Global existence and asymptotic decay for quasilinear second-order symmetric hyperbolic systems of partial differential equations occurring in the relativistic dynamics of dissipative fluids

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Introduction

A vast spectrum of physical phenomena is described in terms of quasilinear symmetric hyperbolic systems of partial differential equations. Hence it has been of great interest for mathematicians and physicists alike to study the properties of such systems. In this regard it is of particular importance to show that the associated Cauchy problem is well-posed, i.e., to show that for given initial data there exists a unique solution depending continuously on the data. Early proofs of well-posedness for scalar equations date back to the 1930's (cf., e.g. [1, 36, 46, 52]) and extensive research on systems was done in the 1950's and 1960's (cf., e.g. [5, 9, 20, 54]). The first general results due to Kato [29, 30] and Lax [42] showed that quasilinear symmetric hyperbolic systems of first order are always locally well-posed in L^2-based Sobolev spaces. I.e., they have smooth solutions on some, possibly short, time interval. These results were generalized to quasilinear second-order symmetric hyperbolic systems by Hughes, Kato and Marsden [24].

However, it is in general not possible to extend those smooth local solutions for all time. This is due to the fact that typically regular solutions to quasilinear symmetric hyperbolic systems cease to exist in finite time even for smooth initial data and shock waves emerge (cf. [27, 32, 38, 40, 42]). To solve this issue one can search for so called weak solutions which are not smooth or not even continuous anymore. Global existence of weak solutions for first-order hyperbolic systems in one space variable has been shown in particular in [1, 21], and some of their properties including asymptotic behavior where studied in, e.g., [10, 11, 22, 39]. As the theory of weak solutions and shock waves constitutes a vast and important area of research in itself which is beyond the scope of this thesis, we refer to the standard work of Dafermos [6] for an in-depth portrayal of this field.

Though in general one cannot expect quasilinear hyperbolic systems of differential equations to have smooth global solutions, this is possible for subclasses of - or classes of systems related to - hyperbolic systems carrying additional structure. In this context one often is not interested in constructing global solutions for arbitrary initial data but only for such data which are sufficiently close to a stationary or quasistationary reference solution such as, in particular, a homogeneous rest state of the underlying system - so called "small solutions". In addition one also wants to prove that the reference state is stable in the sense that solutions decay to it for large time.

The first general results for systems in several space variables are due to Kawashima [33] and Kawashima and Shizuta [53] not for "purely" hyperbolic systems but for systems of a hyperbolic-parabolic composite type where a first-order hyperbolic system and a parabolic system are coupled via their lower order terms. Here local well-posedness is proven by applying the results of Kato [30] to each system individually. To show stability of homogeneous rest states one needs to impose a specific requirement on the linearization at this state, in later publications called *Shizuta-Kawashima condition* or *condition* (K), in such a way that the interplay between the coefficients of higher order and lower order terms ensures the temporal decay of solutions to the linearized problem. The nonlinear system is then treated as a perturbation of the linear one which allows for extending local (small) solutions for all time. In physics, systems of a hyperbolic-parabolic composite type which satisfy condition (K) occur in various fields of continuum mechanics when dissipative effects are taken into account.

In the context of first-order quasilinear symmetric hyperbolic systems, stability of homogeneous rest states can be shown for systems with relaxation (or "source") terms - so called *balance laws* - that satisfy the Shizuta-Kawashima condition as well as *entropy dissipation conditions*. In fact, many hyperbolic-parabolic systems (*viscous conservation laws*) can be derived from balance laws via the *Chapman-Enskog expansion* [3] and it was shown in [35, 58] that, under suitable entropy dissipation conditions, the Shizuta-Kawashima condition holds for a balance law if and only if it holds for the associated hyperbolic-parabolic system. Besides [35, 58], important results in this field of research - building on somewhat different entropy conditions - can be found in, e.g., [4, 50, 51, 57]. Note that in particular, this includes a theory of the relativistic dynamics of dissipative fluids by Müller and Ruggeri, which is derived by methods of extended thermodynamics and leads to an infinite hierarchy of quasilinear first-order hyperbolic systems (cf. [44]).

While stability of rest states of quasilinear first-order symmetric hyperbolic systems with relaxation and symmetric hyperbolic-parabolic systems with dissipation is well understood, far less is known about global existence and asymptotic decay for nonlinear second-order symmetric hyperbolic systems. Among the latter, cases that have been studied are Timoshenko systems, damped wave equations with a nonlinear convection term and systems that occur in the dynamics of viscoelastic materials. Stability of rest states for Timoshenko systems with frictional damping and with memory was shown in [43] and [41], respectively. However, in contrast to the systems considered in this thesis, Timoshenko systems only depend on one space variable and the proofs are based on rewriting these second-order hyperbolic as equivalent first-order hyperbolic systems. The damped wave equations, treated e.g. in [23, 28, 55], are scalar and semilinear whereas the second-order symmetric hyperbolic systems in relativistic fluid dynamics are quasilinear and coupling terms are involved. Closest related to our work are probably the equations of viscoelastic materials for which global existence and asymptotic decay of small solutions was proven in [8]. However, as there occur no spatial first order derivatives of the state variables in the linearization of such systems, the techniques used in [8] do not readily carry over to our situation and we have to use different methods in this thesis for both the linear decay estimates and the transition to the nonlinear system.

There exists also interesting work on hyperbolic extensions of the classical Navier-Stokes equations for incompressible fluids. Paicu and Raugel [45] showed existence after replacing the Laplacian by a d'Alembertian, and Racke and Saal [47, 48] gave a solution theory for a model proposed by Carassi and Morro [2]. While their authors refer to Cattaneo's famous hyperbolic formulation of heat conduction, both models are neither Lorentz nor Galilei invariant. They both also have other fundamental differences from the second-order symmetric hyperbolic Navier-Stokes-Fourier (NSF) equations we consider. On the one hand, in connection with the fact that they deal with incompressible fluids, they do not have finite speed of propagation and have no equation for the conservation of mass; on the other hand their mathematical structure makes in particular the principal parts of their differential operators distinctly easier to treat.

The relativistic hyperbolic NSF equations studied in this thesis were formulated by Freistühler and Temple in their causal theory of relativistic dissipative fluid dynamics (cf. [13, 17, 18, 19]). Since these equations constitute quasilinear second-order symmetric hyperbolic systems they are locally well-posed, as is directly implied by the results of [24]. It is the goal of this thesis to show that they carry additional structure which enables us to extend local solutions globally close to a homogeneous reference state and that those solutions decay to the latter as time tends to infinity. One should note that the theory developed by Freistühler and Temple yields a class of partial differential that is different from both the classical description of relativistic dissipative fluid dynamics by Eckart [12] and Landau [37] and the aforementioned theory of Müller and Ruggeri [44]. In the former it is unclear how to even only classify the resulting equations as systems of partial differential equations and therefore it is unknown if they are - even locally - well-posed, while the latter leads to a hierarchy of quasilinear first-order symmetric hyperbolic systems, where stability of rest states is proven differently.

The strategy to show global existence and asymptotic decay of small solutions for the relativistic hyperbolic NSF equations is quite classical: We first establish energy and decay estimates for the linearization at a homogeneous rest state and then show that for solutions sufficiently close to such a state similar estimates also hold for the nonlinear system. This then allows us to extend local solutions for all positive times.

More specifically, the structure and contents of this thesis are as follows. In Chapter 1, we introduce the concept of covariant second-order hyperbolicity, as defined in [17]. Then we state the Euler equations that describe the relativistic dynamics of perfect fluids. We cite results showing that these equations can always be written as first-order symmetric hyperbolic systems by using Godunov variables and give the explicit form of the Euler equations for the four-field theory of barotropic fluids and the five-field theory of general fluids. Finally, the theory of relativistic hyperbolic NSF equations as formulated by Freistühler and Temple is presented. We give a brief overview of how the requirements of covariant second-order hyperbolicity and sharp causality lead to the construction of the dissipation terms and state the equations explicitly for viscous heat-conductive barotropic fluids (as formulated in [19]) and for general viscous heat-conductive fluids with diffusion (cf. [13]).

In Chapter 2, we prove global existence and asymptotic decay of small solutions to the relativistic hyperbolic NSF equations for viscous heat-conductive barotropic fluids in L^2-based Sobolev spaces. To this end, in Section 2.2, we first analyze the linearization at a homogeneous reference state. Using energy methods in Fourier space similar to the ones employed by Kawashima and Kawashima et al. in [33] and [7], we obtain decay and energy estimates for solutions to the linearized equations. In Sections 2.3 and 2.4, we then consider the nonlinear system: in Section 2.3 for integrable initial data and the case of three space dimensions, and in Section 2.4. for not necessarily integrable initial data for which the results also apply in the case of two space dimensions. In both cases we begin with deriving a priori estimates for small solutions by treating the nonlinear equations as perturbations to the linearized ones and prove global existence by using a standard extension argument on the local solutions that exist due the second-order symmetric hyperbolicity.

In Chapter 3, asymptotic stability of homogeneous rest states is shown for the relativistic hyperbolic NSF equations for non-barotropic viscous heat-conductive fluids with diffusion. We follow the same strategy as in Chapter 2. However, the decay estimates for the linearization in Section 3.2 are not obtained by using energy methods. Instead, we apply perturbation theory for finite dimensional linear operators (cf. [31]) to the pointwise matrix representations of second-order symmetric hyperbolic linear operators in Fourier space. Based on the results for the linearized system the nonlinear system is treated similarly to the argumentation in Section 2.3.

Finally, in Chapter 4, two extensions for barotropic fluids are considered. In Section 4.1 we show that the results of Section 2.3 also apply to the equations of the relativistic electro-magneto-fluid dynamics, in the quasi-neutral approximation. In Section 4.2, we consider the vanishing heat conduction limit for the NSF equations. We establish that the decay and energy estimates proven in Section 2.2 hold uniformly in the coefficient $\chi \geq 0$ of heat conduction for χ sufficiently small. From this we can deduce that solutions to the linearized NSF equations for heat-conductive fluids (i.e. $\chi > 0$) converge to those of the linearized NSF equations for non-heat-conductive fluids (i.e. $\chi = 0$) as $\chi \downarrow 0$ in some L^2-based Sobolev spaces.

We conclude the introduction by presenting some open problems that arise from the work in this thesis.

On the one hand, it is desirable to generalize the results of global existence and asymptotic decay to a broader class of quasilinear second-order symmetric hyperbolic systems that contains the relativistic hyperbolic NSF equations as a special case. To this end it is necessary to establish conditions on the coefficient matrices of the linearized equations that are equivalent to the decay of solutions to the linearization.

On the other hand, not all equations occurring in the causal theory of relativistic fluid dynamics of Freistühler and Temple are second-order hyperbolic systems. In fact, for barotropic fluids which are viscous but not heat-conductive and for general viscous heat-conductive fluids which are not diffusive the NSF equations constitute a symmetric hyperbolic system of mixed order - i.e. the coefficient matrices corresponding to the second-order terms have non-trivial kernel (cf. [13, 19]). One would also like to establish asymptotic stability of homogeneous rest states for such systems. An additional difficulty that may arise here is that the kernel of the coefficient matrices may depend on the state variables. In fact, the NSF equations for non-heat-conductive barotropic fluids are of that type (cf. [19]). It would also be interesting to study whether solutions to the second-order NSF equations for diffusive fluids/heat-conductive barotropic fluids converge for vanishing diffusion/heat conduction to solutions of the mixed-order NSF equations for non-diffusive fluids/non-heat-conductive barotropic fluids.

The problems mentioned above are currently investigated by Freistühler and the author - and some results have been obtained (cf. [16]). Going further, also other standard research questions in the field of partial differential equations, and particularly in the context of hyperbolic systems, could be studied for the aforementioned classes of equations. This includes stability of non-constant reference solutions, in particular traveling waves, as well as existence and behavior of solutions which are not necessarily close to any reference state.

1. The equations of relativistic fluid dynamics

In this chapter we introduce Navier-Stokes-Fourier equations governing the dynamics of relativistic dissipative fluids as proposed by Freistühler and Temple in [13, 17, 18, 19]. The guiding principle in their theory is that these partial differential equations should be directly related in a specific way to a class of systems identified by Hughes, Kato, and Marsden in [24].

Throughout this thesis we will work in the context of special relativity (cf., e.g. [56, Chap. 2]). In particular, we assume a flat space-time with the standard Minkowski metric $g^{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1)$, Greek indices run from 0 to 3 and are raised or lowered by contraction with $g^{\alpha\beta}, g_{\alpha\beta}$, and we use the Einstein summation convention.

1.1. Second-order symmetric hyperbolicity

1.1.1 Definition. We call a second order differential operator

$$B^{a\beta g\delta}(\psi_e)\frac{\partial^2\psi_g}{\psi x^\beta \psi x^\delta},\ a = 0, \ldots, N-1, \tag{1.1}$$

that acts on a state variable ψ_e hyperbolic in the sense of Hughes, Kato and Marsden or *HKM hyperbolic* if the $\beta\delta$-symmetrized coefficients

$$\tilde{B}^{a\beta g\delta} = \frac{1}{2}\left(B^{a\beta g\delta} + B^{a\delta g\beta}\right)$$

have the following properties:

1. $\tilde{B}^{a\beta g\delta}(\psi_e)$ is symmetric in a, g, i.e.

$$\tilde{B}^{a\beta g\delta}(\psi_e) = \tilde{B}^{g\beta a\delta}(\psi_e).$$

2. There exists a time-like H_β such that

$$\tilde{B}^{a\beta g\delta}(\psi_e)H_\beta H_\delta V_a V_g < 0, \quad \text{for all } V_a \neq 0,$$

and

$$\tilde{B}^{a\beta g\delta}(\psi_e)N_\beta N_\delta V_a V_g > 0, \quad \text{for all } V_a \neq 0,$$

for all N_β with $N_\beta H^\beta = 0$.

We call such a 4-vector H^β *direction of hyperbolicity* for (1.1).

If the operator (1.1) is HKM hyperbolic we call a system of partial differential equations of the form

$$B^{a\beta g\delta}(\psi_e)\frac{\partial^2\psi_g}{\psi x^\beta \psi x^\delta} + f^a(\psi_e, \partial\psi_e) = 0, \; a = 0, \dots N-1,$$

HKM hyperbolic.

1.1.2 Remark. 1. The state variable $\psi_e \in \mathbb{R}^N$ represents an ensemble of covariant tensors. We symbolically use Roman indices a, g, e like the Greek ones, including the summation convention. In particular, in case the state is described by one single 4-vector ψ_ϵ, (1.1) reads

$$B^{\alpha\beta\gamma\delta}(\psi_\epsilon)\frac{\partial^2\psi_\gamma}{\psi x^\beta \psi x^\delta}, \; \alpha = 0, \dots, 3.$$

2. In a Lorentz-frame in which $(1,0,0,0)^t$ is the direction of time, the operator (1.1) is of the form - with respect to a fixed frame we use i, j for the spatial indices $1, 2, 3$, again including the summation convention -

$$-A(\psi)\psi_{tt} + B^{ij}(\psi)\psi_{x_i x_j} - D^j(\psi)\psi_{tx_j},$$

where

$$A = (-\tilde{B}^{a0g0})_{0\leq a,g\leq N-1}, \quad B^{ij} = (\tilde{B}^{aigj})_{0\leq a,g\leq N-1},$$
$$D^j = (-2\tilde{B}^{a0gj})_{0\leq a,g\leq N-1}.$$

In this notation, the operator is HKM hyperbolic if A, B^{ij}, D^j are symmetric, $A > 0$ and

$$B^{ij}\xi_i\xi_j > 0 \;\; \text{for all } \xi \in \mathbb{R}^3 \setminus \{0\}.$$

This is the definition of a second order symmetric hyperbolic operator used in [24]. As definiteness of matrices is an open property the same operator will also be HKM hyperbolic in the sense of [24] in other nearby Lorentz frames.

After thus having introduced the general concept of covariant HKM hyperbolicitiy we now turn to the specific equations of relativistic fluid dynamics.

1.2. The relativistic Euler equations

We start by introducing the equations describing the dynamics of relativistic perfect fluids - called the relativistic Euler equations - while effects of dissipation will then be modeled augmenting the Euler equations by additional terms.

In general, the thermodynamic properties of a fluid can be described by its specific internal energy $e = e(n, s)$ as a function of matter density n and specific entropy s. Then other thermodynamic state variables such as internal energy ρ, pressure p and temperature θ can be expressed in terms of $e(n, s)$ as

$$\rho = \rho(n, s) = ne(n, s), \quad p = p(n, s) = n^2 e_n(n, s), \quad \theta = e_s(n, s), \quad (1.2)$$

For describing the temporal dynamics of a fluid one additionally considers the fluid's 4-velocity u^α, with $u^\alpha u_\alpha = -1$.

The relativistic Euler equations for perfect fluids are given by

$$\frac{\partial}{\partial x^\beta}\left(T^{\alpha\beta}\right) = 0, \quad \alpha = 0, 1, 2, 3 \quad (1.3)$$

$$\frac{\partial}{\partial x^\beta}\left(N^{\beta}\right) = 0, \quad (1.4)$$

with energy-momentum tensor

$$T^{\alpha\beta} = (\rho + p)u^\alpha u^\beta + \rho g^{\alpha\beta}$$

and matter current

$$N^\beta = nu^\beta.$$

Setting

$$T^{4\beta} = N^\beta$$

we can write (1.3), (1.4) in the more compact form

$$\frac{\partial}{\partial x^\beta}\left(T^{a\beta}\right) = 0, \quad a = 0, \ldots, 4. \quad (1.5)$$

It is an important principle in continuum mechanics that conservation laws can be formulated in terms of symmetric hyperbolic systems of partial differential equations. Ruggeri and Strumia have shown in [49, 51] that this is the case for (1.5) for a broad class of fluids. More precisely, they proved that the following holds under standard assumptions of thermodynamics: For the variables

$$\psi^\alpha = \frac{u^\alpha}{\theta}, \quad \psi^4 = \frac{\upsilon}{\theta}, \quad (1.6)$$

where $\upsilon = (\rho + p)/n - \theta s$ denotes the chemical potential, there exists a vector $X^\beta(\psi_e)$ such that

$$T^{a\beta} = \frac{\partial X^\beta(\psi_e)}{\partial \psi_a},$$

and the matrix

$$\left(\frac{\partial^2 X^\beta(\psi_e)}{\partial \psi_a \partial \psi_g} H_\beta \right)_{a,g=0,\dots 4} > 0 \quad \text{for all } H^\beta \text{ with } H_\beta H^\beta \leq 0. \tag{1.7}$$

The ψ_a are called *Godunov variables* and X^β is called *4-potential* for (1.5).

In the following we will state explicit formulas for X^β in the special case of barotropic fluids and for a general class of fluids. We will closely follow the arguments in [15], where one can find the proofs of the following propositions as well as the definition of the general concept of Godunov variables and details about the history of their use in fluid dynamics.

We begin with barotropic fluids. A fluid is called barotropic and causal if there exist one-to-one relationships between the internal energy ρ and pressure p and between the internal energy and the temperature θ,

$$p = \hat{p}(\rho), \quad \theta = \hat{\theta}(\rho) \tag{1.8}$$

and the speed of sound $c_s = (\hat{p}'(\rho))^{1/2}$ satisfies

$$0 < c_s < 1. \tag{1.9}$$

In this case the conservation laws (1.3) for energy-momentum constitute a self-consistent system with four degrees of freedom, i.e. the dynamics are governed by four independent variables instead of five as in the general case (1.2).

1.2.1 Proposition. *For a barotropic fluid* (1.8),(1.9)

$$\psi^\alpha = \frac{u^\alpha}{\theta}, \quad \alpha = 0, \dots, 3, \tag{1.10}$$

are Godunov variables for (1.3). *The corresponding* 4*-potential is given by*

$$X^\beta(\psi_\epsilon) = \frac{\partial X(\psi_\epsilon)}{\partial \psi_\beta}, \tag{1.11}$$

where

$$\begin{aligned} X(\psi_\epsilon) &= \hat{X}((-\psi^\alpha \psi_\alpha)^{-\frac{1}{2}}) \\ &= \hat{X}(\theta) = \int \frac{p(\theta)}{\theta^3} d\theta \end{aligned} \tag{1.12}$$

1.2.2 Remark. 1. Note that for reasons of legibility we denote as a barotropic fluid what is more precisely called a thermo-barotropic fluid (cf. [19]). This does not lead to any confusion as no other barotropic fluids are considered in this thesis.

2. For barotropic fluids one can prove that

$$\rho + p = n\theta s.$$

In particular, this yields - see (1.6) -

$$\psi_4 = \frac{\rho + p}{n\theta} - s = 0.$$

3. Computing

$$\frac{\partial^2 X^\beta(\psi_\epsilon)}{\partial\psi_\alpha \partial\psi_\gamma}$$

one finds that for barotropic fluids in Godunov variables (1.10), (1.3) reads

$$A^{\alpha\beta\gamma}(\psi_\epsilon)\frac{\partial\psi_\gamma}{\partial x^\beta} = 0, \quad \alpha = 0, 1, 2, 3,$$

where

$$A^{\alpha\beta\gamma} = ns\theta^2 \left(u^\alpha g^{\beta\gamma} + u^\beta g^{\alpha\gamma} + u^\gamma g^{\alpha\beta} + (3 + c_s^{-2})u^\alpha u^\beta u^\gamma\right). \tag{1.13}$$

4. Due to (1.9), one has

$$A^{\alpha 0\gamma} - \xi_i A^{\alpha i\gamma} > 0$$

for all $\xi \in \mathbb{R}^3$, $|\xi| \leq 1$.

5. The most important examples in this class are "double γ-law" fluids

$$\rho(n, s) = kn^\gamma s^\gamma, \quad 1 < \gamma < 2.$$

For $\gamma = 4/3$ this describes pure radiation [56].

Next, we turn to more general fluids. We assume that all fluids considered here have the following thermodynamic properties:

$$\upsilon_{\theta\theta}(p, \theta) > 0, \tag{1.14}$$

$$0 < p_\rho(\rho, s) < 1. \tag{1.15}$$

For such fluids we get the following result (cf. [15, 51]).

1.2.3 Proposition. *For a fluid satisfying* (1.14), (1.15), (ψ^α, ψ^4) *defined by* (1.6) *are Godunov variables for* (1.5) *and there exists a scalar function* $\hat{X}(\theta, \psi^4)$ *such the corresponding* 4*-potential is given by*

$$X^\beta(\psi_e) = \hat{X}(\theta, \psi_4)\psi^\beta. \tag{1.16}$$

1.2.4 Remark. 1. Computing

$$\frac{\partial^2 X^\beta(\psi_e)}{\partial\psi_a \partial\psi_g}$$

one can write (1.5) as

$$A^{a\beta g}(\psi_e)\frac{\partial\psi_g}{\partial x^\beta} = 0, \quad a = 0, \ldots, 4, \tag{1.17}$$

where

$$A^{\alpha\beta\gamma} = \theta^2 \frac{\partial\hat{X}}{\partial\theta}\left(u^\alpha g^{\beta\gamma} + u^\beta g^{\alpha\gamma} + u^\gamma g^{\alpha\beta}\right) + \frac{\partial}{\partial\theta}\left(\theta^3 \frac{\partial\hat{X}}{\partial\theta}\right) u^\alpha u^\beta u^\gamma, \tag{1.18}$$

$$A^{\alpha\beta 4} = \theta \frac{\partial^2\hat{X}}{\partial\psi^4 \partial\theta} u^\alpha u^\beta + \frac{\partial\hat{X}}{\partial\psi} g^{\beta\gamma} = A^{4\beta\alpha}, \tag{1.19}$$

$$A^{4\beta 4} = \frac{\partial^2\hat{X}}{\partial(\psi^4)^2}\psi^\beta. \tag{1.20}$$

2. Due to (1.15), one has

$$A^{a0b}(\psi_e) - \xi_i A^{aib}(\psi_e) > 0$$

for all $\xi \in \mathbb{R}^3$, $|\xi| \leq 1$.

1.3. Relativistic Navier-Stokes-Fourier equations

Now, we are able to introduce NSF equations that describe the relativistic dynamics of dissipative fluids. As stated above, the general ansatz for modeling dissipative effects in relativistic fluid dynamics consists in augmenting the Euler equations (1.3), (1.4) by additional terms $\Delta T^{\alpha\beta}$ and ΔN^β - called the dissipation tensors - to obtain

$$\frac{\partial}{\partial x^\beta}\left(T^{\alpha\beta} + \Delta T^{\alpha\beta}\right) = 0 \tag{1.21}$$

and

$$\frac{\partial}{\partial x^\beta}\left(N^\beta + \Delta N^\beta\right) = 0. \tag{1.22}$$

Like in classical fluid dynamics, the dissipation terms $\Delta T^{\alpha\beta}$, ΔN^β are required to be linear in the space-time gradients of the state variables and should only depend on four free parameters η, ζ, χ, and μ that quantify shear viscosity, bulk viscosity, heat conduction and diffusion. Then (1.21), (1.22) form a system of quasi-linear second order partial differential equations. The first proposals for such tensors were made by Eckart [12] and Landau [37]. Although having otherwise strong physical justifications the equations of Eckart and Landau violate the principle of causality - i.e. the speed of wave propagation is not bounded. Freistühler and Temple fixed that issue in their work by formulating the dissipation tensors in such a way that (1.21), (1.22) form a system of partial differential equations which is HKM hyperbolic or on the boundary of the HKM class. Their particular choice of equations satisfies the additional conditions that they are first order equivalent to the equations of Eckart and Landau and the signal speeds are sharply casual - i.e. sharply bounded by the speed of light (for precise definitions of those conditions see [18]). Their theory is formulated for general fluids without diffusion in [18], with diffusion in [13] and for barotropic fluids in [19]. In the following we will give an overview of the results of those papers.

We will first state the hyperbolic dissipation tensor for viscous and heat-conductive barotropic fluids. For such fluids, as in the inviscid case, equations (1.21) form a self-consistent system with four degrees of freedom. Let $\chi > 0, \eta \geq 0, \zeta \geq 0$ be the coefficients of heat conduction, shear viscosity, bulk viscosity which quantify the effects of dissipation. Furthermore define $\sigma, \tilde{\zeta}$ by

$$\sigma = (\frac{4}{3}\eta + \zeta)/(1 - c_s^2) - c_s^2\chi\theta, \quad \tilde{\zeta} = \zeta + c_s^2\sigma - c_s^2(1 - c_s^2)\chi\theta.$$

and let

$$\begin{aligned} B^{\alpha\beta\gamma\delta} = \chi\theta^2 u^\alpha u^\gamma g^{\beta\delta} &- \sigma\theta u^\beta u^\delta \Pi^{\alpha\gamma} + \tilde{\zeta}\theta\Pi^{\alpha\beta}\Pi^{\gamma\delta} \\ &+ \eta\theta(\Pi^{\alpha\gamma}\Pi^{\beta\delta} + \Pi^{\alpha\delta}\Pi^{\beta\gamma} - \frac{2}{3}\Pi^{\alpha\beta}\Pi^{\gamma\delta}) \\ &+ \sigma\theta(u^\alpha u^\beta g^{\gamma\delta} - u^\alpha u^\delta g^{\gamma\delta}) + \chi\theta^2(u^\beta u^\gamma g^{\gamma\delta} - u^\gamma u^\delta g^{\gamma\delta}), \end{aligned} \tag{1.23}$$

where

$$\Pi^{\alpha\gamma} = u^\alpha u^\gamma + g^{\alpha\gamma}.$$

Using these definitions one gets the following result (cf. [19]).

1.3.1 Proposition. *For a barotropic fluid*

$$-\Delta T^{\alpha\beta} = B^{\alpha\beta\gamma\delta}\frac{\partial\psi_\gamma}{\partial x^\delta}$$

makes (1.21) *HKM hyperbolic, first order equivalent to the equations of Eckart-Landau, and sharply causal. Furthermore with $V_\beta V^\beta = 1$, $V_\beta u^\beta = 0$, the temporal, respectively spatial, coefficient tensors*

$$B^{\alpha\beta\gamma\delta}u_\beta u_\delta, \quad B^{\alpha\beta\gamma\delta}V_\beta V_\delta$$

have the rest frame matrix representations

$$\begin{pmatrix} -\chi\theta^2 & 0 \\ 0 & -\sigma\theta \end{pmatrix}, \quad \begin{pmatrix} \chi\theta^2 & 0 \\ 0 & \eta\theta\delta^{ij} + (\frac{1}{3}\eta + \tilde{\zeta})\theta V^i V^j \end{pmatrix}.$$

In particular $(1,0,0,0)$ *is a direction of hyperbolicity in the rest frame representation of* (1.21) *.*

We turn to the hyperbolic dissipation tensor for viscous heat-conductive diffusive fluids following [13]. Setting

$$\Delta T^{4\beta} = \Delta N^\beta,$$

(1.21), (1.22) read

$$\frac{\partial}{\partial x^\beta}\left(T^{a\beta} + \Delta T^{a\beta}\right) = 0, \quad a = 0, \ldots, 4. \tag{1.24}$$

With $\chi > 0$, $\eta > 0$, $\tilde{\zeta} \geq 0$ and $\mu > 0$ the coefficients of heat conduction, shear viscosity, modified bulk viscosity (cf. [13]) and diffusion, let

$$\sigma = (4/3)\eta + \tilde{\zeta}$$

and define

$$\begin{aligned} B^{\alpha\beta\gamma\delta} = \chi\theta^2 u^\alpha u^\gamma g^{\beta\delta} - \sigma\theta u^\beta u^\delta \Pi^{\alpha\gamma} + \tilde{\zeta}\theta\Pi^{\alpha\beta}\Pi^{\gamma\delta} \\ + \eta\theta(\Pi^{\alpha\gamma}\Pi^{\beta\delta} + \Pi^{\alpha\delta}\Pi^{\beta\gamma} - \frac{2}{3}\Pi^{\alpha\beta}\Pi^{\gamma\delta}) \\ + \sigma\theta(u^\alpha u^\beta g^{\gamma\delta} - u^\alpha u^\delta g^{\beta\gamma}) + \chi\theta^2(u^\beta u^\gamma g^{\alpha\delta} - u^\gamma u^\delta g^{\alpha\beta}) \end{aligned} \tag{1.25}$$

$$B^{\alpha\beta 4\delta} = 0, \quad B^{4\beta\gamma\delta} = \sigma\theta\left(u^\beta g^{\gamma\delta} - u^\delta g^{\beta\gamma}\right), \tag{1.26}$$

$$B^{4\beta 4\delta} = \mu g^{\beta\delta}. \tag{1.27}$$

1.3.2 Proposition. *For fluids satisfying* (1.14), (1.15)

$$-\Delta T^{a\beta} = B^{a\beta g\delta} \frac{\partial \psi_g}{\partial x^\delta},$$

makes (1.24) *HKM hyperbolic, first order equivalent to the equations of Eckart and Landau, and sharply causal. Furthermore with* $V_\beta V^\beta = 1$, $V_\beta u^\beta = 0$, *the temporal, respectively spatial, coefficient tensors*

$$B^{a\beta g\delta} u_\beta u_\delta, \quad B^{a\beta g\delta} V_\beta V_\delta$$

have the rest frame matrix representations

$$\begin{pmatrix} -\chi\theta^2 & 0 & 0 \\ 0 & -\sigma\theta\delta^{ij} & 0 \\ 0 & 0 & -\mu \end{pmatrix}, \quad \begin{pmatrix} \chi\theta^2 & 0 & 0 \\ 0 & \eta\theta\delta^{ij} + (\frac{1}{3}\eta + \tilde{\zeta})\theta V^i V^j & 0 \\ 0 & 0 & \mu \end{pmatrix}.$$

In particular $(1,0,0,0)$ *is a direction of hyperbolicity for the rest frame representation of* (1.24).

2. Global existence and asymptotic decay of solutions to the relativistic Navier-Stokes-Fourier equations for viscous heat-conductive barotropic fluids

In this chapter we study the NSF equations that describe the relativistic dynamics of viscous heat-conductive barotropic fluids as introduced in Chapter 1. Our goal is to show global existence and asymptotic decay of solutions close to homogeneous reference states. The corresponding arguments and results developed in Sections 2.2 and 2.3 presuppose, in particular, integrability of the initial data. Section 2.4 treats non-integrable data and the case of two space dimensions.

2.1. Preliminaries and main result

We begin by introducing some notation which will also be used in the following chapters. For $p \in [1, \infty]$ and some $m \in \mathbb{N}$ just write L^p for $L^p(\mathbb{R}^3, \mathbb{R}^m)$. For $s \in \mathbb{N}_0$ we denote by H^s the L^2-Sobolev-space of order s, namely

$$H^s := \{u \in L^2 : \forall\ \alpha \in \mathbb{N}_0^n\ (|\alpha| \leq s) : \|\partial_x^\alpha u\|_{L^2} < \infty\}$$

with norm

$$\|u\|_s = \left(\sum_{0 \leq |\alpha| \leq s} \|\partial_x^\alpha u\|_{L^2} \right)^{\frac{1}{2}}.$$

We write $\|u\|$ instead of $\|u\|_0$ and $\partial_x^k u$ for the vector whose entries are the partial derivatives of u of order k.

For $s, k \in \mathbb{N}_0$ and $U = (u, v) \in H^s \times H^k$ set

$$\|U\|_{s,k} = \left(\|u\|_s^2 + \|v\|_k^2 \right)^{\frac{1}{2}}$$

and for $U \in (H^s \times H^k) \cap (L^p \times L^p)$ set

$$\|U\|_{s,k,p} = \|U\|_{s,k} + \|U\|_{(L^p \times L^p)}.$$

For $u \in H^s$, $v \in H^{l-1}$ $(0 \leq l \leq s)$ and $\alpha \in \mathbb{N}_0^n$, $|\alpha| \leq s$, set

$$[\partial_x^\alpha, u]v = \partial_x^\alpha(uv) - u\partial_x^\alpha v.$$

For $l \in \mathbb{N}_0$, B^l denotes the space of bounded continuous functions whose derivatives of order less than or equal to l are also bounded continuous, equipped with the norm

$$|f|_l = \sum_{k=0}^{l} \sup |\partial_x^k f(x)|.$$

For $\delta > 0$ let ϕ_δ denote the Friedrichs mollifier and set

$$[\phi_\delta *, u]v = \phi_\delta * (uv) - u(\phi_\delta * v).$$

Also from now on, we often describe the 4-vector ψ_ϵ, and later the ensemble ψ_e, as ψ.

Consider a viscous heat-conductive barotropic fluid characterized by (1.8), (1.9) and coefficients χ, η, ζ of heat conduction, bulk viscosity and shear viscosity depending on the state variables. In Godunov variables (1.10) the relativistic NSF equations are given in divergence form by

$$\frac{\partial}{\partial x^\beta}\left(-B^{\alpha\beta\gamma\delta}\frac{\partial \psi_\gamma}{\partial x^\delta} + \frac{\partial X^\beta}{\partial \psi_\alpha}\right) = 0, \;\; \alpha = 0, 1, 2, 3, \tag{2.1}$$

where X^β and $B^{\alpha\beta\gamma\delta}$ are defined by (1.11), (1.12) and (1.23). Carrying out the differentiation with respect to x^β yields

$$-B^{\alpha\beta\gamma\delta}(\psi)\frac{\partial^2 \psi_\gamma}{\partial x^\beta \partial x^\delta} + A^{\alpha\beta\gamma}\frac{\partial \psi_\gamma}{\partial x^\beta} - \frac{\partial}{\partial x^\beta}\left(B^{\alpha\beta\gamma\delta}(\psi)\right)\frac{\partial \psi_\gamma}{\partial x^\delta} = 0, \tag{2.2}$$

where $A^{\alpha\beta\gamma}$ is defined by (1.13). As stated in Proposition 1.3.1, (3.2) is HKM hyperbolic with u^α being a direction of hyperbolicity. Thus, for states ψ with u sufficiently close to $(1, 0, 0, 0)$, or, in other words, in almost co-moving coordinates, using

$$B^{\alpha\beta\gamma\delta}(\psi)\frac{\partial^2 \psi_\gamma}{\partial x^\beta \partial x^\delta} = \tilde{B}^{\alpha\beta\gamma\delta}(\psi)\frac{\partial^2 \psi_\gamma}{\partial x^\beta \partial x^\delta}$$

with

$$\begin{aligned}\tilde{B}^{\alpha\beta\gamma\delta}(\psi) &= \frac{1}{2}\left(B^{\alpha\beta\gamma\delta}(\psi) + B^{\alpha\delta\gamma\beta}(\psi)\right)\\ &= \chi\theta^2 u^\alpha u^\gamma g^{\beta\delta} - \sigma\theta u^\beta u^\delta \Pi^{\alpha\gamma} + \tilde{\zeta}\theta\Pi^{\alpha\beta\gamma\delta} + \eta\theta(\Pi^{\alpha\gamma}\Pi^{\beta\delta} + \frac{1}{3}\Pi^{\alpha\beta\gamma\delta}),\end{aligned}$$

where

$$\Pi^{\alpha\beta\gamma\delta} = \frac{1}{2}(\Pi^{\alpha\beta}\Pi^{\gamma\delta} + \Pi^{\alpha\delta}\Pi^{\beta\gamma}),$$

we can write (2.2) as

$$A(\psi)\psi_{tt} - B^{ij}(\psi)\psi_{x_i x_j} + D^j(\psi)\psi_{tx_j} + f(\psi, \psi_t, \partial_x\psi) = 0, \tag{2.3}$$

where

$$\begin{aligned} A &= (-\tilde{B}^{\alpha 0\gamma 0})_{0\le\alpha,\gamma\le 3}, \quad B^{ij} = (\tilde{B}^{\alpha i\gamma j})_{0\le\alpha,\gamma\le 3},\\ D^j &= (-2\tilde{B}^{\alpha 0\gamma j})_{0\le\alpha,\gamma\le 3}\end{aligned} \tag{2.4}$$

are symmetric 4×4 matrices, $A(\psi)$ is positive definite, $\xi_i B^{ij}(\psi)\xi_j$ is positive definite for arbitrary $\xi\in\mathbb{R}^3\setminus\{0\}$, and

$$f^\alpha = \frac{\partial}{\partial x^\beta}T^{\alpha\beta}(\psi) - \frac{\partial}{\partial x^\beta}\left(B^{\alpha\beta\gamma\delta}(\psi)\right)\frac{\partial\psi_\gamma}{\partial x^\delta},\ \alpha = 0,1,2,3. \tag{2.5}$$

Throughout the chapter we will consider the Cauchy problem associated with (2.3):

$$A\psi_{tt} - B^{ij}\psi_{x_i x_j} + D^j\psi_{tx_j} + f = 0 \tag{2.6}$$

$$\psi(0) = {}^0\psi \tag{2.7}$$

$$\psi_t(0) = {}^1\psi, \tag{2.8}$$

To provide a global-in-time solution theory of these equations, we first analyze the linearization of (2.6) at some homogeneous reference state (Section 2.2) and then the nonlinear problem as a perturbation of the linear one (Section 2.3), both with techniques that were developed, or are simililar to techniques developed, by Kawashima and co-authors notably in [7, 33]. The main result is the following.

2.1.1 Theorem. *Let $s \ge 3$, and $\bar{\psi} = (\bar{\theta}^{-1}, 0, 0, 0,)^t$ with a constant temperature $\bar{\theta} > 0$. Then there exist $\delta_0 > 0$, $C_0 = C_0(\delta_0) > 0$ such that for all initial data $({}^0\psi, {}^1\psi) \in \left(H^{s+1}\times H^s\right)\cap\left(L^1\times L^1\right)$ satisfying*

$\|({}^0\psi - \bar{\psi}, {}^1\psi)\|^2_{s+1,s,1} < \delta_0$ *there exists a unique solution* ψ *of the Cauchy problem* (2.6)-(2.8) *such that*

$$\psi - \bar{\psi} \in \bigcap_{j=0}^{s} C^j\left([0,\infty), H^{s+1-j}\right).$$

ψ *satisfies the decay estimates*

$$\|(\psi(t) - \bar{\psi}, \psi_t(t))\|^2_{s+1,s} + \int_0^t \|(\psi(\tau) - \bar{\psi}, \psi_t(\tau))\|^2_{s+1,s} d\tau \\ \leq C_0 \|({}^0\psi - \bar{\psi}, {}^1\psi)\|^2_{s+1,s,1}, \quad (2.9)$$

$$\|(\psi(t) - \bar{\psi}, \psi_t(t))\|_{s,s-1} \leq C_0 (1+t)^{-\frac{3}{4}} \|({}^0\psi - \bar{\psi}, {}^1\psi)\|_{s,s-1,1} \quad (2.10)$$

for all $t \in [0,\infty)$.

2.2. Decay estimates for the linearized system

In this section we study the linearization of (2.3) about a quiescent, isothermal reference state $\bar{\psi} = u/\bar{\theta}$, $u = (1,0,0,0)^t$, $\bar{\theta} > 0$ on a fixed time interval $[0,T]$. The resulting equations read

$$A_{(1)}\psi_{tt} - B^{ij}_{(1)}\psi_{x_i x_j} + a_{(1)}\psi_t + b^j_{(1)}\psi_{x_j} = 0, \quad (2.11)$$

where

$$A_{(1)} = \begin{pmatrix} \chi\bar{\theta}^2 & 0 \\ 0 & \sigma\bar{\theta} I_3 \end{pmatrix}, \quad (2.12)$$

$$B^{ij}_{(1)} = \begin{pmatrix} \chi\bar{\theta}^2\delta^{ij} & 0 \\ 0 & \bar{\theta}\eta I_3 \delta^{ij} + \frac{1}{2}\bar{\theta}(\tilde{\zeta} + \frac{1}{3}\eta)(e^i \otimes e^j + e^j \otimes e^i) \end{pmatrix}, \quad (2.13)$$

$$a_{(1)} = ns\bar{\theta}^2 \begin{pmatrix} c_s^{-2} & 0 \\ 0 & I_3 \end{pmatrix}, \quad b^j_{(1)} = ns\bar{\theta}^2 (e^j \otimes e^0 + e^0 \otimes e^j), \quad (2.14)$$

where $n, s, \chi, c_s, \eta, \tilde{\zeta}$ are evaluated at the reference state. Note that no mixed derivative ψ_{tx_j} occurs here, as

$$\tilde{B}^{\alpha 0 \gamma j} = \tilde{B}^{\alpha j \gamma 0} = 0$$

at the reference state. First, multiply (2.11) by $(ns)^{-1}\theta^{-2}$ and set $\bar{\chi} = \chi(ns)^{-1}$, $\bar{\eta} = \eta(ns\theta)^{-1}$, $\bar{\zeta} = \tilde{\zeta}(ns\theta)^{-1}$, $\bar{\sigma} = \sigma(ns\theta)^{-1}$. We arrive at the equivalent system

$$A_{(2)}\psi_{tt} - B^{ij}_{(2)}\psi_{x_i x_j} + a_{(2)}\psi_t + b^j_{(2)}\psi_{x_j} = 0, \quad (2.15)$$

where

$$A_{(2)} = \begin{pmatrix} \bar{\chi} & 0 \\ 0 & \bar{\sigma} I_3 \end{pmatrix},$$

$$B^{ij}_{(2)} = \begin{pmatrix} \bar{\chi}\delta^{ij} & 0 \\ 0 & \bar{\eta} I_3 \delta^{ij} + \frac{1}{2}\left(\bar{\zeta} + \frac{1}{3}\bar{\eta}\right)(e^i \otimes e^j + e^j \otimes e^i) \end{pmatrix},$$

$$a_{(2)} = \begin{pmatrix} c_s^{-2} & 0 \\ 0 & I_3 \end{pmatrix}, \quad b^j_{(2)} = e^j \otimes e^0 + e^0 \otimes e^j.$$

Finally, multiplying (2.15) by $(A_{(2)})^{-\frac{1}{2}}$ and writing it in variables $(A_{(2)})^{\frac{1}{2}}\psi$ gives

$$\psi_{tt} - \bar{B}^{ij}\psi_{x_i x_j} + a\psi_t + b^j \psi_{x_j} = 0, \tag{2.16}$$

where

$$\bar{B}^{ij} = \begin{pmatrix} \delta^{ij} & 0 \\ 0 & \bar{\sigma}^{-1}\left(\bar{\eta} I_3 \delta^{ij} + \frac{1}{2}\left(\bar{\zeta} + \frac{1}{3}\bar{\eta}\right)(e^i \otimes e^j + e^j \otimes e^i)\right) \end{pmatrix}, \tag{2.17}$$

$$a = \begin{pmatrix} c_s^{-2}\bar{\chi}^{-1} & 0 \\ 0 & \bar{\sigma}^{-1} I_3 \end{pmatrix}, \quad b^j = (\bar{\chi}\bar{\sigma})^{-\frac{1}{2}}(e^j \otimes e^0 + e^0 \otimes e^j). \tag{2.18}$$

The goal is to prove a decay estimate for the Cauchy problem associated with (2.16):

$$\psi_{tt} - \bar{B}^{ij}\psi_{x_i x_j} + a\psi_t + b^j \psi_{x_j} = 0 \tag{2.19}$$

$$\psi(0) = {}^0\psi, \tag{2.20}$$

$$\psi_t(0) = {}^1\psi. \tag{2.21}$$

2.2.1 Proposition. *For some $s \in \mathbb{N}_0$ let $({}^0\psi, {}^1\psi) \in (H^{s+1} \times H^s) \cap (L^1 \times L^1)$ and $(\psi(t), \psi_t(t)) \in H^{s+1} \times H^s$ be a solution of (2.19)-(2.21). Then there exist $c, C > 0$ such that for all integers $0 \le k \le s$ and all $t \in [0, T]$*

$$\begin{aligned} \|\partial_x^k \psi(t)\|_1 + \|\partial_x^k \psi_t(t)\| \le{}& C(1+t)^{-\frac{3}{4}-\frac{k}{2}}\left(\|{}^0\psi\|_{L^1} + \|{}^1\psi\|_{L^1}\right) \\ &+ Ce^{-ct}\left(\|\partial_x^k({}^0\psi)\|_1 + \|\partial_x^k({}^1\psi)\|\right). \end{aligned} \tag{2.22}$$

To prove Proposition 2.2.1 we consider (2.19)-(2.21) in Fourier space, i.e.

$$\hat{\psi}_{tt} + |\xi|^2 B(\check{\xi})\hat{\psi} + a\hat{\psi}_t - i|\xi| b(\check{\xi})\hat{\psi} = 0, \tag{2.23}$$

$$\hat{\psi}(0) = {}^0\hat{\psi}(\xi), \tag{2.24}$$

$$\hat{\psi}_t(0) = {}^1\hat{\psi}(\xi), \tag{2.25}$$

where $\check{\xi} = \xi/|\xi|$,

$$B(\omega) = \omega_i \bar{B}^{ij}\omega_j = \begin{pmatrix} 1 & 0 \\ 0 & \bar{\sigma}^{-1}\left(\bar{\eta} I_3 + \left(\bar{\zeta} + \frac{1}{3}\bar{\eta}\right)(\omega \otimes \omega)\right) \end{pmatrix},$$

$$b(\omega) = b^j \omega_j = (\bar{\chi}\bar{\sigma})^{-\frac{1}{2}} \begin{pmatrix} 0 & \omega^t \\ \omega & 0 \end{pmatrix}, \ \omega \in \mathbb{S}^2.$$

We get the following pointwise decay estimate.

2.2.2 Lemma. *In the situation of Proposition 2.2.1 there exist* $c, C > 0$ *such that for* $(t, \xi) \in [0, T] \times \mathbb{R}^3$

$$\begin{aligned} (1 + |\xi|^2)|\hat{\psi}(t,\xi)|^2 + |\hat{\psi}_t(t,\xi)|^2 \\ \le C \exp(-c\rho(\xi)t)\left((1 + |\xi|^2)|{}^0\hat{\psi}(\xi)|^2 + |{}^1\hat{\psi}(\xi)|^2\right), \end{aligned} \quad (2.26)$$

where $\rho(\xi) = |\xi|^2/(1 + |\xi|^2)$.

Proof. Our goal is to arrive at an expression of the form

$$\frac{1}{2}\frac{d}{dt}E(t,\xi) + F(t,\xi) \le 0, \quad (2.27)$$

where $E(t,\xi)$ is uniformly equivalent to

$$E_0(t,\xi) = (1 + |\xi|)^2|\hat{\psi}(t,\xi)|^2 + |\hat{\psi}_t(t,\xi)|^2$$

and $F \ge c\rho(\xi)E_0$. Then (2.26) follows from Gronwall's Lemma.

W.l.o.g. assume $\xi = (|\xi|, 0, 0)$ (otherwise rotate the coordinate system). Since $(4/3)\bar{\eta} + \bar{\zeta} = \bar{\sigma}$, (2.23) decomposes into the two uncoupled systems

$$w_{tt} + |\xi|^2 w + \tilde{a}w_t - i|\xi|\tilde{b}w = 0, \quad (2.28)$$

$$v_{tt} + \bar{\eta}\bar{\sigma}^{-1}|\xi|^2 v + \bar{\sigma}^{-1}v_t = 0, \quad (2.29)$$

where $w = (\hat{\psi}_0, \hat{\psi}_1)$, $v = (\hat{\psi}_2, \hat{\psi}_3)$,

$$\tilde{a} = \begin{pmatrix} \bar{\chi}^{-1}c_s^{-2} & 0 \\ 0 & \bar{\sigma}^{-1} \end{pmatrix}, \quad \tilde{b} = (\bar{\chi}\bar{\sigma})^{-\frac{1}{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (2.30)$$

Obviously, this allows us to prove estimate (2.26) for w and v independently.

First, we study system (2.28). We take the scalar product of (2.28) with w_t. The real part of the resulting equation reads

$$\frac{1}{2}\frac{d}{dt}\left(|w_t|^2 + |\xi|^2|w|^2\right) + \langle \tilde{a}w_t, w_t\rangle + \Re\langle -i|\xi|\tilde{b}w, w_t\rangle = 0. \quad (2.31)$$

Taking the scalar product of (2.28) with w and considering the real part gives

$$\frac{1}{2}\frac{d}{dt}\left(\langle \tilde{a}w, w\rangle + 2\Re\langle w_t, w\rangle\right) - |w_t|^2 + |\xi|^2|w|^2 = 0. \tag{2.32}$$

Now, set $r = \tilde{a}_{11}\tilde{a}_{22}/(\tilde{a}_{11} + \tilde{a}_{22})$ and add (2.31)+r(2.32). The resulting equation is

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\Big(|w_t|^2 + |\xi|^2|w|^2 + r\langle \tilde{a}w, w\rangle + 2r\Re\langle w_t, w\rangle\Big)\\ &+ \langle(\tilde{a} - Ir)w_t, w_t\rangle + \Re\langle -i|\xi|\tilde{b}w, w_t\rangle + r|\xi|^2|w|^2 = 0. \end{aligned}\tag{2.33}$$

Next define

$$S = \frac{\tilde{b}_{12}}{\tilde{a}_{11} + \tilde{a}_{22}}\begin{pmatrix} 0 & \tilde{a}_{22} - a_{11} \\ \tilde{a}_{11} - \tilde{a}_{22} & 0 \end{pmatrix}.$$

Since iS is Hermitian,

$$\Re\langle iSw, w_t\rangle = \frac{1}{2}\frac{d}{dt}\langle iSw, w\rangle$$

holds and we can write (2.33) as

$$\frac{1}{2}\frac{d}{dt}E^{(1)} + F^{(1)} = 0, \tag{2.34}$$

where

$$E^{(1)} = |w_t|^2 + |\xi|^2|w|^2 + r\langle \tilde{a}w, w\rangle + 2r\Re\langle w_t, w\rangle + \langle -i|\xi|Sw, w\rangle$$

and

$$F^{(1)} = \langle(\tilde{a} - Ir)w_t, w_t\rangle + \Re\langle -i|\xi|(\tilde{b} - S)w, w_t\rangle + r|\xi|^2|w|^2.$$

First, show that $E^{(1)}$ is uniformly equivalent to $E_0^{(1)} = (1+|\xi|^2)|w|^2+|w_t|^2$. Obviously, there exists $C_1 > 0$ such that

$$E^{(1)} \leq C_1 E_0^{(1)}.$$

On the other hand note that

$$\begin{aligned}E^{(1)} = |w_t|^2 + |\xi|^2|w|^2 + r\tilde{a}_{11}|w^1|^2 + r\tilde{a}_{22}|w^2|^2 + 2r\Re(w_t^1\bar{w}^1 + w_t^2\bar{w}^2)\\ + 2\frac{\tilde{b}_{12}\tilde{a}_{11}}{\tilde{a}_{11} + \tilde{a}_{22}}\Re(-i|\xi|w^1\bar{w}^2) + 2\frac{\tilde{b}_{12}\tilde{a}_{22}}{\tilde{a}_{11} + \tilde{a}_{22}}\Re(-i|\xi|\bar{w}^1 w^2).\end{aligned}$$

Since $\tilde{a}_{11}\tilde{a}_{22} > \tilde{b}_{12}^2$, an easy calculation shows that for $j = 1, 2$

$$r\tilde{a}_{jj} - r^2 - \left(\frac{\tilde{b}_{12}\tilde{a}_{jj}}{\tilde{a}_{11} + \tilde{a}_{22}}\right)^2 > 0.$$

Hence using Young's inequality for suitable $\varepsilon_{kj} > 1$, $(k, j = 1, 2)$ we get

$$\begin{aligned} E^{(1)} \geq \sum_{j=1}^{2} (1 - \varepsilon_{1j}^{-1})|w_t^j|^2 + (1 - \varepsilon_{2j}^{-1})|\xi|^2|w^j|^2 \\ + \left(r\tilde{a}_{jj} - \varepsilon_{1j} r^2 - \varepsilon_{2j}\left(\frac{\tilde{b}_{12}\tilde{a}_{jj}}{\tilde{a}_{11} + \tilde{a}_{22}}\right)^2\right)|w^j|^2 \\ \geq C|w|^2 + C(1 + |\xi|^2)|w|^2 \end{aligned}$$

It remains to show that $F^{(1)} \geq c(|w_t|^2 + |\xi|^2|w|^2)$. To this end use

$$\tilde{b} - S = \frac{2\tilde{b}_{12}}{\tilde{a}_{11} + \tilde{a}_{22}} \begin{pmatrix} 0 & \tilde{a}_{11} \\ \tilde{a}_{22} & 0 \end{pmatrix},$$

and write $F^{(1)} = F^{(11)} + F^{(12)}$, where

$$\begin{aligned} F^{(11)} &= (\tilde{a}_{11} - r)|w_t^1|^2 + 2\Re\left(-i|\xi|\frac{\tilde{b}_{12}\tilde{a}_{11}}{\tilde{a}_{11} + \tilde{a}_{22}} w^2 \bar{w}_t^1\right) + r|\xi|^2|w^2|^2, \\ F^{(12)} &= (\tilde{a}_{22} - r)|w_t^2|^2 + 2\Re\left(-i|\xi|\frac{\tilde{b}_{12}\tilde{a}_{22}}{\tilde{a}_{11} + \tilde{a}_{22}} w^1 \bar{w}_t^2\right) + r|\xi|^2|w|^2. \end{aligned}$$

As $\tilde{a}_{11}\tilde{a}_{22} \geq (\tilde{b}_{12})^2$, one can easily check that

$$\begin{aligned} (\tilde{a}_{11} - r)r &> \left(\frac{\tilde{b}_{12}\tilde{a}_{11}}{\tilde{a}_{11} + \tilde{a}_{22}}\right)^2, \\ (\tilde{a}_{22} - r)r &> \left(\frac{\tilde{b}_{12}\tilde{a}_{22}}{\tilde{a}_{11} + \tilde{a}_{22}}\right)^2. \end{aligned}$$

Since $\tilde{a}_{11}, \tilde{a}_{22} > r$, this yields

$$F^{(11)} \geq c(|w_t^1|^2 + |\xi|^2|w^2|^2), \quad F^{(22)} \geq c(|w_t^2|^2 + |\xi|^2|w_t^1|^2).$$

Now, consider (2.29). Again, take the scalar product (in $\mathbb{C}^2$) of (2.29) with $v_t + rv$. The real part reads

$$\frac{1}{2}\frac{d}{dt}E^{(2)} + F^{(2)} = 0,$$

where

$$E^{(2)} = |v_t|^2 + \bar{\eta}\bar{\sigma}^{-1}|\xi|^2|v|^2 + r\sigma^{-1}|v|^2 + 2r\Re\langle v_t, v\rangle, \tag{2.35}$$

and

$$F^{(2)} = (\bar{\sigma}^{-1} - r)|v_t|^2 + \bar{\eta}\bar{\sigma}^{-1}r|\xi|^2|v|^2. \tag{2.36}$$

Since $r < \sigma^{-1}$, it is obvious that $E^{(2)}$ and $F^{(2)}$ have the desired properties. □

Based on Lemma 2.2.2 the proof for Proposition 2.2.1 goes as [7, Proof of Theorem 3.1].

Next consider the inhomogeneous initial-value problem

$$\psi_{tt} - B^{ij}\psi_{x_i x_j} + a\psi_t + b^j\psi_{x_j} = h, \text{ on,} \tag{2.37}$$

$$\psi(0) = {}^0\psi_0, \tag{2.38}$$

$$\psi_t(0) = {}^1\psi. \tag{2.39}$$

for some $h : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^4$. We get the following results:

2.2.3 Proposition. *Let s be a non-negative integer, $({}^0\psi, {}^1\psi) \in (H^{s+1} \times H^s) \cap (L^1 \times L^1)$ and $h \in C([0,T], H^s \cap L^1)$. Then the solution ψ of (2.37)-(2.39) satisfies*

$$\begin{aligned}
\|\partial_x^k \psi(t)\|_1 + \|\partial_x^k \psi_t(t)\| \le{}& C(1+t)^{-\frac{3}{4}-\frac{k}{2}}(\|{}^0\psi\|_{L^1} + \|{}^1\psi\|_{L^1}) \\
&+ Ce^{-ct}(\|\partial_x^k({}^0\psi)\|_1 + \|\partial_x^k({}^1\psi)\|) \\
&+ C\int_0^t (1+t-\tau)^{-\frac{3}{4}-\frac{k}{2}}\|h(\tau)\|_{L^1} \\
&\qquad + Ce^{-c(t-\tau)}\|\partial_x^k h(\tau)\|d\tau
\end{aligned} \tag{2.40}$$

for all $t \in [0,T]$ and $0 \le k \le s$.

Proof. For $t \in [0,T]$ let $T(t)$ be the linear operator which maps $({}^0\psi, {}^1\psi)$ to the solution $(\psi(t)), \psi_t(t))$ of the homogeneous IVP (2.19)-(2.21) at time t. By Duhamel's principle the solution of (2.37)-(2.39) is given as

$$(\psi(t), \psi_t(t)) = T(t)({}^0\psi, {}^1\psi) + \int_0^t T(t-\tau)(0, h(\tau))d\tau.$$

Hence the assertion is an immediate consequence of Proposition 2.2.1. □

2.2.4 Proposition. *Let s be a non-negative integer, $({}^0\psi, {}^1\psi) \in H^{s+1} \times H^s$ and $h \in C([0,T], H^s)$. Then the solution ψ of (2.37)-(2.39) satisfies*

$$\begin{aligned}\|\partial_x^k \psi(t)\|_1^2 + \|\partial_x^k \psi_t(t)\|^2 + \int_0^t \|\partial_x^{k+1}\psi(t)\|^2 + \|\partial_x^k \psi_t(t)\|^2 \\ \leq C(\|\partial_x^k({}^0\psi)\|_1^2 + \|\partial_x^k({}^1\psi)\|^2) \\ + C\left|\int_0^t \left(\partial_x^k h(\tau), \partial_x^k \psi_t(\tau)\right)_{L^2} d\tau\right| \\ C\left|\int_0^t \left(\partial_x^k h(\tau), \partial_x^k \psi(\tau)\right)_{L^2} d\tau\right| \quad (2.41)\end{aligned}$$

for all $t \in [0,T]$ and integers $0 \leq k \leq s$.

Proof. Let $\hat{\psi}$ be the Fourier-transform of the solution to (2.37)-(2.39). We proceed exactly as in the proof of Lemma 2.2.2 to obtain

$$\frac{1}{2}\frac{d}{dt}E + F = \Re\left\langle \hat{h}, \hat{\psi}_t + r\hat{\psi}\right\rangle, \quad (2.42)$$

where E is equivalent to $E_0 = (1+|\xi|^2)|\hat{\psi}|^2 + |\hat{\psi}_t|^2$, $F \geq c(|\hat{\psi}_t|^2 + |\xi|^2|\hat{\psi}|^2)$ and r being the constant from the proof of Lemma 2.2.2. Integrating (2.42) from 0 to t leads to

$$\begin{aligned}(1+|\xi|^2)|\hat{\psi}|^2 + |\hat{\psi}_t|^2 + \int_0^t |\xi|^2|\hat{\psi}|^2 + |\hat{\psi}_t|^2 d\tau \\ \leq C((1+|\xi|^2)|{}^0\hat{\psi}|^2 + |{}^1\hat{\psi}|^2) \\ + C\left|\int_0^t \Re\left\langle \hat{h}, \hat{\psi}_t\right\rangle d\tau\right| + C\left|\int_0^t \Re\left\langle \hat{h}, \hat{\psi}\right\rangle d\tau\right|\end{aligned}$$

Multiplying by $\xi^{2\alpha}$ for $\alpha \in \mathbb{N}_0^n$, $0 \leq |\alpha| = k \leq s$, integrating with respect to ξ and using Plancherel's identity yields the assertion. □

2.3. Global existence and asymptotic decay

The goal of this section is to prove Theorem 2.1.1. We will proceed as follows: First we show a decay estimate for all but the highest order derivatives of a solution, Proposition 2.3.1, and then an energy estimate for the derivatives of highest order, Proposition 2.3.3. Theorem 2.1.1 follows from combining the two, Proposition 2.3.4.

As in Section 2.2 fix $\bar{\theta} > 0$, multiply (2.3) by $(n(\bar{\theta})s(\bar{\theta}))^{-1}\bar{\theta}^{-2}(A^{(2)})^{-\frac{1}{2}}$ and change the variables to $(A^{(2)})^{\frac{1}{2}}\psi$ such that the linearization at $\bar{\psi} =$

$(\bar{\theta}^{-1}, 0, 0, 0)$ is given by (2.16). In addition, consider $\psi - \bar{\psi}$ instead of ψ, ${}^0\psi - \bar{\psi}$ instead of ${}^0\psi$, $A(\cdot + \bar{\psi})$ instead of $A(\cdot)$ and so on, such that the rest state is shifted from $\bar{\psi}$ to 0. In the following, when (2.3) or (2.6)-(2.8) are mentioned, we actually mean these modified equations which we will consider on a fixed time interval $[0, T]$.

Write $U = (\psi, \psi_t)$ and $U_0 = ({}^0\psi, {}^1\psi)$ for a solution to (2.6)-(2.8) and the initial values, respectively. Let $s \geq s_0 + 1$ $(s_0 = [3/2] + 1)$, $T > 0$, $U_0 \in H^{s+1} \times H^s$, and ψ satisfy

$$\psi \in \bigcap_{j=0}^{s} C^j\left([0, T], H^{s+1-j}\right). \tag{2.43}$$

For $0 \leq t \leq t_1 \leq T$ define

$$N_s(t, t_1)^2 = \sup_{\tau \in [t, t_1]} \|U(\tau)\|^2_{s+1,s} + \int_t^{t_1} \|U(\tau)\|^2_{s+1,s} d\tau.$$

We write $N_s(t)$ instead of $N_s(0, t)$. Furthermore assume that $N_s(T) \leq a_0$ for an $a_0 > 0$. Since $s \geq s_0$, $H^s \hookrightarrow L^\infty$ is a continuous embedding. Hence $N_s(T) \leq a_0$ implies that $(\psi, \psi_t, \partial_x \psi)$ takes values in a closed ball $\overline{B(0, r)} \subset \mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{R}^{12}$ for some $r > 0$. In particular, we choose a_0 sufficiently small such that (2.2) can be written as the symmetric hyperbolic system (2.3).

First we prove the decay estimate. To this end it is convenient to rewrite (2.3) as

$$\psi_{tt} - \bar{B}^{ij}\psi_{x_i x_j} + a\psi_t + b^j \psi_{x_j} = h(\psi, \psi_t, \partial_x \psi, \partial_x^2 \psi, \partial_x \psi_t), \tag{2.44}$$

where,

$$\begin{aligned} h(\psi, \psi_t, \partial_x \psi, \partial_x^2 \psi, \partial_x \psi_t) = {} & \left(A(\psi)^{-1} B^{ij}(\psi) - \bar{B}^{ij}\right) \psi_{x_i x_j} \\ & - A(\psi)^{-1} D^j(\psi) \psi_{t x_j} \\ & - A(\psi)^{-1} f(\psi, \psi_t, \partial_x \psi) + a\psi_t + b^j \psi_{x_j}. \end{aligned} \tag{2.45}$$

2.3.1 Proposition. *There exist constants $a_1 (\leq a_0)$, $\delta_1 = \delta_1(a_1)$, $C_1 = C_1(a_1, \delta_1) > 0$ such that the following holds: If $\|U_0\|^2_{s,s-1,1} \leq \delta_1$ and $N_s(T)^2 \leq a_1$ for a solution ψ of (2.6)-(2.8) satisfying (2.43), then*

$$\|U(t)\|_{s,s-1} \leq C_1 (1 + t)^{-\frac{3}{4}} \|U_0\|_{s,s-1,1} \quad (t \in [0, T]). \tag{2.46}$$

Proof. Let $t \in [0,T]$ and ψ be a solution to (2.6)-(2.8). Since $B_{ij}(0) = \bar{B}_{ij}$, $D_j(0) = 0$ and

$$a\psi_t + b^j \psi_{x_j} = Df(0)(\psi, \psi_t, \partial_x \psi),$$

Lemmas A.1.1, A.1.2 show that there exist $C, c > 0$ ($c \le a_0$) such that $h(t) \in H^{s-1} \cap L^1$ and

$$\begin{aligned}
\|h(t)\|_{s-1} &\le C\|\psi(t)\|_{s-1} \left(\|\partial_x^2 \psi(t)\|_{s-1} + \|\partial_x \psi_t(t)\|_{s-1}\right) \\
&\quad + C\|(\psi(t), \psi_t(t), \partial_x \psi(t))\|_{s-1}^2 \\
&\le C\|U(t)\|_{s+1,s}\|U(t)\|_{s,s-1}, \\
\|h(t)\|_{L^1} &\le C\|U(t)\|_{2,1}^2,
\end{aligned}$$

if $N_s(T) \le c$. We will assume this throughout the proof. Then Proposition 2.2.3 yields

$$\begin{aligned}
\|U(t)\|_{s,s-1} &\le C(1+t)^{-\frac{3}{4}}\|U_0\|_{s,s-1,1} \\
&\quad + C\int_0^t e^{-c(t-\tau)}\|h(\tau)\|_{s-1} + (1+t-\tau)^{-\frac{3}{4}}\|h(\tau)\|_{L^1} d\tau.
\end{aligned}$$

This leads to

$$\begin{aligned}
\|U(t)\|_{s-1,s} &\le C(1+t)^{-\frac{3}{4}}\|U_0\|_{s,s-1,1} \\
&\quad + C \sup_{\tau \in [0,t]} \|U(\tau)\|_{s+1,s} \int_0^t e^{-c(t-\tau)}\|U(\tau)\|_{s,s-1} d\tau \\
&\quad + C\int_0^t (1+t-\tau)^{-\frac{3}{4}}\|U(\tau)\|_{s,s-1}^2 d\tau.
\end{aligned}$$

Multiplying with $(1+t)^{\frac{3}{4}}$ gives

$$\begin{aligned}
(1+t)^{\frac{3}{4}}\|U(t)\|_{s,s-1} &\le C\|U_0\|_{s,s-1,1} \\
&\quad + CN_s(t)\mu_1(t) \sup_{\tau \in [0,t]} (1+\tau)^{\frac{3}{4}}\|U(\tau)\|_{s,s-1} \\
&\quad + C\mu_2(t) \sup_{\tau \in [0,t]} (1+\tau)^{\frac{3}{2}}\|U(\tau)\|_{s,s-1}^2,
\end{aligned}$$

where

$$\begin{aligned}
\mu_1(t) &= (1+t)^{\frac{3}{4}} \int_0^t e^{-c(t-\tau)}(1+\tau)^{-\frac{3}{4}} d\tau \\
\mu_2(t) &= (1+t)^{\frac{3}{4}} \int_0^t (1+t-\tau)^{-\frac{3}{4}}(1+\tau)^{-\frac{3}{2}} d\tau.
\end{aligned}$$

Since μ_1, μ_2 are bounded functions on $[0,\infty)$, we get

$$\sup_{\tau\in[0,t]}(1+\tau)^{\frac{3}{4}}\|U(\tau)\|_{s,s-1} \leq C\|U_0\|_{s,s-1,1} \\ + CN_s(t)\sup_{\tau\in[0,t]}(1+\tau)^{\frac{3}{4}}\|U(\tau)\|_{s,s-1} \\ + C\sup_{\tau\in[0,t]}(1+\tau)^{\frac{3}{2}}\|U(\tau)\|^2_{s,s-1}.$$

We can deduce from this equation that there in fact exists $a_1 > 0$ ($a_1 \leq c$), $\delta_1 > 0$ and $C_1 > 0$, such that

$$\sup_{\tau\in[0,t]}(1+\tau)^{\frac{3}{4}}\|U(\tau)\|_{s,s-1} \leq C_1\|U_0\|_{s,s-1,1},$$

whenever $N_s(T)^2 \leq a_1$ and $\|U_0\|^2_{s,s-1,1} \leq \delta_1$. □

2.3.2 Corollary. *In the situation of Proposition 2.3.1 there exists a $C_2 = C_2(a_1,\delta_1) > 0$ such that*

$$N_{s-1}(T)^2 \leq C_2\|U_0\|^2_{s,s-1,1} \tag{2.47}$$

whenever $N_s(T)^2 \leq a_1$ and $\|U_0\|^2_{s,s-1,1} \leq \delta_1$.

Proof. The function $t \mapsto (1+t)^{-\frac{3}{4}}$ is square-integrable on $[0,\infty)$. Therefore the assertion is a direct consequence of Proposition 2.3.1. □

Now it is convenient to write (2.3) as

$$\psi_{tt} - \bar{B}^{ij}\psi_{x_ix_j} + a\psi_t + b^j\psi_{x_j} = L(\psi)\psi + h_2(\psi,\psi_t,\partial_x\psi), \tag{2.48}$$

where

$$L(\psi)\psi = (I - A(\psi))\psi_{tt} - (\bar{B}^{ij} - B^{ij}(\psi))\psi_{x_ix_j} - D^j(\psi)\psi_{tx_j},$$

$$h_2(\psi,\psi_t,\partial_x\psi) = a\psi_t + b^j\psi_{x_j} - f(\psi,\psi_t,\partial_x\psi).$$

2.3.3 Proposition. *There exist constants $a_2(\leq a_0)$ and $C_3 = C_3(a_2) > 0$ such that the following holds: If $N_s(T)^2 \leq a_2$ for a solution ψ of* (2.6)-(2.8) *satisfying* (2.43)*, then*

$$\|\partial_x^s\psi(t)\|_1^2 + \|\partial_x^s\psi_t(t)\|^2 + \int_0^t \|\partial_x^{s+1}\psi(\tau)\|^2 + \|\partial_x^s\psi_t(\tau)\|^2 d\tau \\ \leq C_3\left(\|U_0\|^2_{s,s+1} + N_s(t)^3\right) \quad (t\in[0,T]). \tag{2.49}$$

Proof. We prove the result in two steps.

Step 1: Let $U_0 = ({}^0\psi, {}^1\psi) \in H^{s+1} \times H^s$ and

$$\psi \in \bigcap_{j=0}^{s} C^j \left([0,T], H^{s+2-j}\right) \tag{2.50}$$

be a solution to (2.6)-(2.8). By Lemma A.1.2 there exists a $c > 0$ such that $I - A(\psi), \bar{B}^{ij} - B_{ij}(\psi), D^j(\psi) \in H^{s+1}$ provided $N_s(T) \le c$. We will assume this throughout the proof. Then due to (2.50) and [33, Lemma 2.3] $L(\psi)\psi \in H^s$. Lemma A.1.2 yields $h_2 \in H^s$. Thus we can conclude by Proposition 2.2.4 that

$$\begin{aligned}
\|\partial_x^\alpha \psi(t)\|_1^2 + \|\partial_x^\alpha \psi_t(t)\|^2 + \int_0^t \|\partial_x^\alpha \partial_x \psi(\tau)\|^2 + \|\partial_x^\alpha \psi_t(\tau)\|^2 d\tau& \\
\le C \left(\|\partial_x^\alpha({}^0\psi)\|_1^2 + \|\partial_x^\alpha({}^1\psi)\|^2\right)& \\
+ C \left| \int_0^t \left(\partial_x^\alpha (L(\psi(\tau))\psi(\tau) + h_2(\tau)), \partial_x^\alpha \psi_t(\tau)\right)_{L^2} d\tau \right|& \\
+ C \left| \int_0^t \left(\partial_x^\alpha (L(\psi(\tau))\psi(\tau) + h_2(\tau)), \partial_x^\alpha \psi(\tau)\right)_{L^2} d\tau \right|&
\end{aligned} \tag{2.51}$$

for all $\alpha \in \mathbb{N}_0^3$, $|\alpha| = s$. First obviously

$$|(\partial_x^\alpha h_2, \partial_x^\alpha \psi_t)_{L^2}| + |(\partial_x^\alpha h_2, \partial_x^\alpha \psi)_{L^2}| \le C\|h_2\|_s \|U\|_s \tag{2.52}$$

and integrating by parts gives

$$\begin{aligned}
|(\partial_x^\alpha (L(\psi)\psi), \partial_x^\alpha \psi)_{L^2}| &\le C\|L(\psi)\psi\|_{s-1}\|\psi\|_{s+1} \\
&\le C\|I - A(\psi)\|_s \|\psi_{tt}\|_{s-1}\|\psi\|_{s+1} \\
&+ C\sum_{i,j=1}^{3} \|\bar{B}^{ij} - B^{ij}(\psi)\|_s \|\partial_x^2 \psi\|_{s-1}\|\psi\|_{s+1} \\
&+ C\sum_{j=1}^{3} \|D^j(\psi)\|_s \|\partial_x \psi\|_{s-1}\|\psi\|_{s+1}.
\end{aligned} \tag{2.53}$$

Next write

$$\begin{aligned}
\partial_x^\alpha \left(L(\psi)\psi\right) = L(\psi)\partial_x^\alpha \psi + [\partial_x^\alpha, (I - A(\psi))]\psi_{tt}& \\
- [\partial_x^\alpha, (\bar{B}^{ij} - B^{ij}(\psi))]\psi_{x_i x_j} - [\partial_x^\alpha, D^j(\psi)]\psi_{t x_j}.&
\end{aligned}$$

Since $I - A(\psi), \bar{B}^{ij} - B^{ij}(\psi), D^j(\psi) \in H^s$, [33, Lemma 2.5(i)] yields

$$\begin{aligned} \|[\partial_x^\alpha, (I - A(\psi))]\psi_{tt}\| &\le C\|\partial_x A(\psi)\|_{s-1}\|\psi_{tt}\|_{s-1} \\ \|[\partial_x^\alpha, (\bar{B}^{ij} - B^{ij}(\psi))]\psi_{x_i x_j}\| &\le C\|\partial_x B^{ij}(\psi)\|_{s-1}\|\psi_{x_i x_j}\|_{s-1} \\ \|[\partial_x^\alpha, D^j]\psi_{tx_j}\| &\le C\|\partial_x D^j(\psi)\|_{s-1}\|\psi_{tx_j}\|_{s-1}. \end{aligned} \tag{2.54}$$

Furthermore integration by parts and the symmetry of A, B^{ij} and D^j give

$$\begin{aligned} &\left|\int_0^t (L(\psi)\partial_x^\alpha\psi, \partial_x^\alpha\psi_t)_{L^2}\, d\tau\right| \\ &\qquad \le C\int_0^t \|\partial_t A\|_{L^\infty}\|\partial_x^\alpha(\partial_x\psi, \psi_t)\|^2 \\ &+ \left(\sum_{i,j=1}^3 \|\partial_t B^{ij}\|_{L^\infty} + \|\partial_x B^{ij}\|_{L^\infty} + \sum_{j=1}^3 \|\partial_x D^j\|_{L^\infty}\right)\|\partial_x^\alpha(\partial_x\psi, \psi_t)\|^2 d\tau \\ &\quad + C\left(\|I - A\|_{L^\infty} + \sum_{i,j=1}^3 \|\bar{B}^{ij} - B^{ij}\|_{L^\infty}\right)\|\partial_x^\alpha(\partial_x\psi, \psi_t)\|^2 + \\ &\qquad C\|\partial_x^\alpha(\partial_x{}^0\psi, {}^1\psi)\|^2, \end{aligned} \tag{2.55}$$

In conclusion, (2.51) and the estimates (2.52), (2.53), (2.54), (2.55) lead to

$$\begin{aligned} &\|\partial_x^\alpha\psi(t)\|_1^2 + \|\partial_x^\alpha\psi_t(t)\|^2 + \int_0^t \|\partial_x^\alpha\partial_x\psi(\tau)\|^2 + \|\partial_x^\alpha\psi_t(\tau)\|^2 d\tau \\ &\quad \le C\|U_0\|_{s+1,s}^2 + C\int_0^t \|h_2\|_s\|U\|_{s+1,s} + R_1(\psi)\|U\|_{s+1,s}^2 d\tau \\ &\qquad + C\int_0^t \|I - A(\psi)\|_s\|\psi_{tt}\|_{s-1}\|U\|_{s+1,s} d\tau \\ &\qquad\qquad + CR_2(\psi)\|U(t)\|_{s+1,s}^2, \end{aligned} \tag{2.56}$$

where

$$\begin{aligned} R_1(\psi) = \|\partial_t A(\psi)\|_s + \|I - A(\psi)\|_s + \sum_{i,j=1}^3 \|\partial_t B^{ij}(\psi)\|_s + \|\bar{B}^{ij} - B^{ij}(\psi)\|_s \\ + \sum_{j=1}^3 \|D^j(\psi)\|_s \end{aligned}$$

and

$$R_2(\psi) = \|I - A(\psi)\|_s + \sum_{i,j=1}^3 \|\bar{B}^{ij} - B^{ij}(\psi)\|_s.$$

Step 2: Now let ψ be a solution to (2.6)-(2.8) satisfying (2.43). For $\delta > 0$ set $\psi^\delta = \phi_\delta * \psi$. Applying $\phi_\delta *$ to (2.48) yields

$$\psi^\delta_{tt} - \bar{B}^{ij}\psi^\delta_{x_i x_j} + a\psi^\delta_t + b^j\psi^\delta_{x_j} = L(\psi)\psi^\delta + R^\delta(\psi) + h_2^\delta,$$

where $h_2^\delta = \phi_\delta * h_2$ and

$$R^\delta(\psi) = [\phi_\delta *, (I - A(\psi))]\psi_{tt} - [\phi_\delta *, \bar{B}^{ij} - B^{ij}(\psi)]\psi_{x_i x_j} \\ - [\phi_\delta *, D^j(\psi)]\psi_{t x_j}.$$

Due to [33, Lemma 2.5 (ii)] $R^\delta(\psi) \in H^s$. Hence $L(\psi)\psi^\delta + R^\delta(\psi) + h_2^\delta \in H^s$. Thus proceeding as in step 1 yields

$$\|\partial_x^\alpha \psi^\delta(t)\|_1^2 + \|\partial_x^\alpha \psi_t^\delta(t)\|^2 + \int_0^t \|\partial_x^\alpha \partial_x \psi^\delta(\tau)\|^2 + \|\partial_x^\alpha \psi_t^\delta(\tau)\|^2 d\tau \\ \le C\|U_0^\delta\|_{s+1,s}^2$$

$$+C\int_0^t \|h_2^\delta\|_s\|U^\delta\|_{s+1,s} + R_1(\psi)\|U^\delta\|_{s+1,s}^2 + \|I - A(\psi)\|_s\|\psi_{tt}^\delta\|_{s-1}\|U^\delta\|_{s+1,s} d\tau \\ + C\int_0^t \|R^\delta(\psi)\|_s\|U^\delta\|_{s+1,s} d\tau + CR_2(\psi)\|U^\delta(t)\|_{s+1,s}^2$$

It is easy to see that $U^\delta \to U$ and $h_2^\delta \to h_2$ in $L^\infty([0,T], H^{s+1} \times H^s)$ and in $L^2([0,T], H^s)$, respectively, as $\delta \to 0$. Furthermore $R^\delta(\psi) \to 0$ in $L^2([0,T], H^s)$ as $\delta \to 0$ due to [33, Lemma 2.5(ii)]. Hence we get (2.56) for ψ satisfying (2.43).

Furthermore by Lemma A.1.1

$$\|h_2\|_s \le C\|U\|_{s+1,s}^2.$$

and by Lemma A.1.2

$$R_1(\psi) + R_2(\psi) \le C\|U\|_{s+1,s}$$

for $N_s(T)$ sufficiently small. Finally, since ψ satisfies (2.6),

$$\|\psi_{tt}\|_{s-1} \le C(\|\partial_x^2\psi\|_{s-1} + \|\partial_x\psi_t\|_{s-1} + \|f(\psi, \psi_t, \partial_x\psi)\|_{s-1}) \le C\|U\|_{s+1,s}$$

holds for $N_s(T)$ sufficiently small. Therefore we can deduce from (2.56) that

$$\|\partial_x^\alpha \psi(t)\|_1^2 + \|\partial_x^\alpha \psi_t(t)\|^2 + \int_0^t \|\partial_x^\alpha \partial_x \psi(\tau)\|^2 + \|\partial_x^\alpha \psi_t(\tau)\|^2 d\tau \\ \le C\|U_0\|_{s+1,s}^2 + C\|U(t)\|_{s+1,s}^3 + C\int_0^t \|U(\tau)\|_{s+1,s}^3 d\tau.$$

The assertion is an immediate consequence of this inequality. □

2.3.4 Proposition. *In the situation of Proposition 2.3.1 there exist constants* $a_3(\leq \min\{a_2, a_1\})$, $C_4 = C_4(a_3, \delta_1) > 0$ *(δ_1 being the constant in Proposition 2.3.1) such that the the following holds: If* $N_s(T)^2 \leq a_3$ *and* $\|U_0\|^2_{s,s-1,1} \leq \delta_1$ *for a solution* ψ *of* (2.6)-(2.8) *satisfying* (2.43), *then*

$$N_s(t)^2 \leq C_4 \|U_0\|^2_{s+1,s,1} \quad (t \in [0,T]). \tag{2.57}$$

Proof. This follows directly by adding (2.47) and (2.49). □

Finally we turn to the proof of Theorem 2.1.1.

Proof of Theorem 2.1.1. Let $T_1 > 0, \delta_2 > 0$ such that for all

$$U_0 = ({}^0\psi, {}^1\psi) \in H^{s+1} \times H^s,$$

where $\|U_0\|^2_{s+1,s} < \delta_2$, there exists a solution $U = (\psi, \psi_t)$ of the Cauchy problem (2.6)-(2.8) with

$$\psi \in \bigcap_{j=0}^{s} C^j\left([0,T_1], H^{s+1-j}\right).$$

This is possible due to [24, Theorem III]. Furthermore let a_3, δ_1 and C_4 be the constants in Proposition 2.3.4. Choose $0 < \varepsilon < a_3/(2(1+T_1))$. Due to [24, ibd.] there exists $\delta_3 > 0, (\delta_3 \leq \delta_2)$ such that for all $U_0 = (\psi_0, \psi_1) \in H^{s+1} \times H^s$, where $\|U_0\|^2_{s+1,s} < \delta_3$, the solution U of (2.6)-(2.8) satisfies

$$\sup_{t\in[0,T_1]} \|U(t)\|^2_{s+1,s} < \varepsilon.$$

Now set

$$\delta_0 = \min\{\delta_1, \delta_3, \delta_3/C_4, a_3/(2C_4)\}$$

and choose any $U_0 \in \left(H^{s+1} \times H^s\right) \cap \left(L^1 \times L^1\right)$ for which $\|U_0\|^2_{s+1,s,1} < \delta_0$. Since $\delta_0 \leq \delta_3$, we have

$$N_s(T_1)^2 < \varepsilon + T_1\varepsilon < \frac{a_3}{2}.$$

Hence by Proposition 2.3.4 and $\|U_0\|^2_{s+1,s,1} < \delta_1$

$$N_s(T_1)^2 \leq C_4\|U_0\|^2_{s+1,s,1} < C_4\delta_0 \leq \delta_3. \tag{2.58}$$

Furthermore due to Proposition 2.3.1, (2.10) holds for all $t \in [0, T_1]$. In particular (2.58) yields

$$\|U(T_1)\|^2_{s+1,s} < \delta_3. \tag{2.59}$$

Thus we can solve (2.6) on $[T_1, 2T_1]$ with initial values $(\psi(T_1), \psi_t(T_1))$ and get

$$N_s(T_1, 2T_1)^2 \leq \varepsilon + T_1\varepsilon < \frac{a_3}{2}.$$

Now extend the solution (ψ, ψ_t) continuously on $[0, 2T_1]$. We can conclude

$$N_s(2T_1)^2 \leq N_s(T_1)^2 + N_s(T_1, 2T_1)^2 < \frac{a_3}{2} + \frac{a_3}{2} = a_3.$$

Since we have already assumed $\|U_0\|^2_{s,s-1,1} < \delta_0$, Proposition 2.3.4 yields

$$N_s(2T_1) \leq C_4\delta_0 \tag{2.60}$$

and (2.10) holds for all $t \in [0, 2T_1]$. Due to (2.60) we can repeat the former argument to obtain a solution on $[0, 3T_1]$ and further repetition proves the assertion. $\square$

2.4. Cases of two space dimensions and/or non-integrable initial data

In this section we again examine (2.2), (2.3), respectively. This time we take a slightly different approach, which allows us to also prove global existence and asymptotic in two space dimensions. We cannot use exactly the same methods as in the previous section, since an integral part of the proofs in Section 2.3 was that the decay rate for the linearization $t \mapsto (1+t)^{-\frac{3}{4}}$ is square integrable. In two space dimensions Lemma 2.2.2 still holds, but would only lead to a decay rate $t \mapsto (1+t)^{-\frac{1}{2}}$ in estimate (2.22). However, it turns out that based on Proposition 2.2.4 we are able to prove global existence and asymptotic decay even in two space dimensions. Furthermore this method allows us to drop the requirement that the initial data be integrable. On the other hand we only get decay of solutions in $B^{s-(s_0+1)}$ ($s_0 = [n/2] + 1$) without an explicit rate. We have the following setting: Consider - on a time interval $[0, T]$ -

$$-B^{\alpha\beta\gamma\delta}\frac{\partial\psi_\gamma}{\partial x^\beta \partial x^\delta} + A^{\alpha\beta\gamma}\frac{\partial\psi_\gamma}{\partial x^\beta} - \frac{\partial}{\partial x^\beta}\left(B^{\alpha\beta\gamma\delta}\right)\frac{\partial\psi_\gamma}{\partial x^\delta} = 0, \qquad \alpha = 0, 1, \ldots, n. \tag{2.61}$$

Now, Greek indices run from 0 to n, where $n \in \{2, 3\}$. We still write the Cauchy problem associated with (2.61) as

$$A\psi_{tt} - B^{ij}\psi_{x_ix_j} + D^j\psi_{tx_j} + f = 0, \tag{2.62}$$
$$\psi(0) = {}^0\psi, \tag{2.63}$$
$$\psi_t(0) = {}^1\psi, \tag{2.64}$$

where A, B^{ij}, D^j are symmetric $(n+1) \times (n+1)$ matrices and Latin indices run from 1 to n. More precisely, as in Section 2.3 we fix a constant temperature $\bar{\theta} > 0$ and modify (2.62), such that the rest state $(\bar{\theta}^{-1}, 0) \in \mathbb{R} \times \mathbb{R}^n$ is shifted to $0 \in \mathbb{R}^n$ and

$$\psi_{tt} - \bar{B}^{ij}\psi_{x_ix_j} + a\psi_t + b^j\psi_{x_j} = 0 \text{ on } (0, T], \tag{2.65}$$

is the linearization of (2.62). Again, write $U = (\psi, \psi_t)$ and $U_0 = ({}^0\psi, {}^1\psi)$ for a solution to (2.62)-(2.64) and the initial values, respectively. Let $s \geq s_0 + 1$ ($s_0 = [n/2] + 1$), $T > 0$, $U_0 \in H^{s+1} \times H^s$, and ψ satisfy (2.43). At this point define, for $0 \leq t \leq t_1 \leq T$,

$$N_s(t, t_1)^2 = \sup_{\tau \in [t, t_1]} \|U(\tau)\|^2_{s+1,s} + \int_t^{t_1} \|\partial_x\psi(\tau)\|^2_s + \|\psi_t\|^2_s d\tau,$$

and assume that $N_s(T) \leq a_0$ for an $a_0 > 0$.

Before we start with the proof of the first proposition of this section, note that Proposition 2.2.4 also holds for two space dimensions. Again it is convenient to write (2.62) as

$$\psi_{tt} - \bar{B}^{ij}\psi_{x_ix_j} + a\psi_t + b^j\psi_{x_j} = L(\psi)\psi + h_2(\psi, \psi_t, \partial_x\psi),$$

where

$$L(\psi)\psi = (I - A(\psi))\psi_{tt} - (\bar{B}^{ij} - B^{ij}(\psi))\psi_{x_ix_j} - D^j(\psi)\psi_{tx_j},$$

$$h_2(\psi, \psi_t, \partial_x\psi) = a\psi_t + b^j\psi_{x_j} - f(\psi, \psi_t, \partial_x\psi).$$

2.4.1 Proposition. *There exist constants $a_1 (\leq a_0)$ and $C_1 = C_1(a_1) > 0$ such that the following holds: If $N_s(T)^2 \leq a_1$ for a solution ψ of* (2.62)-(2.64) *satisfying* (2.43)*, then*

$$N_s(t)^2 \leq C_1 \|U_0\|^2_{s,s+1} \quad (t \in [0, T]). \tag{2.66}$$

Proof. Let $U_0 = ({}^0\psi, {}^1\psi) \in H^{s+1} \times H^s$. We only prove the result for ψ satisfying (2.50). The actual assertion follows by using the Friedrichs mollifier similary to the argumentation in the proof of Proposition 2.3.3.

Proposition 2.2.4 yields

$$\begin{aligned} \|\partial_x^k \psi(t)\|_1^2 + \|\partial_x^k \psi_t(t)\|^2 + \int_0^t \|\partial_x^{k+1}\psi(t)\|^2 + \|\partial_x^k \psi_t(t)\|^2 \\ \leq C(\|\partial_x^k({}^0\psi)\|_1^2 + \|\partial_x^k({}^1\psi)\|^2 \\ + C\left|\int_0^t \Big(\partial_x^k(L(\psi(\tau))\psi(\tau) + h_2(\tau)), \partial_x^k\psi_t(\tau)\Big)_{L^2} d\tau\right| \\ + C\left|\int_0^t \Big(\partial_x^k(L(\psi(\tau))\psi(\tau) + h_2(\tau)), \partial_x^k\psi(\tau)\Big)_{H^1} d\tau\right| \end{aligned} \tag{2.67}$$

for $t \in [0,T]$ and integers $0 \leq k \leq s$. For the rest of the proof we assume $N_s(T)$ to be sufficiently small. Our first goal is to estimate the term

$$\left|\Big(\partial_x^k h_2(\psi,\psi_t,\partial_x\psi), \partial_x^k\psi\Big)_{L^2}\right| + \left|\Big(\partial_x^k h_2(\psi,\psi_t,\partial_x\psi,\partial_x^k\psi\Big)_{L^2}\right|.$$

To this end note that

$$|h_2(\psi,\psi_t,\partial_x\psi)| \leq C\left(|\psi|(|\partial_x\psi| + |\psi_t|) + |\partial_x\psi|^2 + |\psi_t|^2\right) \tag{2.68}$$

and (by [33, Lemma 2.3], [33, Lemma 2.4])

$$\|h_2(\psi,\psi,\psi_t)\|_s \leq C\left(\|\psi\|_s(\|\partial_x\psi\|_s + \|\psi_t\|_s) + \|\partial_x\psi\|_s^2 + \|\psi_t\|_s^2\right). \tag{2.69}$$

Due to [33, Corollary 2.2] (with $p = 4, s = 1$) (2.68) yields

$$\begin{aligned} (h_2(\psi,\psi_t,\partial_x\psi),\psi)_{L^2} &\leq C\int |\psi|^2(|\psi_t| + |\partial_x\psi|) + |\psi|(|\psi_t|^2 + |\partial_x\psi|^2)dx \\ &\leq C\|\psi\|_{L^4}^2(\|\psi_t\| + \|\partial_x\psi\|) + C\|\psi\|(\|\psi_t\|^2 + \|\partial_x\psi\|^2) \\ &\leq C\|\psi\|_1(\|\psi_t\|^2 + \|\partial_x\psi\|^2). \end{aligned}$$

Integration by parts and (2.69) give

$$\begin{aligned} \Big(\partial_x^k h_2(\psi,\psi_t,\partial_x\psi), \partial_x^k\psi\Big)_{L^2} &\leq \|h_2\|_{s-1}\|\partial_x\psi\|_s \\ &\leq C\|\psi\|_{s-1}(\|\partial_x\psi\|_s^2 + \|\psi_t\|_{s-1}^2) + C(\|\psi_t\|_{s-1}^3 + \|\partial_x\psi\|_s^3). \end{aligned}$$

for $1 \leq k \leq s$. Obviously,

$$\begin{aligned} \Big(\partial_x^k h_2(\psi,\psi_t,\partial_x\psi), \partial_x^k\psi_t\Big)_{L^2} &\leq \|h_2(\psi)\|_s\|\psi_t\|_s \\ &\leq C\|\psi\|_s(\|\psi_t\|_s^2 + \|\partial_x\psi\|_s^2) + C(\|\psi_t\|_s^3 + \|\partial_x\psi\|_s^3) \end{aligned}$$

for $0 \le k \le s$. In conclusion

$$\left|\left(\partial_x^k h_2(\psi,\psi_t,\partial_x\psi),\partial_x^k\psi\right)_{L^2}\right| + \left|\left(\partial_x^k h_2(\psi,\psi_t,\partial_x\psi),\partial_x^k\psi\right)_{L^2}\right| \\ \le C\|\psi\|_s(\|\psi_t\|_s^2 + \|\partial_x\psi\|_s^2) + C(\|\psi_t\|_s^3 + \|\partial_x\psi\|_s^3) \quad (2.70)$$

for $0 \le k \le s$. Next we prove an estimate for

$$\left|\int_0^t \left(\partial_x^k(L(\psi(\tau)\psi(\tau)),\partial_x^k\psi_t(\tau)\right)_{L^2} d\tau\right| \\ + \left|\int_0^t \left(\partial_x^k(L(\psi(\tau)\psi(\tau)),\partial_x^k\psi(\tau)\right)_{L^2} d\tau\right|$$

Obviously,

$$|L(\psi)\psi| \le C|\psi|(|\psi_{tt}| + |\partial_x^2\psi| + |\partial_x\psi_t|).$$

Therefore

$$((L(\psi)\psi),\psi)_{L^2} \le \|\psi\|_{L^4}^2(\|\psi_{tt}\| + \|\psi_t\|_1 + \|\partial_x^2\psi\|) \\ \le \|\psi\|_1(\|\psi_{tt}\|^2 + \|\partial_x\psi\|_1^2 + \|\psi_t\|_1^2) \quad (2.71)$$

and

$$((L(\psi)\psi),\psi_t)_{L^2} \le C\|\psi\|(\|\partial_x\psi\|_1^2 + \|\psi_t\|_1^2 + \|\psi_{tt}\|^2), \quad (2.72)$$

where again [33, Corollary 2.2] was used to derive inequality (2.71). Now, let $1 \le k \le s$. Integration by parts gives

$$\begin{aligned}\left|\left(\partial_x^k(L(\psi)\psi),\partial_x^k\psi\right)_{L^2}\right| &\le C\|L(\psi)\psi\|_{s-1}\|\partial_x\psi\|_s \\ &\le C\|I - A(\psi)\|_{s-1}\|\psi_{tt}\|_{s-1}\|\partial_x\psi\|_s \\ &+ C\sum_{i,j=1}^n \|\bar{B}^{ij} - B^{ij}(\psi)\|_{s-1}\|\partial_x^2\psi\|_{s-1}\|\partial_x\psi\|_s \\ &+ C\sum_{j=1}^n \|D^j(\psi)\|_{s-1}\|\psi_{tx}\|_{s-1}\|\partial_x\psi\|_s \\ &\le C\|\psi\|_{s-1}(\|\partial_x\psi\|_s^2 + \|\psi_t\|_s^2 + \|\psi_{tt}\|_{s-1}^2).\end{aligned} \quad (2.73)$$

For the next step write

$$\begin{aligned}\partial_x^k\left(L(\psi)\psi\right) &= L(\psi)\partial_x^k\psi + [\partial_x^k, L(\psi)]\psi \\ &= L(\psi)\partial_x^k\psi + [\partial_x^k,(I - A(\psi))]\psi_{tt} - [\partial_x^k,(\bar{B}^{ij} - B^{ij}(\psi))]\psi_{x_ix_j} \\ &\qquad - [\partial_x^k, D^j(\psi)]\psi_{tx_j}.\end{aligned}$$

As $\bar{A}-A, \bar{B}^{ij}-B^{ij}, D^j \in H^s$, [33, Lemma 2.4] and [33, Lemma 2.5(i)] yield

$$\|[\partial_x^k, L(\psi)]\psi\| \leq C\|\psi\|_s(\|\psi_{tt}\|_{s-1} + \|\partial_x^2\psi\|_{s-1} + \|\partial_x\psi_t\|_{s-1}). \tag{2.74}$$

Finally, due to the symmetry of A, B^{ij}, D^j, integration by parts gives (as in the proof of Proposition 2.3.3)

$$\begin{aligned}
&\int_0^t \left(L(\psi)\partial_x^k\psi, \partial_x^k\psi_t\right)_{L^2} d\tau \\
&\qquad\qquad \leq C\int_0^t \|\partial_t A\|_{L^\infty}\|\partial_x^k(\partial_x\psi,\psi_t)\|^2 \\
&+\left(\sum_{i,j=1}^3 \|\partial_t B^{ij}\|_{L^\infty} + \|\partial_x B^{ij}\|_{L^\infty} + \sum_{j=1}^3 \|\partial_x D^j\|_{L^\infty}\right)\|\partial_x^k(\partial_x\psi,\psi_t)\|^2 d\tau \\
&\qquad + C\left(\|I-A\|_{L^\infty} + \sum_{i,j=1}^3 \|\bar{B}^{ij}-B^{ij}\|_{L^\infty}\right)\|\partial_x^k(\partial_x\psi,\psi_t)\|^2 \\
&\qquad\qquad + C\|\partial_x^k(\partial_x{}^0\psi, {}^1\psi)\|^2 \\
&\leq C\|U_0\|_{s+1,s}^2 + C\int_0^t \|\psi_t\|_s^3 + \|\partial_x\psi\|_s^3 d\tau + C\|\psi\|_s(\|\psi_t\|_s^2 + \|\partial_x\psi\|_s^2)
\end{aligned} \tag{2.75}$$

Hence, (2.71), (2.72), (2.73), (2.74) and (2.75) lead to

$$\begin{aligned}
&\left|\int_0^t \left(\partial_x^k(L(\psi(\tau)\psi(\tau)), \partial_x^k\psi_t(\tau)\right)_{L^2} d\tau\right| \\
&\qquad + \left|\int_0^t \left(\partial_x^k(L(\psi(\tau))\psi(\tau)), \partial_x^k\psi_t(\tau)\right)_{L^2} d\tau\right| \\
&\leq C\|U_0\|_{s+1,s}^2 + C\int_0^t (\|\psi\|_{s+1} + \|\psi_t\|_s)\left(\|\partial_x\psi\|_s^2 + \|\psi_t\|_s^2 + \|\psi_{tt}\|_{s-1}^2\right) d\tau \\
&\qquad\qquad + C\|\psi\|(\|\psi_t\|_s^2 + \|\partial_x\psi\|_s^2).
\end{aligned} \tag{2.76}$$

As ψ satisfies (3.3),

$$\|\psi_{tt}\|_{s-1} \leq C(\|\psi_t\|_s + \|\partial_x\psi\|_s).$$

Therefore, inserting estimates (2.70) and (2.76) in (2.67) finally yields

$$
\begin{aligned}
N_s(T)^2 \leq C\|U_0\|_{s+1,s}^2 + &\\
C \sup_{t\in[0,T]} \left(\|\psi\|_{s+1} + \|\psi_t\|_s\right) \int_0^t \left(\|\partial_x\psi\|_s^2 + \|\psi_t\|_s^2 + \|\psi_{tt}\|_{s-1}^2\right) d\tau &\\
+ C\left(\|\psi_t\|_s^3 + \|\partial_x\psi\|_s^3\right) &\\
\leq C\|U_0\|_{s+1,s}^2 + CN_s(T)^3. &
\end{aligned}
$$

The assertion is an immediate consequence of this estimate. □

Now, we are able to prove the main result of this section.

2.4.2 Theorem. *Let $s \geq s_0+1$. Then there exist $\delta_0 > 0$, $C_0 = C_0(\delta_0) > 0$ such that for all initial data $U_0 \in H^{s+1} \times H^s$ satisfying $\|U_0\|_{s+1,s}^2 < \delta_0$ there exists a unique solution*

$$
\psi \in \bigcap_{j=0}^{s} C^j\left([0,\infty], H^{s+1-j}\right) \tag{2.77}
$$

of (2.62)-(2.64) *satisfying the estimate*

$$
\sup_{t\in[0,\infty)} \left(\|\psi(t)\|_{s+1}^2 + \|\psi_t(t)\|_s^2\right) + \int_0^\infty \|\partial_x\psi(t)\|_s^2 + \|\psi_t(t)\|_s^2 dt \leq C_0\|U_0\|_{s+1,s}^2, \tag{2.78}
$$

and

$$
|\psi(t)|_{s-(s_0+1)} + |\psi_t(t)|_{s-(s_0+1)} \to 0 \quad \textit{as} \;\; t \to \infty. \tag{2.79}
$$

Proof. To prove the existence of a global solution satisfying (2.77) and (2.78) for sufficiently small initial data we take a local solution provided by [24, Theorem III] and use Proposition 2.4.1 to extend it for all $t \in [0,\infty)$. This can be done analogously as in the proof of Theorem 2.1.1. So we omit the details. We just show decay law (2.79). To this end set

$$
M(t) = \|\partial_x\psi\|_{s-1}^2 + \|\psi_t\|_{s-1}^2
$$

Then (2.78) and the fact that

$$
\|\psi_{tt}\|_{s-1} + \|\psi_{tx}\|_{s-1} \leq C(\|\psi_t\|_s + \|\partial_x\psi\|_s)
$$

yield

$$
\int_0^\infty |M(t)| + |M_t(t)| dt \leq C\|U\|_{s+1,s}^2.
$$

Therefore $M(t) \to 0$ as $t \to \infty$. Furthermore it follows from [33, Corollary 2.2] (with $p = \infty$ and $a = n/(2s_0)$) that

$$\begin{aligned} |\psi|_{s-(s_0+1)} &\leq C\|\partial_x \psi\|_{s-1}^a \|\psi\|_{s-(s_0+1)}^{1-a}, \\ |\psi_t|_{s-(s_0+1)} &\leq C\|\psi_{tx}\|_{s-1}^a \|\psi_t\|_{s-(s_0+1)}^{1-a}. \end{aligned} \tag{2.80}$$

Due to (2.78), $\|\psi\|_{s-(s_0+1)}$ and $\|\psi_t\|_{s-(s_0+1)}$ are bounded. Thus the assertion is a consequence of (2.80) and the decay of $M(t)$. □

3. Global existence and asymptotic decay of solutions to the relativistic Navier-Stokes-Fourier equations for viscous heat-conductive fluids with diffusion

3.1. Preliminaries

Consider a fluid satisfying (1.14), (1.15) with dissipation quantified by the coefficients $\chi > 0$, $\eta > 0$, $\tilde{\zeta} \geq 0$, and $\mu > 0$ of heat conduction, modified bulk viscosity, shear viscosity and diffusion. The dynamics of such a fluid is governed by (1.24). In the corresponding Godunov variables (1.6), (1.24) reads

$$\frac{\partial}{\partial x^\beta}\left(-B^{a\beta g\delta}\frac{\partial \psi_g}{\partial x^\delta} + \frac{\partial X^\beta}{\partial \psi_a}\right) = 0, \;\; a = 0, \ldots, 4, \tag{3.1}$$

where X^β and $B^{a\beta g\delta}$ are defined by (1.16) and (1.25), (1.26), (1.27). Carrying out the differentiation with respect to x^β yields

$$-B^{a\beta g\delta}(\psi)\frac{\partial^2 \psi_g}{\partial x^\beta \partial x^\delta} + A^{a\beta g}\frac{\partial \psi_g}{\partial x^\beta} - \frac{\partial}{\partial x^\beta}\left(B^{a\beta g\delta}(\psi)\right)\frac{\partial \psi_g}{\partial x^\delta} = 0, \tag{3.2}$$

where $A^{a\beta g}$ is given by (1.18), (1.19), (1.20). As stated in Chapter 1 (Proposition 1.3.2), (3.2) is HKM hyperbolic with u^α being a direction of hyperbolicity. Thus for states ψ with u sufficiently close to $(1, 0, 0, 0)$, using

$$B^{a\beta g\delta}(\psi)\frac{\partial \psi_g}{\partial x^\beta \partial x^\delta} = \tilde{B}^{a\beta g\delta}(\psi)\frac{\partial \psi_g}{\partial x^\beta \partial x^\delta},$$

we can write (3.2) as

$$A(\psi)\psi_{tt} - B^{ij}(\psi)\psi_{x_i x_j} + D^j(\psi)\psi_{tx_j} + f(\psi, \psi_t, \partial_x \psi) = 0, \tag{3.3}$$

where

$$A = (-\tilde{B}^{a0g0})_{0\le a,g\le 4}, \quad B_{ij} = (\tilde{B}^{aigj})_{0\le a,g\le 4},$$
$$D_j = (-2\tilde{B}^{a0gj})_{0\le a,g\le 4}$$

are symmetric 5×5 matrices, $A(\psi)$ is positive definite, $\xi_i B^{ij}(\psi)\xi_j$ is positive definite for arbitrary $\xi \in \mathbb{R}^3 \setminus \{0\}$, and

$$f^a = A^{a\beta g}\frac{\partial \psi_g}{\partial x^\beta} - \frac{\partial}{\partial x^\beta}\left(B^{a\beta g\delta}(\psi)\right)\frac{\partial \psi_g}{\partial x^\delta}, \; a = 0, \dots, 4.$$

Throughout the chapter we will consider the Cauchy problem associated with (3.3):

$$A\psi_{tt} - B^{ij}\psi_{x_i x_j} + D^j\psi_{tx_j} + f = 0 \text{ on } (0,T] \times \mathbb{R}^3, \tag{3.4}$$
$$\psi(0) = {}^0\psi \text{ on } \mathbb{R}^3, \tag{3.5}$$
$$\psi_t(0) = {}^1\psi \text{ on } \mathbb{R}^3, \tag{3.6}$$

As we did for barotropic fluids in chapter 2, we first study the linearization at a homogeneous rest state (Section 3.2) and then the nonlinear equations as perturbations of the linear ones (Section 3.3). However, we did not find a way to use "Kawashima-type" energy methods to obtain a decay estimate for the linearized equations. Instead we apply results of perturbation theory of finite dimensional linear operators to the linear equations in Fourier space, which are proven in Section A.2.

3.2. Decay estimates for the linearized system

In this section we consider the linearization of (3.3) about an arbitrary fixed homogeneous reference state in the fluid's rest frame $\bar{\psi}^\alpha = \bar{u}^\alpha/\bar{\theta}$, $\bar{\psi}^4 = \bar{\upsilon}/\bar{\theta}$, $\bar{u} = (1,0,0,0)^t$ and $\bar{\theta} > 0$, $\bar{\upsilon}$, $\bar{s}$ the constant temperature, chemical potential and specific entropy at the reference state. The resulting equations read

$$A_{(1)}\psi_{tt} - B^{ij}_{(1)}\psi_{x_i x_j} + a_{(1)}\psi_t + b^j_{(1)}\psi_{x_j} = 0, \tag{3.7}$$

where

$$A_{(1)} = \begin{pmatrix} \chi\bar{\theta}^2 & 0 & 0 \\ 0 & \sigma\bar{\theta}I_3 & 0 \\ 0 & 0 & \mu \end{pmatrix},$$

$$B^{ij}_{(1)} = \begin{pmatrix} \chi\bar{\theta}^2\delta^{ij} & 0 & 0 \\ 0 & \eta\bar{\theta}\delta^{ij}I_3 + \frac{1}{2}\bar{\theta}(\frac{1}{3}\eta + \tilde{\zeta})(e^i\otimes e^j + e^j \otimes e^i) & 0 \\ 0 & 0 & \mu\delta^{ij} \end{pmatrix},$$

$$a_{(1)} = \begin{pmatrix} \bar\theta^2 \hat X_\theta + \bar\theta^3 \hat X_{\theta\theta} & 0 & \bar\theta \hat X_{\psi\theta} - \hat X_\psi \\ 0 & \bar\theta^2 \hat X_\theta I & 0 \\ \bar\theta \hat X_{\psi\theta} - \hat X_\psi & 0 & \bar\theta^{-1} \hat X_{\psi\psi} \end{pmatrix},$$

$$b^j_{(1)} = \bar\theta^2 \hat X_\theta (e^0 \otimes e^j + e^j \times e^0) + \hat X_\psi (e^5 \otimes e^j + e^j \otimes e^5).$$

Note that $\chi, \sigma, \eta, \tilde\zeta, \mu$ and the derivatives of $\hat X$ are all evaluated at the reference state. Next multiply (3.7) by $(A_{(1)})^{-\frac{1}{2}}$ and write it in variables $(A_{(1)})^{\frac{1}{2}}\psi$ to obtain

$$\psi_{tt} - \bar B^{ij} + a\psi_{tt} + b^j \psi_{x_j} = 0, \tag{3.8}$$

where

$$\bar B^{ij} = (A_{(1)})^{-\frac{1}{2}} B^{ij}_{(1)} (A_{(1)})^{-\frac{1}{2}},$$

$$a = (A_{(1)})^{-\frac{1}{2}} a_{(1)} A_{(1)}^{-\frac{1}{2}}, \quad b^j = (A_{(1)})^{-\frac{1}{2}} b^j_{(1)} (A_{(1)})^{-\frac{1}{2}}.$$

Our goal is to prove decay and energy estimates for the Cauchy problem associated with (3.7),

$$\psi_{tt} - \bar B^{ij} \psi_{x_i x_j} + a\psi_t + b^j \psi_{x_j} = 0, \tag{3.9}$$

$$\psi(0) = {}^0\psi, \tag{3.10}$$

$$\psi_t(0) = {}^1\psi, \tag{3.11}$$

which will be considered on a fixed time interval $[0, T]$. Again, we start by considering the Cauchy problem (3.9)-(3.11) in Fourier space:

$$\hat\psi_{tt} + |\xi|^2 B(\check\xi)\hat\psi + a\hat\psi_t - i|\xi| b(\check\xi)\hat\psi = 0, \tag{3.12}$$

$$\hat\psi(0) = {}^0\hat\psi(\xi), \tag{3.13}$$

$$\hat\psi_t(0) = {}^1\hat\psi(\xi), \tag{3.14}$$

where $\check\xi = \xi/|\xi|$ and for $\omega \in \mathbb{S}^2$

$$B(\omega) = \omega_i \bar B^{ij} \omega_j = \begin{pmatrix} y1 & 0 & 0 \\ 0 & \sigma^{-1}\left(\eta I_3 + \left(\tilde\zeta + \frac{1}{3}\eta\right)(\omega \otimes \omega)\right) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$b(\omega) = b^j \omega_j = \begin{pmatrix} 0 & (\chi\sigma\bar\theta)^{-\frac{1}{2}} \hat X_\theta \omega^t & 0 \\ (\chi\sigma\bar\theta)^{-\frac{1}{2}} \hat X_\theta \omega & 0 & (\mu\sigma\bar\theta)^{-\frac{1}{2}} \hat X_\psi \omega \\ 0 & (\mu\sigma\bar\theta)^{-\frac{1}{2}} \hat X_\psi \omega^t & 0 \end{pmatrix}.$$

We start with a useful remark about the algebraic structure of the coefficient matrices.

3.2.1 Remark. For each $\xi \in \mathbb{R}^3$, consider the orthogonal decomposition

$$\mathbb{C}^5 = (\mathbb{C} \times \xi\mathbb{C} \times \mathbb{C}) \oplus (\{0\} \times \{\xi\}^\perp \times \{0\}) := J_1(\xi) \oplus J_2(\xi).$$

Setting $\hat{\psi} = w + v$ with $w \in J_1(\xi)$ and $v \in J_2(\xi)$, (3.12) decomposes into the two uncoupled systems

$$w_{tt} + |\xi|^2 w + \tilde{a} w_t - i|\xi|\tilde{b} w = 0, \tag{3.15}$$
$$v_{tt} + \eta\sigma^{-1}|\xi|^2 v + \sigma^{-1}\bar{\theta} X_\theta v_t = 0, \tag{3.16}$$

where

$$\tilde{a} = \begin{pmatrix} \chi^{-1}(\hat{X}_\theta + \bar{\theta}\hat{X}_{\theta\theta}) & 0 & (\chi\mu)^{-\frac{1}{2}}(\hat{X}_{\psi\theta} - \bar{\theta}^{-1}\hat{X}_\psi) \\ 0 & \sigma^{-1}\bar{\theta}\hat{X}_\theta & 0 \\ (\chi\mu)^{-\frac{1}{2}}(\hat{X}_{\psi\theta} - \bar{\theta}^{-1}\hat{X}_\psi) & 0 & (\mu\bar{\theta})^{-1}\hat{X}_{\psi\psi} \end{pmatrix},$$

$$\tilde{b} = \begin{pmatrix} 0 & (\chi\sigma\bar{\theta})^{-\frac{1}{2}}\hat{X}_\theta & 0 \\ (\chi\sigma\bar{\theta})^{-\frac{1}{2}}\hat{X}_\theta & 0 & (\mu\sigma\bar{\theta})^{-\frac{1}{2}}\hat{X}_\psi \\ 0 & (\mu\sigma\bar{\theta})^{-\frac{1}{2}}\hat{X}_\psi & 0 \end{pmatrix}.$$

In particular, it follows from Remark 1.2.4 that $\tilde{a} \pm \tilde{b} > 0$. Furthermore, we have seen in the proof of Lemma 2.2.2 that there exist $C, c > 0$ such that for each $\xi \in \mathbb{R}^3$ a solution $(v(t), v_t(t))$ of (3.16) satisfies

$$\begin{aligned}(1 + |\xi|^2)|v(t,\xi)|^2 &+ |v_t(t,\xi)|^2 \\ &\le C\exp(-c\rho(\xi)t)((1+|\xi|^2)|v(0,\xi)|^2 + |v_t(0,\xi)|^2)\end{aligned} \tag{3.17}$$

for all $t \in [0,T], \xi \in \mathbb{R}^3$.

Next, we show that solutions $(w(t), w_t(t))$ to (3.15) also satisfy a pointwise decay estimate of the form (3.17).

3.2.2 Lemma. *There exist $c, C > 0$, such that for each $\xi \in \mathbb{R}^3$ a solution $(w(t,\xi), w_t(t,\xi))$ to (3.15) satisfies*

$$\begin{aligned}(1 + |\xi|^2)|w(t,\xi)|^2 &+ |w_t(t,\xi)|^2 \\ &\le C\exp(-c\rho(\xi)t)\left((1+|\xi|^2)|w(0,\xi)|^2 + |w_t(0,\xi)|^2\right)\end{aligned} \tag{3.18}$$

for each $t \in [0,T]$, $\xi \in \mathbb{R}^3$.

Proof. Let $W = (w, w_t)$, then we can write (3.15) as

$$W_t = M(|\xi|)W, \quad \xi \in \mathbb{R}^3, \tag{3.19}$$

where

$$M(\kappa) = \begin{pmatrix} 0 & I_3 \\ -\kappa^2 I_3 + i\kappa\tilde{b} & -\tilde{a} \end{pmatrix}, \kappa \in \mathbb{C}. \tag{3.20}$$

By Proposition A.2.2 there exist $r_0 > 0, c_0 > 0$ and a family of holomorphic and invertible 6×6 matrices

$$\{S(\kappa) : |\kappa| \leq r_0\},$$

such that $M_*(\kappa) = (S(\kappa))^{-1}M(\kappa)S(\kappa)$ satisfies

$$\langle (M_*(\kappa) + M_*(\kappa)^*)Z, Z\rangle \leq -c_0\kappa^2|Z|^2, \tag{3.21}$$

for $Z \in \mathbb{C}^6$, and $\kappa \in [0, r_0]$. Now, set $\tilde{W}(\xi) = S^{-1}(|\xi|)W(\xi)$. $\tilde{W}$ satisfies

$$\tilde{W}_t = M_*(|\xi|)\tilde{W}.$$

Taking the scalar product of this equation with $\tilde{W}$, considering the real part and using (3.21) gives

$$\frac{d}{dt}|\tilde{W}(\xi)|^2 = \langle (M_*(|\xi|) + M_*(|\xi|))^*)\tilde{W}(\xi), \tilde{W}(\xi)\rangle \leq -c_0|\xi|^2|\tilde{W}(\xi)|^2$$

for $|\xi| \in [0, r_0]$. Applying Gronwall's Lemma yields

$$|\tilde{W}(t,\xi)|^2 \leq \exp(-c_0|\xi|^2 t)|\tilde{W}(0,\xi)|^2, \ (t, |\xi|) \in [0,T] \times [0, r_0].$$

Since $S(|\xi|)$ is continuous on $[0, r_0]$, there exists a $C > 0$ (independent of ξ) such that

$$|W(t,\xi)|^2 \leq C\exp(-c_0|\xi|^2 t)|W(0,\xi)|^2, \ (t, |\xi|) \in [0,T] \times [0, r_0]. \tag{3.22}$$

Due to

$$\rho(\xi) \leq |\xi|^2 \leq r_0^2, \ |\xi| \in [0, r_0],$$

(3.22) yields

$$\begin{aligned}(1 + |\xi|^2)|w(t,\xi)|^2 &+ |w_t(t,\xi)|^2 \\ &\leq C\exp(-c_0\rho(\xi)t)\left((1 + |\xi|^2)|w(0,\xi)|^2 + |w_t(0,\xi)|^2\right), \\ &\qquad (t, |\xi|) \in [0,T] \times [0, r_0]. \end{aligned}\tag{3.23}$$

Next set $V = (|\xi| w, w_t)$ and write (3.15) as

$$V_t = N(|\xi|)V,$$

$$N(\kappa) = \begin{pmatrix} 0 & \kappa I_3 \\ -\kappa I_3 + i\tilde{b} & -\tilde{a} \end{pmatrix}, \ \kappa \in \mathbb{C}.$$

Due to Corollary A.2.6 there exist $r_\infty, c_\infty > 0$ and a family of bounded and invertible 6×6 matrices

$$\{L(\kappa) : |\kappa| \geq r_\infty\},$$

where $\kappa \mapsto (L(\kappa)^{-1})$ is also bounded, such that $N_*(\kappa) = L(\kappa)^{-1} N(\kappa) L(\kappa)$ satisfies

$$\langle (N_*(\kappa) + (N_*(\kappa))^*) Z, Z \rangle \leq -c_\infty |Z|^2,$$

for all $Z \in \mathbb{C}^6$, $\kappa \in [r_\infty, \infty)$. It follows with the same arguments as in the first part of the proof that there exists a $C > 0$ such that

$$|V(t,\xi)|^2 \leq C \exp(-c_\infty t) |V(0,\xi)|^2, \ (t, |\xi|) \in [0,T] \times [r_\infty, \infty). \tag{3.24}$$

Using $\rho(\xi) \leq 1$, this gives

$$\begin{aligned}(1 + |\xi|^2)|w(t,\xi)|^2 + |w_t(t,\xi)|^2 \\ \leq C \exp(-c_\infty \rho(\xi) t) \left((1 + |\xi|^2)|w(0,\xi)|^2 + |w_t(0,\xi)|^2 \right), \\ (t, |\xi|) \in [0,T] \times [r_\infty, \infty).\end{aligned} \tag{3.25}$$

Lastly, let $\xi_0 \in \mathbb{R}^3$ with $r_0 \leq |\xi_0| \leq r_\infty$. Due to Lemma A.2.8 each eigenvalue of $M(|\xi_0|)$ has negative real part. Therefore there exists an invertible matrix $Q_{\xi_0} \in \mathbb{C}^{6 \times 6}$ and a $c > 0$ such that $M_*(\kappa) = Q_{\xi_0}^{-1} M(\kappa) Q_{\xi_0}$ satisfies

$$\langle (M_*(|\xi_0|) + (M_*(|\xi_0|))^*) Z, Z \rangle \leq -c|Z|^2, \ Z \in \mathbb{C}^6.$$

Since M is continuous there also exist $\delta(\xi_0), c(\xi_0) > 0$ such that for all $\xi \in B_\delta(\xi_0)$

$$\langle (M_*(|\xi|) + (M_*(|\xi|))^*) Z, Z \rangle \leq -c|Z|^2, \ Z \in \mathbb{C}^6.$$

Hence by Gronwall's Lemma there exists a $C > 0$ such that

$$|W(t,\xi)|^2 \leq C e^{-ct} |W(0,\xi)|^2, \ t \in [0,T], \tag{3.26}$$

for $\xi \in B_\delta(\xi_0)$. As $K = \{\xi \in \mathbb{R}^3 | r_0 \leq |\xi| \leq r_\infty\}$ is compact, there exist $c, C > 0$ (independent of ξ) such that (3.26) holds for all $\xi \in K$, and thus

$$\begin{aligned}(1+|\xi|^2)|w(t,\xi)|^2 + |w_t(t,\xi)|^2 \\ \leq C\exp(-c\rho(\xi)t)\left((1+|\xi|^2)|w(0,\xi)|^2 + |w(0,\xi)|^2\right), \\ (t,|\xi|) \in [0,T]\times[r_0,r_\infty].\end{aligned}$$

This, together with (3.23) and (3.25), proves the assertion. □

3.2.3 Proposition. *For some $s \in \mathbb{N}_0$ let $({}^0\psi, {}^1\psi) \in (H^{s+1}\times H^s)\cap(L^1\times L^1)$ and $(\psi(t), \psi_t(t)) \in H^{s+1}\times H^s$ be a solution of (3.9)-(3.11). Then there exist $c, C > 0$ such that for all integers $0 \leq k \leq s$ and all $t \in [0,T]$*

$$\begin{aligned}\|\partial_x^k\psi(t)\|_1 + \|\partial_x^k\psi_t(t)\| \leq C(1+t)^{-\frac{3}{4}-\frac{k}{2}}\left(\|{}^0\psi\|_{L^1} + \|{}^1\psi\|_{L^1}\right) \\ + Ce^{-ct}\left(\|\partial_x^k({}^0\psi)\|_1 + \|\partial_x^k({}^1\psi)\|\right). \quad (3.27)\end{aligned}$$

Proof. Let $\hat{\psi}$ be the Fourier transform of a solution ψ to (3.9)-(3.11). For fixed $\xi \in \mathbb{R}^3$ write $\hat{\psi} = w + v$ with $w \in J_1(\xi)$, $v \in J_2(\xi)$ as in Remark 3.2.1. Then by Remark 3.2.1

$$\begin{aligned}(1+|\xi|^2)|v(t,\xi)|^2 + |v_t(t,\xi)|^2 \\ \leq C\exp(-c\rho(\xi)t)((1+|\xi|^2)|v(0,\xi)|^2 + |v_t(0,\xi)|^2)\end{aligned}$$

and by Lemma 3.2.2

$$\begin{aligned}(1+|\xi|^2)|w(t,\xi)|^2 + |w_t(t,\xi)|^2 \\ \leq C\exp(-c\rho(\xi)t)\left((1+|\xi|^2)|w(0,\xi)|^2 + |w_t(0,\xi)|^2\right),\end{aligned}$$

where the constants are independent of ξ. As $\mathbb{C}^5 = J_1(\xi) \oplus J_2(\xi)$ is an orthogonal decomposition, this yields

$$(1+|\xi|^2)|\hat{\psi}(t,\xi)|^2 + |\hat{\psi}_t(t,\xi)|^2 \leq C\exp(-c\rho(\xi)t)((1+|\xi|^2)|{}^0\hat{\psi}(\xi)|^2 + |{}^1\hat{\psi}(\xi)|^2)$$

for all $(t,\xi) \in [0,T]\times\mathbb{R}^3$. Based on this estimate the assertion is proved as in [7, Proof of Theorem 3.1]. □

Next consider the inhomogeneous initial-value problem

$$\psi_{tt} - \bar{B}^{ij}\psi_{x_i x_j} + a\psi_t + b^j\psi_{x_j} = h, \quad (3.28)$$

$$\psi(0) = {}^0\psi, \quad (3.29)$$

$$\psi_t(0) = {}^1\psi. \quad (3.30)$$

for some $h : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^5$. The following result is an immediate consequence of Proposition 3.2.3.

3.2.4 Proposition. *Let s be a non-negative integer,* $({}^0\psi, {}^1\psi) \in (H^{s+1} \times H^s) \cap (L^1 \times L^1)$ *and* $h \in C([0,T], H^s \cap L^1)$. *Then the solution ψ of* (3.28)-(3.30) *satisfies*

$$\begin{aligned}
\|\partial_x^k \psi(t)\|_1 + \|\partial_x^k \psi_t(t)\| \le C(1+t)^{-\frac{3}{4}-\frac{k}{2}}(\|{}^0\psi\|_{L^1} + \|{}^1\psi\|_{L^1}) \\
+ Ce^{-ct}(\|\partial_x^k({}^0\psi)\|_1 + \|\partial_x^k({}^1\psi)\|) \\
+ C\int_0^t (1+t-\tau)^{-\frac{3}{4}-\frac{k}{2}}\|h(\tau)\|_{L^1} \\
+ Ce^{-c(t-\tau)}\|\partial_x^k h(\tau)\| d\tau \quad (3.31)
\end{aligned}$$

for all $t \in [0,T]$ and $0 \le k \le s$.

Furthermore we can prove the following energy estimate.

3.2.5 Proposition. *Let s be a non-negative integer. There exists a $C > 0$ such that for all $({}^0\psi, {}^1\psi) \in H^{s+1} \times H^s$ and $h \in C([0,T], H^s)$ the solution ψ of* (3.28)-(3.30) *satisfies*

$$\begin{aligned}
&\|\partial_x^k \psi(t)\|_1^2 + \|\partial_x^k \psi_t(t)\|^2 + \int_0^t \|\partial_x^{k+1}\psi(\tau)\|^2 + \|\partial_x^k \psi_t(\tau)\|^2 d\tau \\
&\le C\left(\|\partial_x^k({}^0\psi)\|_1^2 + \|\partial_x^k({}^1\psi)\|^2\right) + C\int_0^t \|\partial_x^k \psi(\tau)\|^2 d\tau \\
&+ C\left|\int_0^t \left(\partial_x^k h(\tau), \frac{a}{2}\partial_x^k \psi(\tau)\right)_{L^2} d\tau\right| + C\left|\int_0^t \left(\partial_x^k h(\tau), \partial_x^k \psi_t(\tau)\right)_{L^2} d\tau\right|, \quad (3.32)
\end{aligned}$$

for all $t \in [0,T]$ and integers $0 \le k \le s$.

Proof. Consider (3.28) in Fourier space, i.e.,

$$\hat{\psi}_{tt} + |\xi|^2 B(\check{\xi})\hat{\psi} + a\hat{\psi}_t - i|\xi|b(\check{\xi})\hat{\psi} = \hat{h}. \quad (3.33)$$

For fixed $\xi \in \mathbb{R}^3$ write (3.33) as

$$w_{tt} + |\xi|^2 w + \tilde{a}w_t - i|\xi|\tilde{b}w = \hat{h}_1, \quad (3.34)$$

$$v_{tt} + \eta\sigma^{-1}|\xi|^2 v + \sigma^{-1}\bar{\theta}X_\theta v_t = \hat{h}_2, \quad (3.35)$$

where $\hat{\psi} = w + v$, $\hat{h} = \hat{h}_1 + \hat{h}_2$ with $w, \hat{h}_1 \in J_1(\xi)$ and $v, \hat{h}_2 \in J_2(\xi)$ (see Remark 3.2.1). Set $d = \sigma^{-1}\bar{\theta}X_\theta$ and take the scalar product of (3.35) with $v_t + (d/2)v$. The real part reads

$$\frac{1}{2}\frac{d}{dt}E^{(2)} + F^{(2)} = \Re\left\langle \hat{h}_2, v_t + \frac{d}{2}v \right\rangle, \tag{3.36}$$

where

$$E^{(2)} = |v_t|^2 + \frac{\eta}{\sigma}|\xi|^2|v|^2 + \frac{d^2}{2}|v|^2 + d\Re\langle v_t, v\rangle,$$

and

$$F^{(2)} = \frac{d}{2}|v_t|^2 + \frac{d\eta}{2\sigma}|\xi|^2|v|^2.$$

Since

$$|d\Re\langle v_t, v\rangle| \leq \frac{d^2}{3}|v|^2 + \frac{3}{4}|v_t|^2,$$

$E^{(2)}$ is uniformly equivalent to $E_0^{(2)} = |v_t|^2 + (1 + |\xi|^2)|v|^2$ and $F^2 \geq c(|v_t|^2 + |\xi|^2|v|^2)$. Thus integrating (3.36) leads to

$$\begin{aligned} C_1\left(|v_t|^2 + (1+|\xi|^2)|v|^2\right) + C_1\int_0^t |v_t|^2 + |\xi|^2|v|^2 d\tau \\ \leq C_2\left(|v_t(0)|^2 + (1+|\xi|^2)|v(0)|^2\right) + \int_0^t \Re\langle \hat{h}_2, v_t + \frac{d}{2}v\rangle d\tau. \end{aligned} \tag{3.37}$$

Next, take the scalar product of (3.34) with $w_t + (\tilde{a}/2)w$. The real part reads

$$\frac{1}{2}\frac{d}{dt}E^{(1)} + F^{(1)} = \Re\langle \hat{h}_1, w_t + \frac{1}{2}\tilde{a}w\rangle, \tag{3.38}$$

where

$$E^{(1)} = |w_t|^2 + |\xi|^2|w|^2 + \frac{1}{2}|\tilde{a}w|^2 + \Re\langle \tilde{a}w_t, w\rangle,$$

and

$$F^{(1)} = \frac{1}{2}\langle \tilde{a}w_t, w_t\rangle + \Re\langle -i|\xi|\tilde{b}w, w_t\rangle + \frac{1}{2}|\xi|^2\langle \tilde{a}w, w\rangle - \frac{1}{2}\Re\langle i|\xi|\tilde{b}w, \tilde{a}w\rangle.$$

Using Young's inequality it is easy to see that $E^{(1)}$ is uniformly equivalent to $|w_t|^2 + (1 + |\xi|^2)|w|^2$. Furthermore

$$F^{(1)} = \frac{1}{2}\langle DY, Y\rangle_{\mathbb{C}^4} - \frac{1}{2}\Re\langle i|\xi|\tilde{b}w, \tilde{a}w\rangle,$$

where

$$D = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{b} & \tilde{a} \end{pmatrix}$$

and $Y = (w_t, -i|\xi|w)$. As

$$\sigma(D) = \sigma(\tilde{a} + \tilde{b}) \cup \sigma(\tilde{a} - \tilde{b})$$

and $\tilde{a} \pm \tilde{b} > 0$ (see Remark 3.2.1), D is positive definite. Hence, there exist $c_1, c_2 > 0$ such that

$$F^{(1)} \geq c_1(|w_t|^2 + |\xi|^2|w|^2) - c_2|\xi||w||w| \geq \frac{c_1}{2}(|w_t|^2 + |\xi|^2|w|^2) - \frac{c_2^2}{2c_1}|w|^2.$$

Thus integrating (3.38) leads to

$$\begin{aligned} &C_1\left(|w_t|^2 + (1+|\xi|^2)|w|^2\right) + C_1\int_0^t |w_t|^2 + |\xi|^2|w|^2 d\tau \\ &\quad \leq C_2\left(|w_t(0)|^2 + (1+|\xi|^2)|w(0)|^2\right) + \int_0^t C_2|w|^2 + \Re\langle \hat{h}_1, w_t + \frac{\tilde{a}}{2}w\rangle d\tau. \end{aligned} \tag{3.39}$$

Adding (3.37) and (3.39) gives

$$\begin{aligned} &|\hat{\psi}_t|^2 + (1+|\xi|^2)|\hat{\psi}|^2 + \int_0^t |\hat{\psi}_t|^2 + |\xi|^2|\hat{\psi}|^2 d\tau \\ &\quad \leq C\left(|{}^1\hat{\psi}|^2 + (1+|\xi|^2)|{}^0\hat{\psi}|^2\right) + \int_0^t C|\hat{\psi}|^2 + c\Re\langle \hat{h}, \hat{\psi}_t + \frac{a}{2}\hat{\psi}\rangle d\tau. \end{aligned} \tag{3.40}$$

Finally the assertion follows by multiplying (3.40) with $\xi^{2\alpha}$ for $\alpha \in \mathbb{N}_0^n$, $0 \leq |\alpha| \leq s$, integrating with respect to ξ, and using Plancherel's identity. $\square$

3.3. Global existence and asymptotic decay

In this section we will prove that for initial values sufficiently close to a homogeneous reference state there exists a global solution to (3.4)-(3.6) which decays asymptotically to this state. The argumentation is similar to the one in Section 2.3: Based on Proposition 3.2.4 we can show a decay estimate for all but the highest order derivatives of a solution, Proposition 3.3.1. Then using Proposition 3.2.5 we prove an energy estimate for the derivatives of highest order, Proposition 3.3.3. Finally, combining these two, Proposition 3.3.4 leads to the main result, Theorem 3.3.5.

As in Section 3.2 fix $\bar{\theta} > 0$, $\bar{v}$ and $\bar{s}$. Then multiply (3.4) by $(A_{(1)})^{-1/2}$ and change the variables to $(A_{(1)})^{1/2}\psi$ such that the linearization at $\bar{\psi} = (\bar{\theta}^{-1}, 0, 0, 0, \bar{v}/\bar{\theta})$ is given by (3.8). In addition, consider $\psi - \bar{\psi}$ instead of

ψ, ${}^0\psi - \bar{\psi}$ instead of ${}^0\psi$, $A(\cdot + \bar{\psi})$ instead of $A(\cdot)$ and so on, such that the rest state is shifted from $\bar{\psi}$ to 0. In the following, when (3.3) or (3.4)-(3.6) are mentioned, we actually mean these modified equations which we be considered on a fixed time interval $[0, T]$.

We use the same notation as in Section 2.3: $U = (\psi, \psi_t)$ and $U_0 = ({}^0\psi, {}^1\psi)$ stand for a solution to (3.4)-(3.6) and the initial values, respectively. We always assume $s \geq s_0 + 1$ ($s_0 = [3/2] + 1$), $T > 0$, $U_0 \in H^{s+1} \times H^s$ and

$$\psi \in \bigcap_{j=0}^{s} C^j \left([0, T], H^{s+1-j} \right). \tag{3.41}$$

Again, for $0 \leq t \leq t_1 \leq T$ let

$$N_s(t, t_1)^2 = \sup_{\tau \in [t, t_1]} \|U(\tau)\|_{s+1,s}^2 + \int_t^{t_1} \|U(\tau)\|_{s+1,s}^2 d\tau.$$

and $N_s(T) \leq a_0$ for an $a_0 > 0$, which implies that $(\psi, \psi_t, \partial_x \psi)$ takes values in a closed ball $\overline{B(0, r)} \subset \mathbb{R}^5 \times \mathbb{R}^5 \times \mathbb{R}^{15}$ for some $r > 0$.

We can write (the modified version of) (3.3) as

$$\psi_{tt} - \bar{B}^{ij} \psi_{x_i x_j} + a\psi_t + b^j \psi_{x_j} = h(\psi, \psi_t, \partial_x \psi, \partial_x^2 \psi), \tag{3.42}$$

where

$$\begin{aligned} h(\psi, \psi_t, \partial_x \psi, \partial_x^2 \psi, \partial_x \psi_t) = & \left(A(\psi)^{-1} B^{ij}(\psi) - \bar{B}^{ij} \right) \psi_{x_i x_j} \\ & - A(\psi)^{-1} D^j(\psi) \psi_{t x_j} \\ & \quad - A(\psi)^{-1} f(\psi, \psi_t, \partial_x \psi) + a\psi_t + b^j \psi_{x_j}. \end{aligned}$$

Note that (3.42) is the same type of equation as in the case of a barotropic fluid (Section 2.3, (2.44)). In the proof of the decay estimate for barotropic fluids, Proposition 2.3.1, we only used that the linear operator on the left-hand-side of (2.44) satisfies decay estimate (2.22) in Proposition 2.2.1 and h goes as

$$h = O(|\psi|(|\partial_x^2 \psi| + |\partial_x \psi_t|) + |\psi_t|^2 + |\partial_x \psi|^2) \text{ for } \psi \to 0.$$

In (3.42) the linear operator on the left-hand-side satisfies decay estimate (3.27) in Proposition 3.2.3 and h has the same asymptotic behavior for $\psi \to 0$ as in (2.45). Thus, we get a decay estimate for (3.3) analogous to Proposition 2.3.1 (and Corollary 3.3.2 as a direct consequence).

3.3.1 Proposition. *There exist constants* $a_1(\leq a_0)$, $\delta_1 = \delta_1(a_1)$, $C_1 = C_1(a_1, \delta_1) > 0$ *such that the following holds: If* $\|U_0\|^2_{s,s-1,1} \leq \delta_1$ *and* $N_s(T)^2 \leq a_1$ *for a solution* ψ *of* (3.4)-(3.6) *satisfying* (3.41), *then*

$$\|U(t)\|_{s,s-1} \leq C_1(1+t)^{-\frac{3}{4}}\|U_0\|_{s,s-1,1} \quad (t \in [0,T]). \tag{3.43}$$

3.3.2 Corollary. *In the situation of Proposition 3.3.1 there exists a* $C_2 = C_2(a_1, \delta_1) > 0$ *such that the following holds: If* $\|U_0\|^2_{s,s-1,1} \leq \delta_1$ *and* $N_s(T)^2 \leq a_1$ *for a solution* ψ *of* (3.4)-(3.6) *satisfying* (3.41), *then*

$$N_{s-1}(T)^2 \leq C_2\|U_0\|^2_{s,s-1,1}. \tag{3.44}$$

Now it is convenient to write (3.4) as

$$\psi_{tt} - \bar{B}^{ij}\psi_{x_i x_j} + a\psi_t + b^j\psi_{x_j} = L(\psi)\psi + h_2(\psi, \psi_t, \partial_x\psi), \tag{3.45}$$

where

$$L(\psi)\psi = (I - A(\psi))\psi_{tt} - (\bar{B}^{ij} - B^{ij}(\psi))\psi_{x_i x_j} - D^j(\psi)\psi_{tx_j},$$

$$h_2(\psi, \psi_t, \partial_x\psi) = a\psi_t + b^j\psi_{x_j} - f(\psi, \psi_t, \partial_x\psi).$$

We get the following energy estimate.

3.3.3 Proposition. *There exist constants* $a_2(\leq a_0)$ *and* $c_3 = c_3(a_2)$, $C_3 = C_3(a_2) > 0$ *such that the following holds: If* $N_s(T)^2 \leq a_2$ *for a solution* ψ *of* (3.4)-(3.6) *satisfying* (3.41), *then for all integers* $(0 \leq k \leq s)$,

$$\begin{aligned}\|\partial_x^k\psi(t)\|_1^2 + \|\partial_x^k\psi_t(t)\|^2 + \int_0^t \|\partial_x^{k+1}\psi(\tau)\|^2 + \|\partial_x^k\psi_t(\tau)\|^2 d\tau \\ - c_3\int_0^t \|\partial_x^k\psi(\tau)\|^2 d\tau \leq C_3\left(\|U_0\|^2_{s,s+1} + N_s(t)^3\right) \quad (t \in [0,T]).\end{aligned} \tag{3.46}$$

Proof. The proof is analogous to the one of Proposition 2.3.3. Hence we will only outline it here. Let $0 \leq k \leq s$. Due to Proposition 3.2.5 a solution

$$\psi \in \bigcap_{j=0}^{s} C^j\left([0,T], H^{s+2-j}\right).$$

of (3.45) satisfies

$$\begin{aligned}
&\|\partial_x^k\psi(t)\|_1^2 + \|\partial_x^k\psi_t(t)\|^2 + \int_0^t \|\partial_x^{k+1}\psi(\tau)\|^2 + \|\partial_x^k\psi_t(\tau)\|^2 d\tau \\
&\qquad \le C\left(\|\partial_x^k({}^0\psi)\|_1^2 + \|\partial_x^k({}^1\psi)\|^2\right) + C\int_0^t \|\partial_x^k\psi(\tau)\|^2 d\tau \\
&\qquad\quad + C\left|\int_0^t \left(\partial_x^k(L(\psi(\tau)\psi(\tau) + h_2(\tau)), \partial_x^k\psi_t(\tau)\right)_{L^2} d\tau\right| \\
&\qquad\qquad + C\left|\int_0^t \left(\partial_x^k(L(\psi(\tau)\psi(\tau) + h_2(\tau)), \partial_x^k\psi(\tau)\right)_{L^2} d\tau\right|,
\end{aligned}$$

As in the proof of Proposition 2.3.3 it can be shown that (for $N_s(T)$ sufficiently small)

$$\begin{aligned}
&\left|\int_0^t \left(\partial_x^k(L(\psi(\tau))\psi(\tau) + h_2(\tau)), \partial_x^k\psi_t(\tau)\right)_{L^2} d\tau\right| \\
&\qquad + \left|\int_0^t \left(\partial_x^k(L(\psi(\tau))\psi(\tau) + h_2(\tau)), \partial_x^k\psi(\tau)\right)_{L^2} d\tau\right| \\
&\qquad \le C\|U_0\|_{s+1,s}^2 + C\int_0^t \|h_2\|_s\|U\|_{s+1,s} + R_1(\psi)\|U\|_{s+1,s}^2 d\tau \\
&\qquad\quad + C\int_0^t \|I - A(\psi)\|_s\|\psi_{tt}\|_{s-1}\|U\|_{s+1,s} d\tau \\
&\qquad\qquad + CR_2(\psi)\|U(t)\|_{s+1,s}^2,
\end{aligned}$$

where

$$\begin{aligned}
R_1(\psi) = \|\partial_t A(\psi)\|_s + \|I - A(\psi)\|_s + \|\partial_t B^{ij}(\psi)\|_s \\
+ \|\bar{B}^{ij} - B^{ij}(\psi)\|_s, + \|D^j(\psi)\|_s,
\end{aligned}$$

and

$$R_2(\psi) = \|I - A(\psi)\|_s + \|\bar{B}^{ij} - B^{ij}(\psi)\|_s.$$

Furthermore

$$\|h_2\|_s \le C\|U\|_{s+1,s}^2,$$

$$R_1(\psi) + R_2(\psi) \le C\|U\|_{s+1,s},$$

and

$$\|\psi_{tt}\|_{s-1} \le C(\|\partial_x^2\psi\|_{s-1} + \|\partial_x\psi_t\|_{s-1} + \|f(\psi, \psi_t, \partial_x\psi)\|_{s-1}) \le C\|U\|_{s+1,s},$$

hold for $N_s(T)$ sufficiently small. Therefore

$$\|\partial_x^k \psi(t)\|_1^2 + \|\partial_x^k \psi_t(t)\|^2 + \int_0^t \|\partial_x^k \partial_x \psi(\tau)\|^2 + \|\partial_x^k \psi_t(\tau)\|^2 d\tau$$
$$\leq C\|U_0\|_{s+1,s}^2 + C\int_0^t \|\partial_x^k \psi(\tau)\|^2 d\tau$$
$$+ C\|U(t)\|_{s+1,s}^3 + C\int_0^t \|U(\tau)\|_{s+1,s}^3 d\tau.$$

Estimate (3.46) is an immediate consequence of this inequality. Lastly, for

$$\psi \in \bigcap_{j=0}^{s} C^j\left([0,T], H^{s+1-j}\right).$$

the assertion follows by applying results on the Friedrichs mollifier as in step 2 of the proof of Proposition 2.3.3. □

3.3.4 Proposition. *In the situation of Proposition 3.3.1 there exist constants $a_3(\leq \min\{a_2, a_1\})$, $C_4 = C_4(a_3, \delta_1) > 0$ (δ_1 being the constant in Proposition 3.3.1) such that the the following holds: If $N_s(T)^2 \leq a_3$ and $\|U_0\|_{s,s-1,1}^2 \leq \delta_1$ for a solution ψ of* (3.4)-(3.6) *satisfying* (3.41)*, then*

$$N_s(t)^2 \leq C_4^2 \|U_0\|_{s+1,s,1}^2 \quad (t \in [0,T]). \tag{3.47}$$

Proof. This follows directly by adding (3.44)+ε(3.46) for ε sufficiently small. □

We can readily state the main Theorem. Based on Propositions 3.3.1 and 3.3.4 its proof goes as the one of Theorem 2.1.1.

3.3.5 Theorem. *Let $s \geq 3$. Then there exist $\delta_0 > 0$, $C_0 = C_0(\delta) > 0$ such that for all initial data $({}^0\psi, {}^1\psi) \in (H^{s+1} \times H^s) \cap (L^1 \times L^1)$ satisfying $\|({}^0\psi, {}^1\psi)\|_{s+1,s,1}^2 < \delta_0$ there exists a unique solution*

$$\psi \in \bigcap_{j=1}^{s} C^j\left([0,\infty), H^{s+1-j}\right).$$

of the Cauchy problem (3.4)-(3.6)*. Furthermore ψ satisfies the estimates*

$$\|(\psi(t), \psi_t(t))\|_{s+1,s}^2 + \int_0^t \|(\psi(\tau), \psi_t(\tau))\|_{s+1,s}^2 d\tau$$
$$\leq C_0 \|({}^0\psi, {}^1\psi)\|_{s+1,s}^2, \tag{3.48}$$

$$\|(\psi(t), \psi_t(t))\|_{s,s-1} \leq C_0(1+t)^{-\frac{3}{4}} \|({}^0\psi, {}^1\psi)\|_{s,s-1,1} \tag{3.49}$$

for all $t \in [0,\infty)$.

4. Two extensions for barotropic fluids

4.1. Application to relativistic electro-magneto-fluid dynamics with dissipation

In this chapter we will show that the results of Sections 2.2 and 2.3 for the NSF equations of viscous and heat-conductive barotropic fluids extend to the equations of relativistic electro-magneto-fluid dynamics, at least in the quasi-neutral approximation. In addition to the 4-velocity u^α and thermodynamic state variables, the relativistic dynamics of charged fluids is described by the electromagnetic field tensor $F^{\alpha\beta}$ which in some Lorentz frame is given by

$$\begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & B_1 & 0 \end{pmatrix},$$

with $E = (E_1, E_2, E_3)$ and $B = (B_1, B_2, B_3)$ the electric and magnetic field. For barotropic fluids the equations [56] are generally given as

$$\frac{\partial}{\partial x^\beta}\left(T^{\alpha\beta} + \Delta T^{\alpha\beta}\right) = F^{\alpha\gamma} J_\gamma, \tag{4.1}$$

$$\frac{\partial F^{\alpha\beta}}{\partial x^\beta} = J^\alpha, \tag{4.2}$$

$$\frac{\partial F^{\beta\gamma}}{\partial x_\alpha} + \frac{\partial F^{\gamma\alpha}}{\partial x_\beta} + \frac{\partial F^{\alpha\beta}}{\partial x_\gamma} = 0, \tag{4.3}$$

$$\frac{\partial J^\alpha}{\partial x^\alpha} = 0, \tag{4.4}$$

with energy-momentum tensor $T^{\alpha\beta}$, dissipation tensor $\Delta T^{\alpha\beta}$ (cf. Propositions 1.2.1 and 1.3.1) and 4-current

$$J^\alpha = qu^\alpha + \sigma u_\beta F^{\alpha\beta},$$

q being the charge and $\sigma > 0$ the electrical conductivity [26].

As stated above we will only consider the quasi-neutral case $q = 0$. In this case (4.1)-(4.3) constitute a self-consistent system of partial differential equations, which allows us to drop (4.4). Note that quasineutrality [26] is an approximation of limited validity, as (4.4) will not general hold for solutions of (4.1)-(4.3) [25, 34].

Now, (4.1)-(4.3) can equivalently be written as the second-order system [14]

$$\frac{\partial}{\partial x^\beta}\left(T^{\alpha\beta} + \Delta T^{\alpha\beta} + F^\alpha_\gamma F^{\beta\gamma} - \frac{1}{4}F_{\gamma\delta}F^{\gamma\delta}\right) = 0,$$
$$g^{\gamma\delta}\frac{\partial^2 F_{\alpha\beta}}{\partial x^\gamma \partial x^\delta} = -\frac{\partial}{\partial x^\alpha}(\sigma u^\mu F_{\beta\mu}) + \frac{\partial}{\partial x^\beta}(\sigma u^\nu F_{\alpha\nu}).$$

Using Godunov variables $\psi^\alpha = u^\alpha/\theta$ to describe the behavior of the fluid, we can rewrite these equations as

$$A\psi_{tt} - B^{ij}\psi_{x_i x_j} + D^j\psi_{tx_j} + f + g_1 = 0, \tag{4.5}$$
$$E_{tt} - \Delta E + g_2 = 0, \tag{4.6}$$
$$B_{tt} - \Delta B + g_3 = 0, \tag{4.7}$$

where A, B^{ij}, D^{ij} and f are given by (2.4) and (2.5), and

$$g_1^\alpha = \frac{\partial}{\partial x^\beta}\left(F^\alpha_\gamma F^{\beta\gamma} - \frac{1}{4}F_{\gamma\delta}F^{\gamma\delta}\right),$$
$$g_2 = \frac{\partial}{\partial t}\Big(\sigma(u^0 E + \mathbf{u} \times B)\Big) + \nabla(\sigma u \cdot E),$$
$$g_3 = -\nabla \times \Big(\sigma(u^0 E + \mathbf{u} \times B)\Big),$$

or as

$$\mathcal{A}(U)U_{tt} - \mathcal{B}^{ij}(U)U_{x_i x_j} + \mathcal{D}^j(U)U_{tx_j} + \mathfrak{f}(U, U_t, \partial_x U) = 0, \tag{4.8}$$

where $U = (\psi, E, B)$,

$$\mathcal{A}(U) = \begin{pmatrix} A(\psi) & 0 \\ 0 & I_6 \end{pmatrix}, \quad \mathcal{B}^{ij} = \begin{pmatrix} B^{ij}(\psi) & 0 \\ 0 & I_6\delta^{ij} \end{pmatrix}, \quad \mathcal{D}^j = \begin{pmatrix} D^j(\psi) & 0 \\ 0 & 0 \end{pmatrix},$$

and $\mathfrak{f} = (f + g_1, g_2, g_3)^t$. It is obvious that as the differential operator

$$A\psi_{tt} - B^{ij}\psi_{x_i x_j} + D^j\psi_{tx_j}$$

is HKM hyperbolic sufficiently close to a homogeneous reference state of the fluid, so is the operator

$$\mathcal{L}U = \mathcal{A}U_{tt} - \mathcal{B}^{ij}U_{x_i x_j} + \mathcal{D}^j U_{tx_j}.$$

Hence (4.8) is locally well-posed due to [24]. We consider the Cauchy problem associated to (4.8)

$$\mathcal{A}U_{tt} - \mathcal{B}^{ij}U_{x_ix_j} + \mathcal{D}^j U_{tx_j} + \mathfrak{f} = 0, \tag{4.9}$$

$$(U(0), U_t(0)) = ({}^0U, {}^1U) \tag{4.10}$$

on a time interval $[0,T]$. As stated above the goal of this section is to show global existence and asymptotic decay for (4.9)-(4.10) close to a homogeneous reference state $\bar{U} = (\bar{\psi}, 0, 0)$, where $\bar{\psi} = (\bar{\theta}^{-1}, 0, 0, 0)$ for $\bar{\theta} > 0$ fixed.

To this end, we first need to rewrite g_3. By the product rule

$$g_3 = -\sigma u^0(\nabla \times E) + E \times (\nabla(\sigma u^0)) - \nabla \times (\sigma \mathbf{u} \times B).$$

Furthermore (4.3) yields

$$\nabla \times E = -B_t$$

and thus

$$g_3 = \sigma u^0 B_t + E \times \nabla(\sigma u^0) - \nabla \times (\sigma \mathbf{u} \times B). \tag{4.11}$$

Next, fix $\bar{\theta} > 0$ and consider the linearization of (4.8) at $\bar{U} = (\bar{\psi}, 0, 0, 0)$. Using (4.11) we get:

$$A_{(1)}\psi_{tt} - B^{ij}_{(1)}\psi_{x_ix_j} + a_{(1)}\psi_t + b^j_{(1)}\psi_{x_j} = 0, \tag{4.12}$$

$$V_{tt} - \Delta V + \sigma V_t = 0, \tag{4.13}$$

where $V = (E, B)$, $A_{(1)}, B^{ij}_{(1)}$, and $a_{(1)}, b^j_{(1)}$ are given by (2.12),(2.13) and (2.14). Clearly (4.12) and (4.13) constitute two self-consistent systems for ψ and (E, B) respectively and can therefore be treated independently.

(4.12) is the linearization of the NSF-equations for barotropic fluids - i.e. equation (2.11). Hence it can equivalently be written as

$$\psi_{tt} - \bar{B}^{ij}\psi_{x_ix_j} + a\psi_t + b^j\psi_{x_j} = 0, \tag{4.14}$$

where $\bar{B}^{ij}$ and a, b are given by (2.17) and (2.18). Furthermore each solution (ψ, ψ_t) to the Cauchy problem associated to (4.14) satisfies Proposition 2.2.1. It is easy to see - in fact we showed this in the proof of Lemma 2.2.2 - that also solutions (V, V_t) to the Cauchy problem associated to (4.13) satisfy a decay estimate as in Proposition 2.2.1. Thus we can formulate an analogous decay estimate for the Cauchy problem associated to the full system (4.12)-(4.13), which we rewrite as

$$U_{tt} - \bar{\mathcal{B}}^{ij}U_{x_ix_j} + \mathfrak{a}U_t + \mathfrak{b}^j U_{x_j} = 0, \tag{4.15}$$

$$(U(0), U_t(0)) = ({}^0U, {}^1U), \tag{4.16}$$

where $U = (\psi, E, B)$,

$$\bar{\mathcal{B}}^{ij} = \begin{pmatrix} \bar{B}^{ij} & 0 \\ 0 & I_6\delta^{ij} \end{pmatrix},$$

and

$$\mathfrak{a} = \begin{pmatrix} a & 0 \\ 0 & \sigma I_6 \end{pmatrix}, \quad \mathfrak{b}^j = \begin{pmatrix} b^j & 0 \\ 0 & 0 \end{pmatrix}.$$

4.1.1 Proposition. *For some $s \in \mathbb{N}_0$ let $({}^0U, {}^1U) \in (H^{s+1} \times H^s) \cap (L^1 \times L^1)$ and $(U(t), U_t(t)) \in H^{s+1} \times H^s$ be a solution of* (4.15)-(4.16). *Then there exist $c, C > 0$ such that for all integers $0 \le k \le s$ and all $t \in [0, T]$*

$$\begin{aligned} \|\partial_x^k U(t)\|_1 + \|\partial_x^k U_t(t)\| \le C(1+t)^{-\frac{3}{4}-\frac{k}{2}} \left(\|{}^0U\|_{L^1} + \|{}^1U\|_{L^1}\right) \\ + Ce^{-ct}\left(\|\partial_x^k({}^0U)\|_1 + \|\partial_x^k({}^1U)\|\right). \end{aligned} \tag{4.17}$$

In the same way we can argue that solutions to the inhomogeneous Cauchy problem

$$U_{tt} - \bar{\mathcal{B}}^{ij} U_{x_i x_j} + \mathfrak{a} U_t + \mathfrak{b}^j U_{x_j} = \mathfrak{h}, \tag{4.18}$$

$$(U(0), U_t(0)) = ({}^0U, {}^1U), \tag{4.19}$$

(for some $h : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^{10}$) satisfy energy estimates analogous to (2.40), (2.41).

4.1.2 Proposition. *Let s be a non-negative integer, $({}^0U, {}^1U) \in (H^{s+1} \times H^s) \cap (L^1 \times L^1)$ and $h \in C([0,T], H^s \cap L^1)$. Then the solution U of* (4.18)-(4.19) *satisfies*

$$\begin{aligned} \|\partial_x^k U(t)\|_1 + \|\partial_x^k U_t(t)\| \le C(1+t)^{-\frac{3}{4}-\frac{k}{2}}(\|{}^0U\|_{L^1} + \|{}^1U\|_{L^1}) \\ + Ce^{-ct}(\|\partial_x^k({}^0U)\|_1 + \|\partial_x^k({}^1U)\|) \\ + C\int_0^t (1+t-\tau)^{-\frac{3}{4}-\frac{k}{2}} \|h(\tau)\|_{L^1} \\ + C\exp(-c(t-\tau))\|\partial_x^k h(\tau)\| d\tau \end{aligned} \tag{4.20}$$

for all $t \in [0,T]$ and $0 \le k \le s$.

4.1.3 Proposition. *Let s be a non-negative integer, $({}^0U, {}^1U) \in H^{s+1} \times H^s$ and $h \in C([0,T], H^s)$. Then the solution U of (4.18)-(4.19) satisfies*

$$\begin{aligned} \|\partial_x^k U(t)\|_1^2 + \|\partial_x^k U_t(t)\|^2 + \int_0^t \|\partial_x^{k+1} U(\tau)\|^2 + \|\partial_x^k U_t(\tau)\|^2 d\tau \\ \leq C(\|\partial_x^k({}^0U)\|_1^2 + \|\partial_x^k({}^1U)\|^2) \\ + C \left| \int_0^t \left(\partial_x^k h(\tau), \partial_x^k U_t(\tau)\right)_{L^2} d\tau \right| \\ C \left| \int_0^t \left(\partial_x^k h(\tau), \partial_x^k U(\tau)\right)_{L^2} d\tau \right| \end{aligned}$$

for all $t \in [0,T]$ and integers $0 \leq k \leq s$.

These properties of the linearization together with the fact that (4.8) is HKM hyperbolic are sufficient to show global existence and asymptotic decay of solutions close to a homogeneous resference state. This can be seen as follows: In Section 2.3 the proof of Proposition 2.3.1 for barotropic fluids did not make use of the particular form of the coefficent matrices A, B_{ij}, D_j or the lower order terms f in (2.3). We just wrote (the modified version of) (2.3) as

$$\psi_{tt} - \bar{B}^{ij} \psi_{x_i x_j} + a\psi_t + b^j \psi_{x_j} = h(\psi, \psi_t, \partial_x \psi, \partial_x^2 \psi, \partial_x \psi_t),$$

and applied Proposition 2.2.3. In the same way we write (the modified version of) (4.8) as

$$U_{tt} - \bar{\mathcal{B}}^{ij} U_{x_i x_j} + \mathfrak{a} U_t + \sum_{j=1}^{3} \mathfrak{b}^j U_{x_j} = \mathfrak{h}(U, U_t, \partial_x U, \partial_x^2 U, \partial_x U_t), \qquad (4.21)$$

where

$$\begin{aligned} \mathfrak{h}(U, U_t, \partial_x U, \partial_x^2 U, \partial_x U_t) = \left((\mathcal{A}(U))^{-1} \mathcal{B}^{ij}(U) - \bar{\mathcal{B}}^{ij}\right) U_{x_i x_j} \\ - (\mathcal{A}(U))^{-1} \mathcal{D}^j(U) \psi_{t x_j} \\ - (\mathcal{A}(U))^{-1} \mathfrak{f}(U, U_t, \partial_x U) + \mathfrak{a} U_t + \mathfrak{b}^j U_{x_j}. \end{aligned}$$

Then applying Proposition 4.1.2 shows that Proposition 2.3.1 also holds for solutions U of (4.9)-(4.10) and the same is true for Corollary 2.3.2. In similar fashion, Proposition 4.1.3 yields that solutions to (4.9)-(4.10) satisfy Proposition 2.3.3. As shown in the proof of Theorem 2.1.1 these Propositions suffice to expand local solutions of (4.9)-(4.10) - that exist due to HKM hyperbolicitiy - globally and prove that they decay asymptotically in time. In conclusion we get the following result.

4.1.4 Theorem. *Let $s \geq 3$, $\bar{\psi} = (\bar{\theta}^{-1}, 0, 0, 0,)^t$ with a constant temperature $\bar{\theta} > 0$ and*

$$\bar{U} = (\bar{\psi}, 0, 0)^t \in \mathbb{R}^4 \times \mathbb{R}^3 \times \mathbb{R}^3.$$

Then there exist $\delta_0 > 0$, $C_0 = C_0(\delta_0) > 0$, such that for all initial data $({}^0U, {}^1U) \in (H^{s+1} \times H^s) \cap (L^1 \times L^1)$ satisfying $\|({}^0U - \bar{U}, {}^1U)\|^2_{s+1,s,1} < \delta_0$ there exists a unique solution U of the Cauchy problem (4.9)-(4.10) such that

$$U - \bar{U} \in \bigcap_{j=0}^{s} C^j\left([0,\infty), H^{s+1-j}\right).$$

U satisfies the estimates

$$\|(U(t) - \bar{U}, U_t(t)\|^2_{s+1,s} + \int_0^t \|(U(\tau) - \bar{U}, U_t(\tau))\|^2_{s+1,s} d\tau \\ \leq C_0 \|({}^0U - \bar{U}, {}^1U)\|^2_{s+1,s}, \quad (4.22)$$

$$\|(U(t) - \bar{\psi}, U_t(t))\|_{s,s-1} \leq C_0 (1+t)^{-\frac{3}{4}} \|({}^0U - \bar{U}, {}^1U)\|_{s,s-1,1} \quad (4.23)$$

for all $t \in [0, \infty)$.

4.2. On the vanishing-heat-conduction limit

In this section, we return to barotropic fluids without electromagnetic fields. It would be interesting to extend the theory of Section 2 to the "purely viscous" case, i.e., fluids that have viscosity but no heat conduction, by sending the coefficient χ of heat conduction to zero. However, this limit is quite singular. The purpose of this chapter is to provide a first step in that direction by establishing results for the linearized equations at a homogeneous reference state.

We consider the Cauchy problem associated to the linearization of the NSF equations for barotropic fluids with heat conduction given by (2.15), i.e.,

$$\chi(\psi^0_{tt} - \Delta\psi^0) + c_s^{-2}\psi^0_t + \nabla \cdot \boldsymbol{\psi} = 0 \quad (4.24)$$

$$\bar{\sigma}\boldsymbol{\psi}_{tt} - \bar{\eta}\Delta\boldsymbol{\psi} - (\bar{\zeta} + \frac{1}{3}\bar{\eta})\nabla(\nabla \cdot \boldsymbol{\psi}) + \boldsymbol{\psi}_t + \nabla\psi^0 = 0, \quad (4.25)$$

$$\psi(0) = {}^0\psi, \quad \psi_t(0) = {}^1\psi, \quad (4.26)$$

- we write χ instead of $\bar{\chi}$ and set $\psi = (\psi^0, \boldsymbol{\psi}) \in \mathbb{R} \times \mathbb{R}^3$ for notational purposes - and without heat conduction

$$c_s^{-2}\psi_t^0 + \nabla \cdot \boldsymbol{\psi} = 0 \tag{4.27}$$

$$\bar{\sigma}\boldsymbol{\psi}_{tt} - \bar{\eta}\Delta\boldsymbol{\psi} - (\bar{\zeta} + \frac{1}{3}\bar{\eta})\nabla(\nabla \cdot \boldsymbol{\psi}) + \boldsymbol{\psi}_t + \nabla\psi^0 = 0, \tag{4.28}$$

$$\psi(0) = {}^0\psi, \quad \boldsymbol{\psi}_t(0) = {}^1\boldsymbol{\psi}. \tag{4.29}$$

The procedure in this chapter is the following: First we prove that the decay estimate for solutions to (4.24)-(4.26) in Proposition 2.2.1 can be obtained uniformly in χ (for sufficiently small χ), Proposition 4.2.1. Then it is shown that solutions to (4.27)-(4.29) satisfy the same decay estimate, Proposition 4.2.3. Finally we use these results to conclude that (for compatible initial values) the solution of (4.24)-(4.26) converges, as $\chi \downarrow 0$, to the one of (4.27)-(4.29) in H^k (for some $k \in \mathbb{N}_0$), Proposition 4.2.5.

a) Uniformity in $\chi \downarrow 0$

4.2.1 Proposition. *Fix an integer $s \in \mathbb{N}$. There exist $\chi_0 > 0$ and $C_1(\chi_0), c_1(\chi_0) > 0$ such that the following holds: Let $({}^0\psi, {}^1\psi) \in (H^{s+1} \times H^s) \cap (L^1 \times L^1)$ and for $0 < \chi \leq \chi_0$ let $(\psi(t), \psi_t(t)) \in H^{s+1} \times H^s$ be the solution of (4.24)-(4.26). Then for all integers $0 \leq k \leq s$ and all $t \in [0, \infty)$*

$$\begin{aligned}\|\partial_x^k\psi(t)\|_1 + \|\partial_x^k\psi_t(t)\| \leq C_1(1+t)^{-\frac{3}{4}-\frac{k}{2}}\left(\|{}^0\psi\|_{L^1} + \|{}^1\psi\|_{L^1}\right)&\\ + C_1 e^{-c_1 t}\left(\|\partial_x^k({}^0\psi)\|_1 + \|\partial_x^k({}^1\psi)\|\right).&\end{aligned} \tag{4.30}$$

As in Section 2.2, we consider the Fourier transform of (4.24)-(4.26):

$$\chi(\hat{\psi}_{tt}^0 + |\xi|^2) + c_s^{-2}\hat{\psi}_t^0 - i\xi \cdot \hat{\boldsymbol{\psi}} = 0 \tag{4.31}$$

$$\bar{\sigma}\hat{\boldsymbol{\psi}}_{tt} + \bar{\eta}|\xi|^2\hat{\boldsymbol{\psi}} + (\bar{\zeta} + \frac{1}{3}\bar{\eta})(\xi \otimes \xi)\hat{\boldsymbol{\psi}} + \hat{\boldsymbol{\psi}}_t - i\xi\hat{\psi}^0 = 0, \tag{4.32}$$

$$\hat{\psi}(0) = {}^0\hat{\psi}, \quad \hat{\psi}_t(0) = {}^1\hat{\psi}. \tag{4.33}$$

We get the following pointwise decay estimate from which we can deduce Proposition 4.2.1 by virtue of [7, Proof of Theorem 3.1].

4.2.2 Lemma. *Fix an integer $s \in \mathbb{N}$. There exist $\chi_0 > 0$ and $C_1 = C_1(\chi_0)$, $c_1 = c_1(\chi_0) > 0$ such that the following holds: Let $({}^0\psi, {}^1\psi) \in H^{s+1} \times H^s$*

and for $0 < \chi < \chi_0$ *let* $(\psi(t), \psi_t(t)) \in H^{s+1} \times H^s$ *be the a solution of* (4.24)-(4.25). *Then for all* $(t, \xi) \in [0, \infty) \times \mathbb{R}^3$,

$$(1 + |\xi|^2)|\hat{\psi}(t, \xi)|^2 + |\hat{\psi}_t(t, \xi)|^2 \\ \leq C \exp(-c\rho(\xi)t) \left((1 + |\xi|^2)|^0\hat{\psi}(\xi)|^2 + |^1\hat{\psi}(\xi)|^2\right). \quad (4.34)$$

Proof. W.l.o.g. assume $\xi = (|\xi|, 0, 0)$ (otherwise rotate the coordinate system). For simplicity of notation we set $\hat{\psi}^0 = u$, $\hat{\psi}^1 = v$ and $(\hat{\psi}^2, \hat{\psi}^3) = w$. Since $(4/3)\bar{\eta} + \bar{\zeta} = \bar{\sigma}$, (4.31)-(4.32) decomposes into the two uncoupled systems

$$\chi(u_{tt} + |\xi|^2 u) + c_s^{-2} u_t - i|\xi|v = 0, \quad (4.35)$$

$$\bar{\sigma}(v_{tt} + |\xi|^2 v) + v_t - i|\xi|u^0 = 0, \quad (4.36)$$

and

$$\bar{\sigma} w_{tt} + \bar{\eta}|\xi|^2 w + w_t = 0. \quad (4.37)$$

We have already shown in the proof of Lemma 2.2.2 that solutions to (4.37) satisfy an estimate of the form (4.34). Hence it is sufficient to study (4.35)-(4.36). The real part of

$$\chi^{-1} c_s^{-2} \bar{u}_t (4.35) + \bar{\sigma}^{-1} \bar{v}_t (4.36)$$

is given by

$$\frac{1}{2}\frac{d}{dt}\left(c_s^{-2}(|u_t|^2 + |\xi|^2|u|^2) + |v_t|^2 + |\xi|^2|v|^2\right) \\ + c_s^{-4}\chi^{-1}|u_t|^2 + \bar{\sigma}^{-1}|v_t|^2 - \Re\left(i|\xi|\left(c_s^{-2}\chi^{-1} v\bar{u}_t + \bar{\sigma}^{-1} u\bar{v}_t\right)\right) = 0. \quad (4.38)$$

Taking the real part of

$$\chi^{-1} i|\xi|\bar{v}(4.35) + \bar{\sigma}^{-1} i|\xi|\bar{u}(4.36),$$

yields

$$\frac{d}{dt}\left(\Re(i|\xi|(u_t\bar{v} + v_t\bar{u}))\right) + \Re(i|\xi|(c_s^{-2}\chi^{-1} u_t\bar{v} + \bar{\sigma}^{-1} v_t\bar{u})) \\ + |\xi|^2(\chi^{-1}|v|^2 + \bar{\sigma}^{-1}|u|^2) = 0. \quad (4.39)$$

And the real part of

$$\bar{u}(4.35) + \bar{v}(4.36)$$

reads

$$\frac{1}{2}\frac{d}{dt}\left(c_s^{-2}|u|^2 + |v|^2 + 2\Re(\chi u_t\bar{u} + \bar{\sigma}v_t\bar{v})\right)$$
$$- \chi|u_t|^2 - \bar{\sigma}|v_t|^2 + |\xi|^2(\chi|u|^2 + \bar{\sigma}|v|^2) = 0. \quad (4.40)$$

We add (4.38)+(4.39)+α(4.40) (for some $\alpha > 0$ to be determined later) and get

$$\frac{1}{2}\frac{d}{dt}\tilde{E}^{(1)} + \tilde{F}^{(1)} = 0, \quad (4.41)$$

where

$$\tilde{E}^{(1)} = c_s^{-2}(|u_t|^2 + |\xi|^2|u|^2) + |v_t|^2 + |\xi|^2|v|^2 + 2\Re(i|\xi|(u_t\bar{v} + v_t\bar{u}))$$
$$+ \alpha\left(c_s^{-2}|u|^2 + |v|^2 + 2\Re(\chi u_t\bar{u} + \bar{\sigma}v_t\bar{v})\right)$$

and

$$\tilde{F}^{(1)} = (c_s^{-4}\chi^{-1} - \alpha\chi)|u_t|^2 + (\bar{\sigma}^{-1} - \alpha\bar{\sigma})|v_t|^2$$
$$- 2\Re\left(i|\xi|\left(c_s^{-2}\chi^{-1}v\bar{u}_t + \bar{\sigma}^{-1}u\bar{v}_t\right)\right)$$
$$+ |\xi|^2(\chi^{-1} + \alpha\bar{\sigma})|v|^2 + |\xi|^2(\bar{\sigma}^{-1} + \alpha\chi)|u|^2.$$

Now use

$$\frac{d}{dt}\Re(i|\xi|v\bar{u}) + \Re(i|\xi|\bar{v}_t u) - \Re(i|\xi|v\bar{u}_t) = 0,$$

set $d = (c_s^{-2} + 1)\bar{\sigma}/2$, and write (4.41) as

$$\frac{1}{2}\frac{d}{dt}E^{(1)} + F^{(1)} = 0, \quad (4.42)$$

where

$$E^{(1)} = \tilde{E}^{(1)} + 2\alpha d\Re(i|\xi|v\bar{u}),$$
$$F^{(1)} = \tilde{F}^{(1)} + \alpha d\Re\left(i|\xi|\left(\bar{v}_t u - v\bar{u}_t\right)\right).$$

Next, write $F^{(1)}$ as

$$F^{(1)} = \chi^{-1}\langle A^1(u_t, -i|\xi|v), (u_t, -i|\xi|v)\rangle$$
$$+ \langle A^2(v_t, -i|\xi|u), (v_t, -i|\xi|u)\rangle,$$

where

$$A^1 = \begin{pmatrix} a_{11} & b_1 \\ b_1 & a_{12} \end{pmatrix} = \begin{pmatrix} (c_s^{-4} - \alpha\chi^2) & c_s^{-2} + \chi\frac{\alpha d}{2} \\ c_s^{-2} + \chi\frac{\alpha d}{2} & 1 + \alpha\chi\bar{\sigma} \end{pmatrix},$$

and

$$A^2 = \begin{pmatrix} a_{21} & b_2 \\ b_2 & a_{22} \end{pmatrix} = \begin{pmatrix} \bar{\sigma}^{-1} - \alpha\bar{\sigma} & \bar{\sigma}^{-1} - \frac{\alpha d}{2} \\ \bar{\sigma}^{-1} - \frac{\alpha d}{2} & \bar{\sigma}^{-1} + \alpha\chi \end{pmatrix}.$$

We have

$$\det(A^1) = \alpha\chi\left(\frac{1}{2}\bar{\sigma}c_s^{-2}(c_s^{-2}-1) - \chi\right) - \alpha^2\chi^2\left(\chi\bar{\sigma} + \frac{d^2}{4}\right)$$

and

$$\det(A^2) = \alpha\left(\frac{1}{2}(c_s^{-2}-1) + \chi\bar{\sigma}^{-1}\right) - \alpha^2\left(\chi\bar{\sigma} + \frac{d^2}{4}\right).$$

Therefore (recall that $c_s < 1$) there exist $\chi_0 > 0$ and $\alpha_0 = \alpha_0(\chi_0, \bar{\sigma}) > 0$ and $c = c(\chi_0, \alpha_0, \bar{\sigma})$ such that for all $0 < \chi < \chi_0$, $0 < \alpha < \alpha_0$

$$\det(A^1) \geq c\alpha\chi, \quad \det(A^2) \geq c\alpha.$$

In particular A^1 and A^2 are positive definite. Furthermore, using

$$b_1 = (a_{11}a_{12} - \det(A^1))^{\frac{1}{2}},$$

gives

$$\begin{aligned}\langle A^1(u_t, -i|\xi|v), (u_t, -i|\xi|v)\rangle &= a_{11}|u_t|^2 - 2b_1\Re(i|\xi|\bar{u}_t v) + a_{12}|\xi|^2|v|^2 \\ &\geq \left(1 - b_1(a_{11}a_{12})^{-\frac{1}{2}}\right)(a_{11}|u_t|^2 + a_{12}|\xi|^2|v|^2) \\ &= \left(1 - \left(1 - \det(A^1)(a_{11}a_{12})^{-1}\right)^{\frac{1}{2}}\right)(a_{11}|u_t|^2 + a_{12}|\xi|^2|v|^2) \\ &\geq \frac{1}{2}\det(A^1)\left(a_{11}^{-1}|u_t|^2 + a_{12}^{-1}|\xi|^2|v|^2\right) \\ &\geq c(\alpha_0, \chi_0, \bar{\sigma})\alpha\chi(|u_t|^2 + |\xi|^2|v|^2).\end{aligned}$$

In the same way we get

$$\begin{aligned}\langle A^2(v_t, -i|\xi|u), (v_t, -i|\xi|u)\rangle &\geq \frac{1}{2}\det(A^2)\left(a_{22}^{-1}|v_t|^2 + a_{21}^{-1}|\xi|^2|u|^2\right) \\ &\geq c(\alpha_0, \chi_0, \bar{\sigma})\alpha(|v_t|^2 + |\xi|^2|u|^2)\end{aligned}$$

for all $\chi \leq \chi_0, \alpha \leq \alpha_0$. Thus

$$F^{(1)} \geq c(\alpha_0, \chi_0, \bar{\sigma})\alpha\rho(\xi)\left(((1 + |\xi|^2)(|u|^2 + |v|^2) + |u_t|^2 + |v_t|^2\right). \quad (4.43)$$

To finish the proof we establish estimates on $E^{(1)}$. It is obvious that there exist $\chi_0, \alpha_0 > 0$ and $C_1 = C_1(\alpha_0, \chi_0, \bar{\sigma})$ such that for all $\alpha < \alpha_0$, $\chi < \chi_0$

$$E^{(1)} \le C_1 \left((1 + |\xi|^2)(|u|^2 + |v|^2) + |u_t|^2 + |v_t|^2 \right).$$

On the other hand, since $c_s < 1$,

$$\begin{aligned} c_s^{-2}(|u_t|^2 + |\xi|^2|u|^2) + |v_t|^2 + |\xi|^2|v|^2 + 2\Re(i\xi(u_t\bar{v} + v_t\bar{u})) \\ \ge C \left(|\xi|^2(|u|^2 + |v|^2) + |u_t|^2 + |v_t|^2 \right). \end{aligned}$$

Furthermore there exist $\beta(\chi_0, \bar{\sigma}), C_\beta(\chi_0, \bar{\sigma}) > 0$ such that

$$\begin{aligned} c_s^{-2}|u|^2 + |v|^2 + 2\Re(\chi u_t\bar{u} + \bar{\sigma} v_t\bar{v}) + 2d\Re(i\xi v\bar{u}) \\ \ge \beta \left(|u|^2 + |v|^2 \right) - C_\beta \left(|u_t|^2 + |v_t|^2 + |\xi|^2(|u|^2 + |v|^2) \right). \end{aligned}$$

Due to the last two estimates there exists $C_2 = C_2(\chi_0, \alpha_0, \bar{\sigma}) > 0$ and $C_3 = C_3(\chi_0, \bar{\sigma}) > 0$ such that

$$E^{(1)} \ge C_2 \left(|\xi|^2(|u|^2 + |v|^2) + |u_t|^2 + |v_t|^2 \right) + C_3\alpha(|u|^2 + |v|^2)$$

for $\chi \le \chi_0$ and $\alpha \le \alpha_0$. In conclusion, we have shown that there exist $c, C_1, C_2 > 0$ only depending on $\chi_0, \bar{\sigma}$ such that

$$C_2 E_0^{(1)} \le E^{(1)} \le C_1 E_0^{(1)}, \quad F^{(1)} \ge c\rho(\xi) E_0^{(1)},$$

for

$$E_0^{(1)} = \left((1 + |\xi|^2)(|u|^2 + |v|^2) + |u_t|^2 + |v_t|^2 \right).$$

The assertion now follows from applying Gronwall's Lemma to (4.42). □

b) Estimates for $\chi = 0$

The next two propositions contain a decay and an energy estimate for solutions to (4.27)-(4.29). We will prove these two estimates simultaneously.

4.2.3 Proposition. *Fix an integer $s \in \mathbb{N}$. There exist $C_2, c_2 > 0$ such that the following holds: Let $({}^0\psi, {}^1\boldsymbol{\psi}) \in (H^{s+1} \times H^s) \cap (L^1 \times L^1)$ and $(\psi(t), \psi_t(t)) \in H^{s+1} \times H^s$ be the solution of (4.27)-(4.29). Then for all integers $0 \le k \le s$ and all $t \in [0, \infty)$*

$$\begin{aligned} \|\partial_x^k \psi(t)\|_1 + \|\partial_x^k \psi_t(t)\| \le C_1(1+t)^{-\frac{3}{4}-\frac{k}{2}} \left(\|{}^0\psi\|_{L^1} + \|{}^1\psi\|_{L^1} \right) \\ + C_1 e^{-c_1 t} \left(\|\partial_x^k({}^0\psi)\|_1 + \|\partial_x^k({}^1\boldsymbol{\psi})\| \right). \quad (4.44) \end{aligned}$$

4.2.4 Proposition. *Fix an integer* $s \in \mathbb{N}$. *Let* $({}^0\psi, {}^1\boldsymbol{\psi}) \in (H^{s+1} \times H^s)$ *and let* $(\psi(t), \psi_t(t)) \in H^{s+1} \times H^s$ *be the solution of* (4.27)-(4.29). *Then for all integers* $0 \leq k \leq s$ *and all* $t \in [0, \infty)$

$$\|\partial_x^k \psi(t)\|_1^2 + \|\partial_x^k \psi_t(t)\|^2 + \int_0^t \|\partial_x^{k+1}\psi(\tau)\|^2 + \|\partial_x^k \psi_t(\tau)\|^2 d\tau$$
$$\leq C_1 \left(\|\partial_x^k({}^0\psi)\|_1^2 + \|\partial_x^k({}^1\boldsymbol{\psi})\|^2 \right). \quad (4.45)$$

Proof. Consider (4.27)-(4.29) in Fourier space, i.e.

$$c_s^{-2}\hat{\psi}_t - i\xi \cdot \hat{\boldsymbol{\psi}} = 0 \quad (4.46)$$

$$\bar{\sigma}\hat{\boldsymbol{\psi}}_{tt} + \bar{\eta}|\xi|^2\hat{\boldsymbol{\psi}} + (\bar{\zeta} + \frac{1}{3}\bar{\eta})(\xi \otimes \xi)\hat{\boldsymbol{\psi}} + \hat{\boldsymbol{\psi}}_t - i\xi\psi^0 = 0, \quad (4.47)$$

$$\hat{\psi}(0) = {}^0\hat{\psi}, \quad \hat{\boldsymbol{\psi}}_t(0) = {}^1\hat{\boldsymbol{\psi}}. \quad (4.48)$$

As in the proof of Lemma 4.2.2 assume $\xi = (|\xi|, 0, 0)$ w.l.o.g. and set $\hat{\psi}^0 = u$, $\hat{\psi}^1 = v$ and $(\hat{\psi}^2, \hat{\psi}^3) = w$. We arrive at the two uncoupled systems

$$c_s^{-2}u_t - i|\xi|v = 0, \quad (4.49)$$

$$\bar{\sigma}(v_{tt} + |\xi|^2 v) + v_t - i|\xi|u = 0, \quad (4.50)$$

and

$$\bar{\sigma}w_{tt} + \bar{\eta}|\xi|^2 w + w_t = 0. \quad (4.51)$$

Note that (4.51) leads to

$$\frac{1}{2}\frac{d}{dt}E^{(2)} + F^{(2)} = 0, \quad (4.52)$$

where

$$C_1 E^{(2)} \leq (1 + |\xi|^2)|w|^2 + |w_t|^2 \leq C_2 E^{(2)}, \quad F^{(2)} \geq c\left(|\xi|^2|w|^2 + |w_t|^2\right).$$

(see Lemma 2.2.2). Hence Gronwall's Lemma yields

$$(1 + |\xi|^2)|w|^2 + |w_t|^2 \leq C\exp(-c\rho(\xi))\left((1 + |\xi|^2)|w(0)|^2 + |w_t(0)|^2\right) \quad (4.53)$$

and integrating (4.52) from 0 to t gives

$$(1 + |\xi|^2)|w|^2 + |w_t|^2 + \int_0^t |\xi|^2|w|^2 + |w_t|^2 d\tau$$
$$\leq C\left((1 + |\xi|^2)|w(0)|^2 + |w_t(0)|^2\right). \quad (4.54)$$

In the next step, multiply (4.50) by $\bar{v}_t$ and take the real part:

$$\frac{\bar{\sigma}}{2}\frac{d}{dt}\left(|v_t|^2 + |\xi|^2|v|^2\right) + |v_t|^2 - \Re(i|\xi|u\bar{v}_t) = 0. \tag{4.55}$$

Note that due to (4.49)

$$\Re(i|\xi|v_{tt}\bar{u}) = \frac{d}{dt}\Re(i|\xi|v_t\bar{u}) - \frac{1}{2}\frac{d}{dt}(c_s^2|\xi|^2|v|^2)$$

and

$$\Re(i|\xi|^3 v\bar{u}) = \frac{1}{2}\frac{d}{dt}(c_s^{-2}|\xi|^2|u|^2).$$

Hence the real part of $(4.50)i|\xi|\bar{u}$ is given by

$$\frac{\bar{\sigma}}{2}\frac{d}{dt}\left(c_s^{-2}|\xi|^2|u|^2 - c_s^2|\xi|^2|v|^2 + 2\Re(i|\xi|v_t\bar{u})\right) \\ - \Re(i|\xi|u\bar{v}_t) + |\xi|^2|u|^2 = 0. \tag{4.56}$$

Multiplying (4.50) by $\bar{v}$ and taking the real part gives

$$\frac{1}{2}\frac{d}{dt}\left(2\bar{\sigma}\Re(v_t\bar{v}) + |v|^2 + c_s^{-2}|u|^2\right) - \bar{\sigma}|v_t|^2 + \bar{\sigma}|\xi|^2|v|^2 = 0, \tag{4.57}$$

where we used that due to (4.49)

$$-\Re(i|\xi|\bar{v}u) = c_s^{-2}\frac{d}{dt}|u|^2.$$

Now, add (4.55)+(4.56)+α(4.57) (for $\alpha > 0$ to be determined later) to get

$$\frac{1}{2}\frac{d}{dt}\tilde{E} + \tilde{F} = 0, \tag{4.58}$$

where

$$\tilde{E} = \bar{\sigma}(\left(|v_t|^2 + (1 - c_s^2)|\xi|^2|v|^2 + c_s^{-2}|\xi|^2|u|^2 + 2\Re(i|\xi|v_t\bar{u})\right) \\ + \alpha\left(|v|^2 + c_s^{-2}|u|^2 + 2\bar{\sigma}\Re(v_t\bar{v})\right)$$

and

$$\tilde{F} = (1 - \alpha\bar{\sigma})|v_t|^2 + |\xi|^2|u|^2 + \alpha\bar{\sigma}|\xi|^2|v|^2 - 2\Re(i|\xi|u\bar{v}_t).$$

Due to (4.49)

$$\Re(-i|\xi|u\bar{v}_t) = \frac{d}{dt}\Re(-i|\xi|u\bar{v}) - \Re(-i|\xi|\bar{v}u) = \frac{d}{dt}\Re(-i|\xi|u\bar{v}) - c_s^2|\xi|^2|v|^2,$$

and thus, (4.58) can be written as

$$\frac{1}{2}\frac{d}{dt}E + F = 0, \tag{4.59}$$

where

$$E = \tilde{E} - \alpha\bar{\sigma}(c_s^{-2}+1)\Re(i|\xi|u\bar{v})$$

and

$$\begin{aligned} F &= \tilde{F} + \frac{1}{2}\alpha\bar{\sigma}(c_s^{-2}+1)(\Re(i|\xi|u\bar{v}_t - c_s^2|\xi|^2|v|^2) \\ &= (1-\alpha\bar{\sigma})|v_t|^2 + |\xi|^2|u|^2 + \frac{1}{2}\alpha\bar{\sigma}(1-c_s^2)|\xi|^2|v|^2 \\ &\qquad - 2\left(1 - \frac{1}{4}\alpha\bar{\sigma}(c_s^{-2}+1)\right)\Re(i|\xi|u\bar{v}_t). \end{aligned}$$

Note that

$$(1-\alpha\bar{\sigma}) - \left(1 - \frac{1}{4}\alpha\bar{\sigma}(c_s^{-2}+1)\right)^2 = \frac{1}{2}\alpha\bar{\sigma}(c_s^{-2}-1) - O(\alpha^2).$$

Hence for α sufficiently small there exists a $c = c(\alpha) > 0$ such that

$$F \geq c\left(|v_t|^2 + |\xi|^2|v|^2 + |\xi|^2|u|^2\right).$$

Furthermore by Young's inequality and $c_s^2 < 1$, there exist $C, C_1 > 0$ such that

$$\begin{aligned} E &\geq C\bar{\sigma}\left(|v_t|^2 + |\xi|^2|v|^2 + |\xi|^2|u|^2\right) \\ &\qquad + \alpha\left(\frac{1}{2}(|u|^2+|v|^2) - C_1(|v_t|^2 + |\xi|^2|v|^2)\right). \end{aligned}$$

Hence for α suitably small there exist a $C_1 > 0$ such that

$$E \geq C_1\left(|v_t|^2 + (1+|\xi|^2)|v|^2 + (1+|\xi|^2)|u|^2\right).$$

Obviously there also exists a $C_2 > 0$ such that

$$E \leq C_2\left(|v_t|^2 + (1+|\xi|^2)|v|^2 + (1+|\xi|^2)|u|^2\right).$$

In conclusion, applying Gronwall's Lemma to (4.59) gives

$$\begin{aligned} &(1+|\xi|^2)(|v(t)|^2 + |u(t)|^2) + |v_t(t)|^2 \\ &\qquad \leq C\exp(-c\rho(\xi))\left((1+|\xi|^2)(|v(0)|^2+|u(0)|^2) + |v_t(0)|^2\right) \end{aligned} \tag{4.60}$$

and by integrating (4.59) from 0 to t, we get

$$(1+|\xi|^2)(|v(t)|^2+|u(t)|^2)+|v_t(t)|^2+\int_0^t |\xi|^2(|v(t)|^2+|u(t)|^2)+|v_t(t)|^2 \\ \leq C\left((1+|\xi|^2)(|v(0)|^2+|u(0)|^2)+|v_t(0)|^2\right). \quad (4.61)$$

Taking into account that by (4.49)

$$c_s^{-2}|u_t|^2=|\xi|^2|v|^2,$$

(4.53) and (4.60) give

$$(1+|\xi|^2)|\hat{\psi}(t,\xi)|^2+|\hat{\psi}_t(t,\xi)|^2 \\ \leq C\exp(-c\rho(\xi)t)\left((1+|\xi|^2)|{}^0\hat{\psi}(\xi)|^2+|{}^1\hat{\psi}(\xi)|^2\right). \quad (4.62)$$

and (4.54), (4.61) yield

$$(1+|\xi|^2)|\hat{\psi}(t,\xi)|^2+|\hat{\psi}_t(t,\xi)|^2+\int_0^t |\xi|^2|\hat{\psi}(t,\xi)|^2+|\hat{\psi}_t(t,\xi)|^2 d\tau \\ \leq C\left((1+|\xi|^2)|{}^0\hat{\psi}(\xi)|^2+|{}^1\hat{\psi}(\xi)|^2\right). \quad (4.63)$$

As previously shown, Proposition 4.2.3 follows from (4.62) and Proposition 4.2.4 follows from (4.63). □

c) Convergence for $\chi \downarrow 0$

Finally, we show convergence of solutions of (4.24)-(4.26) to solutions of (4.27)-(4.29) as $\chi \downarrow 0$. Set $(V^\chi, U^\chi)=(\psi^0, \boldsymbol{\psi})$ and write (4.24)-(4.26) as

$$U_t^\chi = W^\chi \quad (4.64)$$

$$V_t^\chi = Y^\chi \quad (4.65)$$

$$W_t^\chi = \bar{\sigma}^{-1}\left(\bar{\eta}\Delta U^\chi + (\bar{\zeta}+\frac{1}{3}\bar{\eta})\nabla(\nabla\cdot U^\chi) - W^\chi - \nabla V^\chi\right) \quad (4.66)$$

$$\chi Y_t^\chi = \chi\Delta V^\chi - c_s^{-2}Y^\chi - \nabla\cdot U^\chi \quad (4.67)$$

$$(U^\chi(0), V^\chi(0)) = Z_0, \quad (W^\chi(0), Y^\chi(0)) = Z_1 \quad (4.68)$$

In the same spirit write (4.27)-(4.29) as

$$U_t = W \tag{4.69}$$
$$V_t = Y \tag{4.70}$$
$$W_t = \bar{\sigma}^{-1}\left(\bar{\eta}\Delta U + (\bar{\zeta} + \frac{1}{3}\bar{\eta})\nabla(\nabla\cdot U) - W - \nabla V\right) \tag{4.71}$$
$$Y = -c_s^2\nabla\cdot U \tag{4.72}$$
$$(U(0), V(0)) = Z_0, \quad W(0) = \mathbf{Z}_1. \tag{4.73}$$

We view (4.64)-(4.68) as a singular perturbed dynamical system and with corresponding "slow limit system" (4.69)-(4.73). For a solution $(U^\chi, V^\chi, W^\chi, Y^\chi)$ of (4.64)-(4.68), Y^χ is called the fast variable and $X^\chi = (U^\chi, V^\chi, W^\chi)$ the slow variable. Furthermore write $(X, Y) = (U, V, W, Y)$ for a solution of (4.69)-(4.73) and set $(\bar{X}^\chi, \bar{Y}^\chi) = (X^\chi - X, Y^\chi - Y)$. We also need to define $\Lambda^\chi = Y^\chi + c_s^2\nabla\cdot U^\chi$ and $\Lambda_0 = \Lambda^\chi(0)$. Lastly, let $T(t)$ be the semi-group that maps $(Z_0, \mathbf{Z}_1) \in H^{s+1}\times H^s$ to the solution $(U(t), V(t), W(t)) \in H^{s+1}\times H^s$ of (4.69)-(4.73) at time t. We can now prove the main result of this section.

4.2.5 Proposition. *Fix an integer $s \geq 2$. There exists $\chi_0 > 0$ and $c = c(\chi_0)$, $C = C(\chi_0) > 0$ such that the following holds: Let $(Z_0, Z_1) \in H^{s+1}\times H^s \cap (L^1\times L^1)$ with $Z_1 = (\mathbf{Z}_1, Z_1^0)$, $(U(t), V(t), W(t), Y(t)) \in (H^{s+1})^2\times(H^s)^2$ be a solution to (4.69)-(4.73) and for each $0 < \chi \leq \chi_0$ let $(U^\chi(t), V^\chi(t), W^\chi(t), V^\chi(t)) \in (H^{s+1})^2\times(H^s)^2$ be a solution to (4.64)-(4.68). Then for all $t\in[0,\infty)$*

$$\begin{aligned}\|\partial_x^k\Lambda^\chi(t)\| \leq \exp(-c_s^{-2}\chi^{-1}t)\|\partial_x^k\Lambda_0\| \\ + C\chi e^{-ct}(\|\partial_x^{k+1}Z_0\|_1 + \|\partial_x^{k+1}Z_1\|) \\ + C\chi(1+t)^{-\frac{3}{2}-\frac{k+1}{2}}(\|Z_0\|_{L^1} + \|Z_1\|_{L^1}),\end{aligned} \tag{4.74}$$

for integers $0\leq k\leq s-1$, and

$$\begin{aligned}\|\partial_x^k\bar{X}^\chi(t)\|_{1,0} \leq C\chi\|\partial_x^k\Lambda_0\|_1 \\ + C\chi\left(\|\partial_x^k Z_0\|_2 + \|\partial_x^k Z_1\|_1 + \|Z_0\|_{L^1} + \|Z_1\|_{L^1}\right),\end{aligned} \tag{4.75}$$

for integers $0\leq k\leq s-2$.

Proof. We first prove estimate (4.74). (4.64) and (4.67) give

$$\begin{aligned}\Lambda_t^\chi &= Y_t^\chi + c_s^2\nabla\cdot U_t^\chi \\ &= \Delta V^\chi - \chi^{-1}\left(c_s^{-2}Y^\chi + \nabla\cdot U^\chi\right) + c_s^2\nabla\cdot W^\chi \\ &= -\chi^{-1}c_s^{-2}\Lambda^\chi + \Delta V^\chi + c_s^2\nabla\cdot W^\chi.\end{aligned}$$

By variation of constants the solution to this differential equation is given by

$$\begin{aligned}\Lambda^\chi(t) &= \exp(-c_s^{-2}\chi^{-1}t)\Lambda_0 \\ &\quad + \int_0^t \exp(-c_s^{-2}\chi^{-1}(t-\tau))(\Delta V^\chi(\tau) + c_s^2\nabla\cdot W^\chi(\tau))d\tau.\end{aligned}$$

By Proposition 4.2.1,

$$\begin{aligned}\|\partial_x^k \Delta V^\chi(t)\| + \|\partial_x^k(\nabla\cdot W^\chi)(t)\| &\leq Ce^{-ct}(\|\partial_x^{k+1}Z_0\|_1 + \|\partial_x^{k+1}Z_1\|) \\ &\quad + C(1+t)^{-\frac{3}{4}-\frac{k+1}{2}}(\|Z_0\|_{L^1} + \|Z_1\|_{L^1}),\end{aligned}$$

for all integers $0 \leq k \leq s-1$ and $C > 0$ independent of χ. Hence for $0 \leq k \leq s-1$

$$\begin{aligned}\|\partial_x^k\Lambda^\chi(t)\| &\leq \exp(-c_s^{-2}\chi^{-1}t)\|\partial_x^k\Lambda_0\| \\ &\quad + C(\|\partial_x^{k+1}Z_0\|_1 + \|\partial_x^{k+1}Z_1\|)\int_0^t \exp(-c\tau)\exp(-c_s^{-2}\chi^{-1}(t-\tau))d\tau \\ &\quad + C(\|Z_0\|_{L^1} + \|Z_1\|_{L^1})\int_0^t (1+\tau)^{-\frac{3}{4}-\frac{k+1}{2}}\exp(-c_s^{-2}\chi^{-1}(t-\tau))d\tau. \quad (4.76)\end{aligned}$$

We denote the integral in line 2 of (4.76) by I_1 and the integral in line 3 by I_3. A simple computation leads to

$$I_1 \leq C\chi e^{-ct}. \quad (4.77)$$

For I_2 we get

$$\begin{aligned}I_2 &\leq \int_0^{\frac{t}{2}} \exp(-c_s^{-2}\chi^{-1}(t-\tau))d\tau \\ &\quad + \int_{\frac{t}{2}}^t \exp(-c_s^{-2}\chi^{-1}(t-\tau))(1+\tau)^{-\frac{3}{4}-\frac{k+1}{2}}d\tau \\ &\leq C\chi\exp(-c\chi^{-1}t) + (1+t/2)^{-\frac{3}{4}-\frac{k+1}{2}}\int_{\frac{t}{2}}^t \exp(-c_s^{-2}\chi^{-1}(t-\tau))d\tau \\ &\leq C\chi\exp(-c\chi^{-1}t) + C\chi(1+t)^{-\frac{3}{4}-\frac{k+1}{2}}.\end{aligned}$$

This together with (4.76) and (4.77) prove the estimate for Λ^χ. To prove estimate (4.75) start with

$$\begin{aligned}
\bar{U}_t^\chi &= \bar{W}^\chi \\
\bar{V}_t^\chi &= -c_s^2 \nabla \cdot \bar{U}^\chi + \Lambda^\chi \\
\bar{W}_t^\chi &= \bar{\sigma}^{-1} \left(\bar{\eta} \Delta \bar{U}^\chi + (\bar{\zeta} + \frac{1}{3}\bar{\eta}) \nabla (\nabla \cdot \bar{U}^\chi) - \bar{W}^\chi - \nabla \bar{V}^\chi \right) \\
&(\bar{U}^\chi(0), \bar{V}^\chi(0), \bar{W}^\chi(0)) = 0
\end{aligned}$$

By Duhamel's principle the solution $(\bar{U}^\chi, \bar{V}^\chi, \bar{W}^\chi) = \bar{X}^\chi$ of this Cauchy problem is given by

$$\bar{X}^\chi(t) = \int_0^t T(t-\tau) \tilde{\Lambda}^\chi(\tau) d\tau,$$

where $\tilde{\Lambda}^\chi = (0, \Lambda^\chi, 0)^t$. Due to Proposition 4.2.4,

$$\|\partial_x^k (T(t-\tau)\tilde{\Lambda}^\chi(\tau))\|_{1,0} \leq C \|\partial_x^k \Lambda^\chi(\tau)\|_1,$$

for $0 \leq k \leq s-2$. Using (4.74), this yields

$$\begin{aligned}
\|\partial_x^k \bar{X}^\chi(t)\|_{1,0} &\leq C \int_0^t \|\partial_x^k \Lambda^\chi(\tau)\|_1 d\tau \\
&\leq C\chi \left(\|\partial_x^k \Lambda_0\|_1 + \|\partial_x^{k+1} Z_0\|_2 + \|\partial_x^{k+1} Z_1\|_1 \right) \\
&\qquad + C\chi \left(\|Z_0\|_{L^1} + \|Z_1\|_{L^1} \right),
\end{aligned}$$

for $0 \leq k \leq s-2$, which is (4.75). $\square$

A. Appendix

A.1. L^∞-H^s-Estimates

A.1.1 Lemma. *Let $n, N \in \mathbb{N}$, $s \geq s_0 := [\frac{n}{2}] + 1$ and $F \in C^\infty(\mathbb{R}^N)$, $F(0) = 0$. Then there exist $\delta > 0$, $C = C(\delta) > 0$ such that for all $u \in H^s$ with $\|u\|_s \leq \delta$, $F(u) - \partial_u F(0)u \in H^s$ and*

$$\|F(u) - \partial_u F(0)u\|_s \leq C\|u\|_s^2.$$

Proof. Since $s \geq s_0$ there exists a $C_1 > 0$ such that

$$\|u\|_{L^\infty} \leq C_1\|u\|_s$$

for all $u \in H^s$. Furthermore due to $F(0) = 0$ there exist $\delta_1 > 0$, $C_2 = C_2(\delta_1) > 0$ such that

$$|F(y) - \partial_y F(0)y| \leq C_2|y|^2.$$

for all $y \in \mathbb{R}^N$ with $|y| \leq \delta_1$. Now let $u \in H^s$ such that $\|u\|_s \leq \delta_1/C_1$ (i.e. $\|u\|_{L^\infty} \leq \delta_1$). Then

$$\|F(u) - \partial_u F(0)u\| \leq C_2\|u\|_{L^\infty}\|u\| \leq C_1C_2\|u\|_s^2. \tag{A.1}$$

Furthermore for $\alpha \in \mathbb{N}_0^n$ with $1 \leq |\alpha| = j \leq s$ we get

$$\partial_x^\alpha F(u) = \partial_u F(u)\partial_x^\alpha u + R,$$

where

$$R = \sum_{1\leq|\beta|<j} \binom{\alpha}{\beta} \partial_x^\beta u \; \partial_x^{\alpha-\beta} F(u).$$

Since $\partial_x u \in H^{s-1}$ and $\|u\|_{L^\infty} \leq \delta_1$, we get $\partial_x F(u) \in H^{s-1}$ and

$$\|\partial_x F(u)\|_{s-1} \leq C_3\|\partial_x u\|_{s-1}$$

for a $C_3 = C_3(\delta_2) > 0$ by [33, Lemma 2.4]. Therefore [33, Lemma 2.3] yields

$$\|R\| \leq C_4\|\partial_x u\|_{s-1}\|\partial_x F(u)\|_{s-1} \leq C_3C_4\|\partial_x u\|_{s-1}^2$$

for a $C_4 > 0$. On the other hand there exist $\delta_2 > 0$, $C_5 = C_5(\delta_2) > 0$, such that

$$|\partial_y F(y) - \partial_y F(0)| \leq C_5 |y|$$

for all $y \in \mathbb{R}^N$ with $|y| \leq \delta_2$. Assuming $\|u\|_s \leq \delta_2 / C_1$ entails

$$\begin{aligned}
\|\partial_x^\alpha (F(u) - \partial_u F(0) u)\| &\leq \|(\partial_u F(u) - \partial_u F(0)) \partial_x^\alpha u\| + \|R\| \\
&\leq \|\partial_u F(u) - \partial_u F(0)\|_{L^\infty} \|u\|_s + C_3 C_4 \|\partial_x u\|_{s-1} \\
&\leq \max\{C_3 C_4, C_5\} \|u\|_s^2.
\end{aligned}$$

Since α was arbitrary this estimate together with (A.1) leads the assertion for $\delta = \min\{\delta_1, \delta_2\}/C_1$. □

A.1.2 Lemma. *Let $n, N \in \mathbb{N}$, $s \geq s_0$ and $F \in C^\infty(\mathbb{R}^N, \mathbb{R}^{N \times N})$. Then there exist $\delta > 0$, $C = C(\delta) > 0$ such that for all $u \in H^s(\mathbb{R}^n, \mathbb{R}^N)$ with $\|u\|_s \leq \delta$, $(F(u) - F(0))u \in H^s$ and*

$$\|(F(u) - F(0))u\|_s \leq C \|u\|_s^2.$$

Proof. First note that there exist $\delta_1 > 0$, $C_1 = C_1(\delta_1) > 0$ such that

$$|F(y) - F(0)| \leq C_1 |y|$$

for all $y \in \mathbb{R}^N$, $|y| \leq \delta_1$ as well as $C_2 > 0$ such that

$$\|v\|_{L^\infty} \leq C_2 \|v\|_s$$

for all $v \in H^s$. Now let $u \in H^s$, $\|u\|_s \leq \delta_1 / C_2$. Then

$$\|F(u) - F(0)\| \leq C_1 \|u\|_s$$

holds. On the other hand by [33, Lemma 2.4] $\partial_x F(u) \in H^{s-1}$ and

$$\|\partial_x F(u)\|_{s-1} \leq C_3 \|\partial_x u\|_{s-1}$$

for a $C_3 = C_3(\delta_1) > 0$. Hence $F(u) - F(0) \in H^s$ and

$$\|F(u) - F(0)\|_s \leq C_4 \|u\|_s$$

for $\|u\|_s \leq \delta = \delta_1 / C_2$. Now the assertion follows from [33, Lemma 2.4]. □

A.2. Perturbation theory for linear hyperbolic operators

Let a, B be symmetric positive definite $N \times N$ matrices and b an arbitrary symmetric $N \times N$ matrix. Define the matrix family

$$M(\kappa) = \begin{pmatrix} 0 & I \\ -\kappa^2 B + i\kappa b & -a \end{pmatrix}, \quad \kappa \in \mathbb{C}. \tag{A.2}$$

Furthermore set

$$\tilde{B} = a^{-\frac{1}{2}} B a^{-\frac{1}{2}}, \quad \tilde{b} = a^{-\frac{1}{2}} b a^{-\frac{1}{2}}$$

and

$$\tilde{N}(\kappa) = -\kappa^2 \tilde{B} + i\kappa \tilde{b}.$$

A.2.1 Condition. Let here exists a $C > 0$ such that for all $\lambda \in \sigma(\tilde{b})$, each $u \in \mathrm{Eig}_{\tilde{b}}(\lambda)$ satisfies

$$\langle (\tilde{B} - \lambda^2 a^{-1}) u, u \rangle \geq C|u|^2.$$

A.2.2 Proposition. *Let Condition A.2.1 be satisfied. Then there exist* $r_0, c_0 > 0$ *and a family of holomorphic invertible* $2N \times 2N$ *matrices*

$$\{S(\kappa) : |\kappa| \leq r_0\},$$

such that $M_*(\kappa) = (S(\kappa))^{-1} M(\kappa) S(\kappa)$ *satisfies*

$$\langle (M_*(\kappa) + (M_*(\kappa))^*) u, u \rangle \leq -c\kappa^2 |u|^2,$$

for all $u \in \mathbb{C}^{2N}$ *and* $\kappa \in [0, r_0]$.

Proof. First set

$$S_1 = \begin{pmatrix} a^{\frac{1}{2}} & a^{\frac{1}{2}} \\ 0 & -a^{\frac{3}{2}} \end{pmatrix}.$$

and

$$T(\kappa) := S_1^{-1} M(\kappa) S_1 = \begin{pmatrix} \tilde{N}(\kappa) & \tilde{N}(\kappa) \\ -\tilde{N}(\kappa) & -\tilde{N}(\kappa) - a \end{pmatrix}.$$

Then write $T(\kappa) = T^{(0)} + \kappa T^{(1)} + \kappa^2 T^{(2)}$, where

$$T^{(0)} = \begin{pmatrix} 0 & 0 \\ 0 & -a \end{pmatrix}, \; T^{(1)} = i \begin{pmatrix} \tilde{b} & \tilde{b} \\ -\tilde{b} & -\tilde{b} \end{pmatrix}, \; T^{(2)} = \begin{pmatrix} -\tilde{B} & -\tilde{B} \\ \tilde{B} & \tilde{B} \end{pmatrix}.$$

For $\zeta \in \rho(T^{(0)})$ let $R(\zeta)$ denote the resolvent of $T^{(0)}$. Let $0 < \tau < \min \sigma(a)$ and set $\Gamma = \partial B_\tau(0) \subset \mathbb{C}$. Now choose $r > 0$ such that

$$\max_{\kappa \in \overline{B_r(0)}} |A(\kappa)| < \min_{\zeta \in \Gamma} |R(\zeta)|^{-1},$$

where $A(\kappa) := T(\kappa) - T^{(0)}$. Then for $\kappa \in \overline{B_r(0)}$, $\zeta \in \Gamma$ the resolvent $R(\zeta, \kappa)$ of $T(\kappa)$ is well-defined and holomorphic on $\Gamma \times \overline{B_r(0)}$ (cf. [31, Chapter II.1]). Next define

$$P(\kappa) = -\frac{1}{2\pi i} \int_\Gamma R(\zeta, \kappa) d\zeta.$$

Since $\mu_0 = 0$ is the only eigenvalue of $T^{(0)}$ lying inside Γ, $P(\kappa)$ is the total projection for the μ_0-group of eigenvalues of $T(\kappa)$ (cf. [31, chapter II.2]), particularly

$$P := P(0) = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

We now use the so called reduction process (cf. ibd.) to expand $P(\kappa)T(\kappa)P(\kappa)$ in a Taylor series. To this end consider the operator

$$\tilde{T}(\kappa) = \frac{1}{\kappa} T(\kappa) P(\kappa), \; \tilde{T}(0) = PT^{(1)}P.$$

Due to $T(\kappa)R(\zeta, \kappa) = I + \zeta R(\zeta, \kappa)$ and $R(\zeta)$ having a pole of order one at $\zeta = 0$, $\tilde{T}$ is holomorphic. Furthermore

$$\tilde{T}(0) = \begin{pmatrix} i\tilde{b} & 0 \\ 0 & 0 \end{pmatrix}$$

and thus

$$\sigma(\tilde{T}(0)) = i\sigma(\tilde{b}) \cup \{0\}.$$

Let $\lambda_1, \ldots, \lambda_s$ be (the pairwise distinct) eigenvalues of $\tilde{b}$. At this point we may assume w.l.o.g $\lambda_i \neq 0$ for $1 \leq i \leq s$ (otherwise we could have considered $M + i\alpha\kappa I$ instead of M for a suitable $\alpha > 0$). Let

$$\gamma = \min\{|\lambda_j|, |\lambda_i - \lambda_j| : 1 \leq i, j \leq s, i \neq j\} > 0.$$

and $\Gamma_j = \partial B_{\gamma/2}(i\lambda_j)$ for $j = 1, \ldots, s$. Then the resolvent $\tilde{R}(\zeta)$ of $\tilde{T}(0)$ is well defined for $\zeta \in \Gamma_j$ and satisfies

$$|\tilde{R}(\zeta)|^{-1} = \operatorname{dist}(\zeta, \sigma(\tilde{T}(0)) \geq \frac{\gamma}{4}.$$

Now chose $0 < \tilde{r} < r$ such that

$$|\tilde{A}(\kappa)| < \frac{\gamma}{4}, \text{ for all } \kappa \in \overline{B_{\tilde{r}}(0)},$$

where $\tilde{A}(\kappa) = \tilde{T}(\kappa) - \tilde{T}(0)$. Then for $\zeta \in \Gamma_j$, $|\kappa| \leq \tilde{r}$, the resolvent $\tilde{R}(\zeta, \kappa)$ of $\tilde{T}(\kappa)$ is well-defined and holomorphic. Again, the total projection for the $i\lambda_j$-group of eigenvalues of $\tilde{T}(\kappa)$, i.e.

$$P_j(\kappa) = -\frac{1}{2\pi i} \int_{\Gamma_j} \tilde{R}(\zeta, \kappa) d\zeta, \tag{A.3}$$

is itself holomorphic for $|\kappa| \leq \tilde{r}$ and

$$P_j := P_j(0) = \begin{pmatrix} p_j & 0 \\ 0 & 0 \end{pmatrix},$$

where p_j is the eigenprojection corresponding to the eigenvalue λ_j of $\tilde{b}$. Furthermore each eigenvalue of the μ_0-group belongs to some $i\kappa\lambda_j$-group and the total projection for this group is $P_j(\kappa)$. Now, the construction of the transformation matrices $S(\kappa)$ works as follows (cf. [31, Chapter II.4]): Set

$$P_0(\kappa) = I - \sum_{j=1}^{s} P_j(\kappa) \ (P_0 = P_0(0)),$$

$$Q(\kappa) = \sum_{j=0}^{s} (\partial_\kappa P_j(\kappa)) P_j(\kappa)$$

and consider the matrix-valued initial-value problem

$$\partial_\kappa U(\kappa) = Q(\kappa) U(\kappa), \quad U(0) = I. \tag{A.4}$$

The unique solution $S(\kappa)$ is invertible, holomorphic and has the property

$$S(\kappa) P_j (S(\kappa))^{-1} = P_j(\kappa), \ j = 0, \ldots, s.$$

Define $M_*(\kappa) = (S(\kappa))^{-1} T(\kappa) S(\kappa)$. By definition each P_j is $M_*(\kappa)$-invariant. Since the P_j are orthogonal projections, $P_j P_k = \delta^{jk} P_k$, and $\sum_{j=0}^{s} P_j = I$, the assertion is shown if we can prove

$$\begin{aligned} &\langle (M_*(\kappa) + (M_*(\kappa))^*) P_j u, P_j u \rangle \\ &\qquad \leq -c\kappa^2 |P_j u|^2, \ \kappa \in [0, \tilde{r}], \ u \in \mathbb{C}^{2N}, \ j = 0, \ldots, s. \end{aligned}$$

First consider $j = 0$. Since $S(0) = I$ and M_* is holomorphic, we have

$$P_0 M_*(\kappa) P_0 = \begin{pmatrix} 0 & 0 \\ 0 & -a \end{pmatrix} + \kappa P_0 H_1(\kappa) P_0,$$

where $H_1(\kappa)$ is holomorphic. As $a > 0$, this yields

$$\langle (M_*(\kappa) + (M_*(\kappa))^*) \, P_0 u, P_0 u\rangle \ \leq |P_0 u|^2(-c + O(\kappa)), \ \kappa \in [0, \tilde{r}]. \quad \text{(A.5)}$$

Next let $j \in \{1, \dots, s\}$. As $P_j T^{(0)} = T^{(0)} P_j = 0$, we can write

$$P_j M_*(\kappa) P_j = \kappa M_*^{(1j)} + \kappa^2 M_*^{(2j)} + \kappa^3 P_j H_2(\kappa) P_j,$$

where $H_2(\kappa)$ is holomorphic

$$M_*^{(1j)} = P_j T^{(1)} P_j = \begin{pmatrix} i\lambda_j p_j & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$M_*^{(2j)} = P_j T^{(2)} P_j + Z^{(1)} T^{(1)} P_j + P_j T^{(1)} W^{(1)} + Z^{(1)} T^{(0)} W^{(1)},$$

whith $W^1 = \partial_\kappa S(0) P_j$ and $Z^1 = P_j \partial_\kappa ((S(\kappa))^{-1}) \,|_{\kappa=0}$. From (A.4) we get $W^{(1)} = P_j'(0) P_j$ and $Z^{(1)} = P_j P_j'(0)$. The general form of $P_j'(0)$ can be computed by inserting the Neumann-series for the resolvent $\tilde{R}$ of $\tilde{T}$ in (A.3) (which is done for the general case in [31, chapter II.2]). In our case some computations lead to

$$M_*^{(2j)} = \begin{pmatrix} -p_j(\tilde{B} - \lambda_j^2 a^{-1}) p_j & 0 \\ 0 & 0 \end{pmatrix}.$$

Since λ_j is a real number, Condition A.2.1 yields

$$\left\langle \left(M_*(\kappa)) + (M_*^{(j)}(\kappa)))^* \right) P_j u, P_j u \right\rangle \leq |P_j u|^2(-c\kappa^2 + O(\kappa^3))$$

for $\kappa > 0$, which together with (A.5) proves the assertion. □

A.2.3 Corollary. *Assume Condition A.2.1. Let $r_0 > 0$ be the constant in Proposition A.2.2 and $\lambda_1, \dots, \lambda_s$ be the eigenvalues of $\tilde{b}$ with corresponding eigenprojections $p_1, \dots, p_j$. Furthermore for each $j = 1, \dots, s$ let $\beta_{j1}, \dots, \beta_{jn}$ be the eigenvalues of*

$$p_j(\tilde{B} - \lambda_j^2 a^{-1}) p_j$$

Then for $|\kappa| \le r_0$ the spectrum of $M(\kappa)$ is given as

$$\sigma(M(\kappa)) = \{\gamma_{11}, \dots, \gamma_{1n}, \dots, \gamma_{s1}, \dots, \gamma_{sn}, \upsilon_1, \dots, \upsilon_m\}$$

with

$$\gamma_{jk} = i\kappa\lambda_j - \kappa^2\beta_{jk} + o(\kappa^2), \tag{A.6}$$

and

$$\upsilon_k = -\alpha_k + o(1), \tag{A.7}$$

where $\alpha_1, \dots, \alpha_m$ are the eigenvalues of a. In particular there exists a $c > 0$ such that for $\kappa \in [0, r_0]$,

$$\Re(\gamma) \le -c\kappa^2$$

for all $\gamma \in \sigma(M(\kappa))$, $\kappa \in [0, r_0]$.

Proof. Let $\kappa \in [0, r_0]$. Obviously, $M_*(\kappa)$ has the same spectrum as $M(\kappa)$. In the proof of Proposition A.2.2 we have seen that for $j = 1, \dots, s$,

$$P_j M_*(\kappa) P_j = \kappa M_*^{(1j)} + \kappa^2 M_*^{(2j)} + O(\kappa^3)$$

where

$$P_j = \begin{pmatrix} p_j & 0 \\ 0 & 0 \end{pmatrix},$$

$$M_*^{(1j)} = \begin{pmatrix} i\lambda_j p_j & 0 \\ 0 & 0 \end{pmatrix},$$

$$M_*^{(2j)} = \begin{pmatrix} -p_j(\tilde{B} - \lambda_j^2 a^{-1}) p_j & 0 \\ 0 & 0 \end{pmatrix}.$$

By [31, Chapter II.2] the eigenvalues of $P_j M_*(\kappa) P_j$ are given by γ_{jk} ($k = 1, \dots, n$) as defined in (A.6). Furthermore for

$$P_0 = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$$

we have

$$P_0 M_*(\kappa) P_0 = \begin{pmatrix} 0 & 0 \\ 0 & -a \end{pmatrix} + O(\kappa), \quad |\kappa| \le r_0,$$

and hence the eigenvalues of $P_0 M_*(\kappa) P_0$ are given by υ_k as defined by (A.7). Since $P_0, \dots, P_s$ form a complete $M_*(\kappa)$-invariant set of projections the assertion is proven. $\square$

Consider now the matrix family

$$K(\nu) = \begin{pmatrix} 0 & I \\ -B + i\nu b & -\nu a \end{pmatrix}, \quad \nu \in \mathbb{C}.$$

A.2.4 Condition. Let there exist a $C > 0$ such that for each $\mu \in \sigma(B)$, each $u \in \mathrm{Eig}_B(\mu)$ satisfies

$$\langle (a + (\mu)^{-\frac{1}{2}} b)u, u\rangle \geq C|u|^2,$$
$$\langle (a - (\mu)^{-\frac{1}{2}} b)u, u\rangle \geq C|u|^2.$$

A.2.5 Proposition. *Let Condition A.2.4 hold. Then there exist $r_\infty, c_\infty > 0$ and a holomorphic, invertible family of $2N \times 2N$ matrices*

$$\{S(\nu) : |\nu| \leq r_\infty\}$$

such that $K_*(\nu) = (S(\nu))^{-1} K(\nu) S(\nu)$ satisfies

$$\langle (K_*(\nu)) + (K_*(\nu))^*)u, u\rangle \leq -c_\infty \nu |u|^2,$$

for all $u \in \mathbb{C}^{2N}$, $\nu \in [0, r_\infty]$.

Proof. Set

$$S_1 := \begin{pmatrix} B^{-\frac{1}{2}} & B^{-\frac{1}{2}} \\ iI & -iI. \end{pmatrix}$$

and define

$$K_1(\nu) := S_1^{-1} K(\nu) S_1 = K_1^{(0)} + \nu K_1^{(1)},$$

where

$$K_1^{(0)} = \begin{pmatrix} iB^{\frac{1}{2}} & 0 \\ 0 & -iB^{\frac{1}{2}} \end{pmatrix},$$

$$K_1^{(1)} = \frac{1}{2} \begin{pmatrix} -a + bB^{\frac{1}{2}} & -a + bB^{\frac{1}{2}} \\ a - bB^{\frac{1}{2}} & -a - bB^{\frac{1}{2}}. \end{pmatrix}$$

Obviously

$$\sigma(K_1^{(0)}) = \{i\mu_1^{\frac{1}{2}}, \ldots, i\mu_l^{\frac{1}{2}}, -i\mu_1^{\frac{1}{2}}, \ldots, -i\mu_l^{\frac{1}{2}}\},$$

where μ_j $(1 \leq j \leq l)$ are the (strictly positive) eigenvalues of B. Due to Condition A.2.4 it can be shown, as in the proof of Proposition A.2.2, that there exists an $r > 0$ such that for each $j = 1, \ldots, l$ the total projections

$P_{+j}(\nu)$ and $P_{-j}(\nu)$ for the $i\mu_j^{\frac{1}{2}}$- and $-i\mu_j^{\frac{1}{2}}$-group of eigenvalues are well-defined and holomorphic on $\overline{B_r(0)}$. Furthermore,

$$P_{+j} := P_{+j}(0) = \begin{pmatrix} p_j & 0 \\ 0 & 0 \end{pmatrix}, \quad P_{-j} = P_{-j}(0) = \begin{pmatrix} 0 & 0 \\ 0 & p_j \end{pmatrix},$$

where p_j is the eigenprojection for the eigenvalue μ_j of B. Then, again as in the proof of Proposition A.2.2, there exists an invertible holomorphic family of $2N \times 2N$ matrices

$$\{S(\nu) : |\nu| \leq r\},$$

such that

$$S(\nu)P_{\pm j}(S(\nu))^{-1} = P_{\pm j}(\nu),\ j = 0, \dots, l.$$

Now define $K_*(\nu) = (S(\nu))^{-1}K_1(\nu)S(\nu)$. Since $S(0) = I$ and

$$S'(0)K_1^{(0)}P_{\pm j} = P_{\pm j}K_1^{(0)}\frac{d}{d\nu}S(\nu)^{-1}\,|_{\nu=0} = 0$$

(cf. [31, Chapter II.4]), one gets

$$\begin{aligned} P_{+j}K_*(\nu)P_{+j} &= P_{+j}K_1^{(0)}P_{+j} + \nu P_{+j}K_1^{(1)}P_{+j} + \nu^2 P_{-j}H_+(\nu)P_{-j} \\ &= \begin{pmatrix} i\mu^{\frac{1}{2}} - \frac{\nu}{2}p_j\left(a - \mu_j^{-\frac{1}{2}}b\right)p_j & 0 \\ 0 & 0 \end{pmatrix} + \nu^2 P_{+j}H_+(\nu)P_{+j} \end{aligned}$$

and

$$P_{-j}K_*(\nu)P_{-j} = \begin{pmatrix} 0 & 0 \\ 0 & -i\mu^{\frac{1}{2}} - \frac{\nu}{2}p_j\left(a + \mu^{-\frac{1}{2}}b\right)p_j \end{pmatrix} + \nu^2 P_{-j}H_-(\nu)P_{-j},$$

where $H_\pm$ are holomorphic. Thus Condition A.2.4 yields

$$\langle (K_*(\nu) + (K_*(\nu))^*)\, P_{\pm j}u, P_{\pm j}u\rangle \leq |P_{\pm_j}u|^2(-c\nu + 0(\nu^2))$$

for $\nu \in [0, r]$, $u \in \mathbb{C}^{2N}$, $1 \leq j \leq l$. As $\{P_{+j}, P_{-j}\}_{1\leq j\leq l}$ form a complete set of K_*-invariant, orthogonal projections, this proves the assertion. □

Next, define the matrix family

$$\tilde{M}(\kappa) = \kappa K(\kappa^{-1}) = \begin{pmatrix} 0 & \kappa I_N \\ -\kappa B + ib & -a \end{pmatrix}, \quad \kappa \in \mathbb{C}$$

Then the following result is an immediate consequence of Proposition A.2.5.

A.2.6 Corollary. *Let Condition A.2.4 hold. Then there exist* $r_\infty, c_\infty > 0$ *and an invertible holomorphic family of* $2N \times 2N$ *matrices*

$$\{S(\kappa) : |\kappa| \geq r_\infty\}$$

such that $\kappa \mapsto S(\kappa)$, $\kappa \mapsto S^{-1}(\kappa)$ *are bounded on* $[r_\infty, \infty)$ *and* $\tilde{M}_*(\kappa) = (S(\kappa))^{-1}\tilde{M}(\kappa)S(\kappa)$ *satisfies*

$$\langle(\tilde{M}_*(\kappa)) + (\tilde{M}_*(\kappa))^*)u, u\rangle \leq -c_\infty |u|^2,$$

for all $u \in \mathbb{C}^{2N}$, $\kappa \in [r_\infty, \infty)$.

A.2.7 Corollary. *Assume Condition A.2.4. Let* $r_\infty > 0$ *be the constant from Corollary A.2.6,* $\mu_1, \ldots, \mu_l$ *be the eigenvalues of* B *with corresponding eigenprojections* $p_1, \ldots, p_l$. *Furthermore, for each* $j = 1, \ldots, l$ *let* $\eta^\pm_{j1}, \ldots, \eta^\pm_{jn}$ *be the eigenvalues of*

$$p_j \left(a \pm \mu_j^{-\frac{1}{2}} b\right) p_j$$

Then for $\kappa \in [r_\infty, \infty)$ *the spectrum of* $M(\kappa)$ *is given by*

$$\sigma(M(\kappa)) = \{\omega^\pm_{11}, \ldots, \omega^\pm_{1n}, \ldots, \omega^\pm_{l1}, \ldots, \omega^\pm_{ln}\}$$

with

$$\omega_{jk} = \mp i\kappa\mu_j - \frac{1}{2}\eta^\pm_{jk} + h(\kappa), \tag{A.8}$$

where

$$h(\kappa) \to 0, \quad \text{as } \kappa \to \infty$$

In particular there exists a $c > 0$ *such that*

$$\Re(\omega) \leq -c$$

for all $\omega \in \sigma(M(\kappa))$, $\kappa \in [r_\infty, \infty)$.

Proof. As

$$M(\kappa) = (L(\kappa))^{-1}\tilde{M}(\kappa)L(\kappa),$$

where

$$L(\kappa) = \begin{pmatrix} I & 0 \\ 0 & \kappa \end{pmatrix},$$

$M(\kappa)$ and $\tilde{M}(\kappa)$ have the same spectrum for $\kappa \neq 0$. Hence the assertion follows from the proof of Proposition A.2.5 and Corollary A.2.7 in the same way as Corollary A.2.3 followed from the proof of Proposition A.2.2. $\square$

A.2.8 Lemma. *Let $B = I$. Then the following statements are equivalent:*

1. *Condition A.2.1 holds.*
2. *Condition A.2.4 holds.*
3. *The matrices $a + b$ and $a - b$ are positive definite.*

Futhermore if these conditions are satisfied, then each eigenvalue of $M(\kappa)$ has strictly positive real part for all $\kappa \in \mathbb{R}_+$.

Proof. We first prove the equivalence of the three statements. That Condition A.2.4 is equivalent to $a + b > 0$ and $a - b > 0$ for $B = I$ is obvious. Furthermore with $B = I$, Condition A.2.1 reads:

$$\langle a^{-1}(I - \lambda^2)u, u\rangle \geq C|u|^2 \tag{A.9}$$

for all $\lambda \in \sigma(\tilde{b})$, each $u \in \mathrm{Eig}_{\tilde{b}}(\lambda)$. Since

$$\langle a^{-1}(I - \lambda^2)u, u\rangle = (1 - \lambda^2)|a^{-\frac{1}{2}}u|^2$$

and $a > 0$, (A.9) is equivalent to $|\lambda| < 1$. This holds true if and only if

$$I \pm \tilde{b} = I \pm a^{-\frac{1}{2}}ba^{-\frac{1}{2}} > 0.$$

It is obvious that this is the case if and only if $a + b > 0$ and $a - b > 0$.

Due to Corollaries A.2.3 and A.2.7, the second part of the assertion is shown if we can prove that no eigenvalue of $M(\kappa)$ is purely imaginary for $\kappa > 0$. To this end suppose that there exist $\kappa > 0$, $\beta \in \mathbb{R}$ and $U \in \mathbb{C}^{2N} \setminus \{0\}$ such that

$$(i\beta - M(\kappa))U = 0. \tag{A.10}$$

Write $U = (v, w)$ $(v, w \in \mathbb{C}^N)$. Then it follows from (A.10) with $B = I$ that $w = i\beta v$ (in particular $v \neq 0$) and

$$\Big(\kappa^2 - i\kappa b + (i\beta)^2 + i\beta a\Big)\, v = 0.$$

Taking the scalar product of this equation with v and using the symmetry of a and b gives

$$(\kappa^2 - \beta^2)|v|^2 = 0,$$
$$\langle(-\kappa b + \beta a)v, v\rangle = 0.$$

Hence $\kappa = |\beta|$ and

$$\beta\langle(a \pm b)v, v\rangle = 0$$

Since $a \pm b > 0$ and $|\beta| = \kappa > 0$, this is a contradiction. □

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Deutschsprachige Zusammenfassung

In der vorliegenden Dissertation werden Resultate über Systeme partieller Differentialgleichungen, die in der relativistischen Dynamik dissipativer Fluide auftreten, bewiesen. Diese Systeme sind quasilinear symmetrisch hyperbolisch von zweiter Ordnung. Lokale Wohlgestelltheit solcher Probleme folgt aus Resultaten von Hughes, Kato und Marsden von 1977. Die Ergebnisse der vorliegenden Arbeit zeigen, dass für Anfangsdaten nahe homogener Referenzzustände sogar globale Lösungen existieren und diese asymptotisch gegen den Referenzzustand abklingen.

Um diese Resultate zu erhalten, werden zunächst Abkling- und Energieabschätzungen für das linearisierte Problem bewiesen und das nichtlineare Problem wird dann als Störung des linearen betrachtet. Dieses allgemeine Konzept geht auf die Dissertation von S. Kawashima von 1983 zurück.

In Kapitel 2 werden konkret barotrope Fluide behandelt. Hier werden Energiemethoden im Fourierraum verwendet, um die Abklingabschätzungen zu beweisen.

In Kapitel 3 geht es um allgemeine Fluide. Für den Beweis der Abklingabschätzungen nutzt man hier Definitheitseigenschaften der punktweisen Matrixdarstellung hyperbolischer Operatoren im Fourierraum.

In Kapitel 4 werden zwei Erweiterungen für barotrope Fluide betrachtet. Zunächst werden die Resultate aus Kapitel 2 nun auch für elektromagnetische Fluide gezeigt. Schließlich werden Konvergenzresultate für eine singuläre Störung der linearisierten Gleichugen bewiesen.