overline{a} \in Rv$ there is some $\overline{0} \neq \overline{b} \in Rv$ such that either $\overline{ab^2}$ or $-\overline{ab^2}$ is a sum of squares in Rv.*
Then the map

$$\mathcal{O}^{v}_{\mathfrak{q}_v}(R) \to \{-1,1\}^I,\ \leq\ \mapsto\ \eta_{\leq}$$

is a bijection.

Proof. It suffices to show that Rv is uniquely ordered. The claim then follows immediately from Theorem 3.26. Note that Rv is uniquely ordered if and only if $\mathrm{Quot}(Rv) = K\nu$ is uniquely ordered, where $K := \mathrm{Quot}(R/\mathfrak{q}_v)$ and ν denotes the unique extension of v to K. Further note that $K\nu$ is uniquely ordered if and only if for any $\overline{a} \in (K\nu)^{\times}$ either $\overline{a}$ or $-\overline{a}$ is a sum of squares in $K\nu$.

So let $\overline{xy^{-1}} \in (K\nu)^\times$ with $\overline{x}, \overline{y} \in Rv$. Then $\overline{xy} \in Rv$, whence there exists some $\overline{0} \neq \overline{b} \in Rv$ such that (w.l.o.g.) $\overline{xyb^2}$ is a sum of squares in Rv, say $\overline{xyb^2} = \sum \overline{p_i}^2$ for some $\overline{p_i} \in Rv$. Then

$$\overline{xy^{-1}} = \sum_i \left(\overline{p_i(yb)^{-1}}\right)^2$$

is a sum of squares in $K\nu$. Moreover, $-\overline{xy^{-1}}$ is not a sum of squares in $K\nu$, since otherwise 0 would be a sum of non-zero squares in Rv. □

Conversely, if a domain is uniquely ordered, then it satisfies the conditions imposed on Rv in the previous proposition.

In the remainder of this chapter, we give a few applications for Theorem 3.26. The first two deal with the relationship between non-Archimedean orderings and real valuations on a commutative ring R, where a valuation v on R is called *real* if the residue class domain Rv is real, i.e. if Rv admits an ordering.

Proposition 3.29. *Let R be a commutative ring.*

(1) If v is a real C-valuation on R, then $\mathcal{O}^v_{\mathfrak{q}_v}(R) \neq \emptyset$.

(2) If v additionally admits negative values (i.e. $v(a) < 0$ for some $a \in R$), then any $\leq \in \mathcal{O}^v_{\mathfrak{q}_v}(R)$ is non-Archimedean. More precisely, any $a \in R$ with $v(a) < 0$ is infinitely large w.r.t. to any such ordering.

Proof. Statement (1) is an immediate consequence of Theorem 3.26 - take the preimage of the given ordering on Rv and the map $\eta = 1$ under the bijective correspondence of the said theorem.

For the proof of (2) let $\leq \in \mathcal{O}^v_{\mathfrak{q}_v}(R)$ be arbitrary and suppose that there is some $a \in R$ such that $v(a) < 0$. Since $v(a) = v(-a)$ and $\mathfrak{q}_\leq = \mathfrak{q}_v$, we may assume that $0 < a$. Hence, we have $0 < a$, and $v(a) < 0 \leq v(n)$ for all $n \in \mathbb{N}$. So the fact that v is strongly $\leq$-compatible implies $n < a$ for all $n \in \mathbb{N}$. Thus, $\leq$ is a non-Archimedean ordering on R, and a is infinitely large. □

Remark 3.30. We may also prove Proposition 3.29(2) by considering the natural valuation of $\leq$, say w (see Example 2.8). Note that $v(a) < 0$ yields $w(a) < 0$, since w is the finest strongly $\leq$-compatible valuation on R. Now $a \notin R_w$ implies that a is infinitely large.

The next result may be considered as a converse of Proposition 3.29. However, note that the following proposition involves valuations that are possibly not C-valuations.

Proposition 3.31. *Let R be a commutative ring. Then $\mathcal{O}(R)$ is the union of the Archimedean orderings on R and the sets $\mathcal{O}^v(R)$, where $v\colon R \to \Gamma_v \cup \{\infty\}$ is some non-trivial real valuation*

Proof. Let $\leq$ be a non-Archimedean ordering on R, and let v denote the natural valuation of $\leq$. Then v is strongly $\leq$-compatible (Example 2.8). Furthermore, v is non-trivial, since $\leq$ is non-Archimedean (Lemma 2.38). It remains to show that v is real. By Theorem 3.9 and Proposition 2.16 we know that $K\nu$ is real. But then so is the subring $Rv \subseteq K\nu$ (Lemma 3.10). Hence, $\leq \in \mathcal{O}^v(R)$ for some non-trivial real valuation v on R. □

We conclude this section by formulating Proposition 3.22 and Proposition 3.25 for orderings.

Proposition 3.32.

(1) *Let $(R, v) \subseteq (S, w)$ be an immediate extension of commutative valued rings such that w is a C-valuation. Any $\leq_R \in \mathcal{O}^v_{\mathfrak{q}_v}(R)$ admits a unique extension $\leq_S \in \mathcal{O}^w_{\mathfrak{q}_w}(S)$.*

(2) *Let $R \subseteq S$ be an integral extension of commutative rings, v a valuation on R, and $\leq_R \in \mathcal{O}^v_{\mathfrak{q}_v}(R)$ such that $\leq_R$ extends to an ordering $\leq_S$ on S. Then there is a unique valuation w on S extending v such that $\leq_S \in \mathcal{O}^w_{\mathfrak{q}_w}(S)$.*

Proof. Both these results follow immediately from Proposition 3.22, respectively Proposition 3.25, since any extension of an ordering is again an ordering. □

3.4. **A Baer-Krull Theorem for Valued Rings.**

In this final section of Chapter 3 we establish a Baer-Krull theorem for valued rings (Theorem 3.33). Moreover, we derive an analogue statement for Manis valuations (Corollary 3.36). Finally, we establish Proposition 3.22 and Proposition 3.25 also for valuations (Proposition 3.37 and Proposition 3.38).

Theorem 3.33. (*Baer-Krull theorem for commutative valued rings*)
Let R be a commutative ring and v a C-valuation on R. Then the map

$$\psi\colon \mathcal{V}^v_{\mathfrak{q}_v}(R) \to \mathcal{V}_0(Rv),\ w \mapsto w/v$$

is an order-preserving bijection with respect to the coarsening relation.

Proof. Immediately from Theorem 3.13 we obtain that

$$\psi\colon \mathcal{V}^v_{\mathfrak{q}_v}(R) \to \mathcal{V}_0(Rv),\ w \mapsto w^*$$

is a bijection. Moreover, Proposition 2.16(2) implies $w^* = w/v$. Finally, it follows from Proposition 2.17 that ψ is indeed order-preserving. □

Remark 3.34.

(1) According to Proposition 2.7, Theorem 3.33 characterises the refinements w of the given C-valuation v.

(2) Theorem 3.33 was already established for valued fields by Efrat (cf. [16, Proposition 7.2.1(d)])

We may also formulate a Baer-Krull theorem for commutative Manis valued rings. Recall from Definition 1.5 that a valuation v on a ring R is called Manis, if Γ_v is a group, i.e. if Γ_v is closed under additive inverses.

Lemma 3.35. *Let (R, v) be a valued ring, and let $w \in \mathcal{V}^v_{\mathfrak{q}_v}(R)$. Then w is Manis if and only if v and w/v are Manis.*

Proof. First suppose that w is Manis. Then v is Manis as a coarsening of a Manis valuation (cf. [32, Remark II.1.3]). Now let $\gamma \in \Gamma_{w/v}$. Then $\gamma = w/v(\overline{a})$ for some $a \in U_v$, and $w/v(\overline{a}) = w(a)$. Since w is Manis, we find some $b \in R$ such that $w(ab) = 0$. By strong $\preceq_w$-compatibility of v we obtain that also $v(ab) = 0$. Thus, $a \in U_v$, implies $b \in U_v$, whence $w/v(\overline{b}) = w(b) = -\gamma \in \Gamma_{w/v}$.

Now let v and w/v be Manis, and $\gamma \in \Gamma_w$. Then $\gamma = w(a)$ for some $a \in R\backslash\mathfrak{q}_w$, and $a \notin \mathfrak{q}_v$ because $\mathfrak{q}_v = \mathfrak{q}_w$. Since v is Manis, we find some $y \in R$ such that $ay \in U_v$. So $w/v(\overline{ay}) = w(ay) =: \gamma_1 \in \Gamma_{w/v}$. Since w/v is also Manis, there is some $z \in R$ such that $w(z) = w/v(\overline{z}) = -\gamma_1 \in \Gamma_{w/v}$. Therefore, $w(z) = -w(ay) = -w(a) - w(y)$. This yields $-\gamma = -w(a) = w(yz) \in \Gamma_w$. □

Corollary 3.36. (*Baer-Krull theorem for commutative Manis valued rings*) *Let R be a commutative ring and v a Manis valuation on R. Then the map*

$$\psi\colon \{w \in \mathcal{V}^v_{\mathfrak{q}_v}(R)\colon w \text{ Manis}\} \to \{u \in \mathcal{V}_0(Rv)\colon u \text{ Manis}\},\ w \mapsto w/v$$

is an order-preserving bijection with respect to the coarsening relation.

Proof. By Proposition 3.14, v is a C-valuation. Thus, the claim follows immediately from Theorem 3.33 and Lemma 3.35. □

Last but not least, we may of course also establish Proposition 3.22 and Proposition 3.25 for valuations.

Proposition 3.37.

(1) *Let $(R, v) \subseteq (S, w)$ be an immediate extension of commutative valued rings such that w is a C-valuation. Any $v' \in \mathcal{V}^v_{\mathfrak{q}_v}(R)$ (i.e. any refinement of v with same support) admits a unique extension $w' \in \mathcal{V}^w_{\mathfrak{q}_w}(S)$.*

(2) *Let $R \subseteq S$ be an integral extension of commutative rings, v a valuation on R, and $v' \in \mathcal{V}^v_{\mathfrak{q}_v}(R)$ such that v' extends to a valuation w' on S. Then there is a unique valuation w on S extending v such that $w' \in \mathcal{V}^w_{\mathfrak{q}_w}(S)$.*

Proof. This follows immediately from Proposition 3.22 and Proposition 3.25, since any extension of a valuation is again a valuation. □

Part (1) of the previous proposition was proven by Efrat for algebraic extensions of valued fields, cf. [16, Theorem 20.1.1(b)]. We conclude this whole chapter by also transferring this result to integral extensions of commutative valued rings.

Proposition 3.38. *Let $(R, v) \subseteq (S, w)$ be an integral extension of commutative valued rings, and let $v' \in \mathcal{V}^v_{\mathfrak{q}_v}(R)$. Then there is a unique valuation $w' \in \mathcal{V}^w_{\mathfrak{q}_w}(S)$ extending v'.*

Proof. As in the proof of Proposition 3.25, we derive the result by considering quotient fields. Since w extends v from R to S, Lemma 3.23 states that ω extends ν from $K := \mathrm{Quot}(R/\mathfrak{q}_v)$ to $L := \mathrm{Quot}(S/\mathfrak{q}_w)$, where ω and ν denote the unique extensions of w and v to L and K, respectively. Moreover, $\nu' \in \mathcal{V}^\nu(K)$ by Lemma 2.35, where ν' denotes the unique extension of v' to K. Finally, by Remark 3.24, the field extension $L|K$ is algebraic, since $R \subseteq S$ is an integral extension.

Hence, by [16, Theorem 20.1.1(b)], there is a unique valuation ω' on L extending ν' such that $\omega' \in \mathcal{V}^\omega(L)$. Going back from K to R and from L to S via Lemma 2.35 and Lemma 3.23, we obtain a valuation $w' \in \mathcal{V}^w_{\mathfrak{q}_w}(S)$ extending v'. The uniqueness of w' follows from the uniqueness stated in [16, Theorem 20.1.1(b)] and the fact that any other such valuation extends to a valuation on L. □

A similar result was proven by Manis ([52, Proposition 8(iii)]). There, a slightly weaker statement is derived for Manis valuations without the assumption that the ring extension is integral.

4. The Tree Structure of Quasi-Orderings

Let K be a field. In [16, Chapter 7], Efrat defines localities on K to be the disjoint union of all orderings and valuations on K, i.e. localities and quasi-orderings on fields are the same object, except that the former class provides no underlying set of axioms. Any locality λ induces a subgroup G_λ of $K^\times$, namely

$$G_\lambda = \begin{cases} 1 + I_v, & \text{if } \lambda = v \text{ is a valuation,} \\ \\ P \backslash \{0\}, & \text{if } \lambda = P \text{ is an ordering.} \end{cases}$$

Based on this, Efrat defines a coarsening relation $\leq$ on the set of all localities on K by declaring

$$\lambda_1 \leq \lambda_2 :\Leftrightarrow G_{\lambda_2} \leq G_{\lambda_1}.$$

His main result from [16, Chapter 7] states that the set of all localities on a given field, equipped with this coarsening relation, forms a rooted tree (cf. [16, Corollary 7.3.6]), i.e. a partially ordered set admitting a maximum such that for any locality λ the set $\{\lambda' \colon \lambda \leq \lambda'\}$ is linearly ordered.

The aim of the present chapter is to uniformly establish Efrat's tree structure theorem for quasi-ordered rings and to study some of its consequences. Let R be a ring. In Section 4.1, we show that the coarsening relation $\leq$ introduced in Chapter 1 (Definition 1.71) defines a partial ordering on the set of all quasi-orderings on R, which coincides with Efrat's coarsening relation. The main result of Section 4.2 states that the set $\mathcal{Q}_\mathfrak{q}(R)$ of all quasi-orderings on R with fixed support $\mathfrak{q}$ forms a rooted tree with respect to $\leq$, thereby generalising Efrat's result from above. In Section 4.3 we prove that the set of all special quasi-orderings (respectively Manis quasi-orderings) on a ring R is the ordered disjoint union of the rooted trees of all special quasi-orderings (respectively Manis quasi-orderings) with fixed support.

Section 4.4 deals with applications of the tree structure theorem. As main application we obtain that $\mathcal{Q}_\mathfrak{q}(R)$ is a spectral set, i.e. order-isomorphic to the spectrum of some commutative ring partially ordered by inclusion. The same applies to the set of all special quasi-orderings, respectively Manis quasi-orderings on R (Subsection 4.4.1). In Subsection 4.4.2 we introduce a dependency relation on $\mathcal{Q}(R)$ and deduce from the tree structure theorem that it defines an equivalence relation on $\mathcal{Q}_\mathfrak{q}(R)$. Finally, we derive the same result in Subsection 4.4.3 by considering the topology induced by a quasi-ordering.

4.1. **A Coarsening Relation for Quasi-Orderings.**

In this section we first recall the coarsening relation on quasi-orderings that we already introduced in Section 1.4 (Definition 1.71) and verify that it generalises Efrat's respective notion from above (see also [16, Chapter 7.1]). We then show that it defines a partial ordering on the set $\mathcal{Q}(R)$ of all quasi-orderings on a given ring R (Proposition 4.5).

Recall 4.1. Let R be a ring, and let $\preceq_1$ and $\preceq_2$ be quasi-orderings on R. We say that $\preceq_2$ is *coarser* than $\preceq_1$ (or $\preceq_1$ *finer* than $\preceq_2$), written $\preceq_1 \leq \preceq_2$, if

$$\forall x, y \in R \colon 0 \preceq_1 x \preceq_1 y \Rightarrow x \preceq_2 y.$$

Moreover, we call the quasi-orderings $\preceq_1$ and $\preceq_2$ *comparable*, if $\preceq_1 \leq \preceq_2$ or $\preceq_2 \leq \preceq_1$. Otherwise, we call $\preceq_1$ and $\preceq_2$ *incomparable*.

Note that if $\preceq_2$ is induced by a valuation, then this coarsening relation coincides with our notion of strong compatibility between quasi-orderings and valuations (Definition 2.4). Depending on the kind of the quasi-orderings, the coarsening relation yields the following possible cases:

(1) If both quasi-orderings are orderings, we obtain $\preceq_1 \subseteq \preceq_2$, i.e. that the finer ordering $\preceq_1$ is contained in the coarser ordering $\preceq_2$.
(2) If both quasi-orderings are valuations, we obtain that $\preceq_2$ is a coarsening of $\preceq_1$ in the sense of Definition 2.6 (see Proposition 2.7).
(3) The case where $\preceq_1$ is a valuation and $\preceq_2$ is an ordering cannot occur; $0 \preceq_1 0 \preceq_1 -1$ would imply $0 \preceq_2 -1$, a contradiction. Hence, a valuation is never finer than an ordering.
(4) If $\preceq_1$ is an ordering and $\preceq_2$ a valuation, we obtain that $\preceq_2$ is strongly $\preceq_1$-compatible (Definition 2.4).

Therefore, our coarsening relation subsumes three different notions at once. Moreover, it is a uniform generalisation of Efrat's coarsening relation (cf. [16, p.69]). In fact, our definition yields a significant improvement. According to us, an ordering P is coarser than an ordering Q if and only if $Q \subseteq P$, while for Efrat it is the other way round. Since he works over fields, inclusion of orderings implies equality anyway. In the ring case, however, there are proper inclusions of orderings, whence, in analogy to valuation rings, the finer ordering should be contained in the coarser one.

Remark 4.2. Quasi-orderings admit a uniform definition of the set G_λ that Efrat introduced (see the beginning of this chapter, or [16, Chapter 7.1]). Indeed, for a quasi-ordered field $(K, \preceq)$ we have

$$G_{\preceq} = \{x \in K^\times : 1 - x \prec 1\} \leq K^\times.$$

However, this set $G_{\preceq}$ is inconvenient to work with in a uniform way. Moreover, Efrat's approach does not apply to rings for various reasons.

We conclude this section by showing that $\leq$ defines a partial ordering on the set $\mathcal{Q}(R)$ of all quasi-orderings on a given ring R

Lemma 4.3. *Let R be a ring and $\preceq_1, \preceq_2 \in \mathcal{Q}(R)$. Then the following are equivalent:*

(1) $\preceq_1 \leq \preceq_2$,
(2) $\forall x, y \in R \colon 0 \preceq_1 x \preceq_1 y \Rightarrow 0 \preceq_2 x \preceq_2 y$.

Proof. Obviously, (2) implies (1). For the converse let $\preceq_1 \leq \preceq_2$ and $0 \preceq_1 x \preceq_1 y$. Then $x \preceq_2 y$. Moreover, $0 \preceq_1 0 \preceq_1 x$ yields $0 \preceq_2 x$. Thus, $0 \preceq_2 x \preceq_2 y$. □

Lemma 4.4. *Let $(R, \preceq)$ be a quasi-ordered ring and $x, y \in R$. If $x \preceq y \prec 0$, then $0 \prec -y \preceq -x$.*

Proof. Clearly $0 \prec -x, -y$ by Lemma 1.29. It remains to show that $-y \preceq -x$. Assume that $-x \prec -y$. We know that $y \prec 0 \prec -x, -y$, and therefore $-x \not\preceq y$ and $y \not\preceq -y$. Via (QR4), it follows from $x \preceq y$ that $0 \preceq y - x$ and from $-x \preceq -y$ that $y - x \preceq 0$. Thus, $y - x \in \mathfrak{q}_{\preceq}$. This implies $-y \sim -x$ (see Lemma 1.28), a contradiction. Hence, $-y \preceq -x$. □

Proposition 4.5. *Let R be a ring. Then $(\mathcal{Q}(R), \leq)$ is a partially ordered set.*

Proof. Obviously, $\leq$ is reflexive. Next, suppose that $\preceq_1 \leq \preceq_2 \leq \preceq_3$, and let $x, y \in R$ such that $0 \preceq_1 x \preceq_1 y$. From $\preceq_1 \leq \preceq_2$ and Lemma 4.3 it follows $0 \preceq_2 x \preceq_2 y$. Thus, $\preceq_2 \leq \preceq_3$ yields $x \preceq_3 y$. This shows $\preceq_1 \leq \preceq_3$, whence $\leq$ is transitive.

Last but not least, we prove that $\leq$ is anti-symmetric. So suppose that $\preceq_1 \leq \preceq_2$ and $\preceq_2 \leq \preceq_1$. Then Lemma 4.3 implies that $0 \preceq_1 x \preceq_1 y \Leftrightarrow 0 \preceq_2 x \preceq_2 y$ for all $x, y \in R$, in particular $0 \preceq_1 x \Leftrightarrow 0 \preceq_2 x$. So if $\preceq_1 \neq \preceq_2$, there exist some $x, y \in R$ such that (w.l.o.g.) $x \preceq_1 y \prec_1 0$, but $y \prec_2 x \prec_2 0$. Lemma 4.4 tells us that $0 \prec_1 -y \preceq_1 -x$, and $0 \prec_2 -x \preceq_2 -y$. Now if $-x \sim_2 -y$, then Corollary 1.60 implies that $x \sim_2 y$, a contradiction. Hence, $0 \prec_1 -y \preceq_1 -x$ but $-x \prec_2 -y$, contradicting the assumption that $\preceq_1 \leq \preceq_2$. Therefore, $\preceq_1 = \preceq_2$. □

4.2. **The Tree Structure Theorem for Quasi-Ordered Rings.**

In the present section we establish the main theorem of this chapter, stating that for any ring R and any two-sided completely prime ideal $\mathfrak{q}$ of R, the set $(\mathcal{Q}_{\mathfrak{q}}(R), \leq)$ of all quasi-orderings on R with support $\mathfrak{q}$, equipped with the coarsening relation $\leq$, is a rooted tree (Theorem 4.14). From this we deduce a partial ordering $\leq'$, which canonically partitions $\mathcal{Q}(R)$ into the trees $\mathcal{Q}_{\mathfrak{q}}(R)$ (Corollary 4.15). Next, we show that a quasi-ordering $\preceq$ on R is comparable with a quasi-ordering of support $\mathfrak{q}$ if and only if $\mathfrak{q}$ is $\preceq$-convex (Proposition 4.23). This gives rise to the result that the set of all valuations with support $\mathfrak{q}$ and all orderings for which $\mathfrak{q}$ is convex, partially ordered by $\leq$, is a rooted tree (Proposition 4.27).

In this section, let R always be a ring and $\mathfrak{q}$ a two-sided completely prime ideal of R. We use the notation introduced in Notation 3.1(4), i.e. $\mathcal{Q}(R)$ denotes the set of all quasi-orderings on R, and $\mathcal{Q}_{\mathfrak{q}}(R)$ the set of all quasi-orderings on R with support $\mathfrak{q}$. Moreover, $\leq$ always denotes the coarsening relation introduced in Definition 1.71 (see also Recall 4.1).

Definition 4.6.

(1) A partially ordered set $(S, \leq)$ is called a *tree* if for all $s, s_1, s_2 \in S$ with $s \leq s_1, s_2$, the elements s_1 and s_2 are comparable, i.e. $s_1 \leq s_2$ or $s_2 \leq s_1$.

(2) A *rooted tree* is a tree that admits a maximum.

(3) A *branch* S_i of a tree S is a maximal chain in S.

(4) The *length of a branch* S_i is the order type of S_i.

Remark 4.7. Compared to the usual definition of a (rooted) tree we reversed the ordering in Definition 4.6. This is because we want coarse valuations to be large elements with respect to our coarsening relation. In particular, trivial valuations shall be maximal elements.

Unfortunately, as we will see below (Lemma 4.25), the whole set $(\mathcal{Q}(R), \leq)$ is in general neither a tree, nor admits a maximum. We first show that $(\mathcal{Q}_{\mathfrak{q}}(R), \leq)$ is always a rooted tree (see Theorem 4.14 below).

Lemma 4.8. *Let $\preceq_1, \preceq_2 \in \mathcal{Q}_{\mathfrak{q}}(R)$ such that $\preceq_1 \leq \preceq_2$ and $-1 \preceq_2 0$. Then $0 \preceq_1 x$ if and only if $0 \preceq_2 x$.*

Proof. The only-if-implication is trivial. For the if-implication suppose that $0 \preceq_2 x$, and assume for the sake of a contradiction that $x \prec_1 0$. Then $0 \prec_1 -x$, so from the assumption $\preceq_1 \leq \preceq_2$ and the fact that the supports of these quasi-orderings coincide, we obtain $0 \prec_2 -x$. Applying (QR3) on $0 \preceq_2 x$ with $-x$ yields $0 \preceq_2 -1$, a contradiction to the assumption that $-1 \preceq_2 0$. Thus, $0 \preceq_1 x$. □

Lemma 4.9. *Let $0 \prec x \preceq y$ and $0 \preceq a \prec b$. Then $ax \prec by$ and $xa \prec yb$.*

Proof. Using (QR2) and (QR3), we obtain from $a \prec b$ and $0 \prec x$ that $ax \prec bx$. Moreover, from $x \preceq y$ and $0 \preceq b$, it follows via (QR2) that $bx \preceq by$. Therefore, $ax \prec by$. The same arguing shows that also $xa \prec yb$. □

In the first part of the proof of the following proposition, we adapt the proof given for Proposition 2.31.

Proposition 4.10. *Let* $\preceq, \preceq_1, \preceq_2 \in \mathcal{Q}_{\mathfrak{q}}(R)$ *such that* $\preceq \leq \preceq_1, \preceq_2$ *. Then* $\preceq_1$ *and* $\preceq_2$ *are comparable.*

Proof. First suppose that $0 \prec_i -1$ for $i = 1, 2$. By Lemma 1.45 and Corollary 1.46(2), we know that $0 \preceq_i x$ and $x \sim_{\preceq_i} -x$ for all $x \in R$. Assume for the sake of a contradiction that $\preceq_1$ and $\preceq_2$ are incomparable. Then we find $a_i, b_i \in R\backslash\mathfrak{q}$ such that $0 \prec_1 a_1 \preceq_1 a_2$ but $0 \prec_2 a_2 \prec_2 a_1$, and $0 \prec_2 b_1 \preceq_2 b_2$ but $0 \prec_1 b_2 \prec_1 b_1$. Since $0 \prec_i -1$, we may w.l.o.g. assume that $0 \prec a_i, b_i$ (otherwise replace a_i with $-a_i$, respectively b_i with $-b_i$). Lemma 4.9 implies $a_1b_2 \prec_1 a_2b_1$ and $a_2b_1 \prec_2 a_1b_2$. Since all a_i, b_i are positive w.r.t. $\preceq$, the contraposition of $\preceq \leq \preceq_i$ yields $a_1b_2 \prec a_2b_1$ and $a_2b_1 \prec a_1b_2$, a contradiction. Hence, $\preceq_1$ and $\preceq_2$ are comparable.

Next, suppose that $-1 \prec_i 0$ for some i. Then also $-1 \prec 0$, because $\preceq \leq \preceq_i$. Lemma 4.8 implies $0 \preceq x \Leftrightarrow 0 \preceq_i x$. Together with Lemma 1.42 (if $y \notin \mathfrak{q}$), respectively Lemma 1.28 (if $y \in \mathfrak{q}$), this yields

$$x \preceq y \Leftrightarrow x - y \preceq 0 \Leftrightarrow x - y \preceq_i 0 \Leftrightarrow x \preceq_i y,$$

whence $\preceq = \preceq_i$. Since $\preceq \leq \preceq_1, \preceq_2$, this finishes the proof. $\square$

Definition 4.11. A quasi-ordering $\preceq$ on R is called *trivial* if $R/{\sim_{\preceq}}$ has exactly two elements.

Remark 4.12. If $\preceq$ is some trivial quasi-ordering on a ring R, then $E(0) = \mathfrak{q}_{\preceq}$ and $E(1) = R\backslash\mathfrak{q}_{\preceq}$ by (QR1), i.e. $x \sim y$ for all $x, y \in R\backslash\mathfrak{q}_{\preceq}$. There is a unique trivial quasi-ordering with support $\mathfrak{q}$, namely the one induced by the trivial valuation with support $\mathfrak{q}$. We denote it by $\preceq_{\mathfrak{q}}$.

Proposition 4.13. *The trivial quasi-ordering* $\preceq_{\mathfrak{q}}$ *is the maximum of* $(\mathcal{Q}_{\mathfrak{q}}(R), \leq)$.

Proof. Let $\preceq \in \mathcal{Q}_{\mathfrak{q}}(R)$, and let $x, y \in R$ such that $0 \preceq x \preceq y$. If $x \in \mathfrak{q}$, then $x \preceq_{\mathfrak{q}} y$ by Remark 4.12. If $x \notin \mathfrak{q}$, then also $y \notin \mathfrak{q}$ since $0 \preceq x \preceq y$. Consequently, $x \sim_{\preceq_{\mathfrak{q}}} 1 \sim_{\preceq_{\mathfrak{q}}} y$. Therefore, $\preceq \leq \preceq_{\mathfrak{q}}$, so $\preceq_{\mathfrak{q}}$ is the maximum of $\mathcal{Q}_{\mathfrak{q}}(R)$. $\square$

Theorem 4.14. (the tree structure theorem)
Let R be a ring and $\mathfrak{q}$ a two-sided completely prime ideal of R. Then $(\mathcal{Q}_{\mathfrak{q}}(R), \leq)$ is a rooted tree.

Proof. By Proposition 4.5, $(\mathcal{Q}_{\mathfrak{q}}(R), \leq)$ is a partially ordered set. Proposition 4.10 implies that, given a quasi-ordering $\preceq \in \mathcal{Q}_{\mathfrak{q}}(R)$, its coarsenings in $\mathcal{Q}_{\mathfrak{q}}(R)$ are linearly ordered. Moreover, $\mathcal{Q}_{\mathfrak{q}}(R)$ admits a maximum, namely the trivial quasi-ordering with support $\mathfrak{q}$ (Proposition 4.13). Thus, $(\mathcal{Q}_{\mathfrak{q}}(R), \leq)$ is a rooted tree. $\square$

Corollary 4.15. *Let R be a ring. Then the partial ordering*

$$\preceq_1 \leq' \preceq_2 :\Leftrightarrow \preceq_1 \leq \preceq_2 \text{ and } \mathfrak{q}_{\preceq_2} \subseteq \mathfrak{q}_{\preceq_1}$$

yields a partition of $\mathcal{Q}(R)$ into the rooted trees $\mathcal{Q}_{\mathfrak{q}}(R)$, where $\mathfrak{q}$ runs over all two-sided completely prime ideals of R. Moreover, $(\mathcal{Q}(R), \leq')$ is a tree.

Proof. Note that $\preceq_1 \leq' \preceq_2$ if and only if $\preceq_1 \leq \preceq_2$ and $\mathfrak{q}_{\preceq_1} = \mathfrak{q}_{\preceq_2}$. Therefore, both claims easily follow from Theorem 4.14. $\square$

We will see in Section 4.3 that this partition comes naturally, if we only consider special quasi-orderings, respectively Manis quasi-orderings.

In case where R is commutative, and at the cost of uniformity, we could have easily derived Theorem 4.14 from Efrat's result that the set of all localities on a field forms a rooted tree (cf. [16, Corollary 7.3.6]). This is due to the following generalisation of Lemma 2.35:

Lemma 4.16. *Let R be a commutative ring and $\preceq_1, \preceq_2 \in \mathcal{Q}_\mathfrak{q}(R)$. The following are equivalent:*

(1) $\preceq_1 \leq \preceq_2$,

(2) $\preceq_1' \leq \preceq_2'$,

(3) $\trianglelefteq_1 \leq \trianglelefteq_2$.

Proof. Clearly, (1) $\Leftrightarrow$ (2) and (3) $\Rightarrow$ (2). For the proof of (2) $\Rightarrow$ (3) suppose that $\overline{0}/\overline{1} \trianglelefteq_1 \overline{x}/\overline{y} \trianglelefteq_1 \overline{a}/\overline{b}$. Then $\overline{0} \preceq_1' \overline{xyb^2} \preceq_1' \overline{aby^2}$. So $\preceq_1' \leq \preceq_2'$ yields $\overline{xyb^2} \preceq_2' \overline{aby^2}$. This implies $\overline{x}/\overline{y} \trianglelefteq_2 \overline{a}/\overline{b}$, as desired. $\square$

Lemma 4.16 and the fact that the map $\preceq \mapsto \trianglelefteq$ is a bijective correspondence yield an order isomorphism between $(\mathcal{Q}_\mathfrak{q}(R), \leq)$ and $(\mathcal{Q}(\mathrm{Quot}(R/\mathfrak{q})), \leq)$, whence the rooted tree structure is preserved.

Remark 4.17.

(1) Any branch S of $\mathcal{Q}_\mathfrak{q}(R)$ consists of the trivial valuation $v_\mathfrak{q}$, possibly further valuations, and at most one ordering. If such an ordering exists, then it is the least element of S. Moreover, it admits a neighbour in S, namely its natural valuation.

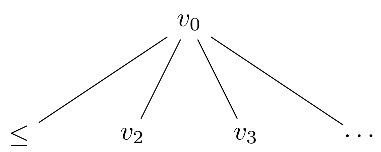
(2) If a branch S of $\mathcal{Q}_\mathfrak{q}(R)$ admits a least element $\preceq$, then the length of S is either $\mathrm{rk}(R, \preceq)+1$ (if $\preceq$ is a valuation), or $\mathrm{rk}(R, \preceq)+2$ (if $\preceq$ is an ordering). In general, the rank tells us how many quasi-orderings are above a given quasi-ordering in S.

(3) Let v be a C-valuation on R. Our Baer-Krull theorems tell us how many orderings (see Theorem 3.26) and how many valuations (see Theorem 3.33) are below v in the tree $\mathcal{Q}_{\mathfrak{q}_v}(R)$.

(4) If R is a commutative ring with characteristic 0, then $(\mathcal{Q}_0(R), \leq)$ admits infinitely many branches. This is because $(\mathcal{Q}_0(R), \leq)$ and $(\mathcal{Q}(\mathrm{Quot}(R)), \leq)$ are order-isomorphic (see Lemma 4.16), the fact that the p-adic valuations on $\mathbb{Q} \subseteq \mathrm{Quot}(R)$ are pairwise incomparable, and Chevalley's extension theorem (cf. [18, Theorem 3.1.1]). The latter result states that any valuation on a field K extends to any field extension of K.

Example 4.18. We give two examples for the tree structure theorem.

(1) The integers $\mathbb{Z}$ admit as support $\{0\}$ quasi-orderings the unique ordering $\leq$, the trivial valuation v_0, and the p-adic valuation v_p for any prime number $p \in \mathbb{N}$. Therefore, the rooted tree $(\mathcal{Q}_0(\mathbb{Z}), \leq)$ is completely described by the following diagram, where a connecting line between two quasi-orderings means that the upper quasi-ordering is coarser than the lower one:



(2) Consider the following quasi-orderings on $\mathcal{Q}_0(\mathbb{Q}[X])$:

(i) the Archimedean ordering P_a defined by $0 \leq f :\Leftrightarrow 0 \leq f(\pi)$ in $\mathbb{R}$,

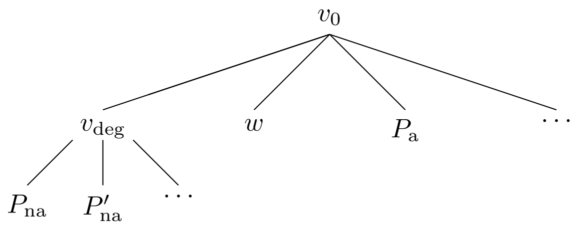
(ii) the non-Archimedean ordering P_{na} defined by $\mathbb{Q} < X$,

(iii) the non-Archimedean ordering P_{na}' defined by $X < \mathbb{Q}$,

(iv) the degree valuation $v_{\deg}\colon \mathbb{Q}[X] \to \mathbb{Z} \cup \{\infty\}$, $f \mapsto -\deg(f)$,

(v) the valuation $w\colon \mathbb{Q}[X] \to \mathbb{Z}$, $0 \neq \sum a_i X^i \mapsto \min\{v_p(a_i)\colon a_i \neq 0\}$ for some p-adic valuation v_p on $\mathbb{Q}$ (see Proposition 1.16).

Then the respective part of $(\mathcal{Q}_0(\mathbb{Q}[X]), \leq)$ looks as follows:



Indeed, P_a is Archimedean, so its only coarsening is the trivial valuation (Corollary 2.41). There is no valuation strictly in-between v_0 and w, since the rank of Γ_w is 1 (Corollary 2.37), and there is no ordering below w, because any p-adic valuation on $\mathbb{Q}$ is incomparable with the ordering on $\mathbb{Q}$. Theorem 3.26 tells us that there are $\#\{-1,1\} \times \mathcal{O}(\mathbb{Q}) = 2$ orderings below v_{deg}, namely P_{na} and P'_{na}. Finally, by Theorem 3.33, the valuations below v_{deg} are in bijective correspondence with $\mathcal{V}(\mathbb{Q})$. The trivial valuation on $\mathbb{Q}$ corresponds to v_{deg} itself, while the p-adic valuations on $\mathbb{Q}$ correspond to some rank 2 valuations w_p on R whose restriction to $\mathbb{Q}$ coincides with the p-adic valuation on $\mathbb{Q}$.

Next, we discuss the failure of the tree structure theorem for $(\mathcal{Q}(R), \leq)$.

The following proposition and Theorem 4.14 particularly imply that $(\mathcal{Q}(R), \leq)$ is a rooted tree if and only if R has exactly one two-sided completely prime ideal.

Proposition 4.19. *Let R be a ring. Then $(\mathcal{Q}(R), \leq)$ admits a maximum if and only if R has exactly one two-sided completely prime ideal.*

Proof. One direction was already shown in Proposition 4.13.

Conversely, suppose that R has at least two different such ideals, and assume that $(\mathcal{Q}(R), \leq)$ admits a maximum, say $\preceq$. Then Proposition 4.13 tells us that $\preceq$ must be trivial, say $\preceq = \preceq_{\mathfrak{q}}$. Let $\mathfrak{q} \neq \mathfrak{p}$ be another two-sided completely prime ideal of R. From $\preceq_{\mathfrak{p}} \leq \preceq_{\mathfrak{q}}$, it follows that $\mathfrak{p} \subsetneq \mathfrak{q}$. Let $y \in \mathfrak{q} \backslash \mathfrak{p}$. Then we have $0 \preceq_{\mathfrak{p}} 1 \preceq_{\mathfrak{p}} y$, but $y \prec_{\mathfrak{q}} 1$, a contradiction to $\preceq_{\mathfrak{p}} \leq \preceq_{\mathfrak{q}}$. Thus, $(\mathcal{Q}(R), \leq)$ admits no maximum. $\square$

Remark 4.20. A commutative ring R has exactly one prime ideal if and only if every $x \in R$ is either nilpotent or a unit.

The previous discussion raises the question whether the subset $\{\preceq \in \mathcal{Q}(R) \colon \preceq \leq \preceq_{\mathfrak{q}}\}$ of $\mathcal{Q}(R)$ is a rooted tree. The notion of convexity yields a precise characterisation of the quasi-orderings in $\mathcal{Q}(R)$ that are finer than $\preceq_{\mathfrak{q}}$.

Recall 4.21. (see Definition 1.56) Let $(R, \preceq)$ be a quasi-ordered ring and $M \subseteq R$ an additive subgroup of R. Then M is *convex*, if $0 \preceq x \preceq y \in M$ implies $x \in M$ for all $x, y \in R$.

Example 4.22.

(1) If $(R, \preceq)$ is a quasi-ordered ring, then $\mathfrak{q}_{\preceq}$ is the smallest convex subgroup of R.

(2) Let $R = \mathbb{R}[X]$, let P be the unique ordering on R with $0 <_P X <_P \mathbb{R}_{>0}$, and let $M = (X)$. Then M is convex, and $\mathfrak{q}_P = \{0\} \subsetneq M$.

(3) Replace M in (2) with $M' = (X+1)$. Then $0 <_P X <_P X+1 \in M'$, but $X \notin M'$. Thus, M' is not convex.

Proposition 4.23. *Let R be a ring, $\preceq$ a quasi-ordering on R and $\mathfrak{q}$ a two-sided completely prime ideal of R. The following are equivalent:*

(1) $\mathfrak{q}$ *is* $\preceq$*-convex,*
(2) $\preceq\, \leq\, \preceq_1$ *for some quasi-ordering* $\preceq_1$ *with support* $\mathfrak{q}$,
(3) $\preceq\, \leq \preceq_{\mathfrak{q}}$.

Proof. We first show that (1) implies (2) and (3). Let $0 \preceq x \preceq y$. We have to show that $x \preceq_{\mathfrak{q}} y$. For $x \in \mathfrak{q}$ this is trivial. If $x \notin \mathfrak{q}$, then $y \notin \mathfrak{q}$ by (1), and therefore $x \sim_{\preceq_{\mathfrak{q}}} 1 \sim_{\preceq_{\mathfrak{q}}} y$.
If $\preceq$ is finer than some quasi-ordering $\preceq_1$ with support $\mathfrak{q}$, then $\preceq\, \leq\, \preceq_1\, \leq\, \preceq_{\mathfrak{q}}$, whence $\preceq\, \leq\, \preceq_{\mathfrak{q}}$. Thus, (2) implies (3).
Finally, we show that (3) implies (1). So suppose that $0 \preceq x \preceq y \in \mathfrak{q}$. Then also $0 \preceq_{\mathfrak{q}} x \preceq_{\mathfrak{q}} y$ by (3) and Lemma 4.3. Therefore, $x \in \mathfrak{q}$. □

Corollary 4.24. *Let R be a ring and $\mathfrak{q}$ a two-sided completely prime ideal of R. Then*

$$\mathcal{Q}_{\mathfrak{q}}(R)' := \{\preceq\, \in \mathcal{Q}(R) \colon \mathfrak{q} \text{ is } \preceq\text{-convex}\},$$

equipped with $\leq$, is a partially ordered set admitting a maximum, namely $\preceq_{\mathfrak{q}}$.

Proof. This follows immediately from Proposition 4.5 and Proposition 4.23. □

However, $(\mathcal{Q}_{\mathfrak{q}}(R)', \leq)$, and therefore a fortiori $(\mathcal{Q}(R), \leq)$, is in general not a tree. We may for instance consider the following valuations (see Proposition 1.16):

$$v \colon \mathbb{R}[X,Y] \to \mathbb{Z}\times\mathbb{Z},\ 0 \neq \textstyle\sum_{i,j} a_{ij}X^iY^j \mapsto \min\{i(1,0)+j(0,1) \colon a_{ij}\neq 0\},$$
$$w \colon \mathbb{R}[X,Y] \to \mathbb{Z},\ 0 \neq \textstyle\sum_{i,j} a_{ij}X^iY^j \mapsto \min\{i+j\infty \colon a_{ij}\neq 0\},$$
$$u \colon \mathbb{R}[X,Y] \to \mathbb{Z},\ 0 \neq \textstyle\sum_{i,j} a_{ij}X^iY^j \mapsto \min\{j \colon a_{ij}\neq 0 \text{ for some } i\},$$

where $\mathbb{Z}\times\mathbb{Z}$ is equipped with the inverse lexicographic ordering.

Lemma 4.25. *Consider the valuations v, w and u on $\mathbb{R}[X,Y]$.*

(1) (Y) is convex w.r.t. v, w and u.
(2) $v \leq w, u$, but neither $w \leq u$, nor $u \leq w$.

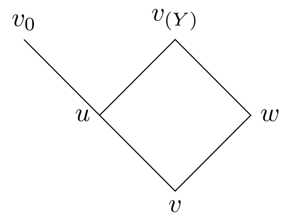
Proof. In this proof, when we consider polynomials in $\mathbb{R}[X,Y]$, we may w.l.o.g. assume that they are monomials. This is due to the definition of v, w and u.

(1) We have to show that if $f \in (Y)$ and $v(f) \leq v(g)$, then also $g \in (Y)$ for any $g \in \mathbb{R}[X,Y]$, and analogously for w and u. For either of these valuations, this is obviously true by the way they are defined.
(2) We first show $v \leq w, u$. Suppose that $u(g) < u(f)$. Then the power of Y in f is bigger than the one in g. Thus, also $v(g) < v(f)$. Hence, $v \leq u$. Likewise, suppose that $w(g) < w(f)$. Then Y does not appear in g, and Y appears in f or the power of X in f is strictly bigger than the power of X in g. Hence, $v(g) < v(f)$. Therefore, also $v \leq w$.

Clearly $w \not\leq u$, since $w \leq u$ would imply that $(Y) = \mathfrak{q}_w \subseteq \mathfrak{q}_u = (0)$, a contradiction. Now let $f = X^2$ and $g = X$. Then $w(g) = 1 < 2 = w(f)$, but $u(g) = u(f) = 0$, i.e. $u(g) \not< u(f)$. Thus, $u \not\leq w$.

□

The failure of both properties of a rooted tree over $(\mathbb{R}[X,Y], \leq)$ can be visualised by the following diagram, where v_0 and $v_{(Y)}$ denote the trivial valuations with support $\{0\}$, respectively (Y).



We conclude this section by showing that the statement of Theorem 4.14 can nevertheless be improved, however, at the cost of uniformity. More precisely, we will show that the set $\mathcal{Q}_{\mathfrak{q}}(R)''$ consisting of all valuations with support $\mathfrak{q}$ and all $\mathfrak{q}$-convex orderings is a rooted tree.

By Proposition 4.5 and Proposition 4.23, it remains to verify that $P \leq \preceq_1, \preceq_2$ implies $\preceq_1 \leq \preceq_2$ or $\preceq_2 \leq \preceq_1$ for all $P, \preceq_1, \preceq_2 \in \mathcal{Q}(R)''_{\mathfrak{q}}$. Note that we wrote P for an ordering, because for a valuation v we would have $\mathfrak{q}_v = \mathfrak{q}_{\preceq_1} = \mathfrak{q}_{\preceq_2}$ by definition of $\mathcal{Q}_{\mathfrak{q}}(R)''$, whence the claim follows from Theorem 4.14. We distinguish whether $\preceq_1$ and $\preceq_2$ are orderings or valuations.

If $\preceq_1$ and $\preceq_2$ are orderings, this is a standard result from real algebra. The result is also known if both, $\preceq_1$ and $\preceq_2$, are valuations (cf. [56, Lemma 4.2]). Therefore, we are left to deal with the case where $P \leq v, Q$ for some valuation v with support $\mathfrak{q}$ and some orderings P, Q such that $\mathfrak{q}$ is convex with respect to the two of them.

Lemma 4.26. *Let R be a ring, v a valuation, and P, Q orderings on R such that $P \leq v, Q$. Then $Q \leq v$ if and only if $\mathfrak{q}_Q \subseteq \mathfrak{q}_v$.*

Proof. The only-if-implication is trivial. For the converse we have to show that $0 \leq_Q x \leq_Q y$ implies $v(y) \leq v(x)$. If $x \in \mathfrak{q}_Q \subseteq \mathfrak{q}_v$ this is clear. If $x \notin \mathfrak{q}_Q$, then also $y \notin \mathfrak{q}_Q$. Since $P \subseteq Q$, we have $Q = P \cup \mathfrak{q}_Q$, whence $x, y \in P$. Thus, $0 \leq_P x \leq_P y$, so $P \leq v$ implies $v(y) \leq v(x)$. □

By definition of $\mathcal{Q}_{\mathfrak{q}}(R)''$, we know that $\mathfrak{q}_Q \subseteq \mathfrak{q}_v$ applies. Thus, we have proven:

Proposition 4.27. *Let R be a ring and $\mathfrak{q}$ a two-sided completely prime ideal of R. Then the set $(\mathcal{Q}_{\mathfrak{q}}(R)'', \leq)$ is a rooted tree.*

Corollary 4.28. *Let R be a ring and $\mathfrak{q}$ a two-sided completely prime ideal of R. Then $(\mathcal{Q}_{\mathfrak{q}}(R)''', \leq)$ is a tree, where $\mathcal{Q}_{\mathfrak{q}}(R)'''$ denotes the set of all valuations with support $\mathfrak{q}$ and all orderings with support contained in $\mathfrak{q}$.*

Proof. This applies, because in the proof of the previous proposition we only used convexity of $\mathfrak{q}$ to ensure that $(\mathcal{Q}_{\mathfrak{q}}(R)'', \leq)$ has a maximum □

Summary 4.29. We briefly summarise our results. Let R be a ring, $\mathfrak{q}$ a two-sided completely prime ideal of R, and $\leq$ the coarsening relation from Definition 1.71.

(1) $(\mathcal{Q}(R), \leq)$ is a partially ordered set, where $\mathcal{Q}(R)$ denotes the set of all quasi-orderings on R (Proposition 4.5).
(2) $(\mathcal{Q}_{\mathfrak{q}}(R), \leq)$ is a tree, where $\mathcal{Q}_{\mathfrak{q}}(R)$ denotes the set of all quasi-orderings on R with support $\mathfrak{q}$ (Theorem 4.14).
(3) The set $(\mathcal{Q}_{\mathfrak{q}}(R)', \leq)$ of all $\mathfrak{q}$-convex quasi-orderings is a partially ordered set admitting $\preceq_{\mathfrak{q}}$ as a maximum (Corollary 4.24).
(4) The set $(\mathcal{Q}_{\mathfrak{q}}(R)'', \leq)$ of all valuations on R with support $\mathfrak{q}$ and all $\mathfrak{q}$-convex orderings on R is a rooted tree (Proposition 4.27)
(5) The set $(\mathcal{Q}_{\mathfrak{q}}(R)''', \leq)$ of all valuations on R with support $\mathfrak{q}$ and all orderings on R with support contained in $\mathfrak{q}$ is a tree. (Corollary 4.28).

We have

$$\mathcal{Q}(R) \supseteq \mathcal{Q}_{\mathfrak{q}}(R)', \mathcal{Q}_{\mathfrak{q}}(R)''' \supseteq \mathcal{Q}_{\mathfrak{q}}(R)'' \supseteq \mathcal{Q}_{\mathfrak{q}}(R).$$

4.3. **The Tree Structure of Special and Manis Quasi-Orderings.**

In Chapter 1 we introduced special quasi-orderings (Definition 1.54) and Manis quasi-orderings (Definition 1.63), that way generalising the respective notions from valuation theory. Here, we establish the result that the set $\mathcal{Q}^S_{\mathfrak{q}}(R)$ of all special quasi-orderings with support $\mathfrak{q}$ on a ring R is a rooted subtree of $\mathcal{Q}_{\mathfrak{q}}(R)$, and that the set $\mathcal{Q}^S(R)$ of all special quasi-orderings on R is the ordered disjoint union of the rooted trees $\mathcal{Q}^S_{\mathfrak{q}}(R)$ (Theorem 4.34). We conclude this section by proving that the same applies to Manis quasi-orderings (Theorem 4.37).

Notation 4.30. Let R be a ring and $\mathfrak{q}$ a two-sided completely prime ideal of R. We denote by $\mathcal{Q}^S(R)$ the set of all special quasi-orderings on R, and by $\mathcal{Q}^S_{\mathfrak{q}}(R)$ the set of all special quasi-orderings on R with support $\mathfrak{q}$.

The sets $\mathcal{Q}^M(R)$ and $\mathcal{Q}^M_{\mathfrak{q}}(R)$ are defined analagously for Manis quasi-orderings.

Definition 4.31. A (rooted) *subtree* of a tree $(S, \leq_S)$ is a (rooted) tree $(T, \leq_T)$ such that

(1) $T \subseteq S$ and $\leq_T = \leq_S \cap T^2$,

(2) T is upward closed under $\leq_S$, i.e. if $s, t \in S$ and $T \ni t \leq_S s$, then $s \in T$.

Condition (1) of the previous Definition is obviously satisfied by the subsets $\mathcal{Q}^S_{\mathfrak{q}}(R)$ and $\mathcal{Q}^M_{\mathfrak{q}}(R)$ of $\mathcal{Q}_{\mathfrak{q}}(R)$ The following Lemma shows that (2) also applies.

Lemma 4.32. *Let R be a ring, and let $\preceq_1 \leq \preceq_2$ be quasi-orderings on R.*

(1) If $\preceq_1$ is special, then $\preceq_2$ is special.

(2) If $\preceq_1$ is Manis, then $\preceq_2$ is Manis.

Proof. We show (1), the proof of (2) being analogue. From $\preceq_1 \leq \preceq_2$ it immediately follows that $\mathfrak{q}_{\preceq_1} \subseteq \mathfrak{q}_{\preceq_2}$. Now let $x \in R\backslash\mathfrak{q}_2$. Then also $x \in R\backslash\mathfrak{q}_1$. So we find some $y \in R$ such that $0 \prec_1 1 \preceq_1 xy$. Thus, $\preceq_1 \leq \preceq_2$ implies that $1 \preceq_2 xy$, i.e. that $\preceq_2$ is special. □

Definition 4.33. If a partially ordered set $(X, \leq_X)$ is the disjoint union of partially ordered sets $(X_i, \leq_{X_i})_{i \in I}$, we say that X is the *ordered disjoint union* of the sets X_i, if for all $x, y \in X$:

$$x \leq_X y \Leftrightarrow \exists i \in I \colon x, y \in X_i \text{ and } x \leq_{X_i} y.$$

Theorem 4.34. *Let R be a ring, and let $\mathfrak{q}$ be a two-sided completely prime ideal of R. Then*

(1) $(\mathcal{Q}^S_{\mathfrak{q}}(R), \leq)$ is a rooted subtree of $(\mathcal{Q}_{\mathfrak{q}}(R), \leq)$.

(2) $(\mathcal{Q}^S(R), \leq)$ is the ordered disjoint union of the rooted trees $(\mathcal{Q}^S_{\mathfrak{q}}(R), \leq)$.

Proof. The set $\mathcal{Q}^S_{\mathfrak{q}}(R) = \mathcal{Q}^S(R) \cap \mathcal{Q}_{\mathfrak{q}}(R)$ is clearly a rooted tree, since $\mathcal{Q}_{\mathfrak{q}}(R)$ is a rooted tree (Theorem 4.14), and since the trivial quasi-ordering $\preceq_{\mathfrak{q}}$ is special. Moreover, by Lemma 4.32(1), $\mathcal{Q}^S_{\mathfrak{q}}(R)$ is upward closed under $\leq$. Hence, $(\mathcal{Q}^S_{\mathfrak{q}}(R), \leq)$ is a rooted subtree of $(\mathcal{Q}_{\mathfrak{q}}(R), \leq)$.

Evidently, $\mathcal{Q}^S(R)$ is the disjoint union of the trees $\mathcal{Q}^S_{\mathfrak{q}}(R)$. Now let $\preceq_1 \leq \preceq_2 \in \mathcal{Q}^S(R)$. We have to show that $\preceq_1, \preceq_2 \in \mathcal{Q}^S_{\mathfrak{q}}(R)$ for some two-sided completely prime ideal $\mathfrak{q}$ of R. By Proposition 1.57, $\mathfrak{q} := \mathfrak{q}_{\preceq_1}$ is the only two-sided completely prime ideal of R that is $\preceq_1$-convex. According to Proposition 4.23, $\preceq_1$ only compares to quasi-orderings with support $\mathfrak{q}$. Thus, $\preceq_1$ and $\preceq_2$ both lie in $\mathcal{Q}^S_{\mathfrak{q}}(R)$. Therefore, $\mathcal{Q}^S(R)$ is the ordered disjoint union of the rooted subtrees $\mathcal{Q}^S_{\mathfrak{q}}(R)$ of $\mathcal{Q}_{\mathfrak{q}}(R)$. □

The last part of the previous proof may also be shown straight-forward (i.e. without exploiting convexity) as follows:

Lemma 4.35. *Let R be a ring and let $\preceq_1, \preceq_2$ be quasi-orderings on R such that $\preceq_1$ is a special quasi-ordering. If $\preceq_1 \leq \preceq_2$, then $\mathfrak{q}_{\preceq_1} = \mathfrak{q}_{\preceq_2}$.*

Proof. Assume not. Then $\mathfrak{q}_{\preceq_1} \subsetneq \mathfrak{q}_{\preceq_2}$. So we find some $x \in \mathfrak{q}_{\preceq_2} \backslash \mathfrak{q}_{\preceq_1}$. Since $\preceq_1$ is special, there is some $y \in R$ such that $0 \prec_1 1 \preceq_1 xy$. Hence, $\preceq_1 \leq \preceq_2$, implies $1 \preceq_2 xy$. On the other hand, $xy \sim_{\preceq_2} 0$, since $x \in \mathfrak{q}_{\preceq_2}$, a contradiction. Therefore, $\mathfrak{q}_{\preceq_1} = \mathfrak{q}_{\preceq_2}$. □

Remark 4.36. Lemma 4.35 particularly implies that special orderings are maximal elements in the real spectrum (cf. [36, Theorem 6], [13, Theorem 13.2.8] also for the converse implication). Note that unlike us, the cited authors do not allow special quasi-orderings to have non-trivial support, whence they consider the ring $R/\mathfrak{q}_{\preceq}$.

With the very same methods as in the proof of Theorem 4.34, we also get:

Theorem 4.37. *Let R be a ring and let $\mathfrak{q}$ be a two-sided completely prime ideal of R. Then*

(1) *$(\mathcal{Q}^M_{\mathfrak{q}}(R), \leq)$ is a rooted subtree of $(\mathcal{Q}^S_{\mathfrak{q}}(R), \leq)$, respectively $(\mathcal{Q}_{\mathfrak{q}}(R), \leq)$.*

(2) *$(\mathcal{Q}^M(R), \leq)$ is the ordered disjoint union of the rooted trees $(\mathcal{Q}^M_{\mathfrak{q}}(R), \leq)$.*

The Theorems 4.34 and 4.37 are an improvement over Corollary 4.15 in the sense that for special and Manis quasi-orderings the partition into the rooted trees $\mathcal{Q}^S_{\mathfrak{q}}(R)$, respectively $\mathcal{Q}^M_{\mathfrak{q}}(R)$, comes naturally. Recall that in the said corollary, the partial ordering $\leq'$ artificially enforced the supports to be equal.

4.4. **Applications of the Tree Structure Theorem.**

As the main application of the tree structure theorem (Theorem 4.14) we obtain that the trees $(\mathcal{Q}_{\mathfrak{q}}(R), \leq), (\mathcal{Q}(R), \leq'), (\mathcal{Q}^S(R), \leq)$, and $(\mathcal{Q}^M(R), \leq)$ are all spectral sets (Corollary 4.41, Corollary 4.42, Theorem 4.44), i.e. isomorphic to the spectrum of a commutative ring, partially ordered by inclusion. Hence, even if we start off with a non-commutative ring R, its quasi-orderings, respectively its special and Manis quasi-orderings, can be realised by the prime ideals of some commutative ring.

Another, very immediate, application of Theorem 4.14 is that dependency is an equivalence relation on each of the sets $\mathcal{Q}_{\mathfrak{q}}(R)$, $\mathcal{Q}^S(R)$ and $\mathcal{Q}^M(R)$ (Proposition 4.46 and Corollary 4.47), where two quasi-orderings are called dependent if they admit a non-trivial common coarsening. We take this as an opportunity to introduce the topology induced by a quasi-ordering. While this is not related to the tree structure theorem, it gives rise to an alternative proof that dependency is an equivalence relation on the set $\mathcal{Q}_{\mathfrak{q}}(R)$.

4.4.1. *On Spectral Sets of Quasi-Orderings.*

Let us first define spectral sets.

Definition 4.38. A partially ordered set $(X, \leq)$ is called *spectral*, if it is order-isomorphic to $(\mathrm{Spec}(S), \subseteq)$ for some commutative ring S.

It is a known result that any finite partially ordered set is spectral (cf. [27], [50, Theorem 2.10]). In [49, Corollary 3.3], Lewis and Ohm even showed that a partially ordered set $(X, \leq)$ is spectral, if for any $x \in X$ there are only finitely many $y \in X$ such that $y \leq x$, respectively $x \leq y$. This immediately implies:

Proposition 4.39. *Let R be a ring and $X \subseteq \mathcal{Q}(R)$ such that any quasi-ordering $\preceq \in X$ has only finitely many coarsenings and refinements. Then $(X, \leq)$ is spectral.*

In the general case, our work relies on a further result by these authors. They also showed that a partially ordered set $(X, \leq)$ is a tree satisfying Kaplansky's properties (cf. [30], [50, Ch. 3], or (K1) and (K2) from Proposition 4.40 below) if and only if $(X, \leq)$ is isomorphic to the spectrum of a Bézout domain ([49, Theorem 4.2]), i.e. a commutative domain in which the sum of two principal ideals is again a principal ideal.

Proposition 4.40. *Let R be a ring and let $\mathfrak{q}$ be a two-sided completely prime ideal of R. The tree $\mathcal{Q}_{\mathfrak{q}}(R)$ has the following properties:*

- (K1) *Every chain in $\mathcal{Q}_{\mathfrak{q}}(R)$ has a supremum and an infimum.*
- (K2) *If $\preceq_1 < \preceq_2 \in \mathcal{Q}_{\mathfrak{q}}(R)$, then we find elements $\preceq_3, \preceq_4 \in \mathcal{Q}_{\mathfrak{q}}(R)$ such that*
  - *(i)* $\preceq_1 \leq \preceq_3 < \preceq_4 \leq \preceq_2$,
  - *(ii) there is no element in $\mathcal{Q}_{\mathfrak{q}}(R)$ lying strictly in-between $\preceq_3$ and $\preceq_4$.*

Proof. We first prove (K1). Let $(\preceq_i)_{i\in I}$ be a chain in $\mathcal{Q}_{\mathfrak{q}}(R)$, w.l.o.g. $\#I \geq 2$. It is easy to see that the supremum is given by the union of all quasi-orderings such that $0 \prec_i -1$ (i.e. such that $\preceq_i$ is a valuation), and the infimum either by the only ordering (if $(\preceq_i)_{i\in I}$ contains an ordering) or else by the intersection of all $\preceq_i$.

For (K2) let $\preceq_1 < \preceq_2 \in \mathcal{Q}_{\mathfrak{q}}(R)$. Define

$$A := \{\preceq \in \mathcal{Q}_{\mathfrak{q}}(R) \colon \preceq_1 < \preceq \leq \preceq_2\}.$$

Then $(A, \leq)$ is a non-empty partially ordered set. Moreover, note that any $\preceq \in A$ is a valuation, since it is coarser than $\preceq_1$. Hence, for any chain in A, its intersection is a lower bound in A, Thus, by Zorn's lemma, there is a minimal element $\preceq \in A$. Consequently, the choice $\preceq_3 = \preceq_1$ and $\preceq_4 = \preceq$ fulfils condition (K2). □

Corollary 4.41. *Let R be a ring and $\mathfrak{q}$ a two-sided completely prime ideal of R. Then $(\mathcal{Q}_{\mathfrak{q}}(R), \leq)$ is a spectral set and anti-isomorphic to the spectrum of a Bézout domain.*

Proof. The second statement follows from Theorem 4.14, Proposition 4.40 and [49, Theorem 4.2]. Recall that in our definition of a tree we reversed the ordering (see Remark 4.7), which is why we obtain an anti-isomorphism.

Now [27, Proposition 8] states that a partially ordered set $(X, \leq)$ is spectral if and only if $(X, \leq^0)$ is spectral, where $x \leq^0 y :\Leftrightarrow y \leq x$. Therefore, the first claim immediately follows from the second one. □

Corollary 4.42. *Let R be a ring Then $(\mathcal{Q}(R), \leq')$ is a spectral set, where $\leq'$ denotes the partial ordering introduced in Corollary 4.15. Moreover, $(\mathcal{Q}(R), \leq')$ is anti-isomorphic to the spectrum of a Bézout domain.*

Proof. This is proven exactly like Corollary 4.41, since Proposition 4.40 obviously also applies to the tree $(\mathcal{Q}(R), \leq')$. □

Example 4.43.

(1) The tree $(\mathcal{Q}_0(\mathbb{Z}), \leq)$ of all quasi-orderings on $\mathbb{Z}$ with support 0 (see Example 4.18(1)) is anti-order-isomorphic to the spectrum $(\mathrm{Spec}(\mathbb{Z}), \subseteq)$ of the Bézout domain $\mathbb{Z}$.

(2) The proof of [50, Theorem 3.1] is constructive, i.e. for any tree $(T, \leq)$ (in the usual sense) satisfying (K1) and (K2), an explicit Bézout domain can be constructed such that its spectrum is order-isomorphic to $(T, \leq)$. The construction is as follows (cf. [50] and [21, Theorem 18.6]):

(a) Define $T^\times := \{t \in T \mid \exists q \in T\colon q << t\}$, where $q << t$ means that there is no element in T, which lies strictly in-between q and t. Then $(T^\times, \leq)$ is also a tree.

(b) Define $A := \{f\colon T^\times \to \mathbb{Z} \,|\, f(t) = 0$ for all but finitely many $t \in T^\times\}$. For $f \in A$, denote

$$\mathrm{MS}(f) := \{t \in T\colon f(t) \neq 0 \text{ and } f(t') = 0 \text{ for all } t' < t\}.$$

Set $A^+ := \{f \in A\colon f(p) > 0$ for all $p \in \mathrm{MS}(f)\}$. Then (A, A^+) is a lattice ordered group.

(c) Choose some field K and let D be the group ring of A over K. Then D is an integral domain. The map

$$w\colon D \to G \cup \{\infty\},\ 0 \neq f = \sum_{i=1}^{n} a_i X^{g_i} \mapsto \inf\{g_i\}_{i=1}^{n}$$

is a demivaluation, whence it admits a unique extension to a demivaluation v on $L := \mathrm{Quot}(D)$.

(d) The demivaluation ring D_v is the Bézout domain we are looking for.

Proposition 4.40 is easily seen to apply to the trees $(\mathcal{Q}^S(R), \leq)$ and $(\mathcal{Q}^M(R), \leq)$ as well. Therefore, we further obtain:

Theorem 4.44. *Let R be a ring. Then the trees $(\mathcal{Q}^S(R), \leq)$ and $(\mathcal{Q}^M(R), \leq)$ are spectral sets and anti-isomorphic to the spectrum of a Bézout domain*

Proof. This is proven just like Corollary 4.41. □

The previous theorem also applies to $(\mathcal{Q}^S_{\mathfrak{q}}(R), \leq)$, respectively $(\mathcal{Q}^M_{\mathfrak{q}}(R), \leq)$. More generally, it applies to any tree of quasi-orderings that fulfils the two Kaplansky conditions (K1) and (K2).

4.4.2. *Dependency of Quasi-Orderings.*

We continue by introducing a dependency relation on the set $\mathcal{Q}(R)$. As an easy consequence of the tree structure theorem (Theorem 4.14) we obtain that it defines an equivalence relation on $\mathcal{Q}_{\mathfrak{q}}(R)$ for any two-sided completely prime ideal $\mathfrak{q}$ of R (Proposition 4.46). The same applies to the sets $\mathcal{Q}^S(R)$ and $\mathcal{Q}^M(R)$ (Corollary 4.47). The dependency relation is further considered in Subsection 4.4.3.

Definition 4.45. Let R be a ring. Two quasi-orderings $\preceq_1, \preceq_2$ on R are called *dependent*, written $\preceq_1 \sim \preceq_2$, if there is a non-trivial quasi-ordering $\preceq$ on R such that $\preceq_1, \preceq_2 \leq \preceq$, or if $\preceq_1 = \preceq_2$ is trivial. Otherwise, we call them *independent*.

Proposition 4.46. *Let R be a ring and $\mathfrak{q}$ a two-sided completely prime ideal of R. Then dependency is an equivalence relation on $\mathcal{Q}_{\mathfrak{q}}(R)$.*

Proof. Reflexivity and symmetry are both trivial. For transitivity suppose that $\preceq_1 \sim \preceq_2$ and $\preceq_2 \sim \preceq_3$, w.l.o.g. all non-trivial. Then there are non-trivial quasi-orderings $\preceq, \preceq' \in \mathcal{Q}_{\mathfrak{q}}(R)$ such that $\preceq_1, \preceq_2 \leq \preceq$ and $\preceq_2, \preceq_3 \leq \preceq'$. Hence, $\preceq_2 \leq \preceq, \preceq'$. Theorem 4.14 implies that (w.l.o.g.) $\preceq \leq \preceq'$. Thus, $\preceq_1, \preceq_3 \leq \preceq'$, i.e. $\preceq_1 \sim \preceq_3$. □

Corollary 4.47. *Let R be a ring. Then dependency is an equivalence relation on $\mathcal{Q}^S(R)$ and $\mathcal{Q}^M(R)$.*

Proof. This follows immediately from Lemma 4.35 and Proposition 4.46. □

4.4.3. *Digression on the Topology Induced by a Quasi-Ordering.*

We conclude Chapter 4 by introducing the topology induced by a quasi-ordering. This yields an alternative proof of Proposition 4.46, if the ring R is commutative (Corollary 4.55). It is perhaps the more usual approach to obtain the said result and was for instance taken by Eggert for Manis valuations on commutative rings (cf. [17, p.187]).

In [19, Remark 4.2], Fakhruddin defines quasi-intervals for quasi-ordered fields. Taking the same definition for rings yields:

Definition 4.48. Let $(R, \preceq)$ be a quasi-ordered ring, and let $a, b \in R$. The *quasi-interval* of length b around a is defined by

$$(a-b, a+b)_{\preceq} = \{x \in R \mid \exists y \in R\colon 0 \preceq y \prec b \text{ and } -y \preceq x-a \preceq y\}.$$

Remark 4.49. If $\preceq = \leq$ is an ordering on R, then

$$(a-b, a+b)_{\leq} = \{x \in R\colon -b < x-a < b\} = \{x \in R\colon a-b < x < a+b\},$$

which is just the usual notion of open intervals in ordered rings, and therefore an open basis for the order topology on R.
If $\preceq = v$ is a valuation on R, then

$$\begin{aligned}(a-b, a+b)_v &= \{x \in R \mid \exists y \in R\colon v(b) < v(y) \text{ and } v(x-a) = v(y)\} \\ &= \{x \in R\colon v(x-a) > v(b)\} \\ &=: B_{v(b)}(a).\end{aligned}$$

As for valued fields, the set $\mathcal{B} := \{B_{v(b)}(a)\colon a, b \in R\}$ forms an open basis for a Hausdorff topology on R (cf. [18, p. 45]).

Lemma 4.50. *Let (R, v) be a valued ring. Then $\mathcal{B}$ forms an open basis for a topology on R. If v has trivial support, then this topology is Hausdorff.*

Proof. Obviously, the elements of $\mathcal{B}$ cover R. Now let $B_{v(b_1)}(a_1), B_{v(b_2)}(a_2) \in \mathcal{B}$ be arbitrary, and denote by B their intersection. We may w.l.o.g. assume that B is non-empty, so in particular that $b_1, b_2 \notin \mathfrak{q}_v$. For $z \in B$ define

$$B_3 := B_\gamma(z) = \{x \in R\colon v(x-z) > \gamma\} \in \mathcal{B},$$

where $\gamma := \max\{v(b_1), v(b_2)\} \in \Gamma_v$. Clearly, $z \in B_3$. So it remains to show that $B_3 \subseteq B$. Let $y \in B_3$. Then $v(y-z) > \gamma \geq v(b_1)$. We compute

$$v(y-a_1) = v((y-z)+(z-a_1)) \geq \min\{v(y-z), v(z-a_1)\} > v(b_1).$$

Thus, $y \in B_{v(b_1)}(a_1)$. Analogously, we obtain $y \in B_{v(b_2)}(a_2)$, so $y \in B$. Hence, $\mathcal{B}$ is indeed an open basis for a topology on R.
Now suppose that v has trivial support, and let $y, z \in R$ such that $y \neq z$. Then $v(y-z) \in \Gamma_v$. It is easy to see that the open sets $B_{v(y-z)}(y)$ and $B_{v(y-z)}(z)$ separate y and z. Therefore, the topology induced by $\mathcal{B}$ is Hausdorff. □

Consequently, Fakhruddin's notion of quasi-intervals provides a uniform open basis for the topology induced by a quasi-ordering on R.

Notation 4.51. Let $(R, \preceq)$ be a quasi-ordered ring. We denote by

$$\mathcal{T}_{\preceq} := \{(a,b)_{\preceq}\colon a, b \in R\}$$

the topology on R induced by $\preceq$.

Unfortunately, these quasi-intervals are very inconvenient to work with - which may be a reason why Fakhruddin never deepened his work on quasi-ordered fields.

In [16, Ch. 8 and 10], Efrat studies the topologies induced by localities on fields, and their interplay with the dependency relation. While his approach is non-uniform right away, he manages to simultaneously formulate results for topologies induced by orderings, respectively valuations.

We conclude this chapter by transferring two of them to commutative quasi-ordered rings via quotient fields. They will give rise to an alternative proof of Proposition 4.46. As usual, given a quasi-ordered ring $(R, \preceq)$, we denote by $(K, \trianglelefteq)$ its associated quasi-ordered field (see Definition 1.39).

Lemma 4.52. *Let R be a commutative ring and let $\preceq_1, \preceq_2 \in \mathcal{Q}_{\mathfrak{q}}(R)$ for some prime ideal $\mathfrak{q}$ of R. The following are equivalent:*

(1) $\preceq_1 \sim \preceq_2$ *in R,*

(2) $\preceq_1' \sim \preceq_2'$ *in $R/\mathfrak{q}$,*

(3) $\trianglelefteq_1 \sim \trianglelefteq_2$ *in K.*

Proof. This is an immediate consequence of Lemma 4.16. □

Proposition 4.53. (cf. [16, Lemma 8.1.3])
Let R be a commutative ring, $\preceq_1, \preceq_2 \in \mathcal{Q}_{\mathfrak{q}}(R)$, and $\preceq \in \mathcal{Q}_{\mathfrak{q}}(R)$ a non-trivial common coarsening for some prime ideal $\mathfrak{q}$ of R. Then $\mathcal{T}_{\trianglelefteq_1} = \mathcal{T}_{\trianglelefteq_2}$.

Proof. By assumption $\preceq_1, \preceq_2 \leq \preceq$. Hence, Lemma 4.16 implies that $\trianglelefteq_1, \trianglelefteq_2 \leq \trianglelefteq$ in $K = \mathrm{Quot}(R/\mathfrak{q})$. By [16, Lemma 8.1.3], we obtain $\mathcal{T}_{\trianglelefteq_1} = \mathcal{T}_{\trianglelefteq} = \mathcal{T}_{\trianglelefteq_2}$. □

Proposition 4.54. (cf. [16, Corollary 10.1.4])
Let R be a commutative ring and let $\preceq_1, \preceq_2$ be non-trivial quasi-orderings on R with same support. If $\mathcal{T}_{\trianglelefteq_1} \subseteq \mathcal{T}_{\trianglelefteq_2}$, then $\preceq_1 \sim \preceq_2$.

Proof. Let $\mathfrak{q}$ be the common support of $\preceq_1$ and $\preceq_2$. Then $\trianglelefteq_1$ and $\trianglelefteq_2$ are non-trivial quasi-orderings on $K := \mathrm{Quot}(R/\mathfrak{q})$ such that $\mathcal{T}_{\trianglelefteq_1} \subseteq \mathcal{T}_{\trianglelefteq_2}$. By [16, Corollary 10.1.4], $\trianglelefteq_1 \sim \trianglelefteq_2$, i.e. we find some non-trivial quasi-ordering $\trianglelefteq$ on K such that $\trianglelefteq_1, \trianglelefteq_2 \leq \trianglelefteq$. Going back to R again yields via Lemma 4.16 that $\preceq_1, \preceq_2 \leq \preceq$. Moreover, $\preceq$ is non-trivial, since $\trianglelefteq$ is non-trivial. Thus, $\preceq_1$ and $\preceq_2$ are dependent. □

Corollary 4.55. *Let R be a commutative ring. Then dependency is an equivalence relation on $\mathcal{Q}_{\mathfrak{q}}(R)$ for any prime ideal $\mathfrak{q}$ of R.*

Proof. Obviously, reflexivity and symmetry are both fulfilled. For transitivity, let $\preceq_1, \preceq_2, \preceq_3 \in \mathcal{Q}_{\mathfrak{q}}(R)$, w.l.o.g. all non-trivial, such that $\preceq_1 \sim \preceq_2$ and $\preceq_2 \sim \preceq_3$. Then $\mathcal{T}_{\trianglelefteq_1} = \mathcal{T}_{\trianglelefteq_2} = \mathcal{T}_{\trianglelefteq_3}$ by Proposition 4.53. Now $\mathcal{T}_{\trianglelefteq_1} = \mathcal{T}_{\trianglelefteq_3}$ implies via Proposition 4.54 that $\preceq_1 \sim \preceq_3$. Hence, $\sim$ is an equivalence relation on $\mathcal{Q}_{\mathfrak{q}}(R)$. □

5. The Quasi-Real Spectrum of a Ring

The subject of Chapter 4 was to study the quasi-orderings on a ring R as a partially ordered set. In the present chapter we consider $\mathcal{Q}(R)$ as a topological space. For that purpose we develop a notion of quasi-real spectrum, that way unifying the notions of real spectrum and valuation spectrum (see Preliminaries below).

In Section 5.1 we first introduce the quasi-real spectrum of a commutative ring R as the set of all tuples $(\mathfrak{q}, \trianglelefteq)$, where $\mathfrak{q}$ is a prime ideal of R and $\trianglelefteq$ a quasi-ordering on the field $\mathrm{Quot}(R/\mathfrak{q})$. We then show that this set is in a natural bijective correspondence with the set $\mathcal{Q}(R)$ of all quasi-orderings on R. The latter result enables us to define the quasi-real spectrum for arbitrary rings.

In Section 5.2 we generalise the Harrison topology $\mathcal{H}$ and the Tychonoff topology $\mathcal{T}$ from the real spectrum to the quasi-real spectrum $\mathcal{Q}(R)$. Our main results are that $(\mathcal{Q}(R), \mathcal{T})$ is a Boolean space, and that $(\mathcal{Q}(R), \mathcal{H})$ is a spectral space. We conclude this section by briefly introducing and discussing a third topology on $\mathcal{Q}(R)$, namely the one induced by our coarsening relation $\leq$ (see Definition 1.71).

The derivation and study of the quasi-real spectrum is achieved in almost complete analogy to the real spectrum, respectively valuation spectrum. Therefore, it is a perfect example of how quasi-orderings provide a uniform treatment of ordered and valued rings.

Preliminaries on the real spectrum and the valuation spectrum.

The real spectrum, related to the Zariski spectrum of a ring, was introduced for commutative rings by Coste and Roy around 1979 ([10]), and later also considered for possibly non-commutative rings by Leung, Marshall, and Y. Zhang ([48]). It plays an important role in real algebraic geometry for the study of semi-algebraic sets, and for establishing Nullstellen- and Positivstellensätze (cf. e.g. [64, Ch. 4], [54, Ch. 5], [33, Kap. 3]).

Definition 5.1. Let R be a commutative ring. The *real spectrum* r-Spec(R) of R is the set

$$\{(\mathfrak{q}, \trianglelefteq) \colon \mathfrak{q} \in \mathrm{Spec}(R),\ \trianglelefteq \in \mathcal{O}(\mathrm{Quot}(R/\mathfrak{q}))\}.$$

The real spectrum of R is in a natural bijective correspondence with the set $\mathcal{O}(R)$ of all orderings on R, and therefore identified with this set. It is equipped with two canonical topologies, namely the Harrison topology and the Tychonoff topology. The former is that of primary interest, while the latter is rather a tool for a better understanding of the Harrison topology - that will also be the case in the quasi-real spectrum, see Section 5.2 below.

Definition 5.2. Let R be a commutative ring.

(1) The *Harrison topology* (or *spectral topology*) $\mathcal{H}$ on $\mathcal{O}(R)$ is the topology generated by the sets $U(a)$ as open subbasis, where

$$U(a) := \{\leq \in \mathcal{O}(R) \colon 0 < a\} \quad (a \in R)$$

(2) The *Tychonoff topology* (or *patch topology*, or *constructible topology*) $\mathcal{T}$ on $\mathcal{O}(R)$ is the topology generated by the sets $U(a)$ and $V(a)$ as open subbasis, where

$$V(a) := \{\leq \in \mathcal{O}(R) \colon 0 \leq a\} \quad (a \in R)$$

It can be shown that $(\mathcal{O}(R), \mathcal{H})$ is a spectral space, and that $(\mathcal{O}(R), \mathcal{T})$ is a Boolean space (cf. e.g. [54, Proposition 6.3.3]).

Definition 5.3. A topological space (X, τ) is called a *Boolean space*, if it is quasi-compact, Hausdorff, and totally disconnected.

Definition 5.4. [27] A topological space (X, τ) is called a *spectral space*, if

(1) it is T_0,
(2) it is quasi-compact,
(3) quasi-compact open subsets are closed under finite intersections,
(4) quasi-compact open subsets form an open basis,
(5) every non-empty irreducible closed subset $A \subseteq X$ has a generic point, i.e. a point whose closure is all of A.

Hochster showed in [27] that a topological space is spectral if and only if it is homeomorphic to the spectrum of a commutative ring equipped with the Zariski topology. Therefore, the notion of spectral spaces corresponds to the one of partially ordered spectral sets (see Definition 4.38). In fact, there is a strong relation between partially ordered sets and topological spaces.

Definition 5.5. Let (X, τ) be a topological space and let $x, y \in X$. Then y is called a *specialisation* of x (or x a *generalisation* of y), denoted by $x \leq_\tau y$, if $y \in \overline{\{x\}}$.

If (X, τ) is a T_0 space, then $\leq_\tau$ defines a partial ordering on X. Conversely, if $\leq$ is a partial ordering on X, then $(X, \tau_\leq)$ is a T_0 space, where $\tau_\leq$ is the topology generated by the closed subbasis $\{y \in X : y \geq x\}$ (cf. [49]).

Remark 5.6. (cf. [49]) If $(X_1, \tau_1), (X_2, \tau_2)$ are homeomorphic T_0 spaces, then $(X_1, \leq_{\tau_1})$ and $(X_2, \leq_{\tau_2})$ are order-isomorphic. The converse of this statement may be false.

Inspired by the real spectrum, the notion of valuation spectrum was independently developed by de la Puente ([65]) and Huber ([28]), and then further investigated by Huber and Knebusch ([29]). While de la Puente was interested in applications related to affine algebraic varieties, Huber and Knebusch were looking for analogies between semi-algebraic geometry and rigid analytic geometry. Nevertheless, they derived the same topologies on the valuation spectrum.

Definition 5.7. Let R be a commutative ring. The *valuation spectrum* v-Spec(R) of R is the set

$$\{(\mathfrak{q}, \nu) : \mathfrak{q} \in \mathrm{Spec}(R),\, \nu \in \mathcal{V}(\mathrm{Quot}(R/\mathfrak{q}))\}.$$

In analogy to the real case, this set is in a natural bijective correspondence with the set $\mathcal{V}(R)$ of all valuations on R, and therefore identified with this set. The valuation spectrum is canonically equipped with the topologies generated by the following sets (cf. [29]):

(1) $\tau := \{v \in \mathcal{V}(R) : v(a) \geq v(b) \neq \infty\} \quad (a, b \in R)$
(2) $\tau' := \{v \in \mathcal{V}(R) : v(a) > v(b)\} \quad (a, b \in R)$
(3) $\tau'' :=$ the topology generated by $\tau \cup \tau'$.

In [29] it is shown that $\mathcal{V}(R)$ is a spectral space with respect to either of these topologies.

Via quasi-orderings, it is easy to see that the topology τ' on the valuation spectrum corresponds to the Harrison topology on the real spectrum. Moreover, the topology τ'' resembles the Tychonoff topology. We will make use of this in Section 5.2.

5.1. **The Quasi-Real Spectrum as a Set.**

Here, we introduce the notion of the quasi-real spectrum qr-Spec(R) of a ring R. As for orderings and valuations, we first define it for commutative rings to be the set of all tuples $(\mathfrak{q}, \trianglelefteq)$, where $\mathfrak{q}$ is a prime ideal of R and $\trianglelefteq$ a quasi-ordering on the field Quot($R/\mathfrak{q}$) (Definition 5.8). Thereafter, we prove that qr-Spec(R) may be identified with the set $\mathcal{Q}(R)$ of all quasi-orderings on R (Corollary 5.10), resulting in a definition of the quasi-real spectrum which also applies to non-commutative rings (Definition 5.11). We conclude this section by describing the quasi-real spectrum of R in terms of ring homomorphisms (Proposition 5.14).

Definition 5.8. Let R be a commutative ring. We call

$$\text{qr-Spec}(R) := \{(\mathfrak{q}, \trianglelefteq) \colon \mathfrak{q} \in \operatorname{Spec}(R),\ \trianglelefteq \in \mathcal{Q}(\operatorname{Quot}(R/\mathfrak{q}))\}$$

the *quasi-real spectrum* of R.

Our first aim is to derive a definition of the quasi-real spectrum that applies to possibly non-commutative rings.

Proposition 5.9. *Let R be a ring, I a two-sided ideal of R and $S \subseteq R$ a multiplicative set.*

(1) There is a bijective correspondence

$$\varphi \colon \{\preceq\, \in \mathcal{Q}(R) \colon \mathfrak{q}_\preceq \supseteq I\} \to \mathcal{Q}(R/I)$$

given by the assignment $\preceq\, \mapsto\, \preceq'$.

(2) There is a bijective correspondence

$$\varphi \colon \mathcal{Q}_{\mathfrak{q}_\preceq}(R) \to \mathcal{Q}_0(R/\mathfrak{q}_\preceq)$$

given by the assignment $\preceq\, \mapsto\, \preceq'$.

(3) If R is commutative, there is a bijective correspondence

$$\psi \colon \{\preceq\, \in \mathcal{Q}(R) \colon \mathfrak{q}_\preceq \cap S = \emptyset\} \to \mathcal{Q}(R_S)$$

given by the assignment $\preceq\, \mapsto\, \trianglelefteq$.

Proof.

(1) By Lemma 1.35, the map φ is well-defined. Moreover, since $x \preceq y \Leftrightarrow \overline{x} \preceq' \overline{y}$, φ is easily seen to be bijective.

(2) This is precisely (1) in the case where $I = \mathfrak{q}_\preceq$.

(3) We have already seen in Proposition 1.37 that the map ψ is well-defined and injective. Moreover, if $\trianglelefteq$ is a quasi-ordering on R_S, then the restriction $\preceq\, := R \cap \trianglelefteq$ is a quasi-ordering on R such that $\psi(\preceq) = \trianglelefteq$ holds. Hence, ψ is bijective.

□

Corollary 5.10. *Let R be a commutative ring. The map*

$$\rho \colon \mathcal{Q}(R) \to \textit{qr-Spec}(R),\ \preceq\, \mapsto (\mathfrak{q}_\preceq, \trianglelefteq)$$

is a bijection.

Proof. Let $\mathfrak{q}$ be a prime ideal of R. Applying Proposition 5.9(2) and (3) with the choice $S = (R/\mathfrak{q})\backslash\{0\}$ implies that $\psi \circ \varphi \colon\ \preceq\, \mapsto\, \trianglelefteq$ is a bijection between $\mathcal{Q}_\mathfrak{q}(R)$ and $\mathcal{Q}(\operatorname{Quot}(R/\mathfrak{q}))$. From this it is easy to see that the map ρ from the claim is also a bijection. □

Because of Corollary 5.10, we may identify the quasi-real spectrum qr-Spec(R) of a commutative ring R with $\mathcal{Q}(R)$. This enables us to generalise the quasi-real spectrum to possibly non-commutative rings.

Definition 5.11. Let R be a ring. We call
$$\mathcal{Q}(R) = \{\preceq \subseteq R^2 \colon \preceq \text{ is a quasi-ordering on } R\}$$
the *quasi-real spectrum* of R.

Remark 5.12. Consequently, Proposition 5.9(1) and (3) yield a description of the quasi-real spectrum of both, the quotient ring R/I and the localisation R_S, as subsets of the quasi-real spectrum of R.

We conclude the current section with a further description of the quasi-real spectrum $\mathcal{Q}(R)$ of a commutative ring R (see Proposition 5.14 below).

Lemma 5.13. *Let R, S be rings, $\varphi\colon R \to S$ a ring homomorphism and $\preceq_S$ a quasi-ordering on S. Then $x \preceq_R y :\Leftrightarrow \varphi(x) \preceq_S \varphi(y)$ defines a quasi-ordering on R with support $\mathfrak{q}_{\preceq_R} = \{x \in R \colon \varphi(x) \in \mathfrak{q}_{\preceq_S}\}$.*

Proof. Showing that $\preceq_R$ defines a quasi-ordering on R with support $\varphi^{-1}(\mathfrak{q}_{\preceq_S})$ is done straight-forward, which is why we only showcase that (QR4) applies. So let $x, y, z \in R$ such that $x \preceq_R y$ and $z \not\sim_{\preceq_R} y$. Then $\varphi(x) \preceq_S \varphi(y)$ and $\varphi(z) \not\sim_{\preceq_S} \varphi(y)$. Via (QR4) it follows that
$$\varphi(x+z) = \varphi(x) + \varphi(z) \preceq_S \varphi(y) + \varphi(z) = \varphi(y+z),$$
and therefore $x + z \preceq_R y + z$. $\square$

By abuse of notation, we denote the quasi-ordering $\preceq_R$ from the previous lemma by $\varphi^{-1}(\preceq_S)$. Evidently, $\varphi^{-1}(\preceq_S)$ is an ordering (respectively a valuation) if and only if $\preceq_S$ is an ordering (respectively a valuation).

Proposition 5.14. *Let R be a commutative ring and $\preceq \subseteq R^2$. The following are equivalent:*

(1) $\preceq \in \mathcal{Q}(R)$,

(2) There is a quasi-ordered field $(K, \trianglelefteq)$ and a ring homomorphism $\varphi\colon R \to K$ such that $\preceq = \varphi^{-1}(\trianglelefteq)$.

Proof. We have already shown in Lemma 5.13 that (2) implies (1). For the converse, let $\preceq \in \mathcal{Q}(R)$. Then $\preceq$ uniquely extends to a quasi-ordering $\trianglelefteq$ on $K := \operatorname{Quot}(R/\mathfrak{q}_{\preceq})$. Now consider the canonical homomorphism $\varphi\colon R \to K,\ x \mapsto \overline{x} \,/\, \overline{1}$. We obtain by Lemma 1.35 and Proposition 1.37 that
$$x \preceq y \Leftrightarrow \overline{x} \preceq' \overline{y} \Leftrightarrow \overline{x} \,/\, \overline{1} \trianglelefteq \overline{y} \,/\, \overline{1},$$
i.e. that $x \preceq y$ if and only if $\varphi(x) \trianglelefteq \varphi(y)$. Hence, $(K, \trianglelefteq)$ is a quasi-ordered field such that $\preceq = \varphi^{-1}(\trianglelefteq)$. $\square$

By imposing a suitable equivalence relation on the set of all such homomorphisms, the real spectrum of a commutative ring can also be introduced via the previous proposition (cf. [8, Ch. 7]).

5.2. **The Quasi-Real Spectrum as a Topological Space.**

In this section we generalise the Harrison topology $\mathcal{H}$ and the Tychonoff topology $\mathcal{T}$ from the real spectrum to the quasi-real spectrum of a ring (Definition 5.15, Definition 5.22), and show that many of their properties are preserved. In fact, almost all results presented here are well-known to hold for the real spectrum of commutative rings with identical proofs.

Our first main result of the present section states that $(\mathcal{Q}(R), \mathcal{T})$ is a Boolean space for any ring R (Theorem 5.24). The latter implies that $(\mathcal{Q}(R), \mathcal{H})$ is a spectral space (Theorem 5.26). This provides a uniform proof of the fact that $(\mathcal{O}(R), \mathcal{H})$

and $(\mathcal{V}(R), \mathcal{H})$ are also spectral spaces (Corollary 5.27). We further show that the Harrison topology is completely determined by the Tychonoff topology and the partial ordering $\leq_{\mathcal{H}}$ (Corollary 5.31, respectively Corollary 5.32). Finally, we briefly discuss a third topology on $\mathcal{Q}(R)$, which we call tree topology (Definition 5.33). It is the topology induced by the coarsening relation $\leq$.

Definition 5.15. Let R be a ring.

(1) For $a, b \in R$ we define the set

$$U(a,b) := \{\preceq \in \mathcal{Q}(R) \colon a \prec b\}.$$

(2) The topology $\mathcal{H}$ on $\mathcal{Q}(R)$ generated by the open subbasis

$$\{U(a,b) \subseteq \mathcal{Q}(R) \colon a, b \in R\}$$

is called the *Harrison topology* (or *spectral topology*).

If not mentioned otherwise, we always equip $\mathcal{Q}(R)$ with this topology $\mathcal{H}$.

Remark 5.16. The restriction of $\mathcal{H}$ to $\mathcal{O}(R)$ is obviously the usual Harrison topology on the real spectrum. If we restrict $\mathcal{Q}(R)$ to $\mathcal{V}(R)$, then $\mathcal{H}$ coincides with the topology τ' introduced in [29, 1.3] (see also the preliminaries above).

Example 5.17.

(1) Exploiting the dichotomy (Theorem 1.50), it is easy to see that $\mathcal{O}(R)$ is open and closed with respect to the Harrison topology. Indeed,

$$\mathcal{O}(R) = U(-1, 0),$$

so $\mathcal{O}(R)$ is open. Moreover, $(\mathcal{O}(R))^C = \mathcal{V}(R) = U(0, -1)$ is also open, whence $\mathcal{O}(R)$ is closed.

(2) The set $\mathcal{V}(R)$ of all valuations on R is also open and closed. This follows immediately from (1).

(3) The bijections from Proposition 5.9 can be shown to be homeomorphisms.

We begin our discussion of the Harrison topology with a small observation on the relationship of the quasi-real spectrum and the spectrum of rings.

Lemma 5.18.

(a) Let R be a commutative ring. The support map

$$\mathit{supp} \colon \mathcal{Q}(R) \to \mathit{Spec}(R),\ \preceq \mapsto \mathfrak{q}_{\preceq}$$

is continuous.

(b) Let R, S be rings and $\varphi \colon R \to S$ a ring homomorphism. Then the map

$$\mathit{q\text{-}Sper}\,\varphi \colon \mathcal{Q}(S) \to \mathcal{Q}(R),\ \preceq \mapsto \varphi^{-1}(\preceq)$$

is continuous.

Proof. First consider the support map. Let $r \in R$ and $D(r) = \{\mathfrak{q} \in \mathrm{Spec}(R) \colon r \notin \mathfrak{q}\}$ be a basic open set of the spectrum of R. Then

$$\begin{aligned} \mathrm{supp}^{-1}(D(r)) &= \{\preceq \in \mathcal{Q}(R) \colon r \notin \mathfrak{q}_{\preceq}\} \\ &= \{\preceq \in \mathcal{Q}(R) \colon 0 \prec r \text{ or } r \prec 0\} \\ &= U(0, r) \cup U(r, 0), \end{aligned}$$

is an open set, whence supp is continuous.

By Lemma 5.13, the map q-Sper φ is well-defined. Now let $A = \{\preceq \in \mathcal{Q}(R) \colon a \prec b\}$ be a subbasic open set for some $a, b \in A$. Then

$$\begin{aligned}(\text{q-Sper}\,\varphi)^{-1}(A) &= \{\preceq \in \mathcal{Q}(S) \colon \text{q-Sper}\,\varphi(\preceq) \in A\} \\ &= \{\preceq \in \mathcal{Q}(S) \colon (a, b) \in \varphi^{-1}(\preceq) \text{ and } (b, a) \notin \varphi^{-1}(\preceq)\} \\ &= \{\preceq \in \mathcal{Q}(S) \colon \varphi(a) \prec \varphi(b)\} \\ &= U(\varphi(a), \varphi(b))\end{aligned}$$

where $(a, b) \in \preceq \,:\Leftrightarrow a \preceq b$. Hence, q-Sper φ is continuous as well. □

Corollary 5.19. *Let R, S be commutative rings, $\varphi \colon R \to S$ a ring homomorphism and $\mathit{Spec}\,\varphi \colon \mathit{Spec}(S) \to \mathit{Spec}(R)$, $\mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p})$. Then*

$$\begin{tikzcd} \mathcal{Q}(S) \arrow[r, "\textit{q-Sper}\,\varphi"] \arrow[d, "\textit{supp}"] & \mathcal{Q}(R) \arrow[d, "\textit{supp}"] \\ \mathit{Spec}(S) \arrow[r, "\mathit{Spec}\,\varphi"] & \mathit{Spec}(R) \end{tikzcd}$$

is a commutative diagram of continuous maps.

Proof. We have already seen that all these maps are continuous. Now let $\preceq \in \mathcal{Q}(S)$. Then

$$\begin{aligned}\operatorname{Spec}\varphi(\operatorname{supp}(\preceq)) &= \operatorname{Spec}\varphi(\mathfrak{q}_{\preceq}) = \varphi^{-1}(\mathfrak{q}_{\preceq}) = \operatorname{supp}(\varphi^{-1}(\preceq)) \\ &= \operatorname{supp}(\text{q-Sper}\,\varphi(\preceq)).\end{aligned}$$

□

We now want to study the properties of the space $(\mathcal{Q}(R), \mathcal{H})$, eventually proving that it is a spectral space (see Theorem 5.26).

Lemma 5.20. *Let R be a ring and $\preceq_1, \preceq_2 \in \mathcal{Q}(R)$. Then*

$$\preceq_2 \in \overline{\{\preceq_1\}} \;\Leftrightarrow\; \preceq_1 \subseteq \preceq_2,$$

i.e. $\leq_{\mathcal{H}} = \subseteq$ (see Definition 5.5).

Proof. Let $\mathcal{N}_{\preceq_2}$ denote the set of all open neighbourhoods of $\preceq_2$. We deduce

$$\begin{aligned}\preceq_1 \subseteq \preceq_2 &\Leftrightarrow \forall a, b \in R (a \preceq_1 b \Rightarrow a \preceq_2 b) \\ &\Leftrightarrow \forall a, b \in R (b \prec_2 a \Rightarrow b \prec_1 a) \\ &\Leftrightarrow \forall N \in \mathcal{N}_{\preceq_2} \colon N \cap \{\preceq_1\} \neq \emptyset \\ &\Leftrightarrow \preceq_2 \in \overline{\{\preceq_1\}}.\end{aligned}$$

□

Corollary 5.21. *Let R be a ring. Then $(\mathcal{Q}(R), \mathcal{H})$ is a T_0 space.*

Proof. Two quasi-orderings $\preceq_1, \preceq_2 \in \mathcal{Q}(R)$ are topologically indistinguishable if and only if $\preceq_1 \in \overline{\{\preceq_2\}}$ and vice versa. Consequently, by Lemma 5.20, if and only if $\preceq_1 = \preceq_2$. Hence, $(\mathcal{Q}(R), \mathcal{H})$ is T_0. □

For a further study of the Harrison topology we introduce the Tychonoff topology on $\mathcal{Q}(R)$.

Definition 5.22. Let R be a ring.

(1) For $a, b \in R$ we define the set
$$V(a,b) := \{\preceq \in \mathcal{Q}(R) \colon a \preceq b\}.$$

(2) The topology $\mathcal{T}$ on $\mathcal{Q}(R)$ generated by the open subbasis
$$\{U(a,b) \subseteq \mathcal{Q}(R) \colon a, b \in R\} \cup \{V(a,b) \subseteq \mathcal{Q}(R) \colon a, b \in R\}$$
is called the *Tychonoff topology* (or *constructible topology*, respectively *patch topology*).

Remark 5.23.

(1) On $\mathcal{O}(R)$, the topology $\mathcal{T}$ obviously coincides with the usual Tychonoff topology on the real spectrum. For the valuation spectrum, however, Huber and Knebusch consider $\{v \in \mathcal{V}(R) \colon v(a) \geq v(b) \neq \infty\}$ instead of $V(a,b)$ for their corresponding topology τ'' (cf. [29, 1.3]).

(2) Immediately by definition, the Tychonoff topology is finer than the Harrison topology, i.e. $\mathcal{H} \subseteq \mathcal{T}$.

Theorem 5.24. *Let R be a ring. Then $(\mathcal{Q}(R), \mathcal{T})$ is a Boolean space.*

Proof. We first show that this space is quasi-compact. Any quasi-ordering $\preceq$ on R is uniquely determined by the map
$$f_{\preceq} \colon R^2 \to \{0,1\},\ (a,b) \mapsto \begin{cases} 1, & a \preceq b \\ 0, & b \prec a \end{cases}$$
Hence, we may consider $\mathcal{Q}(R)$ as a subset of $\mathcal{F} := \{0,1\}^{R^2}$. We equip $\{0,1\}$ with the discrete topology, which makes it a quasi-compact space. So by the Tychonoff theorem, $\mathcal{F}$ is also quasi-compact with respect to the product topology. An open subbasis is given by the sets
$$H_\varepsilon(a,b) := \{f \colon R^2 \to \{0,1\},\ (a,b) \mapsto \varepsilon\} \quad (a, b \in R,\ \varepsilon \in \{0,1\}).$$
Therefore, we may identify $\mathcal{Q}(R)$ as a subspace of $\mathcal{F}$. We show that it is a closed subspace, which implies that $(\mathcal{Q}(R), \mathcal{T})$ is quasi-compact. So let $f \in \mathcal{F}$ be not induced by some quasi-ordering on R, i.e. the induced binary relation on R does not satisfy one of the axioms of a quasi-ordering. If for instance axiom (QR4) fails, then we find some $x, y, z \in R$ such that $f(x,y) = 1$ and $f(y,z) = 0 = f(y+z, x+z)$, respectively $f(z,y) = 0 = f(y+z, x+z)$. Thus, f is separated from $\mathcal{Q}(R)$ by the open set
$$H_1(x,y) \cap (H_0(y,z) \cup H_0(y,z)) \cap H_0(y+z, x+z),$$
whence $s \notin \overline{\mathcal{Q}(R)}$. The same arguing applies to all the other axioms. Therefore, $\mathcal{Q}(R)$ is a closed subspace of $\mathcal{F}$, implying that $(\mathcal{Q}(R), \mathcal{T})$ is quasi-compact.

Next, we show that $(\mathcal{Q}(R), \mathcal{T})$ is Hausdorff. So let $\preceq_1, \preceq_2 \in \mathcal{Q}(R)$ with $\preceq_1 \neq \preceq_2$. Then there exist (w.l.o.g.) some $a, b \in R$ such that $a \preceq_1 b$ and $b \prec_2 a$. Consequently, $\preceq_1 \in V(a,b) \backslash U(b,a)$ and $\preceq_2 \in U(b,a) \backslash V(a,b)$. This shows that $(\mathcal{Q}(R), \mathcal{T})$ fulfils the Hausdorff property.

Finally, assume that there is some connected component $X \subseteq \mathcal{Q}(R)$ such that $|X| \geq 2$, and let $\preceq_1 \neq \preceq_2 \in X$. Then we find some $a, b \in R$ such that (w.l.o.g.) $a \preceq_1 b$ and $b \prec_2 a$. Thus,
$$X = (X \cap V(a,b)) \cup (X \cap U(b,a)),$$
is a union of two disjoint non-empty open sets. This contradicts the connectedness of X. Hence, $(\mathcal{Q}(R), \mathcal{T})$ is totally disconnected. Altogether, we have shown that $(\mathcal{Q}(R), \mathcal{T})$ is a Boolean space. $\square$

Remark 5.25.

(1) Any Boolean space is homeomorphic to the real spectrum of a field (cf. [12] or [11, Theorem 5]; see also [62, Theorem 6.9]).

(2) The Hausdorff property implies that all singletons $\{\preceq\}$ are closed. Hence, the partial ordering on $\mathcal{Q}(R)$ induced by the Tychonoff Topology is equality, i.e. $\preceq_1 \leq_{\mathcal{T}} \preceq_2 \Leftrightarrow \preceq_1 = \preceq_2$.

Theorem 5.26. *Let R be a ring. Then $(\mathcal{Q}(R), \mathcal{H})$ is a spectral space.*

Proof. We have already proved that $(\mathcal{Q}(R), \mathcal{H})$ is a T_0 space (Corollary 5.21), and that $(\mathcal{Q}(R), \mathcal{T})$ is a compact space (Theorem 5.24). Moreover, the sets $U(a, b)$ are both open and closed with respect to $\mathcal{T}$, and an open subbasis of $\mathcal{H}$. Therefore, [27, Proposition 7] tells us that $(\mathcal{Q}(R), \mathcal{H})$ is a spectral space. □

Corollary 5.27. *Let R be a ring. Then $(\mathcal{O}(R), \mathcal{H})$ and $(\mathcal{V}(R), \mathcal{H})$ are both spectral spaces.*

Proof. This is an easy consequence of Theorem 5.26 and Example 5.17, since closed subsets of spectral spaces are again spectral spaces (cf. [13, Theorem 2.1.3]). □

The previous corollary may be as well deduced from Theorem 5.26 via [13, Theorem 2.4.3], since $\mathcal{Q}(R)$ is easily seen to be the topological sum of $\mathcal{O}(R)$ and $\mathcal{V}(R)$. The cited result also yields that Corollary 5.27 implies Theorem 5.26.

Corollary 5.28. *Let R be a ring. Then $(\mathcal{Q}(R), \subseteq)$ is a spectral set.*

Proof. By Theorem 5.26, $(\mathcal{Q}(R), \mathcal{H})$ is a spectral space. Hence, there is a commutative ring S such that $(\mathcal{Q}(R), \mathcal{H})$ is homeomorphic to $(\mathrm{Spec}(S), \mathcal{Z})$, where Z denotes the Zariski topology. According to [49], this implies that $\mathcal{Q}(R)$ and $\mathrm{Spec}(S)$ are order-isomorphic with respect to the induced partial orderings $\leq_{\mathcal{H}}$, respectively $\leq_{\mathcal{Z}}$. Thus, $(\mathcal{Q}(R), \subseteq) \cong (\mathrm{Spec}(S), \subseteq)$, whence $(\mathcal{Q}(R), \subseteq)$ is spectral. □

Next, we generalise the result that $\mathcal{H}$ is completely determined by $\mathcal{T}$ and $\leq_{\mathcal{H}}$ from the real to the quasi-real spectrum (cf. [33, Ch. III.4] or [54, Proposition 6.3.4]).

Lemma 5.29. *Let R be a ring and $(A_i)_{i\in I}$ a family of $\mathcal{T}$-closed subsets of $\mathcal{Q}(R)$ such that $\bigcap_{i\in I} A_i = \emptyset$. Then there is a finite index set $J \subseteq I$ such that $\bigcap_{j\in J} A_j = \emptyset$.*

Proof. From $\bigcap_{i\in I} A_i = \emptyset$ follows that $\mathcal{Q}(R) = \bigcup_{i\in I} A_i^C$ is an open cover of $\mathcal{Q}(R)$. By quasi-compactness, there is a finite $J \subseteq I$ such that $\mathcal{Q}(R) = \bigcup_{j\in J} A_j^C$. Therefore, $\bigcap_{j\in J} A_j = \emptyset$. □

Proposition 5.30. *Let R be a ring and $Y \subseteq \mathcal{Q}(R)$ a closed subset with respect to the Tychonoff topology. Then*

$$\overline{Y} = \bigcup_{\preceq\in Y} \overline{\{\preceq\}} = \{\preceq' \in \mathcal{Q}(R)\colon\ \preceq \leq_{\mathcal{H}} \preceq' \textit{ for some } \preceq \in Y\}$$

with respect to the Harrison topology.

Proof. The second equality follows immediately from Lemma 5.20. Furthermore, clearly $\overline{Y} \supseteq \bigcup_{\preceq\in Y} \overline{\{\preceq\}}$. Conversely, let $\preceq \in \overline{Y}$ be arbitrary and $\{U_i\colon i \in I\}$ the family of all basic open neighbourhoods of $\preceq$ with respect to the Harrison topology. Then $Y \cap U_i$ is non-empty and $\mathcal{T}$-closed for all $i \in I$. Therefore, Lemma 5.29 implies that $\bigcap_{i\in I} Y \cap U_i$ is also non-empty. So let $\preceq' \in \bigcap_{i\in I} Y \cap U_i$. Then $\preceq'$ is contained in every basic open neighbourhood of $\preceq$ with respect to the Harrison topology, whence $\preceq \in \overline{\{\preceq'\}}$. □

Corollary 5.31. *Let R be a ring. A subset $Y \subseteq \mathcal{Q}(R)$ is $\mathcal{H}$-closed if and only if it is $\mathcal{T}$-closed and stable under specialisation w.r.t. $\mathcal{H}$.*

Proof. First let Y be $\mathcal{H}$-closed. Then Y is also $\mathcal{T}$-closed, since $\mathcal{H} \subseteq \mathcal{T}$. Now let $\preceq_1 \in Y$ and $\preceq_1 \leq_{\mathcal{H}} \preceq_2$. Then we obtain $\preceq_2 \in \overline{\{\preceq_1\}} \subseteq \overline{Y} = Y$. Thus, Y is stable under specialisation.

Conversely, suppose that Y is $\mathcal{T}$-closed and stable under specialisation. Then we get

$$\overline{Y} = \{\preceq \in \mathcal{Q}(R) \colon \ \preceq' \leq_{\mathcal{H}} \preceq \text{ for some } \preceq' \in Y\} = Y,$$

where the first equality follows from Proposition 5.30 and the second one from the assumption that Y is stable under specialisation. Hence, Y is $\mathcal{H}$-closed. □

Corollary 5.32. *Let R be a ring. A subset $Y \subseteq \mathcal{Q}(R)$ is $\mathcal{H}$-open if and only if it is $\mathcal{T}$-open and stable under generalisation w.r.t. $\mathcal{H}$.*

Proof. Making use of Corollary 5.31, the proof goes as follows:

$$\begin{aligned} Y \text{ is } \mathcal{H}\text{-open} &\Leftrightarrow Y^C \text{ is } \mathcal{H}\text{-closed} \\ &\Leftrightarrow Y^C \text{ is } \mathcal{T}\text{-closed and stable under specialisation} \\ &\Leftrightarrow Y \text{ is } \mathcal{T}\text{-open and stable under generalisation.} \end{aligned}$$

□

We conclude this chapter by introducing a third topology on $\mathcal{Q}(R)$, which arises from the coarsening relation $\leq$ that we studied in Chapter 4. Recall from Lemma 4.3 that if $\preceq_1, \preceq_2 \in \mathcal{Q}(R)$, then

$$\preceq_1 \leq \preceq_2 \Leftrightarrow \forall a, b \in R \colon 0 \preceq_1 a \preceq_1 b \Rightarrow 0 \preceq_2 a \preceq_2 b.$$

Definition 5.33. Let R be a ring.

(1) For $a, b \in R$ we define the set

$$W(a, b) := \{\preceq \in \mathcal{Q}(R) \colon 0 \preceq a \preceq b\}.$$

(2) We call the topology τ_T on $\mathcal{Q}(R)$ generated by the closed subbasis

$$\{W(a, b) \subseteq \mathcal{Q}(R) \colon a, b \in R\}$$

the *tree topology.*

The corresponding open subbasis of τ_T is given by the sets

$$W(a, b)^C = \{\preceq \in \mathcal{Q}(R) \colon b \prec a \text{ or } a \prec 0\}.$$

Remark 5.34.

(1) If we restrict $\mathcal{Q}(R)$ to valuations, we obtain

$$W(a, b)^C \cap \mathcal{V}(R) = \{v \in \mathcal{V}(R) \colon b \prec_v a\} = U(b, a) \cap \mathcal{V}(R),$$

i.e. for valuations our Harrison and our tree topology coincide.

(2) If we restrict $\mathcal{Q}(R)$ to orderings, we obtain

$$\begin{aligned} U(a, b) \cap \mathcal{O}(R) &= \{\leq \in \mathcal{O}(R) \colon a - b < 0\} \\ &= \{\leq \in \mathcal{O}(R) \colon a - b < 0 \text{ or } 0 < 0\} \\ &= W(0, a - b)^C \cap \mathcal{O}(R), \end{aligned}$$

and conversely

$$W(a, b)^C \cap \mathcal{O}(R) = (U(b, a) \cap \mathcal{O}(R)) \cup (U(a, 0) \cap \mathcal{O}(R)).$$

Hence, the Harrison and the tree topology also coincide on $\mathcal{O}(R)$.

Lemma 5.35. *Let R be a ring, let $\preceq_1, \preceq_2 \in \mathcal{Q}(R)$, and let $\leq$ denote the coarsening relation on $\mathcal{Q}(R)$. Then*

$$\preceq_2 \in \overline{\{\preceq_1\}} \ \Leftrightarrow \ \preceq_1 \leq \preceq_2,$$

w.r.t. τ_T, i.e. $\leq_{\tau_T} = \leq$.

Proof. This is proven just like Lemma 5.20. So let $\mathcal{N}_{\preceq_2}$ denote the set of all open neighbourhoods of $\preceq_2$ w.r.t. τ_T. We obtain

$$\begin{aligned}
\preceq_1 \leq \preceq_2 &\Leftrightarrow \forall x, y \in R\colon (0 \preceq_1 x \preceq_1 y \Rightarrow 0 \preceq_2 x \preceq_2 y)\\
&\Leftrightarrow \forall x, y \in R\colon (y \prec_2 x \text{ or } x \prec_2 0 \Rightarrow y \prec_1 x \text{ or } x \prec_1 0)\\
&\Leftrightarrow \forall N \in \mathcal{N}_{\preceq_2}\colon N \cap \{\preceq_1\} \neq \emptyset\\
&\Leftrightarrow \preceq_2 \in \overline{\{\preceq_1\}}.
\end{aligned}$$

□

Remark 5.36. A singleton $\{\preceq\}$ is τ_T-closed if and only if $\preceq$ is a trivial quasi-ordering. This is an immediate consequence of Lemma 5.35 and the fact that the $\leq$-maximal elements are precisely the trivial quasi-orderings.

So unlike $\mathcal{H}$ and $\mathcal{T}$, the tree topology admits that a valuation lies in the closure of an ordering. Conversely, an ordering cannot be in the closure of a valuation, since an ordering is never coarser than a valuation (see Section 4.1).

We conclude our discussion of the quasi-real spectrum with the following result:

Corollary 5.37. *Let R be a ring. Then $(\mathcal{Q}(R), \tau_T)$ is a T_0 space.*

Proof. We may argue just like in the proof of Corollary 5.21. Two quasi-orderings $\preceq_1, \preceq_2 \in \mathcal{Q}(R)$ are topologically indistinguishable if and only if $\preceq_1 \in \overline{\{\preceq_2\}}$ and $\preceq_2 \in \overline{\{\preceq_1\}}$, i.e. if and only if $\preceq_1 \leq \preceq_2 \leq \preceq_1$ with respect to the coarsening relation $\leq$. Anti-symmetry of $\leq$ yields $\preceq_1 = \preceq_2$. Thus, $(\mathcal{Q}(R), \tau_T)$ is T_0. □

6. Partial Quasi-Orderings on Rings

In the present chapter we develop a notion of partially quasi-ordered rings and establish an analogue of the dichotomy proven for totally quasi-ordered rings in Chapter 1 (see Theorem 1.50). Moreover, we deal with the question under which conditions partial quasi-orderings extend to total quasi-orderings.

While there is a rich theory on various kinds of partial orderings on rings, among others preorderings, preprimes and quadratic modules, little attention has been paid to a development of partial valuations. The only such notion, as far as the author of this thesis can tell, is that of (strict) partial valuations on division rings as elaborated on in [73]. Building upon this one, it turns out that the right counterparts in the sense of a dichotomy are division closed partial orderings, i.e. partial orderings that are compatible with addition and multiplication in the usual sense and admit division, respectively cancellation. These classes are the subject of Section 6.1.

In Section 6.2 we axiomatically introduce partially quasi-ordered rings and prove that all strictly partially valued rings and all division closed partially ordered rings are contained in this class. The subject of Section 6.3 is to show that the converse also applies, i.e. to establish a dichotomy for partial quasi-orderings in analogy to the total case. The proof of this result essentially follows the one that we presented in Section 1.3 for total quasi-orderings. Finally, in Section 6.4, we introduce the notion of quadratic partial quasi-orderings and prove that any quadratic partial quasi-ordering on a field extends to a total quasi-ordering.

6.1. **Partial Valuations and Partial Orderings.**

In this section we briefly introduce the notions of partially and strictly partially valued rings on the one hand (Definition 6.2), and partially ordered rings (Definition 6.6), respectively division closed partially ordered rings (Definition 6.8), on the other hand.

Remark 6.1. Throughout this chapter we only consider partial valuations and partial orderings with support $\{0\}$. For the sake of convenience and by abuse of notation, we will just call them partial valuations and partial orderings, i.e. not mention all over again that the supports are always trivial. The reader should keep this in mind, since otherwise the following is partly inconsistent with our previous work on total quasi-orderings.

Combining the notion of valuations on rings (see Definition 1.5) with the notion of partial valuations on division rings (cf. [73, Definition 1.1.1]) results in the following definition of partially valued rings:

Definition 6.2.

(a) Let $(\Gamma, +, \leq)$ be a partially ordered cancellative monoid. A *partial valuation* on a ring R is a surjective map $v\colon R \to \Gamma \cup \{\infty\}$, where ∞ is a symbol such that $\gamma < \infty$ and $\gamma + \infty = \infty = \infty + \gamma$ for all $\gamma \in \Gamma$, such that for all $x, y, z \in R$:

(PV1) $v(x) = \infty \Leftrightarrow x = 0$,

(PV2) $v(xy) = v(x) + v(y)$,

(PV3) If $v(z) \leq v(x), v(y)$, then $v(z) \leq v(x + y)$.

We call $\Gamma_v := \Gamma$ the *value monoid* of v.

(b) A *strict partial valuation* is a *partial valuation* that satisfies:

(PV4) If $v(z) < v(x), v(y)$, then $v(z) < v(x+y)$.

Remark 6.3.

(1) Any partially valued ring is in fact a domain, for if $xy = 0$ for some $x, y \in R$, then $v(x) + v(y) = \infty$ by (PV1) and (PV2), whence $x = 0$ or $y = 0$.

(2) Definition 6.2 may be generalised by replacing axiom (PV1) with the weaker axiom (PV1') $v(0) = \infty$ and $v(1) = 0$ (see Definition 1.4).
In analogy to (total) valuations, the support $\mathfrak{q}_v := \{x \in R \colon v(x) = \infty\}$ of v then would be a two-sided completely prime ideal of R, and the mapping $v' \colon R/\mathfrak{q}_v \to \Gamma_v$, $\overline{x} \mapsto v(x)$ would be a partial valuation on $R/\mathfrak{q}_{\preceq}$ in the sense of Definition 6.2. Moreover, v is strict if and only if v' is strict.

Lemma 6.4. *Let R be a ring, v a partial valuation on R and $x, y \in R$.*

(1) If $v(x)$ and $v(y)$ are comparable, then $v(x+y) \geq \min\{v(x), v(y)\}$.

(2) If v is strict and if $v(x)$ and $v(y)$ are strictly comparable (i.e. $v(x) < v(y)$ or vice versa), then $v(x+y) = \min\{v(x), v(y)\}$.

Proof. We first show that (1) holds. By assumption, $v(x)$ and $v(y)$ are comparable, whence $\gamma := \min\{v(x), v(y)\}$ exists. Since $\gamma \leq v(x), v(y)$, axiom (PV3) implies $\gamma \leq v(x+y)$. For the proof of (2) suppose that $v(x)$ and $v(y)$ are strictly comparable, say $v(x) < v(y)$. By (1), we obtain $v(x+y) \geq v(x)$. Assume for the sake of a contradiction that $v(x+y) > v(x)$. Then $v(x) < v(y) = v(-y)$ and $v(x) < v(x+y)$. Hence, (PV4) implies $v(x) < v(x+y-y) = v(x)$, a contradiction. Consequently, $v(x+y) = v(x) = \min\{v(x), v(y)\}$. □

Lemma 6.4(1) tells us that a partial valuation is a valuation if and only if its value monoid is totally ordered. The second part of the lemma is crucial to establish that an analogue of axiom (QR4) holds for the relation induced by a partial valuation v via $x \preceq y \Leftrightarrow v(y) \leq v(x)$ (see our proof of Proposition 1.33). For that reason we are eventually only interested in strict partial valuations.

Example 6.5.

(1) Let $p, q \in \mathbb{N}$ be prime, $p \neq q$. Consider the map

$$v := (v_p, v_q) \colon \mathbb{Z} \to \mathbb{N}_0 \times \mathbb{N}_0,\ x \mapsto (v_p(x), v_q(x)),$$

where v_p and v_q denote the p-adic, respectively q-adic valuation, and $\mathbb{N}_0 \times \mathbb{N}_0$ is equipped with the natural (i.e. componentwise/coordinatewise) partial ordering. Then v is a partial valuation, which is not strict. For instance $v(1) < v(p), v(q)$, but $v(1) = v(p+q)$.

If $p = q$, then v as above is a strict partial valuation, the value monoid being the diagonal $\{(n, n) \colon n \in \mathbb{N}_0\} \subseteq \mathbb{N}_0 \times \mathbb{N}_0$.

(2) More generally, it may be shown that any product $(v_i)_{i \in I}$ of valuations (with support $\{0\}$) on a ring R is a partial valuation, which is strict if all v_i coincide.

(3) Let D be a division ring and R a subring of D which is stable, i.e. invariant under inner automorphisms. Then R is local if and only if $R = D_v$ for some strict partial valuation v on D, where $D_v := \{x \in D \colon v(x) \geq 0\}$ denotes the valuation ring induced by v ([73, Proposition 1.1.6]).

(4) ([73, Example 1.1.7]) Consider $R = \mathbb{Z}_{(p)}[[X]]$ as subring of $K = \mathbb{Q}((X))$. Then R is local and stable. Hence, there is a strict partial valuation v on K such that $K_v = R$. Moreover, v is not a valuation. Indeed, since neither $pX^{-1} \in R$ nor $p^{-1}X \in R$ holds, K_v is not a valuation ring.

Definition 6.6. Let R be a ring and $\leq$ a binary, reflexive, transitive and anti-symmetric relation on R. Then we call $(R, \leq)$ a *partially ordered ring* if for all $a, b, x, y, z \in R$:

(PO1) $0 < 1$

(PO2) If $x \leq y$ and $0 \leq a, b$, then $axb \leq ayb$.

(PO3) If $x \leq y$, then $x + z \leq y + z$.

Lemma 6.7. *Let $(R, \leq)$ be a partially ordered ring. Consider the conditions*

(PO4) *If $axb \leq ayb$ and $0 < a, b$, then $x \leq y$ for all $a, b, x, y \in R$.*

(PO4′) *If $0 < xy$ and $0 < x$ (respectively $0 < y$), then $0 < y$ (respectively $0 < x$) for all $x, y \in R$.*

Then (PO4) implies (PO4'). If R is a domain, the converse is also true.

Proof. We first show that (PO4) implies (PO4'). Let $0 < xy$ and (w.l.o.g.) $0 < x$. Then $0 \leq y$ by (PO4). Moreover, $y \neq 0$, since $0 < xy$. Hence, $0 < y$ follows from anti-symmetry of $\leq$.

Conversely, suppose that R is a domain and that (PO4') holds. Let $axb \leq ayb$ and $0 < a, b$. Then $0 \leq a(y - x)b$. If equality holds, then $y - x = 0$, since R is a domain and $a, b \neq 0$. Hence, $x = y$. If $0 < a(y - x)b$, the claim follows from (PO4'). □

Fuchs calls a partially ordered ring division closed, if it fulfils condition (PO4') of Lemma 6.7 (cf. [20, p. 165]). For our purposes, the slightly stronger axiom (PO4) is necessary in order to deal with rings admitting zero-divisors.

Definition 6.8. A partially ordered ring $(R, \leq)$ is called *division closed* if it satisfies condition (PO4) of Lemma 6.7.

Eventually, we are only interested in division closed partial orderings. This is due to the fact that we will need an analogue of axiom (QR3) for partially quasi-ordered rings, and therefore also for partially ordered rings.

Example 6.9.

(1) Let $(R, \leq_R)$ be an ordered ring and $M := M_{n \times n}(R)$ the ring of all $(n \times n)$-matrices with entries in R for some $n \in \mathbb{N}$. We partially order M by defining

$$A \leq_M B :\Leftrightarrow \forall i, j \in \{1, \dots, n\} \colon a_{ij} \leq_R b_{ij}.$$

Then $\leq_M$ defines a partial ordering on M, which is not division closed. Consider for instance

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

Then $0 = XZ \leq YZ = 0$ and $0 < Z$, but X and Y are incomparable.

(2) Let D be a division ring. Any proper quadratic preordering on D is a division closed partial ordering on D. Indeed, since all squares are positive we have $d > 0$ if and only if $d^{-1} > 0$, whence (PO4) follows from (PO2).

6.2. **Partially Quasi-Ordered Rings.**

In the present section we introduce the notion of partially quasi-ordered rings (Definition 6.12) and show that it is consistent with our definition of totally quasi-ordered rings (Proposition 6.15). We further show that any division closed partially ordered ring (Proposition 6.16) and any strictly partially valued ring (Proposition 6.17) is a partially quasi-ordered ring.

Ideally, we would like to derive partially quasi-ordered rings from totally quasi-ordered rings by simply omitting totality of $\preceq$. However, by doing so we encounter a problem with the compatibility of $\preceq$ with addition.

First of all we have to decide what the symbol $\nsim$ in axiom (QR4) is supposed to mean in the partial case.

Notation 6.10. Let $\preceq$ be a binary relation on R. We denote

$$x \sim y :\Leftrightarrow x \preceq y \wedge y \preceq x,$$
$$x \asymp y :\Leftrightarrow x \preceq y \vee y \preceq x.$$

The relation $\sim$ was already defined in Notation 1.21, while $x \asymp y$ expresses that x and y are comparable. The negation of $\asymp$ is denoted by $\not\asymp$.

There are two natural translations of the assumption $z \nsim y$ in (QR4) to the partial case, namely

(i) either $y \prec z \vee z \prec y$,

(ii) or $y \prec z \vee z \prec y \vee y \not\asymp z$.

In what follows we choose (i) over (ii), since we aim for a dichotomy in the sense of Theorem 1.50, and condition (ii) is not fulfilled by some strict partial valuations.

However, as the following example shows, neither of these translations of (QR4) to partial quasi-orderings is strong enough to deduce a dichotomy.

Example 6.11. Let $(K, \leq)$ be some totally ordered field and $R = K[X]$. Define a binary relation $\preceq$ on R by declaring

$$x \preceq y :\Leftrightarrow \begin{cases} x, y \in K \text{ and } x \leq y \\ \\ \quad x, y \notin K \end{cases}$$

(1) $\preceq$ is a reflexive, transitive and non-total relation satisfying the axioms (QR1) - (QR4).

(2) $-1 \prec 0$ and $X \not\asymp X + 1$.

Proof. Claim (2) follows immediately from the definition of $\preceq$. So let us prove (1). Reflexive and non-total is clear. Next, we show transitivity. So suppose that $x \preceq y$ and $y \preceq z$. By definition of $\preceq$, either $x, y, z \in K$ or $x, y, z \notin K$. In the former case the claim follows by transitivity of $\leq$, while the latter case is trivial.

It remains to check the axioms (QR1) - (QR4). (QR1) is obviously satisfied. For (QR2) suppose that $x \preceq y$ and $0 \preceq z$. The crucial observation is that $0 \preceq z$ yields $z \in K$, whence clearly $xz \preceq yz$. The same reasoning applies to (QR3). For the proof of (QR4) suppose that $x \preceq y$ and that $z \prec y$ or $y \prec z$. The latter implies that y and z are in K, because all elements in $R \backslash K$ are $\preceq$-equivalent. But then also $x \in K$, so we obtain $x + z \preceq y + z$.

To cover the second interpretation of (QR4) from above, let us finally assume that y and z are not comparable. Then $y \in K$ and $z \notin K$ or vice versa. Moreover, from $x \preceq y$ follows that $x \in K$ if and only if $y \in K$. Hence, either $x, y \in K$ and $z \notin K$, or $x, y \notin K$ and $z \in K$. Either way, we obtain that $x + z$, $y + z \notin K$, whence $x + z \sim y + z$. □

Now (2) implies that $\preceq$, even though it satisfies all axioms of a quasi-ordered ring but totality, is neither induced by a partial ordering, nor by a partial valuation. Indeed, for partial valuations $0 \prec -1$ applies, while for partial orderings $X < X + 1$ follows from $0 < 1$ and compatibility with addition.

Hence, we need further axioms for the compatibility of $\preceq$ with addition.

Definition 6.12. Let R be a ring and $\preceq$ a binary, reflexive and transitive relation on R. Then $(R, \preceq)$ is a *partially quasi-ordered ring* if for all $x, y, z, a, b \in R$:

(PQ1) If $x \sim 0$, then $x = 0$.

(PQ2) $1 \asymp 0, -1$

(PQ3) If $0 \prec a, b$, then $x \preceq y$ if and only if $axb \preceq ayb$.

(PQ4) If $x \preceq y$ and $z \prec y$ or $y \prec z$, then $x + z \preceq y + z$.

(PQ5) If $x \preceq y$ and $z \not\preceq 0$, then $x + z \preceq y + z$.

(PQ6) If $0 \preceq -1$ and $x, y \preceq z$, then $x + y \preceq z$.

In this section, let $(R, \preceq)$ always denote a partially quasi-ordered ring. Our first aim is to show that our notion of partially quasi-ordered rings is consistent with the total one as introduced in Definition 1.24 (see also Remark 6.1).

Lemma 6.13. *If* $x \preceq 0$, *then* $0 \preceq -x$.

Proof. If $-x \not\preceq 0$, then $x \preceq 0$ implies via (PQ5) that $0 \preceq -x$, a contradiction. On the other hand, if $-x \prec 0$, then $x \preceq 0$ implies via (PQ4) that $0 \preceq -x$, again a contradiction. Hence, $0 \preceq -x$. □

Corollary 6.14. $0 \prec 1$.

Proof. By (PQ1) and (PQ2), either $0 \prec 1$ or $1 \prec 0$. But if $1 \prec 0$, then Lemma 6.13 tells us that $0 \preceq -1$, and applying (PQ3) yields $0 \preceq 1$, a contradiction. □

Proposition 6.15.

(1) If $(R, \preceq)$ *is a partially quasi-ordered ring such that* $\preceq$ *is total, then* $(R, \preceq)$ *is a quasi-ordered ring* (*with support* $\{0\}$).

(2) If $(R, \preceq)$ *is a quasi-ordered ring* (*with support* $\{0\}$), *then* $(R, \preceq)$ *is a partially quasi-ordered ring.*

Proof. We first show that (1) holds. Let $(R, \preceq)$ be a partially quasi-ordered ring such that $\preceq$ is total. Axiom (QR1) is precisely Corollary 6.14, while (PQ3) subsumes the axioms (QR2) and (QR3). Finally, the totality of $\preceq$ implies that (PQ4) coincides with (QR4). Hence, $(R, \preceq)$ is a quasi-ordered ring. Its support is trivial by (PQ1).

Conversely, let $(R, \preceq)$ be a quasi-ordered ring with support $\{0\}$. Then (PQ1) is obviously fulfilled. The axioms (PQ2) and (PQ5) follow immediately from the totality of $\preceq$, while (PQ3) is implied from (QR2) and (QR3). Moreover, (PQ4) is clearly weaker than (QR4). Last but not least, (PQ6) was proven in greater generality in Lemma 1.48. □

We conclude this chapter by showing that all division closed partial orderings, as well as all strict partial valuations, are partial quasi-orderings.

Proposition 6.16. *Any division closed partially ordered ring is a partially quasi-ordered ring.*

Proof. (PQ1) is an easy consequence of anti-symmetry, while (PQ2) is implied by (PO1) and (PO3). The axiom (PQ3) follows from (PO2) and (PO4). Furthermore, (PQ4) and (PQ5) are both immediately implied by (PO3). Finally, (PQ6) is vacuously true. □

Proposition 6.17. *Let* R *be a ring and* v *a strict partial valuation on* R. *Then* $(R, \preceq)$ *is a partially quasi-ordered ring, where*

$$\forall x, y \in R \colon x \preceq y :\Leftrightarrow v(y) \leq v(x).$$

Proof. We obtain $x \sim 0 \Leftrightarrow v(x) = v(0) = \infty \Leftrightarrow x = 0$ by (PV1). Thus, (PQ1) applies. Clearly, $0 \prec 1 \sim -1$, whence (PQ2) holds. Next, we verify (PQ3). So let $0 \prec a, b$. If $x \preceq y$, then $v(y) \leq v(x)$. Therefore,

$$v(ayb) = v(a) + v(y) + v(b) \leq v(a) + v(x) + v(b) = v(axb),$$

which means $axb \preceq ayb$. For the converse, suppose that $0 \prec a, b$ and $axb \preceq ayb$. The former yields $v(a), v(b) \in \Gamma_v$, whence we may apply that Γ_v is cancellative. So $v(a) + v(y) + v(b) \leq v(a) + v(x) + v(b)$ implies $v(y) \leq v(x)$, i.e. $x \preceq y$.
For (PQ4), let $x, y, z \in R$ such that $x \preceq y$, and $y \prec z$ or $z \prec y$, i.e. $v(y) \leq v(x)$ and $v(z) < v(y)$ or $v(y) < v(z)$. We first deal with the case $v(y) < v(z)$. Then

$$v(y + z) = v(y) \leq v(x), v(z)$$

by Lemma 6.4(2). Thus, $v(y + z) \leq v(x + z)$ by (PV3), i.e. $x + z \preceq y + z$.
Likewise, if $v(z) < v(y)$, we obtain

$$v(y + z) = v(z) \leq v(x), v(z),$$

so by (PV3) we get $v(y + z) \leq v(x + z)$, i.e. again $x + z \preceq y + z$.
The axiom (PQ5) is vacuously true, while (PQ6) and (PV3) coincide. □

6.3. **The Dichotomy of Partially Quasi-Ordered Rings.**

The aim of this section is to generalise the dichotomy from the total case (Theorem 1.50) to partially quasi-ordered rings (Theorem 6.28). For that purpose we show that any partially quasi-ordered ring with $-1 \prec 0$ is a division closed partially ordered ring (Proposition 6.23), and that any partially quasi-ordered ring with $0 \prec -1$ is a strictly partially valued ring (Proposition 6.27). The proof essentially follows the one that we gave in Section 1.3.

Let $(R, \preceq)$ again always denote a partially quasi-ordered ring.

Lemma 6.18. *Either* $-1 \prec 0$ *or* $0 \prec -1$.

Proof. Assume that $-1 \not\asymp 0$. Then $0 \prec 1$ (Corollary 6.14) implies via (PQ5) that $-1 \preceq 0$, a contradiction. Hence, $-1 \asymp 0$. Furthermore, (PQ1) yields $-1 \not\sim 0$. □

We first deal with the ordered case, i.e. the case $-1 \prec 0$ (see Proposition 6.23 below).

Lemma 6.19. *Let* $x \in R$. *Then* $x \asymp 0$ *if and only if* $-x \asymp 0$.

Proof. Let $x \asymp 0$, and assume that $-x \not\asymp 0$. Then it follows from $x \preceq 0$ or $0 \preceq x$ that $0 \preceq -x$ or $-x \preceq 0$. Hence, $-x \asymp 0$, a contradiction. □

Corollary 6.20. *Let* $x \in R$ *such that* $x \not\asymp 0$. *Then* $E(x) = \{x\}$.

Proof. Let $y \in E(x)$. By the previous lemma $-x \not\asymp 0$, whence (PQ5) implies that $0 \preceq y - x$ follows from $x \preceq y$, and that $y - x \preceq 0$ follows from $y \preceq x$. Therefore, $y - x = 0$ by (PQ1), i.e. $y = x$. Thus, $E(x) = \{x\}$. □

Lemma 6.21. *Let* $x \in R$, *and suppose that* $-1 \prec 0$. *Then* $0 \preceq x$ *if and only if* $-x \preceq 0$. *Moreover,* $-x \prec x$ *or* $x \prec -x$ *for any* $0 \neq x \in R$ *such that* $x \asymp 0$.

Proof. If $x \preceq 0$, then $0 \preceq -x$ by Lemma 6.13. Conversely, if $0 \prec x$, then $-x \preceq 0$ by the assumption $-1 \prec 0$ and axiom (PQ3). The moreover statement follows via (PQ1), because either $-x \prec 0 \prec x$ or $x \prec 0 \prec -x$. □

Corollary 6.22. *Let* $x \in R$ *such that* $x \asymp 0$, *and suppose that* $-1 \prec 0$. *Then* $E(x) = \{x\}$.

Proof. If $x = 0$, this is precisely the statement of (PQ1). So let $x \neq 0$, and let $y \in E(x)$. Then also $y \asymp 0$, and x and y have the same sign, which is contrary to the ones of $-x$ and $-y$ (Lemma 6.21). Therefore, it follows via (PQ4) that $y \preceq x$ implies $y - x \preceq 0$, and that $x \preceq y$ implies $0 \preceq y - x$. Thus, (PQ1) yields $y - x = 0$, i.e. $y = x$. Hence, $E(x) = \{x\}$. $\square$

Proposition 6.23. *Let $(R, \preceq)$ be a partially quasi-ordered ring such that $-1 \prec 0$. Then $(R, \preceq)$ is a division closed partially ordered ring.*

Proof. We only have to verify that $\preceq$ is anti-symmetric and that $(R, \preceq)$ satisfies axiom (PO3). Anti-symmetry of $\preceq$ was already shown in Corollary 6.20 and Corollary 6.22. Now suppose that $x \preceq y$. Then we get $x - y \preceq 0$. Indeed, either $-y \not\asymp 0$, and then we may add $-y$ via (PQ5); or $-y \asymp 0$, and then y and $-y$ have different signs (Lemma 6.21; w.l.o.g. $y \neq 0$), whence we may add $-y$ via (PQ4). Thus, $x - y \preceq 0$. Now if $y + z \not\asymp 0$, then $x + z \preceq y + z$ by (PQ5). On the other hand, if $y + z \asymp 0$, the claim follows either immediately (if $y + z = 0$), or else again via (PQ4). $\square$

It remains to prove that any partial quasi-ordering $\preceq$ on R with $0 \prec -1$ is induced by a strict partial valuation on R.

Lemma 6.24. *Suppose that $0 \prec -1$. Then $0 \preceq x$ for all $x \in R$.*

Proof. We first show that $x \asymp 0$ for all $x \in R$. From $x, x \preceq x$ follows $2x \preceq x$ by (PQ6). So if $x \not\asymp 0$, then Lemma 6.19 and (PQ5) imply $x \preceq 0$, a contradiction. Hence, $x \asymp 0$ for any $x \in R$. It remains to show that this means $0 \preceq x$ for any $x \in R$. Lemma 6.13 yields $0 \preceq x$ or $0 \preceq -x$ for all $x \in R$. From the assumption $0 \prec -1$ and (PQ3) it follows that $0 \preceq x$ for all $x \in R$. $\square$

So if $0 \prec -1$, then Lemma 6.24 and (PQ3) imply that $x \preceq y$ if and only if $axb \preceq ayb$ for all $a, b \in R\backslash\{0\}$. Consequently, $\sim$ is preserved under multiplication and cancellation.

Lemma 6.25. *Suppose that $0 \prec -1$. Then $x \sim -x$ for all $x \in R$.*

Proof. Evidently, we may w.l.o.g. assume that $\mathrm{char}(R) \neq 2$. Since $\sim$ is preserved under multiplication for $0 \prec -1$, it suffices to show that $-1 \sim 1$. If not, then $-1 \prec 1$ or $1 \prec -1$ by axiom (PQ2). If $1 \prec -1$, then we obtain by (PQ4), (PQ1) and $\mathrm{char}(R) \neq 2$ that $2 \prec 0$, a contradiction to Lemma 6.24. If $-1 \prec 1$, then it follows from $0 \prec -1$ and (PQ3) that $1 \preceq -1$, again a contradiction. $\square$

The rest of the proof is essentially the same as in the total case. Define

$$H := (R\backslash\{0\})/\sim, \quad [x] + [y] := [xy], \quad \text{and} \quad [x] \leq [y] :\Leftrightarrow y \preceq x.$$

Lemma 6.26. *$(H, +, \leq)$ is a partially ordered cancellative monoid.*

Proof. The proof is completely analogue to the one Lemma 1.47. $\square$

We claim that the following map is a strict partial valuation that induces $\preceq$:

$$v \colon R \to H \cup \{\infty\},\, x \mapsto \begin{cases} [x], & x \neq 0 \\ \infty, & x = 0 \end{cases}$$

Proposition 6.27. *Let $(R, \preceq)$ be a partially quasi-ordered ring such that $0 \prec -1$. Then there is a strict partial valuation v on R such that $x \preceq y$ if and only if $v(y) \leq v(x)$ for all $x, y \in R$.*

Proof. We prove that the map v that we just defined fulfils the claim. Evidently, v is surjective and satisfies (PV1). Now let $x, y \in R$, w.l.o.g. $x, y \neq 0$. Then

$$v(xy) = [xy] = [x] + [y] = v(x) + v(y),$$

so (PV2) is also fulfilled. Since $0 \prec -1$, (PV3) is precisely (PQ6). Next, we prove (PV4). So we have to show that $v(z) < v(x), v(y)$ implies $v(z) < v(x+y)$, i.e. that $x, y \prec z$ implies $x + y \prec z$. Because of $0 \prec -1$, (PQ6) yields $x + y \preceq z$. Assume for the sake of a contradiction that $x + y \sim z$. Clearly, $-x \prec -z$ by Lemma 6.25 and the assumption that $x \prec z$. It suffices to show that $x + z \prec -z$. Then we may deduce from $-x \prec -z$ via (PQ4) that

$$z = -x + (x + z) \preceq -z + (x + z) = x,$$

the desired contradiction, whence (PV4) holds. So let us verify that $x + z \prec -z$. We know that $-z \sim z \sim x + y$. Moreover, $-x \sim x \prec z \sim -z$. So we have $-z \sim x + y$ and $-x \prec -z, x + y$. So (PQ4) tells us that $-x - z \sim y$. Hence, indeed $x + z \sim -x - z \sim y \prec z \sim -z$. Thus, v is a strict partial valuation.
Moreover, if $x, y \in R \backslash \{0\}$, then

$$x \preceq y \Leftrightarrow [y] \leq [x] \Leftrightarrow v(y) \leq v(x).$$

The same obviously applies if $x = 0$ or $y = 0$. □

Our results give rise to the following dichotomy in the sense of Fakhruddin:

Theorem 6.28. *Let R be a ring and $\preceq$ a binary relation on R. Then $(R, \preceq)$ is a partially quasi-ordered ring if and only if it is either a division closed partially ordered ring or there is a strict partial valuation v on R such that $x \preceq y$ if and only if $v(y) \leq v(x)$ for all $x, y \in R$.*

Proof. This is an immediate consequence of Proposition 6.16, Proposition 6.17, Lemma 6.18, Proposition 6.23 and Proposition 6.27. □

6.4. **Extending Partial Quasi-Orderings.**

In this section we introduce quadratic partially quasi-ordered rings (Definition 6.29). Theorem 6.28 yields the dichotomy that any quadratic partially quasi-ordered field is either a preordered field or a strictly partially valued field (Theorem 6.31). From this result we derive that any quadratic partial quasi-ordering on a field K extends to a quasi-ordering on K (Theorem 6.34).

Definition 6.29. We call a partially quasi-ordered ring $(R, \preceq)$ *quadratic*, if $0 \preceq x^2$ for all $x \in R$.

Example 6.30.

(1) Any strict partial valuation on a ring R is a quadratic partial quasi-ordering. This was shown in Proposition 6.17 and Lemma 6.24.
(2) The partial ordering from Example 6.9(2) is quadratic.
(3) Any quadratic partial ordering on a division ring is a preordering and vice versa, where in this section *preordering* shall always mean proper quadratic preordering. This follows from Theorem 6.28 and the fact that preorderings over division rings are division closed (see Example 6.9(3)).

As an immediate consequence of Theorem 6.28 and Example 6.30(1) and (3), we obtain the following dichotomy:

Theorem 6.31. *Let K be a field and $\preceq$ a binary relation on K. Then $(K, \preceq)$ is a quadratic partially quasi-ordered field if and only if it is either a preordered field or there is a strict partial valuation v on K such that $x \preceq y$ if and only if $v(y) \leq v(x)$ for all $x, y \in K$.*

In what follows let $(K, \preceq)$ always denote a quadratic partially quasi-ordered field. We define the set

$$\Sigma_{\preceq} := \{\preceq' \subseteq K^2 \colon \preceq' \text{ is a quadratic partial quasi-ordering s.t.} \, \forall x, y \in K \colon \\ x \preceq y \Rightarrow x \preceq' y \text{ and } x \prec y \Rightarrow x \prec' y\}.$$

We claim that $\Sigma_{\preceq}$ admits a maximal element and that any of its maximal elements is a quasi-ordering on K extending $\preceq$. For the proof, we need the notion of dominance of local rings.

Definition 6.32. Let A and B be local rings with maximal ideals $\mathfrak{m}_A$, respectively $\mathfrak{m}_B$. Then B *dominates* A, written $A \leq B$, if

$$A \subseteq B \,\text{and}\, \mathfrak{m}_B \cap A = \mathfrak{m}_A.$$

If K is a field and v a strict partial valuation on K, then $K_v = \{x \in K \colon v(x) \geq 0\}$ is a local ring with maximal ideal $I_v = \{x \in K \colon v(x) > 0\}$ ([73, Proposition 1.1.6]). Moreover, the valuation rings on K are precisely the maximal local subrings of K w.r.t. $\leq$. (cf. [57, Theorem 10.2]).

Proposition 6.33. *Let K be a field, and let v and w be strict partial valuations on K. The following two conditions are equivalent:*

(1) (i) If $v(x) \leq v(y)$, then $w(x) \leq w(y)$ for all $x, y \in K$ and
(ii) If $v(x) < v(y)$, then $w(x) < w(y)$ for all $x, y \in K$.

(2) K_w dominates K_v.

Proof. We first show that (1) implies (2). From (i) it is easy to see that $K_v \subseteq K_w$. Now let $x \in I_w \cap K_v$. Then $v(x) \geq 0$. If $v(x) = 0$, then $w(x) = 0$ by (i), contradicting the fact that $x \in I_w$. Hence, $v(x) > 0$, i.e. $x \in I_v$. Conversely, let $x \in I_v \subseteq K_v$. Then $v(x) > 0$, whence also $w(x) > 0$ by (ii). Thus, $x \in I_w \cap K_v$.

For the converse, first suppose that $v(x) < v(y)$. Then $y/x \in I_v \subseteq I_w$ by (2). Therefore, $w(x) < w(y)$, so (ii) is satisfied. Next, suppose that $v(x) \leq v(y)$. The case $v(x) < v(y)$ was already dealt with, so we may assume that $v(x) = v(y)$. Then x/y and $y/x \in K_v \subseteq K_w$. Consequently, also $w(x) = w(y)$. □

By exploiting the dichotomy from Theorem 6.31, we may prove:

Theorem 6.34. *Let $(K, \preceq)$ be a quadratic partially quasi-ordered field. Then $\preceq$ extends to a quasi-ordering on K.*

Proof. Consider the set $\Sigma_{\preceq}$ from above. Obviously, $\preceq \in \Sigma$, whence $\Sigma_{\preceq} \neq \emptyset$. We partially order $\Sigma_{\preceq}$ by inclusion. For any chain $(\preceq_i)_{i \in I}$ in $\Sigma_{\preceq}$, it is routine to check that $\bigcup_{i \in I} \preceq_i \in \Sigma$ is an upper bound for this chain. Hence, Zorn's lemma states that $\Sigma_{\preceq}$ has a maximal element, say $\preceq'$. By definition of $\Sigma_{\preceq}$, we know that $\preceq'$ is a quadratic partial quasi-ordering on K extending $\preceq$.

If $-1 \prec 0$, then $\preceq$ is a preordering and $\preceq'$ a maximal preordering containing $\preceq$. Hence, $\preceq'$ is an ordering on K extending $\preceq$ (cf. e.g. [64, Proposition 4.1.4]).

If $0 \prec -1$, then $\preceq$ and $\preceq'$ are induced by strict partial valuations on K, say $\preceq = \preceq_v$ and $\preceq' = \preceq_w$ for some strict partial valuations v, w on K ([73, Proposition 1.1.6]). By definition of $\Sigma_{\preceq}$ and Proposition 6.33, the valuation ring K_v is dominated by K_w. We show that K_w is a maximal element under the dominance relation. Then K_w is a valuation ring ([57, Theorem 10.2]) containing K_v, whence w is a valuation on K

extending v. So let A be a local ring dominating K_w. Then A also dominates K_v. However, by Proposition 6.33, the choice of $\preceq'$ and the fact that local subrings of K are in bijective correspondence with strict partial valuations on K ([73, Proposition 1.1.6]), we know that K_w is a maximal element among all local rings dominating K_v, whence $A = K_w$. Thus, w is a valuation on K extending v. □

Remark 6.35.

(1) We considered fields in order to exploit the fact that maximal local rings are valuation rings. Moreover, the proof of Theorem 6.34 relies on the result that for division rings there is a bijective correspondence between strict partial valuations and stable local subrings (cf. [73, Proposition 1.1.6]).

(2) For an alternative, possibly uniform proof of Theorem 6.34 it would suffice to show that the maximal element $\preceq'$ of $\Sigma_{\preceq}$ is total. We may then conclude by applying Proposition 6.15.

7. Quasi-Real Closed Fields and Quantifier Elimination

In the present chapter we first introduce quasi-real closed fields (Definition 7.2, see also Proposition 7.8), that way subsuming the classes of algebraically closed fields and real closed fields (Corollary 7.4). By referring to the respective results for algebraically closed valued fields and real closed valued fields, we then deduce that the theory of quasi-real closed fields adjoined with a non-trivial compatible valuation admits quantifier elimination (Theorem 7.9). As an immediate consequence, we obtain that this theory is model complete. Moreover, it is the model companion of the theory of quasi-ordered fields adjoined with a non-trivial compatible valuation (Corollary 7.11).

To begin with, we introduce a notion of quasi-real closed fields. This is achieved by generalising the characterisation of real closed fields from Artin and Schreier.

Remark 7.1. ([4]) An ordered field (K, P) is real closed if and only if one of the following equivalent conditions is fulfilled:

(1) P does not extend to any proper algebraic extension of K.

(2) (K, P) satisfies:

 (i) every $a \in P$ has a square root in K,

 (ii) every polynomial $f \in K[X]$ of odd degree has a root in K.

We will exploit both of these conditions. The first condition is suitable to develop a notion of quasi-real closure, while (2) has the crucial advantage that it is expressible in the language of quasi-ordered fields.

Definition 7.2. A quasi-ordered field $(K, \preceq)$ is called *quasi-real closed*, if $\preceq$ does not extend to any proper algebraic extension of K.

Lemma 7.3. *Let K be a field. The following are equivalent:*

(1) K is algebraically closed,

(2) (K, v) is quasi-real closed for some valuation v on K,

(3) (K, v) is quasi-real closed for any valuation v on K.

Proof. If K is algebraically closed, then K admits no proper algebraic extension, whence (K, v) is quasi-real closed for any valuation v on K. Consequently, (1) implies (3). Moreover, since any field admits a valuation, clearly (3) implies (2).

Finally, suppose that (2) holds. Note that the valuation v on K extends to any field extension L of K by Chevalley's Extension Theorem (cf. [18, Theorem 3.1.1]). Since (K, v) is quasi-real closed, this means that K does not admit any proper algebraic extension, i.e. that K is algebraically closed. Thus, (2) implies (1). □

Corollary 7.4. *A quasi-ordered field $(K, \preceq)$ is quasi-real closed if and only if $\preceq$ is either an ordering and $(K, \preceq)$ a real closed field, or $\preceq$ is a valuation and K an algebraically closed field.*

Proof. This follows from the dichotomy of quasi-ordered fields (Theorem 1.23), Remark 7.1(1) (if $\preceq$ is an ordering), and Lemma 7.3 (if $\preceq$ is a valuation). □

Definition 7.5. Let $(K, \preceq)$ be a quasi-ordered field. A quasi-ordered field $(L, \preceq')$ is called a *quasi-real closure* of $(K, \preceq)$, if

(1) $(L, \preceq')$ is a quasi-real closed field,

(2) $L|K$ is an algebraic field extension,

(3) $\preceq'$ is an extension of $\preceq$, i.e. $K \cap \preceq' = \preceq$.

Proposition 7.6. *Any quasi-ordered field* $(K, \preceq)$ *admits a quasi-real closure.*

Proof. Applying Zorn's lemma to the set

$$\{(L, \preceq') \colon (L, \preceq') \text{ is a quasi-ordered field}, L|K \text{ algebraic}, K \cap \preceq' = \preceq\},$$

partially ordered by

$$(L_1, \preceq_1') \leq (L_2, \preceq_2') :\Leftrightarrow L_1 \subseteq L_2 \text{ and } L_1 \cap \preceq_2' = \preceq_1',$$

yields a quasi-ordered field $(L, \preceq')$ that is a maximal algebraic field extension of $(K, \preceq)$. Hence, $(L, \preceq')$ is quasi-real closed and a quasi-real closure of $(K, \preceq)$. □

Exploiting Fakhruddin's dichotomy (Theorem 1.23), we could further deduce that the quasi-real closure of a quasi-ordered field is unique up to K-isomorphism.

Our next aim is to establish that quasi-real closed fields adjoined with a compatible valuation admit quantifier elimination. To this end, we refer to the respective results that are known for real closed valued and algebraically closed valued fields.

Theorem 7.7. (cf. [63, Theorem 4.4.2, Theorem 4.5.1]).

(1) *The theory of algebraically closed fields adjoined with a non-trivial valuation* v *admits quantifier elimination in the language of fields adjoined with* $\preceq_v$.

(2) *The theory of real closed fields adjoined with a non-trivial compatible valuation* v *admits quantifier elimination in the language of ordered fields adjoined with* $\preceq_v$.

In order to subsume these results, we now work with a different characterisation of quasi-real closed fields.

Proposition 7.8. *A quasi-ordered field* $(K, \preceq)$ *is quasi-real closed if and only if the following two conditions are satisfied:*

(i) *every* $x \in K$ *with* $0 \preceq x$ *has a square root in* K.

(ii) *every polynomial* $f \in K[X]$ *of odd degree has a root in* K.

Proof. If $\preceq$ is an ordering, this follows from Remark 7.1(2) and Corollary 7.4. So let $\preceq$ be a valuation. If $(K, \preceq)$ is quasi-real closed, then K is algebraically closed according to Corollary 7.4, whence (i) and (ii) are both fulfilled.

Conversely, suppose that (i) and (ii) hold. Since $\preceq$ is a valuation, we obtain by (i) that all elements in K have a square root in K. In particular, $\sqrt{-1} \in K$. Along with condition (ii), this implies that K is algebraically closed according to [70, Theorem 2]. Hence, $(K, \preceq)$ is quasi-real closed. □

Proposition 7.8 enables us to formalise quasi-real closed fields in a first order language. As our language of quasi-ordered fields with a compatible valuation v we fix $\mathcal{L} = \{+, \cdot, -, \preceq, \preceq_v, 0, 1\}$. Moreover, we denote by Σ the following set of $\mathcal{L}$-sentences:

(1) the axioms that $(K, +, \cdot, -)$ is a field.

(2) the axioms that $(K, \preceq)$ is a quasi-real closed field.

(3) the axioms that $(K, \preceq_v)$ is a non-trivially valued field.

(4) the following sentences that determine the relationship of $\preceq$ and $\preceq_v$:

   (i) $0 \prec -1 \to (\forall x, y \colon 0 \preceq x \preceq y \leftrightarrow x \preceq_v y)$

   (ii) $-1 \prec 0 \to (\forall x, y \colon 0 \preceq x \preceq y \to x \preceq_v y)$

The last two sentences state that $\preceq = \preceq_v$ if $\preceq$ is a valuation, and that v is $\preceq$-compatible if $\preceq$ is an ordering. That way, we have unified the languages of real closed valued fields and algebraically closed valued fields (see Theorem 7.7).

We refer to the theory of Σ as the *theory of quasi-real closed fields adjoined with a non-trivial compatible valuation.* Exploiting the dichotomy of quasi-ordered fields and Theorem 7.7, we obtain:

Theorem 7.9. *The theory of quasi-real closed fields adjoined with a non-trivial compatible valuation admits quantifier elimination.*

Proof. Let Σ denote this theory. Then any model of Σ is either an algebraically closed field adjoined with a non-trivial valuation or a real closed field adjoined with a non-trivial compatible valuation (Corollary 7.4). Therefore, the claim follows from the fact that these two classes admit quantifier elimination, see [63, Theorem 4.4.2], respectively [63, Theorem 4.5.1]. $\square$

Definition 7.10. Let $\Sigma \subseteq \mathrm{Sent}(\mathcal{L})$ for a given language $\mathcal{L}$. A set $\Sigma^* \subseteq \mathrm{Sent}(\mathcal{L})$ is called a *model companion* of Σ, if

(1) every model of Σ^* is a model of Σ,
(2) every model of Σ can be extended to a model of Σ^*,
(3) Σ^* is model complete.

Corollary 7.11. (cf. [63, Corollary 4.4.3], [63, Corollary 4.5.4])
The theory of quasi-real closed fields adjoined with a non-trivial compatible valuation is model complete. It is the model companion of the theory of quasi-ordered fields adjoined with a non-trivial compatible valuation.

Proof. The model completeness follows immediately from the quantifier elimination that we obtained in Theorem 7.9. It remains to show that condition (2) of Definition 7.10 holds in our setting, both (1) and (3) being obviously satisfied.

So let $(K, \preceq, \preceq_v)$ be some quasi-ordered field adjoined with a non-trivial compatible valuation, and let $(L, \preceq')$ be a quasi-real closure of $(K, \preceq)$ (Proposition 7.6). Then $(L, \preceq')$ is a quasi-real closed field. Moreover, $L|K$ is algebraic, whence Proposition 3.25 yields a unique extension $\preceq'_\nu$ of $\preceq_v$ from K to L such that ν is compatible with $\preceq'$. If $\preceq$ is a valuation (i.e. $\preceq_v = \preceq$), then the uniqueness tells us that also $\preceq'_\nu = \preceq'$. Furthermore, ν is non-trivial since v is non-trivial. Hence, $(L, \preceq', \preceq'_\nu)$ is an extension of $(K, \preceq, \preceq_v)$ and a model of the theory of quasi-real closed fields adjoined with a non-trivial compatible valuation. $\square$

Remark 7.12.

(1) The theory of quasi-real closed fields adjoined with a non-trivial compatible valuation is easily seen not to be complete. For example the sentence $\exists x \colon x^2 + 1 = 0$ is false for any real closed field, but true for any algebraically closed field. Consequently, not all models of our theory Σ are elementary equivalent.
(2) Our notion of quasi-real closure might also provide a partly uniform proof of Theorem 7.9. In fact, the existence (see Proposition 7.6) and uniqueness of the algebraic closure, respectively real closure, plays an important role in the proof of [63, Theorem 4.4.2], respectively [63, Theorem 4.5.1].

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List of Symbols

Let R always denote a ring.

Quasi-Orderings:

$\preceq$	standard symbol for (possibly partial) quasi-orderings
$\sim_{\preceq}$	equivalence relation given by $x \sim_{\preceq} y \Leftrightarrow x \preceq y \wedge y \preceq x$
$E_{\preceq}(x)$	equivalence class of x w.r.t. $\sim_{\preceq}$
$\asymp_{\preceq}$	comparability relation given by $x \asymp_{\preceq} y \Leftrightarrow x \preceq y \vee y \preceq x$
$\mathfrak{q}_{\preceq}$	support of $\preceq$
$\preceq_v$	quasi-ordering induced by a valuation v
$\preceq'$	induced quasi-ordering on R/I for an ideal $I \subseteq \mathfrak{q}_{\preceq}$ of R
$\trianglelefteq$	induced quasi-ordering on the localisation R_S for a multiplicative subset $S \subset R$. Sometimes also used for quasi-orderings on fields.
$\preceq^*$	induced quasi-ordering on the residue class domain Rv for a $\preceq$-compatible valuation v on R
$\mathcal{T}_{\preceq}$	topology induced by $\preceq$

Orderings:

$\leq$	standard symbol for (possibly partial) orderings
$P_{\leq}$	positive cone corresponding to $\leq$
$\mathfrak{q}_{\leq}$	support of $\leq$
$\mathfrak{q}_P$	support of P

Valuations:

v, w, u	standard symbols for (possibly partial) valuations
$\mathfrak{q}_v$	support of v
Γ_v	value monoid/value group of v
R_v	valuation ring of a valuation v on R
I_v	valuation ideal of v
Rv	residue class domain R_v/I_v of a valuation v on R
v'	induced valuation on R/I for an ideal $I \subseteq \mathfrak{q}_{\preceq}$ of R
ν	induced valuation on the localisation R_S for a multiplicative subset $S \subset R$. Sometimes also used for valuations on fields.
w/v	quotient valuation (also called compositum) of w and v
$v_{\preceq}$	natural valuation of a quasi-ordering $\preceq$

Relationship of quasi-orderings:

$\preceq_1 \leq \preceq_2$ $\preceq_2$ is coarser than $\preceq_1$ (or $\preceq_1$ finer than $\preceq_2$)

$\preceq_1 \sim \preceq_2$ $\preceq_1$ and $\preceq_2$ are dependent

Spaces of quasi-orderings, orderings and valuations:

qr-Spec(R)	quasi-real spectrum of R
$\mathcal{Q}(R)$	set of all quasi-orderings on R
$\mathcal{O}(R)$	set of all orderings on R
$\mathcal{V}(R)$	set of all valuations on R
$\mathcal{Q}_{\mathfrak{q}}(R)$	set of all quasi-orderings with support $\mathfrak{q}$ on R
$\mathcal{Q}^v(R)$	set of all quasi-orderings $\preceq$ on R such that v is strongly $\preceq$-compatible
$\mathcal{Q}^v_{\mathfrak{q}}(R)$	set of all quasi-orderings $\preceq$ on R with support $\mathfrak{q}$ such that v is strongly $\preceq$-compatible

The sets $\mathcal{O}_{\mathfrak{q}}(R), \mathcal{V}_{\mathfrak{q}}(R), \mathcal{O}^v(R), \mathcal{V}^v(R), \mathcal{O}^v_{\mathfrak{q}}(R)$ and $\mathcal{V}^v_{\mathfrak{q}}(R)$ are defined analogously.

Topologies on spaces of quasi-orderings:

$\mathcal{H}$	Harrison topology (or spectral topology)
$\mathcal{T}$	Tychonoff topology (or constructible topology, or patch topology)
τ_T	tree topology