# Conditional Variational Analysis and Path-dependent Optimization

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# Abstract

Optimization and variational problems are considered in a conditional setting. To enlarge the classical deterministic case we optimize utilities conditioned on information given by a $\sigma$-algebra. Naturally, these optimization problems can be described by conditional analysis, formally using conditional sets or $L^0$-modules. In $L^0$-modules, $\mathbb{P}$-almost sure convergence can be induced by a conditional metric, thus, conditional sets give a different approach to optimization in stochastics.

Assumptions on the utility in conditional analysis are weaker than in the theory of random sets. Therefore, conditional variational analysis provides directly the existence of optimizers. Based on conditional sets we derive conditional topological results and give a conditional integral dependent on the information. Further, conditional versions of standard theorems in measure theory are stated.

As in classical variational analysis, a conditional set convergence is derived and applied to conditional hypographs which is used to control maximizers. Then, we give a conditional version of a saddle point problem. A conditional version of Brouwers fixed point theorem gives the existence and stability of a Walras equilibrium, an economy randomly driven by offer and demand. Finally, the optimization problem is solved in multiple periods by the Bellman principle where the utility function additionally may depend on observed history.

With a fixed probability measure conditional variational analysis provides all results directly without a lot of technical assumptions known from variational analysis for random sets an measurable selection. These are, for example, topological properties or integrability conditions on the utility. The setting also works for infinite dimensional spaces. Thus, conditional variational analysis contributes to stochastic optimization.

# Zusammenfassung

Optimierungs- und Variationsprobleme werden in einem bedingten Zusammenhang untersucht. Den klassischen, deterministischen Fall erweitern wir, indem wir Nutzenfunktionen auf durch eine $\sigma$-Algebra gegebene Information bedingt optimieren. Natürlicherweise können diese Optimierungsprobleme mittels bedingter Analysis beschrieben werden, formell durch die Verwendung von bedingten Mengen oder $L^0$-Modulen. In $L^0$-Modulen wird $\mathbb{P}$-fast sichere Konvergenz durch eine bedingte Metrik induziert, daher gibt bedingte Theorie einen alternativen Ansatz zur stochastischen Optimierung.

Die an die Nutzenfunktion gestellten Bedingungen sind schwächer als bei zufälligen mengenwertigen Abbildungen. Deshalb liefert bedingte Variationsrechnung direkt die Existenz von Optima. Auf Grundlage bedingter Mengen leiten wir bedingte topologische Resultate her und führen ein bedingtes informationsabhängiges Integral ein. Weiterhin werden bedingte Versionen von wesentlichen Sätzen in der Maßtheorie ausgewiesen.

Wie in klassischer Variationsrechnung wird eine bedingte Mengenkonvergenz hergeleitet und auf bedingte Hypographen angewendet, um Maximierer zu kontrollieren. Danach geben wir eine bedingte Version eines Sattelpunktproblemes. Eine bedingte Version des Brouwerschen Fixpunktsatzes liefert die Existenz und Stabilität eines Walras-Gleichgewichtes, ein zufällig durch Angebot und Nachfrage getriebener Markt. Schließlich wird das Optimierungsproblem in mehreren Zeitschritten mittels des Bellman-Prinzips gelöst, wobei die Nutzenfunktion zusätzlich von der beobachteten Vergangenheit abhängen kann.

Mittels eines festgesetzten Wahrscheinlichkeitsmaßes liefert bedingte Variationsrechnung alle Ergebnisse auf direkte Weise und ohne einige technische Prämissen der Variationsrechnung für mengenwertige Funktionen. Zum Beispiel sind dies topologische Eigenschaften oder eine Integrabilitätsbedingung an die Nutzenfunktion. Dieses funktioniert in endlich- und unendlich-dimensionalen Räumen. Auf diese Weise trägt bedingte Variationsrechnung zur stochastischen Optimierung bei.

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# Contents

# Introduction

The main subject of this thesis is the study of utility optimization in a conditional setting. With information given by a $\sigma$-algebra, the utility is optimized with respect to the information. In a natural way, conditional sets or $L^0$-theory consider this class of problems.

In stochastics, $\mathbb{P}$-almost sure convergence on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is not induced by a metric. Convergence in probabilty is induced by a metric, but stronger. In $L^0$-theory, a conditional metric inducing $\mathbb{P}$-almost sure convergence can be introduced which allows for a different approach to optimization in stochastics. Thus, classical optimization is done pointwisely, in a conditional setting we optimize with topological methods in a measurable way.

In multiple time steps, conditional theory allows for measurable and not only expected utilities when passing from one time step to another. Also, utilities may depend on the observed history and not only to current information.

There are different ways to evaluate assets in stochastics: preferences, risk measures or utility functions. For preference and risk measures, the conditional setting is natural and has been applied to optimization. Risk measures have been introduced by Artzner et al. [ADEH99] and put in a dynamically setting, for example in Cheridito et al. [CDK06] or Detlefsen and Scandolo [DS05]. A conditional setting for risk measures has been given by Filipović et al. [FKV12]. Preferences in a dynamic setting have been introduced by Kreps and Porteus [KP78, KP79] and preferences in a conditional setting have been studied by Bielecki et al. [BCDK13], Karliczek [Kar14] or Drapeau and Jamneshan [DJ16].

To involve the information given by the $\sigma$-algebra, the optimization problems have been regarded in $L^0$-modules. Early applications of the theory of $L^0$-modules can be found in Cheridito et al. [CKV15], Filipović et al. [FKV09] and Guo [Guo10b], [Guo11]. The concept of $\sigma$-stability which is a local property is very important for $L^0$-theory. Therefore, conditional theory has been introduced as a generalization of $L^0$-theory by Drapeau et al. [DJKK16] and Jamneshan [Jam14]. For further references to conditional analysis in $L^0$-modules and applications, we refer to Bachhoff and Horst [BH16], Cerreira et al. [CVKM+16], Cheridito et al.[CKV15], Frittelli and Maggis [FM14], Guo [Guo10a], Orihuela and Zapata [OZ16b, OZ16a, Zap], and for optimization in conditional analysis we refer to Jamneshan et al. [JKZ18].

In variational analysis or stochastic control theory, measurable optimization problems are regarded pointwisely as set-valued maps. In order to find an optimal solution, the problem is optimized pointwisely for each $\omega \in \Omega$. The existence of a global solution which is measurable is obtained by the application of a measurable selection theorem, cf., for example, Kuratowski and Ryll-Nardzeski [KRN65] or Castaing [Cas67], with applications provided in Rockafellar and Wets [RW09] and Pham [Pha09]. To avoid to verify the assumptions of the measurable selection, or simply, the necessity to consider measurablity additionally, we make use of the approach that is proposed by $L^0$-modules, namely, we make use of the conditional metric that induces $\mathbb{P}$-almost sure convergence and covers the classical pointwise results simultaneously.

In Chapter 1, based on the definition of conditional sets we sum up the main results of conditional topologies from [DJKK16]. Then, a conditional integral is derived analogously to that in standard measure theory, cf. Elstrodt [Els96] as well as applications like a conditional Radon-Nikodým or Fubini theorem. A discussion of conditional expectation in the context of conditional theory is posponed to Chapter 4.

In Chapter 2, a conditional variational analysis setting is introduced whose classical equivalent can be found in Rockafellar and Wets [RW09]. The concept of set convergence is transferred to conditional sets. It originally dates back to Painlevé, Hausdorff [Hau27] and Kuratowski [Kur33], and for economic application see for example in Debreu [Deb67]. Then, we introduce conditional versions of hypo- and lopsided convergence in the context of conditional sets, classically, the former was introduced by Wijsman [Wij64], [Wij66], the latter appears in Attouch [Att84], Attouch and Wets [AW83] and Aubin and Frankowska [AF90]. Convex optimization problems and dual characterizations in $L^0$-Theory can be found in Filipović et al. [FKV09].

The Brouwer fixed point theorem is a fundamental theorem in mathematics. We give the introduction as in the paper by Drapeau et al. [DKKS13] in Chapter 2.11. Its application in game theory suits to the intentions of $L^0$-theory. Equivalent to the Brouwer fixed point theorem is the Ky Fan inequality. The setting in Aubin and Ekeland [AE06] is put into a conditional context and its equivalence to the Brouwer fixed point theorem is shown.

The conditional variational analysis setting is applied to solve a Walras equilibrium problem in Chapter 3. Here, in the context of random variables, the measurable selection theorem is crucial and its preconditions cannot be easily verified. The classical Walras problem is described in Jofré and Wets [JW02] where the Walrasian is introduced for an economic, a set-valued bivariate function similar to the Hamiltonian in a Calculus of Variations setting which we put it into a conditional setting. The stability of the equilibrium is also described in terms of hypo- and lopsided convergence. Other authors deal with slightly different assumptions, cf. Flåm [Flå94] or Lucchetti and Patrone [LP86], or with dependency on the endowment, cf. Mas-Colell [MC85], or Balasko [Bal88], [Bal03].

Finally, in Chapter 4, we apply the Walras equilibrium in multiple periods in order to optimize utilities. Conditional sets on a filtration of $\sigma$-algebras is a family of $L^0$-modules over different rings which are nested. For a first examination in a martingale context, see [Hei14]. The effects on basic concepts in conditional theory are discussed, such as conditional topology and conditional functions along with their continuity and semicontinuity. Here, we also consider the conditional expectation in the context of conditional theory. To solve the utility optimization problem in multiple periods, it is decomposed in one-step models where utilities are connected by generators. This procedure is as in dynamic programming principle, but the conditions on the generator are somewhat different. Optimal utilities at time $t-1$ are obtained by maximizing the generator over trading strategies where the generator depends on the maximal utility at time $t$ and trading strategies from time $t-1$ to time $t$. Together with the conditional integral definition, the setting also gives an alternative approach to normal integrands, cf. Rockafellar and Wets [RW09].

## Basic Notation

In this section we sum up some notation which will be used in the sequel. By doing so we generally follow Bauer [Bau92].

By $\mathbb{N}$ and $\mathbb{R}$ we denote the sets of natural numbers and of real numbers, respectively. Extending the real line by $\pm\infty$ to make it compact we define $\overline{\mathbb{R}}$ to be this extended real line. Addition with $\pm\infty$ is defined as follows: $a + (\pm\infty) = \pm\infty$ for all $a \in \mathbb{R}$, $(+\infty)+(+\infty) = +\infty$ and $(-\infty)+(-\infty) = -\infty$. Both, $(-\infty)+(+\infty)$ and $(+\infty)+(-\infty)$ are not defined. Multiplication with $\pm\infty$ is defined as follows: $a\cdot(\pm\infty) = \pm\infty$ for $a \in \mathbb{R}$ which are positive and $a = \infty$, $a\cdot(\pm\infty) = \mp\infty$ for $a \in \mathbb{R}$ which are negative and $a = -\infty$. Additionally, we define $0 \cdot (\pm\infty) = 0$. The relation $<$ extends to $\overline{\mathbb{R}}$ via $-\infty < a < \infty$ for all $a \in \mathbb{R}$, and $-\infty < \infty$.

We write $m \in M$ if $m$ is an element of a set $M$. A set $M^{'}$ is a subset of $M$ if for any $m \in M^{'}$ it holds $m \in M$ and we write $M^{'} \subset M$. For operations on sets we use the symbol $\bigcup$ for union, the symbol $\bigcap$ for intersection and the symbol $\cdot^c$ for the set complement. Sets are called disjoint if their intersection is $\emptyset$.

A mapping of a set $L$ into a set $M$ is denoted by $f\colon L \to M$. The mapping $f$ is real-valued if $M = \mathbb{R}$. A sequence in a set $M$ is a mapping $f\colon \mathbb{N} \to M$. For $f(n)$ we usually write $a_n$, and for the mapping $f$ we write $(a_n)_{n\in\mathbb{N}}$. More generally, if we have a mapping $f\colon I \to M$ for some index set $I$, we write $(a_i)_{i\in I}$ for a family of elements in $M$.

Next we consider measurable spaces. A measurable space $(\Omega, \mathcal{F})$ is a pair consisting of a set $\Omega$ and a $\sigma$-algebra $\mathcal{F}$ on $\Omega$. A $\sigma$-algebra $\mathcal{F}$ on $\Omega$ is a family of subsets of $\Omega$ such that $\emptyset \in \mathcal{F}$ and for any elements of $\mathcal{F}$ their complements and countable unions are also elements of $\mathcal{F}$. For two $\sigma$-algebras $\mathcal{F}$ and $\mathcal{G}$ we say $\mathcal{F}$ is finer than $\mathcal{G}$ if any element of $\mathcal{G}$ is an element of $\mathcal{F}$ and we write $\mathcal{G} \subset \mathcal{F}$. The trace of the $\sigma$-algebra $\mathcal{F}$ on $A \in \mathcal{F}$ is denoted by $\mathcal{F}^A$, i. e. $\mathcal{F}^A = \{A \cap F \mid F \in \mathcal{F}\}$. If $(\Omega, \mathcal{F})$ and $(M, \mathcal{M})$ are measurable spaces then a mapping $X\colon \Omega \to M$ is measurable if $f^{-1}(M^*) \in \mathcal{F}$ for all $M^* \in \mathcal{M}$, where $f^{-1}(M^*)$ denotes the pre-image of $M^*$. If $M = \mathbb{R}$ is endowed with the Borel-$\sigma$-algebra, the $\sigma$-algebra generated by the open intervals in $\mathbb{R}$, then $X$ is a real-valued $\mathcal{F}$-measurable function, a random variable.

Next, we consider probability spaces. A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a triplet consisting of a set $\Omega$, a $\sigma$-algebra $\mathcal{F}$ on $\Omega$ and a probability measure $\mathbb{P}$ on the measurable space $(\Omega, \mathcal{F})$. The probability measure $\mathbb{P}$ is a function on $\mathcal{F}$ with values in $[0, 1]$. It holds that $\mathbb{P}(\emptyset) = 0$ and $\mathbb{P}(\bigcup_{n\in\mathbb{N}} F_n) = \sum_{n\in\mathbb{N}} \mathbb{P}(F_n)$ for pairwise disjoint $F_n \in \mathcal{F}$, $n \in \mathbb{N}$. By $\mathcal{F}_+$ we denote all elements of $\mathcal{F}$ which have positive measure.

A partially ordered set $P$ is a set endowed with a partial order $\leq$. A partial order $\leq$ is a binary relation which is reflexive, antisymmetric and transitive. That means, for all $a$, $b$, $c \in P$, we have $a \leq a$, if $a \leq b$ and $b \leq a$ then $a = b$, and if $a \leq b$ and $b \leq c$ then $a \leq c$, respectively. A lattice $(P, \leq)$ is a partially ordered set which in addition fulfills the following condition: if $a$, $b \in P$ then $a \wedge b$, $a \vee b \in P$, where $a \wedge b$ denotes the supremum or least upper bound of $a$ and $b$ and $a \vee b$ denotes the infimum or greatest lower bound of $a$ and $b$. For an introduction into rings, see Jacobson [Jac09] or Lang [Lan02].

## Basic Notation in $L^0$-theory

We fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $L^0 := L^0(\Omega, \mathcal{F}, \mathbb{P})$ and $\overline{L}^0$ be the space of all $\mathcal{F}$-measurable random variables with values in $\mathbb{R}$ and $\overline{\mathbb{R}}$, respectively, where $\mathbb{P}$-almost surely equal random variables are identified. In particular, we identify $A, B \in \mathcal{F}$ if $\mathbb{P}((A \backslash B) \cup (B \backslash A)) = 0$ and we define $\mathcal{F}_+ = \{A \in \mathcal{F} \mid \mathbb{P}(A) > 0\}$. For $X, Y \in$

$L^0$, the relations $X \geq Y$ and $X > Y$ are understood $\mathbb{P}$-almost surely. The set $L^0$ with the almost everywhere order is a lattice ordered ring, and every nonempty subset $\mathcal{C} \subset \overline{L}^0$ has a least upper bound $\operatorname{ess\,sup} \mathcal{C}$ and a greatest lower bound $\operatorname{ess\,inf} \mathcal{C}$ (cf. [FKV09],[FS04]). For a subset $\mathcal{M} \subset L^0$ we further denote $\mathcal{M}_+ := \{X \in \mathcal{M} \mid X \geq 0\}$ and $\mathcal{M}_{++} := \{X \in \mathcal{M} \mid X > 0\}$ as well as $\underline{L}^0 := \left\{X \in \overline{L}^0 \mid X < \infty\right\}$. For $m \in \mathbb{R}$, we denote the constant random variable $m \cdot \mathbb{1}_\Omega$ by $m$. The $L^0$-scalar product and $L^0$-norm on $\left(L^0\right)^d := \left\{(X_1, \ldots, X_d) \mid X_i \in L^0\right\}$ are defined as

$$\langle X, Y \rangle = \sum_{i=1}^{d} X_i Y_i \quad \text{and} \quad \|X\| = \langle X, X \rangle^{\frac{1}{2}} .$$

We call $\mathcal{C} \subset \left(L^0\right)^d$ bounded if $\operatorname{ess\,sup}_{X \in \mathcal{C}} \|X\| \in L^0$. We introduce the balls $\mathcal{B}^\varepsilon(X) := \left\{Y \in L^0 \mid \|X - Y\| < \varepsilon\right\}$ of random variables centered at $X$ with radius $\varepsilon > 0$. We recall that $L^0$ is an $L^0$-module.

A sequence $(X_n)_{n \in \mathbb{N}}$ of random variables converges $\mathbb{P}$-almost surely to a random variable $X \in \overline{L}^0$ if $\mathbb{P}\left(\omega \in \Omega \mid \lim_{n \to \infty} X^n(\omega) = X(\omega)\right) = 1$. A function $f \colon L^0 \to L^0$ is called $\mathbb{P}$-almost surely continuous if $f(X_n)$ converges $\mathbb{P}$-almost surely to $f(X)$ whenever $X_n$ coverges $\mathbb{P}$-almost surely to $X$.

In many optimization problems, continuity, in this case, $\mathbb{P}$-almost sure continuity, is relaxed to semicontinuity when considering the extended-valued reals. Thus, a function $f \colon \left(L^0\right)^d \to \left(L^0\right)^d$ is called upper semicontinuous if $\operatorname{ess\,lim\,sup}_{n \to \infty} f(X_n) \leq f(X)$ for every $\mathbb{P}$-almost surely converging sequence $X_n \to X$, where $\operatorname{ess\,lim\,sup}_{n \to \infty} f(X_n) := \operatorname{ess\,inf}_{n \to \infty} \operatorname{ess\,sup}_{m > n} f(X_m)$.

Classically, $\mathbb{P}$-almost sure convergence is not induced by a metric. So, one can pass to convergence in probability, induced by the metric $d(X, Y) := \mathbb{E}\left[\min(1, |X - Y|)\right]$ on the space of random variables. If we consider $L^0$-modules as in Filipović et al. [FKV09] and Cheridito et al. [CKV15] the conditional distance has many properties of metrics, but we do not need convergence in probability.

The concept of $\sigma$-stability introduced in Filipović et al. [FKV09] turns out to be crucial. The $\sigma$-stable hull of a set $\mathcal{C} \subset L^0$ is defined as

$$\sigma(\mathcal{C}) = \left\{ \sum_{i \in \mathbb{N}} \mathbb{1}_{A_i} X_i \mid X_i \in \mathcal{C},\ (A_i)_{i \in \mathbb{N}} \text{ is a partition} \right\},$$

where a partition is a countable family $(A_i)_{i \in \mathbb{N}} \subset \mathcal{F}$ such that $\mathbb{P}(A_i \cap A_j) = 0$ for $i \neq j$ and $\mathbb{P}\left(\bigcup_{i \in \mathbb{N}} A_i\right) = 1$. We call a nonempty set $\mathcal{C}$ $\sigma$-stable if it is equal to $\sigma(\mathcal{C})$.

# 1 Conditional theory

## 1.1 Introduction to conditional sets

Stochastic optimal control problems will be solved by methods of conditional theory which fundamentally has been introduced in [DJKK16] with some ideas already presented in $L^0$-theory such as in [CKV15], [DKKS13] or [FKV09]. Classically, these optimization problems are regarded pointwisely. In the end a measurable solution is obtained by applying a measurable selection theorem. The measurable selection approach can be found in [RW09, Chapter 14] or the preceeding work by [Cas67]. To present an alternative approach in conditional theory, we give an introduction to conditional sets for our purpose.

### 1.1.1 Conditional sets

Let $\mathcal{A} := (\mathcal{A}, \vee, \wedge, ^c, 0, 1)$ be a complete Boolean Algebra. The probabilistic example is the $\sigma$-algebra $\mathcal{F} = (\mathcal{F}, \cup, \cap, ^c, \emptyset, \Omega)$, where $A, B \in \mathcal{F}$ are identified if $\mathbb{P}(A \triangle B) = 0$, other examples with their pecularities can be found in [DJKK16]. With the relation $a \leq b$ if and only if $a \wedge b = a$ for $a, b \in \mathcal{A}$, the pair $(\mathcal{A}, \leq)$ is a complete complemented distributive lattice, particularly this holds for the example $(\mathcal{F}, \subset)$. A partition of $a \in \mathcal{A}$ is a family $(a_i)_{i \in I}$ in $\mathcal{A}$ such that $\vee_{i \in I} a_i = a$ and $a_i \wedge a_j = 0$ if $i \neq j$. In $\mathcal{F}$, a partition of $A \in \Omega$ is a family $(A_i)_{i \in \mathbb{N}}$ in $\mathcal{F}$ such that $\bigcup_{i \in \mathbb{N}} A_i = \Omega$ and $\mathbb{P}(A_i \cap A_j) = 0$ for $i \neq j$. By the well-ordering theorem, for every family $(a_i)_{i \in I}$ there exists a partition $(b_i)_{i \in I}$ of $\vee_{i \in I} a_i$ such that $b_i \leq a_i$ for all $i \in I$ [DJKK16]. Thus, in the sequel, we assume all families $(a_i)_{i \in I}$ to be partitions of $\vee_{i \in I} a_i$. Now, we can define conditional sets.

**Definition 1.1** (Conditional set, [DJKK16])**.** Let $\mathfrak{X}$ be a nonempty set and let $\mathcal{A} := (\mathcal{A}, \vee, \wedge, ^c, 0, 1)$ a Boolean Algebra. The set $\mathbf{X} := (\mathfrak{X}, \mathcal{A})$ is a conditional set if it is a nonempty collection of objects of the form $X|a$ for $X \in \mathfrak{X}$ and $a \in \mathcal{A}$ such that

(i). $X|a = Y|b$ implies $a = b$ for all $X, Y \in \mathfrak{X}$ and $a, b \in \mathcal{A}$,

(ii). $X|a = Y|a$ implies $X|b = Y|b$ for all $X, Y \in \mathfrak{X}$ and $a, b \in \mathcal{A}$ with $b \leq a$, (iii). there exists exactly one $X \in \mathfrak{X}$ such that $X|a_i = X_i|a_i$ for all $X_i \in \mathfrak{X}$, $i \in I$ and every partition $(a_i)_{i\in I}$ of $1 \in \mathcal{A}$ (and thus also for any partition of any element in $\mathcal{A}$, cf. [DJKK16]).

In (iii), the element $X$ is denoted by $\sum_{i\in I} X_i|a_i$.

We remark here that in (i), if we set $b = 1$, we have $\mathbf{X} = \{X|1 \mid X \in \mathbf{X}\}$. That is why we also write $\mathbf{X}$ for the pair $(\mathfrak{X}, \mathcal{A})$. Furthermore, by (iii), the set $\{X|0 \mid X \in \mathbf{X}\}$ consists of exactly one element, denoted by $\mathbf{X}|0$. There is no further meaning of this element.

**Example 1.2.** The main example in the sequel is the set of random variables. We write $L^0_A := L^0\left(\Omega, \mathcal{F}^A, \mathbb{P}|_A\right)$ for $\mathbb{P}|_A(B) := \mathbb{P}(A \cap B)$ and consider the sets $X|A = \left\{Y \in L^0_A \mid \mathbb{P}(\omega \in A \mid X(\omega) = Y(\omega)) = \mathbb{P}(A)\right\}$ for any $X \in L^0$ and $A \in \mathcal{F}_+$ which form a conditional set.

**Remark 1.3.** The element $\mathbf{X}|0$ plays the role of a random variable that is defined on a $\mathbb{P}$-nullset, or, equivalently, on the empty set. Thus, it may be identified with the empty set, with the necessity to modify condition (iii). For simplicity of condition (iii) we regard conditional sets as stated. This connection will be clear when defining the conditional inclusion, where $\{\mathbf{X}|0\}$ is conditionally subset of any conditional subset of $\mathbf{X}$.

**Definition 1.4** (Conditional inclusion, [DJKK16])**.** Let $\mathbf{X}$ be a conditional set. A subset $\mathcal{Y}|a \subset \mathbf{X}$ is called stable, if $\mathcal{Y} = \left\{\sum_{i\in I} Y_i|a_i \text{ for a partition } (a_i)_{i\in I} \text{ of } a \text{ and } Y_i \in \mathcal{Y}\right\}$. Let $(\mathbf{X}_1, \mathcal{A}_1)$ and $(\mathbf{X}_2, \mathcal{A}_2)$ be two stable conditional sets. We say that $\mathbf{X}_2 \sqsubset \mathbf{X}_1$ is a conditional subset of $\mathbf{X}_1$ if there exists $\overline{a} \in \mathcal{A}_1$ such that $\mathcal{A}_2 = \{\overline{a} \wedge a_1 \mid a_1 \in \mathcal{A}_1\}$ and $\mathbf{X}_2|\overline{a} \subset \mathbf{X}_1|\overline{a}$. The conditional power set is $\mathbf{P}(\mathbf{X}_1) = \{\mathbf{X}_2 \sqsubset \mathbf{X}_1 \mid \mathbf{X}_2 \text{ is a conditional set}\}$. Typically, we write $\mathcal{X} \sqsubset \mathbf{X}$ for a conditional subset $\mathcal{X}$ of a given conditional set $\mathbf{X}$.

By Definition 1.1, every conditional set $\mathbf{X}$ is stable. Thus, the conditional inclusion as a relation between conditional sets inherits stability. There may be further subsets of a conditional set $\mathbf{X}$. In $L^0$, any stable subset of $L^0$ is a conditional subset of $L^0$. Instead of conditional subset or stable, we say $\sigma$-stable subset of $L^0$ for the underlying algebra is a $\sigma$-algebra.

**Theorem 1.5.** *Let $\mathbf{X}$ be a conditional set. Then $(\mathbf{P}(\mathbf{X}), \sqsubset)$ is a complete complemented distributive lattice.*

*Proof.* The proof can be found in [DJKK16]. The main difficulty is to ensure that there are the supremum and infimum of conditional subsets with respect to conditional inclusion that also are conditional sets. $\square$

**Definition 1.6** (Conditional operations on conditional sets, [DJKK16])**.** Let $(\mathcal{X}_i, a_i)_{i\in I}$ be conditional subsets of a conditional set $\mathbf{X}$. Then, we define the conditional intersection $\sqcap_{i\in I}\mathcal{X}_i := (\{X \in \mathbf{X} \mid X \in \mathcal{X}_i \text{ for all } i \in I\}, \wedge_{i\in I} a_i)$ which is a conditional set and the largest conditional set with respect to conditional inclusion that is a conditional subset of all $\mathcal{X}_i$, $i \in I$. The conditional union $\mathcal{Y} = \bigsqcup_{i\in I}\mathcal{X}_i$ is the smallest conditional set with respect to conditional inclusion such that $\mathcal{X}_i \sqsubset \mathcal{Y}$ for all $i \in I$, or $\bigsqcup_{i\in I}\mathcal{X}_i = \left\{\sum_{i\in I} X_i|b_i \text{ for a partition } (b_i)_{i\in I} \text{ of } \vee_{i\in I} a_i \text{ and } X_i \in \mathcal{X}_i\right\}$. The conditional complement of a conditional subset $\mathcal{X} \sqsubset \mathbf{X}$ is $\mathcal{X}^{\sqsubset} := \bigsqcup\{\mathcal{Y} \sqsubset \mathbf{X} \mid \mathcal{X} \sqcap \mathcal{Y} = \mathbf{X}|0\}$.

We say that some $\mathcal{X} \sqsubset \mathbf{X}$ lives or is on $a \in \mathcal{A}$ if $\mathcal{X} \subset \mathbf{X}|a$ for $a \in \mathcal{A}$. For $\mathcal{X}, \mathcal{Y} \in \mathbf{X}$, it holds that $\mathcal{X} \cap \mathcal{Y} \neq \emptyset$ if and only if $\mathcal{X} \sqcap \mathcal{Y}$ lives on $1 \in \mathcal{A}$. We remark that also $\sum_{i\in I} X_i|a_i = \bigsqcup_{i\in I} X_i|a_i$ for all $X_i \in \mathbf{X}$, $i \in I$ and every partition $(a_i)_{i\in I}$ of $1 \in \mathcal{A}$.

**Theorem 1.7.** *Let $\mathbf{X}$ be a conditional set. Then $\mathbf{P}(\mathbf{X}) = (\mathbf{P}(\mathbf{X}), \sqcup, \sqcap, ^{\sqsubset}, \mathbf{X}|0, \mathbf{X})$ is a complete Boolean algebra.*

*Proof.* The proof in all detail can be found in [DJKK16]. □

**Definition 1.8** (Conditional partial order, [DJKK16])**.** Let $(\mathbf{X}_i)_{i\in I}$ be a nonempty conditionally countable family of conditional sets and their conditional product $\prod_{i\in I}\mathbf{X}_i := \left\{\left((X_i)_{i\in I}|a\right)_{a\in\mathcal{A}}, X_i \in \mathbf{X}_i\right\}$ which is a conditional set. A conditional relation $\trianglelefteq$ on $\mathbf{X}_1 \times \mathbf{X}_2$ is a conditional subset of $\mathbf{X}_1 \times \mathbf{X}_2$ that lives on $1 \in \mathcal{A}$. A conditional partial order on $\mathbf{X} \times \mathbf{X}$ is a conditional relation that is antisymmetric, reflexive, symmetric and transitive. It is conditionally total if for all $X, Y \in \mathbf{X}$ there are $a_1, a_2 \in \mathcal{A}$ such that $a_1 \vee a_2 = 1$, $X|a_1 \trianglelefteq Y|a_1$ and $X|a_2 \trianglerighteq Y|a_2$. A conditional set is conditionally directed if it is closed with respect to supremum or infimum of $\mathbf{2}$ elements.

For simplicity, for all conditional sets $\mathcal{C} \sqsubset \mathbf{X}$, we write $\mathcal{C}_+ := \{X \in \mathcal{C} \mid X \geq 0\}$ and $\mathcal{C}_{++} := \{X \in \mathcal{C} \mid X > 0\}$.

Having introduced conditional sets we discuss how to generate a conditional set $\mathbf{E}$ from a given nonempty set $E$ and a given Boolean algebra $\mathcal{A}$. The elements $e \in E$ are isomorphic to the elements $e|1 \in \mathbf{E}$. Next, we consider the stable hull generated by $e|1$ for $e \in E$ for all partitions of $1 \in \mathcal{A}$ and identify elements $\sum_{i\in I} e_i|a_i$ and $\sum_{i\in I} e_i'|a_i'$ if they are equal, that is, for all $e \in E$, it holds that $\vee_{i\in I}\{a_i \in \mathcal{A} \mid e_i|a_i = e|a_i\} = \vee_{i\in I}\{a_i' \in \mathcal{A} \mid e_i'|a_i' = e|a_i'\}$. Indeed, this is an equivalence relation. Now, the constructed conditional set $\mathbf{E}$ contains the objects $\sum_{i\in I}(e_i|a_i)|a$ for all $a \in \mathcal{A}$ under the identification of the equivalence classes. This conditionel set $\mathbf{E}$ is the conditional set on the Boolean algebra $\mathcal{A}$ with values in $E$. Examples are the random variables with values in $\mathbb{N}$, denoted by $\mathbf{N}$, or with values in $\mathbb{Q}$, denoted by $\mathbf{Q}$. A natural conditional partial order on $\mathbf{Q} \times \mathbf{Q}$ order is given by $\sum_{i \in I} X_i | a_i \trianglelefteq \sum_{j \in J} Y_J | b_j$ if $X_i \leq Y_j$ in $\mathbb{Q}$ whenever $a_i \wedge b_j > 0$.

For the construction of $\mathbf{R}$, the random variables with values in $\mathbb{R}$, one usually introduces conditional relation, Cauchy sequences and completeness. The proof is done as for the construction of $\mathbb{R}$, we do not give it explicitely and refer to [DJKK16] for the details.

This construction yields the same result as passing from $\mathbb{R}$ to $L^0$ via equivalence classes of $\mathbb{P}$-almost sure equal random variables, as shown in [DJKK16]. In the sequel, we only write $\mathbf{R}$ instead of $L^0$ to stress the conditional point of view, not the pointwise approach as classically for $L^0$, although, as been shown in [DJKK16], the extension of the natural conditional partial order from $\mathbf{Q}$ to $\mathbf{R}$ yields a conditional partial order that coincides with almost-sure dominated order. In order to maintain this, we write essential supremum, infimum and limit, and, as classically, $X \leq Y$ for random variables, elements of $\mathbf{R}$. The conditional set $\overline{\mathbf{R}}$ is then the extended conditional set $\mathbf{R}$ which includes also $\infty$ and $-\infty$ by the same construction starting from $\mathbb{N} \cup \{\infty\}$.

**Definition 1.9** (Conditional function)**.** Let $\mathbf{X}$ and $\mathbf{Y}$ be conditional sets over the same Boolean algebra $\mathcal{A}$. Let $f\colon \mathbf{X} \to \mathbf{Y}$ be a function. It is a conditional function if $f\left(\sum_{i \in I} X_i | a_i\right) = \sum_{i \in I} f(X_i) | a_i$.

Conditional functions appear naturally, for example, if a function from $L^0$ to $L^0$ is $L^0$-convex, it is a conditional function, cf. [FKV09]. Convexity is discussed in Section 2.8.

**Remark 1.10.** With the observation that any element $X \in \mathbf{X}$ that lives on 1 is up to equivalence classes an element of $\mathfrak{X}$ and vice versa, the property $f\left(\sum_{i \in I} X_i | a_i\right) = \sum_{i \in I} f(X_i) | a_i$ for a conditional function $f\colon \mathbf{X} \to \mathbf{Y}$ is induced by the same porperty but for a stable function $f\colon \mathfrak{X} \to \mathfrak{Y}$. This identification allows for writing all the theorems in the setting of conditional sets. We will do so in the sequel.

Since the most important example of conditional sets is $\mathbf{R}$, to consider the space of random variables covers already a large subclass of conditional sets. Then passing to a Boolean algebra is barely an algebraic question. Thus, for the economic examples considered here we put rather emphasis on the probability and measurable discussion. Similarly, the $\mathbf{R}$-valuedness as a restriction of general conditional sets is reasoned by practicability and economic applicability. That is why in Chapter 1 we put the more general case of arbitrary conditional sets to demonstrate generality, and in the preceeding chapters we restrict ourselves to $\mathbf{R}$. Additionally, semicontinuity which will be discussed later is classically only defined in topological spaces or in the space of random variables. The example of $\mathbf{R}$ is the main example for all the properties discussed in conditional sets.

**Definition 1.11** (Conditionally countable, [DJKK16])**.** Let $\mathbf{X}$ be a conditional set. It is conditionally countable if there exists a injective conditional function $f\colon \mathbf{X} \to \mathbf{N}$. It is conditionally finite if there exists $N \in \mathbf{N}$ and a bijective conditional function $f\colon \mathbf{X} \to \{1, \dots, N\}$.

In a conditional set, conditional sequences are images of conditional functions of conditionally countable index sets.

**Definition 1.12** (Conditional family and sequence, [DJKK16])**.** Let $\mathbf{X}$ be a conditional set. The set $(X_J)_{J\in\mathcal{J}}$ is a conditional family if it is the image of a conditional function $\iota\colon \mathcal{J} \to \mathbf{X}$. It is a conditional sequence if $\mathcal{J} \sqsubset \mathbf{N}$ and $\iota$ is injective.

### 1.1.2 Conditional topology

**Definition 1.13** (Conditional topology, [DJKK16])**.** Let $\mathbf{X}$ be a conditional set. A family $\mathfrak{T} = (\mathcal{O}_J)_{J\in\mathcal{J}}$ of conditional subsets $\mathcal{O}_J$ of $\mathbf{X}$ is a conditional topology if

(i). $\mathbf{X}|0, \mathbf{X}, \mathcal{O}|a \in \mathfrak{T}$ for all $a \in \mathcal{A}$ and $\mathcal{O} \in \mathfrak{T}$

(ii). $\mathcal{O}_1 \sqcap \mathcal{O}_2 \in \mathfrak{T}$ for $\mathcal{O}_1, \mathcal{O}_2 \in \mathfrak{T}$

(iii). $\bigsqcup_{J\in\mathcal{J}'} \mathcal{O}_J \in \mathfrak{T}$ for $\mathcal{O}_J \in \mathfrak{T}$ for all $J \in \mathcal{J}' \sqsubset \mathcal{J}$.

Elements of $\mathfrak{T}$ are conditionally open. The conditional complement of a conditional open set is conditionally closed. A conditional topology generated by $(\mathcal{T}'_J)_{J\in\mathcal{J}}$ is the smallest conditional topology with respect to conditional inclusion that contains $(\mathcal{T}'_J)_{J\in\mathcal{J}}$. A conditional topological space is the pair $(\mathbf{X}, \mathfrak{T})$. Further, we denote the set of conditional neighbourhoods of $X$ by $\mathfrak{U}(X)$ where a conditional set $\mathcal{U} \sqsubset \mathbf{X}$ is called a neighbourhood of $X$ if there exists a conditionally open conditional set $\mathcal{O}$ such that $X \in \mathcal{O}$ and $\mathcal{O} \sqsubset \mathcal{U}$.

**Definition 1.14** (Conditional closure, conditional interior)**.** Let $(\mathbf{X}, \mathfrak{T})$ be a conditional topological space. The conditional set $\operatorname{cl}(\mathcal{C}) := \bigsqcap \{\mathcal{D} \sqsubset \mathbf{X} \mid \mathcal{C} \sqsubset \mathcal{D},\, \mathcal{D}^{\sqsubset} \in \mathfrak{T}\}$ of a conditional subset $\mathcal{C} \sqsubset \mathbf{X}$ is called the conditional closure of $\mathcal{C}$. The conditional set $\operatorname{int}(\mathcal{C}) := \bigsqcap \{\mathcal{D} \sqsubset \mathbf{X} \mid \mathcal{D} \sqsubset \mathcal{C},\, \mathcal{D} \in \mathfrak{T}\}$ is called the conditional interior of $\mathcal{C}$.

**Remark 1.15.** By definition, $\operatorname{int}(\mathcal{C}^{\sqsubset}) = \operatorname{cl}(\mathcal{C})^{\sqsubset}$ and $\operatorname{cl}(\mathcal{C}^{\sqsubset}) = \operatorname{int}(\mathcal{C})^{\sqsubset}$. A conditional subset $\mathbf{Y} \sqsubset \mathbf{X}$ is conditionally dense if $\operatorname{cl}(\mathbf{Y}) = \mathbf{X}$.

In the sequel, as in [DJKK16], we assume all conditional sets to be conditionally Hausdorff, that is, for $X, Y \in \mathbf{X}$ with $X \sqcap Y = \mathbf{X}|0$, there a conditional neighbourhoods $\mathcal{U}_X \in \mathfrak{U}(X)$ and $\mathcal{U}_Y \in \mathfrak{U}(Y)$ such that $\mathcal{U}_X \sqcap \mathcal{U}_Y = \mathbf{X}|0$. In the sequel, we assume all sets to be conditionally Hausdorff and include it in every definition without further mentioning. Since the conditional set is conditionally Hausdorff, all limit points will be unique [DJKK16].

**Definition 1.16** (Conditional compactness)**.** Let $(\mathbf{X}, \mathfrak{T})$ be a conditional topological space. A cover of $\mathbf{X}$ is a family of conditional subsets $(\mathcal{O}_J)_{J\in\mathcal{J}}$ such that $\bigsqcup_{J\in\mathcal{J}} \mathcal{O}_J = \mathbf{X}$. It is a conditionally open cover, if each $\mathcal{O}_J$ is conditionally open. The conditional set $\mathbf{X}$ is conditionally compact, if for any conditionally open covering $(\mathcal{O}_J)_{J\in\mathcal{J}}$ of $\mathbf{X}$ there exists a conditionally finite subcover $(\mathcal{O}_{J'})_{1\leq J'\leq \overline{J}} \sqsubset (\mathcal{O}_J)_{J\in\mathcal{J}}$ of $\mathbf{X}$, $\overline{J} \in \mathbf{N}$.

**Remark 1.17.** In $\mathbf{R}$, a conditional subset $\mathcal{C} \sqsubset \mathbf{R}$ is conditionally compact if and only if is is conditionally closed and bounded, that is, there exists $X \in \mathbf{R}$ such that $-X \leq Y \leq X$ for all $Y \in \mathcal{C}$. The proof is the same as in $\mathbb{R}$, we do not give it explicitely.

Conditional topologies can be induced by conditional distances.

**Definition 1.18** (Conditional distance, [DJKK16])**.** Let $\mathbf{X}$ be a conditional set. A conditional distance is a conditional function $\mathrm{d}\colon \mathbf{X}\times\mathbf{X} \to \mathbf{R}_+$ such that for all $X, Y, Z \in \mathbf{X}$ holds that $\mathrm{d}(X,Y) = 0$ if and only if $X = Y$, $\mathrm{d}(X,Y) = \mathrm{d}(Y,X)$ and $\mathrm{d}(X,Y) \leq \mathrm{d}(X,Z) + \mathrm{d}(Z,Y)$. The pair $(\mathbf{X}, \mathrm{d})$ is a conditional metric space.

**Example 1.19.** Consider $\mathbf{R}^d$. The balls $\mathcal{B}^\varepsilon(X) := \left\{Y \in \mathbf{R}^d \mid \|Y - X\| < \varepsilon\right\}$ for $X \in \mathbf{R}$ and $\varepsilon \in \mathbf{R}_{++}$ generate a conditional topology [DJKK16] and is called the conditional Euclidean distance on $\mathbf{R}$.

We now give the definition of continuity of a conditional sequence in terms of the conditional Euclidean distance.

**Definition 1.20** (Continuity in $\mathbf{R}$)**.** In $\mathbf{R}$, a conditional sequence $(X_J)_{J\in\mathbf{N}}$ converges to $X \in \mathbf{R}$ if vor all $\varepsilon \in \mathbf{R}_{++}$ there exists $N \in \mathbf{N}$ such that $X_J \in \mathcal{B}^\varepsilon(X)$ for all $J \geq N$. We also write $X = \lim_{J\in\mathbf{N}} X_J$

## 1.2 Conditional-valued integration and applications

The following section (including Remark 2.3) is joint work with Asgar Jamneshan and Michael Kupper [JKS18].

### 1.2.1 Conditional measurable structures

**Definition 1.21.** Let $\mathbf{X}$ be a conditional set. A conditional $\sigma$-algebra on $\mathbf{X}$ is a conditional family $\mathfrak{F}$ of conditional subsets of $\mathbf{X}$ satisfying

(i). $\mathbf{X} \in \mathfrak{F}$,

(ii). if $\mathcal{C} \in \mathfrak{F}$, then $\mathcal{C}^{\sqsubset} \in \mathfrak{F}$,

(iii). if $(\mathcal{C}_N)_{N \in \mathbf{N}}$ is a conditional sequence in $\mathfrak{F}$, then $\bigsqcup_{N \in \mathbf{N}} \mathcal{C}_N \in \mathfrak{F}$.

The pair $(\mathbf{X}, \mathfrak{F})$ is a conditional measurable space. Given two conditional measurable spaces $(\mathbf{X}, \mathfrak{F})$ and $(\mathbf{Y}, \mathfrak{H})$, a conditional function $f \colon \mathbf{X} \to \mathbf{Y}$ is conditionally measurable whenever $f^{-1}(\mathcal{D}) \in \mathfrak{F}$ for every $\mathcal{D} \in \mathfrak{H}$.

A conditional $\sigma$-algebra $\mathfrak{F}$ is a $\sigma$-complete Boolean subalgebra of the conditional power set $\mathbf{P}(\mathbf{X})$. Thus the conditional version of those properties of a classical $\sigma$-algebra which are due to Boolean arithmetic (see for the elementary arithmetic of Boolean algebras Chapter 1 of [MKB89]) are automatically fulfilled by a conditional $\sigma$-algebra. For instance, one can weaken Definition 1.21 by replacing the conditional sequence $(\mathcal{C}_N)_{N \in \mathbf{N}}$ by conditionally pairwise disjoint conditional sequences in (iii). Since $\mathbf{X} \in \mathfrak{F}$, it follows from (ii) that $\mathbf{X}|0 = \mathbf{X}^{\sqsubset} \in \mathfrak{F}$. Thus by stability also $\mathcal{C}|a + (\mathbf{X}|0)\,|a^c \in \mathfrak{F}$ for every $\mathcal{C} \in \mathfrak{F}$ and $a \in \mathcal{A}$.

The conditional intersection of any non-empty family $\left(\mathfrak{F}^i\right)_{i \in I}$ of conditional $\sigma$-algebras on some fixed conditional set $\mathbf{X}$ coincides with their classical intersection since $\mathbf{X} \in \mathfrak{F}^i$ for all $i \in I$, and thus it is a conditional $\sigma$-algebra on $\mathbf{X}$. For a conditional set $\mathfrak{E}$ of subsets of some conditional set $\mathbf{X}$ let $\Sigma(\mathfrak{E})$ denote the conditional intersection of all conditional $\sigma$-algebras $\mathfrak{F}$ such that $\mathfrak{E} \sqsubset \mathfrak{F}$. Given a conditional $\sigma$-algebra $\mathfrak{F}$, we say that $\mathfrak{E} \sqsubset \mathfrak{F}$ generates $\mathfrak{F}$ whenever $\Sigma(\mathfrak{E}) = \mathfrak{F}$.

Let $\mathbf{X}$ be a conditional set, $(\mathbf{Y}, \mathfrak{H})$ a conditional measurable space and $f \colon \mathbf{X} \to \mathbf{Y}$ a conditional function. Then $f^{-1}(\mathfrak{H}) := \left\{ f^{-1}(D) \mid D \in \mathfrak{H} \right\}$ is a conditional $\sigma$-algebra on $\mathbf{X}$ by Proposition 2.19 in [DJKK16]. Moreover, if $\mathfrak{E} \sqsubset \mathbf{P}(\mathbf{Y})$ then $\Sigma\left(f^{-1}(\mathfrak{E})\right) = f^{-1}(\Sigma(\mathfrak{E}))$. Indeed, since $f^{-1}(\mathfrak{E}) \sqsubset f^{-1}(\Sigma(\mathfrak{E}))$ and since $f^{-1}(\Sigma(\mathfrak{E}))$ is a conditional $\sigma$-algebra, it follows that $\Sigma\left(f^{-1}(\mathfrak{E})\right) \sqsubset f^{-1}(\Sigma(\mathcal{E}))$. Conversely, it follows from Proposition 2.19 in [DJKK16] that $\left\{ \mathcal{D} \sqsubset \mathbf{Y} \mid f^{-1}(\mathcal{D}) \in f^{-1}(\Sigma(\mathfrak{E})) \right\}$ is a conditional $\sigma$-algebra including $\mathfrak{E}$, and thus it also holds that $f^{-1}(\Sigma(\mathcal{E})) \sqsubset \Sigma\left(f^{-1}(\mathfrak{E})\right)$.

**Example 1.22.** (i). The conditional trivial $\sigma$-algebra on some conditional set $\mathbf{X}$ is the conditional set $\{\mathbf{X}|a \mid a \in \mathcal{A}\}$.

(ii). Let $N \in \mathbf{N}$ where $N = \sum_{n \in \mathbb{N}} m_n|a_n$, and let $\mathbf{R}^N = \sum_{n \in \mathbb{N}} \mathbf{R}^{m_n}|a_n$ be the $N$-dimensional conditional Euclidean space. Denote by $\mathcal{B}^N$ the conditional $\sigma$-algebra conditionally generated by the conditional set of conditionally open sets in $\mathbf{R}^N$ and call it the conditional Borel $\sigma$-algebra on $\mathbf{R}^N$. Let $K \in \mathbf{N}$ be such that $K = \sum_{n\in\mathbb{N}} k_n|a_n$ where $1 \leq k_n \leq m_n$. The conditional $K$-th projection $\pi_K \colon \mathbf{R}^N \to \mathbf{R}$, mapping $\sum_{n\in\mathbb{N}} (x_1, \ldots, x_{m_n}) |a_n$ to $\sum_{n\in\mathbb{N}} x_{k_n}|a_n$, is conditionally measurable.

(iii). We define $\overline{\mathcal{B}}$ to be the conditional Borel $\sigma$-algebra on $\overline{\mathbf{R}}$ generated by the conditional topology of conditionally open conditional sets on $\overline{\mathbf{R}}$.

**Definition 1.23.** Let $(\mathbf{X}, \mathfrak{F})$ be a conditional measurable space. A conditional measure is a conditional function $\mu \colon \mathfrak{F} \to [0, \infty]$ such that

(i). $\mu(\mathbf{X}|0) = 0$,

(ii). $\mu\left(\bigsqcup_{N\in\mathbf{N}} \mathcal{C}_N\right) = \sum_{N\in\mathbf{N}} \mu(\mathcal{C}_N)_{N\in\mathbf{N}}$ for every conditional sequence $(\mathcal{C}_N)_{N\in\mathbf{N}}$ of conditionally pairwise disjoint conditional sets in $\mathfrak{F}$.

The triplet $(\mathbf{X}, \mathfrak{F}, \mu)$ is called a conditional measure space. Let $a = \vee\{\overline{a} \mid \mu(\mathbf{X}|\overline{a}) = \infty\}$. Then the conditional measure $\mu$ is finite if $a = 0$; it is conditionally infinite if $a > 0$; it is a conditional probability measure if $\mu(\mathbf{X}) = 1$. If $X = \bigsqcup_{N\in\mathbf{N}} \mathcal{C}_N$ for some conditional sequence of conditional sets in $\mathfrak{F}$ satisfying $\mu(\mathcal{C}_N) < \infty$ for each $N \in \mathbf{N}$, then $\mu$ is conditionally $\sigma$-finite.

**Remark 1.24.** We collect some properties of a conditional measure space $(\mathbf{X}, \mathfrak{F}, \mu)$.

(i). Due to stability and (i) in Definition 1.23,

$$\mu(\mathcal{C}|a + (\mathbf{X}|0)|a^c) = \mu(\mathcal{C})|a + \mu(\mathbf{X}|0)|a^c = \mu(\mathcal{C})|a + 0|a^c,$$

for every $\mathcal{C} \in \mathfrak{F}$ and $a \in \mathcal{A}$.

(ii). For conditionally finite and pairwise disjoint conditional sets $(\mathcal{C}_M)_{1\leq M\leq N} \sqsubset \mathfrak{F}$, it holds that

$$\mu\left(\bigsqcup_{1\leq M\leq N} \mathcal{C}_M\right) = \sum_{1\leq M\leq N} \mu(\mathcal{C}_M).$$

(iii). If $\mathcal{C} \sqsubset \mathcal{D}$, then $\mu(\mathcal{C}) \leq \mu(\mathcal{D})$. Indeed, let $a = \vee\{\overline{a} \mid \mu(\mathcal{D})|\overline{a} = +\infty\}$. Then $\mu(\mathcal{C})|a \leq \mu(\mathcal{D})|a$. So assume without loss of generality that $a = 0$. Then

$$\mu(\mathcal{D}) = \mu\left(\mathcal{C} \sqcup \left(\mathcal{D} \sqcap \mathcal{C}^{\sqsubset}\right)\right) = \mu(\mathcal{C}) + \mu\left(\mathcal{D} \sqcap \mathcal{C}^{\sqsubset}\right) \geq \mu(\mathcal{C}).$$

(iv). For all $\mathcal{C}, \mathcal{D} \in \mathfrak{F}$ such that $\mu(\mathcal{C}), \mu(\mathcal{D}) < +\infty$, it holds that

$$\mu(\mathcal{C} \sqcup \mathcal{D}) = \mu(\mathcal{C}) + \mu(\mathcal{D}) - \mu(\mathcal{C} \sqcap \mathcal{D}).$$

Indeed, write $\mathcal{C}\sqcup\mathcal{D} = (\mathcal{C}\sqcap\mathcal{D})\sqcup(\mathcal{C}\sqcap\mathcal{D}^{\sqsubset})\sqcup(\mathcal{D}\sqcap\mathcal{C}^{\sqsubset})$. Then $\mu(\mathcal{C}\sqcup\mathcal{D}) = \mu(\mathcal{C}\sqcap\mathcal{D}) + \mu(\mathcal{C}\sqcap\mathcal{D}^{\sqsubset}) + \mu(\mathcal{D}\sqcap\mathcal{C}^{\sqsubset})$. Subtracting twice $\mu(\mathcal{C}\sqcap\mathcal{D}) < +\infty$ yields the result. By conditional induction one extends the previous equation in order to obtain the conditional version of the inclusion-exclusion formula (see e.g. Section 10 in [Bil86]).

(v). Let $(\mathcal{C}_N)_{N\in\mathbf{N}}$ be an increasing, that is $\mathcal{C}_N \sqsubset \mathcal{C}_M$ for $N \leq M$, conditional sequence in $\mathfrak{F}$ such that $\mathcal{C} = \bigsqcup_{N\in\mathbf{N}} C_N$. Let $\mathcal{D}_1 = \mathcal{C}_1$ and $\mathcal{D}_n = \mathcal{C}_n \sqcap \mathcal{C}_{n-1}^{\sqsubset}$. Then

$$Y_n := \mu(\mathcal{C}_n) = \mu\left(\bigsqcup_{k=1}^{n} \mathcal{D}_k\right) = \sum_{k=1}^{n} \mu(\mathcal{D}_k) \leq \mu(\mathcal{C}).$$

Let $a := \vee\{\overline{a} \mid \mu(\mathcal{C})|\overline{a} = \infty\}$. Define $X_N = \sum_{n\in\mathbb{N}} Y_{m_n}|a_n$ for $N = \sum_{n\in\mathbb{N}} m_n|a_n \in \mathbf{N}$. Then $(X_N|a^c)_{N\in\mathbf{N}}$ is a conditionally increasing and bounded sequence in $\mathbf{R}$. By Lemma 5.2.9 in [Jam14] the sequence has an essential supremum. Thus $\mu(\mathcal{C}_N)_{N\in\mathbf{N}}$ converges to $\mu(\mathcal{C})$ from below.

(vi). For a conditional sequence $(\mathcal{C}_N)_{N\in\mathbf{N}}$ in $\mathfrak{F}$, we have $\mu\left(\bigsqcup_{N\in\mathbf{N}} \mathcal{C}_N\right) \leq \sum_{N\in\mathbf{N}} \mu(\mathcal{C}_N)$. Indeed, define $\mathcal{D}_1 = \mathcal{C}_1$ and $\mathcal{D}_N = \mathcal{C}_N \sqcap (\mathcal{C}_1 \sqcup \ldots \sqcup \mathcal{C}_{N-1})^{\sqsubset}$. By monotonicity $\mu(\mathcal{D}_N) \leq \mu(\mathcal{C}_N)$ and $\mu\left(\bigsqcup_{N\in\mathbf{N}} \mathcal{C}_N\right) = \mu\left(\bigsqcup_{N\in\mathbf{N}} \mathcal{D}_N\right) = \sum_{N\in\mathbf{N}} \mu(\mathcal{D}_N) \leq \sum_{N\in\mathbf{N}} \mu(\mathcal{C}_N)$.

**Example 1.25.** (i). Let $\nu$ be a measure on $\mathbf{N}$. Let $\mathcal{C} \sqsubset \mathbf{N}$ and for every non-empty $I \subset \mathbb{N}$ let $a_I := \vee\{\overline{a} \mid \mathcal{C}|\overline{a} = I\}$. Then $\vee_{I\in\mathbb{N}} a_I = 1$. Then $\mu(\mathcal{C}) := \sum_{n\in\mathbb{N}} \nu(I_n)|a_n$, defines a conditional measure $\mu\colon \mathbf{P}(\mathbf{N}) \to [0,+\infty]$.

(ii). Let $(\mathbf{X},\mathfrak{F})$ be a conditional measurable space. A conditional measure $\mu$ on $(\mathbf{X},\mathfrak{F})$ is conditionally discrete if there are conditional sequences $(X_N)_{N\in\mathbf{N}}$ in $\mathbf{X}$ and $(m_N)_{N\in\mathbf{N}}$ in $[0,+\infty]$ such that for any $\mathcal{C} \in \mathfrak{F}$,

$$\mu(\mathcal{C}) = \sum_{N\in\mathbf{N}} (m_N|a_N + 0|a_N^c), \quad a_N = \vee\{\overline{a} \mid X_N|\overline{a} \in \mathcal{C}|\overline{a}\}.$$

In particular, for $X \in \mathbf{X}$ and $m = 1$ we obtain the conditional Dirac measure on $X \in \mathbf{X}$.

**Definition 1.26.** Let $\mathbf{X}$ be a conditional set. A conditional $\pi$-system on $\mathbf{X}$ is a conditional family of conditional subsets of $\mathbf{X}$ which is closed under conditionally finite conditional intersections. A conditional $\lambda$-system on $\mathbf{X}$ is a conditional family $\mathfrak{D}$ of conditional subsets of $\mathbf{X}$ satisfying

(i). $\mathbf{X} \in \mathfrak{D}$,

(ii). if $\mathcal{C} \in \mathfrak{D}$, then $\mathcal{C}^{\sqsubset} \in \mathfrak{D}$,

(iii). if $(\mathcal{C}_N)_{N\in\mathbf{N}}$ is a conditional sequence of conditionally pairwise disjoint sets in $\mathfrak{D}$, then $\bigsqcup_{N\in\mathbf{N}} \mathcal{C}_N \in \mathfrak{D}$.

**Remark 1.27.** Since $\mathbf{X} \in \mathfrak{D}$, the conditional intersection of any non-empty family of conditional $\lambda$-systems is a conditional $\lambda$-system. Let $\Delta(\mathfrak{E})$ denote the conditional $\lambda$-system generated by some conditional set $\mathfrak{E}$ of conditional subsets of $\mathbf{X}$.

**Theorem 1.28.** *Let $\mathbf{X}$ be a conditional set and $\mathfrak{E} \sqsubset \mathbf{P}(\mathbf{X})$ a $\pi$-system. Then $\Sigma(\mathfrak{E}) = \Delta(\mathfrak{E})$.*

*Proof.* The proof of Dynkin's $\pi$-$\lambda$ theorem (see e.g. proof of Theorem 3.2 in [Bil86]) relies only on Boolean arithmetic. Since Boolean arithmetic is valid in conditional set theory by Theorem 2.8 in [DJKK16], a proof of a conditional version of Dynkin's $\pi$-$\lambda$ theorem follows analogously to the classical proof. □

**Theorem 1.29.** *Let $\mathfrak{E}$ be a conditional $\pi$-system on some conditional set $\mathbf{X}$, and let $\mu_1$ and $\mu_2$ be two conditional measures on $\Sigma(\mathfrak{E})$. Suppose there exist two conditional sequences $(\mathcal{C}_N)_{N\in\mathbf{N}}$ and $(\mathcal{D}_N)_{N\in\mathbf{N}}$ in $\mathfrak{E}$ such that $\mu_1(\mathcal{C}_N), \mu_2(\mathcal{D}_N) < +\infty$ for all $N \in \mathbf{N}$ and $\bigsqcup_{N\in\mathbf{N}} \mathcal{C}_N = \bigsqcup_{N\in\mathbf{N}} \mathcal{D}_N = \mathbf{X}$. If $\mu_1(\mathcal{C}) = \mu_2(\mathcal{C})$ for all $\mathcal{C} \in \mathfrak{E}$, then $\mu_1(\mathcal{D}) = \mu_2(\mathcal{D})$ for all $\mathcal{D} \in \Sigma(\mathfrak{E})$.*

*Proof.* The theorem follows analogous to a proof of the respective classical statement (e.g. proof of Theorem 10.3 in [Bil86]) by Theorem 1.28 and Properties (iv) and (v) of conditional measure spaces. □

**Definition 1.30.** Let $\mathbf{X}$ be a conditional set. A conditional function $\mu^*\colon \mathbf{P}(\mathbf{X}) \to [0,\infty]$ is a conditional outer measure if

(i). $\mu^*(\mathbf{X}|0) = 0$,

(ii). $\mu^*(\mathcal{C}) \leq \mu^*(\mathcal{D})$ for $\mathcal{C} \sqsubset \mathcal{D}$,

(iii). $\mu^*\left(\bigsqcup_{N\in\mathbf{N}} \mathcal{C}_N\right) \leq \sum_{N\in\mathbf{N}} \mu^*(\mathcal{C}_N)$ for any conditional sequence $(\mathcal{C}_N)_{N\in\mathbf{N}}$ in $\mathbf{X}$.

A conditional set $\mathcal{C} \sqsubset \mathbf{X}$ is conditionally $\mu^*$-measurable if

$$\mu^*(\mathcal{W} \sqcap \mathcal{C}) + \mu^*\left(\mathcal{W} \sqcap \mathcal{C}^{\sqsubset}\right) \leq \mu^*(\mathcal{W}), \quad \text{for all } \mathcal{W} \sqsubset \mathbf{X}. \tag{1.1}$$

Denote by $\mathfrak{F}(\mu^*)$ the class of conditionally $\mu^*$-measurable conditional sets.

**Theorem 1.31.** *If $\mu^*$ is a conditional outer measure, then $\mathfrak{F}(\mu^*)$ is a conditional $\sigma$-algebra, and $\mu^*$ restricted to $\mathfrak{F}(\mu^*)$ a conditional measure.*

*Proof.* From Properties (i) and (ii) of Definition 1.30 it follows that $\mathbf{X} \in \mathfrak{F}(\mu^*)$. Since (1.1) is symmetric in $\mathcal{C}$ and $\mathcal{C}^{\sqsubset}$, for every $\mathcal{C} \in \mathfrak{F}(\mu^*)$ it follows also that $\mathcal{C}^{\sqsubset} \in \mathfrak{F}(\mu^*)$. To show that for all $\mathcal{C}, \mathcal{D} \in \mathfrak{F}(\mu^*)$, it follows that $\mathcal{C} \sqcup \mathcal{D} \in \mathfrak{F}(\mu^*)$, one can proceed as in the classical argument by applying (1.1) twice and using the conditional subadditivity of $\mu^*$ and the distributivity and De Morgan's law on the conditional power set. For conditionally disjoint $\mathcal{C}, \mathcal{D} \in \mathfrak{F}(\mu^*)$ it follows from (1.1) that $\mu^*(\mathcal{W} \sqcap (\mathcal{C} \sqcup \mathcal{D})) \geq \mu^*(\mathcal{W} \sqcap \mathcal{C}) + \mu^*(\mathcal{W} \sqcap \mathcal{D})$, and thus by induction for $\mathcal{C}_1, \ldots, \mathcal{C}_n \in \mathfrak{F}(\mu^*)$, pairwise conditionally disjoint, that

$$\mu^*\left(\mathcal{W} \sqcap \left(\bigsqcup_{k=1}^{n} \mathcal{C}_k\right)\right) \geq \sum_{k=1}^{n} \mu^*(\mathcal{W} \sqcap \mathcal{C}_k). \tag{1.2}$$

The inequation (1.2) extends by stability of the conditional set operations and of $\mu^*$ to every conditionally finite family of conditionally pairwise disjoint sets. Thus for a conditionally finite and pairwise disjoint family $(\mathcal{C}_M)_{1 \leq M \leq \sum_{n\in\mathbb{N}} m_n | a_n}$ in $\mathfrak{F}(\mu^*)$ it holds that

$$\begin{aligned}\mu^*(\mathcal{W}) &\geq \mu^*\left(\mathcal{W} \sqcap \left(\bigsqcup_{M\in\mathbf{N}} \mathcal{C}_M\right)\right) + \mu^*\left(\mathcal{W} \sqcap \left(\bigsqcup_{M\in\mathbf{N}} \mathcal{C}_M\right)^{\sqsubset}\right) \\ &\geq \sum_{n\in\mathbb{N}}\left(\sum_{k=1}^{m_n} \mu^*(\mathcal{W} \sqcap \mathcal{C}_k) | a_n\right) + \mu^*\left(\mathcal{W} \sqcap \left(\bigsqcup_{M\in\mathbf{N}} \mathcal{C}_M\right)^{\sqsubset}\right).\end{aligned} \tag{1.3}$$

Given a conditionally pairwise disjoint conditional sequence $(\mathcal{C}_N)_{N\in\mathbf{N}}$, we take in (1.3) the limit for $M \in \mathbf{N}$ and by applying conditional subadditivity twice it follows that

$$\mu^*(\mathcal{W}) \geq \sum_{N\in\mathbf{N}} \mu^*(\mathcal{W} \sqcap \mathcal{C}_N) + \mu^*\left(\mathcal{W} \sqcap \left(\bigsqcup_{N\in\mathbf{N}} \mathcal{C}_N\right)^{\sqsubset}\right)$$

which implies that $\mathfrak{F}(\mu^*)$ is a conditional $\sigma$-algebra by (1.1) and also that $\mu^*$ on $\mathfrak{F}(\mu^*)$ is a conditional measure. □

**Definition 1.32.** Let $\mathbf{X}$ be a conditional set. A conditional semiring $\mathfrak{R}$ on $\mathbf{X}$ is a conditional family of conditional subsets of $\mathbf{X}$ such that

(i). $\mathbf{X}|0 \in \mathfrak{R}$,

(ii). If $\mathcal{C}, \mathcal{D} \in \mathfrak{R}$, then $\mathcal{C} \sqcap \mathcal{D} \in \mathfrak{R}$,

(iii). If $\mathcal{C}, \mathcal{D} \in \mathfrak{R}$ and $\mathcal{C} \sqsubset \mathcal{D}$, then there exists a conditionally finite family $(\mathcal{W}_M)_{1\leq M\leq N}$ of conditional subsets in $\mathfrak{R}$ such that $\mathcal{D} \sqcap \mathcal{C}^{\sqsubset} = \bigsqcup_{1\leq M\leq N} \mathcal{W}_M$.

**Example 1.33.** Let $\mathbf{R}^N$ be the conditional $N$-dimensional Euclidean space. Then

$$\mathfrak{R}^N = \left\{ ]P|a, Q|a] \mid P, Q \in \mathbf{Q}^N, P \leq Q, a \in \mathcal{A} \right\}$$

is a conditional semiring.

**Theorem 1.34.** *Let* $\mathbf{X}$ *be a conditional set and let* $\mathfrak{R}$ *be a conditional semi-ring on* $\mathbf{X}$*. Let* $\mu\colon \mathfrak{R} \to [0, \infty]$ *be a conditional function such that*

*(i).* $\mu(\mathbf{X}|0) = 0$,

*(ii).* $\mu\left(\bigsqcup_{1 \leq N \leq \overline{N}} \mathcal{C}_N\right) = \sum_{1 \leq N \leq \overline{N}} \mu(\mathcal{C}_N)$ *for every conditionally finite and pairwise disjoint conditional sequence* $(\mathcal{C}_N)_{1 \leq N \leq \overline{N}}$ *in* $\mathfrak{R}$ *such that* $\bigsqcup_{1 \leq N \leq \overline{N}} \mathcal{C}_N \in \mathfrak{R}$,

*(iii).* $\mu\left(\bigsqcup_{N \in \mathbf{N}} \mathcal{C}_N\right) \leq \sum_{N \in \mathbf{N}} \mu(\mathcal{C}_N)$ *for every conditional sequence* $(\mathcal{C}_N)_{N \in \mathbf{N}}$ *in* $\mathfrak{R}$ *such that* $\bigsqcup_{N \in \mathbf{N}} \mathcal{C}_N \in \mathfrak{R}$.

*Then* $\mu$ *extends to a conditional measure on* $\Sigma(\mathfrak{R})$.

*Proof.* Let $a_{\mathcal{C}} := \vee \left\{ \overline{a} \in \mathcal{A} \mid \exists\, (\mathcal{D}_N)_{N \in \mathbf{N}} \subset \mathfrak{R},\, \mathcal{C}|\overline{a} \sqsubset \bigsqcup_{N \in \mathbf{N}} \mathcal{D}_N \right\}$ for every $\mathcal{C} \sqsubset \mathbf{X}$, and define

$$\mu^*(\mathcal{C}) := \inf \left\{ \sum_{N \in \mathbf{N}} \mu(\mathcal{D}_N) \mid \mathcal{D}_N \in \mathfrak{R},\, \mathcal{C} \sqsubset \bigsqcup_{N \in \mathbf{N}} \mathcal{D}_N \right\} |a_{\mathcal{C}} + \infty|a_{\mathcal{C}}^c. \tag{1.4}$$

We need to show that $\mu^*$ is a conditional outer measure. First, we show that $\mu^*$ is a conditional function. To this end let $(a_n)_{n \in \mathbb{N}}$ be a partition in $\mathcal{A}$ and $(\mathcal{C}_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbf{X}$. For each $n$, it holds that $a_{a_n \mathcal{C}_n} = a_n \wedge a_{\mathcal{C}_n}$ implying $a_{\sum_{n \in \mathbb{N}} a_n \mathcal{C}_n} = \vee_{n \in \mathbb{N}} (a_n \wedge a_{\mathcal{C}_n})$. Thus $\mu^*\left(\sum_{n \in \mathbb{N}} \mathcal{C}_n | a_n\right) = \sum_{n \in \mathbb{N}} \mu^*(\mathcal{C}_n)|a_n$. Second, we verify the axioms of a conditional outer measure. Since $a(\mathbf{X}|0) = 1$, it follows that $\mu^*(\mathbf{X}|0) = \mu(\mathbf{X}|0) = 0$ by assumption. Let $\mathcal{C}, \mathcal{D} \sqsubset \mathbf{X}$ and such that $\mathcal{C} \sqsubset \mathcal{D}$. Since every conditional cover of $\mathcal{D}$ is a conditional cover of $\mathcal{C}$ by Boolean arithmetic, $a_{\mathcal{D}} \leq a_{\mathcal{C}}$, and thus $\mu^*(\mathcal{C}) \leq \mu^*(\mathcal{D})$ by definition of $\mu^*$. Finally, let $(\mathcal{C}_N)_{N \in \mathbf{N}}$ be a conditional sequence in $\mathbf{X}$. By Proposition 2.25 in [DJKK16] $\mathbf{N}^{\mathbf{N}}$ is conditionally countable. Thus $\vee_{N \in \mathbf{N}} a_{\mathcal{C}_N} = a_{\bigsqcup_{N \in \mathbf{N}} \mathcal{C}_N}$. On $a_{\bigsqcup_{N \in \mathbf{N}} \mathcal{C}_N}$ we argue with sequences. That is to say, by [DJKK16, Theorem 4.4], that for every $n \in \mathbb{N}$ there exists a conditional cover $(\mathcal{D}_{n,k})_{k \in \mathbb{N}}$ for $\mathcal{C}_n$ such that $\sum_{k \in \mathbb{N}} \mu(\mathcal{D}_{n,k}) < \mu^*(\mathcal{C}_n) + \varepsilon/2^n$, and then $\mu^*\left(\bigsqcup_{N \in \mathbf{N}} \mathcal{C}_N\right) = \mu^*\left(\bigsqcup_{n \in \mathbb{N}} \mathcal{C}_n\right) \leq \sum_{(n,k) \in \mathbb{N} \times \mathbb{N}} \mu(\mathcal{D}_{n,k}) < \sum_{n \in \mathbb{N}} \mu^*(\mathcal{C}_n) + \varepsilon = \sum_{N \in \mathbf{N}} \mu^*(\mathcal{C}_N) + \varepsilon$. This shows that $\mu^*$ defines a conditional outer measure.

Next we show that $\mathfrak{R} \sqsubset \mathfrak{F}(\mu^*)$. Let $\mathcal{C} \in \mathfrak{R}$ and $\mathcal{W} \sqsubset \mathbf{X}$ and

$$a = \vee \{\overline{a} \in \mathcal{A} \mid \mu^* (\mathcal{W}|\overline{a}) = \infty\}$$

The relation (1.1) is trivially true on $a$. We argue on $a^c$ in the following. For every $\varepsilon > 0$, there exists a conditionally sequence $(\mathcal{D}_N)_{N \in \mathbf{N}}$ in $\mathcal{R}$ such that $\mathcal{W} \sqsubset \bigsqcup_{N \in \mathbf{N}} \mathcal{D}_N$ and $\sum_{N \in \mathbf{N}} \mu^* (\mathcal{D}_N) < \mu^* (\mathcal{W}) + \varepsilon$ [DJKK16, Theorem 4.4]. Since $\mathfrak{R}$ is a conditional semi-ring, $\mathcal{E}_N := \mathcal{C} \sqcap \mathcal{D}_N \in \mathfrak{R}$ and there exist $(\mathcal{E}'_{N,K})_{1 \leq K \leq M(N)}$ in $\mathfrak{R}$ such that $\mathcal{C}^{\sqsubset} \sqcap \mathcal{D}_N = \mathcal{D}_N \sqcap \mathcal{E}_N^{\sqsubset} = \bigsqcup_{1 \leq K \leq M(N)} \mathcal{E}'_{N,K}$ for every $N$. Thus $\mathcal{D}_N = \bigsqcup_{1 \leq K \leq M(N)} \mathcal{E}'_{N,K} \sqcup \mathcal{E}_N$ is a conditionally disjoint conditional union and it holds that $\mathcal{C} \sqcap \mathcal{W} \sqsubset \bigsqcup_{N \in \mathbf{N}} \mathcal{E}_N$ and $\mathcal{C} \sqcap \mathcal{W}^{\sqsubset} \sqsubset \bigsqcup_{N \in \mathbf{N}} \bigsqcup_{1 \leq K \leq M(N)} \mathcal{E}'_{N,K}$. By definition of $\mu^*$ and conditional finite additivity of $\mu$,

$$\begin{aligned} \mu^* (\mathcal{C} \sqcap \mathcal{W}) + \mu^* \left(\mathcal{C}^{\sqsubset} \sqcap \mathcal{W}\right) &\leq \sum_{N \in \mathbf{N}} \mu (\mathcal{E}_N) + \sum_{N \in \mathbf{N}} \sum_{1 \leq K \leq M(N)} \mu \left(\mathcal{E}'_{N,K}\right) \\ &= \sum \mu (\mathcal{C}_N) < \mu^* (\mathcal{W}) + \varepsilon. \end{aligned}$$

Letting $\varepsilon$ going to 0 yields (1.1) on $a^c$, and thus $\mathcal{C}$ is $\mu^*$-measurable by stability.

It remains to show that $\mu$ and $\mu^*$ coincide on $\mathfrak{R}$. Let $\mathcal{C}, \mathcal{D} \in \mathfrak{R}$ be such that $\mathcal{C} \sqsubset \mathcal{D}$. Then there exists a conditionally finite family $(\mathcal{W}_M)_{1 \leq M \leq N}$ such that $\mathcal{D} \sqcap \mathcal{C}^{\sqsubset} = \bigsqcup_{1 \leq M \leq N} \mathcal{W}_M$ since $\mathfrak{R}$ is a conditional semi-ring. By Boolean arithmetic $\bigsqcup_{1 \leq M \leq N} \mathcal{W}_M \sqcap \mathcal{C} = \mathbf{X}|0$. Given $N = \sum_{n \in \mathbb{N}} m_n | a_n$, by stability and associativity,

$$\begin{aligned} \left( \bigsqcup_{1 \leq M \leq N} \mathcal{W}_M \right) \sqcup \mathcal{C} &= \left( \sum_{n \in \mathbb{N}} \left( \bigsqcup_{k=1}^{m_n} \mathcal{W}_k | a_n \right) \right) \sqcup \mathcal{C} = \sum_{n \in \mathbb{N}} \left( \bigsqcup_{k=1}^{m_n} (\mathcal{W}_k \sqcup \mathcal{C}) | a_n \right) \\ &= \sum_{n \in \mathbb{N}} \left( \bigsqcup_{k=1}^{m_n+1} \mathcal{W}_k | a_n \right) = \bigsqcup_{1 \leq M \leq N+1} \mathcal{W}_M \end{aligned}$$

where $\mathcal{W}_{m_n+1} = \mathcal{C}$ for all $n$. By conditionally finite additivity,

$$\mu (\mathcal{D}) = \mu \left( \bigsqcup_{1 \leq M \leq N+1} \mathcal{W}_M \right) = \sum_{n \in \mathbb{N}} \left( \sum_{k=1}^{m_n+1} \mu (\mathcal{W}_k) | a_n \right) \geq \mu (\mathcal{C}). \tag{1.5}$$

Let $\mathcal{C} \in \mathfrak{R}$ and $(\mathcal{C}_N)_{N \in \mathbf{N}}$ in $\mathfrak{R}$ be a conditional cover of $\mathcal{C}$. By (1.5), it holds that $\mu (\mathcal{C}) \leq \sum_{N \in \mathbf{N}} \mu (\mathcal{C} \sqcap \mathcal{C}_N) \leq \sum_{N \in \mathbf{N}} \mu (\mathcal{C}_N)$. Thus $\mu (\mathcal{C}) \leq \mu^* (\mathcal{C})$. Since for $C \in \mathcal{R}$, it holds that $a_C = 1$ and since $C$ is a conditional cover of $C$, it holds also that $\mu^*(C) \leq \mu(C)$, and thus $\mu(C) = \mu^*(C)$ for all $C \in \mathcal{R}$.

Since $\mathfrak{R} \sqsubset \mathfrak{F}(\mu^*)$ and since $\mathfrak{F}(\mu^*)$ is a conditional $\sigma$-algebra by Theorem 1.31, it holds that $\Sigma(\mathfrak{R}) \sqsubset \mathfrak{F}(\mu^*)$. Since $\mu^*$ is countably additive on $\mathfrak{F}(\mu^*)$ by Theorem 1.31, $\mu^*$ restricted to $\Sigma(\mathfrak{R})$ is a conditional extension of $\mu$ on $\mathfrak{R}$. □

We show that every kernel on $d$-dimensional Euclidean space $\mathbb{R}^d$ extends naturally to a conditional measure on the $d$-dimensional conditional Euclidean space $\mathbf{R}^d$ and from there by stability to $\mathbf{R}^D$ for every $D \in \mathbf{N}$. In particular, every measure on $\mathbb{R}^d$ extends to a conditional measure on $\mathbf{R}^D$.

**Example 1.35.** Recall that $(\Omega, \mathcal{F}, \mathbb{P})$ is the underlying probability space. Let $\mathcal{B}^d$ denote the Borel $\sigma$-field on $\mathbb{R}^d$. Let $\nu\colon \Omega \times \mathcal{B}^d \to \overline{\mathbb{R}}_+$ satisfy

(i). for all $\omega \in \Omega$, $\nu(\omega, \cdot)$ is a measure on $\mathcal{B}^d$,

(ii). for every $A \in \mathcal{B}^d$, $\nu(\cdot, A)$ is a measurable function.

Let $\mathfrak{R}$ be the conditional semi-ring of bounded conditional rational rectangles. For $]P, Q]|a \in \mathfrak{R}$, where $a = A$, define

$$\mu(]P, Q]|a)(\omega) := \nu(\omega, ]P, Q](\omega)), \quad \mathbb{P}\text{-a.e. } \omega \in A.$$

Measurability implies that $\mu\colon \mathfrak{R} \to [0, \infty]$ is a conditional function. Inspection shows that $\mu$ satisfies the assumptions of Theorem 1.34. Thus $\mu$ can be extended to the conditional Borel $\sigma$-field $\mathfrak{B}^d$ on $\mathbf{R}^d$. Note that if $\nu(\omega, \cdot) = \rho(\cdot)$ for all $\omega$ where $\rho$ is a $\sigma$-finite measure on $\mathbb{R}^d$, then its conditional extension $\mu$ to $\mathbf{R}^d$ is conditionally $\sigma$-finite, and thus by Theorem 1.29 it is unique. We call the conditional extension of the Lebesgue measure the conditional Lebesgue measure. The extension of the conditional Borel $\sigma$-field to $\overline{\mathbf{R}}^d$ is denoted by $\overline{\mathfrak{B}}^d$.

### 1.2.2 Conditional integrals

The conditional Borel $\sigma$-algebra is generated by either of the following conditional systems; the conditionally open conditional balls, the conditionally closed conditional balls, the conditionally compact conditional balls, or the bounded conditional rational rectangles. This is a result of the properties of Boolean arithmetic in [DJKK16].

**Remark 1.36.** A conditional function $f\colon (\mathbf{X}, \mathfrak{F}) \to (\overline{\mathbf{R}}, \overline{\mathfrak{B}})$ is $\mathfrak{F}$-$\overline{\mathfrak{B}}$-conditionally measurable if $\{X \sqsubset \mathbf{X} \mid f(X) \sqsubset [-\infty, \alpha]\} \in \mathfrak{F}$ for all $\alpha \in \overline{\mathbf{R}}$. This coincides with the more general definition of measurability as in Definition 1.21.

Being consistent with the standard literature on measure and integration theory, we consider the short notation $\{f \leq g\} := \{X \in \mathbf{X} \mid f(X) \leq g(X)\}|a$ with the definition

$a := \vee \{\overline{a} \in \mathcal{A} \mid \text{there exists}\, X \in \mathbf{X} \,\text{such that}\, f(X)|a \leq g(X)|a\}$ for $\mathfrak{F}$-$\overline{\mathfrak{B}}$-conditionally measurable conditional functions $f, g\colon (\mathbf{X}, \mathfrak{F}) \to \left(\overline{\mathbf{R}}, \overline{\mathfrak{B}}\right)$.

**Theorem 1.37.** *The conditional function $f\colon (\mathbf{X}, \mathfrak{F}) \to \left(\overline{\mathbf{R}}, \overline{\mathfrak{B}}\right)$ is $\mathfrak{F}$-$\overline{\mathfrak{B}}$-conditionally measurable if and only if one of the following conditions holds true*

$$\begin{aligned} &\{f \geq \alpha\} \in \mathfrak{F} \quad \forall \alpha \in \mathbf{R}, &&\{f > \alpha\} \in \mathfrak{F} \quad \forall \alpha \in \mathbf{R} \\ &\{f \leq \alpha\} \in \mathfrak{F} \quad \forall \alpha \in \mathbf{R}, &&\{f < \alpha\} \in \mathfrak{F} \quad \forall \alpha \in \mathbf{R}. \end{aligned}$$

*Furthermore, it is equivalent if quantification ranges over $\alpha \in \overline{\mathbf{R}}$.*

*Proof.* Clearly, the conditional intervals generate the conditional $\sigma$-algebra $\overline{\mathfrak{B}}$. Further, we observe that $\{f > \alpha\} = \bigsqcup_{N \in \mathbf{N}} \left\{f \geq \alpha + \frac{1}{N}\right\}$, $\{f \leq \alpha\} = \{f > \alpha\}^{\sqsubset}$, $\{f < \alpha\} = \bigsqcup_{N \in \mathbf{N}} \left\{f \leq \alpha + \frac{1}{N}\right\}$ and $\{f \geq \alpha\} = \{f < \alpha\}^{\sqsubset}$. □

**Theorem 1.38.** *Let $f, g\colon (\mathbf{X}, \mathfrak{F}) \to \left(\overline{\mathbf{R}}, \overline{\mathfrak{B}}\right)$ be $\mathfrak{F}$-$\overline{\mathfrak{B}}$-conditionally measurable conditional functions. Then $\{f < g\}, \{f \leq g\}, \{f = g\}, \{f \neq g\} \in \mathfrak{F}$.*

*Proof.* This follows by Theorem 1.37, $\{f < g\} = \bigsqcup_{Q \in \mathbf{Q}} (\{f < Q\} \sqcap \{Q < g\})$ and that $\mathbf{Q}$ is conditionally dense in $\mathbf{R}$. □

**Theorem 1.39.** *Let $f, g\colon (\mathbf{X}, \mathfrak{F}) \to \left(\overline{\mathbf{R}}, \overline{\mathfrak{B}}\right)$ be $\mathfrak{F}$-$\overline{\mathfrak{B}}$-conditionally measurable conditional functions. Then, $f \cdot g, f + g, f - g$ are $\mathfrak{F}$-$\overline{\mathfrak{B}}$-conditionally measurable conditional functions if they are well-defined.*

*Proof.* The conditional sets $\{f = \pm\infty\}$ and $\{g = \pm\infty\}$ are in $\mathfrak{F}$ by Theorem 1.37. On their conditional complement, by Theorem 1.37, $\beta + \gamma g$ is $\mathfrak{F}$-$\overline{\mathfrak{B}}$-conditionally measurable for $\beta, \gamma \in \mathbf{R}$ if $g$ is $\mathfrak{F}$-$\overline{\mathfrak{B}}$-conditionally measurable. Hence, with $\{f + g \geq \alpha\} = \{f \geq \alpha - g\}$, $f + g$ and $f - g$ are $\mathfrak{F}$-$\overline{\mathfrak{B}}$-conditionally measurable. Considering $f \cdot g = \frac{1}{4}(f + g)^2 - \frac{1}{4}(f - g)^2$, it suffices to show that $f^2$ is $\mathfrak{F}$-$\overline{\mathfrak{B}}$-conditionally measurable. To this end, observe $\left\{f^2 \geq \alpha\right\} = \mathbf{X}|a + (\{f \geq \sqrt{\alpha}\} \sqcup \{f \leq -\sqrt{\alpha}\})|a^c$ where we use the definition $a = \vee \{\overline{a} \in \mathcal{A} \mid \alpha|\overline{a} \leq 0\}$. □

**Theorem 1.40.** *Let $(f_N)_{N \in \mathbf{N}}$ be a conditional sequence of $\mathfrak{F}$-$\overline{\mathfrak{B}}$-conditionally measurable conditional functions $f_N\colon (\mathbf{X}, \mathfrak{F}) \to \left(\overline{\mathbf{R}}, \overline{\mathfrak{B}}\right)$, $N \in \mathbf{N}$. Then the conditional functions $\operatorname{ess\,inf}_{N \in \mathbf{N}} f_N$, $\operatorname{ess\,sup}_{N \in \mathbf{N}} f_N$, $\operatorname{ess\,lim\,inf}_{N \in \mathbf{N}} f_N$ and $\operatorname{ess\,lim\,sup}_{N \in \mathbf{N}} f_N$ are $\mathfrak{F}$-$\overline{\mathfrak{B}}$-conditionally measurable conditional functions, and, too $\lim_{N \in \mathbf{N}} f_N$ if it exists.*

*Proof.* We observe that $\{\operatorname{ess\,sup}_{N\in\mathbf{N}} f_N \leq \alpha\} = \bigsqcap_{N\in\mathbf{N}} \{f_N \leq \alpha\}$, hence, $\operatorname{ess\,sup}_{N\in\mathbf{N}} f_N$ is $\mathfrak{F}$-$\overline{\mathfrak{B}}$-conditionally measurable. With Theorem 1.39 and by definition, $\operatorname{ess\,inf}_{N\in\mathbf{N}} f_N = -\operatorname{ess\,sup}_{N\in\mathbf{N}} (-f_N)$, $\operatorname{ess\,lim\,inf}_{N\in\mathbf{N}} f_N = \operatorname{ess\,sup}_{N\in\mathbf{N}} (\operatorname{ess\,inf}_{M>N} f_M)$ and also finally $\operatorname{ess\,lim\,sup}_{N\in\mathbf{N}} f_N = \operatorname{ess\,inf}_{N\in\mathbf{N}} (\operatorname{ess\,sup}_{M>N} f_M)$ are $\mathfrak{F}$-$\overline{\mathfrak{B}}$-conditionally measurable. □

**Theorem 1.41.** *The conditional function $f\colon (\mathbf{X},\mathfrak{F}) \to (\overline{\mathbf{R}},\overline{\mathfrak{B}})$ is $\mathfrak{F}$-$\overline{\mathfrak{B}}$-conditionally measurable if and only if its positive part $f^+ := \operatorname{ess\,sup}\{f,0\}$ and its negative part $f^- := \operatorname{ess\,inf}\{f,0\}$ are $\mathfrak{F}$-$\overline{\mathfrak{B}}$-conditionally measurable. If so, the absolute value $|f| := \operatorname{ess\,sup}\{f,-f\}$ is $\mathfrak{F}$-$\overline{\mathfrak{B}}$-conditionally measurable.*

*Proof.* Considering $f = f^+ - f^-$ and $|f| = f^+ + f^-$, Theorem 1.39 and Theorem 1.40 yield the claims. □

We define a conditional sequence $(f_N)_{N\in\mathbf{N}}$ of $\mathfrak{F}$-$\overline{\mathfrak{B}}$-conditionally measurable conditional functions $f_N\colon (\mathbf{X},\mathfrak{F}) \to (\overline{\mathbf{R}},\overline{\mathfrak{B}})$ to be increasing if $f_N \geq f_M$ whenever $N \geq M$ and strictly increasing if $f_N > f_M$ whenever $N > M$.

**Definition 1.42.** Let $\mathfrak{F}$ be a conditional $\sigma$-algebra on a conditional set $\mathbf{X}$. Let $X \in \mathbf{X}$ and $\mathcal{C} \in \mathfrak{F}$. The conditional indicator function $\chi_{\mathcal{C}}(X) : \mathbf{X} \times \mathfrak{F} \to \{0,1\} \sqsubset \mathbf{R}$ is defined by $\chi_{\mathcal{C}}(X) := (1|a + 0|a^c)\,|b$ where $a := \vee\{\overline{a} \in \mathcal{A} \mid X|\overline{a} \in \mathcal{C}|\overline{a}\} \leq b$ and $\{X\} \sqcap \mathcal{C}$ lives on $b$. A conditional function $f\colon (\mathbf{X},\mathfrak{F}) \to (\overline{\mathbf{R}},\overline{\mathfrak{B}})$ is called elementary if there exist conditional finite conditional families $(\alpha_N)_{1\leq N\leq\overline{N}}$ in $\mathbf{R}$ and $(\mathcal{C}_N)_{1\leq N\leq\overline{N}}$ in $\mathfrak{F}$ such that $f = \sum_{1\leq N\leq\overline{N}} \alpha_N \chi_{\mathcal{C}_N}$. Further, if $\mu\colon \mathfrak{F} \to [0,\infty]$ is a conditional measure and $f\colon (\mathbf{X},\mathfrak{F}) \to (\overline{\mathbf{R}},\overline{\mathfrak{B}})$ an elementary conditional function, $f = \sum_{1\leq N\leq\overline{N}} \alpha_N \chi_{\mathcal{C}_N}$, we define the integral $\int f d\mu := \sum_{1\leq N\leq\overline{N}} \alpha_N \mu(\mathcal{C}_N)$ which is independent of the elementary representation as the following lemma shows.

**Lemma 1.43.** *Let $\mu\colon \mathfrak{F} \to [0,\infty]$ be a conditional measure. A conditional function $f\colon (\mathbf{X},\mathfrak{F}) \to (\overline{\mathbf{R}},\overline{\mathfrak{B}})$ may have two elementary representations, $f = \sum_{1\leq N\leq\overline{N}} \alpha_N \chi_{\mathcal{C}_N} = \sum_{1\leq M\leq\overline{M}} \beta_M \chi_{\mathcal{D}_M}$. Then, it holds that $\sum_{1\leq N\leq\overline{N}} \alpha_N \mu(\mathcal{C}_N) = \sum_{1\leq M\leq\overline{M}} \beta_M \mu(\mathcal{D}_M)$.*

*Proof.* We observe that $\alpha_N = \beta_M$ on $\mathcal{C}_N \sqcap \mathcal{D}_M$. Then,

$$\begin{aligned}\sum_{1\leq N\leq\overline{N}} \alpha_N \mu(\mathcal{C}_N) &= \sum_{1\leq N\leq\overline{N}} \alpha_N \sum_{1\leq M\leq\overline{M}} \mu(\mathcal{C}_N \sqcap \mathcal{D}_M)\\ &= \sum_{1\leq M\leq\overline{M}} \beta_M \sum_{1\leq N\leq\overline{N}} \mu(\mathcal{C}_N \sqcap \mathcal{D}_M) = \sum_{1\leq M\leq\overline{M}} \beta_M \mu(\mathcal{D}_M),\end{aligned}$$

which shows the claim. □

**Remark 1.44.** The integral is $\mathbf{R}$-linear and monotone, that is, $\int \alpha f + g d\mu = \alpha \int f d\mu + \int g d\mu$ and $\int f d\mu \leq \int g d\mu$ if $f \leq g$ for $\alpha \in \mathbf{R}$, elementary conditional functions $f, g$ and a conditional measure $\mu$.

**Theorem 1.45.** *Let* $\mu\colon \mathfrak{F} \to [0,\infty]$ *be a conditional measure and* $f, f_N\colon (\mathbf{X}, \mathfrak{F}) \to (\overline{\mathbf{R}}, \overline{\mathfrak{B}})$ *elementary conditional functions. Then* $\int f d\mu \leq \operatorname{ess\,sup}_{N\in\mathbf{N}} \int f_N d\mu$ *whenever* $f \leq \operatorname{ess\,sup}_{N\in\mathbf{N}} f_N$.

*Proof.* Let $f = \sum_{1\leq N\leq \overline{N}} \alpha_N \chi_{\mathcal{C}_N}$ and $\alpha \in \left]0,1\right[$. By Theorem 1.39, $\mathcal{D}_N := \{f_N \geq \alpha f\} \in \mathfrak{F}$. Then, $\int f_N d\mu \geq \alpha \int f \chi_{\mathcal{D}_N} d\mu$ by Remark 1.44. Since $\mu$ is a conditional measure, it holds that $\int f d\mu = \sum_{1\leq N\leq \overline{N}} \alpha_N \mu(\mathcal{C}_N) = \lim_{M\in\mathbf{N}} \sum_{1\leq N\leq \overline{N}} \alpha_N \mu(\mathcal{C}_N \sqcap \mathcal{D}_M) = \lim_{M\in\mathbf{N}} \int f \chi_{\mathcal{D}_M} d\mu$. To conclude,

$$\operatorname*{ess\,sup}_{N\in\mathbf{N}} \int f_N d\mu \geq \operatorname*{ess\,sup}_{N\in\mathbf{N}} \alpha \int u \chi_{\mathcal{D}_N} d\mu = \alpha \lim \int f \chi_{\mathcal{D}_M} d\mu = \alpha \int u d\mu.$$

Since $\alpha \in \left]0,1\right[$ has been chosen arbitrarily, the claim follows. $\square$

**Theorem 1.46.** *Let* $\mu\colon \mathfrak{F} \to [0,\infty]$ *be a conditional measure. Let* $f\colon (\mathbf{X}, \mathfrak{F}) \to (\overline{\mathbf{R}}, \overline{\mathfrak{B}})$ *be a nonnegative* $\mathfrak{F}$-$\overline{\mathfrak{B}}$*-conditionally measurable conditional function. Then there exists an increasing conditional sequence* $(f_N)_{N\in\mathbf{N}}$ *of elementary conditional functions* $f_N\colon (\mathbf{X}, \mathfrak{F}) \to (\overline{\mathbf{R}}, \overline{\mathfrak{B}})$ *such that* $\lim_{N\in\mathbf{N}} f_N = f$.

*Proof.* We approximate every nonnegative $\mathfrak{F}$-$\overline{\mathfrak{B}}$-conditionally measurable conditional function by dyadic conditional functions in $\mathbf{R}$. To that end, we define conditional sets $\mathcal{C}_{kn} \in \mathfrak{F}$ by

$$\mathcal{C}_{kn} := \begin{cases} \left\{f \geq \frac{k}{2^n}\right\} \sqcap \left\{f < \frac{k+1}{2^n}\right\}, & 0 \leq k \leq n2^n - 1, \quad n \in \mathbb{N}, k \in \mathbf{N}, \\ \{f \geq n\}, & k = n2^n. \end{cases}$$

We have that $\mathbf{X} = \bigsqcup_{0\leq k\leq n2^n} \mathcal{C}_{kn}$. Next, we define $f_n := \sum_{0\leq k\leq n2^n} \frac{k}{2^n} \chi_{\mathcal{C}_{kn}}$ and $f_N := \sum_{n\in\mathbb{N}} a_n f_{m_n}$ for $N = \sum_{n\in\mathbb{N}} a_n m_n$ which is an elementary conditional function. The conditional sequence $(f_N)_{N\in\mathbf{N}}$ is increasing and $f = \operatorname{ess\,sup}_{N\in\mathbf{N}} f_N$ by construction. $\square$

**Definition 1.47.** Let $\mu\colon \mathfrak{F} \to [0,\infty]$ be a conditional measure. Let $f, g\colon (\mathbf{X}, \mathfrak{F}) \to (\overline{\mathbf{R}}, \overline{\mathfrak{B}})$ be $\mathfrak{F}$-$\overline{\mathfrak{B}}$-conditionally measurable conditional functions. A nonnegative $f$ may be represented as an increasing conditional sequence $(f_N)_{N\in\mathbf{N}}$ of elementary conditional functions. Then, the representation-independent

$$\int f d\mu := \operatorname*{ess\,sup}_{N\in\mathbf{N}} \int f_N d\mu \geq 0 \tag{1.6}$$

is the $\mu$-conditional integral of $f$. The conditional function $g$ is called $\mu$-conditionally integrable if the integrals $\int g^+ d\mu$ and $\int g^- d\mu$ are in $\mathbf{R}$. Then,

$$\int g d\mu := \int g^+ d\mu - \int g^- d\mu \tag{1.7}$$

is the $\mu$-condtional integral of $g$.

**Remark 1.48.** The $\mu$-conditional integral is $\mathbf{R}$-linear, monotone and satisfies $\left|\int f d\mu\right| \leq \int |f|\, d\mu$. Furthermore, the $\mu$-conditional integral of $f$ over $\mathcal{C} \in \mathcal{F}$ is defined by $\int_{\mathcal{C}} f d\mu := \int f \chi_{\mathcal{C}} d\mu$ if $f$ is $\mu$-conditionally integrable or a nonnegative $\mathfrak{F}$-$\overline{\mathfrak{B}}$-conditionally measurable conditional function. Plenty of properties of this integral can be derived from properties of the conditional indicator functions.

**Theorem 1.49** (Monotone convergence, Fatou's Lemma, Dominated convergence)**.** *Let $\mu\colon \mathfrak{F} \to [0,\infty]$ be a conditional measure. Let $(f_N)_{N\in\mathbf{N}}$ be a conditional sequence of nonnegative $\mathfrak{F}$-$\overline{\mathfrak{B}}$-conditionally measurable conditional functions $f_N\colon (\mathbf{X},\mathfrak{F}) \to \left(\overline{\mathbf{R}},\overline{\mathfrak{B}}\right)$. Then $\operatorname{ess\,sup}_{N\in\mathbf{N}} f_N$ is a nonnegative $\mathfrak{F}$-$\overline{\mathfrak{B}}$-conditionally measurable conditional function and $\int \operatorname{ess\,sup}_{N\in\mathbf{N}} f_N d\mu = \operatorname{ess\,sup}_{N\in\mathbf{N}} \int f_N d\mu$. Further,*

$$\int \operatorname*{ess\,lim\,inf}_{N\in\mathbf{N}} f_N d\mu \leq \operatorname*{ess\,lim\,inf}_{N\in\mathbf{N}} \int f_N d\mu. \tag{1.8}$$

*If additionally $\lim_{N\in\mathbf{N}} f_N = f$ for an $\mathfrak{F}$-$\overline{\mathfrak{B}}$-conditionally measurable conditional function $f\colon (\mathbf{X},\mathfrak{F}) \to \left(\overline{\mathbf{R}},\overline{\mathfrak{B}}\right)$ and there is a $\mu$-conditionally integrable conditional function $g\colon (\mathbf{X},\mathfrak{F}) \to \left(\overline{\mathbf{R}},\overline{\mathfrak{B}}\right)$ with $|f_N| \leq g$ then $f$ and $f_N$ are $\mu$-conditionally integrable and $\lim_{N\in\mathbf{N}} \int |f - f_N|\, d\mu = 0$.*

*Proof.* Let $f^* := \operatorname{ess\,sup}_{N\in\mathbf{N}} f_N$. The claim is to find an increasing conditional sequence $(g_N)_{N\in\mathbf{N}}$ of elementary conditional functions with $\operatorname{ess\,sup}_{N\in\mathbf{N}} g_N = f^*$. To this end by definition of $f_N$, there are elementary conditional functions $g_{MN}$ with $\operatorname{ess\,sup}_{M\in\mathbf{N}} g_{MN} = f_N$. Then $g_M := \operatorname{ess\,sup}_{0\leq K\leq M} g_{MK}$ is a elementary conditional function. The conditional sequence $(g_M)_{M\in\mathbf{N}}$ is increasing. It follows that $g_M \leq f_M$ and $\operatorname{ess\,sup}_{M\in\mathbf{N}} g_M \leq f^*$. Further, $g_{MN} \leq g_M$, hence, $\operatorname{ess\,sup}_{M\in\mathbf{N}} g_{MN} = f_N \leq \operatorname{ess\,sup}_{M\in\mathbf{N}} g_M$, and finally, $\operatorname{ess\,sup}_{M\in\mathbf{N}} g_M = f^*$, hence, the conditional sequence $(g_M)_{M\in\mathbf{N}}$ has all the required properties.

To show Fatou's Lemma, by monotone convergence, $\int \operatorname{ess\,sup}_{N\in\mathbf{N}} \operatorname{ess\,inf}_{M\geq N} f_M = \operatorname{ess\,sup}_{N\in\mathbf{N}} \int \operatorname{ess\,inf}_{M\geq N} f_M d\mu$. Further, $\int \operatorname{ess\,inf}_{M\geq N} f_N d\mu \leq \int f_M d\mu$. Taking the essential infimum on both sides yields $\int \operatorname{ess\,inf}_{M\geq N} f_N d\mu \leq \operatorname{ess\,inf}_{M\geq N} \int f_M d\mu$. Hence finally, it holds that $\int \operatorname{ess\,sup}_{N\in\mathbf{N}} \operatorname{ess\,inf}_{M\geq N} f_M = \operatorname{ess\,sup}_{N\in\mathbf{N}} \int \operatorname{ess\,inf}_{M\geq N} f_M d\mu \leq \operatorname{ess\,sup}_{N\in\mathbf{N}} \operatorname{ess\,inf}_{M\geq N} \int f_M d\mu$.

To show Dominated convergence, we define $g_N := |f - f_N|$ and we will show that $\lim_{N\in\mathbf{N}} \int g_N d\mu = 0$. To this end, consider $0 \leq g_N \leq g + |f|$ where the latter is $\mu$-conditionally integrable since $|f| \leq g$ and by Theorem 1.41, Theorem 1.46 and Remark 1.48. Then $g_N$ is $\mu$-conditionally integrable for the same reasons. We apply Fatou's Lemma to $g + |f| - g_N$ and obtain, since $\lim_{N\in\mathbf{N}} g_N = \operatorname{ess\,lim\,inf}_{N\in\mathbf{N}} g_N = 0$,

$$\begin{aligned}\int g + |f|\, d\mu &= \int \operatorname{ess}\liminf_{N\in\mathbf{N}} \left(g + |f| - g_N\right) d\mu \leq \operatorname{ess}\liminf_{N\in\mathbf{N}} \int \left(g + |f| - g_N\right) d\mu \\ &= \int g + |f|\, d\mu - \operatorname{ess}\limsup_{N\in\mathbf{N}} \int g_N d\mu\end{aligned}$$

which yields $\operatorname{ess\,lim\,sup}_{N\in\mathbf{N}} \int g_N d\mu \leq 0$. But $g_N$ is nonnegative, thus we have established that $\lim_{N\in\mathbf{N}} \int g_N d\mu = 0$ which by definition of $g_N$ yields the claim. □

### 1.2.3 Radon-Nikodym theorem

In the sequel, let $\mu\colon \mathfrak{F} \to [0,\infty]$ be a conditional measure and let $f\colon (\mathbf{X},\mathfrak{F}) \to \left(\overline{\mathbf{R}},\mathfrak{B}\right)$ be a nonnegative $\mathfrak{F}$-$\mathfrak{B}$-conditionally measurable conditional function.

**Theorem 1.50.** *For all nonnegative $\mathfrak{F}$-$\mathfrak{B}$-conditionally measurable conditional functions $f\colon (\mathbf{X},\mathfrak{F}) \to \left(\overline{\mathbf{R}},\mathfrak{B}\right)$, we define a conditional measure $\nu\colon \mathfrak{F} \to [0,\infty]$ by*

$$\nu\left(\mathcal{C}\right) := \int_{\mathcal{C}} f d\mu := \int f \chi_{\mathcal{C}} d\mu. \tag{1.9}$$

*Proof.* By definition, $\nu\left(\mathbf{X}|0\right) = 0$ and $\mu\left(\mathcal{C}\right) \geq 0$. For a conditional sequence $\left(\mathcal{C}_N\right)_{N\in\mathbf{N}}$ of conditionally pairwise disjoint conditional sets in $\mathfrak{F}$, it holds that $f\chi_{\bigsqcup_{N\in\mathbf{N}} \mathcal{C}_N} = \sum_{N\in\mathbf{N}} f\chi_{\mathcal{C}_N}$, hence, by **R**-conditional linearity and monotone convergence (Theorem 1.49), $\nu\left(\mathcal{C}\right) = \sum_{N\in\mathbf{N}} \nu\left(\mathcal{C}_N\right)$, thus the claim holds. □

**Theorem 1.51.** *Let $\mu\colon \mathfrak{F} \to [0,\infty]$ be a conditional measure and let $f\colon (\mathbf{X},\mathfrak{F}) \to \left(\overline{\mathbf{R}},\mathfrak{B}\right)$ be a nonnegative $\mathfrak{F}$-$\mathfrak{B}$-conditionally measurable conditional function. Then, $\int f d\mu = 0$ if and only if $\mu\left(\{f = 0\}\right) = 0$. A conditional function $g\colon (\mathbf{X},\mathfrak{F}) \to \left(\overline{\mathbf{R}},\mathfrak{B}\right)$ is $\mu$-conditionally integrable over every $\mathcal{C} \in \mathcal{F}$ with $\mu\left(\mathcal{C}\right) = 0$.*

*Proof.* Clearly, $\{f \neq 0\} = \{f > 0\} \in \mathfrak{F}$, since $f$ is $\mathfrak{F}$-$\mathfrak{B}$-conditionally measurable and by Theorem 1.37. So, let $\int f d\mu = 0$. Further, let $\mathcal{C}_N := \left\{f \geq \frac{1}{2^N}\right\} \in \mathfrak{F}$. Then, $f \geq \frac{1}{2^N}\chi_{\mathcal{C}_N}$ implies $0 = \int f d\mu \geq \frac{1}{2^N}\mu(\mathcal{C}_N) \geq 0$. But, $\mu(\{f > 0\}) = \lim_{N\in\mathbf{N}} \mu(\mathcal{C}_N) = 0$.

On the other hand, let $\mu(\{f > 0\}) = 0$. Then, $f \leq \operatorname{ess\,sup}_{N\in\mathbf{N}} N\chi_{\{f>0\}}$, and $0 \leq \int f d\mu \leq \int \operatorname{ess\,sup}_{N\in\mathbf{N}} N\chi_{\{f>0\}} d\mu = \operatorname{ess\,sup}_{N\in\mathbf{N}} \int N\chi_{\{f>0\}} d\mu = 0$ by monotone convergence. But, for all $N \in \mathbf{N}$ it holds that $\int N\chi_{\{f>0\}} = 0$ by construction.

If $g \geq 0$ then consider $f := g\chi_{\mathcal{C}}$ which fulfills the conditions of the first part of the theorem. For arbitrary $g$, we apply this to $g^+$ and $-g^-$ to show the last claim. $\square$

**Definition 1.52.** For all nonnegative $\mathfrak{F}$-$\mathfrak{B}$-conditionally measurable conditional functions $f\colon (\mathbf{X}, \mathfrak{F}) \to (\overline{\mathbf{R}}, \mathfrak{B})$, the conditional measure $\nu\colon \mathfrak{F} \to [0, \infty]$ defined by (1.9) is called the conditional measure with density $f$ with respect to $\mu$. We write $\nu = f\mu$.

**Lemma 1.53.** *Let $f, g, \varphi\colon (\mathbf{X}, \mathfrak{F}) \to (\overline{\mathbf{R}}, \mathfrak{B})$ be nonnegative $\mathfrak{F}$-$\mathfrak{B}$-conditionally measurable conditional functions. We define $\nu := f\mu$ and $\nu^* := g\nu$. Then,*

$$\int \varphi d\nu = \int \varphi f d\mu \quad \textit{and} \quad \nu^* = (gf)\,\mu. \tag{1.10}$$

*Proof.* The first part follows directly for simple conditional functions and by monotone convergence on their supremum. The second part follows from the first part and $\nu^*(\mathcal{C}) = \int g\chi_{\mathcal{C}} d\nu = \int gf\chi_{\mathcal{C}} d\mu$ which is the claim. $\square$

**Definition 1.54.** A conditional measure $\nu\colon \mathfrak{F} \to [0, \infty]$ is called continuous with respect to a conditional measure $\mu\colon \mathfrak{F} \to [0, \infty]$ or $\mu$-conditionally continuous if $\mu(\mathcal{C})|a = 0$ implies $\nu(\mathcal{C})|a = 0$ for $\mathcal{C} \in \mathfrak{F}$.

**Theorem 1.55** (Radon-Nikodym)**.** *Let $\mu, \nu\colon \mathfrak{F} \to [0, \infty]$ be conditional measures and let $\mu$ be conditionally $\sigma$-finite. Then, $\nu$ has a density with respect to $\mu$ if and only if $\nu$ is $\mu$-continuous.*

*Proof.* If $\nu$ has a density with respect to $\mu$, then, by Theorem 1.51, $\nu$ is $\mu$-continuous. For the inverse implication, we consider the cases, that $\mu, \nu$ are finite, then only that $\mu$ is finite and finally, $\mu$ is conditionally $\sigma$-finite.

First, let $\mu$ and $\nu$ be finite. We define the set $G$ of nonnegative $\mathfrak{F}$-$\mathfrak{B}$-conditionally measurable conditional functions $g$ with $g\mu \leq \nu$. The conditional function $g \equiv 0$ is in $G$, hence, $G$ is nonempty. Further, define $\gamma := \operatorname{ess\,sup}\left\{\int g d\mu \mid g \in G\right\} \in \mathbf{R}$ which exists since $\nu$ is finite. By [DJKK16, Theorem 4.5] und monotonicity of the integral, there exists an nondecreasing conditional sequence $(g_N)_{N\in\mathbf{N}}$ such that $\lim_{N\in\mathbf{N}} \int g_N d\mu =$

$\gamma$. By monotone convergence, $\operatorname{ess\,sup}_{N\in\mathbf{N}} g_N \in G$ and $\int_{N\in\mathbf{N}} \operatorname{ess\,sup} g_N d\mu = \gamma$. Thus, $\operatorname{ess\,sup}_{N\in\mathbf{N}} g_N$ is a maximizer of $g \mapsto \int g d\mu$. We show that $\operatorname{ess\,sup}_{N\in\mathbf{N}} g_N \mu = \nu$. To this end, let $\overline{\nu} := \nu - \operatorname{ess\,sup}_{N\in\mathbf{N}} g_N \mu$ which is $\mu$-conditionally continuous by assumption and assume that $\overline{\nu}(\mathbf{X})|a > 0$, since if $\overline{\nu}(\mathbf{X}) = 0$ we are done. By $\mu$-conditional continuity, $\mu(\mathbf{X})|a > 0$, thus, we can define $\beta := \frac{\overline{\nu}(\mathbf{X})}{2\cdot\mu(\mathbf{X})}|a + 0|a^c$ with $\beta|a > 0$. Applying Lemma 1.57, we obtain $\mathcal{C}_0 \in \mathfrak{F}$ such that $\overline{\nu}(\mathcal{C}_0) - \beta\mu(\mathcal{C}_0) \geq \overline{\nu}(\mathbf{X}) - \beta\mu(\mathbf{X})$ with $(\overline{\nu}(\mathcal{C}_0) - \beta\mu(\mathcal{C}_0))|a > 0$ and $\overline{\nu}(\mathcal{D})|a \geq \beta\mu(\mathcal{D})|a$ for all $\mathcal{D} \sqsubset \mathcal{C}_0 \sqcap \mathfrak{F}$. We show that $\operatorname{ess\,sup}_{N\in\mathbf{N}} g_N + \beta\chi_{\mathcal{C}_0} \in G$. First, it is $\mathfrak{F}$-$\mathfrak{B}$-conditionally measurable. For all $\mathcal{C} \in \mathfrak{F}$, it holds that

$$\begin{aligned}\int_{\mathcal{C}} \operatorname*{ess\,sup}_{N\in\mathbf{N}} g_N + \beta\chi_{\mathcal{C}_0} d\mu &\leq \int_{\mathcal{C}} \operatorname*{ess\,sup}_{N\in\mathbf{N}} g_N d\mu + \beta\mu(\mathcal{C}_0 \sqcap \mathcal{C}) \\ &\leq \int_{\mathcal{C}} \operatorname*{ess\,sup}_{N\in\mathbf{N}} g_N d\mu + \overline{\nu}(\mathcal{C}_0 \sqcap \mathcal{C}) \leq \nu(\mathcal{C}).\end{aligned}$$

Since $\nu(\mathcal{C}_0)|a = (\overline{\nu}(\mathcal{C}_0) - \beta\mu(\mathcal{C}_0))|a > 0$ and $\nu$ is $\mu$-conditional continuous, it holds that $\mu(\mathcal{C}_0)|a > 0$. Then,

$$\int \operatorname*{ess\,sup}_{N\in\mathbf{N}} g_N + \beta\chi_{\mathcal{C}_0} d\mu|a = \gamma|a + \beta\mu(\mathcal{C}_0)|a > \gamma|a,$$

in contradiction to the maximality of $\gamma$ and $\operatorname{ess\,sup}_{N\in\mathbf{N}} g_N + \beta\chi_{\mathcal{C}_0} \in G$. Hence, $\overline{\nu} \equiv 0$, as required.

In the second part, we consider the case that only $\mu$ is finite. We construct $\mathcal{C}_0 \in \mathfrak{F}$ and a conditional sequence $(C_N)_{N\in\mathbf{N}}$ such that $\bigsqcup_{N\in\mathbf{N}} \mathcal{C}_N \sqcup \mathcal{C}_0 = \mathbf{X}$, $\nu(\mathcal{C}_N)$ is finite, and for all $\mathcal{D} \sqsubset \mathcal{C}_0 \sqcap \mathfrak{F}$, there exists $a_{\mathcal{D}} \in \mathcal{A}$ such that $\mu(\mathcal{D})|a_{\mathcal{D}} = \nu(\mathcal{D})|a_{\mathcal{D}} = 0$ and $0 < \mu(\mathcal{D})|a_{\mathcal{D}}^c < \nu(\mathcal{D})|a_{\mathcal{D}}^c = \infty$. To this end, we define $\alpha := \operatorname{ess\,sup}\{\mu(\mathcal{D}) \mid \mathcal{D} \in \mathfrak{F},\ \nu(\mathcal{D}) < \infty\} \in \mathbf{R}$ since $\mu$ is finite. Again, by [DJKK16, Theorem 4.5], there exists an nondecreasing conditional sequence $(\mathcal{D}_N)_{N\in\mathbf{N}}$ with $\lim_{N\in\mathbf{N}} \mu(\mathcal{D}_N) = \alpha$. Thus, $\bigsqcup_{N\in\mathbf{N}} D_N \in \mathfrak{F}$, and by monotone convergence, $\mu\left(\bigsqcup_{N\in\mathbf{N}} \mathcal{D}_N\right) = \alpha$. We consider $\mathcal{C}_1 := \mathcal{D}_1$, $\mathcal{C}_{N+1} := \mathcal{D}_{N+1} \sqcap \mathcal{D}_N^{\sqsubset}$ and $\mathcal{C}_0 := \left(\bigsqcup_{N\in\mathbf{N}} \mathcal{D}_N\right)^{\sqsubset}$. For all $\mathcal{D} \sqsubset \mathcal{C}_0 \sqcap \mathfrak{F}$, let $a_{\mathcal{D}} := (\vee\{a \in \mathcal{A} \mid \nu(\mathcal{D})|a = \infty\})^c$. We show that $\mu(\mathcal{D})|a_{\mathcal{D}} = 0$. Since $\nu(\mathcal{D})|a_{\mathcal{D}} < \infty$, it holds that $\nu(\mathcal{D}_N \sqcap \mathcal{D})|a_{\mathcal{D}} < \infty$, thus, $\mu(\mathcal{D}_N \sqcap \mathcal{D})|a_{\mathcal{D}} \leq \alpha|a_{\mathcal{D}}$, and by monotone convergence, $\mu\left(\bigsqcup_{N\in\mathbf{N}} \mathcal{D}_N \sqcap \mathcal{D}\right)|a_{\mathcal{D}} = \lim_{N\in\mathbf{N}} \mu(\mathcal{D}_N \sqcap \mathcal{D})|a_{\mathcal{D}} \leq \alpha|a_{\mathcal{D}}$. Since by construction $\bigsqcup_{N\in\mathbf{N}} \mathcal{D}_N \sqcap \mathcal{D} = \mathbf{X}|0$, it holds that $\mu\left(\bigsqcup_{N\in\mathbf{N}} \mathcal{D}_N \sqcap \mathcal{D}\right)|a_{\mathcal{D}} = \alpha|a_{\mathcal{D}} + \mu(\mathcal{D})|a_{\mathcal{D}}$. Thus, $\mu(\mathcal{D})|a_{\mathcal{D}} = 0$.

We make use of this decomposition by considering $\mu$ and $\nu$ restricted on the conditional $\sigma$-algebras $\mathcal{C}_N \sqcap \mathfrak{F}$ denoted by $\mu_N$ and $\nu_N$. By assumption, $\nu_N$ is $\mu_N$-conditionally continuous and $\mu$ and $\nu$ are finite. By the first part of the proof, there are nonnegative $\mathcal{C}_N \sqcap \mathfrak{F}$-$\mathfrak{B}$-conditionally measurable conditional functions $f_N$ such that $\nu_N = f_N\mu_N$. By the construction of the decomposition, $\mu_0$ and $\nu_0$ as restrictions of $\mu$ and $\nu$ on $\mathcal{C}_0 \sqcap \mathfrak{F}$ that fulfill $\nu_0 = f_0\mu_0$ with $f_0 \equiv \infty$. Finally, $f$ defined as $f_N$ on $\mathcal{C}_N$ satisfies $\nu = f\mu$.

For the third part, assume $\mu$ to be conditionally $\sigma$-finite. By Lemma 1.56, there exists a $\mu$-conditionally integrable conditional function $f\colon (\mathbf{X}, \mathfrak{F}) \to (\mathbf{R}, \mathfrak{B})$ with $f > 0$. Thus, we can define the conditional measure $\overline{\mu} := f\mu$ with $\mu(\mathcal{C}) = 0$ if and only if $\overline{\mu}(\mathcal{C}) = 0$ for all $\mathcal{C} \in \mathfrak{F}$. Hence, $\nu$ is $\overline{\mu}$-conditionally continuous. By the second part of the proof, there exists a nonnegative $\mathfrak{F}$-$\mathfrak{B}$-conditionally measurable conditional function $g$ with $\nu = g\overline{\mu}$. Then, $\nu = (gf)\mu$ by Lemma 1.53 which yields the claim. □

**Lemma 1.56.** *Let $\mu\colon \mathfrak{F} \to [0, \infty]$ be a conditional measure. It is $\sigma$-conditionally finite if and only if there exists a $\mu$-conditionally integrable conditional function $f\colon (\mathbf{X}, \mathfrak{F}) \to (\mathbf{R}, \mathfrak{B})$ with $f > 0$.*

*Proof.* By definition of a conditionally $\sigma$-finite conditional measure, there exists a conditional sequence $(C_N)_{N\in\mathbf{N}}$ in $\mathbf{X}$ such that $\mu(\mathcal{C}_N) \geq 0$ and $\bigsqcup_{N\in\mathbf{N}} \mathcal{C}_N = \mathbf{X}$. We define $a_N := \vee\{\overline{a} \in \mathcal{A} \mid \mu(\mathcal{C}_N)|\overline{a} = 0\}$ and $\alpha_N := \frac{1}{2^N}|a_N + \left(\frac{1}{2^N} \wedge \frac{1}{\mu(\mathcal{C}_N)\cdot 2^N}\right)|a_N^c > 0$. Further, define $f := \sum_{N\in\mathbf{N}} \alpha_N \chi_{\mathcal{C}_N}$. By definition, $f$ is a $\mathfrak{F}$-$\mathfrak{B}$-conditionally measurable conditional function with $0 \leq f \leq 1$ and $\int f d\mu \leq 1$. If $f > 0$, we are done. If $f|a = 0$, we have that $\mu(\mathcal{C}_n)|a = 0$, hence $\mu = 0$, in turn, any $f$ is as required. □

**Lemma 1.57.** *Let $\mu, \nu\colon \mathfrak{F} \to [0, \infty]$ be finite conditional measures. Let $\overline{\mu} := \nu - \mu$. Then, there is $\mathcal{C}_0 \in \mathfrak{F}$ such that $\overline{\mu}(\mathcal{C}_0) \geq \overline{\mu}(\mathbf{X})$ and $\overline{\mu}(\mathcal{D}) \geq 0$ for all $\mathcal{D} \sqsubset \mathcal{C}_0 \sqcap \mathfrak{F}$.*

*Proof.* First, we proof that for every $\varepsilon > 0$ there exists $\mathcal{C}_\varepsilon \in \mathfrak{F}$ such that $\overline{\mu}(\mathcal{C}_\varepsilon) \geq \overline{\mu}(\mathbf{X})$ and $\overline{\mu}(\mathcal{D}) \geq -\varepsilon$ for all $\mathcal{D} \sqsubset \mathcal{C}_\varepsilon \sqcap \mathfrak{F}$. We may assume that $\overline{\mu}(\mathbf{X}) \geq 0$. For if $a := \sup\{\overline{a} \in \mathcal{A} \mid \overline{\mu}(\mathbf{X})|\overline{a} \geq 0\}$ then any $\mathcal{C}_\varepsilon \in \mathfrak{F}$ with $\overline{\mu}(\mathcal{C}_\varepsilon)|a \geq \overline{\mu}(\mathbf{X})|a$ and $\overline{\mu}(\mathcal{D})|a \geq -\varepsilon|a$ for all $\mathcal{D} \sqsubset \mathcal{C}_\varepsilon \sqcap \mathfrak{F}$ yields $\mathcal{C}_\varepsilon|a \in \mathfrak{F}$ fulfilling all the properties of the claim.

We define $a_0 := (\vee\{\overline{a} \in \mathcal{A} \mid \overline{\mu}(\mathcal{D})|\overline{a} \geq -\varepsilon|\overline{a}\ \forall \mathcal{D} \in \mathfrak{F}\})^c$. If $a_0 = 0$, we choose $\mathcal{C}_\varepsilon := \mathbf{X}$ to fulfill the claim. Otherwise, there exists $\mathcal{D}_1 \in \mathfrak{F}$ with $\overline{\mu}(\mathcal{D}_1)|a_0 \leq -\varepsilon|a_0$. By definition of $\overline{\mu}$, it holds that $\overline{\mu}(\mathcal{D}_1^{\sqsubset})|a_0 \geq (\overline{\mu}(\mathbf{X}) + \varepsilon)|a_0 > \overline{\mu}(\mathbf{X})|a_0$. Define $a_1 := (\vee\{\overline{a} \in \mathcal{A} \mid \overline{\mu}(\mathcal{D})|\overline{a} \geq -\varepsilon|\overline{a}\ \forall \mathcal{D} \in \mathcal{D}_1^{\sqsubset} \sqcap \mathfrak{F}\})^c \wedge a_0$. If $a_1 = 0$, we choose $\mathcal{C}_\varepsilon := \mathcal{D}_1^{\sqsubset}|a_0 + \mathbf{X}|a_0^c$ to fulfill the claim. Otherwise, there exists $\mathcal{D}_2 \in \mathcal{D}_1^{\sqsubset} \sqcap \mathfrak{F}$ with $\overline{\mu}(\mathcal{D}_2)|a_2 \leq -\varepsilon|a_2$. Assume $(\mathcal{D}_N)_{N\in\mathbf{N}}$ to be constructed pairwise conditionally disjoint with $\overline{\mu}(\mathcal{D}_N) \leq -\varepsilon$ for $\mathcal{D}_N \in \mathcal{D}_{N-1}^{\sqsubset} \sqcap \mathfrak{F}$. Then, on $a_N$,

$$\overline{\mu}\left(\left(\bigsqcup_{1\leq N\leq K} \mathcal{D}_N\right)^{\sqsubset}\right) = \overline{\mu}(\mathbf{X}) - \sum_{1\leq N\leq K} \overline{\mu}(\mathcal{D}_N) \geq \overline{\mu}(\mathbf{X}) + K\varepsilon > \overline{\mu}(\mathbf{X}).$$

Thus, $\sum_{1\leq N\leq K}\overline{\mu}\left(\mathcal{D}_N\right)|a_K \leq -K\varepsilon|a_K$ and, hence, $\sum_{N\in\mathbf{N}}\overline{\mu}\left(\mathcal{D}_N\right)|\wedge_{K\in\mathbf{N}}a_K = -\infty$. But also $\sum_{N\in\mathbf{N}}\overline{\mu}\left(\mathcal{D}_N\right)|\wedge_{K\in\mathbf{N}}a_K = \nu\left(\bigsqcup_{N\in\mathbf{N}}^{\infty}\mathcal{D}_N\right)|\wedge_{K\in\mathbf{N}}a_K - \mu\left(\bigsqcup_{N\in\mathbf{N}}\mathcal{D}_N\right)|\wedge_{K\in\mathbf{N}}a_K > -\infty$. Thus, $\wedge_{K\in\mathbf{N}}a_K = 0$. Then, $\mathcal{C}_\varepsilon := \sum_{N\in\mathbf{N}}\left(a_N - a_{N-1}\right)\mathcal{D}_{N+1}^{\sqsubset} + a_0\mathcal{D}_1 + a_0^c\mathbf{X}$ fulfills the claim.

We conclude with the original claim. We consider $\varepsilon := \frac{1}{N}$, and choose the conditional sequence $\left(\mathcal{C}_N\right)_{N\in\mathbf{N}}$ such that $\mathcal{C}_N \sqsubset \mathcal{C}_{N-1}$ in $\mathfrak{F}$. Consider, $\mathcal{C}_0 := \bigsqcap_{N\in\mathbf{N}}\mathcal{C}_N$. Then, it holds that $\overline{\mu}\left(\mathcal{D}\right) \geq -\varepsilon$ for all $\mathcal{D}\in\mathcal{C}_0\sqcap\mathfrak{F}$ and all $\varepsilon > 0$, hence, $\overline{\mu}\left(\mathcal{D}\right)\geq 0$. Further, since $\overline{\mu}\left(\mathcal{C}_n\right)\geq\overline{\mu}\left(\mathbf{X}\right)$ we can apply the limit and obtain $\overline{\mu}\left(\mathcal{C}_0\right) = \lim_{N\in\mathbf{N}}\overline{\mu}\left(\mathcal{C}_N\right)\geq\overline{\mu}\left(\mathbf{X}\right)$. The claim then follows by Remark 2.3. □

### 1.2.4 Product measures

**Definition 1.58.** Let $\mathfrak{F}_i$ be a conditional $\sigma$-algebra on $\mathbf{X}_i$. The conditional $\sigma$-algebra $\Sigma\left(pr_i^{-1}\left(\mathfrak{F}_i\right)\right)$ on the conditional product of $\left(\mathbf{X}^i\right)_{i\in I}$ generated by the projection mappings $pr_i\colon \prod_{i\in I}\mathbf{X}_i\to\mathbf{X}_i$ is the conditional product $\sigma$-algebra $\otimes_{i\in I}\mathcal{F}_i$.

**Lemma 1.59.** *Let $\mathfrak{C}_i$ be a generator of $\mathfrak{F}_i$ with $\left(\mathcal{C}_{ik}\right)_{k\in\mathbb{N}}$ in $\mathfrak{C}_i$ such that $\bigsqcup_{k\in\mathbb{N}}\mathcal{C}_{ik} = \mathbf{X}_i$. Then, $\otimes\mathfrak{F}_i = \Sigma\left(\times_{i\in I}\mathcal{C}_i\right)$ for $\mathcal{C}_i\in\mathfrak{C}_i$.*

*Proof.* That follows directly from the definitions. □

**Theorem 1.60.** *Let $\mathfrak{C}_i$ be a conditional $\pi$-system and a generator of $\mathfrak{F}_i$ with $\left(\mathcal{C}_{ik}\right)_{k\in\mathbb{N}}$ in $\mathfrak{C}_i$ such that $\bigsqcup_{k\in\mathbb{N}}\mathcal{C}_{ik} = \mathbf{X}_i$ and $\mu_i\left(\mathcal{C}_{ik}\right) < \infty$. Then, there is at most one conditional measure $\mu$ on $\otimes\mathfrak{F}_i$ such that $\mu\left(\times_{i\in I}\mathcal{C}_i\right) = \prod_{i\in I}\mu\left(\mathcal{C}_i\right)$ for all $\mathcal{C}_i\in\mathfrak{C}_i$.*

*Proof.* Clearly, $\otimes\mathfrak{F}_i$ is a conditional $\pi$-system and $\mathcal{C}_k := \times_{i\in I}\mathcal{C}_{ik}$ are such that $\bigsqcup_{k\in\mathbb{N}}\mathcal{C}_k = \prod_{i\in I}X_i$. The claim follows by Lemma 1.59 and Theorem 1.29. □

For the sake of simplicity, we consider the conditional product of two conditional sets $\mathbf{X}_1$ and $\mathbf{X}_2$. By induction, the generalisations of the following theorems can easlily be derived for condtional countable conditional products by Remark 2.3.

**Lemma 1.61.** *Let $\mathcal{C}\sqsubset\mathbf{X}_1\times\mathbf{X}_2$ and define $\mathcal{C}_{x_1} := \{X_2\in\mathbf{X}_2 \mid (X_1,X_2)\in\mathcal{C}\}$ and $\mathcal{C}_{X_2} := \{X_1\in\mathbf{X}_1 \mid (X_1,X_2)\in\mathcal{C}\}$, the $X_1$-section of $\mathcal{C}$ and the $X_2$-section of $\mathcal{C}$. If $\mathcal{C}\in\mathfrak{F}_1\otimes\mathfrak{F}_2$ then $\mathcal{C}_{X_1}\in\mathfrak{F}_2$ and $\mathcal{C}_{X_2}\in\mathfrak{F}_1$.*

*Proof.* We observe that

$$\left(\mathcal{C}_1\times\mathcal{C}_2\right)_{X_1} = \mathcal{C}_2|b, \qquad b := \vee\left\{b^*\in\mathcal{A} \mid X_1|b^*\in\mathcal{C}_1|b^*\right\}.$$

Thus, the conditional sets $\{\mathcal{C} \mid \mathcal{C}_{X_1} \in \mathfrak{F}\}$ are a conditional $\sigma$-algebra for fixed $X_1$. Every connditional set $\mathcal{C}_1 \times \mathcal{C}_2$ for $\mathcal{C}_1 \in \mathfrak{F}_1$ and $\mathcal{C}_2 \in \mathfrak{F}_2$ is in this conditional $\sigma$-algebra which is the smallest conditional $\sigma$-algebra containing all these sets by Lemma 1.59. $\square$

**Lemma 1.62.** *Let $\mu_1$ and $\mu_2$ be conditionally $\sigma$-finite. For $\mathcal{C} \in \mathfrak{F}_1 \otimes \mathfrak{F}_2$, the conditional functions $X_1 \mapsto \mu_2(\mathcal{C}_{X_1})|a_{\mathcal{C}}^{X_1}$ and $X_2 \mapsto \mu_1(\mathcal{C}_{X_2})|a_{\mathcal{C}}^{X_2}$ are $\mathfrak{F}_1$-$\mathfrak{B}$-measurable and $\mathfrak{F}_2$-$\mathfrak{B}$-measurable, respectively, where $a_{\mathcal{C}}^{X_i} := \vee\{b^* \in \mathcal{A} \mid X_i|b^* \in pr_i(\mathcal{C})|b^*\}$.*

*Proof.* We define $s_{\mathcal{C}}(X_1) := \mu_2(\mathcal{C}_{X_1})|a_{\mathcal{C}}^{X_1}$. First, let $\mu_2$ be totally finite. The collection of conditional sets $\mathfrak{D} := \{\mathcal{C} \in \mathfrak{F}_1 \otimes \mathfrak{F}_2 \mid s_{\mathcal{C}}$ is $\mathfrak{F}_1$-$\mathfrak{B}$-measurable$\}$ is a conditional $\lambda$-system for which $\mathcal{C}_1 \times \mathcal{C}_2$ for $\mathcal{C}_i \in \mathfrak{F}_i$ are a conditional $\pi$-system since $s_{\mathcal{C}}(\mathcal{C}_1 \times \mathcal{C}_2) = \mu_2(\mathcal{C}_2)\chi_{\mathcal{C}_1}$. Thus, $\mathfrak{D} = \mathfrak{F}_1 \otimes \mathfrak{F}_2$ by Theorem 1.28. For conditionally $\sigma$-finite conditional measures $\mu_2$, let $(\mathcal{D}_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbf{X}_2$ such that $\bigsqcup_{n\in\mathbb{N}} \mathcal{D}_n = \mathbf{X}_2$ with $\mu_2(\mathcal{D}_n) < \infty$. Then, $\mu_{2,n}(\mathcal{C}_2 \sqcap \mathcal{D}_n)$ is a totally finite conditional measure on $\mathfrak{F}_2$. Finally, $\mu_2(\mathcal{C}_{X_2}) = \lim_{n\in\mathbb{N}} \mu_{2,n}(\mathcal{C}_{X_2})$ is also $\mathfrak{F}_1$-$\mathfrak{B}$-measurable by Property (v) following Definition 1.23 and Theorem 1.49. $\square$

**Theorem 1.63.** *Let $\mathbf{X}_1$ and $\mathbf{X}_2$ be conditional measure spaces where $\mu_1$ and $\mu_2$ are conditionally $\sigma$-finite. Then, a conditional measure $\mu$ on $\mathfrak{F}_1 \otimes \mathfrak{F}_2$ with $\mu(\mathcal{C}_1 \times \mathcal{C}_2) = \mu_1(\mathcal{C}_1)\mu_2(\mathcal{C}_2)$ exists and is unique. It is further conditionally $\sigma$-finite and it holds that*

$$\mu(\mathcal{C}) = \int \left(\mu_2(\mathcal{C}_{X_1})|a_{\mathcal{C}}^{X_1}\right)\mu_1(dX_1) = \int \left(\mu_1(\mathcal{C}_{X_2})|a_{\mathcal{C}}^{X_2}\right)\mu_2(dX_2). \tag{1.11}$$

*Proof.* We define $s_{\mathcal{C}}(X_1) := \mu_2(\mathcal{C}_{X_1})|a_{\mathcal{C}}^{X_1}$ and $\mu(\mathcal{C}) := \int s_{\mathcal{C}} d\mu_1$. Then, $\mu\colon \mathbf{X}_1 \times \mathbf{X}_2 \to [0,\infty]$ is a conditional measure by the properties of the integral. For $s_{\mathcal{C}_1 \times \mathcal{C}_2} = \mu_2(\mathcal{C}_2)\chi_{\mathcal{C}_1}$, by integration with respect to $\mu_1$, it holds that $\mu(\mathcal{C}_1 \times \mathcal{C}_2) = \mu_1(\mathcal{C}_1)\mu_2(\mathcal{C}_2)$. By Theorem 1.60, it is unique. The conditional measure $\mu^*(\mathcal{C}) := \int \mu_1(\mathcal{C}_{X_2})|a_{\mathcal{C}}^{X_2}\mu_2(dX_2)$ is thus equal to $\mu$, thus, (1.11) holds. Conditional $\sigma$-finiteness of $\mu$ follows from the same property of $\mu_1$ and $\mu_2$ since $\mu(\mathcal{C}_1 \times \mathcal{C}_2) = \mu_1(\mathcal{C}_1)\mu_2(\mathcal{C}_2) < \infty$ for the same exhausting sequences of $\mu_1$ and $\mu_2$. $\square$

**Definition 1.64.** The conditional measure constructed in Theorem 1.63 on conditionally $\sigma$-finite conditional measure spaces is the product measure and denoted by $\mu_1 \otimes \mu_2$.

For the $X_1$-section and $X_2$-section of conditional functions $f\colon \mathbf{X}_1 \times \mathbf{X}_2 \to \overline{\mathbf{R}}$, we consider conditional functions $f\colon \mathbf{X}_2 \to \overline{\mathbf{R}}$ with $f_{X_1}(X_2) := f(X_1, X_2)$ for fixed $X_1$ and $f\colon \mathbf{X}_1 \to \overline{\mathbf{R}}$ with $f_{X_2}(X_1) := f(X_1, X_2)$ for fixed $X_2$.

**Lemma 1.65.** *Let $f\colon \mathbf{X}_1 \times \mathbf{X}_2 \to \overline{\mathbf{R}}$ be a $\mathfrak{F}_1 \otimes \mathfrak{F}_2$-$\mathfrak{B}$-measurable conditional function. Then, $f_{X_1}$ is $\mathfrak{F}_2$-$\mathfrak{B}$-measurable and $f_{X_2}$ is $\mathfrak{F}_1$-$\mathfrak{B}$-measurable.*

*Proof.* By $\mathfrak{F}_1 \otimes \mathfrak{F}_2$-$\mathfrak{B}$-measurability of $f$, it holds that $f_{X_1}^{-1}(\mathcal{C}) = \left(f^{-1}(\mathcal{C})\right)_{X_1}$ and the claim follows from Lemma 1.61. $\square$

**Theorem 1.66.** *Let $\mathbf{X}_1$ and $\mathbf{X}_2$ be conditional measure spaces where $\mu_1$ and $\mu_2$ are conditionally $\sigma$-finite and let $f\colon \mathbf{X}_1 \times \mathbf{X}_2 \to [0,\infty]$ be a $\mathfrak{F}_1 \otimes \mathfrak{F}_2$-$\mathfrak{B}$-measurable conditional function. Then, the conditional functions $X_2 \mapsto \int f_{X_2} d\mu_1 | a_{\mathbf{X}}^{X_2}$ and $X_1 \mapsto \int f_{X_1} d\mu_2 | a_{\mathbf{X}}^{X_1}$ are $\mathfrak{F}_2$-$\mathfrak{B}$-measurable and $\mathfrak{F}_1$-$\mathfrak{B}$-measurable, respectively. It further holds that*

$$\int f d\left(\mu_1 \otimes \mu_2\right) = \int \left(\int f_{X_2} d\mu_1 | a_{\mathbf{X}}^{X_2}\right) \mu_2\left(dX_2\right) = \int \left(\int f_{X_1} d\mu_2 | a_{\mathbf{X}}^{X_2}\right) \mu_1\left(dX_1\right). \tag{1.12}$$

*Proof.* We first consider elementary conditional functions $f = \sum_{1\le n\le \overline{n}} \alpha_n \chi_{\mathcal{C}_n}$. By Lemma 1.65 and Lemma 1.62, it holds that $\int f_{X_2} d\mu_1 = \sum_{1\le n\le \overline{n}} \alpha_n \mu_1\left(\mathcal{C}_{n,X_2}\right)$ is $\mathfrak{F}_2$-$\mathfrak{B}$-measurable. By Theorem 1.63, we may intregrate with respect to $\mu_2$ and obtain $\int \left(\int f_{X_2} d\mu_1 | a_{\mathbf{X}}^{X_2}\right) d\mu_2 = \sum_{1\le n\le \overline{n}} \alpha_n \mu\left(\mathcal{C}_n\right) = \int f d\mu$. For nonnegative $f$, let $\left(f^m\right)_{m\in\mathbb{N}}$ be an increasing sequence of simple conditional functions with limit $f$. Then, $g^m\left(X_2\right) := \int f_{X_2}^m d\mu_1 | a_{\mathbf{X}}^{X_2}$ is $\mathfrak{F}_2$-$\mathfrak{B}$-measurable and $\left(g^m\right)_{m\in\mathbb{N}}$ increases to $\int f_{X_2} d\mu_1 | a_{\mathbf{X}}^{X_2}$. Finally,

$$\int \left(\int f_{X_2} d\mu_1 | a_{\mathbf{X}}^{X_2}\right) \mu_2\left(dX_2\right) = \operatorname*{ess\,sup}_{m\in\mathbb{N}} \int g^m d\mu_2 = \operatorname*{ess\,sup}_{m\in\mathbb{N}} \int f^m d\mu = \int f d\mu$$

by monotone convergence. $\square$

**Corollary 1.67** (Fubini)**.** *Let $\mathbf{X}_1$ and $\mathbf{X}_2$ be conditional measure spaces where $\mu_1$ and $\mu_2$ are conditionally $\sigma$-finite and let $f\colon \mathbf{X}_1 \times \mathbf{X}_2 \to [0,\infty]$ be a $\mu_1 \otimes \mu_2$-integrable conditional function. Then, the conditional functions $X_2 \mapsto \int f_{X_2} d\mu_1 | a_{\mathbf{X}}^{X_2}$ and $X_1 \mapsto \int f_{X_1} d\mu_2 | a_{\mathbf{X}}^{X_1}$ are $\mu_1$-integrable and $\mu_2$-integrable, respectively and* (1.12) *holds true.*

*Proof.* By (1.12), it holds that

$$\begin{aligned}\int \left(\int |f|_{X_2} d\mu_1 | a_{\mathbf{X}}^{X_2}\right) \mu_2\left(dX_2\right) &= \int \left(\int |f|_{X_1} d\mu_2 | a_{\mathbf{X}}^{X_1}\right) \mu_1\left(dX_1\right) \\ &= \int |f| d\left(\mu_1 \otimes \mu_2\right) < \infty.\end{aligned}$$

Thus, $X_1 \mapsto \int |f|_{X_1} d\mu_2 | a_{\mathbf{X}}^{X_1}$ is $\mathfrak{F}_1$-$\mathfrak{B}$-measurable, $\mu_1$-integrable and conditionally real-valued by Theorem 1.51. Thus, $f_{X_1}$ is $\mu_2$-integrable with $\int f_{X_1} d\mu_2 | a_{\mathbf{X}}^{X_1} = \int f_{X_1}^{+} d\mu_2 | a_{\mathbf{X}}^{X_1} -$

$\int f_{X_1}^{-} d\mu_2|a_{\mathbf{X}}^{X_1}$. By Theorem 1.66, $X_1 \mapsto \int f_{X_1} d\mu_2|a_{\mathbf{X}}^{X_1}$ is $\mathfrak{F}_1$-$\mathfrak{B}$-measurable and $X_1 \mapsto \int f_{X_1}^{+} d\mu_2|a_{\mathbf{X}}^{X_1}$ and $X_1 \mapsto \int f_{X_1}^{-} d\mu_2|a_{\mathbf{X}}^{X_1}$ are $\mu_1$-integrable. Consequently,

$$\begin{aligned}
&\int \left( \int f_{X_1} d\mu_2|a_{\mathbf{X}}^{X_1} \right) \mu_1 (dX_1) \\
&\quad = \int \left( \int f_{X_1}^{+} d\mu_2|a_{\mathbf{X}}^{X_1} \right) \mu_1 (dX_1) - \int \left( \int f_{X_1}^{-} d\mu_2|a_{\mathbf{X}}^{X_1} \right) \mu_1 (dX_1) \\
&\quad = \int f^{+} d (\mu_1 \otimes \mu_2) - \int f^{-} d (\mu_1 \otimes \mu_2) = \int f d (\mu_1 \otimes \mu_2)
\end{aligned}$$

The roles of $X_1$ and $X_2$ can be interchanged, thus the claim is proven. □

### 1.2.5 A conditional version of the Daniell-Stone theorem and Riesz representation theorem

In this section we will prove a conditional version of the Daniell-Stone theorem thanks to which conditional versions of the Riesz representation theorem on the conditionally $n$-dimensional Euclidean space are established.

**Definition 1.68.** Given a conditional set $\mathbf{X}$, a conditional family $\mathcal{L}$ of conditional functions $f\colon \mathbf{X} \to \mathbf{R}$ is called a conditional Stone vector lattice whenever $f + g, Rf$ and $\min\{f, 1\}$ are elements of $\mathcal{L}$ for all $f, g \in \mathcal{L}$ and $R \in \mathbf{R}$ and there exist $f \in \mathcal{L}$ and $X \in \mathbf{X}$ such that $f(X)|a \neq 0|a$ for all $a > 0$.

**Definition 1.69.** For a conditional sequence $(f_N)_{N\in\mathbf{N}}$ of conditional functions $f_N\colon \mathbf{X} \to \mathbf{R}$ and a conditional function $f\colon \mathbf{X} \to \mathbf{R}$ we write $f_N \downarrow f$ if $(f_N)_{N\in\mathbf{N}}$ is decreasing and $\lim_{N\in\mathbf{N}} f_N(X) = f(X)$ for all $X \in \mathbf{X}$.

**Theorem 1.70.** *Let $\mathcal{L}$ be a conditional Stone vector lattice and $L\colon \mathcal{L} \to \mathbf{R}$ a linear conditional function such that $L(f) \geq 0$ whenever $f \geq 0$ and $L(f_N) \downarrow 0$ whenever $f_N \downarrow 0$. Then there exists a conditional measure $\Phi$ on $\Sigma(\mathcal{L})$ such that $L(f) = \int_{\mathbf{X}} f d\Phi$ for all $f$ in $\mathcal{L}$.*

*Proof.* For $f, g$ in $\mathcal{L}$ we define $[f, g[ \, := \{(X, R) \in \mathbf{X} \times \mathbf{R} \mid f(X) \leqslant R < g(X)\}$. The collection $\mathcal{X}$ of all conditional unions of conditional finite families $([f_m, g_m[\,)_{1\leq m\leq n}$ of pairwise disjoint elements is a conditional ring on $\mathbf{X} \times \mathbf{R}$. The conditional function $\Psi\colon \mathcal{X} \to \overline{\mathbf{R}}_+$ given by

$$\Psi\left( \bigsqcup_{1\leq m\leq n} [f_m, g_m[ \right) := \sum_{1\leq m\leq n} L(g_m - f_m)$$

is a conditional pre-measure which by Theorem 1.31 extends to a conditional measure on $\Sigma(\mathcal{X})$. By inspection we have $\mathfrak{M}' := \Sigma\left(\left\{f^{-1}(]1,\infty[) \mid f \in \mathcal{L}\right\}\right) = \Sigma(\mathcal{L})$. For $f \in \mathcal{L}$ let $f^{-1}(]1,\infty[)$ live on $d$, and for $X \in \mathbf{X}$ and $N \in \mathbf{N}$ set

$$a_X = \vee\{\tilde{a} \mid f(X)|\tilde{a} \leqslant 1|\tilde{a}\}, \qquad b_X = \vee\left\{\tilde{b} \mid 1|\tilde{b} < f(X)|\tilde{b} < \frac{N+1}{N}|\tilde{b}\right\},$$
$$c_X = \vee\left\{\tilde{c} \mid f(X)|\tilde{c} \geqslant \frac{N+1}{N}|\tilde{c}\right\}, \quad g_N(X) := 0|(a_X \vee d) + N(f(X)-1)|b_X + 1|c_X.$$

Since $\bigsqcup_{N\in\mathbf{N}}[0, g_N[\ = f^{-1}(]1,\infty[) \times [0,1[$ it holds that the conditional function $\Phi(\mathcal{C}) := \Psi(\mathcal{C} \times [0,1[)$ is a conditional measure on $\mathfrak{M}'$. The representation $L(f) = \int_{\mathbf{X}} f d\Phi$ for all $f \in \mathcal{L}$ follows from (M8) and Theorem 1.49. $\square$

We give a conditional version of Dini's lemma:

**Lemma 1.71.** *Let $(\mathbf{X}, \mathfrak{T})$ be a conditionally compact topological space and $(f_N)_{N\in\mathbf{N}}$ a decreasing conditional sequence of continuous conditional functions $f_N\colon \mathbf{X} \to \mathbf{R}$ converging to a continuous conditional function $f$. Then for all $R > 0$ there exists $N_0$ in $\mathbf{N}$ such that $\sup_{X\in\mathbf{X}}|f_N(X) - f(X)| \leqslant R$ for all $N \geqslant N_0$.*

*Proof.* The proof is similar to the classical proof by using the definition of conditional compactness. $\square$

For $d \in \mathbf{N}$ let $c\left(\mathbf{R}^d, \mathbf{R}\right)$ denote the conditional family of all continuous conditional functions $f\colon \mathbf{R}^d \to \mathbf{R}$. The conditional function $f \in c\left(\mathbf{R}^d, \mathbf{R}\right)$ has conditionally compact support whenever $\mathrm{cl}\left(f^{-1}\left(\{0\}^{\sqsubset}\right)\right)$ is conditionally compact. We denote by $c_c\left(\mathbf{R}^d, \mathbf{R}\right)$ the conditional family of $c\left(\mathbf{R}^d, \mathbf{R}\right)$ of all functions with conditionally compact supported. Both $c\left(\mathbf{R}^d, \mathbf{R}\right)$ and $c_c\left(\mathbf{R}^d, \mathbf{R}\right)$ are conditional Stone vector lattices.

A finite conditional measure $\Phi$ on the conditional Borel $\sigma$-algebra $\mathfrak{B}^N$ is called conditionally tight whenever

$$\Phi(\mathcal{C}) = \sup\{\Phi(\mathcal{D}) \mid \mathcal{D} \sqsubset \mathcal{C} \text{ conditionally compact}\}$$

for all $\mathcal{C} \in \mathfrak{B}^N$.

**Corollary 1.72.** *Let $L\colon c\left(\mathbf{R}^d, \mathbf{R}\right) \to \mathbf{R}$ be an $\mathbf{R}$-linear conditional function such that $L(f) \geq 0$ whenever $f \geq 0$. Then there exists a finite conditionally tight measure $\Phi$ on $\mathfrak{B}^n$ such that $L(f) = \int_{\mathbf{R}^d} f d\Phi$ for all $f \in c\left(\mathbf{R}^d, \mathbf{R}\right)$.*

*Proof.* Let $(f_M)_{M\in\mathbf{N}}$ be a conditional sequence in $c\left(\mathbf{R}^d,\mathbf{R}\right)$ with $f_N\downarrow 0$. Let $\mathcal{C}_K=\left\{X\in\mathbf{R}^d \mid \|X\|\le K\right\}$, $K\in\mathbf{N}$ which is conditionally compact by Remark 1.17. Put $g_K(X)=\max\{1-\min_{Y\in\mathcal{C}_K}\|X-Y\|,0\}$ and $h_{KM}=g_Kf_M+(1-g_K)f_1\|\cdot\|/2K$. One has $f_M\le h_{KM}$ for all $K,M\in\mathbf{N}$. Fix $R>0$. Now choose $K$ such that $1/(2K)\,L(f_1(1-g_1)\|\cdot\|)<R/2$. Next, choose $M$ such that $L(g_Kf_M)<R/2$ by Lemma 1.71. We have $L(f_M)\le L(g_Kf_M)+1/(2K)\,L((1-g_1)f_1\|\cdot\|)<R$. By Theorem 1.70 there exists a finite conditional measure $\Phi$ on $\mathfrak{B}^d$ representing $L$. The regularity condition follows from an adaptation of the arguments in the proof of [CKT15, Proposition 1.5]. □

**Corollary 1.73.** *Let $L\colon c_c\left(\mathbf{R}^d,\mathbf{R}\right)\to\mathbf{R}$ be an $\mathbf{R}$-linear conditional function such that $L(f)\ge 0$ whenever $f\ge 0$. Then there exists a conditional measure $\Phi$ on $\mathfrak{B}^d$ such that $L(f)=\int_{\mathbf{R}^d} f d\Phi$ for all $f\in c_c\left(\mathbf{R}^d,\mathbf{R}\right)$. Moreover, one has $\Phi(\mathcal{K})<\infty$ for all conditionally compact intervals $\mathcal{K}$ and $\Phi(\mathcal{C})=\sup\{\Phi(\mathcal{D})\mid \mathcal{D}\sqsubset\mathcal{C}$ conditionally compact$\}$ for all $\mathcal{C}\in\mathfrak{B}^d$ with $\Phi(\mathcal{C})<\infty$.*

*Proof.* In order to obtain the assumptions of Theorem 1.70 apply Lemma 1.71 to the conditional sequence $(\mathbb{1}_{\mathcal{K}}f_N)_{N\in\mathbb{N}}$ where $\mathcal{K}$ denotes the support of $f_1$. For a conditionally compact interval $\mathcal{K}$ and $f(X)=\max\{1-\min_{Y\in\mathcal{K}}\|X-Y\|,0\}$ one has $\Phi(\mathcal{K})=\int_{\mathbf{R}^d} f d\Phi\le L(f)$. The conditional regularity condition follows similarly to Corollary 1.72. □

## 1.3 Partition of unity

We close this Chapter with some topological theorem. In the sequel, we follow an approach suggested in [AB06] and [Dug75], adapted to conditional theory, where we show that $\mathbf{R}$ is conditionally normal and fulfills Urysohn's characterization of normality in a conditional setting.

**Definition 1.74** (Partition of unity)**.** A conditional family $(f_I)_{I\in\mathcal{I}}$ in $\mathbf{R}$ of conditional functions is a partition of unity if $f_I\colon\mathbf{R}\to[0,1]$, $I\in\mathcal{I}$, such that $f_I(X)\in\mathbf{R}_{++}$ for all $X\in\mathbf{R}$ only for conditionally finitely many $I\in\mathcal{I}$ and $\sum_{I\in\mathcal{I}}f_I(X)=1$. (An arbitrary sum of zeros is zero.)

A partition of unity $(f_I)_{I\in\mathcal{I}}$ is subordinated to a cover $\mathfrak{B}$ of conditional balls $(\mathcal{B}_J)_{J\in\mathcal{J}}$ in $\mathbf{R}$ if for all $I\in\mathcal{I}$ there exists $J\in\mathcal{J}$ such that $f_I|_{\mathcal{B}_J^{\sqsubset}}=0$. It is continuous if $f_I$ is continuous for all $I\in\mathcal{I}$.

**Definition 1.75** (Normality)**.** Let $(\mathbf{X}, \mathfrak{T})$ be a conditional topological space which is conditionally Hausdorff. It is normal if for all conditionally closed conditional sets $\mathcal{C}, \mathcal{D} \sqsubset \mathcal{M}$ which are conditionally disjoint, that is, $\mathcal{C} \sqcap \mathcal{D} = \mathbf{X}|0$, there exist conditionally disjoint conditionally open conditional sets $\mathcal{O}_1$ and $\mathcal{O}_2$ in $\mathbf{M}$ such that $\mathcal{C} \sqsubset \mathcal{O}_1$ and $\mathcal{D} \sqsubset \mathcal{O}_2$.

**Example 1.76.** The conditional set $\mathbf{R}$ is conditionally Hausdorff, cf. [DJKK16] and normal. Given conditionally closed conditional balls $\operatorname{cl} \mathcal{B}^{\varepsilon_1}(X_1)$ and $\operatorname{cl} \mathcal{B}^{\varepsilon_2}(X_2) \sqsubset \mathbf{R}$ which are conditionally disjoint, we choose $Y_1 := X_1$, $Y_2 := X_2$, $\delta_1 := \varepsilon_1 + \frac{\mathrm{d}(X_1,X_2)-\varepsilon_1-\varepsilon_2}{3} > \varepsilon_1$ and $\delta_2 := \varepsilon_2 + \frac{\mathrm{d}(X_1,X_2)-\varepsilon_1-\varepsilon_2}{3} > \varepsilon_2$. By construction, $\operatorname{cl} \mathcal{B}^{\varepsilon_1}(X_1) \sqsubset \mathcal{B}^{\delta_1}(Y_1)$, $\operatorname{cl} \mathcal{B}^{\varepsilon_2}(X_2) \sqsubset \mathcal{B}^{\delta_2}(Y_2)$ and $\mathcal{B}^{\delta_1}(Y_1) \sqcap \mathcal{B}^{\delta_2}(Y_2) = \mathbf{R}|0$.

**Theorem 1.77** (Urysohn's characterization of normality)**.** *In a conditional topological space $(\mathbf{X}, \mathfrak{T})$, for conditionally disjoint conditionally closed conditional balls $\mathcal{C}, \mathcal{D} \sqsubset \mathbf{X}$ there exists a continuous conditional function $f \colon \mathbf{X} \to [0,1]$ such that $0 \leq f(X) \leq 1$ for all $X \in \mathbf{X}$ and*

$$
\begin{aligned}
f(X) = 0 \text{ on } a_0 \quad &\text{for} \quad a_0 := \vee \{a \mid X|a \in \mathcal{C}|a\}, \\
f(X) = 1 \text{ on } a_1 \quad &\text{for} \quad a_1 := \vee \{a \mid X|a \in \mathcal{D}|a\}.
\end{aligned}
$$

*We call $f$ a conditional Urysohn function for $\mathcal{C}$ and $\mathcal{D}$.*

*Proof.* Let $\mathcal{C}$ and $\mathcal{D}$ be conditionally disjoint and conditionally closed conditional sets in $\mathbf{X}$. We show the existence of a conditional Urysohn function for $\mathcal{C}$ and $\mathcal{D}$. We consider $\mathcal{R} := \left\{\frac{K}{2^n} \mid 0 \leq K \leq 2^n, K \in \mathbf{N}, n \in \mathbb{N}\right\}$, a conditional countable conditional dense conditional subset of $[0,1] \sqsubset \mathbf{R}$.

In an inductive procedure, we construct conditionally open $\mathcal{B}(R) := \mathcal{B}^{\bar{\varepsilon}_R}\left(\overline{X}_R\right) \sqsubset \mathbf{X}$ for every $R \in \mathcal{R}$ such that

$$
\begin{aligned}
\mathcal{C} \sqsubset \mathcal{B}(R), \quad &\mathcal{B}(R) \sqcap \mathcal{D} = \mathbf{X}|0, \quad \text{and} \\
R' - R \in \mathcal{R}_{++} \quad &\text{implies} \quad \operatorname{cl} \mathcal{B}(R) \sqsubset \mathcal{B}(R').
\end{aligned}
$$

First, we do the construction for $k \in \mathbb{N}$. Let $\mathfrak{D}_n := \left\{\mathcal{B}\left(\frac{k}{2^n}\right) \mid k = 0, 1, \ldots, 2^n\right\}$. Let $\mathcal{B}(1) := (\mathcal{D})^{\sqsubset}$ and $\mathcal{B}(0) \sqsubset \mathbf{X}$ be such that $\mathcal{C} \sqsubset \mathcal{B}(0) \sqsubset \operatorname{cl} \mathcal{B}(0) \sqsubset \mathcal{D}^{\sqsubset}$ which exists by the normality assumption. Define $\mathfrak{D}_0 := \{\mathcal{B}(0), \mathcal{B}(1)\}$. Assuming $\mathcal{D}_{m-1}$ already being constructed we note that $\mathcal{B}\left(\frac{k}{2^m}\right)$ for even $k \in \mathbb{N}$ is already defined. For $k \in \mathbb{N}$ odd, by normality, there exists $\mathcal{B}\left(\frac{k}{2^n}\right)$ such that $\operatorname{cl} \mathcal{B}\left(\frac{k-1}{2^n}\right) \sqsubset \mathcal{B}\left(\frac{k}{2^n}\right) \sqsubset \operatorname{cl} \mathcal{B}\left(\frac{k}{2^n}\right) \sqsubset \mathcal{B}\left(\frac{k+1}{2^n}\right)$. Now, let $K \in \mathbf{N}$ be arbitrarily. Then, for all $R \in \mathcal{R}$, we define $\mathcal{B}(R) := \sum_{R' \in \mathcal{R}} \mathcal{B}(R')|a_{r'}$ where $a_{r'} := \vee \{a \mid R|a = R'|a\}$ for $R' = \frac{k}{2^n}$ with $k \in \mathbb{N}$.

We continue with the definition of the conditional function $f$. Now, let $\mathcal{B}'(1) := \mathbf{X}$, and for $R \in \mathcal{R}$, $\mathcal{B}'(R) := \mathbf{X}|a_1 + \mathcal{B}(R)|a_1^c$ where $a_1 := \vee\{a \mid R|a = 1|a\}$, thus, passing from $\mathcal{B}(R)$ to $\mathcal{B}'(R)$ is the identity but $\mathcal{B}(R)|a$ is replaced by $\mathbf{R}|a$ if $R|a = 1|a$. We define $f(X) := \operatorname{ess\,inf}\{R \in \mathcal{R} \mid X \in \mathcal{B}'(R)\}$ which is well defined by the conditional density of $\mathcal{R}$ in $[0,1]$. By definition, $0 \leq f(X) \leq 1$ for all $X \in \mathbf{X}$. We observe that $X|a \in \mathcal{C}|a$ for some $a$ implies that $X|a \in \mathcal{B}'(0)|a$, thus, $f(X)|a = 0|a$. If $X|a \in \mathcal{D}|a$ for some $a$ we observe that $X|a' \in \mathcal{D}^{\sqsubset}|a'$ for $R|a < 1|a$ and $X|a \in \mathcal{B}'(1)|a = \mathbf{X}|a$, hence, $f(X)|a = 1|a$.

To proof continuity of the conditional function $f$, let $X_0 \in \mathbf{X}$ and $R_0 \in \mathcal{R}$ such that $f(X_0) = R_0$. Let $\varepsilon \in \mathbf{R}_{++}$. If $R_0|a = 0|a$ for fixed $a$ there exists $\overline{R} \in \mathcal{R}$ such that $\overline{R}|a \in \,]0,(R_0+\varepsilon)[\,|a$ for which holds that $\mathcal{B}'\left(\overline{R}\right)|a$ is a neighbourhood of $X_0|a$ with $f\left(\mathcal{B}'\left(\overline{R}\right)\right)|a \sqsubset [0, R_0+\varepsilon[\,|a$ since by definition of $f$ it holds that $f(X)|a \leq \overline{R}|a$ for all $X|a \in \mathcal{B}'\left(\overline{R}\right)|a$. If $R_0|a = 1|a$ there exists $\underline{R} \in \mathcal{R}$ such that $\underline{R}|a \in \,]R_0-\varepsilon, 1[\,|a$ for which holds that $(\operatorname{cl}\mathcal{B}'(\underline{R}))^{\sqsubset}|a$ is a neighbourhood of $X_0|a$ with $f\left((\operatorname{cl}\mathcal{B}'(\underline{R}))^{\sqsubset}\right)|a \sqsubset \,]R_0-\varepsilon, 1]\,|a$ since $f(X)|a \geq \underline{R}|a$ for all $X|a$ in $(\operatorname{cl}\mathcal{B}'(\underline{R}))^{\sqsubset}|a$ by definition of $f$. Combining this whenever $0|a < R_0|a < 1|a$ there exist $\overline{R}, \underline{R} \in \mathcal{R}$ such that $\overline{R}|a \in \,]R_0, R_0+\varepsilon[\,|a$ and $\underline{R}|a \in \,]R_0-\varepsilon, R_0[\,|a$ for which hold that $\mathcal{B}'|a := \mathcal{B}'\left(\overline{R}\right) \sqcap (\operatorname{cl}\mathcal{B}'(\underline{R}))^{\sqsubset}|a$ is a neighbourhood of $X_0|a$ with $f(\mathcal{B})|a \sqsubset \,]R_0-\varepsilon, R_0+\varepsilon[\,|a$ since by definition of $f$ it holds that $f(X)|a \leq \overline{R}|a$ for all $X|a \in \mathcal{B}'\left(\overline{R}\right)|a$ and $f(X)|a \geq \underline{R}|a$ for all $X|a \in (\operatorname{cl}\mathcal{B}'(\underline{R}))^{\sqsubset}|a$. Hence, $f$ is continuous. $\square$

**Lemma 1.78.** *Let $\mathfrak{B} := (\mathcal{B}_I)_{I \in \mathcal{I}}$ be a conditionally open cover of $\mathbf{X}$. Then, there is a continuous partition of unity which is subordinated to $\mathfrak{B}$.*

*Proof.* For $X \in \mathbf{X}$, define $a_{X,I} := \vee\{a \mid \{X\} \sqcap \mathcal{B}_I|a = \{X\}|a\}$. Since $\mathfrak{B}$ is a cover of $\mathbf{X}$, it holds that $\bigcup_{I \in \mathcal{I}} a_{X,I} = 1$ for all $X \in \mathbf{X}$. We fix a partition $(a_I)_{I \in \mathcal{I}}$ of 1 subordinated to $(a_{X,I})_{I \in \mathcal{I}}$, that is, without loss of generality, $a_I \leq a_{X,I}$ and the conditionally open conditional ball $\mathcal{B}_I \in \mathfrak{B}$ with $X|a_I \in \mathcal{B}_I|a_I$. By Urysohns characterization of normality in Lemma 1.77 on $a_I$ for conditionally closed $\{X|A_I\}$ and $\mathcal{B}_I^{\sqsubset}$ there is a continuous conditional function $f_X\colon \mathbf{X} \to [0,1]$ with $f_X(Y)|a = 1|a$ for any $a \leq a_I$ such that $Y|a = X|a$ and $f_X(Y)|a' = 0|a'$ for any $a' \leq a_I$ such that $Y|a' \in \mathcal{B}_I^{\sqsubset}|a'$. We consider the conditional set $\mathcal{C}_X := \{Y \in \mathbf{X} \mid f_X(Y) \in \mathbf{R}_{++}\}$. It is a conditionally open conditional set of $X$ since $f_X$ is continuous. Thus, $(\mathcal{C}_X)_{X \in \mathbf{X}}$ is a conditional open cover of $\mathbf{X}$. We choose a conditional finite subcover $(\mathcal{C}_{X_J})_{1 \leq J \leq N}$ by conditional compactness. On $a \leq a_I$, we observe that $f_{X_J}(Y)|a > 0|a$ for all $Y|a \in \mathcal{C}_{X_J}|a$ for all $1 \leq J \leq N$ and $f_{X_J}(Y)|a = 0|a$ for all $Y|a \in \mathcal{C}_{X_J}^{\sqsubset}|a$. Now, we define $\overline{f}(Y) := \sum_{1 \leq J \leq N} f_{X_J}(Y)$ and observe that $\overline{f}(Y)|a > 0|a$ for all $Y \in \mathbf{X}$ on $a \leq a_I$. We normalize $f_X$ by dividing by $\overline{f}$ and assume that $\sum_{1\leq J\leq N} f_{X_J}(Y) = 1$ for all $Y \in \mathbf{X}$ which now, having again constructed $f := \sum_{I\in\mathcal{I}} f_{X_I}|a_I$, has all the proterties of the claim. $\square$

# 2 Variational analysis in a conditional setting

In this chapter, we generalize the concepts of variational analysis on $\mathbb{R}^d$ provided by [RW09] to the $L^0(\mathcal{F})$-module $L^0(\mathcal{F})^d$. To that end, we make use of the basics of conditional theory as explained in Chapter 1. We give all the results within $L^0$-theory. A generalization to conditional theory can be derived in the same way.

First, we give the details of set convergence in $L^0(\mathcal{F})$ followed by its application to hypographs. After that, we explain the relation of hypoconvergence and maximization. Further, the concept of lopsided convergence is introduced in an $L^0$-theory setting. Finally, we present the results for a KY FAN-inequality and its relation to the Brouwer fixed point theorem in $L^0(\mathcal{F})$.

## 2.1 Conditional subsequences

Having introduced conditional sequences in Definition 1.12 we also consider conditional subsequences. Therefore, we examine subsets of $\mathbb{N}(\mathcal{F})$. Let

$$\mathbb{N}(\mathcal{F})_\infty^\# := \left\{\mathcal{N} \sqsubset \mathbb{N}(\mathcal{F}) \mid \forall \overline{N} \in \mathcal{N},\ \exists N' \in \mathcal{N},\ \forall a \in \mathcal{A}_+ \colon N'|a > \overline{N}|a\right\}$$

be the conditional subset of $\mathbb{N}(\mathcal{F})$ containing strictly increasing conditional sequences and

$$\mathbb{N}(\mathcal{F})_\infty := \left\{\mathcal{N} \sqsubset \mathbb{N}(\mathcal{F}) \mid \exists \overline{N} \in \mathbb{N}(\mathcal{F}),\ \forall N' \geq \overline{N} \colon N' \in \mathcal{N}\right\}$$

be the conditional subset of $\mathbb{N}(\mathcal{F})$ containing all elements beyond some $\overline{N} \in \mathbb{N}(\mathcal{F})$ and naturally, strictly increasing conditional sequences. Equivalent definitions are, that $\mathbb{N}(\mathcal{F})_\infty^\#$ are all conditional countable subsets of $\mathbb{N}(\mathcal{F})$ and $\mathbb{N}(\mathcal{F})_\infty$ are all sets of the type $\mathbb{N}(\mathcal{F}) \sqcap \mathcal{M}^\sqsubset$ where $\mathcal{M}$ is conditionally finite.

**Lemma 2.1.** *There is a natural duality given by the relations*

$$\mathbb{N}(\mathcal{F})_{\infty}^{\#} = \left\{\mathcal{N} \sqsubset \mathbb{N}(\mathcal{F}) \mid \forall \mathcal{N}' \in \mathbb{N}(\mathcal{F})_{\infty},\ \mathcal{N} \sqcap \mathcal{N}' \textit{ lives on } \Omega\right\},$$
$$\mathbb{N}(\mathcal{F})_{\infty} = \left\{\mathcal{N} \sqsubset \mathbb{N}(\mathcal{F}) \mid \forall \mathcal{N}' \in \mathbb{N}(\mathcal{F})_{\infty}^{\#},\ \mathcal{N} \sqcap \mathcal{N}' \textit{ lives on } \Omega\right\}.$$

*Proof.* Let $\mathcal{N} \sqsubset \mathbb{N}(\mathcal{F})$ be a conditional subset such that for all $\mathcal{N}' \in \mathbb{N}(\mathcal{F})_{\infty}^{\#}$ holds that $\mathcal{N} \sqcap \mathcal{N}'$ lives on $\Omega$. Assume for all $\overline{N} \in \mathcal{N}$ exists $N' \in \mathbb{N}(\mathcal{F})$, $N' \notin \mathcal{N}$ with $N'|a > \overline{N}|a$ for all $a \in \mathcal{A}_+$. This yields a strictly increasing conditional sequence in $\mathbb{N}(\mathcal{F})_{\infty}^{\#}$ which is a conditional subset of $\mathcal{N}$. This contradicts that $\mathcal{N} \sqcap \mathcal{N}'$ lives on $\Omega$, thus $\mathcal{N} \in \mathbb{N}(\mathcal{F})_{\infty}$. Now, let $\mathcal{N} \sqsubset \mathbb{N}(\mathcal{F})$ be a conditional subset such that for all $\mathcal{N}' \in \mathbb{N}(\mathcal{F})_{\infty}$ holds that $\mathcal{N} \sqcap \mathcal{N}'$ lives on $\Omega$. Let $\mathcal{N}_0' := \mathbb{N}(\mathcal{F})$. By assumption, there exist $\overline{N}_k \in \mathcal{N} \sqcap \mathcal{N}_k'$ on 1 with $\mathcal{N}_k' := \{\tilde{N} \mid \tilde{N} \geq \overline{N}_{k-1} + 1\}$, $k \in \mathbb{N}$. By construction, $\overline{N}_{k+1}|a > \overline{N}_k|a$, hence $\mathcal{N} \in \mathbb{N}(\mathcal{F})_{\infty}^{\#}$. □

**Definition 2.2** (Conditional subsequence)**.** Let $X$ be a conditional set. A conditional subsequence of $(X_J)_{J\in\mathbb{N}(\mathcal{F})}$ is $(X_J)_{J\in\mathcal{M}}$ where $\mathcal{M} \in \mathbb{N}(\mathcal{F})_{\infty}^{\#}$.

**Remark 2.3.** Let $P$ be a property about a conditional set which holds for all $n \in \mathbb{N}$. Then by stability $P$ already holds for all $N \in \mathbb{N}(\mathcal{F})$ since each $N \in \mathbb{N}(\mathcal{F})$ is of the form $N = \sum_{n\in\mathbb{N}} m_n|a_n$. We illustrate this with the following definition. Let $(\mathcal{C}_n)_{n\in\mathbb{N}}$ be a sequence of conditional subsets in $L^0(\mathcal{F})$. Then $(\mathcal{C}_n)_{n\in\mathbb{N}}$ is pairwise conditionally disjoint if $\mathcal{C}_n \sqcap \mathcal{C}_m = L^0(\mathcal{F})|0$ whenever $m \neq n$. Now define $\mathcal{C}_N := \sum_{n\in\mathbb{N}} \mathcal{C}_{m_n}|a_n$ for every $N = \sum_{n\in\mathbb{N}} m_n|a_n \in \mathbb{N}(\mathcal{F})$. Then $(\mathcal{C}_N)_{N\in\mathbb{N}(\mathcal{F})}$ is a conditional sequence such that $\mathcal{C}_N \sqcap \mathcal{C}_M = L^0(\mathcal{F})|0$ whenever $N \sqcap M = L^0(\mathcal{F})|0$.

To illustrate, we give the connection between almost sure convergence in $L^0(\Omega, \mathcal{F}, \mathbb{P})$ and conditional convergence in $L^0(\mathcal{F})$ if the underlying Boolean algebra is a $\sigma$-algebra.

**Example 2.4.** Consider the standard example of $\mathbb{R}$-valued random variables on the measurable space $(\Omega, \mathcal{F}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with $\mathbb{P}$ being the Lebesgue-measure. It is known that the sequence $(X_n)_{n\in\mathbb{N}}$ defined by $X_n := \mathbb{1}_{[0,\frac{1}{n}]}$ converges $\mathbb{P}$-almost surely to 0. Further, the property that

$$\text{for any}\,\varepsilon \in L^0_{++}\,\text{there exists}\,n_0 \in \mathbb{N}\,\text{such that}\,X_n < \varepsilon\,\text{for all}\,n \geq n_0 \tag{2.1}$$

does not hold since for any $\varepsilon \in L^0_{++}$ with $\varepsilon < \frac{1}{2}$, we observe that $X_n = \mathbb{1}_{[0,\frac{1}{n}]} \geq \frac{1}{2}\mathbb{1}_{[0,\frac{1}{n}]} \geq \varepsilon\mathbb{1}_{A_n}$ for all $n \in \mathbb{N}$. But, the slightly different condition that

$$\text{for any}\,\varepsilon \in L^0_{++}\,\text{there exists}\,N_0 \in \mathbb{N}(\mathcal{F})\,\text{such that}\,X_N < \varepsilon\,\text{for all}\,N \geq N_0 \tag{2.2}$$

does hold, since with the definition $N_0 := \sum_{n=1}^{\infty} n \mathbb{1}_{(\frac{1}{n},\frac{1}{n-1}]} \in \mathbb{N}(\mathcal{F})$ we have that $0 \leq X_N \leq X_{N_0} = \sum_{n=1}^{\infty} \mathbb{1}_{\{N=n\}} X_n = \sum_{n=1}^{\infty} \mathbb{1}_{(\frac{1}{n},\frac{1}{n-1}]} \mathbb{1}_{[0,\frac{1}{n}]} = 0$ for all $N \geq N_0$, thus, the property (2.2) holds.

**Lemma 2.5.** *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence in $L^0(\Omega,\mathcal{F},\mathbb{P})$ and $(X_N)_{N\in\mathbb{N}(\mathcal{F})}$ be the same family regarded as a conditional sequence in $L^0(\mathcal{F})$. Then the following statements are equivalent.*

$$\text{The sequence } (X_n)_{n\in\mathbb{N}} \text{ converges } \mathbb{P}\text{-almost surely to } X. \tag{2.3}$$

$$\begin{gathered}\text{The conditional sequence } (X_N)_{N\in\mathbb{N}(\mathcal{F})} \text{ converges to } X \\ \text{with respect to the conditional topology.}\end{gathered} \tag{2.4}$$

*Proof.* We first show that (2.4) implies (2.3). Let $\delta > 0$ for $\delta \in \mathbb{R}$ be fixed. We show that there exists $n_0 \in \mathbb{N}$ such that $\mathbb{P}(\omega \mid |X_n(\omega) - X(\omega)| > \delta) < \delta$ for all $n \geq n_0$. By assumption, there exists $N_0 \in \mathbb{N}(\mathcal{F})$ such that $\|X_N - X\| < \delta \mathbb{1}_\Omega$ for all $N \geq N_0$. Then, for $\overline{n} \in \mathbb{N}$, we define $A_{\overline{n}} := \{\overline{n} \geq N_0\}$. Since the conditional distance is a conditional function and thus stable we first observe that, on $A_{\overline{n}}$, for all $n \geq \overline{n}$, $\|X_n - X\| \mathbb{1}_{A_{\overline{n}}} < \delta \mathbb{1}_{A_{\overline{n}}} + \mathbb{1}_{A_{\overline{n}}^c}$. Thus, $\mathbb{P}(\omega \mid |X_n(\omega) - X(\omega)| > \delta) \leq \mathbb{P}(A_{\overline{n}}^c)$. By construction, it holds that $\mathbb{P}(A_{\overline{n}}^c) \to 0$ for $n \to \infty$. Thus, there is $n_0 \in \mathbb{N}$ such that $\mathbb{P}\left(A_{n_0}^c\right) < \delta$, for which the claim holds.

Now, we show that (2.3) implies (2.4). We show that there exists $N_0 \in \mathbb{N}(\mathcal{F})$ such that $X_N < \varepsilon$ for all $N \geq N_0$. Let $\varepsilon \in L^0(\mathcal{F})_{++}$. Then, there is $\varepsilon' \in \mathbb{Q}(\mathcal{F})_{++}$ such that $\varepsilon' \leq \varepsilon$ since $\mathbb{Q}(\mathcal{F}) \sqsubset L^0(\mathcal{F})$ is conditionally dense. We proof (2.4) for $\varepsilon'$. Since $\varepsilon'$ is in $\mathcal{Q}(\mathcal{F})_{++}$ it can be written as $\varepsilon' = \sum_{k=1}^{\infty} \mathbb{1}_{A_k} \delta_k$ for a partition $(A_k)_{k\in\mathbb{N}}$ of $\Omega$ and $\delta_k \in \mathbb{R}$. For each $k \in \mathbb{N}$, by assumption, there exists $n_0^k \in \mathbb{N}$ such that $\mathbb{1}_{A_k}\mathbb{P}(\omega \mid |X_n(\omega) - X(\omega)| > \delta_k) < \delta_k \mathbb{1}_{A_k}$ for all $n \geq n_0^k$. Define $N_0 := \sum_{k=1}^{\infty} n_0^k \mathbb{1}_{A_k} \in \mathbb{N}(\mathcal{F})$. Since $X_n = \sum_{k=1}^{\infty} X_n \mathbb{1}_{A_k}$ and $X = \sum_{k=1}^{\infty} X \mathbb{1}_{A_k}$, we further observe that $\|X_N - X\| = \sum_{k=1}^{\infty} \mathbb{1}_{A_k} \|X_N - X\| < \sum_{k=1}^{\infty} \mathbb{1}_{A_k} \delta_k = \varepsilon'$ for all $N \geq N_0$. $\square$

Frequently, we apply this Lemma 2.5 without further mentioning. For example, we write $\lim_{N\in\mathbb{N}(\mathcal{F})} X_N$ in a conditional setting for $\lim_{n\in\mathbb{N}} X_n$, the $\mathbb{P}$-almost sure limit of the sequence $(X_n)_{n\in\mathbb{N}}$ of random variables. By the indexing the version of the convergence is made clear, by using conditional or classical index sets.

## 2.2 Hypographs and semicontinuity

For optimization problems, it has been turned out that it is useful to consider epigraphs and hypographs, and by closedness characterizations the conditional functions are semi-continuous. Traditionally, semicontinuity rather may be formulated for an arbitrary conditional topological space. In context with extended-valued conditional functions, we restrict ourselves to $L^0(\mathcal{F})$. Further, all relations are clear from $L^0(\mathcal{F})$, if not, a conditional relation along with conditional directness, that is, a conditional set is closed with respect to the supremum or infimum of $\mathbf{2}$ elements of the conditional set, should be imposed. That is as for the construction of $\mathbf{R}$ by closing $\mathbf{Q}$. Then, theorems for the existence of an essential supremum and infimum with respect to the conditional relation can be derived just as in [FS04].

For the following definition we recall that, by [DJKK16], the product space $L^0(\mathcal{F}) \times L^0(\mathcal{F})$ is a conditional set.

**Definition 2.6** (Epigraph and hypograph of an extended-valued conditional function)**.** Let $f\colon L^0(\mathcal{F}) \to \overline{L}^0(\mathcal{F})$ be a conditional function. Then, the epigraph $\operatorname{epi} f$ of $f$ is

$$\operatorname{epi} f := \left\{(X,Y) \in L^0(\mathcal{F}) \times L^0(\mathcal{F}) \mid f(X) \leq Y\right\}$$

and the hypograph $\operatorname{hypo} f$ is

$$\operatorname{hypo} f := \left\{(X,Y) \in L^0(\mathcal{F}) \times L^0(\mathcal{F}) \mid f(X) \geq Y\right\}.$$

**Definition 2.7.** Let $f\colon L^0(\mathcal{F}) \to \overline{L}^0(\mathcal{F})$ be a conditional function. The upper limit of $f$ at $\overline{X}$ is defined by

$$\operatorname*{ess\,lim\,sup}_{X \to \overline{X}} f(X) := \operatorname*{ess\,inf}_{\delta > 0} \left( \operatorname*{ess\,sup}_{X \in \mathcal{B}^\delta(\overline{X})} f(X) \right).$$

The conditional function $f$ is upper semicontinuous if $\operatorname{ess\,lim\,sup}_{X \to \overline{X}} f(X) \leq f\left(\overline{X}\right)$ and it is lower semicontinuous if $-f$ is upper semicontinuous.

**Lemma 2.8** (Characterization of upper limits)**.** *Let $f\colon L^0(\mathcal{F}) \to L^0(\mathcal{F})$ be a conditional function. It holds that*

$$\operatorname*{ess\,lim\,sup}_{X \to \overline{X}} f(X) = \max \left\{ Y \in L^0(\mathcal{F}) \mid \exists \left(X_J\right)_{J \in \mathbb{N}(\mathcal{F})} \to \overline{X} \colon \lim_{J \in \mathbb{N}(\mathcal{F})} f(X_J) = Y \right\}.$$

*Proof.* Let $(X_J)_{J\in\mathbb{N}(\mathcal{F})} \to \overline{X}$ be a conditional sequence with $\lim_{J\in\mathbb{N}(\mathcal{F})} f(X_J) = Y$. We show $Y \le \overline{Y} := \operatorname{ess\,lim\,sup}_{X\to\overline{X}} f(X)$. Let $\delta \in L^0(\mathcal{F})_{++}$. If $X_J \in \mathcal{B}^\delta\left(\overline{X}\right)$, $J \in \mathbb{N}(\mathcal{F})$, then $\operatorname{ess\,sup}\left\{f(X) \mid X \in \mathcal{B}^\delta\left(\overline{X}\right)\right\} \ge \lim_{J\in\mathbb{N}(\mathcal{F})} f(X_J) = Y$. Since $\delta \in L^0(\mathcal{F})_{++}$ has been chosen arbitrarily, $Y \le \overline{Y}$. We show the existence of a maximizing conditional sequence. For $J \in \mathbb{N}(\mathcal{F})$, let $\overline{Y}_J := \operatorname{ess\,sup}\{f(X) \mid X \in \mathcal{B}^{\delta_J}\left(\overline{X}\right)\}$ for $\delta_J \in L^0(\mathcal{F})_{++}$ with $\delta_J - \delta_{J+1} \in L^0(\mathcal{F})_{++}$ and $\lim_{J\in\mathbb{N}(\mathcal{F})} \delta_J = 0$. By definition of $\overline{Y}$, $\lim_{J\in\mathbb{N}(\mathcal{F})} \overline{Y}_J = \overline{Y}$. By definition of the conditional sequence $\left(\overline{Y}_J\right)_{J\in\mathbb{N}(\mathcal{F})}$, there are $X_J \in \mathcal{B}^{\delta_J}\left(\overline{X}\right)$ such that $\overline{Y}_J \ge f(X_J) \ge \overline{Y}_J - \delta_J$. Then $\lim_{J\in\mathbb{N}(\mathcal{F})} X_J = X$ and $\overline{Y} \ge \lim_{J\in\mathbb{N}(\mathcal{F})} f(X_J) \ge \lim_{J\in\mathbb{N}(\mathcal{F})} \overline{Y}_J - \lim_{J\in\mathbb{N}(\mathcal{F})} \delta_J = \overline{Y}$. □

Since we always refer to the sequential characterization we write $\operatorname{ess\,lim\,sup}_{J\in\mathbb{N}(\mathcal{F})} f(X_J)$ for $\operatorname{ess\,lim\,sup}_{X\to\overline{X}} f(X)$ whenever $\lim_{J\in\mathbb{N}(\mathcal{F})} X_J = \overline{X}$, similarly, for the essential limes inferior.

Since the epigraph of a conditional function is a conditional set, it holds that $\bigsqcap_{i\in I} \operatorname{epi} f_i = \operatorname{epi}(\operatorname{ess\,sup}_{i\in I} f_i)$ and $\bigsqcup_{i\in I} \operatorname{epi} f_i = \operatorname{epi}(\operatorname{ess\,inf}_{i\in I} f_i)$ for a family $(f_i)_{i\in I}$ of conditional functions $f_i \colon L^0(\mathcal{F}) \to \overline{L}^0(\mathcal{F})$, $i \in I$. Consequently, $\bigsqcap_{i\in I} \operatorname{hypo} f_i = \operatorname{hypo}(\operatorname{ess\,inf}_{i\in I} f_i)$ and $\bigsqcup_{i\in I} \operatorname{hypo} f_i = \operatorname{hypo}(\operatorname{ess\,sup}_{i\in I} f_i)$. That directly implies that the essential infimum of a family of upper semicontinuous conditional functions is an upper semicontinuous conditional function since the conditional intersection of arbitrarily many hypographs is conditionally closed by the definition of a conditional topology.

For equivalent characterizations of epigraphs and hypographs, we need the following concepts of level sets.

**Definition 2.9** (Level set)**.** Let $f\colon L^0(\mathcal{F}) \to \overline{L}^0(\mathcal{F})$ be a conditional function. We define the lower level set $\operatorname{lev}_{\le Y} f := \left\{X \in L^0(\mathcal{F}) \mid f(X) \le Y\right\}$ and the upper level set $\operatorname{lev}_{\ge Y} f := \left\{X \in L^0(\mathcal{F}) \mid f(X) \ge Y\right\}$ for all $Y \in \overline{L}^0(\mathcal{F})$.

**Theorem 2.10.** *Let $f\colon L^0(\mathcal{F}) \to L^0(\mathcal{F})$ be a conditional function. Then, the following is equivalent.*

*(i). $f$ is upper semicontinuous.*

*(ii).* hypo $f$ *is conditionally closed in $L^0(\mathcal{F}) \times L^0(\mathcal{F})$.*

*(iii). The level sets* $\operatorname{lev}_{\ge Y} f$ *are conditionally closed in $L^0(\mathcal{F})$.*

*Proof.* We show that (i) implies (ii). Assume that $(X_J, Y_J)_{J\in\mathbb{N}(\mathcal{F})}$ is a conditional sequence in hypo $f$ with $\lim_{J\in\mathbb{N}(\mathcal{F})} X_J = \overline{X}$ and $\lim_{J\in\mathbb{N}(\mathcal{F})} Y_J = \overline{Y}$. We show that $\overline{Y} \le f\left(\overline{X}\right)$. In $L^0(\mathcal{F})$, any conditional sequence has a converging conditional subsequence, thus, there is some $\mathcal{M} \in \mathbb{N}(\mathcal{F})^{\#}_{\infty}$ with $\lim_{J\in\mathcal{M}} f(X_J) = Y^*$. Then, by construction, $\overline{Y} \leq Y^*$, and $Y^* \leq \operatorname{ess\,lim\,sup}_{X \to \overline{X}} f(X)$ by Lemma 2.8. Hence, $\overline{Y} \leq f\left(\overline{X}\right)$ by assumption that $f$ is upper semicontinuous.

We show that (ii) implies (iii). If $\operatorname{hypo} f$ is conditionally closed in $L^0(\mathcal{F}) \times L^0(\mathcal{F})$ then also $\operatorname{hypo} f \sqcap \left(L^0(\mathcal{F}), Y\right) \sqsubset L^0(\mathcal{F}) \times L^0(\mathcal{F})$ is conditionally closed for all $Y \in L^0(\mathcal{F})$ which is the level set $\operatorname{lev}_{\geq Y} f$ for $Y \in L^0(\mathcal{F})$. If $Y \in L^0(\mathcal{F})^{\sqsubset}$ then the level set $\operatorname{lev}_{\geq Y} f = \bigsqcap_{J \in \mathbb{N}(\mathcal{F})} \operatorname{lev}_{\geq Y_J} f$ is for some conditional sequence $(Y_J)_{J \in \mathbb{N}(\mathcal{F})}$ with $\lim_{J \in \mathbb{N}(\mathcal{F})} Y_J = Y$ and $Y_J \leq Y$ for all $J \in \mathbb{N}(\mathcal{F})$, thus, conditionally closed, or if $Y_J \geq Y$ for all $J \in \mathbb{N}(\mathcal{F})$ then the level set $\operatorname{lev}_{\geq Y} f = \bigsqcup_{J \in \mathbb{N}(\mathcal{F})} \operatorname{lev}_{\geq Y_J} f$, nevertheless, is conditionally closed.

We show that (iii) implies (i). Fix $\overline{X} \in L^0(\mathcal{F})$ and let $\overline{Y} := \operatorname{ess\,lim\,sup}_{X \to \overline{X}} f(X)$. We show that $f\left(\overline{X}\right) \geq \overline{Y}$. By Lemma 2.8, there is is a conditional sequence $(X_J)_{J \in \mathbb{N}(\mathcal{F})}$ with $\lim_{J \in \mathbb{N}(\mathcal{F})} X_J = X$ and $\lim_{J \in \mathbb{N}(\mathcal{F})} f(X_J) = \overline{Y}$. If there is $Y < \overline{Y}$ (if $Y \geq \overline{Y}$ on $A$, we are done on this $A$) then there is $\mathcal{M} \in \mathbb{N}(\mathcal{F})^{\#}_{\infty}$ such that $f(X_J) \geq Y$ for all $J \in \mathcal{M}$. That is, $X_J \in \operatorname{lev}_{\geq Y} f$ which is conditionally closed by assumption. Since $\lim_{J \in \mathcal{M}} X_J = \overline{X}$ it holds that $\overline{X} \in \operatorname{lev}_{\geq Y} f$. Hence, $f\left(\overline{X}\right) \geq Y$ for all $Y < \overline{Y}$, thus, $f\left(\overline{X}\right) \geq \overline{Y}$. □

For any conditional function $f\colon L^0(\mathcal{F}) \to L^0(\mathcal{F})$, we can defined $\operatorname{cl}(\operatorname{hypo} f)$. This is a hypograph of some upper semicontinuous conditional function, denoted by $\operatorname{cl} f$ for which holds that $f \leq clf$. It is the lowest of all upper semicontinuous conditional functions larger than $f$.

**Definition 2.11.** The domain of a conditional function $f\colon L^0(\mathcal{F}) \to \overline{L}^0(\mathcal{F})$ is defined by $\operatorname{dom} f := \left\{X \in L^0(\mathcal{F}) \mid f(X) < \infty\right\}$. The conditional function $f$ is called proper if there is some $X \in L^0(\mathcal{F})$ such that $f(X) \in L^0(\mathcal{F})$ and $f(X) < \infty$ for all $X \in L^0(\mathcal{F})$. It is level-bounded if the conditional sets $\operatorname{lev}_{\geq \alpha} f := \left\{X \in L^0(\mathcal{F}) \mid f(X) \geq \alpha\right\}$ are bounded for all $\alpha \in L^0(\mathcal{F})$.

**Theorem 2.12.** *Let $f\colon L^0(\mathcal{F}) \to L^0(\mathcal{F})$ be an upper semicontinuous, level-bounded and proper conditional function. Then* $\operatorname{ess\,sup} f \in L^0(\mathcal{F})$ *and* $\operatorname{argmin} f$ *lives on* $\Omega$ *and is conditionally compact.*

*Proof.* Define $\overline{a} := \operatorname{ess\,sup} f > -\infty$ since $f$ is proper. For $\alpha < \overline{\alpha}$, the level sets $\operatorname{lev}_{\geq \alpha}$ are nonempty, conditionally closed by Theorem 2.10 and bounded, thus conditionally compact. Their conditional intersection lives on $\Omega$ by [FKV09], or [DJKK16, Proposition 3.25], and is equal to $\operatorname{argmax} f$ which is nonempty, conditionally compact and nowhere $\infty$ since $f$ is proper. □

## 2.3 Set convergence

The most general concept in variational analysis is the convergence of sets. Its application to hypographs or epigraphs of functions is useful to find their optimal points. These approximations of optimal points sometimes fail if the problem is considered pointwisely, namely continuity is not preserved for the limit of a sequence of continuous functions, however, semicontinuity has this property.

In this section we characterize set convergence. The main result is a condition for the existence of a limit of a conditional sequence of sets in Theorem 2.20.

**Definition 2.13** (limit set, inner limit set, outer limit set)**.** Let $(\mathcal{C}_J)_{J\in\mathbb{N}(\mathcal{F})}$ be a conditional sequence of conditional subsets of $L^0(\mathcal{F})$ where each $\mathcal{C}_J$ lives on $A_J$ for $J \in \mathbb{N}(\mathcal{F})$. Then, the outer limit is

$$\limsup_{J\in\mathbb{N}(\mathcal{F})} \mathcal{C}_J := \left\{ X \in L^0(\mathcal{F}) \mid \exists \mathcal{N} \in \mathbb{N}(\mathcal{F})^{\#}_{\infty},\, \exists X_J \in \mathcal{C}_J,\, \lim_{J\in\mathcal{N}} X_J = X \right\} \tag{2.5}$$

which lives on

$$A_o := \operatorname{ess\,sup} \left\{ A \in \mathcal{F} \mid \exists \mathcal{N} \subset \mathbb{N}(\mathcal{F})^{\#}_{\infty},\, \forall J \in \mathcal{N} \colon A \subset A_J \right\} \tag{2.6}$$

whereas the inner limit is

$$\liminf_{J\in\mathbb{N}(\mathcal{F})} \mathcal{C}_J := \left\{ X \in L^0(\mathcal{F}) \mid \exists \mathcal{N} \in \mathbb{N}(\mathcal{F})_{\infty},\, \exists X_J \in \mathcal{C}_J,\, \lim_{J\in\mathcal{N}} X_J = X \right\} \tag{2.7}$$

which lives on

$$A_i := \operatorname{ess\,sup} \left\{ A \in \mathcal{F} \mid \exists \mathcal{N} \subset \mathbb{N}(\mathcal{F})_{\infty},\, \forall J \in \mathcal{N} \colon A_J \subset A \right\}. \tag{2.8}$$

If outer limit and inner limit are equal on some $A \in \mathcal{F}$ the limit exists

$$\lim_{J\in\mathbb{N}(\mathcal{F})} \mathcal{C}_J := \limsup_{J\in\mathbb{N}(\mathcal{F})} \mathcal{C}_J = \liminf_{J\in\mathbb{N}(\mathcal{F})} \mathcal{C}_J \text{ on } A.$$

In general, it holds that $A_i \subset A_o$. The definition may be generelized to arbitrary index families $\mathcal{M} \sqsubset \mathbb{N}(\mathcal{F})^{\#}_{\infty}$. Then, the sets of conditional subsequences in 2.5 and 2.7 are conditional subsets of $\mathcal{M}$, additionally.

We observe that the outer and inner limit are in $L^0(\mathcal{F})$ and defined almost surely, thus, it may be convenient to write essential outer and inner limit. Clearly, since there is no need to consider a pointwise outer and inner limit, we simply write limes superior and inferior for conditional sets. When we consider random variables, we write essential limes, explicitely.

For the next characterization, we will use the conditional distance of an element $X \in L^0(\mathcal{F})$ to a conditional subset $\mathcal{C} \sqsubset L^0(\mathcal{F})$. We define $\|X-\mathcal{C}\| := \operatorname{ess\,inf}_{Y\in\mathcal{C}}\|X-Y\|$.

**Lemma 2.14** (Equivalent characterizations of outer and inner limit)**.** *Let* $(\mathcal{C}_J)_{J\in\mathbb{N}(\mathcal{F})}$ *be a conditional sequence of subsets of* $L^0(\mathcal{F})$. *Then, on* $A_o$, *given by* (2.6),

$$\limsup_{J\in\mathbb{N}(\mathcal{F})}\mathcal{C}_J = \left\{X\in L^0(\mathcal{F}) \mid \forall\,\mathcal{V}\in\mathfrak{U}(X)\ \exists\,\mathcal{N}\in\mathbb{N}(\mathcal{F})^{\#}_{\infty},\ \forall\,J\in\mathcal{N}\colon \mathcal{V}\cap\mathcal{C}_J\neq\emptyset\right\} \tag{2.9}$$

$$= \left\{X\in L^0(\mathcal{F}) \mid \forall\delta\in L^0(\mathcal{F})_{++},\ \exists\,\mathcal{N}\in\mathbb{N}(\mathcal{F})^{\#}_{\infty},\ \forall\,J\in\mathcal{N}\colon \|X-\mathcal{C}_J\|\le\delta\right\} \tag{2.10}$$

$$= \left\{X\in L^0(\mathcal{F}) \mid \operatorname{ess}\liminf_{J\in\mathbb{N}(\mathcal{F})}\|X-\mathcal{C}_J\| = 0\right\} \tag{2.11}$$

$$= \bigsqcap_{\mathcal{N}\in\mathbb{N}(\mathcal{F})_{\infty}} \operatorname{cl}\bigsqcup_{J\in\mathcal{N}}\mathcal{C}_J \tag{2.12}$$

*and, on* $A_i$, *given by* (2.8),

$$\liminf_{J\in\mathbb{N}(\mathcal{F})}\mathcal{C}_J = \left\{X\in L^0(\mathcal{F}) \mid \forall\,\mathcal{V}\in\mathfrak{U}(X)\ \exists\,\mathcal{N}\in\mathbb{N}(\mathcal{F})_{\infty},\ \forall\,J\in\mathcal{N}\colon \mathcal{V}\cap\mathcal{C}_J\neq\emptyset\right\} \tag{2.13}$$

$$= \left\{X\in L^0(\mathcal{F}) \mid \forall\delta\in L^0(\mathcal{F})_{++},\ \exists\,\mathcal{N}\in\mathbb{N}(\mathcal{F})_{\infty},\ \forall\,J\in\mathcal{N}\colon \|X-\mathcal{C}_J\|\le\delta\right\} \tag{2.14}$$

$$= \left\{X\in L^0(\mathcal{F}) \mid \operatorname{ess}\limsup_{J\in\mathbb{N}(\mathcal{F})}\|X-\mathcal{C}_J\| = 0\right\} \tag{2.15}$$

$$= \bigsqcap_{\mathcal{N}\in\mathbb{N}(\mathcal{F})^{\#}_{\infty}} \operatorname{cl}\bigsqcup_{J\in\mathcal{N}}\mathcal{C}_J \tag{2.16}$$

*Proof.* We observe that, on $A_i$, (2.13) is just a reformulation of convergence of (2.7) in terms of conditional neighborhoods instead of conditional sequences, equivalently, (2.14) by the use of the conditional distance. Moreover, (2.15) is a reformulation of conditional subsequences, and (2.16) by means of the limes superior. The same holds for the outer limit. □

In the sequel, we assume that $A_i = \Omega$ since it is not important where the objects live.

**Proposition 2.15.** *For conditional sequences $(\mathcal{C}_J)_{J\in\mathbb{N}(\mathcal{F})}$, $(\mathcal{D}_J)_{J\in\mathbb{N}(\mathcal{F})}$ in $L^0(\mathcal{F})$, the outer and inner limits are conditionally closed. Further, if $\operatorname{cl}\mathcal{C}_J = \operatorname{cl}\mathcal{D}_J$ for all $J \in \mathbb{N}(\mathcal{F})$ then $\liminf_{J\in\mathbb{N}(\mathcal{F})} \mathcal{C}_J = \liminf_{J\in\mathbb{N}(\mathcal{F})} \mathcal{D}_J$ and $\limsup_{J\in\mathbb{N}(\mathcal{F})} \mathcal{C}_J = \limsup_{J\in\mathbb{N}(\mathcal{F})} \mathcal{D}_J$.*

*Proof.* We apply the equivalent characterizations of outer and inner limit in (2.12) and (2.16). □

**Example 2.16.** Let $\left(\mathcal{C}^1_J\right)_{J\in\mathbb{N}(\mathcal{F})}$, $\left(\mathcal{C}^2_J\right)_{J\in\mathbb{N}(\mathcal{F})}$ and $(\mathcal{C}_J)_{J\in\mathbb{N}(\mathcal{F})}$ be conditional sequences in $L^0(\mathcal{F})$. It holds that $\lim_{J\in\mathbb{N}(\mathcal{F})} \mathcal{C}_J = \operatorname{cl}\bigsqcup_{J\in\mathbb{N}(\mathcal{F})} \mathcal{C}_J$ if $\mathcal{C}_J \sqsubset \mathcal{C}_{J+1}$ and $\lim_{J\in\mathbb{N}(\mathcal{F})} \mathcal{C}_J = \bigsqcap_{J\in\mathbb{N}(\mathcal{F})} \operatorname{cl}\mathcal{C}_J$ if $\mathcal{C}_{J+1} \sqsubset \mathcal{C}_J$.
If $\mathcal{C}^1_J \sqsubset \mathcal{C}_J \sqsubset \mathcal{C}^2_J$ for all $J \in \mathbb{N}(\mathcal{F})$ then $\lim_{J\in\mathbb{N}(\mathcal{F})} \mathcal{C}^1_J = \mathcal{C} = \lim_{J\in\mathbb{N}(\mathcal{F})} \mathcal{C}^2_J$ implies $\lim_{J\in\mathbb{N}(\mathcal{F})} \mathcal{C}_J = \mathcal{C}$.

**Example 2.17.** A conditional sequence $(\mathcal{C}_J)_{J\in\mathbb{N}(\mathcal{F})}$ in $L^0(\mathcal{F})$ with $\mathcal{C}_J = 1\!\!1_{A_J}\mathcal{D}_1 + 1\!\!1_{A_J^c}\mathcal{D}_2$ for $A_J := \operatorname{ess\,sup}\{A \in \mathcal{F} \mid 1\!\!1_A J = 1\!\!1_A(2k+1) \text{ for some } k \in \mathbb{N}\}$ has the inner limit $\mathcal{D}_1 \sqcap \mathcal{D}_2$ and the outer limit $\mathcal{D}_1 \sqcup \mathcal{D}_2$, hence, does not necessariliy setconverge.

**Example 2.18.** In $L^0(\mathcal{F})$, a conditional sequence of balls $\left(\mathcal{B}^{\delta_J}(X_J)\right)_{J\in\mathbb{N}(\mathcal{F})}$ setconverges to the conditionally closed ball $\operatorname{cl}\mathcal{B}^{\delta}(X)$ if for all $\varepsilon \in L^0(\mathcal{F})_{++}$ there exists some $\overline{J} \in \mathbb{N}(\mathcal{F})$ such that $\|X_J - X\| < \varepsilon$ and $\|\delta_J - \delta\| < \varepsilon$ for all $J > \overline{J}$. The constant conditional sequence $\mathcal{C}_J \equiv \mathbb{Q}(\mathcal{F})$ setconverges to $L^0(\mathcal{F})$. Further, we recall from [DJKK16] that a conditional subset $\mathcal{M} \sqsubset L^0(\mathcal{F})$ is conditionally dense if $\operatorname{cl}\mathcal{M} = L^0(\mathcal{F})$ and it is conditionally countable if there is an bijective conditional function $g\colon \mathcal{M} \to \mathbb{N}(\mathcal{F})$. It is conditionally seperable if it is conditionally dense and conditionally countable. Thus, $\mathbb{Q}(\mathcal{F}) \sqsubset L^0(\mathcal{F})$ is conditionally separable since it is conditionally dense in $L^0(\mathcal{F})$ and $\mathbb{Q}(\mathcal{F})$ is conditionally countable.

In the next theorem, we describe set convergence in $L^0(\mathcal{F})$ in terms of a conditionally separable conditional subset.

**Theorem 2.19** (Hit-and-miss-criteria)**.** *Let $\mathcal{C}, \mathcal{C}_J$ be conditional subsets of $L^0(\mathcal{F})$ for all $J \in \mathbb{N}(\mathcal{F})$ with $\mathcal{C}$ being conditionally closed. Let $\mathcal{D} \sqsubset L^0(\mathcal{F})$ be conditionally separable. Then, it holds that*

*(i). $\mathcal{C} \sqsubset \liminf_{J\in\mathbb{N}(\mathcal{F})} \mathcal{C}_J$ if and only if for every conditionally open set $\mathcal{O} \sqsubset L^0(\mathcal{F})$ with $\mathcal{C} \sqcap \mathcal{O}$ living on $\Omega$ there exists $\mathcal{N} \in \mathbb{N}(\mathcal{F})_\infty$ such that $\mathcal{C}_J \sqcap \mathcal{O}$ lives on $\Omega$ for all $J \in \mathcal{N}$.*

*(ii). It suffices in (i) to consider the collection of conditional balls $\mathcal{B}^{\delta}(Y)$ with $Y \in \mathcal{D}$ and $\delta \in \mathcal{D}_{++}$.*

*(iii).* $\limsup_{J\in\mathbb{N}(\mathcal{F})}\mathcal{C}_J \sqsubset \mathcal{C}$ *if and only if for every conditionally compact set* $\mathcal{B}\sqsubset L^0(\mathcal{F})$ *with* $\mathcal{C}\sqcap\mathcal{B}=L^0(\mathcal{F})|\emptyset$ *there exists* $\mathcal{N}\in\mathbb{N}(\mathcal{F})_\infty$ *such that* $\mathcal{C}_J\sqcap\mathcal{B}=L^0(\mathcal{F})|\emptyset$ *for all* $J\in\mathcal{N}$.

*(iv). It suffices in (iii) to consider the collection of conditional closures of conditional balls* $\mathcal{B}^\delta(Y)$ *with* $Y\in\mathcal{D}$ *and* $\delta\in\mathcal{D}_{++}$.

*Proof.* (i) and (ii). Let $X\in\mathcal{C}$ and $\delta\in\mathcal{D}_{++}$. We show that $X\in\liminf_{J\in\mathbb{N}(\mathcal{F})}\mathcal{C}_J$. Since $\mathcal{D}\sqsubset L^0(\mathcal{F})$ is conditionally dense, there is $X'\in\mathcal{D}$ with $X'\in\mathcal{B}^{\frac{\delta}{2}}(X)$. By assumption, there exists $\mathcal{N}\in\mathbb{N}(\mathcal{F})_\infty$ such that $\mathcal{C}_J\sqcap\mathcal{B}^{\frac{\delta}{2}}(X)$ lives on $\Omega$ for all $J\in\mathcal{N}$. Thus, $\|X'-\mathcal{C}_J\|\le\frac{\delta}{2}$ and $\|X-\mathcal{C}_J\|\le\|X'-\mathcal{C}_J\|+\|X-X'\|\le\frac{\delta}{2}+\frac{\delta}{2}=\delta$ for all $J\in\mathcal{N}$ which by Lemma 2.14 yields that $X\in\liminf_{J\in\mathbb{N}(\mathcal{F})}\mathcal{C}_J$. The other direction is clear by definition.

(iii) and (iv). Let $X\in\mathcal{C}^{\sqsubset}$. We show that $X\in\left(\limsup_{J\in\mathbb{N}(\mathcal{F})}\mathcal{C}_J\right)^{\sqsubset}$. That implies directly $\limsup_{J\in\mathbb{N}(\mathcal{F})}\mathcal{C}_J\sqsubset\mathcal{C}$. Since $\mathcal{C}$ is conditionally closed, there exists $\delta\in\mathcal{D}_{++}$ such that $\mathcal{C}\sqcap\mathcal{B}^{2\delta}(X)=L^0(\mathcal{F})|\emptyset$. We find $X'\in\mathcal{B}^\delta(X)\sqcap\mathcal{D}$. Then $X\in\mathcal{B}^\delta(X')$ and $\mathcal{C}\sqcap\mathcal{B}^\delta(X')=L^0(\mathcal{F})|\emptyset$. By assumption, there exists $\mathcal{N}\in\mathbb{N}(\mathcal{F})_\infty$ such that $\mathcal{C}_J\sqcap\mathcal{B}^\delta(X')=L^0(\mathcal{F})|\emptyset$ for all $J\in\mathcal{N}$. Since $X\in\operatorname{int}\mathcal{B}^\delta(X')$, it holds that $\{X\}\sqcap\limsup_{J\in\mathbb{N}(\mathcal{F})}\mathcal{C}_J=L^0(\mathcal{F})|\emptyset$. The other direction is clear by definition. $\square$

**Theorem 2.20** (Setconvergent conditional subsequences)**.** *Let* $\mathcal{D}\sqsubset L^0(\mathcal{F})$ *be conditionally separable. A conditional sequence* $(\mathcal{C}_J)_{J\in\mathbb{N}(\mathcal{F})}$ *of subsets of* $L^0(\mathcal{F})$ *has an outer limit that lives on* $A\in\mathcal{F}$. *Then there is some index set* $\mathcal{N}\in\mathbb{N}(\mathcal{F})^\#_\infty$ *such that the sequence* $(\mathcal{C}_J)_{J\in\mathcal{N}}$ *is setconverging to a set* $\mathcal{C}\sqsubset L^0(\mathcal{F})$ *that lives on* $A\in\mathcal{F}$.

*Proof.* We suppose that $A=\Omega$. If not, we may consider the proof in $L^0(\mathcal{F})$ and all sets living on $A\in\mathcal{F}$. Since there is $\overline{X}\in L^0(\mathcal{F})$ in the outer limit, there exists $N^0\in\mathbb{N}(\mathcal{F})^\#_\infty$ such that

$$\lim_{J\in N^0}X_J=\overline{X}\in\limsup_{J\in\mathbb{N}(\mathcal{F})}\mathcal{C}_J.$$

Next, consider the conditionally countable collection of conditionally open balls $\mathcal{B}^\delta(Y)$ for $Y\in\mathcal{D}$ and $\delta\in\mathcal{D}_{++}$ in Theorem 2.19 (ii) writing it as a conditional sequence $\left(\mathcal{B}^I\right)_{I\in\mathbb{N}(\mathcal{F})}$. We construct a conditional sequence of index sets $\left(N^I\right)_{I\in\mathbb{N}(\mathcal{F})}$ with $N^I\sqsubset$

$N^{I'}$ if $I' \leq I$ by defining

$$\overline{A} := \operatorname{ess\,sup}\left\{A \in \mathcal{F} \mid \left\{J \in N^{I-1} \mid \mathcal{C}_J \sqcap \mathcal{B}^I \text{ lives on } A\right\} \in \mathbb{N}\left(\mathcal{F}\right)^{\#}_{\infty}\right\} \tag{2.17}$$

$$N^I := 1\!\!1_{\overline{A}}\left\{J \in N^{I-1} \mid \mathcal{C}_J \sqcap \mathcal{B}^I \text{ lives on } \overline{A}\right\} + 1\!\!1_{\overline{A}^c}\left\{J \in N^{I-1} \mid \mathcal{C}_J \sqcap \mathcal{B}^I = L^0\left(\mathcal{F}\right)|\emptyset\right\} \tag{2.18}$$

for $I \geq 1$. We observe that, on $\overline{A}^c$, the conditional set $\left\{J \in N^{I-1} \mid \mathcal{C}_J \sqcap \mathcal{B}^I = L^0\left(\mathcal{F}\right)|\emptyset\right\}$ is in $\mathbb{N}\left(\mathcal{F}\right)_{\infty}$, that is if the definition for $\overline{A}$ does not hold.

Let $I' \leq I$. We show that $N^I \sqsubset N^{I'}$. Since $I, I' \in \mathbb{N}\left(\mathcal{F}\right)$, we write $I - I' = \sum_{k\in\mathbb{N}} 1\!\!1_{A_k} k$ where $A_k = \operatorname{ess\,sup}\left\{A \in \mathcal{F} \mid 1\!\!1_A\left(I - I'\right) = 1\!\!1_A k\right\}$ for all $k \in \mathbb{N}$. Restricting (2.18) on each $A_k$ yields $N^I \sqsubset N^{I'}$ by definition.

Next, define the conditional set $\mathcal{N} = \left(\overline{N}_K\right)_{K\in\mathbb{N}(\mathcal{F})}$ such that $\inf N^0 \in \mathcal{N}$ and $\overline{N}_K := \operatorname{ess\,inf}\left\{\overline{N} \in \left(N^I\right)_{I\geq K} \mid \overline{N} > M \ \forall M \in \overline{N}_{K'},\ K' < K\right\}$. By construction, $\mathcal{N} \in \mathbb{N}\left(\mathcal{F}\right)^{\#}_{\infty}$. For each $K \in \mathbb{N}\left(\mathcal{F}\right)$, we observe that

$$1\!\!1_{\overline{A}}\left\{J \mid \mathcal{C}_J \sqcap \mathcal{B}^K \text{ lives on } \overline{A}\right\} + 1\!\!1_{\overline{A}^c}\left\{J \mid \mathcal{C}_J + \mathcal{B}^K = L^0\left(\mathcal{F}\right)|\emptyset\right\} \in \mathbb{N}\left(\mathcal{F}\right)_{\infty} \tag{2.19}$$

with $\overline{A} := \operatorname{ess\,sup}\left\{A \in \mathcal{F} \mid \left\{J \mid \mathcal{C}_J \sqcap \mathcal{B}^K \text{ lives on } A\right\} \in \mathbb{N}\left(\mathcal{F}\right)^{\#}_{\infty}\right\}$.

Let $\mathcal{C} := \limsup_{J\in\mathcal{N}} \mathcal{C}_J$. Clearly, $\overline{X} \in \mathcal{C}$. Suppose, for a ball $\mathcal{B}^K$ where $K \in \mathbb{N}\left(\mathcal{F}\right)$, it holds that $\mathcal{C} \sqcap \mathcal{B}^K$ lives on $\Omega$. By the definition of the outer limit, it cannot hold that $\mathcal{C}_J \sqcap \mathcal{B}^I = L^0\left(\mathcal{F}\right)|\emptyset$ for all $J \in \mathcal{N}$. Thus, by (2.19), $\mathcal{C}_J \sqcap \mathcal{B}^I$ lives on $\Omega$ for all $J \in \mathcal{N}$ and, by Theorem 2.19(ii), it holds that $\mathcal{C} \sqsubset \liminf_{J\in\mathcal{N}} \mathcal{C}_J$. □

## 2.4 Outer and inner semicontinuity

An application of the outer and inner limits in Section 2.3 is the definition of outer und inner semicontinuity, where a conditional function has subsets of $L^0\left(\mathcal{F}\right)$ as values. We write $\left(X_J\right)_{J\in\mathbb{N}(\mathcal{F})} \to X$ for some converging conditional sequence with limit $X$. For a conditional function $f\colon L^0(\mathcal{F}) \to L^0(\mathcal{F})$, we define

$$\begin{aligned}
\limsup_{X\to\overline{X}} f(X) &:= \bigsqcup_{X_J\to\overline{X}} \limsup_{J\in\mathbb{N}(\mathcal{F})} f(X_J)\\
&= \Big\{Y\in L^0(\mathcal{F}) \mid \exists\, (X_J)_{J\in\mathbb{N}(\mathcal{F})}\to\overline{X},\, \exists\, (Y_J)_{J\in\mathbb{N}(\mathcal{F})}\to\overline{Y},\, Y_J\in f(X_J)\Big\},\\
\liminf_{X\to\overline{X}} f(X) &:= \bigsqcap_{X_J\to\overline{X}} \liminf_{J\in\mathbb{N}(\mathcal{F})} f(X_J)\\
&= \Big\{Y\in L^0(\mathcal{F}) \mid \forall\, (X_J)_{J\in\mathbb{N}(\mathcal{F})}\to\overline{X}, \exists\mathcal{N}\in\mathbb{N}(\mathcal{F})_\infty,\\
&\qquad\qquad (Y_J)_{J\in\mathcal{N}}\to\overline{Y},\, Y_J\in f(X_J)\Big\}.
\end{aligned}$$

**Definition 2.21.** A conditional function $f\colon L^0(\mathcal{F}) \to L^0(\mathcal{F})$ is

$$\begin{aligned}
&\text{outer semicontinuous if } \limsup_{X\to\overline{X}} f(X) \sqsubset f\left(\overline{X}\right),\\
&\text{inner semicontinuous if } f\left(\overline{X}\right) \sqsubset \liminf_{X\to\overline{X}} f(X).
\end{aligned}$$

It is called continuous if it is outer and inner semicontinuous.

We note that inner semicontinuity of $f$ at $\overline{X}\in\operatorname{dom} f$ means that $\overline{X}\in\operatorname{int}\operatorname{dom} f$.

## 2.5 Hypoconvergence

In this section, we examine the consequences of setconvergence applied to hypographs of conditional functions. We see that the set limit of hypographs itself is a hypograph and give characterizations of hypoconvergence by essential limes inferior and superior of their respective conditional functions.

First, we consider a conditional sequence $(\mathcal{C}_J)_{J\in\mathbb{N}(\mathcal{F})}$ of hypographs in $L^0(\mathcal{F})$ and show that their outer and inner limit set are again hypographs. To that end, let $(X,Y)\in \limsup_{J\in\mathbb{N}(\mathcal{F})}\mathcal{C}_J$. We show that $(X,Y')\in\limsup_{J\in\mathbb{N}(\mathcal{F})}\mathcal{C}_J$ for all $Y'\leq Y$. By definition, there are $\mathcal{N}\in\mathbb{N}(\mathcal{F})^{\#}_\infty$, $(X_J)_{J\in\mathcal{N}}$ and $(Y_J)_{J\in\mathcal{N}}$ such that $\lim_{J\in\mathcal{N}} X_J = X$, $\lim_{J\in\mathcal{N}} Y_J = Y$ and $(X_J,Y_J)\in\mathcal{C}_J$ for all $J\in\mathcal{N}$. Define $Y'_J := Y_J\wedge Y'$. By the definition of hypograph, $(X_J,Y'_J)\in\mathcal{C}_J$. Now, $\operatorname{ess\,lim\,sup}_{J\in\mathcal{N}}(Y'_J) = \operatorname{ess\,lim\,sup}_{J\in\mathcal{N}}(Y_J)\wedge Y' = \lim_{J\in\mathcal{N}}(Y_J)\wedge Y' = Y\wedge Y'$ and $\operatorname{ess\,lim\,inf}_{J\in\mathcal{N}}(Y'_J) = \operatorname{ess\,lim\,inf}_{J\in\mathcal{N}}(Y_J)\wedge Y' = \lim_{J\in\mathcal{N}}(Y_J)\wedge Y' = Y\wedge Y'$. Thus, $\lim_{J\in\mathcal{N}}(Y'_J) = Y\wedge Y' = Y'$, or, $(X,Y')\in\limsup_{J\in\mathbb{N}(\mathcal{F})}\mathcal{C}_J$. The same holds for the inner limit, where $\mathcal{N}$ is chosen in $\mathbb{N}(\mathcal{F})_\infty$. By Proposition 2.15, outer and inner limit of $(\mathcal{C}_J)_{J\in\mathbb{N}(\mathcal{F})}$ are conditionally closed. Finally, $\left(\{X\}\times L^0(\mathcal{F})\right)\sqcap\limsup_{J\in\mathbb{N}(\mathcal{F})}\mathcal{C}_J$

and $\left(\{X\} \times L^0(\mathcal{F})\right) \sqcap \liminf_{J\in\mathbb{N}(\mathcal{F})} \mathcal{C}_J$ are conditional intervals that are left-unbounded right-conditionally-closed, thus, hypographs.

Thus, hypoconvergence and semicontinuity is closed under the limits of conditional sequences, a fact that does not hold for continuity, see the example given in [RW09]. Therefore, many optimization problems are given only for semicontinuity.

**Definition 2.22** (Lower and upper hypolimit)**.** Let $(\operatorname{hypo} f_J)_{J\in\mathbb{N}(\mathcal{F})}$ be a conditional sequence of hypographs of conditional functions $f_J\colon L^0 \to L^0(\mathcal{F})$, $J \in \mathbb{N}(\mathcal{F})$. The upper hypolimit $\text{h-}\limsup_{J\in\mathbb{N}(\mathcal{F})} f_J$ is the conditional function $f\colon L^0(\mathcal{F}) \to L^0(\mathcal{F})$ with hypograph which is the outer limit of the conditional sequence $(\operatorname{hypo} f_J)_{J\in\mathbb{N}(\mathcal{F})}$, or

$$\operatorname{hypo}\left(\text{h-}\limsup_{J\in\mathbb{N}(\mathcal{F})} f_J\right) := \limsup_{J\in\mathbb{N}(\mathcal{F})} (\operatorname{hypo} f_J).$$

The lower hypolimit $\text{h-}\liminf_{J\in\mathbb{N}(\mathcal{F})} f_J$ is the conditional function $f\colon L^0(\mathcal{F}) \to L^0(\mathcal{F})$ with hypograph which is the inner limit of the conditional sequence $(\operatorname{hypo} f_J)_{J\in\mathbb{N}(\mathcal{F})}$, or

$$\operatorname{hypo}\left(\text{h-}\liminf_{J\in\mathbb{N}(\mathcal{F})} f_J\right) := \liminf_{J\in\mathbb{N}(\mathcal{F})} (\operatorname{hypo} f_J).$$

If lower and upper hypolimit are equal the hypolimit exists and is

$$\text{h-}\lim_{J\in\mathbb{N}(\mathcal{F})} f_J := \text{h-}\limsup_{J\in\mathbb{N}(\mathcal{F})} f_J = \text{h-}\liminf_{J\in\mathbb{N}(\mathcal{F})} f_J.$$

Indeed, the upper hypolimit conditional function $f\colon L^0(\mathcal{F}) \to L^0(\mathcal{F})$ is a conditional function since

$$\begin{aligned} f\left(\sum_{k\in\mathbb{N}} \mathbb{1}_{A_k} X_k\right) &= \text{h-}\limsup_{J\in\mathbb{N}(\mathcal{F})} f_J\left(\sum_{k\in\mathbb{N}} \mathbb{1}_{A_k} X_k^J\right) \\ &= \sum_{k\in\mathbb{N}} \mathbb{1}_{A_k}\, \text{h-}\limsup_{J\in\mathbb{N}(\mathcal{F})} f_J\left(X_k^J\right) = \sum_{k\in\mathbb{N}} \mathbb{1}_{A_k} f(X_k) \end{aligned}$$

for all partitions $(A_k)_{k\in\mathbb{N}}$ of $\Omega$ in $\mathcal{F}$ and $X_k, X_k^J \in L^0(\mathcal{F})$ for all $J \in \mathbb{N}(\mathcal{F})$ and $k \in \mathbb{N}$. The same holds for the lower hypolimit function.

We remark here, in advantage to the setting in [RW09], that the conditional definition of setconvergence and conditional functions that the definitions yield directly $A \in \mathcal{F}$ where the objects may live. So, there is no need to define a pointwise hypolimit by upper and lower hypolimit values as done there. In the sequel we assume that all sets live on $\Omega$.

**Lemma 2.23** (Characterizations of hypolimits)**.** *Let* $(\operatorname{hypo} f_J)_{J\in\mathbb{N}(\mathcal{F})}$ *be a conditional sequence of hypographs of conditional functions* $f_J \colon L^0(\mathcal{F}) \to L^0(\mathcal{F})$*,* $J \in \mathbb{N}(\mathcal{F})$ *and* $X \in L^0(\mathcal{F})$*. Then*

$$\underset{J\in\mathbb{N}(\mathcal{F})}{\text{h-}\limsup} f_J(X) = \max\left\{Y \in \overline{L}^0(\mathcal{F}) \mid \exists (X_J)_{J\in\mathbb{N}(\mathcal{F})} \to X \colon \underset{J\in\mathbb{N}(\mathcal{F})}{\operatorname{ess}\limsup} f_J(X_J) = Y\right\}, \tag{2.20}$$

$$\underset{J\in\mathbb{N}(\mathcal{F})}{\text{h-}\liminf} f_J(X) = \max\left\{Y \in \overline{L}^0(\mathcal{F}) \mid \exists (X_J)_{J\in\mathbb{N}(\mathcal{F})} \to X \colon \underset{J\in\mathbb{N}(\mathcal{F})}{\operatorname{ess}\liminf} f_J(X_J) = Y\right\}. \tag{2.21}$$

*Proof.* By Definition 2.22, $Y \le \text{h-}\limsup_{J\in\mathbb{N}(\mathcal{F})} f_J(X)$ if and only if for some $\mathcal{N} \in \mathbb{N}(\mathcal{F})^{\#}_{\infty}$ there are conditional sequences $(X_J)_{J\in\mathcal{N}} \to X$, $(Y_J)_{J\in\mathcal{N}} \to Y$ with $Y_J \le f_J(X_J)$ for all $J \in \mathcal{N}$. The conditional set

$$\mathcal{M} := \left\{Y \in \overline{L}^0 \mid \exists (X_J)_{J\in\mathbb{N}(\mathcal{F})} \to X \colon \underset{J\in\mathbb{N}(\mathcal{F})}{\operatorname{ess}\limsup} f_J(X_J) = Y\right\}$$

is upwards directed since for $Y, Y' \in \mathcal{M}$ there are $(X_J)_{J\in\mathbb{N}(\mathcal{F})} \to X$ and $(X'_J)_{J\in\mathbb{N}(\mathcal{F})} \to X$ such that $\operatorname{ess}\limsup_{J\in\mathbb{N}(\mathcal{F})} f_J(X_J) = Y$ and $\operatorname{ess}\limsup_{J\in\mathbb{N}(\mathcal{F})} f_J(X'_J) = Y'$ for which hold that $\left(\mathbb{1}_{\{Y\ge Y'\}}X_J + \mathbb{1}_{\{Y<Y'\}}X'\right)_{J\in\mathbb{N}(\mathcal{F})} \to X$ with

$$\underset{J\in\mathbb{N}(\mathcal{F})}{\operatorname{ess}\limsup} f_J\left(\mathbb{1}_{\{Y\ge Y'\}}X_J + \mathbb{1}_{\{Y<Y'\}}X'\right) = Y \wedge Y',$$

thus, $Y \wedge Y' \in \mathcal{M}$. Then, there exists a nondecreasing conditional sequence $(Y_K)_{K\in\mathbb{N}(\mathcal{F})}$ such that $\lim_{K\in\mathbb{N}(\mathcal{F})} Y_K = \operatorname{ess\,sup}\mathcal{M}$. Let $\varepsilon \in L^0(\mathcal{F})_{++}$. For each $k \in \mathbb{N}$ there is $\overline{Y}_k \in L^0(\mathcal{F})$ such that $\|\overline{Y}_k - \operatorname{ess\,sup}\mathcal{M}\| < \frac{\varepsilon}{2^{k+1}}$ and $\overline{X}_k \in L^0(\mathcal{F})$ such that $\|X - \overline{X}_k\| < \frac{\varepsilon}{2^{k+1}}$ and $\|f_k(\overline{X}_k) - \overline{Y}_k\| < \frac{\varepsilon}{2^{k+1}}$. Now, for the corresponding conditional sequences $(\overline{X}_K)_{K\in\mathbb{N}(\mathcal{F})}$ and $(\overline{Y}_K)_{K\in\mathbb{N}(\mathcal{F})}$, it holds that $\lim_{K\in\mathbb{N}(\mathcal{F})} \overline{X}_K = X$ and $\|\operatorname{ess\,sup}\mathcal{M} - f_K(\overline{X}_K)\| \le \|\operatorname{ess\,sup}\mathcal{M} - \overline{Y}_K\| + \|\overline{Y}_K - f_K(\overline{X}_K)\| \le \frac{\varepsilon}{2^{K+1}} + \frac{\varepsilon}{2^{K+1}}$. Hence, the maximum in (2.20) is justified, also for $\overline{L}^0(\mathcal{F})$. The same holds for $\mathcal{N} \in \mathbb{N}(\mathcal{F})_{\infty}$ and the lower hypolimit. $\square$

**Lemma 2.24** (Characterization of hypoconvergence)**.** *Let* $(f_J)_{J\in\mathbb{N}(\mathcal{F})}$ *be a conditional sequence of conditional functions* $f_J \colon L^0(\mathcal{F}) \to L^0(\mathcal{F})$*,* $J \in \mathbb{N}(\mathcal{F})$*. Let* $f \colon L^0(\mathcal{F}) \to L^0(\mathcal{F})$ *be a conditional function. Then,* $\text{h-}\lim_{J\in\mathbb{N}(\mathcal{F})} f_J = f$ *if and only if for all* $X \in$

*$L^0(\mathcal{F})$ it holds that*

$$\exists\,(X_J)_{J\in\mathbb{N}(\mathcal{F})} \to X\colon\ \operatorname*{ess\,lim\,inf}_{J\in\mathbb{N}(\mathcal{F})} f_J(X_J) \geq f(X),$$

*and*

$$\forall\,(X_J)_{J\in\mathbb{N}(\mathcal{F})} \to X\colon\ \operatorname*{ess\,lim\,sup}_{J\in\mathbb{N}(\mathcal{F})} f_J(X_J) \leq f(X).$$

*Proof.* That is a consequence of Lemma 2.23. □

**Proposition 2.25** (Properties of hypolimits). *Let $(f_J)_{J\in\mathbb{N}(\mathcal{F})}$ be a conditional sequence of conditional functions $f_J\colon L^0(\mathcal{F}) \to L^0(\mathcal{F})$, $J\in\mathbb{N}(\mathcal{F})$. Then the following properties hold.*

*(i). The conditional functions $\text{h-}\limsup_{J\in\mathbb{N}(\mathcal{F})} f_J$, $\text{h-}\liminf_{J\in\mathbb{N}(\mathcal{F})} f_J$ are upper semi-continuous, and $\text{h-}\lim_{J\in\mathbb{N}(\mathcal{F})} f_J$, too, if it exists.*

*(ii). Let $(g_J)_{J\in\mathbb{N}(\mathcal{F})}$ be a conditional sequence of conditional functions $g_J\colon L^0(\mathcal{F}) \to L^0(\mathcal{F})$, $J\in\mathbb{N}(\mathcal{F})$. If $\operatorname{cl} f_J = \operatorname{cl} g_J$ for all $J\in\mathbb{N}(\mathcal{F})$ then $\text{h-}\liminf_{J\in\mathbb{N}(\mathcal{F})} f_J = \text{h-}\liminf_{J\in\mathbb{N}(\mathcal{F})} g_J$ and $\text{h-}\limsup_{J\in\mathbb{N}(\mathcal{F})} f_J = \text{h-}\limsup_{J\in\mathbb{N}(\mathcal{F})} g_J$.*

*(iii). If $f_J \geq f_{J'}$ for $J\leq J'$ then $\text{h-}\lim_{J\in\mathbb{N}(\mathcal{F})} f_J = \operatorname{cl}\left(\operatorname{ess\,inf}_{J\in\mathbb{N}(\mathcal{F})} f_J\right)$.*

*(iv). If $f_J \leq f_{J'}$ for $J\leq J'$ then $\text{h-}\lim_{J\in\mathbb{N}(\mathcal{F})} f_J = \operatorname{ess\,sup}_{J\in\mathbb{N}(\mathcal{F})}\left(\operatorname{cl} f_J\right)$.*

*(v). If $f_J^{\mathbb{1}} \leq f_J \leq f_J^2$ for all $J\in\mathbb{N}(\mathcal{F})$, it holds that if $\text{h-}\lim_{J\in\mathbb{N}(\mathcal{F})} f_J^{\mathbb{1}} = f = \text{h-}\lim_{J\in\mathbb{N}(\mathcal{F})} f_J^2$ then $\text{h-}\lim_{J\in\mathbb{N}(\mathcal{F})} f_J = f$.*

*Proof.* The inner and outer limit of conditional subsets is conditionally closed by Proposition 2.15, this is equivalent to lower semicontinuity of the conditional function by Theorem 2.10. This implies (i). Directly, Proposition 2.15 yields (ii), and (iii), (iv) and (v) are consequences of Example 2.16. □

**Theorem 2.26** (Hypoconvergent conditional subsequences). *Let $(f_J)_{J\in\mathbb{N}(\mathcal{F})}$ be a conditional sequence of conditional functions $f_J\colon L^0(\mathcal{F}) \to L^0(\mathcal{F})$, $J\in\mathbb{N}(\mathcal{F})$. Let the outer limit of $(\operatorname{hypo} f_J)_{J\in\mathbb{N}(\mathcal{F})}$ live on $A\in\mathcal{F}$. Then, there is some $\mathcal{N}\in\mathbb{N}(\mathcal{F})^{\#}_{\infty}$ such that $(f_J)_{J\in\mathcal{N}}$ is hypoconverging to a conditional function $f\colon L^0(\mathcal{F})\to L^0(\mathcal{F})$ on $A\in\mathcal{F}$.*

*Proof.* The upper and lower limit of conditional sequences of hypographs are hypographs. Thus, the claim follows by Theorem 2.20. □

## 2.6 Hypoconvergence and maximization

The main benefit of the theory of hypoconvergence is that upper semicontinuity of the conditional functions is maintained under to closure with respect to limits. In this section, we give the details connected with the maximization problem and focus on conditionally compact domains.

**Theorem 2.27** (Characterization of hypoconvergence via maximization)**.** *Let $(f_J)_{J\in\mathbb{N}(\mathcal{F})}$ be a conditional sequence of conditional functions $f_J\colon L^0(\mathcal{F}) \to L^0(\mathcal{F})$, $J \in \mathbb{N}(\mathcal{F})$. Let $f\colon L^0(\mathcal{F}) \to L^0(\mathcal{F})$ be an upper semicontinuous conditional function. Then it holds that*

*(i).* $\operatorname{h-lim\,sup}_{J\in\mathbb{N}(\mathcal{F})} f_J \leq f$ *if and only if* $\operatorname{ess\,lim\,sup}_{J\in\mathbb{N}(\mathcal{F})} (\operatorname{ess\,sup}_{\mathcal{C}} f_J) \leq \operatorname{ess\,sup}_{\mathcal{C}} f$ *for all conditionally compact conditional subsets $\mathcal{C} \sqsubset L^0(\mathcal{F})$, and*

*(ii).* $\operatorname{h-lim\,inf}_{J\in\mathbb{N}(\mathcal{F})} f_J \geq f$ *if and only if* $\operatorname{ess\,lim\,inf}_{J\in\mathbb{N}(\mathcal{F})} (\operatorname{ess\,sup}_{\mathcal{O}} f_J) \geq \operatorname{ess\,sup}_{\mathcal{O}} f$ *for all conditionally open conditional subsets $\mathcal{O} \sqsubset L^0(\mathcal{F})$.*

*Proof.* The proof relies on the hit-and-miss-criteria in Theorem 2.19. We may use cylinders $\mathcal{C}(X,Y,\delta) := \operatorname{cl}\mathcal{B}^{\delta}(X) \times [Y-\delta, Y+\delta]$ for $X, Y \in L^0(\mathcal{F})$ and $\delta \in L^0(\mathcal{F})_{++}$ instead of balls to fulfill the hit-and-miss-creteria. Then, $\mathcal{B}^{\delta}(X,Y) \sqsubset \mathcal{C}(X,Y,\delta) \sqsubset \sqrt{2}\mathcal{B}^{\delta}(X,Y)$. We assume that $\operatorname{hypo} f$ lives on $\Omega$.

In (i), by definition, $\operatorname{h-lim\,sup}_{J\in\mathbb{N}(\mathcal{F})} f_J \leq f$ if and only if $\limsup_{J\in\mathbb{N}(\mathcal{F})} \operatorname{hypo} f_J \sqsubset \operatorname{hypo} f$. Assuming this, we show that $\limsup_{J\in\mathbb{N}(\mathcal{F})} (\operatorname{ess\,sup}_{\mathcal{C}} f_J) \leq \operatorname{ess\,sup}_{\mathcal{C}} f$ for conditionally compact $\mathcal{C} \sqsubset L^0(\mathcal{F})$. Let $\mathcal{C} \sqsubset L^0(\mathcal{F})$ be conditionally compact and $Y'$ such that $\operatorname{ess\,sup}_{\mathcal{C}} f < Y'$. By construction, $(\mathcal{C}, Y') \sqcap \operatorname{hypo} f = L^0(\mathcal{F})|0$ in $L^0(\mathcal{F}) \times L^0(\mathcal{F})$. By Thereom 2.19 (iii), there is $\mathcal{N} \in \mathbb{N}(\mathcal{F})_{\infty}$ such that $\operatorname{hypo} f_J \sqcap (\mathcal{C}, Y') = L^0(\mathcal{F})|0$ for all $J \in \mathcal{N}$. Then, $\operatorname{ess\,sup}_{\mathcal{C}} f \leq Y'$. Thus, $\operatorname{ess\,lim\,sup}_{J\in\mathbb{N}(\mathcal{F})} (\operatorname{ess\,sup}_{\mathcal{C}} f_J) \leq Y'$ for all $Y' > \operatorname{ess\,sup}_{\mathcal{C}} f$, hence, finally, $\operatorname{ess\,lim\,sup}_{J\in\mathbb{N}(\mathcal{F})} (\operatorname{ess\,sup}_{\mathcal{C}} f_J) \leq \operatorname{ess\,sup}_{\mathcal{C}} f$.

For the reverse implication in (i), let $\operatorname{ess\,lim\,sup}_{J\in\mathbb{N}(\mathcal{F})} (\operatorname{ess\,sup}_{\mathcal{C}} f_J) \leq \operatorname{ess\,sup}_{\mathcal{C}} f$ for conditionally compact $\mathcal{C} \sqsubset L^0(\mathcal{F})$. We show that $\limsup_{J\in\mathbb{N}(\mathcal{F})} \operatorname{hypo} f_J \sqsubset \operatorname{hypo} f$. Suppose a cylinder $\mathcal{C}(X,Y,\delta)$ such that $\mathcal{C}(X,Y,\delta) \sqcap \operatorname{hypo} f = L^0(\mathcal{F})|0$. By assumption, the conditional function $f$ is upper semicontinuous, hence, $\operatorname{hypo} f$ is conditionally closed. This implies $\operatorname{ess\,sup}_{\mathcal{B}^{\delta}(X)} f < Y - \delta$ by the definition of the cylinder. By assumption, $\operatorname{ess\,lim\,inf}_{J\in\mathbb{N}(\mathcal{F})} \left(\operatorname{ess\,sup}_{\mathcal{B}^{\delta}(X)} f_J\right) < Y - \delta$ for the conditionally compact ball $\operatorname{cl}\mathcal{B}^{\delta}(X)$. That means, there exists $\mathcal{N} \in \mathbb{N}(\mathcal{F})_{\infty}$ such that $\operatorname{ess\,sup}_{\mathcal{B}^{\delta}(X)} f_J < Y - \delta$ for all $J \in \mathcal{N}$. Then, $\mathcal{C}(X,Y,\delta) \sqcap \operatorname{hypo} f_J = L^0(\mathcal{F})|0$ for all $J \in \mathcal{N}$. Hence, $\limsup_{J\in\mathbb{N}(\mathcal{F})} \operatorname{hypo} f_J \sqsubset \operatorname{hypo} f$ by Theorem 2.19 (iv).

In (ii), by definition, $\text{h-}\liminf_{J\in\mathbb{N}(\mathcal{F})} f_J \geq f$ if and only if $\operatorname{hypo} f \sqsubset \liminf_{J\in\mathbb{N}(\mathcal{F})} \operatorname{hypo} f_J$. Assuming this, we show that $\operatorname{ess\,lim\,inf}_{J\in\mathbb{N}(\mathcal{F})} \left(\operatorname{ess\,sup}_{\mathcal{O}} f_J\right) \geq \operatorname{ess\,sup}_{\mathcal{O}} f$ for conditionally open $\mathcal{O} \sqsubset L^0(\mathcal{F})$. Let $\mathcal{O} \sqsubset L^0(\mathcal{F})$ be conditionally open and $Y'$ such that $\operatorname{ess\,sup}_{\mathcal{O}} f > Y'$. Now, $\mathcal{O}\times\{Y \mid Y < Y'\}$ is conditionally open and $(\mathcal{O}\times\{Y \mid Y < Y'\})\sqcap \operatorname{hypo} f$ lives on $\Omega$. By Theorem 2.19 (i), there is $\mathcal{N} \in \mathbb{N}(\mathcal{F})_\infty$ such that the conditional set $\operatorname{hypo} f_J \sqcap \mathcal{O}\times\{Y \mid Y < Y'\}$ lives on $\Omega$ for all $J \in \mathcal{N}$. Then, $\operatorname{ess\,sup}_{\mathcal{O}} f_J > Y'$. Thus, $\operatorname{ess\,lim\,inf}_{J\in\mathbb{N}(\mathcal{F})} \left(\operatorname{ess\,sup}_{\mathcal{O}} f_J\right) \geq Y'$ for all $Y' > \operatorname{ess\,sup}_{\mathcal{O}} f$, hence, we have that $\operatorname{ess\,lim\,inf}_{J\in\mathbb{N}(\mathcal{F})} \left(\operatorname{ess\,sup}_{\mathcal{O}} f_J\right) \geq \operatorname{ess\,sup}_{\mathcal{O}} f$.

For the reverse implication in (ii), let $\operatorname{ess\,lim\,inf}_{J\in\mathbb{N}(\mathcal{F})} \left(\operatorname{ess\,sup}_{\mathcal{O}} f_J\right) \geq \operatorname{ess\,sup}_{\mathcal{O}} f$ for conditionally open $\mathcal{O} \sqsubset L^0(\mathcal{F})$. We show that $\operatorname{hypo} f \sqsubset \liminf_{J\in\mathbb{N}(\mathcal{F})} \operatorname{hypo} f_J$. Suppose a conditionally open cylinder $\operatorname{int}\mathcal{C}(X, Y, \delta)$ such that $\operatorname{int}\mathcal{C}(X, Y, \delta) \sqcap \operatorname{hypo} f$ lives on $\Omega$. This yields $\operatorname{ess\,sup}_{\mathcal{B}^\delta(X)} f > Y - \delta$. By assumption, $\operatorname{ess\,lim\,inf}_{J\in\mathbb{N}(\mathcal{F})} \left(\operatorname{ess\,sup}_{\mathcal{B}^\delta(X)} f_J\right) > Y - \delta$ for the conditionally open ball $\mathcal{B}^\delta(X)$. That means, there exists $\mathcal{N} \in \mathbb{N}(\mathcal{F})_\infty$ such that $\operatorname{ess\,sup}_{\mathcal{B}^\delta(X)} f_J > Y - \delta$ for all $J \in \mathcal{N}$. Then, $\mathcal{C}(X, Y, \delta) \sqcap \operatorname{hypo} f_J$ lives on $\Omega$ for all $J \in \mathcal{N}$. Hence, $\operatorname{hypo} f \sqsubset \liminf_{J\in\mathbb{N}(\mathcal{F})} \operatorname{hypo} f_J$ by Theorem 2.19 (ii). $\square$

**Definition 2.28** ($\varepsilon$-optimality)**.** Let $f\colon L^0(\mathcal{F}) \to L^0(\mathcal{F})$ be a conditional function. We define

$$\begin{aligned}
\varepsilon\text{-argmin}\, f &:= \left\{X \in L^0(\mathcal{F}) \mid f(X) \leq \operatorname{ess\,inf} f + \varepsilon\right\}, \\
\varepsilon\text{-argmax}\, f &:= \left\{X \in L^0(\mathcal{F}) \mid f(X) \geq \operatorname{ess\,sup} f - \varepsilon\right\},
\end{aligned}$$

the $\varepsilon$-minimizer and $\varepsilon$-maximizer of $f$ for all $\varepsilon \in L^0(\mathcal{F})_{++}$.

**Proposition 2.29** (Outer limit of maxima is maximum)**.** *Let $(f_J)_{J\in\mathbb{N}(\mathcal{F})}$ be a conditional sequence of conditional functions $f_J\colon L^0(\mathcal{F}) \to L^0(\mathcal{F})$, $J \in \mathbb{N}(\mathcal{F})$. Let $f\colon L^0(\mathcal{F}) \to L^0(\mathcal{F})$ be an upper semicontinuous conditional function. If* $\text{h-}\liminf_{J\in\mathbb{N}(\mathcal{F})} f_J \geq f$ *then* $\operatorname{ess\,lim\,inf}_{J\in\mathbb{N}(\mathcal{F})} \left(\operatorname{ess\,sup} f_J\right) \geq \operatorname{ess\,sup} f$. *Furthermore, if $(\varepsilon_J)_{J\in\mathbb{N}(\mathcal{F})}$ is a conditional sequence in $L^0(\mathcal{F})_{++}$ with* $\lim_{J\in\mathbb{N}(\mathcal{F})} \varepsilon_J = 0$, *then*

$$\limsup_{J\in\mathbb{N}(\mathcal{F})} \left(\varepsilon_J\text{-argmax}\, f_J\right) \sqsubset \operatorname{argmax} f$$

*if, for $\mathcal{N} \in \mathbb{N}(\mathcal{F})^\#_\infty$ and every conditional sequence $(X_J)_{J\in\mathcal{N}}$ in $L^0(\mathcal{F})$ with* $\lim_{J\in\mathcal{N}} X_J = X$ *and* $X_J \in \varepsilon_J\text{-argmax}\, f_J$ *it holds that* $\lim_{J\in\mathcal{N}} f_J(X_J) = f(X)$.

*Proof.* By Theorem 2.27 (ii), $\operatorname{ess\,lim\,inf}_{J\in\mathbb{N}(\mathcal{F})} \left(\operatorname{ess\,sup} f_J\right) \geq \operatorname{ess\,sup} f$ since $L^0(\mathcal{F})$ is conditionally open. Now, assume a conditional sequence $(\varepsilon_J)_{J\in\mathbb{N}(\mathcal{F})}$ in $L^0(\mathcal{F})_{++}$ with

$\lim_{J\in\mathbb{N}(\mathcal{F})}\varepsilon_J = 0$ and for $\mathcal{N}\in\mathbb{N}(\mathcal{F})^{\#}_{\infty}$ and every conditional sequence $(X_J)_{J\in\mathcal{N}}$ in $L^0(\mathcal{F})$ with $\lim_{J\in\mathcal{N}} X_J = X$ and $X_J \in \varepsilon_{\mathrm{J}}\text{-argmax}\, f_J$ it holds that $\lim_{J\in\mathcal{N}} f_J(X_J) = f(X)$. Then,

$$f(X) = \lim_{J\in\mathcal{N}} f(X_J) \geq \lim_{J\in\mathcal{N}} (\operatorname{ess\,sup} f_J - \varepsilon_J) \geq \operatorname{ess}\liminf_{J\in\mathbb{N}(\mathcal{F})} (\operatorname{ess\,sup} f_J) \geq \operatorname{ess\,sup} f.$$

Hence, $X$ maximizes $f$. □

**Proposition 2.30.** *Let $f\colon L^0(\mathcal{F})\to L^0(\mathcal{F})$ be a conditional function. Let $(f_J)_{J\in\mathbb{N}(\mathcal{F})}$ be a conditional sequence of conditional functions $f_J\colon X\to L^0(\mathcal{F})$, $J\in\mathbb{N}(\mathcal{F})$. Let* $\text{h-}\lim_{J\in\mathbb{N}(\mathcal{F})} f_J = f$ *and* $\operatorname{ess\,sup} f \in L^0(\mathcal{F})$.

*(i).* $\lim_{J\in\mathbb{N}(\mathcal{F})}(\operatorname{ess\,sup} f_J) = \operatorname{ess\,sup} f$ *if and only if for every $\varepsilon\in L^0(\mathcal{F})_{++}$ there is a conditionally compact $\mathcal{C}\sqsubset L^0(\mathcal{F})$ and $\mathcal{N}\in\mathbb{N}(\mathcal{F})_{\infty}$ such that* $\operatorname{ess\,sup}_{\mathcal{C}} f_J \geq \operatorname{ess\,sup} f_J - \varepsilon$ *for all $J\in\mathcal{N}$.*

*(ii).* $\limsup_{J\in\mathbb{N}(\mathcal{F})}\left(\varepsilon\text{-argmax}\, f_J\right) \sqsubset \varepsilon\text{-argmax}\, f$ *for all $\varepsilon\in L^0(\mathcal{F})_{+}$ and if $(\varepsilon_J)_{J\in\mathbb{N}(\mathcal{F})}$ is a conditional sequence in $L^0(\mathcal{F})_{++}$ with* $\lim_{J\in\mathbb{N}(\mathcal{F})}\varepsilon_J = 0$ *then*

$$\limsup_{J\in\mathbb{N}(\mathcal{F})} (\varepsilon_{\mathrm{J}}\text{-argmax}\, f_J) \sqsubset \operatorname{argmax} f.$$

*Proof.* To show (i), let $\varepsilon\in L^0(\mathcal{F})_{++}$, $\mathcal{C}\sqsubset L^0(\mathcal{F})$ conditionally compact and $\mathcal{N}\in\mathbb{N}(\mathcal{F})_{\infty}$ such that $\operatorname{ess\,sup}_{\mathcal{C}} f_J \geq \operatorname{ess\,sup} f_J - \varepsilon$ for all $J\in\mathcal{N}$. Then, by Theorem 2.27 (i), it holds that

$$\operatorname{ess}\limsup_{J\in\mathcal{N}}(\operatorname{ess\,sup} f_J - \varepsilon) \leq \operatorname{ess}\limsup_{J\in\mathcal{N}}\left(\operatorname*{ess\,sup}_{\mathcal{C}} f_J\right) \leq \operatorname*{ess\,sup}_{\mathcal{C}} f \leq \operatorname{ess\,sup} f.$$

Since $\varepsilon\in L^0(\mathcal{F})_{++}$ has been chosen arbitrarily, we have $\operatorname{ess}\limsup_{J\in\mathcal{N}}(\operatorname{ess\,sup} f_J)\leq \operatorname{ess\,sup} f$. Further, it holds that $\operatorname{ess}\liminf_{J\in\mathbb{N}(\mathcal{F})}\left(\operatorname{ess\,sup} f_J\right)\geq \operatorname{ess\,sup} f$ by assumption that $\text{h-}\lim_{J\in\mathbb{N}(\mathcal{F})} f_J = f$ and Proposition 2.29, hence, $\lim_{J\in\mathbb{N}(\mathcal{F})}(\operatorname{ess\,sup} f_J) = \operatorname{ess\,sup} f$.

To continue with (i), let $\lim_{J\in\mathbb{N}(\mathcal{F})}(\operatorname{ess\,sup} f_J) = \operatorname{ess\,sup} f$. Further, let $\varepsilon\in L^0(\mathcal{F})_{++}$. Fix $X\in L^0(\mathcal{F})$ such that $f(X)\geq \operatorname{ess\,sup} f - \varepsilon$. Since $\text{h-}\lim_{J\in\mathbb{N}(\mathcal{F})} f_J = f$, by Lemma 2.23, there exists a conditional sequence $(X_J)_{J\in\mathbb{N}(\mathcal{F})}$ that converges to $X$ such that $\operatorname{ess}\liminf_{J\in\mathbb{N}(\mathcal{F})} f_J(X_J)\geq f(X)$. By the convergence, there is a conditionally compact $\mathcal{C}\sqsubset L^0(\mathcal{F})$ containing all $X_J$, $J\in\mathbb{N}(\mathcal{F})$. Then, it holds that $\operatorname{ess\,sup}_{\mathcal{C}} f_J \geq f_J(X_J)$ for all $J\in\mathbb{N}(\mathcal{F})$. By $\operatorname{ess}\liminf_{J\in\mathbb{N}(\mathcal{F})} f_J\left(X_J\right)\geq f(X)$, there is $\mathcal{N}\in\mathbb{N}(\mathcal{F})_{\infty}$ such that $f_J(X_J)\geq f(X)$ for all $J\in\mathcal{N}$. Hence, for all $J\in\mathcal{N}$, $\operatorname{ess\,sup}_{\mathcal{C}} f_J \geq \operatorname{ess\,sup} f - \varepsilon$.

Next, we show (ii). Let $\varepsilon\in L^0(\mathcal{F})_{+}$ and $X_J\in\varepsilon\text{-argmax}\, f_J$. If $\overline{X}\in L^0(\mathcal{F})$ is a cluster point of the conditional sequence $(X_J)_{J\in\mathbb{N}(\mathcal{F})}$, that is, there is $\mathcal{N}\in\mathbb{N}(\mathcal{F})^{\#}_{\infty}$ such that $\lim_{J\in\mathcal{N}} X_J=\overline{X}$, then, by h-$\lim_{J\in\mathcal{N}} f_J=f$ and Theorem 2.27 (i), it holds that $f\left(\overline{X}\right)\geq \operatorname{ess\,lim\,sup}_{J\in\mathcal{N}} f_J(X_J)$. Directly, $\operatorname{ess\,lim\,inf}_{J\in\mathcal{N}} f_J\left(X_J\right)\leq \operatorname{ess\,lim\,sup}_{J\in\mathcal{N}} f_J\left(X_J\right)$. By the assumption that $X_J\in\varepsilon$-argmax $f_J$, it further holds that $\operatorname{ess\,lim\,inf}_{J\in\mathcal{N}} f_J\left(X_J\right)\geq \operatorname{ess\,lim\,inf}_{J\in\mathcal{N}}\left(\operatorname{ess\,sup} f_J-\varepsilon\right)$. Furthermore, by Proposition 2.30 (i), it also holds that $\operatorname{ess\,lim\,inf}_{J\in\mathcal{N}}\left(\operatorname{ess\,sup} f_J-\varepsilon\right)\geq \operatorname{ess\,sup} f-\varepsilon$. Thus, $f\left(\overline{X}\right)\geq \operatorname{ess\,sup} f-\varepsilon$, $\overline{X}\in$ $\varepsilon$-argmin $f$ and $\limsup_{J\in\mathbb{N}(\mathcal{F})}\left(\varepsilon\text{-argmax} f_J\right)\sqsubset\varepsilon$-argmax $f$. Since $\varepsilon_1\leq\varepsilon_2$ for $\varepsilon_1,\varepsilon_2\in L^0(\mathcal{F})_+$ implies $\varepsilon_1$-argmax $f_J\sqsubset\varepsilon_2$-argmax $f_J$ for all $J\in\mathbb{N}(\mathcal{F})$ it holds for a conditional sequence $(\varepsilon_J)_{J\in\mathbb{N}(\mathcal{F})}$ in $L^0(\mathcal{F})_{++}$ with $\lim_{J\in\mathbb{N}(\mathcal{F})}\varepsilon_J=0$ that

$$\limsup_{J\in\mathbb{N}(\mathcal{F})}\left(\varepsilon_{\mathrm{J}}\text{-argmax} f_J\right)\sqsubset\bigsqcap_{J\in\mathbb{N}(\mathcal{F})}\varepsilon_{\mathrm{J}}\text{-argmax} f=\operatorname{argmax} f$$

which yields the claim. □

**Corollary 2.31.** *Let $f\colon L^0(\mathcal{F})\to\overline{L}^0(\mathcal{F})$ be a conditional function. Let $(f_J)_{J\in\mathbb{N}(\mathcal{F})}$ be a conditional sequence of conditional functions $f_J\colon L^0(\mathcal{F})\to L^0(\mathcal{F})$, $J\in\mathbb{N}(\mathcal{F})$. Let* h-$\lim_{J\in\mathbb{N}(\mathcal{F})} f_J=f$ *and* $\operatorname{ess\,sup} f\in L^0(\mathcal{F})$. *If there is $\mathcal{C}\sqsubset L^0(\mathcal{F})$ conditionally compact and $\overline{N}\in\mathbb{N}(\mathcal{F})$ such that* $\operatorname{ess\,sup}_{\mathcal{C}} f_J=\operatorname{ess\,sup} f_J$ *for all $J\geq\overline{N}$ then* $\lim_{J\in\mathbb{N}(\mathcal{F})}(\operatorname{ess\,sup} f_J)=\operatorname{ess\,sup} f$.

*Proof.* That is Proposition 2.30 (ii). □

## 2.7 Lopsided convergence

For the characterization of saddle points, we introduce the concept of lopsided convergence. The optimal points here are maxinf- or minsup-points. We characterize lopsided convergence by hypoconvergence in one variable and the values of the optimal points.

A bivariate conditional function $F\colon L^0(\mathcal{F})\times L^0(\mathcal{F})\to\overline{L}^0(\mathcal{F})$ maps each $(X,Y)\in L^0(\mathcal{F})\times L^0(\mathcal{F})$ to an element in $\overline{L}^0(\mathcal{F})$ and is a conditional function, that is, if $F\left(\sum_{i\in I}\mathbb{1}_{A_i}(X_i,Y_i)\right)=\sum_{i\in I}\mathbb{1}_{A_i}F(X_i,Y_i)$ for every partition $(A_i)_{i\in I}$ of $\Omega$ in $\mathcal{F}$ and all families $(X_i)_{i\in I}$ and $(Y_i)_{i\in I}$ in $L^0(\mathcal{F})$.

**Definition 2.32** (Max/essinf-, min/esssup-points)**.** Let $F\colon L^0(\mathcal{F})\times L^0(\mathcal{F})\to\overline{L}^0(\mathcal{F})$ be a conditional function. If $\overline{X}\in\operatorname{argmax}_{X\in L^0(\mathcal{F})}\Big(\operatorname{ess\,inf}_{Y\in L^0(\mathcal{F})} F(X,Y)\Big)$ we call $\overline{X}\in L^0(\mathcal{F})$ a max/inf-point. Furthermore, we call $\overline{X}\in L^0(\mathcal{F})$ a min/sup-point if $\overline{X}\in\operatorname{argmin}_{X\in L^0(\mathcal{F})}\Big(\operatorname{ess\,sup}_{Y\in L^0(\mathcal{F})} F(X,Y)\Big)$.

**Definition 2.33** (Lopsided convergence)**.** A conditional sequence $(F_J)_{J\in\mathbb{N}(\mathcal{F})}$ of bivariate conditional functions $F_J\colon L^0(\mathcal{F})\times L^0(\mathcal{F})\to\overline{L}^0(\mathcal{F})$, $J\in\mathbb{N}(\mathcal{F})$, converges lopsided to an $\overline{L}^0(\mathcal{F})$-valued bivariate conditional function $F\colon L^0(\mathcal{F})\times L^0(\mathcal{F})\to\overline{L}^0(\mathcal{F})$ if for all $(X,Y)\in L^0(\mathcal{F})\times L^0(\mathcal{F})$ holds that

$$\forall\,(X_J)_{J\in\mathbb{N}(\mathcal{F})}\to X\quad\exists\,(Y_J)_{J\in\mathbb{N}(\mathcal{F})}\to Y\colon\ \operatorname*{ess\,lim\,sup}_{J\in\mathbb{N}(\mathcal{F})}F_J(X_J,Y_J)\le F(X,Y)\,,\tag{2.22}$$

$$\exists\,(X_J)_{J\in\mathbb{N}(\mathcal{F})}\to X\quad\forall\,(Y_J)_{J\in\mathbb{N}(\mathcal{F})}\to Y\colon\ \operatorname*{ess\,lim\,inf}_{J\in\mathbb{N}(\mathcal{F})}F_J(X_J,Y_J)\ge F(X,Y)\,.\tag{2.23}$$

**Theorem 2.34.** *Let $\mathcal{C}\sqsubset L^0(\mathcal{F})$ be conditionally compact. A conditional sequence $(F_J)_{J\in\mathbb{N}(\mathcal{F})}$ of bivariate conditional functions $F_J\colon L^0(\mathcal{F})\times\mathcal{C}\to\overline{L}^0(\mathcal{F})$, $J\in\mathbb{N}(\mathcal{F})$, converges lopsided to a bivariate conditional function $F\colon L^0(\mathcal{F})\times\mathcal{C}\to\overline{L}^0(\mathcal{F})$ such that $\operatorname{ess\,inf}_{Y\in\mathcal{C}}F(X,Y)<\infty$ for all $X\in L^0(\mathcal{F})$. Then,* $\operatorname{h-lim}_{J\in\mathbb{N}(\mathcal{F})}(\operatorname{ess\,inf}_{Y\in\mathcal{C}}F_J(\cdot,Y))=\operatorname{ess\,inf}_{Y\in\mathcal{C}}F(\cdot,Y)$.

*Proof.* We define $g_J(X):=\operatorname{ess\,inf}_{Y\in\mathcal{C}}F_J(X,Y)$ and $g(X):=\operatorname{ess\,inf}_{Y\in\mathcal{C}}F(X,Y)$. As given in Lemma 2.24, we show $\operatorname{ess\,lim\,sup}_{J\in\mathbb{N}(\mathcal{F})}g_J\left(X_J\right)\le g(X)$ if $\lim_{J\in\mathbb{N}(\mathcal{F})}X_J=X$. We define $A_X:=\operatorname{ess\,sup}\{A\in\mathcal{F}\mid 1\!\!1_A g(X)=-\infty\}$. Thus, for $1\!\!1_{A_X^c}g(X)\in L^0(\mathcal{F})$, $\varepsilon\in L^0(\mathcal{F})_{++}$ and $Y_\varepsilon\in\varepsilon\text{-argmax}\,F(X,\cdot)$, by assumption of lopsided convergence (2.22), there exists a conditional sequence $(Y_J)_{J\in\mathbb{N}(\mathcal{F})}$ that converges to $Y_\varepsilon$ with the property finally $1\!\!1_{A_X^c}\operatorname{ess\,lim\,sup}_{J\in\mathbb{N}(\mathcal{F})}F_J\left(X_J,X_J\right)\le 1\!\!1_{A_X^c}F(X,Y_\varepsilon)$. Then

$$1\!\!1_{A_X^c}\operatorname*{ess\,lim\,sup}_{J\in\mathbb{N}(\mathcal{F})}g(X_J)\le 1\!\!1_{A_X^c}\operatorname*{ess\,lim\,sup}_{J\in\mathbb{N}(\mathcal{F})}F_J(X_J,Y_J)\le 1\!\!1_{A_X^c}F(X,Y_\varepsilon)\le 1\!\!1_{A_X^c}(g(X)+\varepsilon)$$

hence, since $\varepsilon$ has been chosen arbitrarily, it holds that $1\!\!1_{A_X^c}\operatorname{ess\,lim\,sup}_{J\in\mathbb{N}(\mathcal{F})}g_J\left(X_J\right)\le 1\!\!1_{A_X^c}g(X)$. Now, on $A_X$, for any $\overline{N}\in\mathbb{N}(\mathcal{F})$ there is $Y_{\overline{N}}$ such that $1\!\!1_{A_X}F\left(X,Y_{\overline{N}}\right)\le -1\!\!1_{A_X}\overline{N}$. Then, by the same assumption on lopsided convergence (2.22), there exists a conditional sequence $(Y_J)_{J\in\mathbb{N}(\mathcal{F})}\to Y_{\overline{N}}$ with $1\!\!1_{A_X}\operatorname{ess\,lim\,sup}_{J\in\mathbb{N}(\mathcal{F})}F_J\left(X_J,Y_J\right)\le 1\!\!1_{A_X}F\left(X,Y_{\overline{N}}\right)$. Then

$$1\!\!1_{A_X}\operatorname*{ess\,lim\,sup}_{J\in\mathbb{N}(\mathcal{F})}g(X_J)\le 1\!\!1_{A_X}\operatorname*{ess\,lim\,sup}_{J\in\mathbb{N}(\mathcal{F})}F_J(X_J,Y_J)\le 1\!\!1_{A_X}F\left(X,Y_{\overline{N}}\right)\le -1\!\!1_{A_X}\overline{N},$$

which holds for any $\overline{N}\in\mathbb{N}(\mathcal{F})$, thus, $1\!\!1_{A_X}\operatorname{ess\,lim\,sup}_{J\in\mathbb{N}(\mathcal{F})}g\left(X_J\right)=-1\!\!1_{A_X}\infty$.

To prove (2.23), we show that there exists a conditional sequence $(X_J)_{J\in\mathbb{N}(\mathcal{F})}\to X$ such that $\liminf_{J\in\mathbb{N}(\mathcal{F})}g_J(X_J)\ge g(X)$. On $A_X$, there is nothing to show. Thus, we assume $g(X)\in L^0(\mathcal{F})$. By the assumption of lopsided convergence (2.23), there exists a conditional sequence $(X_J)_{J\in\mathbb{N}(\mathcal{F})} \to X$ such that for all conditional sequences $(Y_J)_{J\in\mathbb{N}(\mathcal{F})} \to Y$ holds that $\operatorname{ess\,lim\,inf}_{J\in\mathbb{N}(\mathcal{F})} F_J(X_J, Y_J) \geq F(X,Y)$ implying that $\operatorname{ess\,lim\,inf}_{J\in\mathbb{N}(\mathcal{F})} F_J(X_J,\cdot) \geq F(X,\cdot)$. Again, $\operatorname{ess\,lim\,sup}_{J\in\mathbb{N}(\mathcal{F})} F_J\left(X_J, X_J\right) \leq F(X,Y)$ by lopsided convergence (2.22). Thus, the conditional sequence $(F_J(X_J,\cdot))_{J\in\mathbb{N}(\mathcal{F})}$ of conditional functions $F_J(X_J,\cdot) : L^0(\mathcal{F}) \to \overline{L}^0(\mathcal{F})$, $J\in\mathbb{N}(\mathcal{F})$ hypoconverges to $F(X,\cdot)$ by Lemma 2.24. By Proposition 2.30 it holds that $\lim_{J\in\mathbb{N}(\mathcal{F})} \operatorname{ess\,inf}_{Y\in\mathcal{C}} F_J(X_J,Y) = \operatorname{ess\,inf}_{Y\in\mathcal{C}} F(X,Y)$, and since $\mathcal{C} \sqsubset L^0(\mathcal{F})$ is conditionally compact, we finally conclude that $\operatorname{ess\,lim\,inf}_{J\in\mathbb{N}(\mathcal{F})} g_J(X_J) \geq g(X)$. Hence, the theorem holds. □

**Theorem 2.35.** *Let $\mathcal{B} \sqsubset L^0(\mathcal{F})$ and $\mathcal{C} \sqsubset L^0(\mathcal{F})$ be conditionally compact. A conditional sequence of bivariate conditional functions $F_J\colon \mathcal{B}\times\mathcal{C} \to \overline{L}^0(\mathcal{F})$, $J\in\mathbb{N}(\mathcal{F})$, converges lopsided to a bivariate conditional function $F\colon \mathcal{B}\times\mathcal{C} \to \overline{L}^0(\mathcal{F})$ such that $\operatorname{ess\,inf}_{Y\in\mathcal{C}} F(X,Y) < \infty$ for all $X\in\mathcal{B}$. If for all $J\in\mathbb{N}(\mathcal{F})$, $X_J$ is a max/essinf-point of $F_J$ and there is $\mathcal{N}\in\mathbb{N}(\mathcal{F})^{\#}_{\infty}$ such that $\lim_{J\in\mathcal{N}} X_J = \overline{X}$ then $\overline{X}$ is a max/essinf-point of $F$. Moreover, there is convergence of the values of the max/essinf-points*

$$\lim_{J\in\mathcal{N}} \left( \operatorname*{ess\,inf}_{X\in\mathcal{C}} F_J(X_J, Y) \right) = \operatorname*{ess\,inf}_{Y\in\mathcal{C}} F\left(\overline{X}, Y\right).$$

*Proof.* Defining $g_J(X) := \operatorname{ess\,inf}_{Y\in L^0(\mathcal{F})} F_J(X,Y)$ and $g(X) := \operatorname{ess\,inf}_{Y\in L^0(\mathcal{F})} F(X,Y)$, we observe that $\text{h-lim}_{J\in\mathbb{N}(\mathcal{F})} g_J = g$ by Theorem 2.34. Max/essinf-points of $F_J$ and $F$ are maximizers of $g_J$ and $g$. It holds that $\lim_{J\in\mathcal{N}} X_J = \overline{X}$, $X_J \in \operatorname{argmax} g_J$, thus, by Proposition 2.30, $\lim_{J\in\mathcal{N}} g_J(X_J) = g\left(\overline{X}\right)$. That shows the claim. □

## 2.8 Convexity

**Definition 2.36** (Conditionally convex sets, adapted from [DJKK16])**.** Let $\mathcal{C} \sqsubset L^0(\mathcal{F})$ be a conditional set. The conditional convex hull of $\mathcal{C}$ is

$$\operatorname{conv}(\mathcal{C}) := \left\{ \lambda X + (1-\lambda)\overline{X} \mid X, \overline{X} \in \mathcal{C},\ \lambda \in L^0(\mathcal{F}),\ 0 \leq \lambda \leq 1 \right\}.$$

If $\mathcal{C} = \operatorname{conv}(\mathcal{C})$ then $\mathcal{C}$ is called conditionally convex.

**Definition 2.37** (Conditional convex conditional functions, adapted from [CKV15])**.** Let $\mathcal{C} \sqsubset L^0(\mathcal{F})$ be a conditionally convex conditional subset. A conditional function $f\colon \mathcal{C} \to \overline{L}^0(\mathcal{F})$ is conditionally convex if

$$f\left(\lambda X + (1-\lambda)\overline{X}\right) \leq \lambda f(X) + (1-\lambda) f\left(\overline{X}\right) \tag{2.24}$$

for $\lambda \in L^0(\mathcal{F})$ with $0 \leq \lambda \leq 1$. The conditional function $f\colon \mathcal{C} \to \overline{L}^0(\mathcal{F})$ is conditionally strict convex if

$$f\left(\lambda X + (1-\lambda)\overline{X}\right) < \lambda f(X) + (1-\lambda) f\left(\overline{X}\right) \tag{2.25}$$

for $\lambda \in L^0(\mathcal{F})$ with $0 < \lambda < 1$. The conditional function is conditionally concave if $-f$ is conditionally convex.

In [FKV09], it is shown that any conditionally convex function $f\colon \mathcal{C} \to L^0(\mathcal{F})$ is a conditional function.

In the definition of conditional strict convexity we only consider strict inequality for $\lambda \in (0,1)$ since a stable combination of $X$ and $\overline{X}$ leads to equality for a conditional function $f$, thus, a strict inequality is never fullfilled.

**Proposition 2.38.** *Let $\mathcal{C} \sqsubset L^0(\mathcal{F})$ be a conditionally convex conditional subset. A conditional function $f\colon \mathcal{C} \to \overline{L}^0(\mathcal{F})$ is conditionally concave if and only if its hypograph* $\operatorname{hypo} f$ *is conditionally convex in $L^0(\mathcal{F}) \times L^0(\mathcal{F})$.*

*Proof.* Clearly, $\operatorname{hypo} f$ is conditionally convex if and only if $(X,Y), \left(\overline{X},\overline{Y}\right) \in \operatorname{hypo} f \sqcap \mathcal{C}$ and $\lambda \in [0,1]$ implies $\left(X^\lambda, Y^\lambda\right) := \lambda(X,Y) + (1-\lambda)\left(\overline{X},\overline{Y}\right) \in \operatorname{hypo} f \sqcap \mathcal{C}$. That is $f(X) \geq Y$ and $f\left(\overline{X}\right) \geq \overline{Y}$ implies $f\left(X^\lambda\right) \geq Y^\lambda$ which is the definition of conditional concavity of $f$ on $\mathcal{C}$. □

**Theorem 2.39.** *Let $f\colon L^0(\mathcal{F}) \to \overline{L}^0(\mathcal{F})$ be a conditionally concave conditional function. Then, the set* $\operatorname{argmax} f$ *is conditionally convex. Additionally, if $f$ is conditionally strict concave and there exists $X \in L^0(\mathcal{F})$ such that $f(X) < \infty$, the set* $\operatorname{argmax} f$ *consists of at maximum one point.*

*Proof.* If $\operatorname{ess\,sup} f > -\infty$ we consider $X, \overline{X} \in L^0(\mathcal{F})$ such that $f(X) = f\left(\overline{X}\right) = \operatorname{ess\,sup} f$. With $\lambda \in [0,1]$, we define $X^\lambda := \lambda X + (1-\lambda)\overline{X}$ and by conditional concavity, it holds that $f\left(X^\lambda\right) \geq \lambda \operatorname{ess\,sup} f + (1-\lambda)\operatorname{ess\,sup} f = \operatorname{ess\,sup} f$. Thus, $X^\lambda \in \operatorname{argmax} f$ and $\operatorname{argmax} f$ is conditionally convex.

If $f$ is conditionally strict concave assume that $X \in \operatorname{argmax} f$. Then, for any $\overline{X} \in L^0(\mathcal{F})$, $f\left(X^\lambda\right) > \lambda f\left(\overline{X}\right) + (1-\lambda) f(X) \geq f\left(\overline{X}\right)$. Thus, $\overline{X}$ is not a maximum, hence, $\operatorname{argmax} f$ is at maximum a singleton. □

**Proposition 2.40.** *Let $\left(\mathcal{C}_J\right)_{J \in \mathbb{N}(\mathcal{F})}$ be a conditional sequence of conditionally convex sets in $L^0(\mathcal{F})$. The inner limit set $\liminf_{J \in \mathbb{N}(\mathcal{F})} \mathcal{C}_J$ is conditionally convex, and, too, the limit set if it exists.*

*Proof.* Let $X, \overline{X} \in \liminf_{J\in\mathbb{N}(\mathcal{F})} \mathcal{C}_J$. Then, there exists $\mathcal{N} \in \mathbb{N}(\mathcal{F})_\infty$ with $\lim_{J\in\mathcal{N}} X_J = X$ and $\lim_{J\in\mathcal{N}} \overline{X}_J = \overline{X}$. For $\lambda \in [0,1]$, we define $X^\lambda := \lambda X + (1-\lambda)\overline{X}$ and $X_J^\lambda := \lambda X_J + (1-\lambda)\overline{X}_J$ for $J \in \mathcal{N}$. Then, $\lim_{J\in\mathcal{N}} X_J^\lambda = X^\lambda$, hence, $X^\lambda \in \liminf_{J\in\mathbb{N}(\mathcal{F})} \mathcal{C}_J$. □

**Theorem 2.41.** *Let $(f_J)_{J\in\mathbb{N}(\mathcal{F})}$ be a conditional sequence conditionally convex conditional functions $f_J\colon L^0(\mathcal{F}) \to L^0(\mathcal{F})$, $J \in \mathbb{N}(\mathcal{F})$. Then, the function* h-lim inf$_{J\in\mathbb{N}(\mathcal{F})} f_J$ *is conditionally concave, and, too,* h-lim$_{J\in\mathbb{N}(\mathcal{F})} f_J$ *if it exists.*

*Proof.* That is the definition 2.22 of the lower hypo limit and Proposition 2.40 for the inner limit of a conditional sequence of conditional sets. □

## 2.9 Equilibrium

In [AE06], an approach to deriving a KY FAN-inequality is presented. Here, we extend it to conditional theory. We examine conditions on bivariate condtional functions which yield to max/essinf-points and approximate max/essinf-points by continuous conditional functions.

**Lemma 2.42.** *Let $F\colon L^0(\mathcal{F}) \times L^0(\mathcal{F}) \to \overline{L}^0(\mathcal{F})$ be a conditional function. Then,*

$$\operatorname*{ess\,inf}_{Y\in L^0(\mathcal{F})} \operatorname*{ess\,sup}_{X\in L^0(\mathcal{F})} F(X,Y) \geq \operatorname*{ess\,sup}_{X\in L^0(\mathcal{F})} \operatorname*{ess\,inf}_{Y\in L^0(\mathcal{F})} F(X,Y).$$

*Proof.* Let $\mathcal{D} \sqsubset L^0(\mathcal{F})$ be a conditionally finite conditional subset of $L^0(\mathcal{F})$. Let $\mathcal{S} := \{\mathcal{D} \sqsubset L^0(\mathcal{F}) \mid \mathcal{D} \text{ is conditionally finite}\}$. Then, by definition,

$$\begin{aligned}\operatorname*{ess\,inf}_{Y\in L^0(\mathcal{F})} \operatorname*{ess\,sup}_{X\in L^0(\mathcal{F})} F(X,Y) &= \operatorname*{ess\,inf}_{Y'\in L^0(\mathcal{F})} \operatorname*{ess\,sup}_{X\in L^0(\mathcal{F})} \operatorname*{ess\,inf}_{Y\in\{Y'\}} F(X,Y)\\ &\geq \operatorname*{ess\,inf}_{\mathcal{D}\in\mathcal{S}} \operatorname*{ess\,sup}_{X\in L^0(\mathcal{F})} \operatorname*{ess\,inf}_{Y\in\mathcal{D}} F(X,Y).\end{aligned}$$

By $\operatorname{ess\,inf}_{Y\in\mathcal{D}} F(X,Y) \geq \operatorname{ess\,inf}_{Y\in L^0(\mathcal{F})} F(X,Y)$ for conditionally finite $\mathcal{D} \sqsubset L^0(\mathcal{F})$, it holds that

$$\operatorname*{ess\,sup}_{X\in L^0(\mathcal{F})} \operatorname*{ess\,inf}_{Y\in\mathcal{D}} F(X,Y) \geq \operatorname*{ess\,sup}_{X\in L^0(\mathcal{F})} \operatorname*{ess\,inf}_{Y\in L^0(\mathcal{F})} F(X,Y)$$

and thus,

$$\operatorname*{ess\,inf}_{\mathcal{D}\in\mathcal{S}} \operatorname*{ess\,sup}_{X\in L^0(\mathcal{F})} \operatorname*{ess\,inf}_{Y\in\mathcal{D}} F(X,Y) \geq \operatorname*{ess\,sup}_{X\in L^0(\mathcal{F})} \operatorname*{ess\,inf}_{Y\in L^0(\mathcal{F})} F(X,Y).$$

Hence, $\operatorname{ess\,inf}_{Y\in L^0(\mathcal{F})}\operatorname{ess\,sup}_{X\in L^0(\mathcal{F})} F(X,Y) \geq \operatorname{ess\,sup}_{X\in L^0(\mathcal{F})}\operatorname{ess\,inf}_{Y\in L^0(\mathcal{F})} F(X,Y)$. □

**Definition 2.43** (Conditionally inf-compact)**.** Let $f\colon L^0(\mathcal{F}) \to L^0(\mathcal{F})$ be a conditional function. It is conditionally inf-compact if the sets $\operatorname{cl}\left\{X \in L^0(\mathcal{F}) \mid f(X) \geq \alpha\right\}$ are conditionally compact for all $\alpha \in L^0(\mathcal{F})$.

That is the usual setting, for example as in Theorem 2.12.

**Theorem 2.44.** *Let $F\colon L^0(\mathcal{F}) \times L^0(\mathcal{F}) \to \overline{L}^0(\mathcal{F})$ be a bivariate conditional function. Assume that*

*(i). $\exists\, Y_0 \in L^0(\mathcal{F})$ such that $X \mapsto F(X,Y_0)$ is conditionally inf-compact,*

*(ii). $\forall\, Y \in L^0(\mathcal{F})$ the mapping $X \mapsto F(X,Y)$ is upper semicontinuous.*

*Then, again with $\mathcal{S} := \left\{\mathcal{D} \sqsubset L^0(\mathcal{F}) \mid \mathcal{D} \text{ is conditionally finite}\right\}$, it holds that*

$$\operatorname*{ess\,inf}_{\mathcal{D}\in\mathcal{S}}\ \operatorname*{ess\,sup}_{X\in L^0(\mathcal{F})}\ \operatorname*{ess\,inf}_{Y\in\mathcal{D}} F(X,Y) = \operatorname*{ess\,sup}_{X\in L^0(\mathcal{F})}\ \operatorname*{ess\,inf}_{Y\in L^0(\mathcal{F})} F(X,Y) \tag{2.26}$$

*and there exists $X' \in L^0(\mathcal{F})$ living on $\Omega$, such that*

$$\operatorname*{ess\,inf}_{Y\in L^0(\mathcal{F})} F\left(X',Y\right) = \operatorname*{ess\,inf}_{\mathcal{D}\in\mathcal{S}}\ \operatorname*{ess\,sup}_{X\in L^0(\mathcal{F})}\ \operatorname*{ess\,inf}_{Y\in\mathcal{D}} F(X,Y). \tag{2.27}$$

*Proof.* We introduce level sets depending on $Y \in L^0(\mathcal{F})$. Thus, we define

$$\mathcal{C}_Y := \left\{X \in L^0(\mathcal{F}) \mid F(X,Y) \geq \operatorname*{ess\,inf}_{\mathcal{D}\in\mathcal{S}}\ \operatorname*{ess\,sup}_{X\in L^0(\mathcal{F})}\ \operatorname*{ess\,inf}_{Y'\in\mathcal{D}} F\left(X,Y'\right)\right\}.$$

We first show a conditional finite intersection property. Let $\mathcal{D} \sqsubset L^0(\mathcal{F})$ be conditionally finite with $Y_0 \in \mathcal{D}$ living on $\Omega$. We will show that $\bigsqcap_{Y\in\mathcal{D}} \mathcal{C}_Y$ lives on $\Omega$. To see that, consider the conditional function $g_{\mathcal{D}}\colon L^0(\mathcal{F}) \to L^0(\mathcal{F})$ defined by $g_{\mathcal{D}}(X) := \min_{Y\in\mathcal{D}} F(X,Y)$. The minimum is attained since $\mathcal{D}$ is conditionally finite. By assumption (ii) for the attained minimum, $g_{\mathcal{D}}$ is upper semicontinuous. By Theorem 2.10, the level sets $\operatorname{lev}_{\geq\alpha} g_{\mathcal{D}}$ for $\alpha \in L^0(\mathcal{F})$ are conditionally closed. Since $\operatorname{cl}\operatorname{lev}_{\geq\alpha} F(X,Y_0) \sqsubset \operatorname{lev}_{\geq\alpha} g_{\mathcal{D}}$ by definition of $g$ and by assumption (i), the set $\operatorname{lev}_{\geq\alpha} g_{\mathcal{D}}$ is conditionally compact. For $\alpha \leq \operatorname{ess\,sup} g_{\mathcal{D}}$ that both live on $\Omega$, the level sets $\operatorname{lev}_{\geq\alpha} g_{\mathcal{D}}$ live on $\Omega$, and if $\alpha' \leq \alpha$ then $\operatorname{lev}_{\geq\alpha} g_{\mathcal{D}} \sqsubset \operatorname{lev}_{\geq\alpha'} g_{\mathcal{D}}$. The conditional intersection of conditionally compact nested sets is nonempty and lives on $\Omega$, cf. [DJKK16, Proposition 3.25]. Thus, $\bigsqcap_{\alpha\leq\operatorname{ess\,sup} g_{\mathcal{D}}} \operatorname{lev}_{\geq\alpha} g_{\mathcal{D}} = \operatorname{lev}_{\geq\operatorname{ess\,sup}_{g_{\mathcal{D}}}} g_{\mathcal{D}} = \operatorname{argmax} g_{\mathcal{D}}$ lives on $\Omega$. For any $X \in \operatorname{argmax} g_{\mathcal{D}}$, it holds that $X \in \mathcal{C}_Y$, hence, $\bigsqcap_{Y\in\mathcal{D}} \mathcal{C}_Y$ lives on $\Omega$.

Next, we show that $\bigsqcap_{Y\in L^0(\mathcal{F})}\mathcal{C}_Y$ lives on $\Omega$. For the proof, we follow an approach applied in [DJKK16]. Assume the contrary, $\bigsqcap_{Y\in L^0(\mathcal{F})}\mathcal{C}_Y = L^0(\mathcal{F})|0$ on $A\in\mathcal{F}$. Then, we observe that $\mathbb{1}_A\bigsqcup_{Y\in L^0(\mathcal{F})}(\mathcal{C}_Y\sqcap\mathcal{C}_{Y_0}) = \mathbb{1}_A\mathcal{C}_{Y_0}$. Since $\mathcal{C}_{Y_0}$ is conditionally compact from the first part of the proof, there exists a conditionally finite set $\mathcal{D}'$ such that $\mathbb{1}_A\bigsqcup_{Y\in\mathcal{D}'}(\mathcal{C}_Y\sqcap\mathcal{C}_{Y_0}) = \mathbb{1}_A\mathcal{C}_{Y_0}$, and further, $\mathbb{1}_A\bigsqcap_{Y\in(\mathcal{D}'\sqcup\{Y_0\})}\mathcal{C}_Y = L^0(\mathcal{F})|0$ in contradiction to the conditionally finite intersection property. For any $X'\in\bigsqcap_{Y\in L^0(\mathcal{F})}\mathcal{C}_Y$, it further holds that

$$\operatorname*{ess\,inf}_{Y\in L^0(\mathcal{F})} F\left(X',Y\right)\geq \operatorname*{ess\,inf}_{\mathcal{D}\in\mathcal{S}}\ \operatorname*{ess\,sup}_{X\in L^0(\mathcal{F})}\ \operatorname*{ess\,inf}_{Y\in\mathcal{D}} F(X,Y).$$

by definition of $\mathcal{C}_Y$. Finally, we observe that $\operatorname{ess\,sup}_{X\in L^0(\mathcal{F})}\operatorname{ess\,inf}_{Y\in L^0(\mathcal{F})}F(X,Y)\geq \operatorname{ess\,inf}_{Y\in L^0(\mathcal{F})}F(X',Y)\geq \operatorname{ess\,sup}_{X\in L^0(\mathcal{F})}\operatorname{ess\,inf}_{Y\in L^0(\mathcal{F})}F(X,Y)$ by Lemma 2.42. That shows the claim. □

**Definition 2.45.** We define the set $c\left(L^0(\mathcal{F}),L^0(\mathcal{F})\right)$ to the set of continuous conditional functions $f\colon L^0(\mathcal{F})\to L^0(\mathcal{F})$. The set $f\left(L^0(\mathcal{F}),L^0(\mathcal{F})\right)$ is the set of conditional functions $f\colon L^0(\mathcal{F})\to L^0(\mathcal{F})$.

**Lemma 2.46.** *Let $F\colon L^0(\mathcal{F})\times L^0(\mathcal{F})\to \overline{L}^0(\mathcal{F})$ be a bivariate conditional function. Then,*

$$\operatorname*{ess\,inf}_{g\in f(L^0(\mathcal{F}),L^0(\mathcal{F}))}\ \operatorname*{ess\,sup}_{X\in L^0(\mathcal{F})} F(X,g(X)) = \operatorname*{ess\,sup}_{X\in L^0(\mathcal{F})}\ \operatorname*{ess\,inf}_{Y\in L^0(\mathcal{F})} F(X,Y).$$

*Proof.* For all $\varepsilon\in L^0(\mathcal{F})_{++}$ and $X'\in L^0(\mathcal{F})$ there exists a conditional function $g\colon L^0(\mathcal{F})\to L^0(\mathcal{F})$ such that

$$\begin{aligned}\operatorname*{ess\,sup}_{X\in L^0(\mathcal{F})}\ \operatorname*{ess\,inf}_{Y\in L^0(\mathcal{F})} F(X,Y) &\geq \operatorname*{ess\,sup}_{Y\in L^0(\mathcal{F})} F\left(X',Y\right)\geq F\left(X',g\left(X'\right)\right)-\varepsilon\\ &\geq \operatorname*{ess\,inf}_{g\in f(L^0(\mathcal{F}),L^0(\mathcal{F}))}\ \operatorname*{ess\,sup}_{X'\in L^0(\mathcal{F})} F\left(X',g\left(X'\right)\right)-\varepsilon.\end{aligned}$$

Furthermore, it also holds that $\operatorname{ess\,inf}_{g\in f(L^0(\mathcal{F}),L^0(\mathcal{F}))}\operatorname{ess\,sup}_{X'\in L^0(\mathcal{F})}F(X',g(X'))\geq \operatorname{ess\,sup}_{X\in L^0(\mathcal{F})}\operatorname{ess\,inf}_{Y\in L^0(\mathcal{F})}F(X,Y)$ by Lemma 2.42. Since $\varepsilon\in L^0(\mathcal{F})_{++}$ has been chosen arbitrarily, the claim is proven. □

**Theorem 2.47.** *Let $F\colon L^0(\mathcal{F})\times L^0(\mathcal{F})\to\overline{L}^0(\mathcal{F})$ be a bivariate conditional function. Assume that*

*(i). $\exists\, Y_0\in L^0(\mathcal{F})$ such that $X\mapsto F(X,Y_0)$ is conditionally inf-compact,*

*(ii).* $\forall\, Y \in L^0(\mathcal{F}) : X \mapsto F(X,Y)$ *is upper semicontinuous,*

*(iii).* $\forall\, X \in L^0(\mathcal{F}) : Y \mapsto F(X,Y)$ *is conditionally convex.*

*Then,*

$$\operatorname*{ess\,inf}_{h\in c(L^0(\mathcal{F}),L^0(\mathcal{F})))}\ \operatorname*{ess\,sup}_{X\in L^0(\mathcal{F})} F(X,h(X)) = \operatorname*{ess\,sup}_{X\in L^0(\mathcal{F})}\ \operatorname*{ess\,inf}_{Y\in L^0(\mathcal{F})} F(X,Y).$$

*Proof.* We begin with the observation that

$$\begin{aligned}\operatorname*{ess\,inf}_{Y\in L^0(\mathcal{F})}\ \operatorname*{ess\,sup}_{X\in L^0(\mathcal{F})} F(X,Y) &\geq \operatorname*{ess\,inf}_{h\in c(L^0(\mathcal{F}),L^0(\mathcal{F}))}\ \operatorname*{ess\,sup}_{X\in L^0(\mathcal{F})} F(X,h(X))\\ &\geq \operatorname*{ess\,sup}_{X\in L^0(\mathcal{F})}\ \operatorname*{ess\,inf}_{Y\in L^0(\mathcal{F})} F(X,Y).\end{aligned}$$

Indeed, the left hand inequality follows from the fact that $\operatorname{ess\,sup}_{X\in L^0(\mathcal{F})} F(X,Y) \geq \operatorname{ess\,sup}_{X\in L^0(\mathcal{F})} F(X,h(X))$ for $h \in c\left(L^0(\mathcal{F}), L^0(\mathcal{F})\right)$, thus, $\operatorname{ess\,sup}_{X\in L^0(\mathcal{F})} F(X,Y) \geq \operatorname{ess\,inf}_{h\in c(L^0(\mathcal{F}),L^0(\mathcal{F}))} \operatorname{ess\,sup}_{X\in L^0(\mathcal{F})} F(X,h(X))$ and the application of the supremum of $Y \in L^0(\mathcal{F})$ on the left hand side since the right hand side is independent of $Y$. The right hand inequality follows by $F(X,h(X)) \geq \operatorname{ess\,inf}_{Y\in L^0(\mathcal{F})} F(X,Y)$ and applying first the infimum of $X \in L^0(\mathcal{F})$, then the supremum of $h \in c\left(L^0(\mathcal{F}), L^0(\mathcal{F})\right)$, where again the right hand side is independent of $h$.

To show the inequality

$$\operatorname*{ess\,sup}_{X\in L^0(\mathcal{F})}\ \operatorname*{ess\,inf}_{Y\in L^0(\mathcal{F})} F(X,Y) \geq \operatorname*{ess\,inf}_{h\in c(L^0(\mathcal{F}),L^0(\mathcal{F}))}\ \operatorname*{ess\,sup}_{X\in L^0(\mathcal{F})} F(X,h(X)),$$

let $\varepsilon \in L^0(\mathcal{F})_{++}$. Then, by Lemma 2.46, there exists a conditional function $g\colon L^0(\mathcal{F}) \to L^0(\mathcal{F})$ with $\operatorname{ess\,sup}_{X\in L^0(\mathcal{F})} F(X,g(X)) \leq \operatorname{ess\,sup}_{X\in L^0(\mathcal{F})} \operatorname{ess\,inf}_{Y\in L^0(\mathcal{F})} F(X,Y) + \varepsilon$. By (ii) and the definition of upper semicontinuity, there exist conditionally open neighbourhoods $(\mathcal{U}(X))_{X\in L^0(\mathcal{F})}$ such that $F(X,g(X)) \leq F(X',g(X)) + \varepsilon$ for all $X' \in \mathcal{U}(X)$. We consider $\mathcal{C}_0 := \left\{X \in L^0(\mathcal{F}) \mid F(X,Y_0) \geq \operatorname{ess\,sup}_{X\in L^0(\mathcal{F})} \operatorname{ess\,inf}_{Y\in L^0(\mathcal{F})} F(X,Y)\right\}$ which is a conditional subset of $L^0(\mathcal{F})$. It is conditionally compact by the same reasoning as in the proof of Theorem 2.44. Since $X \mapsto F(X,Y)$ is upper semicontinuous, the level set $\operatorname{lev}_{\geq \operatorname{ess\,sup}_{X\in L^0(\mathcal{F})} \operatorname{ess\,inf}_{Y\in L^0(\mathcal{F})} F(X,Y)} F(\cdot,Y_0)$ is conditionally closed (Theorem 2.10), and thus conditionally compact since $Y \mapsto F(X,Y)$ is condionally inf-compact. Thus, from the conditional open covering $(\mathcal{U}(X))_{X\in L^0(\mathcal{F})}$ of $\mathcal{C}_0$, that is, each $\mathcal{U}(X)$ is conditionally open and $\mathcal{C}_0 \sqsubset \bigsqcup_{X\in L^0(\mathcal{F})} \mathcal{U}_X$ we can choose a conditionally finite conditional open covering $(\mathcal{U}_J)_{1\leq J\leq N} := (\mathcal{U}(X_J))_{1\leq J\leq N}$ of $\mathcal{C}_0$ for some $N \in \mathbb{N}(\mathcal{F})$. With $\mathcal{U}_0 := \mathcal{C}_0^{\sqsubset}$, we have a conditional finite conditional open covering $(\mathcal{U}_J)_{0\leq J\leq N}$ of $L^0(\mathcal{F})$.

Now, we consider a continuous partition of unity $(p_J)_{0\leq J\leq N}$ (cf. Section 1.3) subordinate to this conditional finite covering which exists by Lemma 1.78. The conditional function $\overline{g}\colon L^0(\mathcal{F})\to L^0(\mathcal{F})$ defined by $\overline{g}(X):=p_0(X)Y_0+\sum_{1\leq J\leq N}p_J(X)g(X_J)$ is continuous since we have a continuous partition of unity and $F(X,g(X))\leq F(X',g(X))+\varepsilon$ for all $X'\in\mathcal{U}(X)$ for a neighbourhood $\mathcal{U}(X)$ of $X\in L^0(\mathcal{F})$ as already shown. By $p_J(X)\geq 0$, $\sum_{0\leq J\leq N}p_J(X)=1$ and since $Y\mapsto F(X,Y)$ is conditionally convex it holds that $F(X,\overline{g}(X))\leq p_0(X)F(X,Y_0)+\sum_{1\leq J\leq N}p_J(X)F(X,g(X_J))$. If we define $A_0:=\operatorname{ess\,sup}\{A\in\mathcal{F}\mid p_0(X)|A>0\}$ it holds that $\mathbb{1}_{A_0}X\in\mathbb{1}_{A_0}\mathcal{U}_0=\mathbb{1}_{A_0}\mathcal{C}_0^{\sqsubset}$ and therefore

$$\mathbb{1}_{A_0}F(X,Y_0)\leq\mathbb{1}_{A_0}\operatorname*{ess\,sup}_{X\in L^0(\mathcal{F})}\operatorname*{ess\,inf}_{Y\in L^0(\mathcal{F})}F(X,Y)<\mathbb{1}_{A_0}\operatorname*{ess\,sup}_{X\in L^0(\mathcal{F})}\operatorname*{ess\,inf}_{Y\in L^0(\mathcal{F})}F(X,Y)+\varepsilon. \tag{2.28}$$

On the other hand, for $A_J:=\operatorname{ess\,sup}\{A\in\mathcal{F}\mid p_J(X)|A>0\}$, it holds that $\mathbb{1}_{A_J}X\in\mathbb{1}_{A_J}\mathcal{U}_J$ and

$$\mathbb{1}_{A_J}F(X,g(X_J))\leq\mathbb{1}_{A_J}F(X_J,g(X_J))+\varepsilon\geq\mathbb{1}_{A_J}\operatorname*{ess\,sup}_{X\in L^0(\mathcal{F})}\operatorname*{ess\,inf}_{Y\in L^0(\mathcal{F})}F(X,Y)+\varepsilon.$$

By these inequalities and since $\sum_{0\leq J\leq N}p_J(X)=1$, it holds that

$$\begin{aligned}F(X,\overline{g}(X))&\leq p_0(X)F(X,Y_0)+\sum_{1\leq J\leq N}p_J(X)F(X,g(X_J))\\&\leq\sum_{0\leq J\leq N}p_J(X)\left(\operatorname*{ess\,sup}_{X\in L^0(\mathcal{F})}\operatorname*{ess\,inf}_{Y\in L^0(\mathcal{F})}F(X,Y)+\varepsilon\right)\\&=\operatorname*{ess\,sup}_{X\in L^0(\mathcal{F})}\operatorname*{ess\,inf}_{Y\in L^0(\mathcal{F})}F(X,Y)+\varepsilon.\end{aligned}$$

Consequently,

$$\operatorname*{ess\,inf}_{g\in c(L^0(\mathcal{F}),L^0(\mathcal{F}))}F(X,g(X))\leq F(X,\overline{g}(X))\leq\operatorname*{ess\,sup}_{X\in L^0(\mathcal{F})}\operatorname*{ess\,inf}_{Y\in L^0(\mathcal{F})}F(X,Y)+\varepsilon,$$

which shows the claim by letting $\varepsilon$ converge to zero. □

## 2.10 Ky Fan inequality in a conditional setting

**Theorem 2.48** (Conditional version of Brouwer Fixed Point Theorem)**.** *A continuous conditional function $f\colon\mathcal{K}\to\mathcal{K}$ such that $\mathcal{K}$ is a conditionally compact and $L^0(\mathcal{F})$-convex* *subset of* $L^0(\mathcal{F})^d$ *has a fixed point, that is there exists* $X \in \mathcal{K}$ *such that* $f(X) = X$.

*Proof.* The proof will be given in Section 2.11. □

**Theorem 2.49.** *Let* $\mathcal{C} \sqsubset L^0(\mathcal{F})^d$ *be a conditionally convex, conditionally compact conditional subset and* $\Phi\colon \mathcal{C} \times \mathcal{C} \to \overline{L}^0(\mathcal{F})$ *be a conditional function with* $\Phi(X,X) \geq 0$ *and*

*(i).* $X \mapsto \Phi(X,Y)$ *is upper semicontinuous for all* $Y \in \mathcal{C}$,

*(ii).* $Y \mapsto \Phi(X,Y)$ *is conditionally concave.*

*Then, there exists some* $\overline{X} \in \mathcal{C}$ *such that* $\operatorname{ess\,inf}_{Y\in\mathcal{C}} \Phi\left(\overline{X},Y\right) \geq \operatorname{ess\,inf}_{Y\in\mathcal{C}} \Phi(Y,Y)$.

*Proof.* We apply Theorem 2.44 and Theorem 2.47. Then, there exists $\overline{X} \in \mathcal{C}$ such that

$$\operatorname*{ess\,sup}_{X\in\mathcal{C}} \operatorname*{ess\,inf}_{Y\in\mathcal{C}} \Phi(X,Y) = \operatorname*{ess\,inf}_{Y\in\mathcal{C}} \Phi\left(\overline{X},Y\right) = \operatorname*{ess\,inf}_{g\in c(\mathcal{C},\mathcal{C})} \operatorname*{ess\,sup}_{X\in\mathcal{C}} \Phi(X, g(X)),$$

Since $\mathcal{C}$ is conditionally compact and $g\colon \mathcal{C} \to \mathcal{C}$ is a continuous conditional function there exists a fixed point $X' \in \mathcal{C}$ of $g$ by Chapter 2.11, hence,

$$\operatorname*{ess\,sup}_{X\in\mathcal{C}} \Phi(X, g(X)) \geq \Phi\left(X', g\left(X'\right)\right) = \Phi\left(X', X'\right) \geq \operatorname*{ess\,inf}_{Y\in\mathcal{C}} \Phi(Y,Y),$$

which shows the claim. □

The Ky Fan inequality 2.49 also implies the Brouwer Fixed Point Theorem 2.69. Let $\psi\colon \mathcal{C} \to \mathcal{C}$ be a continuous conditional function for a conditionally compact $\mathcal{C} \sqsubset L^0(\mathcal{F})^d$. Then, define $\Phi(X,Y) := \langle \psi(X) - X, Y - X\rangle$ and observe that $\Phi$ satisfies the conditions of Theorem 2.49. Hence, there exists $\overline{X} \in \mathcal{C}$ such that $\operatorname{ess\,sup}_{Y\in\mathcal{C}} \langle \psi\left(\overline{X}\right) - \overline{X}, Y - \overline{X}\rangle \leq 0$. Putting $Y = \psi\left(\overline{X}\right) \in \mathcal{C}$, we obtain that $\|\psi\left(\overline{X}\right) - \overline{X}\|^2 \leq 0$ and $\psi\left(\overline{X}\right) = \overline{X}$.

**Theorem 2.50** (Nash equilibrium in $L^0(\mathcal{F})$)**.** *There are* $n \in \mathbb{N}$ *agents. For all* $i \in \mathbb{N}$, *let there are conditional functions* $f_i\colon \mathcal{C}_i \times \prod_{j\neq i} \mathcal{C}_j \to L^0(\mathcal{F})$ *for conditionally compact conditional conditionally convex set* $\mathcal{C}_i \sqsubset L^0(\mathcal{F})^d$ *such that*

*(i).* $(X_i, Y_i) \mapsto f(X_i, Y_i)$ *is continuous*

*(ii).* $X_i \mapsto f(X_i, Y_i)$ *is conditionally convex for all* $Y_i \in \prod_{j\neq i} \mathcal{C}_j$.

*Then, there exists* $\overline{X} \in \prod_{i\leq n} \mathcal{C}_i$ *such that* $\operatorname{ess\,sup}_{Y\in L^0(\mathcal{F})^{d\times n}} \Phi\left(\overline{X},Y\right) = 0$ *and, furthermore,*

$$f_i\left(\overline{X}_i, \left(\overline{X}_1, \ldots, \overline{X}_{i-1}, \overline{X}_{i+1}, \ldots, \overline{X}_n\right)\right) = \min_{X_i\in\mathcal{C}_i} f_i\left(X_i, \left(\overline{X}_1, \ldots, \overline{X}_{i-1}, \overline{X}_{i+1}, \ldots, \overline{X}_n\right)\right)$$

*for all $i \leq n$.*

*Proof.* The conditional set $\prod_{i \leq n} \mathcal{C}_i$ is conditionally compact and conditionally convex by (i). Let $\Phi\colon \prod_{i \leq n} \mathcal{C}_i \times \prod_{i \leq n} \mathcal{C}_i \to L^0(\mathcal{F})$ defined by $\Phi\left(\left(X_1, \ldots, X_n\right), \left(Y_1, \ldots, Y_n\right)\right) := \sum_{i \leq n} \left(f_i\left(X_i, \left(X_1, \ldots, X_{i-1}, X_{i+1} \ldots, X_n\right)\right) - f_i\left(Y_i, \left(X_1, \ldots, X_{i-1}, X_{i+1} \ldots, X_n\right)\right)\right)$. For a conditional sequence $\left(X_1^J, \ldots, X_n^J\right)_{J \in \mathbb{N}(\mathcal{F})}$ in $\prod_{i \leq n} \mathcal{C}_i$ with $\lim_{J \in \mathbb{N}(\mathcal{F})} \left(X_1^J, \ldots, X_n^J\right) = \left(X_1, \ldots, X_n\right)$ and for all $Y \in \prod_{j \neq i} \mathcal{C}_j$, it holds that

$$
\begin{aligned}
&\operatorname*{ess\,lim\,inf}_{J \in \mathbb{N}(\mathcal{F})} \Phi\left(\left(X_1^J, \ldots, X_n^J\right), \left(Y_1, \ldots, Y_n\right)\right) \\
&= \operatorname*{ess\,lim\,inf}_{J \in \mathbb{N}(\mathcal{F})} \sum_{i \leq n} \left(f_i\left(X_i^J, \left(X_1^J, \ldots, X_{i-1}^J, X_{i+1}^J \ldots, X_n^J\right)\right)\right. \\
&\qquad \left. - f_i\left(Y_i, \left(X_1^J, \ldots, X_{i-1}^J, X_{i+1}^J \ldots, X_n^J\right)\right)\right) \\
&\geq \sum_{i \leq n} \operatorname*{ess\,lim\,inf}_{J \in \mathbb{N}(\mathcal{F})} f_i\left(X_i^J, \left(X_1^J, \ldots, X_{i-1}^J, X_{i+1}^J \ldots, X_n^J\right)\right) \\
&\qquad - \sum_{i \leq n} \operatorname*{ess\,lim\,sup}_{J \in \mathbb{N}(\mathcal{F})} f_i\left(Y_i, \left(X_1^J, \ldots, X_{i-1}^J, X_{i+1}^J \ldots, X_n^J\right)\right) \\
&= \sum_{i \leq n} f_i\left(X_i, \left(X_1, \ldots, X_{i-1}, X_{i+1} \ldots, X_n\right)\right) - \sum_{i \leq n} f_i\left(Y_i, \left(X_1, \ldots, X_{i-1}, X_{i+1} \ldots, X_n\right)\right) \\
&= \Phi\left(\left(X_1, \ldots, X_n\right), \left(Y_1, \ldots, Y_n\right)\right)
\end{aligned}
$$

by (i), that is, the conditional mapping $(X_1, \dots, X_n) \mapsto \Phi((X_1, \dots, X_n), (Y_1, \dots, Y_n))$ is lower semicontinuous for all $(Y_1, \dots, Y_n) \in \prod_{i \le n} \mathcal{C}_i$, and

$$\begin{aligned}
&\Phi\left((X_1, \dots, X_n), \lambda(Y_1, \dots, Y_n) + (1-\lambda)\left(Y_1', \dots, Y_n'\right)\right) \\
&= \sum_{i \le n} f_i\left(X_i, (X_1, \dots, X_{i-1}, X_{i+1} \dots, X_n)\right) \\
&\quad - \sum_{i \le n} f_i\left(\lambda Y_i + (1-\lambda) Y_i', (X_1, \dots, X_{i-1}, X_{i+1} \dots, X_n)\right) \\
&\le \lambda \sum_{i \le n} f_i\left(X_i, (X_1, \dots, X_{i-1}, X_{i+1} \dots, X_n)\right) \\
&\quad - \sum_{i \le n} \lambda f_i\left(Y_i, (X_1, \dots, X_{i-1}, X_{i+1} \dots, X_n)\right) \\
&\quad + (1-\lambda) \sum_{i \le n} f_i\left(X_i, (X_1, \dots, X_{i-1}, X_{i+1} \dots, X_n)\right) \\
&\quad - \sum_{i \le n} (1-\lambda) f_i\left(Y_i', (X_1, \dots, X_{i-1}, X_{i+1} \dots, X_n)\right) \\
&= \lambda \Phi\left((X_1, \dots, X_n), (Y_1, \dots, Y_n)\right) + (1-\lambda)\Phi\left((X_1, \dots, X_n), \left(Y_1', \dots, Y_n'\right)\right),
\end{aligned}$$

by (ii), that is, the conditional mapping $(Y_1, \dots, Y_n) \mapsto \Phi((X_1, \dots, X_n), (Y_1, \dots, Y_n))$ is conditionally concave. By Theorem 2.49, there exists $\overline{X} \in \prod_{i \le n} \mathcal{C}_i$ such that

$$\operatorname*{ess\,sup}_{Y \in L^0(\mathcal{F})^{d \times n}} \Phi\left(\overline{X}, Y\right) \le \operatorname*{ess\,sup}_{Y \in L^0(\mathcal{F})^{d \times n}} \Phi(Y, Y) = 0$$

by definition of $\Phi$. Furthermore, for fixed $i \le n$, we have that

$$\begin{aligned}
&f_i\left(\overline{X}_i, \left(\overline{X}_1, \dots, \overline{X}_{i-1}, \overline{X}_{i+1}, \dots, \overline{X}_n\right)\right) - f_i\left(X, \left(\overline{X}_1, \dots, \overline{X}_{i-1}, \overline{X}_{i+1}, \dots, \overline{X}_n\right)\right) \\
&= \Phi\left(\overline{X}, \left(\overline{X}_1, \dots, \overline{X}_{i-1}, X_i, \overline{X}_{i+1}, \dots, \overline{X}_n\right)\right) \le 0
\end{aligned}$$

if we choose $X = \left(\overline{X}_1, \dots, \overline{X}_{i-1}, X_i, \overline{X}_{i+1}, \dots, \overline{X}_n\right)$. That is the last part of the claim. □

## 2.11 Conditional Brouwer fixed point theorem

The following chapter is from [DKKS13]. It is given in in classical notation of random variables. It states a conditional version of Brouwers Fixed Point Theorem. The application we have in mind will be presented in Chapter 2.10. In view of Lemma 2.5, here, we state all results in almost-sure convergence of sequences.

### 2.11.1 Conditional simplex

We give the introduction from [DKKS13].

In the theory of real vector spaces the Brouwer fixed point theorem and corollaries are very useful tools in analysis. A continuous function from a simplex, or a compact and convex set in $\mathbb{R}^d$, to itself has a fixed point which is a point $x$ such that $f(x) = x$.

Cheridito, Filipović, Kupper and Vogelpoth ([CKV15], [FKV09]) examined properties of $(L^0)^d$ discussing concepts like linear independence, convex hull and sequential continuity of functions on $L^0$-modules. Consequently, given affine independence a conditional simplex can be defined in $(L^0)^d$. We obtain a fixed point for functions on conditional simplexes using a result analogue to Sperner's Lemma. To maintain a lot of nice uniform properties a simplex is subdivided barycentrically. Labeled in a measurable way we ensure that there exists a completely labeled simplex contained in the original one. Thus, we can construct a sequence of simplexes and we show that this converges to a point which has to be a fixed point. Working with a measurable labeling function the fixed point is measurable by construction. Hence, despite mainly following ideas and techniques from $\mathbb{R}^d$ (cf. [Bor99]) we do not need any measurable selection argument.

The fixed point theorem for conditional simplexes by hand we prove a fixed point result for $L^0$-convex, bounded and sequentially closed sets in $(L^0)^d$. At the end we present the implication of nice topological results, which are known from the real-valued case; the incontractibility of a ball to a sphere in $(L^0)^d$ and an intermediate value theorem in $L^0$.

In Probabilistic Analysis the problem of finding random fixed points of random operators is an important issue. Let $\mathcal{C}$ be a compact, convex set of a Banach space and $R\colon \Omega\times\mathcal{C} \to \mathcal{C}$ be a function such that

- $R(.,x)\colon \Omega \to \mathcal{C}$ is a random variable for any fixed $x \in \mathcal{C}$,
- $R(\omega,.)\colon \mathcal{C} \to \mathcal{C}$ is a continuous function for any fixed $\omega \in \Omega$, [1]

which is denoted by saying $R$ is a continuous random operator. Then there exists a random fixed point of $R$ which is a random variable $\xi\colon \Omega \to \mathcal{C}$ such that $\xi(\omega) = R(\omega, \xi(\omega))$ for any $\omega$ (cf. [BR76], [Sha01], [FMM09]). Our approach is completely within the theory of $L^0$ and hence all objects are defined in that language and proofs are done with $L^0$-methods. Therefore, conditional simplexes or sequentially bounded and closed sets are defined using elements of $L^0$-theory and not via fixing $\omega$. Moreover, although it is clear that a conditional function is a continuous random operator it is not clear that the opposite holds true. Also it is not certain that a conditional simplex $\mathcal{S}$

[1] There exist versions in which $\mathcal{C}$ depends on $\omega$ with the property $\omega \mapsto \mathcal{C}(\omega)$ is measurable.

can always be represented via normal simplexes $\mathcal{S}(\omega)$.

We introduce some terminology. The convex hull of $X_1, \dots, X_N \in L^0(\mathcal{F})^d$, $N \in \mathbb{N}$, is defined as

$$\operatorname{conv}(X_1, \dots, X_N) = \left\{ \sum_{i=1}^N \lambda_i X_i \mid \lambda_i \in L^0(\mathcal{F})_+, \sum_{i=1}^N \lambda_i = 1 \right\}.$$

An element $Y \in \operatorname{conv}(X_1, \dots, X_N)$ such that $\lambda_i > 0$ for all $i \in I \subset \{1, \dots, N\}$ is called a strict convex combination of $(X_i)_{i \in I}$. The diameter of $\mathcal{C} \subset L^0(\mathcal{F})^d$ is defined as $\operatorname{diam}(\mathcal{C}) = \operatorname{ess\,sup}_{X,Y \in \mathcal{C}} \|X - Y\|$.

**Definition 2.51.** Elements $X_1, \dots, X_N$ of $L^0(\mathcal{F})^d$, $N \in \mathbb{N}$, are said to be affinely independent, if either $N = 1$ or $N > 1$ and $\{X_i - X_N\}_{i=1}^{N-1}$ are linearly independent, that is

$$\sum_{i=1}^{N-1} \lambda_i (X_i - X_N) = 0 \quad \text{implies} \quad \lambda_1 = \dots = \lambda_{N-1} = 0, \tag{2.29}$$

where $\lambda_1, \dots, \lambda_{N-1} \in L^0(\mathcal{F})$.

The definition of affine independence is equivalent to

$$\sum_{i=1}^N \lambda_i X_i = 0 \ \text{ and } \sum_{i=1}^N \lambda_i = 0 \quad \text{implies} \quad \lambda_1 = \dots = \lambda_N = 0. \tag{2.30}$$

Indeed, first, we show that (2.29) implies (2.30). Let $\sum_{i=1}^N \lambda_i X_i = 0$ and $\sum_{i=1}^N \lambda_i = 0$. Then, $\sum_{i=1}^{N-1} \lambda_i (X_i - X_N) = \lambda_N X_N + \sum_{i=1}^{N-1} \lambda_i X_i = 0$. By assumption (2.29), $\lambda_1 = \dots = \lambda_{N-1} = 0$, thus also $\lambda_N = 0$. To see that (2.30) implies (2.29), let $\sum_{i=1}^{N-1} \lambda_i (X_i - X_N) = 0$. With $\lambda_N = -\sum_{i=1}^{N-1} \lambda_i$, it holds $\sum_{i=1}^N \lambda_i X_i = \lambda_N X_N + \sum_{i=1}^{N-1} \lambda_i X_i = \sum_{i=1}^{N-1} \lambda_i (X_i - X_N) = 0$. By assumption (2.30), $\lambda_1 = \dots = \lambda_N = 0$.

**Remark 2.52.** We observe that if $(X_i)_{i=1}^N \subset L^0(\mathcal{F})^d$ are affinely independent then $(\lambda X_i)_{i=1}^N$, for $\lambda \in L^0(\mathcal{F})_{++}$, and $(X_i + Y)_{i=1}^N$, for $Y \in L^0(\mathcal{F})^d$, are affinely independent. Moreover, if a family $X_1, \dots, X_N$ is affinely independent then also $1_B X_1, \dots, 1_B X_N$ are affinely independent on $B \in \mathcal{F}_+$, which means from $\sum_{i=1}^N 1_B \lambda_i X_i = 0$ and $\sum_{i=1}^N 1_B \lambda_i = 0$ always follows $1_B \lambda_i = 0$ for all $i = 1, \dots, N$.

**Definition 2.53.** A conditional simplex in $L^0(\mathcal{F})^d$ is a set of the form

$$\mathcal{S} = \operatorname{conv}(X_1, \dots, X_N)$$

such that $X_1, \dots, X_N \in L^0(\mathcal{F})^d$ are affinely independent. We call $N \in \mathbb{N}$ the dimension of $\mathcal{S}$.

**Remark 2.54.** The coefficients of convex combinations in a conditional simplex $\mathcal{S} = \operatorname{conv}(X_1, \dots, X_N)$ are unique in the sense that

$$\sum_{i=1}^N \lambda_i X_i = \sum_{i=1}^N \mu_i X_i \text{ and } \sum_{i=1}^N \lambda_i = \sum_{i=1}^N \mu_i = 1 \quad \text{implies} \quad \lambda_i = \mu_i \text{ for all } i = 1, \dots, N. \tag{2.31}$$

Indeed, assume the given convex combinations. Then $\sum_{i=1}^N (\lambda_i - \mu_i) X_i = 0$ with $\sum_{i=1}^N (\lambda_i - \mu_i) = 0$, and hence, by (2.30), $\lambda_i - \mu_i = 0$ for all $i$ since $X_1, \dots, X_N$ are affinely independent.

Since a conditional simplex is a convex hull it is in particular $\sigma$-stable. In contrast to a simplex in $\mathbb{R}^d$ the representation of $\mathcal{S}$ as a convex hull of affinely independent elements is unique but up to $\sigma$-stability.

**Proposition 2.55.** *Let $(X_i)_{i=1}^N$ and $(Y_i)_{i=1}^N$ be families in $L^0(\mathcal{F})^d$ with $\sigma(X_1, \dots, X_N) = \sigma(Y_1, \dots, Y_N)$. Then $\operatorname{conv}(X_1, \dots, X_N) = \operatorname{conv}(Y_1, \dots, Y_N)$. Moreover, $(X_i)_{i=1}^N$ are affinely independent if and only if $(Y_i)_{i=1}^N$ are affinely independent.*
*If $\mathcal{S}$ is a conditional simplex such that $\mathcal{S} = \operatorname{conv}(X_1, \dots, X_N) = \operatorname{conv}(Y_1, \dots, Y_N)$, then it holds $\sigma(X_1, \dots, X_N) = \sigma(Y_1, \dots, Y_N)$.*

*Proof.* Suppose $\sigma(X_1, \dots, X_N) = \sigma(Y_1, \dots, Y_N)$. For $i = 1, \dots, N$, it holds that

$$X_i \in \sigma(X_1, \dots, X_N) = \sigma(Y_1, \dots, Y_N) \subset \operatorname{conv}(Y_1, \dots, Y_N).$$

Therefore, $\operatorname{conv}(X_1, \dots, X_N) \subset \operatorname{conv}(Y_1, \dots, Y_N)$ and the reverse inclusion holds analogously.
Now, let $(X_i)_{i=1}^N$ be affinely independent and $\sigma(X_1, \dots, X_N) = \sigma(Y_1, \dots, Y_N)$. We want to show that $(Y_i)_{i=1}^N$ are affinely independent. To that end, we define the affine hull

$$\operatorname{aff}(X_1, \dots, X_N) = \left\{ \sum_{i=1}^N \lambda_i X_i \mid \lambda_i \in L^0(\mathcal{F}), \sum_{i=1}^N \lambda_i = 1 \right\}.$$

First, let $Z_1, \dots, Z_M \in L^0(\mathcal{F})^d$, $M \in \mathbb{N}$, such that $\sigma(X_1, \dots, X_N) = \sigma(Z_1, \dots, Z_M)$. We will show that if $X_1, \dots, X_N$ are affinely independent and $\mathbb{1}_A \operatorname{aff}(X_1, \dots, X_N) \subset \mathbb{1}_A \operatorname{aff}(Z_1, \dots, Z_M)$ for $A \in \mathcal{F}_+$ then $M \geq N$.

Since $X_i \in \sigma(X_1,\dots,X_N) = \sigma(Z_1,\dots,Z_M) \subset \operatorname{aff}(Z_1,\dots,Z_M)$, we conclude that $\operatorname{aff}(X_1,\dots,X_N) \subset \operatorname{aff}(Z_1,\dots,Z_M)$. Further, it holds that $X_1 = \sum_{i=1}^M 1_{B_i^1} Z_i$ for a partition $\left(B_i^1\right)_{i=1}^M$ and hence there exists at least one $B_{k_1}^1$ such that $A_{k_1}^1 := B_{k_1}^1 \cap A \in \mathcal{F}_+$, and $1_{A_{k_1}^1} X_1 = 1_{A_{k_1}^1} Z_{k_1}$. Therefore,

$$1_{A_{k_1}^1} \operatorname{aff}(X_1,\dots,X_N) \subset 1_{A_{k_1}^1} \operatorname{aff}(Z_1,\dots,Z_M) = 1_{A_{k_1}^1} \operatorname{aff}(\{X_1, Z_1,\dots,Z_M\} \setminus \{Z_{k_1}\}).$$

For $X_2 = \sum_{i=1}^M 1_{A_i^2} Z_i$ we find a set $A_k^2$, such that $A_{k_2}^2 = A_k^2 \cap A_{k_1}^1 \in \mathcal{F}_+$, $1_{A_{k_2}^2} X_2 = 1_{A_{k_2}^2} Z_{k_2}$ and $k_1 \neq k_2$. Assume to the contrary $k_2 = k_1$, then there exists a set $B \in \mathcal{F}_+$, such that $1_B X_1 = 1_B X_2$ which is a contradiction to the affine independence of $(X_i)_{i=1}^N$. Hence, we can again substitute $Z_{k_2}$ by $X_2$ on $A_{k_2}^2$. Inductively, we find $k_1,\dots,k_N$ such that

$$1_{A_{k_N}} \operatorname{aff}(X_1,\dots,X_N) \subset 1_{A_{k_N}} \operatorname{aff}(\{X_1,\dots,X_N, Z_1,\dots,Z_M\} \setminus \{Z_{k_1},\dots Z_{k_N}\})$$

which shows $M \geq N$. Now suppose $Y_1,\dots,Y_N$ are not affinely independent. This means, there exist $(\lambda_i)_{i=1}^N$ such that $\sum_{i=1}^N \lambda_i Y_i = \sum_{i=1}^N \lambda_i = 0$ but not all coefficients $\lambda_i$ are zero, without loss of generality, $\lambda_1 > 0$ on $A \in \mathcal{F}_+$. Thus, $1_A Y_1 = -1_A \sum_{i=2}^N \frac{\lambda_i}{\lambda_1} Y_i$ and it holds $1_A \operatorname{aff}(Y_1,\dots,Y_N) = 1_A \operatorname{aff}(Y_2,\dots,Y_N)$. To see this, consider $1_A Z = 1_A \sum_{i=1}^N \mu_i Y_i \in 1_A \operatorname{aff}(Y_1,\dots,Y_N)$ which means $1_A \sum_{i=1}^N \mu_i = 1_A$. Thus, inserting for $Y_1$,

$$1_A Z = 1_A \left[\sum_{i=2}^N \mu_i Y_i - \mu_1 \sum_{i=2}^N \frac{\lambda_i}{\lambda_1} Y_i\right] = 1_A \left[\sum_{i=2}^N \left(\mu_i - \mu_1 \frac{\lambda_i}{\lambda_1}\right) Y_i\right].$$

Moreover,

$$\begin{aligned} 1_A \left[\sum_{i=2}^N \left(\mu_i - \mu_1 \frac{\lambda_i}{\lambda_1}\right)\right] &= 1_A \left[\sum_{i=2}^N \mu_i\right] + 1_A \left[-\frac{\mu_1}{\lambda_1} \sum_{i=2}^N \lambda_i\right] \\ &= 1_A (1-\mu_1) + 1_A \frac{\mu_1}{\lambda_1} \lambda_1 = 1_A. \end{aligned}$$

Hence, $1_A Z \in 1_A \operatorname{aff}(Y_2,\dots,Y_N)$. It follows that

$$1_A \operatorname{aff}(X_1,\dots,X_N) = 1_A \operatorname{aff}(Y_1,\dots,Y_N) = 1_A \operatorname{aff}(Y_2,\dots,Y_N).$$

This is a contradiction to the former part of the proof (because $N-1 \ngeq N$).

Next, we characterize extremal points in $\mathcal{S} = \operatorname{conv}(X_1,\dots,X_N)$. We show $X \in \sigma(X_1,\dots,X_N)$ if and only if there do not exist $Y$ and $Z$ in $\mathcal{S} \setminus \{X\}$ and $\lambda \in (0,1)$ such that $\lambda Y + (1-\lambda) Z = X$. Consider $X \in \sigma(X_1, \dots, X_N)$ which is $X = \sum_{k=1}^N 1\!\!1_{A_k} X_k$ for a partition $(A_k)_{k\in\mathbb{N}}$. Now assume to the contrary that we find $Y = \sum_{k=1}^N \lambda_k X_k$ and $Z = \sum_{k=1}^N \mu_k X_k$ in $\mathcal{S} \setminus \{X\}$ such that $X = \lambda Y + (1-\lambda) Z$. This means that $X = \sum_{k=1}^N (\lambda\lambda_k + (1-\lambda)\mu_k) X_k$. Due to uniqueness of the coefficients (cf. (2.31)) in a conditional simplex we have $\lambda\lambda_k + (1-\lambda)\mu_k = 1\!\!1_{A_k}$ for all $k = 1 \dots, N$. By means of $0 < \lambda < 1$, it holds that $\lambda\lambda_k + (1-\lambda)\mu_k = 1\!\!1_{A_k}$ if and only $\lambda_k = \mu_k = 1\!\!1_{A_k}$. Since the last equality holds for all $k$ it follows that $Y = Z = X$. Therefore, we cannot find $Y$ and $Z$ in $\mathcal{S} \setminus \{X\}$ such that $X$ is a strict convex combination of them. On the other hand, consider $X \in \mathcal{S}$ such that $X \notin \sigma(X_1, \dots, X_N)$. This means, $X = \sum_{k=1}^N \nu_k X_k$, such that there exist $\nu_{k_1}$ and $\nu_{k_2}$ and $B \in \mathcal{F}_+$ with $0 < \nu_{k_1} < 1$ on $B$ and $0 < \nu_{k_2} < 1$ on $B$. Define $\varepsilon := \operatorname{ess\,inf}\{\nu_{k_1}, \nu_{k_2}, 1-\nu_{k_1}, 1-\nu_{k_2}\}$. Then define $\mu_k = \lambda_k = \nu_k$ if $k_1 \neq k \neq k_2$ and $\lambda_{k_1} = \nu_{k_1} - \varepsilon$, $\lambda_{k_2} = \nu_{k_2} + \varepsilon$, $\mu_{k_1} = \nu_{k_1} + \varepsilon$ and $\mu_{k_2} = \nu_{k_2} - \varepsilon$. Thus, $Y = \sum_{k=1}^N \lambda_k X_k$ and $Z = \sum_{k=1}^N \mu_k X_k$ fulfill $0.5Y + 0.5Z = X$ but both are not equal to $X$ by construction. Hence, $X$ can be written as a strict convex combination of elements in $\mathcal{S} \setminus \{X\}$. To conclude, consider $X \in \sigma(X_1, \dots, X_N) \subset \mathcal{S} = \operatorname{conv}(X_1, \dots, X_N) = \operatorname{conv}(Y_1, \dots, Y_N)$. Since $X \in \sigma(X_1, \dots, X_N)$ it is not a strict convex combinations of elements in $\mathcal{S}\setminus\{X\}$, in particular, of elements in $\operatorname{conv}(Y_1, \dots, Y_N)\setminus\{X\}$. Therefore, $X$ is also in $\sigma(Y_1, \dots, Y_N)$. Hence, $\sigma(X_1, \dots, X_N) \subset \sigma(Y_1, \dots, Y_N)$. With the same argumentation the other inclusion follows. □

As an example consider $[0,1]$. For an arbitrary $A \in \mathcal{F}$, it holds that $1\!\!1_A$ and $1\!\!1_{A^c}$ are affinely independent and $\operatorname{conv}(1\!\!1_A, 1\!\!1_{A^c}) = \{\lambda 1\!\!1_A + (1-\lambda) 1\!\!1_{A^c} : 0 \le \lambda \le 1\} = [0,1]$. Thus, the simplex $[0,1]$ can be written as a convex combination of different affinely independent elements of $L^0(\mathcal{F})$. This is due to the fact that $\sigma(0,1) = \{1\!\!1_B \mid B \in \mathcal{F}\} = \sigma(1\!\!1_A, 1\!\!1_{A^c})$ for any $A \in \mathcal{F}$.

**Remark 2.56.** In $L^0(\mathcal{F})^d$, let $e_i$ be the random variable which is 1 in the $i$-th component and 0 in any other. Then the family $0, e_1, \dots, e_d$ is affinely independent and $L^0(\mathcal{F})^d = \operatorname{aff}(0, e_1, \dots, e_d)$. Hence, the maximal number of affinely independent elements in $L^0(\mathcal{F})^d$ is $d+1$.

The characterization of $X \in \sigma(X_1, \dots, X_N)$ leads to the following definition.

**Definition 2.57.** Let $\mathcal{S} = \operatorname{conv}(X_1, \dots, X_N)$ be a conditional simplex. We define the set $\operatorname{ext}(\mathcal{S}) = \sigma(X_1, \dots, X_N)$ of extremal points. For an index set $I$ and a collection $\mathfrak{S} = (\mathcal{S}_i)_{i\in I}$ of simplexes we denote $\operatorname{ext}(\mathfrak{S}) = \sigma(\operatorname{ext}(\mathcal{S}_i) \mid i \in I)$.

**Remark 2.58.** Let $\mathcal{S}^j = \operatorname{conv}\left(X_1^j, \dots, X_N^j\right)$, $j \in \mathbb{N}$, be conditional simplexes of the same dimension $N$ and $(A_j)_{j\in\mathbb{N}}$ a partition. Then $\sum_{j\in\mathbb{N}} 1\!\!1_{A_j} \mathcal{S}^j$ is again a simplex.

To that end, we define $Y_k = \sum_{j\in\mathbb{N}} 1\!\!1_{A_j} X_k^j$. Then $\sum_{j\in\mathbb{N}} 1\!\!1_{A_j}\mathcal{S}^j = \operatorname{conv}(Y_1,\dots,Y_N)$. Indeed,

$$\sum_{k=1}^N \lambda_k Y_k = \sum_{k=1}^N \lambda_k \sum_{j\in\mathbb{N}} 1\!\!1_{A_j} X_k^j = \sum_{j\in\mathbb{N}} 1\!\!1_{A_j} \sum_{k=1}^N \lambda_k X_k^j \in \sum_{j\in\mathbb{N}} 1\!\!1_{A_j}\mathcal{S}^j, \tag{2.32}$$

shows $\operatorname{conv}(Y_1,\dots,Y_N) \subset \sum_{j\in\mathbb{N}} 1\!\!1_{A_j}\mathcal{S}^j$. Consider $\sum_{k=1}^N \lambda_k^j X_k^j$ in $\mathcal{S}^j$ and define $\lambda_k = \sum_{j\in\mathbb{N}} 1\!\!1_{A_j}\lambda_k^j$ yields the other inclusion.
To show that $Y_1,\dots,Y_N$ are affinely independent consider $\sum_{k=1}^N \lambda_k Y_k = 0 = \sum_{k=1}^N \lambda_k$. Then by (2.32), it holds $1\!\!1_{A_j}\sum_{k=1}^N \lambda_k X_k^j = 0$ and since $\mathcal{S}^j$ is a simplex, $1\!\!1_{A_j}\lambda_k = 0$ for all $j \in \mathbb{N}$ and $k = 1,\dots,N$. From the fact that $(A_j)_{j\in\mathbb{N}}$ is a partition, it follows that $\lambda_k = 0$ for all $k = 1,\dots,N$.

We will prove the Brouwer fixed point theorem in our setting using an analogue version of Sperner's Lemma. As in the unconditional case we have to subdivide a simplex in smaller ones. For our argumentation we cannot use arbitrary subdivisions and need very special properties of the simplexes in which we subdivide. This leads to the following definition.

**Definition 2.59.** Let $\mathcal{S} = \operatorname{conv}(X_1,\dots,X_N)$ be a conditional simplex and $\mathrm{S_N}$ the group of permutations of $\{1,\dots,N\}$. Then for $\pi \in \mathrm{S_N}$ we define

$$\mathcal{C}_\pi = \operatorname{conv}\left(X_{\pi(1)}, \frac{X_{\pi(1)}+X_{\pi(2)}}{2},\dots,\frac{X_{\pi(1)}+\dots+X_{\pi(k)}}{k},\dots,\frac{X_{\pi(1)}+\dots+X_{\pi(N)}}{N}\right).$$

We call $(\mathcal{C}_\pi)_{\pi\in\mathrm{S_N}}$ the barycentric subdivision of $\mathcal{S}$, and denote $Y_k^\pi = \frac{1}{k}\sum_{i=1}^k X_{\pi(i)}$.

**Lemma 2.60.** *The barycentric subdivision is a collection of finitely many conditionally simplexes satisfying the following properties*

*(i).* $\sigma\left(\bigcup_{\pi\in\mathrm{S_N}} \mathcal{C}_\pi\right) = \mathcal{S}$.

*(ii).* $\mathcal{C}_\pi$ *has dimension* $N$, $\pi \in \mathrm{S_N}$.

*(iii).* $\mathcal{C}_\pi \cap \mathcal{C}_{\overline{\pi}}$ *is a conditional simplex of dimension* $r \in \mathbb{N}$ *and* $r < N$ *for* $\pi, \overline{\pi} \in \mathrm{S_N}$, $\pi \neq \overline{\pi}$.

*(iv). For* $s = 1,\dots,N-1$, *let* $\mathcal{B}_s := \operatorname{conv}(X_1,\dots,X_s)$. *All simplexes* $\mathcal{C}_\pi \cap \mathcal{B}_s$, $\pi \in \mathrm{S_N}$, *of dimension* $s$ *subdivide* $\mathcal{B}_s$ *barycentrically.*

*Proof.* We show the affine independence of $Y_1^\pi, \dots, Y_N^\pi$ in $\mathcal{C}_\pi$. It holds

$$\lambda_{\pi(1)} X_{\pi(1)} + \lambda_{\pi(2)} \frac{X_{\pi(1)} + X_{\pi(2)}}{2} + \dots + \lambda_{\pi(N)} \frac{\sum_{k=1}^N X_{\pi(k)}}{N} = \sum_{i=1}^N \mu_i X_i,$$

with $\mu_i = \sum_{k=\pi^{-1}(i)}^N \frac{\lambda_{\pi(k)}}{k}$.

Since $\sum_{i=1}^N \mu_i = \sum_{i=1}^N \lambda_i$, the affine independence of $Y_1^\pi, \dots, Y_N^\pi$ is obtained by the affine independence of $X_1, \dots, X_N$. Therefore all $\mathcal{C}_\pi$ are conditional simplexes.

The intersection of two simplexes $\mathcal{C}_\pi$ and $\mathcal{C}_{\overline{\pi}}$ can be expressed in the following manner. Let $J = \{j \mid \{\pi(1), \dots, \pi(j)\} = \{\overline{\pi}(1), \dots, \overline{\pi}(j)\}\}$ be the set of indexes up to which both $\pi$ and $\overline{\pi}$ have the same set of images. Then,

$$\mathcal{C}_\pi \cap \mathcal{C}_{\overline{\pi}} = \operatorname{conv}\left( \frac{\sum_{k=1}^j X_{\pi(k)}}{j} \mid j \in J \right). \tag{2.33}$$

To show $\mathcal{C}_\pi \cap \mathcal{C}_{\overline{\pi}} \supset \operatorname{conv}\left( \frac{\sum_{k=1}^j X_{\pi(k)}}{j} \mid j \in J \right)$, let $j \in J$. It holds that $\frac{\sum_{k=1}^j X_{\pi(k)}}{j}$ is in both $\mathcal{C}_\pi$ and $\mathcal{C}_{\overline{\pi}}$ because $\{\pi(1), \dots, \pi(j)\} = \{\overline{\pi}(1), \dots, \overline{\pi}(j)\}$. Since the intersection of convex sets is convex, we get this implication.

For the reverse inclusion, let $X \in \mathcal{C}_\pi \cap \mathcal{C}_{\overline{\pi}}$. Since $X \in \mathcal{C}_\pi$, it is of the form $X = \sum_{i=1}^N \lambda_i \left( \sum_{k=1}^i \frac{X_{\pi(k)}}{i} \right)$ and for $X \in \mathcal{C}_{\overline{\pi}}$, it can be written as $X = \sum_{i=1}^N \mu_i \left( \sum_{k=1}^i \frac{X_{\overline{\pi}(k)}}{i} \right)$. Consider $j \notin J$. By definition of $J$, there exist $p, q \le j$ with $\overline{\pi}^{-1}(\pi(p)), \pi^{-1}(\overline{\pi}(q)) \notin \{1, \dots, j\}$. By (2.31), the coefficients of $X_{\pi(p)}$ are equal: $\sum_{i=p}^N \frac{\lambda_i}{i} = \sum_{i=\overline{\pi}^{-1}(\pi(p))}^N \frac{\mu_i}{i}$. The same holds for $X_{\pi(q)}$: $\sum_{i=q}^N \frac{\mu_i}{i} = \sum_{i=\pi^{-1}(\overline{\pi}(q))}^N \frac{\lambda_i}{i}$. Put together

$$\sum_{i=j+1}^N \frac{\mu_i}{i} \le \sum_{i=q}^N \frac{\mu_i}{i} = \sum_{i=\pi^{-1}(\overline{\pi}(q))}^N \frac{\lambda_i}{i} \le \sum_{i=j+1}^N \frac{\lambda_i}{i} \le \sum_{i=p}^N \frac{\lambda_i}{i} = \sum_{i=\overline{\pi}^{-1}(\pi(p))}^N \frac{\mu_i}{i} \le \sum_{i=j+1}^N \frac{\mu_i}{i}$$

which is only possible if $\mu_j = \lambda_j = 0$ since $p, q \le j$.

Furthermore, if $\mathcal{C}_\pi \cap \mathcal{C}_{\overline{\pi}}$ is of dimension $N$ by (2.33) follows that $\pi = \overline{\pi}$. This shows (iii).

As for Condition (i), it clearly holds $\sigma(\cup_{\pi \in \mathrm{S_N}} \mathcal{C}_\pi) \subset \mathcal{S}$. On the other hand, let $X = \sum_{i=1}^N \lambda_i X_i \in \mathcal{S}$. Then, cf. [Dra10], we find a partition $(A_n)_{n \in \mathbb{N}}$ such that on every $A_n$ the indexes are completely ordered which is $\lambda_{i_1^n} \ge \lambda_{i_2^n} \ge \dots \ge \lambda_{i_N^n}$ on $A_n$. This means, that $X \in 1\!\!1_{A_n} \mathcal{C}_{\pi^n}$ with $\pi^n(j) = i_j^n$. Indeed, we can rewrite $X$ on $A_n$ as

$$X = \left(\lambda_{i_1^n} - \lambda_{i_2^n}\right) X_{i_1^n} + \dots + (N-1)\left(\lambda_{i_{N-1}^n} - \lambda_{i_N^n}\right) \frac{\sum_{k=1}^{N-1} X_{i_k^n}}{N-1} + N \lambda_{i_N^n} \frac{\sum_{k=1}^N X_{i_k^n}}{N},$$

which shows that $X \in \mathcal{C}_{\pi^n}$ on $A_n$.

Further, for $\mathcal{B}_s = \operatorname{conv}(X_1, \ldots, X_s)$ the elements $\mathcal{C}_{\pi'} \cap \mathcal{B}_s$ of dimension $s$ are exactly the ones with $\{\pi(i) \mid i = 1, \ldots, s\} = \{1, \ldots, s\}$. Therefore, $(\mathcal{C}_{\pi'} \cap \mathcal{B}_s)_{\pi'}$ is exactly the barycentric subdivision of $\mathcal{B}_s$, which has been shown to fulfill the properties (i)-(iii). $\square$

**Remark 2.61.** If we subdivide the conditional simplex $\mathcal{S} = \operatorname{conv}(X_1, \ldots, X_N)$ barycentrically, we can consider an arbitrary $\mathcal{C}_\pi = \operatorname{conv}(Y_1^\pi, \ldots, Y_N^\pi)$, $\pi \in \mathrm{S_N}$. Then

$$\operatorname{diam}(\mathcal{C}_\pi) \leq \operatorname*{ess\,sup}_{i=1,\ldots,N} \|Y_i^\pi - Y_N^\pi\| \leq \frac{1}{N} \operatorname*{ess\,sup}_{i=1,\ldots,N} \left\| \sum_{k=1}^N (X_i^n - X_k^n) \right\| \leq \frac{N-1}{N} \operatorname{diam}(\mathcal{S}).$$

If we now subdivide $\mathcal{C}_\pi$ barycentrically and continue in that way, we obtain a chain of simplexes $\mathcal{S}^n$, with $\mathcal{S}^0 = \mathcal{S}$. For the diameter of $\mathcal{S}^n$, it holds that $\operatorname{diam}(\mathcal{S}^n) \leq \left(\frac{N-1}{N}\right)^n \operatorname{diam}(\mathcal{S})$. Since $\operatorname{diam}(\mathcal{S}) < \infty$ and $\left(\frac{N-1}{N}\right)^n \to 0$, for $n \to \infty$, it follows that $\operatorname{diam}(\mathcal{S}^n) \to 0$, $n \to \infty$.

### 2.11.2 Brouwer fixed point theorem

**Definition 2.62.** Let $\mathcal{S} = \operatorname{conv}(X_1, \ldots, X_N)$ be a conditional simplex, barycentrically subdivided in $\mathfrak{S} = (\mathcal{C}_\pi)_{\pi \in \mathrm{S_N}}$. A stable function $\phi\colon \operatorname{ext}(\mathfrak{S}) \to \{1, \ldots, N\}$ is called a labeling function of $\mathcal{S}$. For fixed $X_1, \ldots, X_N \in \operatorname{ext}(\mathcal{S})$, the labeling function is called proper, if for any $Y \in \operatorname{ext}(\mathfrak{S})$ it holds that

$$\mathbb{P}(\{\phi(Y) = i\} \subset \{\lambda_i > 0\}) = 1,$$

for $i = 1, \ldots, N$, where $Y = \sum_{i=1}^N \lambda_i X_i$. A conditional simplex $\mathcal{C} = \operatorname{conv}(Y_1, \ldots, Y_N) \subset \mathcal{S}$, with $Y_j \in \operatorname{ext}(\mathfrak{S}), j = 1, \ldots, N$, is said to be completely labeled by $\phi$ if this is a proper labeling function of $\mathcal{S}$ and

$$P\left(\bigcup_{j=1}^N \{\phi(Y_j) = i\}\right) = 1,$$

for all $i \in \{1, \ldots, N\}$.

**Lemma 2.63.** *Let $\mathcal{S} = \operatorname{conv}(X_1, \ldots, X_N)$ be a conditional simplex and $f\colon \mathcal{S} \to \mathcal{S}$ a stable function. Let $\phi\colon \operatorname{ext}(\mathfrak{S}) \to \{0, \ldots, N\}$ be a stable function such that*

*(i).* $\mathbb{P}(\{\phi(X) = i\} \subset \{\lambda_i > 0\} \cap \{\lambda_i \geq \mu_i\}) = 1$, *for all* $i = 1, \ldots, N$,

*(ii).* $\mathbb{P}\left(\bigcup_{i=1}^N (\{\lambda_i > 0\} \cap \{\lambda_i \geq \mu_i\}) \subset \bigcup_{i=1}^N \{\phi(X) = i\}\right) = 1$,

*where* $X = \sum_{i=1}^N \lambda_i X_i$ *and* $f(X) = \sum_{i=1}^N \mu_i X_i$. *Then,* $\phi$ *is a proper labeling function. Moreover, the set of functions fulfilling these properties is non-empty.*

*Proof.* First we show that $\phi$ is a labeling function. Since $\phi$ is stable we just have to prove that $\phi$ actually maps to $\{1, \ldots, N\}$. Due to (ii), we have to show that $\mathbb{P}\left(\bigcup_{i=1}^N \{\lambda_i \geq \mu_i : \lambda_i > 0\}\right) = 1$. Assume to the contrary, $\mu_i > \lambda_i$ on $A \in \mathcal{F}_+$, for all $\lambda_i$ with $\lambda_i > 0$ on $A$. Then it holds that $1 = \sum_{i=1}^N \lambda_i \mathbb{1}_{\{\lambda_i > 0\}} < \sum_{i=1}^N \mu_i \mathbb{1}_{\{\mu_i > 0\}} = 1$ on $A$ which yields a contradiction. Thus, $\phi$ is a labeling function. Moreover, due to (i) it holds that $\mathbb{P}(\{\phi(X) = i\} \subset \{\lambda_i > 0\}) = 1$ which shows that $\phi$ is proper.

To prove the existence, for $X \in \operatorname{ext}(\mathfrak{S})$ with $X = \sum_{i=1}^N \lambda_i X_i$, $f(X) = \sum_{i=1}^N \mu_i X_i$ let $B_i := \{\lambda_i > 0\} \cap \{\lambda_i \geq \mu_i\}$, $i = 1, \ldots, N$. Then we define the function $\phi$ at $X$ as $\{\phi(X) = i\} = B_i \setminus \left(\bigcup_{k=1}^{i-1} B_k\right)$, $i = 1, \ldots, N$. By the former part of the proof it follows that $\phi$ maps to $\{1, \ldots, N\}$ and is proper. It remains to show that $\phi$ is stable. To this end, consider $X = \sum_{j \in \mathbb{N}} \mathbb{1}_{A_j} X^j$ where $X^j = \sum_{i=1}^N \lambda_i^j X_i$ and $f\left(X^j\right) = \sum_{i=1}^N \mu_i^j X_i$. Due to uniqueness of the coefficients in a simplex it holds that $\lambda_i = \sum_{j \in \mathbb{N}} \mathbb{1}_{A_j} \lambda_i^j$ and due to stableity of $f$ it follows that $\mu_i = \sum_{j \in \mathbb{N}} \mathbb{1}_{A_j} \mu_i^j$. Therefore it holds that

$$B_i = \{\lambda_i > 0\} \cap \{\lambda_i \geq \mu_i\} = \bigcup_{j \in \mathbb{N}} \left(\left\{\lambda_i^j > 0\right\} \cap \left\{\lambda_i^j \geq \mu_i^j\right\} \cap A_j\right) = \bigcup_{j \in \mathbb{N}} B_i^j \cap A_j.$$

Hence, $\phi(X) = i$ on $B_i \setminus \left(\bigcup_{k=1}^{i-1} B_k\right) = \left(\bigcup_{j \in \mathbb{N}} \left(B_i^j \cap A_j\right)\right) \setminus \left(\bigcup_{k=1}^{i-1} \left(\bigcup_{j \in \mathbb{N}} B_k^j \cap A_j\right)\right) = \bigcup_{j \in \mathbb{N}} \left(\left(B_i^j \setminus \bigcup_{k=1}^{i-1} B_k^j\right) \cap A_j\right)$. On the other hand, we see that $\sum_{j \in \mathbb{N}} \mathbb{1}_{A_j} \phi\left(X^j\right)$ is $i$ on any $A_j \cap \left\{\phi\left(X^j\right) = i\right\}$, hence it is $i$ on $\bigcup_{j \in \mathbb{N}} \left(B_i^j \setminus \bigcup_{k=1}^{i-1} B_k^j\right) \cap A_j$. Thus, finally, $\sum_{j \in \mathbb{N}} \mathbb{1}_{A_j} \phi\left(X^j\right) = \phi\left(\sum_{j \in \mathbb{N}} \mathbb{1}_{A_j} X^j\right)$ which shows that $\phi$ is stable. $\square$

The reason to demand stableity of a labeling function is exactly because we want to label by the rule explained in the last lemma and hence keep stable information with it. For example consider a conditional simplex $\mathcal{S} = \operatorname{conv}(X_1, X_2, X_3, X_4)$ and $\Omega = \{\omega_1, \omega_2\}$. Let $Y \in \operatorname{ext}(\mathfrak{S})$ be given by $Y = \frac{1}{3} \sum_{i=1}^3 X_i$. Now consider a function $f$ on $\mathcal{S}$ such that

$$f(Y)(\omega_1) = \frac{1}{4} X_1(\omega_1) + \frac{3}{4} X_3(\omega_1), \quad f(Y)(\omega_2) = \frac{2}{5} X_1(\omega_2) + \frac{2}{5} X_2(\omega_2) + \frac{1}{5} X_4(\omega_2).$$

If we label $Y$ by the rule explained in Lemma 2.63, $\phi$ takes the values $\phi(\omega_1) \in \{1, 2\}$ and $\phi(\omega_2) = 3$. Therefore, we can really express on which set $\lambda_i \geq \mu_i$ and on which not. Using a deterministic labeling of $Y$, we would loose this information. For example bearing the label 3 would not mean anything on $\omega_1$ for $Y$. Moreover, it would be impossible to label properly by a deterministic labeling function following the rule of the last lemma since there is no $i$ such that $\lambda_i \geq \mu_i$.

**Theorem 2.64.** *Let $\mathcal{S} = \operatorname{conv}(X_1, \ldots, X_N)$ be a conditional simplex in $L^0(\mathcal{F})^d$. Let $f \colon \mathcal{S} \to \mathcal{S}$ be a stable, sequentially continuous function. Then there exists $Y \in \mathcal{S}$ such that $f(Y) = Y$.*

*Proof.* We consider the barycentric subdivision $(\mathcal{C}_\pi)_{\pi \in \mathrm{S_N}}$ of $\mathcal{S}$ and a proper labeling function $\phi$ on $\operatorname{ext}(\mathfrak{S})$. First, we show that we can find a completely labeled conditional simplex in $\mathcal{S}$. By induction on the dimension of $\mathcal{S} = \operatorname{conv}(X_1, \ldots, X_N)$, we show that there exists a partition $(A_k)_{k=1,\ldots,K}$ such that on any $A_k$ there is an odd number of completely labeled $\mathcal{C}_\pi$. The case $N = 1$ is clear, since a point can be labeled with the constant index 1, only.

Suppose the case $N-1$ is proven. Since the number of $Y_i^\pi$ of the barycentric subdivision is finite and $\phi$ can only take finitely many values, it holds for all $V \in (Y_i^\pi)_{i=1,\ldots,N,\pi \in \mathrm{S_N}}$ there exists a partition $\left(A_k^V\right)_{k=1,\ldots,K}$, $K < \infty$, where $\phi(V)$ is constant on any $A_k^V$. Therefore, we find a partition $(A_k)_{k=1,\ldots,K}$, such that $\phi(V)$ on $A_k$ is constant for all $V$ and $A_k$. Fix $A_k$ now.

In the following, we denote by $\mathcal{C}_{\pi^b}$ these simplexes for which $\mathcal{C}_{\pi^b} \cap \mathcal{B}_{N-1}$ are $N-1$-dimensional (cf. Lemma 2.60 (iv)), therefore $\pi^b(N) = N$. Further we denote by $\mathcal{C}_{\pi^c}$ these simplexes which are not of the type $\mathcal{C}_{\pi^b}$, that is $\pi^c(N) \neq N$. If we use $\mathcal{C}_\pi$ we mean a simplex of arbitrary type. We define

(i). $\mathfrak{C} \subset (\mathcal{C}_\pi)_{\pi \in \mathrm{S_N}}$ to be the set of $\mathcal{C}_\pi$ which are completely labeled on $A_k$.

(ii). $\mathfrak{A} \subset (\mathcal{C}_\pi)_{\pi \in \mathrm{S_N}}$ to be the set of the almost completely labeled $\mathcal{C}_\pi$, which is the property $\{\phi(Y_k^\pi) \mid k = 1, \ldots, N\} = \{1, \ldots, N-1\}$ on $A_k$.

(iii). $\mathfrak{E}_\pi$ to be the set of the intersections $(\mathcal{C}_\pi \cap \mathcal{C}_{\pi_l})_{\pi_l \in \mathrm{S_N}}$ which are $N-1$-dimensional and completely labeled on $A_k$.

(iv). $\mathfrak{B}_{\pi^b}$ to be the set of the intersections $\mathcal{C}_{\pi^b} \cap \mathcal{B}_{N-1}$ which are completely labeled on $A_k$.

We know that $\mathcal{C}_\pi \cap \mathcal{C}_{\pi_l}$ is $N-1$-dimensional on $A_k$ if and only this holds on whole $\Omega$ (cf. Lemma 2.60 (ii)) and $\mathcal{C}_{\pi^b} \cap \mathcal{B}_{N-1} \neq \emptyset$ on $A_k$ if and only if this also holds on whole $\Omega$ (cf. Lemma 2.60 (iv)). So it does not play any role if we look at these sets which are intersections on $A_k$ or on $\Omega$ since they are exactly the same sets.

If $\mathcal{C}_{\pi^c} \in \mathfrak{C}$ then $|\mathfrak{E}_{\pi^c}| = 1$ and if $\mathcal{C}_{\pi^b} \in \mathfrak{C}$ then $|\mathfrak{E}_{\pi^b} \cup \mathfrak{B}_{\pi^b}| = 1$. If $\mathcal{C}_{\pi^c} \in \mathfrak{A}$ then $|\mathfrak{E}_{\pi^c}| = 2$ and if $\mathcal{C}_{\pi^b} \in \mathfrak{A}$ then $|\mathfrak{E}_{\pi^b} \cup \mathfrak{B}_{\pi^b}| = 2$. Therefore it holds $\sum_{\pi \in \mathrm{S_N}} |\mathfrak{E}_\pi \cup \mathfrak{B}_\pi| = |\mathfrak{C}| + 2\,|\mathfrak{A}|$.

If we pick an $E_\pi \in \mathfrak{E}_\pi$ we know there always exists another $\pi_l$ such that $E_\pi \in \mathfrak{E}_{\pi_l}$ (Lemma 2.60(ii)). Therefore $\sum_{\pi \in \mathrm{S_N}} |\mathfrak{E}_\pi|$ is even. Moreover $(\mathcal{C}_{\pi^b} \cap \mathcal{B}_{N-1})_{\pi^b}$ subdivides $\mathcal{B}_{N-1}$ barycentrically[2] and hence we can apply the hypothesis (on $\operatorname{ext}(\mathcal{C}_{\pi^b} \cap \mathcal{B}_{N-1})$). This means that the number of completely labeled simplexes is odd on a partition of $\Omega$ but since $\phi$ is constant on $A_k$ it also has to be odd there. This means that $\sum_{\pi^b} |\mathfrak{B}_{\pi^b}|$ has to be odd. Hence, we also have that $\sum_\pi |\mathfrak{E}_\pi \cup \mathfrak{B}_\pi|$ is the sum of an even and an odd number and thus odd. So we conclude $|\mathfrak{C}| + 2|\mathfrak{A}|$ is odd and hence also $|\mathfrak{C}|$. Thus, we find for any $A_k$ a completely labeled $\mathcal{C}_{\pi_k}$.

By $\sigma$-stability of $\mathcal{S}$ and stableity of $\phi$ we can paste completely labeled simplexes. If we do so, we obtain $\mathcal{S}^1 := \sum_{k=1}^K \mathbb{1}_{A_k} \mathcal{C}_{\pi_k}$, which by Remark 2.58 is indeed a simplex and by Remark 2.61 has a diameter which is less then $\frac{N-1}{N} \operatorname{diam}(\mathcal{S})$. So we finally got a simplex $\mathcal{S}^1 \subset \mathcal{S}$ which is completely labeled on whole $\Omega$.

This holds for any proper labeling function hence also for a $\phi$ of the type as in Lemma 2.63.

Now, we extract a sequence $(\mathcal{S}^n)_{n\in\mathbb{N}}$ of completely labeled simplexes contained in $\mathcal{S}$, fulfilling the diameter property $\operatorname{diam}(\mathcal{S}^n) \to 0$ as in Remark 2.61. By [CKV15, Theorem 3.8]) it holds that $\bigcap_{n\in\mathbb{N}} \mathcal{S}^n \neq \emptyset$. The intersection consists of one element $Y = \sum_{l=1}^N \alpha_l X_l$ by the diameter property. Let $f(Y) = \sum_{l=1}^N \beta_l X_l$. Thus, all $\operatorname{ext}(\mathcal{S}^n)$ of the sequence of simplexes $\mathcal{S}^n$ also converge $\mathbb{P}$-almost surely to $Y$, which then preserves the properties of the index function. That is, for each $i = 1, \dots, N$, there exist $V_k^n \in \operatorname{ext}(C_\pi^n)$ of $\mathcal{S}^n$, $k = 1, \dots, N$, $\pi \in \mathrm{S_N}$, with $\mathbb{P}\left(\{\phi(V_k^n) = i\} \subset \left\{\lambda_i^{n,k} \geq \mu_i^{n,k}\right\}\right) = 1$ (cf. Lemma 2.63), where $V_k^n = \sum_{i=1}^N \lambda_i^{n,k} X_i$ and $f(V_k^n) = \sum_{i=1}^N \mu_i^{n,k} X_i$. Then $\mathbb{P}\left(\bigcap_{n\in\mathbb{N}} \left\{\lambda_i^{n,k} \geq \mu_i^{n,k}\right\} \subset \{\alpha_i \geq \beta_i\}\right) = 1$ for all $k = 1, \dots, N$ by stableity of $f$, $V^n \to Y$ $\mathbb{P}$-almost surely, and $f(V^n) \to f(Y)$ $\mathbb{P}$-almost surely. But,

$$\mathbb{P}\left(\bigcup_{k=1}^N \bigcap_{n\in\mathbb{N}} \left\{\lambda_i^{n,k} \geq \mu_i^{n,k}\right\}\right) = \mathbb{P}\left(\bigcap_{n\in\mathbb{N}} \bigcup_{k=1}^N \left\{\lambda_i^{n,k} \geq \mu_i^{n,k}\right\}\right) = 1$$

by the complete labeling of $\mathcal{S}^n$. Hence, $\alpha_i \geq \beta_i$ for all $i = 1, \dots, N$. This is possible only if $\alpha_i = \beta_i$ for all $i = 1, \dots, N$ which is the condition of a fixed point. $\square$

**Corollary 2.65.** *Let $(\mathcal{S}^n)_{n\in\mathbb{N}}$ be conditional simplexes, $(A_n)_{n\in\mathbb{N}}$ a partition and $\mathcal{S} := \sum_{n\in\mathbb{N}} \mathbb{1}_{A_n} \mathcal{S}^n$. Then a stable, sequentially continuous function $f\colon \mathcal{S} \to \mathcal{S}$ has a fixed point.*

[2] The boundary of $\mathcal{S}$ is a $\sigma$-stable set so if it is partitioned by the labeling function into $A_k$ we know that $\mathcal{B}_{N-1}(\mathcal{S}) = \sum_{k=1}^K \mathbb{1}_{A_k} \mathcal{B}_{N-1}(\mathbb{1}_{A_k}\mathcal{S})$ and by Lemma 2.60 (iv) we can apply the induction hypothesis also on $A_k$.

*Proof.* Since $f$ is stable, we have $f(\mathcal{S}) = \sum_{n\in\mathbb{N}} \mathbb{1}_{A_n} f(\mathcal{S}^n)$ and $f$ restricted on $\mathcal{S}^n$ is still sequentially continuous. Therefore we find $Y_n \in \mathcal{S}^n$ with $\mathbb{1}_{A_n} f(Y_n) = \mathbb{1}_{A_n} Y_n$. Defining $Y = \sum_{n\in\mathbb{N}} \mathbb{1}_{A_n} Y_n$ we have

$$f(Y) = f\left(\sum_{n\in\mathbb{N}} \mathbb{1}_{A_n} Y_n\right) = \sum_{n\in\mathbb{N}} \mathbb{1}_{A_n} f(Y_n) = \sum_{n\in\mathbb{N}} \mathbb{1}_{A_n} Y_n = Y.$$

Thus, $f$ has a fixed point. □

**Remark 2.66.** The $\mathcal{S}^n$ can be of different dimension. If $\mathcal{S}^n = \operatorname{conv}\left(Y_1^n, \dots, Y_{N_n}^n\right)$ is of dimension $N_n$, the object $\mathcal{S}$ can be considered as to be of conditional dimension $\sum_{n\in\mathbb{N}} \mathbb{1}_{A_n} N_n$. This conditional dimension is hence in $\{N_n \mid n \in \mathbb{N}\}$, in particular a measurable object.

### 2.11.3 Fixed point theorem for sequentially closed and bounded sets in $L^0(\mathcal{F})^d$

**Proposition 2.67.** *Let $\mathcal{K}$ be a conditionally convex, sequentially closed and bounded conditional subset of $L^0(\mathcal{F})^d$ and $f\colon \mathcal{K} \to \mathcal{K}$ a sequentially continuous conditional function. Then $f$ has a fixed point.*

*Proof.* Since $\mathcal{K}$ is bounded, there exists a conditional simplex $\mathcal{S}$ such that $\mathcal{K} \subset \mathcal{S}$. Now define the function $h\colon \mathcal{S} \to \mathcal{K}$ by

$$h(X) = \begin{cases} \mathbb{1}_A X, & \text{if } \mathbb{1}_A X \in \mathbb{1}_A \mathcal{K}, \\ \operatorname{argmin}\{\|X - Y\| \mid Y \in \mathcal{K}\}, & \text{else.} \end{cases}$$

This means, that $h$ is the identity on $\mathcal{K}$ and a projection towards $\mathcal{K}$ for the elements in $\mathcal{S} \setminus \mathcal{K}$. Due to [CKV15, Corollary 4.5] this minmium exists and is unique. Therefore $h$ is well-defined.

We can characterize $h$ by

$$Y = h(X) \Leftrightarrow \langle X - Y, Z - Y\rangle \le 0, \text{ for all } Z \in \mathcal{K}. \tag{2.34}$$

Indeed, let $\langle X - Y, Z - Y\rangle \le 0$ for all $Z \in \mathcal{K}$. Then

$$\begin{aligned} \|X - Z\|^2 &= \|(X - Y) + (Y - Z)\| \\ &= \|X - Y\|^2 + 2\langle X - Y, Y - Z\rangle + \|Y - Z\|^2 \ge \|X - Y\|^2, \end{aligned}$$

which shows the minimizing property of $h$. On the other hand, let $Y = h(X)$. Since $\mathcal{K}$ is conditionally convex, $\lambda Z + (1-\lambda) Y \in \mathcal{K}$ for any $\lambda \in (0,1]$ and $Z \in \mathcal{K}$. By standard calculation,

$$\|X - (\lambda Z + (1-\lambda) Y)\|^2 \geq \|X - Y\|^2$$

yields $0 \geq -2\lambda \langle X, -Y\rangle + \left(2\lambda - \lambda^2\right) \langle Y, Y\rangle + 2\lambda \langle X, Z\rangle - \lambda^2 \|Z\|^2 - 2\lambda (1-\lambda) \langle Z, Y\rangle$. Any term can be divided by $\lambda > 0$. We do so and let $\lambda \downarrow 0$ afterwards. We obtain

$$0 \geq -2\langle X, -Y\rangle + 2\langle Y, Y\rangle + 2\langle X, Z\rangle - 2\langle Z, Y\rangle = 2\langle X - Y, Z - Y\rangle,$$

which is the claim.

Furthermore, for any $X, Y \in \mathcal{S}$ holds

$$\|h(X) - h(Y)\| \leq \|X - Y\|.$$

Indeed, $X - Y = (h(X) - h(Y)) + X - h(X) + h(Y) - Y =: (h(X) - h(Y)) + c$ which means

$$\|X - Y\|^2 = \|h(X) - h(Y)\|^2 + \|c\|^2 + 2\langle c, h(X) - h(Y)\rangle. \tag{2.35}$$

Since $\langle c, h(X) - h(Y)\rangle = -\langle X - h(X), h(Y) - h(X)\rangle - \langle Y - h(Y), h(X) - h(Y)\rangle$, by (2.34), it follows that $\langle c, h(X) - h(Y)\rangle \geq 0$. Then (2.35) yields $\|X - Y\|^2 \geq \|h(X) - h(Y)\|^2$. Using this we see that $h$ is a sequentially continuous conditional function, for if $\|X_n - X\| \to 0$ then also $\|h(X_n) - h(X)\| \to 0$.

The function $f \circ h$ is a sequentially continuous function mapping from $\mathcal{S}$ to $\mathcal{S}$, more precisely to $\mathcal{K}$. Hence, there exists a fixed point $f \circ h(Z) = Z$. But since $f \circ h$ maps to $\mathcal{K}$, this $Z$ has to be in $\mathcal{K}$. Therefore we know $h(Z) = Z$ and hence $f(Z) = Z$ which ends the proof. $\square$

**Remark 2.68.** In [DJKK16] a concept of conditional compactness is introduced and it is shown that there is an equivalence between conditional compactness and conditional closed- and boundedness in $L^0(\mathcal{F})^d$. In this concept we can formulate the conditional Brouwer fixed point theorem as follows.

**Theorem 2.69.** *A sequentially continuous conditional function $f\colon \mathcal{K} \to \mathcal{K}$ such that $\mathcal{K}$ is a conditionally compact and $L^0(\mathcal{F})$-convex subset of $L^0(\mathcal{F})^d$ has a fixed point.*

### 2.11.4 Applications in analysis on $L^0(\mathcal{F})^d$

Working in $\mathbb{R}^d$ the Brouwer fixed point theorem can be used to prove several topological properties and is even equivalent to some of them. In the theory of $L^0(\mathcal{F})^d$ we will show that the conditional Brouwer fixed point theorem has several implications as well. Define the conditional unit ball in $L^0(\mathcal{F})^d$ by $\mathcal{B}(d) = \left\{X \in L^0(\mathcal{F})^d \mid \|X\| \leq 1\right\}$. Then by the former theorem any sequentially continuous conditional function $f\colon \mathcal{B}(d) \to \mathcal{B}(d)$ has a fixed point. The unit sphere is defined as $\mathcal{S}(d-1) = \left\{X \in L^0(\mathcal{F})^d \mid \|X\| = 1\right\}$.

**Definition 2.70.** Let $\mathcal{C}$ and $\mathcal{D}$ be conditional subsets of $L^0(\mathcal{F})^d$. A conditional homotopy of two stable, sequentially continuous conditional functions $f, g\colon \mathcal{C} \to \mathcal{D}$ is a sequentially continuous bivariate conditional function $H\colon \mathcal{C} \times [0,1] \to \mathcal{D}$ such that $H(C,0) = f(C)$ and $H(C,1) = g(C)$.

**Lemma 2.71.** *The identity function of the sphere is not conditionally homotop to a constant function.*

The proof is a consequence of the following lemma.

**Lemma 2.72.** *A sequentially continuous conditional function $f\colon \mathcal{B}(d) \to \mathcal{S}(d-1)$ which is the identity on $\mathcal{S}(d-1)$ does not exist.*

*Proof.* Suppose there is a sequentially continuous conditional function $f$ as assumed in the lemma. We define $g\colon \mathcal{S}(d-1) \to \mathcal{S}(d-1)$ by $g(X) = -X$. Then the composition $g \circ f\colon \mathcal{B}(d) \to \mathcal{B}(d)$, which actually maps to $\mathcal{S}(d-1)$, is a sequentially continuous conditional function. Therefore, this has a fixed point $Y$ which has to be in $\mathcal{S}(d-1)$, since this is the image of $g \circ f$. But we know $f(Y) = Y$ and $g(Y) = -Y$ and hence $g \circ f(Y) = -Y$. Therefore, $Y$ cannot be a fixed point (since $0 \in \mathcal{S}(d-1)^{\sqsubset}$) which is a contradiction. $\square$

Directly follows that the identity on the sphere is not conditionally homotop to a constant function. In the case $d = 1$ we get the following result which is the conditional version of a intermediate value theorem.

**Lemma 2.73.** *Let $X, \overline{X} \in L^0(\mathcal{F})$ with $X \leq \overline{X}$. Let $f\colon \left[X, \overline{X}\right] \to L^0(\mathcal{F})$ be a sequentially continuous conditional function. Define $A := \left\{f(X) \leq f\left(\overline{X}\right)\right\}$. Then for every $Y \in \left[\mathbb{1}_A f(X) + \mathbb{1}_{A^c} f\left(\overline{X}\right), \mathbb{1}_A f\left(\overline{X}\right) + \mathbb{1}_{A^c} f(X)\right]$ there exists $\overline{Y} \in \left[X, \overline{X}\right]$ with $f\left(\overline{Y}\right) = Y$.*

*Proof.* It is sufficient to prove the case for $f(X) \leq f\left(\overline{X}\right)$ that is $A = \Omega$. In the general case we consider $A$ and $A^c$ separately, obtain $1\!\!1_A f\left(\overline{Y}_1\right) = 1\!\!1_A Y$, $1\!\!1_{A^c} f\left(\overline{Y}_2\right) = 1\!\!1_{A^c} Y$ and, by stableity, $f\left(1\!\!1_A \overline{Y}_1 + 1\!\!1_{A^c} \overline{Y}_2\right) = Y$. Suppose, now $Y \in \left[f(X), f\left(\overline{X}\right)\right]$.

Let first $f(X) < Y < f\left(\overline{X}\right)$. Define the conditional function $g\colon \left[X, \overline{X}\right] \to \left[X, \overline{X}\right]$ by

$$g(V) := p\left(V - f(V) + Y\right) \quad \text{with} \quad p(Z) = 1\!\!1_{\{Z \leq X\}} X + 1\!\!1_{\{X \leq Z \leq \overline{X}\}} Z + 1\!\!1_{\{\overline{X} \leq Z\}} \overline{X}.$$

Therefore $g$ has a fixed point $\overline{Y}$ by Proposition 2.67. If $\overline{Y} = X$, it holds $X - f(X) + Y \leq X$ which means $Y \leq f(X)$ which is a contradiction. If $\overline{Y} = \overline{X}$, it follows that $f\left(\overline{X}\right) \leq Y$, which is also a contradiction. Hence, $\overline{Y} = \overline{Y} - f\left(\overline{Y}\right) + Y$ which means $f\left(\overline{Y}\right) = Y$.

If $Y = f(X)$ on $B$ and $Y = f\left(\overline{X}\right)$ on $C$, it holds $f(X) < Y < f\left(\overline{X}\right)$ on $(B \cup C)^c =: D$. Then we find $\overline{Y}$ such that $f\left(\overline{Y}\right) = Y$ on $D$. In total $f\left(1\!\!1_B X + 1\!\!1_{C \setminus B} \overline{X} + 1\!\!1_D \overline{Y}\right) = 1\!\!1_B f(X) + 1\!\!1_{C \setminus B} f\left(\overline{X}\right) + 1\!\!1_D f\left(\overline{Y}\right) = Y$. This is the claim for arbitrary $Y \in \left[f(X), f(\overline{X})\right]$. □

# 3 Walras equilibrium

Motivated by the work of Jofré and Wets [JW02] we enlarge their setting in the context of conditional sets. We describe the market setting, agents' preferences dependent on the price in the market, and the Walrasian as a price-dependent conditional function whose max/essinf-points describe a Walras equilibrium price. Furthermore, we discuss stability properties of the Walras equilibrium and converging economies.

## 3.1 Market setting

The market setting is introduced as follows. The set $\mathfrak{A}$ of agents who trade in the market is assumed to be finite. There are $d$ goods which can be exchanged. At the beginning, an agent $\mathfrak{a} \in \mathfrak{A}$ has the endowment $E^{\mathfrak{a}} \in L^0(\mathcal{F})^d_+$. The endowment can be exchanged for allocations $X \in L^0(\mathcal{F})^d$ of goods. Their utility is described by a conditional utility function $u^{\mathfrak{a}} \colon L^0(\mathcal{F})^d \to [-\infty; \infty[$. It is upper semicontinuous and conditionally concave on the nonempty domain $\operatorname{dom} u^{\mathfrak{a}} \sqsubset L^0(\mathcal{F})^d$ of the conditional utility function. The assumption of concavity is discussed in remark 3.5. For the domain we further suppose that it is conditionally closed and its interior $\operatorname{int}\operatorname{dom} u^{\mathfrak{a}}$ is nonempty. Then, a natural consistency assumption is that $E^{\mathfrak{a}} \in \operatorname{int}\operatorname{dom} u^{\mathfrak{a}}$. In the sequel, we reduce the problem to the case that $\operatorname{int}\operatorname{dom} u^{\mathfrak{a}}$ lives on $\Omega$, the rest is of no futher interest mathematically.

Further, we impose criteria when utility functions are disturbed. In detail, we assume a conditional sequence $(u^{\mathfrak{a}}_J)_{J \in \mathbb{N}(\mathcal{F})}$ of conditional utility functions $u^{\mathfrak{a}}_J \colon L^0(\mathcal{F})^d \to [-\infty; \infty[$ disturbing $u^{\mathfrak{a}}$ in a hypoconverging sense, that is $\text{h-lim}_{J \in \mathbb{N}(\mathcal{F})} u^{\mathfrak{a}}_J = u^{\mathfrak{a}}$. This implies that $u^{\mathfrak{a}}$ is upper semicontinuous, cf. Proposition 2.25.

For the optimization problem we consider the exchange of goods at prices $P \in L^0(\mathcal{F})^d_+$. The exchange is limited by $\langle P, X\rangle \leq \langle P, E^{\mathfrak{a}}\rangle$, thus, the feasible good allocations are $\mathcal{C}^{\mathfrak{a}}(P) := \operatorname{dom} u^{\mathfrak{a}} \sqcap \left\{X \in L^0(\mathcal{F})^d \mid \langle P, X\rangle \leq \langle P, E^{\mathfrak{a}}\rangle\right\}$. Clearly, $\mathcal{C}^{\mathfrak{a}}(P)$ is a conditional set, since if $X^1, X^2 \in \mathcal{C}^{\mathfrak{a}}(P)$ and $A \in \mathcal{F}$, it holds that $\mathbb{1}_A X^1 + \mathbb{1}_{A^c} X^2 \in \operatorname{dom} u^{\mathfrak{a}}$ and $\left\langle P, \mathbb{1}_A X^1 + \mathbb{1}_{A^c} X^2\right\rangle \leq \langle P, E^{\mathfrak{a}}\rangle$. Now, as a utility maximizer, each agent acquires

$$\operatorname{argmax}_{X \in \mathcal{C}} \{u^{\mathfrak{a}}(X) \mid X \in \mathcal{C}^{\mathfrak{a}}(P)\} \tag{3.1}$$

for some conditionally compact conditional subset $\mathcal{C} \sqsubset L^0(\mathcal{F})^d$ which lives on $\Omega$ and describes trading constraints. Conditions for a conditionally compact $\mathcal{C} \sqsubset L^0(\mathcal{F})^d$ are discussed in Section 3.2. We define the conditional set of maximizers $\mathcal{D}^{\mathfrak{a}}(P) := \left\{X \in L^0(\mathcal{F})^d \mid X \in \operatorname{argmax}_{X \in \mathcal{C}} \{u^{\mathfrak{a}}(X) \mid X \in \mathcal{C}^{\mathfrak{a}}(P)\}\right\}$. Since if $X^1, X^2 \in \mathcal{D}^{\mathfrak{a}}(P)$ and $A \in \mathcal{F}$, it holds that $\mathbb{1}_A X^1 + \mathbb{1}_{A^c} X^2 \in \mathcal{C}^{\mathfrak{a}}(P)$ since $\mathcal{C}^{\mathfrak{a}}(P)$ is a conditional set and $u^{\mathfrak{a}}\left(\mathbb{1}_A X^1 + \mathbb{1}_{A^c} X^2\right) = \mathbb{1}_A u^{\mathfrak{a}}\left(X^1\right) + \mathbb{1}_{A^c} u^{\mathfrak{a}}\left(X^2\right) = u^{\mathfrak{a}}\left(X^1\right)$, thus, $\mathbb{1}_A X^1 + \mathbb{1}_{A^c} X^2 \in \mathcal{D}^{\mathfrak{a}}(P)$. The mapping $P \mapsto \mathcal{D}^{\mathfrak{a}}(P)$ is a conditional function since for $P_1, P_2 \in L^0(\mathcal{F})^d_+$ and $A \in \mathcal{F}$ it holds that

$$
\begin{aligned}
&\mathcal{D}^{\mathfrak{a}}(\mathbb{1}_A P_1 + \mathbb{1}_{A^c} P_2) \\
&= \left\{X \in L^0(\mathcal{F})^d \mid X \in \operatorname{argmax}_{X \in \mathcal{C}} \{u^{\mathfrak{a}}(X) \mid X \in \mathcal{C}^{\mathfrak{a}}(\mathbb{1}_A P_1 + \mathbb{1}_{A^c} P_2)\}\right\} \\
&= \mathbb{1}_A \left\{X \in L^0(\mathcal{F})^d \mid X \in \operatorname{argmax}_{X \in \mathcal{C}} \{u^{\mathfrak{a}}(X) \mid X \in \mathbb{1}_A \mathcal{C}^{\mathfrak{a}}(P_1)\}\right\} \\
&\qquad + \mathbb{1}_{A^c} \left\{X \in L^0(\mathcal{F})^d \mid X \in \operatorname{argmax}_{X \in \mathcal{C}} \{u^{\mathfrak{a}}(X) \mid X \in \mathbb{1}_{A^c} \mathcal{C}^{\mathfrak{a}}(P_2)\}\right\} \\
&= \mathbb{1}_A \mathcal{D}^{\mathfrak{a}}(P_1) + \mathbb{1}_{A^c} \mathcal{D}^{\mathfrak{a}}(P_2).
\end{aligned}
$$

A solution to the maximization problem in (3.1) exists since $u^{\mathfrak{a}}$ is conditionally concave on a conditionally compact conditional set, and $u^{\mathfrak{a}} < \infty$, cf. Theorem 2.39.

We continue with the definition of Walras prices. Since $D^{\mathfrak{a}}(P) = D^{\mathfrak{a}}(\lambda P)$ for all $D^{\mathfrak{a}}(P) \in \mathcal{D}^{\mathfrak{a}}(P)$ and $\lambda \in L^0(\mathcal{F})_{++}$ we assume that $P \in \Sigma := \{P \in L^0(\mathcal{F})^d_+ \mid \sum_{i \le d} P_i = 1\}$. Now, we can define the excess supply

$$
\begin{aligned}
S(P) &:= \sum_{\mathfrak{a} \in \mathfrak{A}} \left(E^{\mathfrak{a}} - D^{\mathfrak{a}}(P)\right) \text{ for } D^{\mathfrak{a}}(P) \in \mathcal{D}^{\mathfrak{a}}(P), \\
\mathcal{S}(P) &:= \left\{S(P) \in L^0(\mathcal{F}) \mid D^{\mathfrak{a}}(P) \in \mathcal{D}^{\mathfrak{a}}(P)\right\}.
\end{aligned}
$$

By definition, $S\colon \Sigma \to L^0(\mathcal{F})$ is a conditional function, and, thus, $\mathcal{S}(P)$ is a conditional set for each $P \in \Sigma$. The price vector $P$ is a Walras equilibrium price if $S(P) \ge 0$. If there exists some $S' \in \mathcal{S}(P)$ such that $S' \ge 0$, then $S \ge 0$ for all $S \in \mathcal{S}(P)$. To see that, assume $S \in \mathcal{S}(P)$ and $A \in \mathcal{F}_+$ such that $S|A < 0$. By definition, there exist $(D^{\mathfrak{a}}(P))_{\mathfrak{a} \in \mathfrak{A}}$ in $(\mathcal{D}^{\mathfrak{a}}(P))_{\mathfrak{a} \in \mathfrak{A}}$ such that $S = \sum_{\mathfrak{a} \in \mathfrak{A}} (E^{\mathfrak{a}} - D^{\mathfrak{a}}(P))$. Then, on $A$, $0 > \left\langle P, \sum_{\mathfrak{a} \in \mathfrak{A}} E^{\mathfrak{a}} - D^{\mathfrak{a}} \right\rangle = \sum_{\mathfrak{a} \in \mathfrak{A}} \langle P, E^{\mathfrak{a}} - D^{\mathfrak{a}} \rangle \ge 0$ since $D^{\mathfrak{a}}(P) \in \mathcal{C}^{\mathfrak{a}}(P)$. This contradicts $A \in \mathcal{F}_+$.

Finally, the Walrasian is a conditional function $W\colon \Sigma\times\Sigma\to L^0(\mathcal{F})$ which is defined by

$$W(P,Q):=\operatorname*{ess\,sup}_{S\in\mathcal{S}(P)}\langle Q,S\rangle .$$

The max/inf-point of the Walrasian will be our equilibrium price. Therefore, we sum up some useful properties of the described setting.

**Lemma 3.1.** *The following properties hold for the Walrasian.*

$$\text{For all } Q\in\Sigma\colon\quad P\mapsto W(P,Q) \text{ is upper semicontinuous.} \tag{3.2}$$

$$\text{For all } P\in\Sigma\colon\quad Q\mapsto W(P,Q) \text{ is conditionally convex.} \tag{3.3}$$

$$\text{For all } Q\in\Sigma\colon\quad W(Q,Q)\geq 0. \tag{3.4}$$

*Proof.* To show (3.2), first, we consider the conditional set $\mathcal{C}^{\mathfrak{a}}(P)\sqsubset L^0(\mathcal{F})$. For all $P\in\Sigma$, it holds that $\mathcal{C}^{\mathfrak{a}}(P)$ is conditionally closed and conditionally convex by definition. Further, $\operatorname{int}\mathcal{C}^{\mathfrak{a}}(P)$ lives on $\Omega$ since the conditional set $\left\{X\in L^0(\mathcal{F})^d \mid \langle P,X\rangle\leq\langle P,E^{\mathfrak{a}}\rangle\right\}$ is a half plane whose interior lives on $\Omega$ and $E^{\mathfrak{a}}\in\operatorname{int}\operatorname{dom}u^{\mathfrak{a}}$. We will show that $\limsup_{J\in\mathbb{N}(\mathcal{F})}\mathcal{C}^{\mathfrak{a}}\left(P_J\right)\sqsubset\mathcal{C}^{\mathfrak{a}}(P)$ for a conditional sequence $(P_J)_{J\in\mathbb{N}(\mathcal{F})}$ of prices with $\lim_{J\in\mathbb{N}(\mathcal{F})}P_J=P$, that is, the mapping $P\mapsto\mathcal{C}^{\mathfrak{a}}(P)$ is outer semicontinuous. Clearly, for conditional sequences $(P_J)_{J\in\mathbb{N}(\mathcal{F})}\to P$ in $L^0(\mathcal{F})^d$ and $(X_J)_{J\in\mathbb{N}(\mathcal{F})}\to X$ in $L^0(\mathcal{F})^d$ it holds that $(P_{J,i}(X_{J,i}-E^{\mathfrak{a}}_i))_{J\in\mathbb{N}(\mathcal{F})}\to P_i(X_i-E^{\mathfrak{a}}_i)$, thus, $\limsup_{J\in\mathbb{N}(\mathcal{F})}\mathcal{C}^{\mathfrak{a}}\left(P_J\right)\sqsubset\mathcal{C}^{\mathfrak{a}}(P)$ since $\operatorname{dom}u^{\mathfrak{a}}$ is conditionally closed. Next, we show that the conditional function $P\mapsto\mathcal{C}^{\mathfrak{a}}(P)$ is inner semicontinuous, that is, by Definition 2.21, whenever $(P_J)_{J\in\mathbb{N}(\mathcal{F})}\to P$ there exists $\mathcal{M}\in\mathbb{N}(\mathcal{F})_\infty$ such that there exists a conditional subsequence $(X_J)_{J\in\mathcal{M}}\to X$ with $X_J\in\mathcal{C}^{\mathfrak{a}}(P_J)$. For the proof consider $X\in\operatorname{int}\mathcal{C}^{\mathfrak{a}}(P)$. That is, there exist $\delta\in L^0(\mathcal{F})_{++}$ and a conditional ball $\mathcal{B}^\delta(X)\sqsubset\mathcal{C}^{\mathfrak{a}}(P)$ on $\Omega$. For all $X'\in\mathcal{B}^\delta(X)$ it holds that $\langle P,X'\rangle\leq\langle P,E^{\mathfrak{a}}\rangle$ by definition of $\mathcal{C}^{\mathfrak{a}}(P)$. Now, there exist $\varepsilon\in L^0(\mathcal{F})_{++}$ and a conditional ball $\overline{\mathcal{B}}^\varepsilon(P)\sqsubset\Sigma$ on $\Omega$ such that $\left\langle\overline{P},X'\right\rangle\leq\left\langle\overline{P},E^{\mathfrak{a}}\right\rangle$ for all $\overline{P}\in\overline{\mathcal{B}}^\varepsilon(P)$ and $X'\in\mathcal{B}^\delta(X)$. To continue, consider the conditional subsequence $(P_J)_{J\in\mathcal{M}}$ of $(P_J)_{J\in\mathbb{N}(\mathcal{F})}$ that is in $\overline{\mathcal{B}}^\varepsilon(P)$. We observe that $\mathcal{M}\in\mathbb{N}(\mathcal{F})_\infty$ since $(P_J)_{J\in\mathbb{N}(\mathcal{F})}\to P$. By construction, we find a conditional subsequence $(X_J)_{J\in\mathcal{M}}$ with $X_J\in\mathcal{B}^\delta(X)\sqsubset\mathcal{C}^{\mathfrak{a}}(P_J)$, that is inner semicontinuity of $P\mapsto\mathcal{C}^{\mathfrak{a}}(P)$, and thus, $P\mapsto\mathcal{C}^{\mathfrak{a}}(P)$ is continuous.

Second, we continue with the properties of $P\mapsto\mathcal{D}^{\mathfrak{a}}(P)$. As in (3.1), we define

$$\mathcal{D}^{\mathfrak{a}}_J\left(P_J\right):=\operatorname{argmax}_{X\in\mathcal{C}}\left\{u^{\mathfrak{a}}_J(X)\mid X\in\mathcal{C}^{\mathfrak{a}}\left(P_J\right)\right\}$$

and we claim that $\limsup_{J\in\mathbb{N}(\mathcal{F})}\mathcal{D}^{\mathfrak{a}}_J\left(P_J\right)\sqsubset\mathcal{D}^{\mathfrak{a}}(P)$ whenever $(P_J)_{J\in\mathbb{N}(\mathcal{F})}\to P$.

To that end, for all $A \in \mathcal{F}$, we define

$$\mathbb{1}_A v(X) := \begin{cases} \mathbb{1}_A u(X) & \text{if } \langle P, X\rangle \leq \langle P, E^{\mathfrak{a}}\rangle, \\ -\mathbb{1}_A \infty & \text{else,} \end{cases} \tag{3.5}$$

$$\mathbb{1}_A v_J(X) := \begin{cases} \mathbb{1}_A u_J(X) & \text{if } \langle P, X\rangle \leq \langle P, E^{\mathfrak{a}}\rangle, \\ -\mathbb{1}_A \infty & \text{else.} \end{cases} \tag{3.6}$$

and show that $\text{h-lim}_{J\in\mathbb{N}(\mathcal{F})} v_J = v$. By Lemma 2.24, we have to show that

$$\exists\ (X_J)_{J\in\mathbb{N}(\mathcal{F})} \to X\colon\ \operatorname*{ess\,lim\,inf}_{J\in\mathbb{N}(\mathcal{F})} v_J(X_J) \geq v(X), \tag{3.7}$$

$$\forall\ (X_J)_{J\in\mathbb{N}(\mathcal{F})} \to X\colon\ \operatorname*{ess\,lim\,sup}_{J\in\mathbb{N}(\mathcal{F})} v_J(X_J) \leq v(X). \tag{3.8}$$

for all $X \in L^0(\mathcal{F})^d$. Inequality (3.8) is obvious if $\mathbb{1}_A \operatorname{ess\,lim\,sup}_{J\in\mathbb{N}(\mathcal{F})} v_J\left(X_J\right) = -\mathbb{1}_A\infty$. In turn, if $\operatorname{ess\,lim\,sup}_{J\in\mathbb{N}(\mathcal{F})} v_J\left(X_J\right)|A > -\infty$ for some $A \in \mathcal{F}$ there exists $\mathcal{M} \in \mathbb{N}(\mathcal{F})_\infty$ such that $\mathbb{1}_A X_J \in \mathbb{1}_A \mathcal{D}^{\mathfrak{a}}(P_J)$ for all $J \in \mathcal{M}$. Then, since $(P_J)_{J\in\mathbb{N}(\mathcal{F})} \to P$ and, by hypoconvergence of the utility functions $u_J^{\mathfrak{a}}$, $J \in \mathcal{M}$, it follows that $\mathbb{1}_A X \in \mathbb{1}_A \mathcal{D}^{\mathfrak{a}}(P)$.

To proof inequality (3.7), we consider $A_1 := \operatorname{ess\,sup}\left\{A \in \mathcal{F} \mid \mathbb{1}_A X \in \mathbb{1}_A\left(\mathcal{D}^{\mathfrak{a}}(P)\right)^{\sqsubset}\right\}$, $A_2 := \operatorname{ess\,sup}\left\{A \in \mathcal{F} \mid \mathbb{1}_A X \in \mathbb{1}_A \operatorname{int}\mathcal{D}^{\mathfrak{a}}(P)\right\}$ and the complement $A_3 := (A_1 \cup A_2)^c = \operatorname{ess\,sup}\left\{A \in \mathcal{F} \mid \mathbb{1}_A X \in \mathbb{1}_A\left(\operatorname{cl}\mathcal{D}^{\mathfrak{a}}(P) \sqcap \left(\operatorname{int}\mathcal{D}^{\mathfrak{a}}(P)\right)^{\sqsubset}\right)\right\}$ for $X \in L^0(\mathcal{F})^d$. On $A_1$ we choose $\mathbb{1}_{A_1} X_J = \mathbb{1}_{A_1} X$ for all $J \in \mathbb{N}(\mathcal{F})$. Then, $\mathbb{1}_{A_1} \operatorname{ess\,lim\,inf}_{J\in\mathbb{N}(\mathcal{F})} v_J(X_J) = \mathbb{1}_{A_1} v(X)$ by (3.5) and the characterization of hypoconvergence in Lemma 2.23. On $A_2$, there exists $\mathcal{M} \in \mathbb{N}(\mathcal{F})_\infty$ such that $\mathbb{1}_{A_2} X \in \mathbb{1}_{A_2}\mathcal{D}^{\mathfrak{a}}(P_J)$ by definition of $\mathcal{D}^{\mathfrak{a}}$ and $(P_J)_{J\in\mathcal{M}} \to P$. Again, choosing $\mathbb{1}_{A_2} X_J = \mathbb{1}_{A_2} X$ for all $J \in \mathcal{M}$ yields that $\mathbb{1}_{A_2} \operatorname{ess\,lim\,inf}_{J\in\mathcal{M}} v_J(X_J) = \mathbb{1}_{A_2} v(X)$. On $A_3$, we apply the inner semicontinuity of the conditional mapping $P \mapsto \mathcal{D}^{\mathfrak{a}}(P)$ which yields the existence of a conditional sequence $(X_J)_{J\in\mathbb{N}(\mathcal{F})} \to X$ with $X_J \in \operatorname{int}\mathcal{D}^{\mathfrak{a}}(P)$.

Thus, $\text{h-lim}_{J\in\mathbb{N}(\mathcal{F})} v_J = v$. Since, by definition, $\mathcal{D}_J^{\mathfrak{a}}\left(P_J\right) = \operatorname{argmax}_{X\in\mathcal{C}} v_J(X)$ and $\mathcal{D}^{\mathfrak{a}}(P) = \operatorname{argmax}_{X\in\mathcal{C}} v(X)$, it holds that $\limsup_{J\in\mathbb{N}(\mathcal{F})} \mathcal{D}_J^{\mathfrak{a}}\left(P_J\right) \sqsubset \mathcal{D}^{\mathfrak{a}}(P)$ by Proposition 2.30 (ii).

Third, we investigate the conditional mapping $P \mapsto \mathcal{S}(P)$. For the definition $S_J(P_J) := \sum_{\mathfrak{a}\in\mathfrak{A}}\left(E^{\mathfrak{a}} - \mathcal{D}^{\mathfrak{a}}(P_J)\right)$, it holds that

$$\limsup_{J\in\mathbb{N}(\mathcal{F})} \mathcal{S}_J\left(P_J\right) = \sum_{\mathfrak{a}\in\mathfrak{A}} E^{\mathfrak{a}} - \sum_{\mathfrak{a}\in\mathfrak{A}} \limsup_{J\in\mathbb{N}(\mathcal{F})} \mathcal{D}_J^{\mathfrak{a}}\left(P_J\right) \sqsubset \sum_{\mathfrak{a}\in\mathfrak{A}} E^{\mathfrak{a}} - \sum_{\mathfrak{a}\in\mathfrak{A}} \mathcal{D}^{\mathfrak{a}}(P) = \mathcal{S}(P) \tag{3.9}$$

showing that $P \mapsto \mathcal{S}(P)$ is outer semicontinuous.

To finally prove (3.2), we observe that by (3.9)

$$\limsup_{J\in\mathbb{N}(\mathcal{F})} W(P_J, Q) = \limsup_{J\in\mathbb{N}(\mathcal{F})} \operatorname*{ess\,sup}_{S_J\in\mathcal{S}(P_J)} \langle Q, S_J\rangle \tag{3.10}$$

$$= \operatorname*{ess\,sup}_{S\in\limsup_{J\in\mathbb{N}(\mathcal{F})}\mathcal{S}(P_J)} \langle Q, S\rangle \leq \operatorname*{ess\,sup}_{S\in\mathcal{S}(P)} \langle Q, S\rangle = W(P, Q) \tag{3.11}$$

for all $Q \in \Sigma$ which shows that $P \mapsto W(P,Q) : \Sigma \to L^0(\mathcal{F})$ is upper semicontinuous.

To conclude with (3.3), or that $Q \mapsto W(P,Q)$ is conditionally convex, we observe that

$$\begin{aligned} W\left(P, \lambda Q + (1-\lambda) Q'\right) &= \operatorname*{ess\,sup}_{S\in\mathcal{S}(P)} \left\langle \lambda Q + (1-\lambda) Q', S\right\rangle \\ &\leq \lambda \operatorname*{ess\,sup}_{S\in\mathcal{S}(P)} \langle Q, S\rangle + (1-\lambda) \operatorname*{ess\,sup}_{S\in\mathcal{S}(P)} \left\langle Q', S\right\rangle \\ &= \lambda W(P,Q) + (1-\lambda) W\left(P, Q'\right) \end{aligned}$$

for all $\lambda \in L^0(\mathcal{F})^d$ with $0 \leq \lambda \leq 1$, hence, $Q \mapsto W(P,Q)$ is conditionally convex.

Finally, (3.4) holds since $\langle Q, D^{\mathfrak{a}}(Q)\rangle \leq \langle Q, E^{\mathfrak{a}}\rangle$ for all $D^{\mathfrak{a}}(P) \in \mathcal{D}^{\mathfrak{a}}(P)$ implies that $\langle Q, S(Q)\rangle \geq 0$ for all $S(Q) \in \mathcal{S}(Q)$. □

**Theorem 3.2** (Existence of an Equilibrium Price)**.** *The Walrasian has a max/essinf point $\overline{P} \in \Sigma$ such that $0 \leq \operatorname{ess\,inf}_{Q\in\Sigma} W\left(\overline{P}, Q\right) = \operatorname{ess\,sup}_{P\in\Sigma} \operatorname{ess\,inf}_{Q\in\Sigma} W(P,Q)$. Moreover, this point $\overline{P}$ is an equilibrium price.*

*Proof.* Since the Walrasian is a Ky Fan function, by Theorem 2.49, the existence of the max/essinf-point follows. By (3.4), it follows that $0 \leq \operatorname{ess\,inf}_{Q\in\Sigma} W(Q,Q) \leq \operatorname{ess\,inf}_{Q\in\Sigma} W\left(\overline{P}, Q\right)$. Thus, $0 \leq \operatorname{ess\,inf}_{Q\in\Sigma} \operatorname{ess\,sup}_{S\in\mathcal{S}(\overline{P})} \langle Q, S\rangle$.

Next, we show that there exists $\overline{S} \in L^0(\mathcal{F})^d$ such that $\operatorname{ess\,sup}_{S\in\mathcal{S}(\overline{P})} \langle Q, S\rangle = \left\langle Q, \overline{S}\right\rangle$ for each $Q \in \Sigma$. Since $\operatorname{ess\,sup}_{S\in\mathcal{S}(\overline{P})} \langle Q, S\rangle \in L^0(\mathcal{F})$, there exists a conditional sequence $(S_J)_{J\in\mathbb{N}(\mathcal{F})}$ in $\mathcal{S}\left(\overline{P}\right)$ with $\operatorname{ess\,sup}_{S\in\mathcal{S}(\overline{P})} \langle Q, S\rangle = \operatorname{ess\,lim\,sup}_{J\in\mathbb{N}(\mathcal{F})} \left\langle Q, S_J\right\rangle$ by the definition of an essential supremum.

Since $\mathcal{S}(P) \sqsubset E^{\mathfrak{a}} - \mathcal{C}$ lives on $\Omega$, the conditional set $\limsup_{J\in\mathbb{N}(\mathcal{F})} S_J$ lives on $\Omega$. Thus, there exists $\mathcal{M} \sqsubset \mathbb{N}(\mathcal{F})^{\#}_{\infty}$ such that $\lim_{J\in\mathcal{M}} S_J$ exists and is denoted by $\overline{S} \in \mathcal{S}(P)$. Since $S_J = E^{\mathfrak{a}} - X_J$ for $X_J \in \operatorname{argmax}_{X\in\mathcal{C}} \left\{u^{\mathfrak{a}}(X) \mid \left\langle \overline{P}, X\right\rangle \leq \left\langle \overline{P}, E^{\mathfrak{a}}\right\rangle\right\}$ by definition we now consider the conditional family $(X_J)_{J\in\mathcal{M}}$ in $\mathcal{C} \sqsubset L^0(\mathcal{F})^d$ which contains a converging conditional subsequence $(X_J)_{J\in\mathcal{M}'} \to \overline{X}$ since $\mathcal{C}$ is conditionally compact.

We observe that $\langle \overline{P}, \overline{X} \rangle = \lim_{J\in\mathcal{M}'} \langle \overline{P}, X_J \rangle \leq \langle \overline{P}, E^{\mathfrak{a}} \rangle$. Also, by upper semicontinuity uf $u$, $u\left(\overline{X}\right) = u\left(\lim_{J\in\mathcal{M}'} X_J\right) \geq \operatorname{ess\,lim\,sup}_{J\in\mathcal{M}} u(X_J) = u\left(X_J\right)$ for all $J \in \mathcal{M}'$, thus $\overline{X}$ is a maximizer. That means, $\overline{S} = E^{\mathfrak{a}} - \overline{X}$ is such that $\operatorname{ess\,sup}_{S\in\mathcal{S}(\overline{P})}\langle Q, S\rangle = \operatorname{ess\,lim\,sup}_{J\in\mathbb{N}(\mathcal{F})} \langle Q, S_J \rangle = \langle Q, \overline{S}\rangle$. For fixed $Q \in \Sigma$, we denote $\overline{S}_Q\left(\overline{P}\right) := \overline{S}$ as just defined.

Hence, $\left\langle Q, \overline{S}_Q\left(\overline{P}\right)\right\rangle \geq 0$ for all $Q \in \Sigma$. In particular, this holds for the $j$-th unit vector on $A \in \mathcal{F}$, that is $\mathbb{1}_A Q_j = (0, \dots, 0, \mathbb{1}_A, 0, \dots, 0) \in \Sigma$. If $\overline{S}_{Q_j,k}$ denotes the $k$-th component of $\overline{S}_{Q_j}$, it holds that $\overline{S}_{Q_j} \geq 0$. Now, we define $\lambda_j := \frac{1}{d-1}\left(1 - \frac{\overline{P}_j\overline{S}_{Q_j,j}}{\sum_{j\leq d}\overline{P}_j\overline{S}_{Q_j,j}}\right)$ and observe that $\sum_{j\leq d}\lambda_j = 1$ and $0 \leq \lambda_j \leq 1$ for all $j \in \mathbb{N}$. Furthermore, let $\overline{S}_{\lambda,k} := \sum_{j\leq d}\lambda_j\overline{S}_{Q_j,k}$. Since $\overline{S}_\lambda \in \mathcal{S}\left(\overline{P}\right)$ by conditional convexity of $\mathcal{S}\left(\overline{P}\right)$, it holds that $\overline{P}_k\overline{S}_{Q_j,k} \geq -\sum_{i\neq k}\overline{P}_i\overline{S}_{Q_j,i}$. Then,

$$
\begin{aligned}
\overline{S}_{\lambda,k} &= \sum_{j\leq d}\lambda_j\overline{S}_{Q_j,k} \\
&= \frac{1}{d-1}\sum_{j\leq d}\left(1 - \frac{\overline{P}_j\overline{S}_{Q_j,j}}{\sum_{j\leq d}\overline{P}_j\overline{S}_{Q_j,j}}\right)\overline{S}_{Q_j,k} \\
&= \frac{1}{d-1}\sum_{j\neq k}\left(1 - \frac{\overline{P}_j\overline{S}_{Q_j,j}}{\sum_{j\leq d}\overline{P}_j\overline{S}_{Q_j,j}}\right)\overline{S}_{Q_j,k} + \frac{1}{d-1}\left(1 - \frac{\overline{P}_k\overline{S}_{Q_k,k}}{\sum_{j\leq d}\overline{P}_j\overline{S}_{Q_j,j}}\right)\overline{S}_{Q_k,k} \\
&= \frac{1}{d-1}\sum_{j\neq k}\left(\frac{\sum_{i\neq j}\overline{P}_i\overline{S}_{Q_i,i}}{\sum_{j\leq d}\overline{P}_j\overline{S}_{Q_j,j}}\right)\overline{S}_{Q_j,k} + \frac{1}{d-1}\left(1 - \frac{\overline{P}_k\overline{S}_{Q_k,k}}{\sum_{j\leq d}\overline{P}_j\overline{S}_{Q_j,j}}\right)\overline{S}_{Q_k,k} \\
&\geq \frac{1}{d-1}\sum_{j\neq k}\left(\frac{-\overline{P}_j\overline{S}_{Q_j,j}}{\sum_{j\leq d}\overline{P}_j\overline{S}_{Q_j,j}}\right)\overline{S}_{Q_j,k} + \frac{1}{d-1}\left(1 - \frac{\overline{P}_k\overline{S}_{Q_k,k}}{\sum_{j\leq d}\overline{P}_j\overline{S}_{Q_j,j}}\right)\overline{S}_{Q_k,k} \\
&= \frac{1}{d-1}\left(1 - \sum_{j\leq d}\left(\frac{\overline{P}_j\overline{S}_{Q_j,j}}{\sum_{j\leq d}\overline{P}_j\overline{S}_{Q_j,j}}\right)\overline{S}_{Q_j,k}\right) \\
&= 0
\end{aligned}
$$

Thus, $\overline{S}_\lambda\left(\overline{P}\right) \geq 0$, hence, $\overline{P}$ is an equilibrium price. □

In general, these equilibrium prices are not unique, mainly because of Proposition 2.30 if the maximizer is not unique by some given external information. Here we have only established that if $\varepsilon$-optimal prices always converge to the same price for $\varepsilon \to 0$, this limit price is the only optimal solution. Another approach to ensure this is the translation-invariance of the utilities, as for example in [CHKP16]. How this can be applied here is subject to further research.

## 3.2 Compactness of allocations

The assumption that the conditional set of allocations of goods is conditionally compact is rather technical and motivated mathematically. Its reasoning although might be done in an economic way.

According to [JW02], Jofré and Wets' assumption is translated into the conditional setting by assuming that $u^{\mathfrak{a}}$ is conditionally sup-compact, that is, $\operatorname{lev}_{\geq \alpha} u^{\mathfrak{a}} = \left\{X \in L^0(\mathcal{F})^d \mid u^{\mathfrak{a}}(X) \geq \alpha\right\}$ is conditionally compact for all $\alpha \in L^0(\mathcal{F})$. Since $u^{\mathfrak{a}}$ is assumed to be upper semicontinuous, this assumption means that only a boundedness restriction is added to attainable good allocations. Since there exits $X \in L^0(\mathcal{F})^d$ with $X \in \operatorname{int}\operatorname{dom} u^{\mathfrak{a}}$, the conditional set $\mathcal{C} := \operatorname{lev}_{\geq u^{\mathfrak{a}}(X)} u^{\mathfrak{a}}$ fulfills the model assumptions. This is also convenient with the standard optimization problem as in Theorem 2.12.

Another approach is a slight generalization of sensivity to large losses suggested by [DLVM97], also used in [CHKP16] or [FS04].

**Definition 3.3.** A conditional utility function $u\colon L^0(\mathcal{F})^d \to L^0(\mathcal{F})$ is sensitive to large losses if $\lim_{\lambda \to \infty} u(\lambda X) = -\infty$ for all $X \in L^0(\mathcal{F})^d$ with $\mathbb{1}_A X \in -\mathbb{1}_A L^0(\mathcal{F})^d_{++}$ for some $A \in \mathcal{F}_+$.

**Lemma 3.4.** *The level sets of a conditionally concave, upper semicontinuous, proper utility function are conditionally closed if the conditional utility function is sensitive to large losses.*

*Proof.* We assume that $u(E^{\mathfrak{a}}) > -\infty$ for some $E^{\mathfrak{a}} \in L^0(\mathcal{F})^d$ as in the model and remark that if $\lim_{\lambda \to \infty} u(\lambda X) = -\infty$ then $\lim_{\lambda \to \infty} u(\lambda X + E^{\mathfrak{a}}) = -\infty$ for $E^{\mathfrak{a}} \in L^0(\mathcal{F})^d$. We observe that $\left\{X \in L^0(\mathcal{F})^d_+ \mid \langle P, X\rangle \leq \langle P, E^{\mathfrak{a}}\rangle\right\}$ is conditionally bounded, thus conditionally compact. Suppose that there is $\lambda \in L^0(\mathcal{F})$ such that $\left\{X \in L^0(\mathcal{F})^d \mid u(X) \geq \lambda\right\}$ is not conditionally compact. Then, there exists a conditional sequence $(X_N)_{N \in \mathbb{N}(\mathcal{F})}$ in $L^0(\mathcal{F})^d$ with $\langle P, X_N\rangle \leq \langle P, E^{\mathfrak{a}}\rangle$, $u(X_N) \geq \lambda$ such that $\lim_{N \in \mathbb{N}(\mathcal{F})} \|X_N\| = \infty$ on $\Omega$, if only on $A$, on $A^c$ we are done. Then, the conditional sequence $(X_N - E^{\mathfrak{a}}) / \|X_N\|$ converges to $Y \in L^0(\mathcal{F})^d$, if necessary, we pass to a conditional subsequence. By construction, $\|Y\| = 1$, thus, $\mathbb{1}_A Y \in -\mathbb{1}_A L^0(\mathcal{F})^d_{++}$ for some $A \in \mathcal{F}_+$ due to the price restriction. Hence, for any $\lambda > 0$ it holds that

$$\begin{aligned} u(\lambda Y) &\geq \operatorname*{ess\,lim\,sup}_{N\in\mathbb{N}(\mathcal{F})} u\left(\lambda \frac{X_N - E^{\mathfrak{a}}}{\|X_N\|}\right) = \operatorname*{ess\,lim\,sup}_{N\in\mathbb{N}(\mathcal{F})} u\left(\frac{\lambda}{\|X_N\|} X_N + \left(1 - \frac{\lambda}{\|X_n\|}\right) E^{\mathfrak{a}}\right) \\ &\geq \operatorname*{ess\,lim\,sup}_{N\in\mathbb{N}(\mathcal{F})} \left(\frac{\lambda}{\|X_N\|} u(X_N) + \left(1 - \frac{\lambda}{\|X_n\|}\right) u(E^{\mathfrak{a}})\right) \\ &\geq \operatorname*{ess\,lim\,sup}_{N\in\mathbb{N}(\mathcal{F})} \left(\frac{\lambda}{\|X_N\|} \lambda + \left(1 - \frac{\lambda}{\|X_n\|}\right) u(E^{\mathfrak{a}})\right) = u(E^{\mathfrak{a}}) > -\infty \end{aligned}$$

in contradiction to that $u$ is sensitive to large losses. $\square$

**Remark 3.5.** We provide criteria for $\mathcal{D}^{\mathfrak{a}}(P)$ to consist of exactly one point for fixed $P \in \Sigma$, and consequences of this. Among other, we consider strict conditional concavity of the condtitional utility functions.

To that end, let $u^{\mathfrak{a}} \colon L^0(\mathcal{F})^d \to [-\infty; \infty[$ be conditionally strict concave. Let $X, Y \in \mathcal{D}^{\mathfrak{a}}(P)$ and $0 < \lambda < 1$. Then, $u^{\mathfrak{a}}(\lambda X + (1-\lambda) Y) > \lambda u^{\mathfrak{a}}(X) + (1-\lambda) u^{\mathfrak{a}}(Y) \in \mathcal{D}^{\mathfrak{a}}(P)$ in contradiction to the maximality in the definition of $\mathcal{D}^{\mathfrak{a}}(P)$. Thus, $X = Y$. By the proof of Lemma 3.1, it holds that $\limsup_{J\in\mathbb{N}(\mathcal{F})} \mathcal{D}^{\mathfrak{a}}_J(P_J) \sqsubset \mathcal{D}^{\mathfrak{a}}(P)$ whenever $(P_J)_{J\in\mathbb{N}(\mathcal{F})} \to P$. Thus, with $\mathcal{D}^{\mathfrak{a}}(P) = \{D^{\mathfrak{a}}(P)\}$, we have continuity, that is, $(P_J)_{J\in\mathbb{N}(\mathcal{F})} \to P$ implies $D^{\mathfrak{a}}(P_J) \to D^{\mathfrak{a}}(P)$.

## 3.3 Converging economies

We now consider continuity properties of the Walras equilibrium prices. Therefore, we consider a lopsidedly disturbed Ky Fan inequality and its application to a converging economy.

**Theorem 3.6.** *Let $\mathcal{C} \sqsubset L^0(\mathcal{F})^d$ be a conditionally convex, conditionally compact conditional subset. Let $F_J \colon L^0(\mathcal{F})^d \times L^0(\mathcal{F})^d \to \overline{L}^0(\mathcal{F})$, $J \in \mathbb{N}(\mathcal{F})$, be conditional functions with $0 \leq F_J(X,X) \leq \infty$ for $X \in \mathcal{C}$ and*

*(I). $X \mapsto F_J(X,Y)$ is lower semicontinuous for all $Y \in L^0(\mathcal{F})^d$,*

*(II). $Y \mapsto F_J(X,Y)$ is conditionally concave for all $X \in L^0(\mathcal{F})^d$,*

*for all $J \in \mathbb{N}(\mathcal{F})$. Assume the conditional sequence $(F_J)_{J\in\mathbb{N}(\mathcal{F})}$ to converge lopsided to a conditional function $F$. Then, it holds that $F(X,X) \geq 0$ for $X \in \mathcal{C}$ and*

*(i). $X \mapsto F(X,Y)$ is lower semicontinuous for all $Y \in L^0(\mathcal{F})^d$,*

*(ii). $Y \mapsto F(X,Y)$ is conditionally concave for all $Y \in L^0(\mathcal{F})^d$.*

*If* $X \in \limsup_{J\in\mathbb{N}(\mathcal{F})} \operatorname{argmax}_{X\in L^0(\mathcal{F})^d}\left(\operatorname{ess\,inf}_{Y\in L^0(\mathcal{F})^d} F_J(X,Y)\right)$ *is a cluster point of max/essinf-points of* $F_J$ *then* $X \in \operatorname{argmax}_{X\in L^0(\mathcal{F})^d}\left(\operatorname{ess\,inf}_{Y\in L^0(\mathcal{F})^d} F(X,Y)\right)$ *also is a max/essinf-point of* $F$.

*Proof.* First, we observe that max/essinf-points of $F_J$ are in $\mathcal{C}\times\mathcal{C}$ by Theorem 2.49 and Proposition 2.30. Since the conditional sequence $(F_J)_{J\in\mathbb{N}(\mathcal{F})}$ of conditional functions $F_J\colon L^0(\mathcal{F})^d \times L^0(\mathcal{F})^d \to \overline{L}^0(\mathcal{F})$ converges lopsided to the conditional function $F\colon L^0(\mathcal{F})^d \times L^0(\mathcal{F})^d \to \overline{L}^0(\mathcal{F})$ for all conditional sequences $\left(X_K^J\right)_{K\in\mathbb{N}(\mathcal{F})} \to X_J$ there exists a conditional sequence $\left(Y_K\right)_{K\in\mathbb{N}(\mathcal{F})} \to Y$ in $L^0(\mathcal{F})^d$ with the property that $\operatorname{ess\,lim\,sup}_{K\in\mathbb{N}(\mathcal{F})} F_K\left(X_K^J, Y_K\right) \leq F\left(X_J, Y\right)$.

Then, we observe that by lopsided convergence and (I),

$$\begin{aligned}
\operatorname*{ess\,lim\,inf}_{J\in\mathbb{N}(\mathcal{F})} F\left(X_J, Y\right) &\geq \operatorname*{ess\,lim\,sup}_{K\in\mathbb{N}(\mathcal{F})} \operatorname*{ess\,lim\,inf}_{J\in\mathbb{N}(\mathcal{F})} F_K\left(X_K^J, Y_K\right) \\
&\geq \operatorname*{ess\,lim\,inf}_{K\in\mathbb{N}(\mathcal{F})} \operatorname*{ess\,lim\,inf}_{J\in\mathbb{N}(\mathcal{F})} F_K\left(X_K^J, Y_K\right) \\
&\geq \operatorname*{ess\,lim\,inf}_{K\in\mathbb{N}(\mathcal{F})} F_K\left(X_K, Y_K\right).
\end{aligned}$$

Thus, there exists a conditional sequence $(X_K)_{K\in\mathbb{N}(\mathcal{F})} \to X$ such that for all conditional sequences $(Y_K)_{K\in\mathbb{N}(\mathcal{F})} \to Y$ it holds that $\operatorname{ess\,lim\,inf}_{K\in\mathbb{N}(\mathcal{F})} F_K(X_K, Y_K) \geq F(X,Y)$. Hence, finally, $\operatorname{ess\,lim\,inf}_{J\in\mathbb{N}(\mathcal{F})} F\left(X_J, Y\right) \geq F(X,Y)$, thus (i) holds.

Further, for $\lambda\in[0,1]$ and conditional sequences $(X_J)_{J\in\mathbb{N}(\mathcal{F})} \to X$, $(Y_J)_{J\in\mathbb{N}(\mathcal{F})} \to Y$ and $(Y_J')_{J\in\mathbb{N}(\mathcal{F})} \to Y'$ in $L^0(\mathcal{F})^d$, we observe that, by (II),

$$\begin{aligned}
F\left(X, \lambda Y + (1-\lambda)Y'\right) &\leq \operatorname*{ess\,lim\,inf}_{J\in\mathbb{N}(\mathcal{F})} F_J\left(X_J, \lambda Y_J + (1-\lambda) Y_J'\right) \\
&\leq \operatorname*{ess\,lim\,sup}_{J\in\mathbb{N}(\mathcal{F})} \lambda F_J(X_J, Y_J) + (1-\lambda) \operatorname*{ess\,lim\,sup}_{J\in\mathbb{N}(\mathcal{F})} F_J\left(X_J, Y_J'\right) \\
&\leq \lambda F_J(X,Y) + (1-\lambda) F\left(X, Y'\right)
\end{aligned}$$

which shows (ii).

For the last claim, observe that $\operatorname{ess\,inf}_{Y\in L^0(\mathcal{F})^d} F(X,Y) \geq 0$ for a cluster point $X$ of max/essinf-points $X_J$ of $F_J$ by Theorem 2.35. □

**Theorem 3.7.** *Let* $\mathrm{ECO} = (u^{\mathfrak{a}}, e^{\mathfrak{a}})_{\mathfrak{a}\in\mathfrak{A}}$ *be an economy satisfying* $e^{\mathfrak{a}} \in \operatorname{int}(\operatorname{dom} u^{\mathfrak{a}})$. *Let* $(\mathrm{ECO}_J)_{J\in\mathbb{N}(\mathcal{F})}$ *be a conditional sequence of economics* $\mathrm{ECO}^J = (u_J^{\mathfrak{a}}, e^{\mathfrak{a}})_{\mathfrak{a}\in\mathfrak{A}}$, $J\in\mathbb{N}(\mathcal{F})$, *that disturb the economy* ECO. *Assume that* $\operatorname{dom} u_J^{\mathfrak{a}} = \operatorname{dom} u^{\mathfrak{a}}$ *for all* $J\in\mathbb{N}(\mathcal{F})$ *and*

$\mathfrak{a} \in \mathfrak{A}$. *Assume further that* $\text{h-lim}_{J\in\mathbb{N}(\mathcal{F})} u_J^{\mathfrak{a}} = u^{\mathfrak{a}}$ *for all* $\mathfrak{a} \in \mathfrak{A}$. *Then, all the economies* ECO, $\text{ECO}_J$, $J \in \mathbb{N}(\mathcal{F})$ *have an equilibrium price* $\overline{P}, \overline{P}_J$ *in* $\Sigma$, $\limsup_{J\in\mathbb{N}(\mathcal{F})} \overline{P}_J$ *lives on* $\Omega$ *and* $P \in \limsup_{J\in\mathbb{N}(\mathcal{F})} \overline{P}_J$ *is market equilibrium of the economy* ECO.

*Proof.* We introduce Walrasians for each economy and prove their lopsided convergence. By $\mathcal{S}, \mathcal{S}_J$, we denote the excess supply conditional sets of the economies $\text{ECO}, \text{ECO}_J$ and by $W, W_J$ their Walrasians. We show that the conditional sequence $(W_J)_{J\in\mathbb{N}(\mathcal{F})}$ converges lopsided to $W$, that is, for all $(P,Q) \in \Sigma \times \Sigma$, it holds that

$$\forall\ (P_J)_{J\in\mathbb{N}(\mathcal{F})} \to P\ \exists\ (Q_J)_{J\in\mathbb{N}(\mathcal{F})} \to Q\colon\ \operatorname*{ess\,lim\,sup}_{J\in\mathbb{N}(\mathcal{F})} W_J(P_J, Q_J) \leq W(P,Q), \tag{3.12}$$

$$\exists\ (P_J)_{J\in\mathbb{N}(\mathcal{F})} \to P\ \forall\ (Q_J)_{J\in\mathbb{N}(\mathcal{F})} \to Q\colon\ \operatorname*{ess\,lim\,inf}_{J\in\mathbb{N}(\mathcal{F})} W_J(P_J, Q_J) \geq W(P,Q) \tag{3.13}$$

for $P_J, Q_J \in \Sigma$ and $J \in \mathbb{N}(\mathcal{F})$. To show (3.12), consider $Q_J = Q$ for all $J \in \mathbb{N}(\mathcal{F})$ and

$$\operatorname*{ess\,lim\,sup}_{J\in\mathbb{N}(\mathcal{F})} W_J(Q_J, P_J) = \operatorname*{ess\,lim\,sup}_{J\in\mathbb{N}(\mathcal{F})} W_J(Q, P_J) \leq W(P,Q)$$

as in the proof of Lemma 3.1. To show (3.13), choose $P_J = P$ for all $J \in \mathbb{N}(\mathcal{F})$. Then, for all conditional sequences $(Q_J)_{J\in\mathbb{N}(\mathcal{F})} \to Q$ in $\Sigma$, it holds that

$$\begin{aligned}\operatorname*{ess\,lim\,inf}_{J\in\mathbb{N}(\mathcal{F})} W_J(P_J, Q_J) &= \operatorname*{ess\,lim\,inf}_{J\in\mathbb{N}(\mathcal{F})} W_J(P, Q_J)\\ &= \operatorname*{ess\,sup}_{S\in\mathcal{S}(P)} \left\langle \operatorname*{ess\,lim\,inf}_{J\in\mathbb{N}(\mathcal{F})} Q_J, S \right\rangle = W(P,Q)\end{aligned}$$

which shows (3.13). Now, the claim follows as a consequence of Theorem 3.6. □

# 4 Path-dependent conditional optimization

In this chapter, we introduce an approach to optimization in a conditional setting dependent on the observed path. The utility function may depend on history, for example, dependent on the recent change of a stock market index oder changes in budget constraints which will be our main example derived from the setting in Chapter 3. Conceivably, it may be applicated to increasing risk aversion over time.

First, we give an introduction to conditional functions between different underlying $\sigma$-algebras. Then, we discuss the Euclidean conditional topologies with respect to these different $\sigma$-algebras. This is applied to convergence properties of conditional sequences and their images under conditional function with respect to different $\sigma$-algebras. To concentrate on the methodology, we present the setting of the path-dependent conditional optimization in discrete time in the continuous case. This will be generalized to the common porperties of utility functions, semicontinuity and convexity.

## 4.1 Conditional functions with respect to different $\sigma$-algebras

We consider mappings $f\colon L^0(\mathcal{G}) \to L^0(\mathcal{F})$, first for $\mathcal{G} \subset \mathcal{F}$.

**Definition 4.1.** A mapping $f\colon L^0(\mathcal{G}) \to L^0(\mathcal{F})$ is called $\mathcal{G}$-stable if

$$f(\mathbb{1}_A X + \mathbb{1}_{A^c} Y) = \mathbb{1}_A f(X) + \mathbb{1}_{A^c} f(Y)$$

for all $X, Y \in L^0(\mathcal{G})$ and $A \in \mathcal{G}$. We call $f$ a $\mathcal{G}$-stable conditional function.

We observe that $L^0(\mathcal{F})$ is a $\mathcal{G}$-conditional set, and thus, endowed with a $\mathcal{G}$-conditional topology, a conditional topological space.

**Definition 4.2.** A $\mathcal{G}$-stable mapping $f\colon L^0(\mathcal{G}) \to L^0(\mathcal{F})$ is called $\mathcal{G}$-continuous if for any conditional sequence $(X_N)_{N\in\mathbb{N}(\mathcal{G})}$ that converges to $X$ with respect to the Euclidean $\mathcal{G}$-conditional topology the conditional sequence $(f(X_N))_{N\in\mathbb{N}(\mathcal{G})}$ converges to $f(X)$ with respect to the Euclidean $\mathcal{G}$-conditional topology.

**Remark 4.3.** We observe that the Euclidean $\mathcal{F}$-conditional topology is finer than the Euclidean $\mathcal{G}$-conditional topology on the conditional set $L^0(\mathcal{F})$.

**Example 4.4.** An example of such a conditional function is motivated by the utility function from Chapter 3. This example is discussed in detail in Section 4.4. Ordinarily, let $\Delta\pi \in L^0(\mathcal{F})^d$ be differences of prices and $\vartheta \in L^0(\mathcal{G})^d$ be trading strategies. Then, the value $V(\vartheta) := \vartheta\Delta\pi$ is a $\mathcal{G}$-conditional function. If the prices are bounded, then $V(\cdot)$ is a $\mathcal{G}$-continuous conditional function.

Next, we consider mappings $g\colon L^0(\mathcal{F}) \to L^0(\mathcal{G})$ for $\mathcal{G} \subset \mathcal{F}$.

**Definition 4.5.** A mapping $g\colon L^0(\mathcal{F}) \to L^0(\mathcal{G})$ is called $\mathcal{G}$-stable if

$$g(\mathbb{1}_A X + \mathbb{1}_{A^c} Y) = \mathbb{1}_A g(X) + \mathbb{1}_{A^c} g(Y)$$

for all $X, Y \in L^0(\mathcal{G})$ and $A \in \mathcal{G}$. We call $g$ a $\mathcal{G}$-stable conditional function.

**Definition 4.6.** A $\mathcal{G}$-stable mapping $g\colon L^0(\mathcal{F}) \to L^0(\mathcal{G})$ is called continuous if for any conditional sequence $(X_N)_{N\in\mathbb{N}(\mathcal{G})}$ that converges to $X$ with respect to the Euclidean $\mathcal{G}$-conditional topology the conditional sequence $(f(X_N))_{N\in\mathbb{N}(\mathcal{G})}$ converges to $f(X)$ with respect to the Euclidean $\mathcal{G}$-conditional topology.

**Example 4.7.** The conditional expectation $\mathbb{E}[\cdot \mid \mathcal{G}] : L^0(\mathcal{F}) \to L^0(\mathcal{G})$ is $\mathcal{G}$-stable and $\mathcal{G}$-continuous.

In Definition 4.2 and 4.5, we write $\lim_{n\in\mathbb{N}} f(X_n)$ or $\lim_{n\in\mathbb{N}} g(X_n)$ if the limit exists for the $\mathcal{G}$-stable conditional function $f$ or $g$, respectively, and for the $\mathcal{G}$-conditional sequence $(X_n)_{n\in\mathbb{N}}$ and do not name the conditional topology explicitely. It is indicated by the stability property.

In Section 4.6, we examine $\mathcal{G}$-conditional functions with respect to arbitrary conditional topologies. For the model the properties of the conditional Euclidean topology on $L^0$ presented in Section 4.1 are sufficient. To construct the conditional expectation with respect to arbitrary conditional measures and arbitrary sub-$\sigma$-algebras and to present general ideas of the time-shifts we will give an overview in Section 4.6.

## 4.2 Introduction to the setting

The time steps are $0, \dots, t, \dots, T$. Information is given by a filtration $(\mathcal{F}_t)_{t\in\{0,\dots,T\}}$, where the initial $\sigma$-algebra $\mathcal{F}_0$ is trivial. Additionally, history gives information by observed values $x_0, \dots, x_{t-1}$ in $\mathbb{R}$ up to time $t-1$, denoted by $x = (x_0, \dots, x_{t-1})$. In the future, these values are random, $X_s \in L^0(\mathcal{F}_s)$ for $t \le s \le T$. Dependent on this history, the agent has trading constraints, given by a conditionally compact conditional set $\mathcal{C} \sqsubset L^0(\mathcal{F}_{t-1})^d$. The trading strategies are denoted by $\vartheta_{t-1} \in \mathcal{C}$. We remark that this conditionally compact set may also depend on the history, for simplicity, we assume that it is constant. We want to optimize the utilities $(u_s)_{t \le s \le T}$ where each utility is defined by

$$\begin{aligned} u_s \colon \mathbb{R}^{s-1} \times L^0(\mathcal{F}_s) \times L^0(\mathcal{F}_{s-1})^d &\to L^0(\mathcal{F}_s) \\ (x, X_s, \vartheta_{s-1}) &\mapsto u_t(x, X_s, \vartheta_{s-1}). \end{aligned} \tag{4.1}$$

**Example 4.8.** This is a dynamic version of the utility in Chapter 3 where history is given by real endowments. We give the details in Section 4.4.

### 4.2.1 General construction idea

For the continuous case, we often apply the following lemma. To formalize, we introduce the conditional set

$$E(\mathcal{F}) := \Bigg\{ X \in L^0(\mathcal{F}) \mid \exists \text{ a finite representation } X = \sum_{1 \le k \le \overline{k}} \mathbb{1}_{A_k} x_k, \\ x_k \in \mathbb{R}, (A_k)_{1 \le k \le k'} \text{ is a finite partition of } \Omega \text{ in } \mathcal{F},\, \overline{k} \in \mathbb{N} \Bigg\}$$

as a conditional subset of $L^0(\mathcal{F})$ of elementary conditional real numbers, and we call $X = \sum_{1 \le k \le \overline{k}} \mathbb{1}_{A_k} x_k$ for $x_k \in \mathbb{R}$ and $\overline{k} \in \mathbb{N}$ a normal representation, always for a finite partition $(A_k)_{1 \le k \le \overline{k}}$ of $\Omega$ in $\mathcal{F}$.

In the sequel, we assume every function to be nonnegative. If this is not the case, we do the proof for the positive and negative part seperately as in standard measure theory.

For the following lemma, we recall that $L^0(\mathcal{F})$ as a conditional topological space is regarded as an $\mathbb{R}$-module and that $\mathbb{R}$ is the measurable functions with respect to the trivial $\sigma$-algebra $\mathcal{F}_0$.

**Lemma 4.9.** *Let $f \colon \mathbb{R} \times \mathbb{R} \times L^0(\mathcal{F}) \to L^0(\mathcal{F})$ be a mapping with the following properties.*

$$\begin{aligned} &f\left(x, y, \mathbb{1}_A \vartheta + \mathbb{1}_{A^c} \vartheta'\right) = \mathbb{1}_A f(x, y, \vartheta) + \mathbb{1}_{A^c} f\left(x, y, \vartheta'\right) \\ &\qquad \textit{for all } (x, y) \in \mathbb{R}^2,\ \vartheta, \vartheta' \in L^0(\mathcal{F}) \textit{ and } A \in \mathcal{F}, \end{aligned} \tag{4.2}$$

*that is,* $\vartheta \mapsto f(x,y,\vartheta)$ *is* $\mathcal{F}$*-stable for fixed* $(x,y) \in \mathbb{R}\times\mathbb{R}$*, and*

$$\begin{aligned}&\lim_{n\in\mathbb{N}} f(x^n,y^n,\vartheta^n) = f(x,y,\vartheta) \text{ in } L^0(\mathcal{F})\\ &\qquad \text{for all } \lim_{n\in\mathbb{N}} x^n = x \text{ in } \mathbb{R}, \lim_{n\in\mathbb{N}} y^n = y \text{ in } \mathbb{R} \text{ and } \lim_{n\in\mathbb{N}} \vartheta^n = \vartheta \text{ in } L^0(\mathcal{F})\end{aligned} \tag{4.3}$$

*that is,* $(x,y,\vartheta) \mapsto f(x,y,\vartheta)$ *is jointly* $\mathcal{F}_0\times\mathcal{F}_0\times\mathcal{F}$*-continuous.*
*Further, let* $F\colon \mathbb{R}\times L^0(\mathcal{F})\times L^0(\mathcal{F}) \to L^0(\mathcal{F})$ *be defined by*

$$\begin{aligned}&F\left(x, \mathbb{1}_A y + \mathbb{1}_{A^c} y', \mathbb{1}_A\vartheta + \mathbb{1}_{A^c}\vartheta'\right) = \mathbb{1}_A F(x,y,\vartheta) + \mathbb{1}_{A^c} F\left(x,y',\vartheta'\right)\\ &\qquad \text{for all} x\in\mathbb{R},\ y,y'\in L^0(\mathcal{F}),\ \vartheta,\vartheta'\in L^0(\mathcal{F}) \text{ and} A\in\mathcal{F}\end{aligned} \tag{4.4}$$

*that is,* $(y,\vartheta)\mapsto F(x,y,\vartheta)$ *is jointly* $\mathcal{F}$*-stable for fixed* $x\in\mathbb{R}$*,*

$$F(x, c\cdot\mathbb{1}_\Omega, \vartheta) = f(x,c,\vartheta) \text{ for} c\in\mathbb{R} \tag{4.5}$$

*that is,* $F$ *and* $f$ *are identical on* $\mathbb{R}\times\mathbb{R}\times L^0(\mathcal{F})$*, and*

$$F(x,Y,\vartheta) := \operatorname*{ess\,lim\,sup}_{K\in\mathbb{N}} \operatorname*{ess\,lim\,sup}_{\substack{(Y^n)_{n\in\mathbb{N}}\subset E(\mathcal{F})\\ \lim_{n\in\mathbb{N}} Y^n = Y}} F(x, Y^n\wedge K\vee -K, \vartheta). \tag{4.6}$$

*It is the unique mapping with* (4.4)*,* (4.5) *and*

$$\begin{aligned}&\lim_{n\in\mathbb{N}} F(x^n,Y^n,\vartheta^n) = F(x,Y,\vartheta) \text{ in } L^0(\mathcal{F})\\ &\qquad \text{for all } \lim_{n\in\mathbb{N}} x^n = x \text{ in } \mathbb{R}, \lim_{n\in\mathbb{N}} Y^n = Y \text{ in } L^0(\mathcal{F}) \text{ and } \lim_{n\in\mathbb{N}} \vartheta^n = \vartheta \text{ in } L^0(\mathcal{F}),\end{aligned} \tag{4.7}$$

*thus, the unique mapping with* (4.4)*,* (4.5) *and such that* $(x,Y,\vartheta)\mapsto F(x,Y,\vartheta)$ *is jointly* $\mathcal{F}_0\times\mathcal{F}\times\mathcal{F}$*-continuous.*

The proof will be given in section 4.5 and we recall that all limits are with respect to the Euclidean conditional topology on $L^0(\mathcal{F})$ or in $\mathbb{R}$.

### 4.2.2 Utility's and generator's dependence on the path

We continue with the model and give the properties of the utilities and generator dependent on the observed history with the notation $x = (x_0,\dots,x_{t-1})$ as before. We assume that $u_T \equiv 0$ and that we have already an optimal utility at time $t$ that is as follows. The utility

$$
\begin{aligned}
u_t \colon \mathbb{R}^t \times L^0\left(\mathcal{F}_{t-1}\right)^d &\to L^0\left(\mathcal{F}_t\right) \\
\left(x, x_t, \vartheta_{t-1}\right) &\mapsto u_t\left(x, x_t, \vartheta_{t-1}\right)
\end{aligned}
\tag{4.8}
$$

is a mapping that has the following properties,

$$
\begin{aligned}
&u_t\left(x, x_t, \mathbb{1}_A \vartheta_{t-1} + \mathbb{1}_{A^c} \vartheta'\right) = \mathbb{1}_A u_t\left(x, x_t, \vartheta_{t-1}\right) + \mathbb{1}_{A^c} u_t\left(x, x_t, \vartheta'_{t-1}\right) \\
&\qquad \text{for all } \left(x, x_t\right) \in \mathbb{R}^{t-1} \times \mathbb{R},\ \vartheta_{t-1}, \vartheta'_{t-1} \in L^0\left(\mathcal{F}_{t-1}\right)^d \text{ and } A \in \mathcal{F}_{t-1},
\end{aligned}
\tag{4.9}
$$

that is, $\vartheta \mapsto u_t\left(x, x_t, \vartheta\right)$ is $\mathcal{F}_{t-1}$-stable for all $\left(x, x_t\right) \in \mathbb{R}^{t-1} \times \mathbb{R}$, and

$$
\begin{aligned}
&\lim_{n\in\mathbb{N}} u_t\left(x^n, x_t^n, \vartheta_{t-1}^n\right) = u_t\left(x, x_t, \vartheta_{t-1}\right) \\
&\qquad \text{for } \lim_{n\in\mathbb{N}} x^n = x \text{ in } \mathbb{R}^{t-1},\ \lim_{n\in\mathbb{N}} x_t^n = x_t \text{ in } \mathbb{R} \text{ and } \lim_{n\in\mathbb{N}} \vartheta_{t-1}^n = \vartheta_{t-1} \text{ in } L^0\left(\mathcal{F}_{t-1}\right)^d,
\end{aligned}
\tag{4.10}
$$

that is, $\left(x, x_t, \vartheta\right) \mapsto u_t\left(x, x_t, \vartheta\right)$ is jointly $\mathcal{F}_0 \times \mathcal{F}_0 \times \mathcal{F}_{t-1}$-continuous. We recall that the equation $\lim_{n\in\mathbb{N}} u_t\left(x^n, x_t^n, \vartheta_{t-1}^n\right) = u_t\left(x, x_t, \vartheta_{t-1}\right)$ in (4.10) is in the $L^0\left(\mathcal{F}_{t-1}\right)$-module $L^0\left(\mathcal{F}_t\right)$.

We remark here that for an optimal utility at time $t$ the history is known up to time $t$. We know want to derive an optimal utility $u_{t-1}$ at time $t-1$, there, the history is not known, thus, $X_t$ is random. That means, in view of Lemma 4.9, regarded from time $t-1$ the utility at time $t$ depends on the path $x$ up to time $t-1$, a random path step $X_t \in L^0\left(\mathcal{F}_t\right)$ and the trading strategies $\vartheta \in L^0\left(\mathcal{F}_{t-1}\right)^d$. The generator is then to eliminate the random path step to derive the optimal utility $u_{t-1}$ and its properties. An example for that is conditional expectation in Example 4.23. The generator plays the role of an aggregator in the theory of recursive utilities, see for example [Ski98].

Thus, with Lemma 4.9 and the assumptions (4.8), (4.9) and (4.10) on the utility $u_t$ we construct

$$
\begin{aligned}
\tilde{u}_t \colon \mathbb{R}^{t-1} \times L^0\left(\mathcal{F}_t\right) \times L^0\left(\mathcal{F}_{t-1}\right)^d &\to L^0\left(\mathcal{F}_t\right) \\
\left(x, X, \vartheta\right) &\mapsto \tilde{u}_t\left(x, X, \vartheta\right),
\end{aligned}
$$

such that

$$\begin{aligned}&\tilde{u}_t\left(x, x_t \mathbb{1}_\Omega, \vartheta_{t-1}\right) = u_t\left(x, x_t, \vartheta_{t-1}\right),\\&\qquad \text{for all}\, x \in \mathbb{R}^{t-1}, x_t \in \mathbb{R}\,\text{and}\, \vartheta_{t-1} \in L^0\left(\mathcal{F}_{t-1}\right)^d,\end{aligned} \tag{4.11}$$

$$\begin{aligned}&\tilde{u}_t\left(x, \mathbb{1}_A X_t + \mathbb{1}_{A^c}\overline{X}_t, \mathbb{1}_A \vartheta_{t-1} + \mathbb{1}_{A^c}\overline{\vartheta}_t\right) = \mathbb{1}_A \tilde{u}_t\left(x, X_t, \vartheta_{t-1}\right) + \mathbb{1}_{A^c}\tilde{u}_t\left(x, \overline{X}_t, \overline{\vartheta}_{t-1}\right)\\&\qquad \text{for all}\, x \in \mathbb{R}^{t-1}, X_t, \overline{X}_t \in L^0\left(\mathcal{F}_t\right), \vartheta_{t-1}, \overline{\vartheta}_{t-1} \in L^0\left(\mathcal{F}_{t-1}\right)^d \text{ and } A \in \mathcal{F}_{t-1}, \textit{and}\end{aligned} \tag{4.12}$$

$$\begin{aligned}&\lim_{n\in\mathbb{N}} \tilde{u}_t\left(x^n, X_t^n, \vartheta_{t-1}^n\right) = \tilde{u}_t\left(x, X_t, \vartheta_{t-1}\right)\\&\qquad \text{for all } \lim_{n\in\mathbb{N}} x^n = x \,\text{in}\, \mathbb{R}^{t-1}, \lim_{n\in\mathbb{N}} X_t^n = X_t \,\text{in}\, L^0\left(\mathcal{F}_t\right), \lim_{n\in\mathbb{N}} \vartheta_{t-1}^n = \vartheta_{t-1} \,\text{in}\, L^0\left(\mathcal{F}_{t-1}\right)^d.\end{aligned} \tag{4.13}$$

Finally, the generator is defined as follows. The mapping

$$\begin{aligned}G_{t-1}\colon L^0\left(\mathcal{F}_t\right) \times \mathbb{R}^{t-1} \times L^0\left(\mathcal{F}_t\right) &\to L^0\left(\mathcal{F}_{t-1}\right)\\ (u, x, X) &\mapsto G_{t-1}\left(u, x, X\right)\end{aligned}$$

is jointly $\mathcal{F}_{t-1}$-stable in the sense that

$$\begin{aligned}&G_{t-1}\left(\mathbb{1}_A u + \mathbb{1}_{A^c}u', x, \mathbb{1}_A X + \mathbb{1}_{A^c}X'\right) = \mathbb{1}_A G_{t-1}\left(u, x, X\right) + \mathbb{1}_{A^c}G_{t-1}\left(u', x, X'\right)\\&\qquad \text{for all}\, u, u' \in L^0\left(\mathcal{F}_t\right), x \in \mathbb{R}^{t-1}, X, X' \in L^0\left(\mathcal{F}_t\right) \text{ and } A \in \mathcal{F}_{t-1}\end{aligned} \tag{4.14}$$

and it is jointly $\mathcal{F}_{t-1} \times \mathcal{F}_0 \times \mathcal{F}_{t-1}$-continuous, that is,

$$\begin{aligned}&\lim_{n\in\mathbb{N}} G_{t-1}\left(u^n, x^n, X^n\right) = G_{t-1}\left(u, x, X\right)\\&\qquad \text{for all } \lim_{n\in\mathbb{N}} u^n = u \,\text{in}\, L^0\left(\mathcal{F}_{t-1}\right), \lim_{n\in\mathbb{N}} x^n = x, \text{ and } \lim_{n\in\mathbb{N}} X^n = X \,\text{in}\, L^0\left(\mathcal{F}_{t-1}\right).\end{aligned} \tag{4.15}$$

We remark here that the generator is not defined first for known history up to time $t$ and then enlarged as proposed in Lemma 4.9. First, Lemma 4.9 cannot be applied technically because of nonmatching spaces. Second, a pointwise definition of the generator does not match known and random time steps.

Now, we discuss how the next random path step is dependent on the observed path. Here, we assume that

$$\begin{aligned}X_t\colon \mathbb{R}^{t-1} &\to L^0\left(\mathcal{F}_t\right)\\ x &\mapsto X_t\left(x\right)\end{aligned} \tag{4.16}$$

is continuous, that is,

$$\lim_{n\in\mathbb{N}} X_t\left(x^n\right) = X_t\left(x\right) \text{ for all } \lim_{n\in\mathbb{N}} x^n = x. \tag{4.17}$$

This is mainly motivated by the example in Section 4.4 and its mathematical applications. An enhancement of the model here would be appreciated.

We sum up some properties of utilities and generator. To that end, we define the following auxiliary functions. The mapping

$$\begin{aligned} \hat{u}_t \colon \mathbb{R}^{t-1} \times L^0\left(\mathcal{F}_{t-1}\right)^d &\to L^0\left(\mathcal{R}_t\right) \\ (x, \vartheta) &\mapsto \hat{u}_t\left(x, \vartheta\right) \end{aligned}$$

defined by

$$\hat{u}_t\left(x, \vartheta\right) := \tilde{u}_t\left(x, X_t\left(x\right), \vartheta\right) \tag{4.18}$$

is jointly $\mathcal{F}_0 \times \mathcal{F}_{t-1}$-continuous, that is,

$$\lim_{n\in\mathbb{N}} \hat{u}_t\left(x^n, \vartheta^n_{t-1}\right) = \hat{u}_t\left(x, \vartheta\right) \text{ for all } \lim_{n\in\mathbb{N}} x^n = x \,\text{and}\, \lim_{n\in\mathbb{N}} \vartheta^n_{t-1} = \vartheta_{t-1} \tag{4.19}$$

by (4.17) and (4.13). Futher, define the mapping

$$\begin{aligned} \hat{G}_{t-1} \colon \mathbb{R}^{t-1} \times L^0\left(\mathcal{F}_{t-1}\right)^d &\to L^0\left(\mathcal{F}_{t-1}\right) \\ (x, \vartheta_{t-1}) &\mapsto \hat{G}_{t-1}\left(x, \vartheta_{t-1}\right) \end{aligned}$$

by

$$\hat{G}_{t-1}\left(x, \vartheta_{t-1}\right) := G_{t-1}\left(\hat{u}_t\left(x, \vartheta_{t-1}\right), x, X\left(x\right)\right).$$

We examine the properties of the mapping $\hat{G}_{t-1}$, the generator that provides the recursiveness of the utilities, thus,

$$u_{t-1}\left(x\right) = \operatorname*{ess\,sup}_{\vartheta\in\mathcal{C}} \hat{G}_{t-1}\left(x, \vartheta\right).$$

We want to show that $\left(\hat{G}^n_{t-1}\right)_{n\in\mathbb{N}}$ with $\hat{G}^n_{t-1}\left(x, \vartheta_{t-1}\right) = \hat{G}_{t-1}\left(x^n, \vartheta_{t-1}\right)$ is a conditional sequence of conditional functions that hypoconverges. Now, let $\lim_{n\in\mathbb{N}} \vartheta^n_{t-1} = \vartheta_{t-1}$. In view of Corollary 2.31 we want to show that $\left(\vartheta_{t-1} \mapsto \hat{G}^n_{t-1}\left(x, \vartheta_{t-1}\right)\right)_{n\in\mathbb{N}}$ hypoconverges.

By definition, we observe that

$$\begin{aligned}
&\hat{G}_{t-1}\left(x, \mathbb{1}_A \vartheta_{t-1} + \mathbb{1}_{A^c}\vartheta'_{t-1}\right)\\
&= G_{t-1}\left(\hat{u}_t\left(x, \mathbb{1}_A \vartheta_{t-1} + \mathbb{1}_{A^c}\vartheta'_{t-1}\right), x, X(x)\right)\\
&= G_{t-1}\left(\tilde{u}_t\left(x, X(x), \mathbb{1}_A \vartheta_{t-1} + \mathbb{1}_{A^c}\vartheta'_{t-1}\right), x, X(x)\right)\\
&= G_{t-1}\left(\mathbb{1}_A \tilde{u}_t\left(x, X(x), \vartheta_{t-1}\right) + \mathbb{1}_{A^c}\tilde{u}_t\left(x, X(x), \vartheta'_{t-1}\right), x, X(x)\right)\\
&= G_{t-1}\left(\mathbb{1}_A \hat{u}_t\left(x, \vartheta_{t-1}\right) + \mathbb{1}_{A^c}\hat{u}_t\left(x, \vartheta'_{t-1}\right), x, X(x)\right)\\
&= \mathbb{1}_A G_{t-1}\left(\hat{u}_t\left(x, \vartheta_{t-1}\right), x, X(x)\right) + \mathbb{1}_{A^c} G_{t-1}\left(\hat{u}_t\left(x, \vartheta'_{t-1}\right), x, X(x)\right)\\
&= \mathbb{1}_A \hat{G}_{t-1}\left(x, \vartheta_{t-1}\right) + \mathbb{1}_{A^c}\hat{G}_{t-1}\left(x, \vartheta'\right)
\end{aligned}$$

for all $A \in \mathcal{F}_{t-1}$ by (4.12) and (4.14). Thus, $\vartheta_{t-1} \mapsto \hat{G}_{t-1}(x, \vartheta_{t-1})$ is a conditional function.

Further, let $\lim_{n\in\mathbb{N}} x^n = x$ and $\lim_{n\in\mathbb{N}} \vartheta^n_{t-1} = \vartheta$. Then, with the definition $\hat{G}^n_{t-1}(x, \vartheta) = \hat{G}_{t-1}(x^n, \vartheta)$ it holds that

$$\begin{aligned}
&\lim_{n\in\mathbb{N}} \hat{G}^n_{t-1}\left(x, \vartheta^n_{t-1}\right)\\
&= \lim_{n\in\mathbb{N}} \hat{G}_{t-1}\left(x^n, \vartheta^n_{t-1}\right)\\
&= \lim_{n\in\mathbb{N}} G_{t-1}\left(\tilde{u}_t\left(x^n, X(x^n), \vartheta^n_{t-1}\right), x^n, X(x^n)\right)\\
&= G_{t-1}\left(\lim_{n\in\mathbb{N}} \tilde{u}_t\left(x^n, X(x^n), \vartheta^n_{t-1}\right), \lim_{n\in\mathbb{N}} x^n, \lim_{n\in\mathbb{N}} X(x^n)\right)\\
&= G_{t-1}\left(\tilde{u}_t\left(\lim_{n\in\mathbb{N}} x^n, \lim_{n\in\mathbb{N}} X(x^n), \lim_{n\in\mathbb{N}} \vartheta^n_{t-1}\right), x, X\left(\lim_{n\in\mathbb{N}} x^n\right)\right)\\
&= G_{t-1}\left(\tilde{u}_t\left(x, X\left(\lim_{n\in\mathbb{N}} x^n\right), \vartheta_{t-1}\right), x, X(x)\right)\\
&= G_{t-1}\left(\tilde{u}_t\left(x, X(x), \vartheta_{t-1}\right), x, X(x)\right)
\end{aligned}$$

by (4.15), (4.13), (4.17), $\lim_{n\in\mathbb{N}} x^n = x$ and $\lim_{n\in\mathbb{N}} \vartheta^n_{t-1} = \vartheta$. Hence, in particular, $\operatorname{ess\,lim\,sup}_{n\in\mathbb{N}} \hat{G}^n_{t-1}\left(x, \vartheta^n_{t-1}\right) \leq G_{t-1}\left(\tilde{u}_t\left(x, X(x), \vartheta_{t-1}\right), x, X(x)\right)$. Since the conditional set $\mathcal{C} \sqsubset L^0(\mathcal{F}_{t-1})^d$ is conditionally compact we can apply Corollary 2.31 and obtain

$$\lim_{n\in\mathbb{N}} u_{t-1}(x^n) = \lim_{n\in\mathbb{N}} \operatorname*{ess\,sup}_{\vartheta\in\mathcal{C}} \hat{G}^n_{t-1}(\vartheta) = \operatorname*{ess\,sup}_{\vartheta\in\mathcal{C}} \hat{G}_{t-1}(\vartheta) = u_{t-1}(x). \tag{4.20}$$

**Lemma 4.10.** *Let $u_t$ be a utility mapping given by* (4.8), (4.9) *and* (4.10). *Then, for a conditionally compact $\mathcal{C} \sqsubset L^0(\mathcal{F}_{t-1})^d$, the mapping $x \mapsto u_{t-1}(x)$ defined by*

$u_{t-1}(x) := \operatorname{ess\,sup}_{\vartheta\in\mathcal{C}} \hat{G}_{t-1}(x,\vartheta)$ *is* $\mathcal{F}_0$*-continuous, that is,* $\lim_{n\in\mathbb{N}} u_{t-1}(x^n) = u_{t-1}(x)$ *for* $\lim_{n\in\mathbb{N}} x^n = x$.

*Proof.* These are the assumptions from Section 4.2.2 and it has been shown in (4.20). □

**Example 4.11.** A typical generator is conditional expectation with respect to a conditional measure in its generalized form as in Example 4.23 by what has been discussed in Section 4.6.

## 4.3 Hypoconvergent generators and semicontinuous and conditionally concave utilities

In order to obtain (4.20) the assumptions on generator and utilities may be relaxed, and thus, the model will be generalized. We replace them by a semicontinuous assumption on the utility and a Fatou type assumption on the generator in order to propose a slightly different approach to normal integrands, as or example in [RW09]. We recall that $\underline{L}^0$ denotes the random variables with values in $\mathbb{R}\cup\{-\infty\}$.
Again, we assume that $X_t\colon \mathbb{R}^{t-1}\to L^0(\mathcal{F}_t)$ is continuous, that is

$$\lim_{n\in\mathbb{N}} X_t(x^n) = X_t(x) \text{ for all } \lim_{n\in\mathbb{N}} x^n = x, \tag{4.21}$$

again, for the $\mathbb{R}$-module $L^0(\mathcal{F})$.
The utility

$$\begin{aligned} u_t\colon \mathbb{R}^{t-1}\times\mathbb{R}\times L^0(\mathcal{F}_{t-1})^d &\to \underline{L}^0(\mathcal{F}_t)\\ (x,x_t,\vartheta_{t-1}) &\mapsto u_t(x,x_t,\vartheta_{t-1}) \end{aligned} \tag{4.22}$$

is a mapping that has the following properties,

$$\begin{aligned} &u_t\left(x,x_t,\mathbb{1}_A\vartheta_{t-1}+\mathbb{1}_{A^c}\vartheta'\right) = \mathbb{1}_A u_t(x,x_t,\vartheta_{t-1}) + \mathbb{1}_{A^c} u_t\left(x,x_t,\vartheta'_{t-1}\right)\\ &\qquad \text{for all } (x,x_t)\in\mathbb{R}^{t-1}\times\mathbb{R},\ \vartheta_{t-1},\vartheta'_{t-1}\in\mathcal{C} \text{ and } A\in\mathcal{F}_{t-1}, \end{aligned} \tag{4.23}$$

that is, $\vartheta\mapsto u_t(x,x_t,\vartheta)$ is $\mathcal{F}_{t-1}$-stable for all $(x,x_t)$, and

$$\begin{aligned} &\operatorname{ess}\limsup_{n\in\mathbb{N}} u_t\left(x^n,x_t^n,\vartheta_{t-1}^n\right) \le u_t(x,x_t,\vartheta_{t-1})\\ &\qquad \text{for } \lim_{n\in\mathbb{N}} x^n = x \text{ in } \mathbb{R}^{t-1},\ \lim_{n\in\mathbb{N}} x_t^n = x_t \text{ in } \mathbb{R} \text{ and } \lim_{n\in\mathbb{N}} \vartheta_{t-1}^n = \vartheta_{t-1} \text{ in } L^0(\mathcal{F}_{t-1})^d, \end{aligned} \tag{4.24}$$

thus, $\mathcal{F}_{t-1}$-upper semicontinuous (that is, upper semicontinuous w.r.t. the conditional Euclidean topology for the $L^0(\mathcal{F}_{t-1})$-module $L^0(\mathcal{F}_t)$) for converging history and converging trading strategies, and

$$\begin{aligned} &u_t\left(x, x_t, \lambda\vartheta_{t-1} + (1-\lambda)\vartheta'\right) \geq \lambda u_t(x, x_t, \vartheta_{t-1}) + (1-\lambda)\, u_t\left(x, x_t, \vartheta'_{t-1}\right) \\ &\qquad \text{for all } (x, x_t) \in \mathbb{R}^t,\ \vartheta_{t-1}, \vartheta'_{t-1} \in \mathcal{C} \text{ and } \lambda \in [0,1], \end{aligned} \tag{4.25}$$

thus, conditionally concave in the trading strategies for constant history. For recursiveness of the utilities the aim is to construct a mapping

$$\begin{aligned} u_{t-1}\colon \mathbb{R}^{t-1} &\to \underline{L}^0(\mathcal{F}_{t-1}) \\ x &\mapsto u_{t-1}(x) \end{aligned} \tag{4.26}$$

which is upper semicontinuous and independent of trading strategies, the control variables, at time $t$.

The extension $\tilde{u}_t$ of $u_t$ will fullfill

$$\begin{aligned} &\tilde{u}_t(x, x_t, \vartheta_{t-1}) = u_t(x, x_t, \vartheta_{t-1}), \\ &\qquad \text{for all } x \in \mathbb{R}^{t-1}, X_t \in L^0(\mathcal{F}_t) \text{ and } \vartheta_{t-1} \in L^0(\mathcal{F}_{t-1})^d, \end{aligned} \tag{4.27}$$

$$\begin{aligned} &\tilde{u}_t\left(x, \mathbb{1}_A X_t + \mathbb{1}_{A^c}\overline{X}_t, \mathbb{1}_A\vartheta_{t-1} + \mathbb{1}_{A^c}\overline{\vartheta}_t\right) = \mathbb{1}_A\tilde{u}_t(x, X_t, \vartheta_{t-1}) + \mathbb{1}_{A^c}\tilde{u}_t\left(x, \overline{X}_t, \overline{\vartheta}_{t-1}\right) \\ &\qquad \text{for all } x \in \mathbb{R}^{t-1}, X_t, \overline{X}_t \in L^0(\mathcal{F}_t)\ \vartheta_{t-1}, \overline{\vartheta}_{t-1} \in L^0(\mathcal{F}_{t-1})^d \text{ and } A \in \mathcal{F}_{t-1}, and \end{aligned} \tag{4.28}$$

$$\begin{aligned} &\operatorname{ess}\limsup_{n\in\mathbb{N}} \tilde{u}_t\left(x^n, X_t^n, \vartheta_{t-1}^n\right) \leq \tilde{u}_t(x, X_t, \vartheta_{t-1}) \\ &\qquad \text{for all } \lim_{n\in\mathbb{N}} x^n = x \text{ in } \mathbb{R}^{t-1}, \lim_{n\in\mathbb{N}} X_t^n = X_t \text{ in } L^0(\mathcal{F}_t), \lim_{n\in\mathbb{N}} \vartheta_{t-1}^n = \vartheta_{t-1} \text{ in } L^0(\mathcal{F}_{t-1})^d \end{aligned} \tag{4.29}$$

according to the construction in Lemma 4.13 at the end of this section.

Analaguously, the mapping

$$\begin{aligned} \hat{u}_t\colon \mathbb{R}^{t-1} \times L^0(\mathcal{F}_{t-1})^d &\to \underline{L}^0(\mathcal{F}_t) \\ (x, \vartheta) &\mapsto \hat{u}_t(x, \vartheta) \end{aligned}$$

defined by

$$\hat{u}_t(x, \vartheta) := \tilde{u}_t(x, X_t(x), \vartheta) \tag{4.30}$$

is jointly $\mathcal{F}_0 \times \mathcal{F}_{t-1}$-upper semicontinuous, that is,

$$\begin{aligned} &\operatorname*{ess\,lim\,sup}_{n\in\mathbb{N}} \hat{u}_t\left(x^n, \vartheta_{t-1}^n\right) \leq \hat{u}_t\left(x, \vartheta\right) \\ &\qquad \text{for all } \lim_{n\in\mathbb{N}} x^n = x \,\text{in}\, \mathbb{R}^{t-1} \,\text{and}\, \lim_{n\in\mathbb{N}} \vartheta_{t-1}^n = \vartheta_{t-1} \,\text{in}\, L^0\left(\mathcal{F}_{t-1}\right)^d \end{aligned} \tag{4.31}$$

by (4.21) and (4.29).

The generator mapping

$$\begin{aligned} G_{t-1} \colon \underline{L}^0\left(\mathcal{F}_t\right) \times \mathbb{R}^{t-1} \times L^0\left(\mathcal{F}_t\right) &\to \underline{L}^0\left(\mathcal{F}_{t-1}\right) \\ (u, x, X) &\mapsto G_{t-1}\left(u, x, X\right) \end{aligned}$$

is jointly $\mathcal{F}_{t-1}$-stable in the sense that

$$\begin{aligned} &G_{t-1}\left(\mathbb{1}_A u + \mathbb{1}_{A^c} u', x, \mathbb{1}_A X + \mathbb{1}_{A^c} X'\right) = \mathbb{1}_A G_{t-1}\left(u, x, X\right) + \mathbb{1}_{A^c} G_{t-1}\left(u', x, X'\right) \\ &\quad \text{for all}\, u, u' \in \underline{L}^0\left(\mathcal{F}_t\right),\, x \in \mathbb{R}^{t-1},\, X, X' \in L^0\left(\mathcal{F}_t\right) \text{ and } A \in \mathcal{F}_{t-1}, \end{aligned} \tag{4.32}$$

it is monotone for fixed history, that is,

$$G_{t-1}\left(u, x, X\right) \leq G_{t-1}\left(u', x, X\right) \text{ for}\, u \leq u' \,\text{and fixed}\, x \in \mathbb{R}^{t-1}, X \in L^0\left(\mathcal{F}_t\right) \tag{4.33}$$

and it has a Fatou property for converging history, that is,

$$\begin{aligned} &\operatorname*{ess\,lim\,sup}_{n\in\mathbb{N}} G_{t-1}\left(u^n, x^n, X^n\right) \leq G_{t-1}\left(\operatorname*{ess\,lim\,sup}_{n\in\mathbb{N}} u^n, x, X\right) \\ &\qquad \text{for all } \lim_{n\in\mathbb{N}} x^n = x \,\text{in}\, \mathbb{R}^{t-1} \,\text{and}\, \lim_{n\in\mathbb{N}} X^n = X \,\text{in}\, L^0\left(\mathcal{F}_{t-1}\right). \end{aligned} \tag{4.34}$$

We remark that the property (4.33) yields a time-consistency.

Again, finally, we define the mapping

$$\begin{aligned} \hat{G}_{t-1} \colon \mathbb{R}^{t-1} \times L^0\left(\mathcal{F}_{t-1}\right)^d &\to L^0\left(\mathcal{F}_{t-1}\right) \\ \left(x, \vartheta_{t-1}\right) &\mapsto \hat{G}_{t-1}\left(x, \vartheta_{t-1}\right) \end{aligned}$$

by

$$\hat{G}_{t-1}\left(x, \vartheta_{t-1}\right) := G_{t-1}\left(\hat{u}_t\left(x, \vartheta_{t-1}\right), x, X\left(x\right)\right).$$

All stable properties remain as in Section 4.2.2, thus,

$$\hat{G}_{t-1}\left(x, \mathbb{1}_A\vartheta_{t-1} + \mathbb{1}_{A^c}\vartheta'_{t-1}\right) = \mathbb{1}_A\hat{G}_{t-1}\left(x, \vartheta_{t-1}\right) + \mathbb{1}_{A^c}\hat{G}_{t-1}\left(x, \vartheta'\right)$$

for all $x \in \mathbb{R}^{t-1}$, $\vartheta_{t-1}, \vartheta'_{t-1} \in L^0\left(\mathcal{F}_{t-1}\right)^d$ and $A \in \mathcal{F}_{t-1}$.
Further,

$$\begin{aligned}
&\operatorname{ess}\limsup_{n\in\mathbb{N}} \hat{G}^n_{t-1}\left(x, \vartheta^n_{t-1}\right) \\
&= \operatorname{ess}\limsup_{n\in\mathbb{N}} \hat{G}_{t-1}\left(x^n, \vartheta^n_{t-1}\right) \\
&= \operatorname{ess}\limsup_{n\in\mathbb{N}} G_{t-1}\left(\tilde{u}_t\left(x^n, X\left(x^n\right), \vartheta^n_{t-1}\right), x^n, X\left(x^n\right)\right) \\
&\leq G_{t-1}\left(\operatorname{ess}\limsup_{n\in\mathbb{N}} \tilde{u}_t\left(x^n, X\left(x^n\right), \vartheta^n_{t-1}\right), \lim_{n\in\mathbb{N}} x^n, \lim_{n\in\mathbb{N}} X\left(x^n\right)\right) \\
&\leq G_{t-1}\left(\tilde{u}_t\left(\lim_{n\in\mathbb{N}} x^n, \lim_{n\in\mathbb{N}} X\left(x^n\right), \lim_{n\in\mathbb{N}} \vartheta^n_{t-1}\right), x, X\left(\lim_{n\in\mathbb{N}} x^n\right)\right) \\
&= G_{t-1}\left(\tilde{u}_t\left(x, X\left(\lim_{n\in\mathbb{N}} x^n\right), \vartheta_{t-1}\right), x, X\left(x\right)\right) \\
&= G_{t-1}\left(\tilde{u}_t\left(x, X\left(x\right), \vartheta_{t-1}\right), x, X\left(x\right)\right).
\end{aligned} \tag{4.35}$$

by (4.34), (4.33), 4.29, (4.17), $\lim_{n\in\mathbb{N}} x^n = x$ and $\lim_{n\in\mathbb{N}} \vartheta^n_{t-1} = \vartheta_{t-1}$.
The result is stated in the following lemma.

**Lemma 4.12.** *Let $u_t$ be a utility mapping given by* (4.22)*,* (4.23) *and* (4.24)*. Then, for a conditionally compact conditional set $\mathcal{C} \sqsubset L^0\left(\mathcal{F}_{t-1}\right)^d$, the mapping $x \mapsto u_{t-1}\left(x\right)$ defined by $u_{t-1}\left(x\right) := \operatorname{ess\,sup}_{\vartheta\in\mathcal{C}} \hat{G}_{t-1}\left(x, \vartheta\right)$ is upper semicontinuous for converging history, that is, similarly to the assumption* (4.24)*, $\operatorname{ess}\limsup_{n\in\mathbb{N}} u_{t-1}\left(x^n\right) \leq u_{t-1}\left(x\right)$ for $\lim_{n\in\mathbb{N}} x^n = x$.*

*Proof.* These are the assumptions presented in Section 4.3 where the result has been proven in (4.35). □

**Lemma 4.13.** *Let $f\colon \mathbb{R} \times \mathbb{R} \times L^0\left(\mathcal{F}\right) \to \underline{L}^0\left(\mathcal{F}\right)$ be a mapping with the following properties.*

$$\begin{aligned}
&f\left(x, y, \mathbb{1}_A\vartheta + \mathbb{1}_{A^c}\vartheta'\right) = \mathbb{1}_A f\left(x, y, \vartheta\right) + \mathbb{1}_{A^c} f\left(x, y, \vartheta'\right) \\
&\qquad \text{for all } (x, y) \in \mathbb{R}^2,\ \vartheta, \vartheta' \in L^0\left(\mathcal{F}\right) \text{ and } A \in \mathcal{F},
\end{aligned} \tag{4.36}$$

*that is,* $\vartheta \mapsto f(x,y,\vartheta)$ *is* $\mathcal{F}$*-stable for fixed* $(x,y) \in \mathbb{R}\times\mathbb{R}$*, and*

$$\begin{aligned}&\operatorname*{ess\,lim\,sup}_{n\in\mathbb{N}} f(x^n,y^n,\vartheta^n) \le f(x,y,\vartheta)\\ &\qquad \text{for all } \lim_{n\in\mathbb{N}} x^n = x \text{ in } \mathbb{R}, \lim_{n\in\mathbb{N}} y^n = y \text{ in } \mathbb{R} \text{ and } \lim_{n\in\mathbb{N}} \vartheta^n = \vartheta \text{ in } L^0(\mathcal{F})\end{aligned} \tag{4.37}$$

*that is,* $(x,y,\vartheta) \mapsto f(x,y,\vartheta)$ *is jointly* $\mathcal{F}_0 \times \mathcal{F}_0 \times \mathcal{F}$*-upper semicontinuous. Further, let* $F\colon \mathbb{R}\times L^0(\mathcal{F})\times L^0(\mathcal{F}) \to \underline{L}^0(\mathcal{F})$ *be defined by*

$$\begin{aligned}&F\left(x, \mathbb{1}_A y + \mathbb{1}_{A^c} y', \mathbb{1}_A\vartheta + \mathbb{1}_{A^c}\vartheta'\right) = \mathbb{1}_A F(x,y,\vartheta) + \mathbb{1}_{A^c} F\left(x,y',\vartheta'\right)\\ &\qquad \text{for all} x \in \mathbb{R},\ y,y' \in L^0(\mathcal{F}),\ \vartheta,\vartheta' \in L^0(\mathcal{F}) \text{ and} A \in \mathcal{F}\end{aligned} \tag{4.38}$$

*that is,* $(y,\vartheta) \mapsto F(x,y,\vartheta)$ *is jointly* $\mathcal{F}$*-stable for fixed* $x \in \mathbb{R}$*,*

$$F(x, c\cdot \mathbb{1}_\Omega, \vartheta) = f(x,c,\vartheta) \text{ for} c \in \mathbb{R} \tag{4.39}$$

*that is,* $F$ *and* $f$ *are identical on* $\mathbb{R}\times\mathbb{R}\times L^0(\mathcal{F})$*, and*

$$F(x,Y,\vartheta) := \operatorname*{ess\,lim\,sup}_{K\in\mathbb{N}} \operatorname*{ess\,lim\,sup}_{\substack{(Y^n)_{n\in\mathbb{N}}\subset E(\mathcal{F})\\ \lim_{n\in\mathbb{N}} Y^n = Y}} F(x, Y^n \wedge K \vee -K, \vartheta). \tag{4.40}$$

*It is the unique mapping with* (4.38)*,* (4.39) *and*

$$\begin{aligned}&\operatorname*{ess\,lim\,sup}_{n\in\mathbb{N}} F(x^n,Y^n,\vartheta^n) \le F(x,Y,\vartheta)\\ &\qquad \text{for all } \lim_{n\in\mathbb{N}} x^n = x \text{ in } \mathbb{R}, \lim_{n\in\mathbb{N}} Y^n = Y \text{ in } L^0(\mathcal{F}) \text{ and } \lim_{n\in\mathbb{N}} \vartheta^n = \vartheta \text{ in } L^0(\mathcal{F}),\end{aligned} \tag{4.41}$$

*thus, the unique mapping with* (4.38)*,* (4.39) *and such that* $(x,Y,\vartheta) \mapsto F(x,Y,\vartheta)$ *is jointly* $\mathcal{F}_0\times\mathcal{F}\times\mathcal{F}$*-upper semicontinuous.*

The proof is presented in Section 4.5.

## 4.4 Example

We return to the setting of Walras equilibrium prices from Section 3.1. In a multi-period model, the equilibrium may be solved by considering a one period model from the initial time to time $T$ given the filtration $(\mathcal{F}_t)_{t=0,\dots,T}$ of $\sigma$-algebras. Here, we propose a stepwise maximization similar to the Dynamic Programming Principle making use of Chapter 4.

The endowments $E_t^{\mathfrak{a}} \in L^0(\mathcal{F}_t)_+^d$ constitute the observed history. An additional endowment may depend continuously on the history, for example, a constant production, that is,

$$\begin{aligned} E_t^{\mathfrak{a}}\colon L^0(\mathcal{F}_1)^d \times \ldots \times L^0(\mathcal{F}_{t-1})^d &\to L^0(\mathcal{F}_t)^d \\ \left(E_1^{\mathfrak{a}}, \ldots, E_{t-1}^{\mathfrak{a}}\right) &\mapsto E_t^{\mathfrak{a}}\left(\left(E_1^{\mathfrak{a}}, \ldots, E_{t-1}^{\mathfrak{a}}\right)\right) \end{aligned} \tag{4.42}$$

is continuous, that is,

$$\lim_{n\in\mathbb{N}} E_t^{\mathfrak{a}}\left(E^{\mathfrak{a},n}\right) = E_t^{\mathfrak{a}}\left(E^{\mathfrak{a}}\right) \text{ for all } \lim_{n\in\mathbb{N}} E^{\mathfrak{a},n} = E^{\mathfrak{a}} \tag{4.43}$$

for $E^{\mathfrak{a}} = \left(E_1^{\mathfrak{a}}, \ldots, E_{t-1}^{\mathfrak{a}}\right)$ and $E^{\mathfrak{a},n} = \left(E_1^{\mathfrak{a},n}, \ldots, E_{t-1}^{\mathfrak{a},n}\right)$. The trading strategies are $\vartheta_t \in L^0(\mathcal{F}_t)^d$, the ways, the endowment can be exchanged. For the notation $\vartheta^{\mathfrak{a}} = \left(\vartheta_1^{\mathfrak{a}}, \ldots, \vartheta_{t-1}^{\mathfrak{a}}\right)$ and $\vartheta^{\mathfrak{a},n} = \left(\vartheta_1^{\mathfrak{a},n}, \ldots, \vartheta_{t-1}^{\mathfrak{a},n}\right)$, the utility

$$\begin{aligned} u_t\colon L^0(\mathcal{F}_1)^d \times \ldots \times L^0(\mathcal{F}_{t-1})^d \times L^0(\mathcal{F}_t)^d \times L^0(\mathcal{F}_1)^d \times \ldots \times L^0(\mathcal{F}_t)^d \\ \to \underline{L}^0(\mathcal{F}_t) \\ \left(E_1^{\mathfrak{a}}, \ldots, E_t^{\mathfrak{a}}, \vartheta_1^{\mathfrak{a}}, \ldots, \vartheta_t^{\mathfrak{a}}\right) \mapsto u_t\left(E_1^{\mathfrak{a}}, \ldots, E_t^{\mathfrak{a}}, \vartheta_1^{\mathfrak{a}}, \ldots, \vartheta_t^{\mathfrak{a}}\right) \end{aligned} \tag{4.44}$$

is jointly $\mathcal{F}_1 \times \ldots \times \mathcal{F}_{t-1} \times \mathcal{F}_t \times \mathcal{F}_1 \times \ldots \times \mathcal{F}_t$-stable, upper semicontinuous for converging history and converging trading strategies, cf. (4.24), and conditionally concave in the trading strategies for constant history, cf. (4.25).

The generator may be jointly $\mathcal{F}_1 \times \ldots \times \mathcal{F}_{t-1} \times \mathcal{F}_t \times \mathcal{F}_1 \times \ldots \times \mathcal{F}_t$-stable in utility and trading strategies, (4.14), monotone for fixed history, (4.33) and may have the Fatou property for converging history, (4.34). These assumptions are fullfilled in the economy described in Chapter 3, now, in multiple period. Further, the generator may be such that a bunch of goods if either useless or useful in all times, that is, for any fixed path and trading strategies $(E_{t-1}, E_t(E_{t-1}), \vartheta_{t-1}, \vartheta_t(\vartheta_{t-1}))$, it holds that $G_{t-1}(u_t, E_{t-1}, E_t(E_{t-1}), \vartheta_{t-1}, \vartheta_t(\vartheta_{t-1})) = -\infty$ if and only if $u_t = -\infty$, that is if the utility at $t$ is proper, then also the utility at $t-1$. This is, for example, the case if $E_{t-1}^{\mathfrak{a}} + \vartheta_{t-1}^{\mathfrak{a}} + E_t^{\mathfrak{a}}$ is the bunch of goods that is traded at time $t$.

Now, our induction assumption is, that under all the given assumptions in Section 4.4, there exists an equilibrium price $\hat{P}_t \in \Sigma_t = \left\{P \in L^0(\mathcal{F}_t)_+^d \mid \sum_{i\leq d} P_i = 1\right\}$, that is, there exists

$$\hat{\vartheta}_t^{\mathfrak{a}} \in \mathcal{D}_t^{\mathfrak{a}}\left(\hat{P}\right) = \left\{Y \in L^0(\mathcal{F}_t)^d \mid Y \in \operatorname{argmax}_{Y\in\mathcal{C}}\left\{u^{\mathfrak{a}}(Y) \mid Y \in \mathcal{C}_t^{\mathfrak{a}}\left(\hat{P}\right)\right\}\right\}$$

for $\mathcal{C}_t^{\mathfrak{a}}(P) = \operatorname{dom} u^{\mathfrak{a}} \sqcap \left\{ Y \in L^0(\mathcal{F}_t)^d \mid \langle P, Y \rangle \leq \langle P, E^{\mathfrak{a}} \rangle \right\}$ and a conditionally compact conditional set $\mathcal{C}_t \in L^0(\mathcal{F}_t)^d$ with $\sum_{\mathfrak{a} \in \mathfrak{A}} \hat{\vartheta}_t^{\mathfrak{a}} \geq 0$.

We will show that there is an equilibrium price $\hat{P}_{t-1} \in \Sigma_{t-1}$. As in Chapter 3, we define $v_t$ and $v_{t-1}$. Let

$$\begin{aligned}
&\mathbb{1}_A v_t\left(E_{t-1}^{\mathfrak{a}}, E_t^{\mathfrak{a}}\left(E_{t-1}^{\mathfrak{a}}\right), \vartheta_{t-1}^{\mathfrak{a}}, \vartheta_t^{\mathfrak{a}}\left(\vartheta_{t-1}^{\mathfrak{a}}\right)\right) \\
&\quad := \begin{cases} \mathbb{1}_A u_t\left(E_{t-1}^{\mathfrak{a}}, E_t^{\mathfrak{a}}\left(E_{t-1}^{\mathfrak{a}}\right), \vartheta_{t-1}^{\mathfrak{a}}, \vartheta_t^{\mathfrak{a}}\left(\vartheta_{t-1}^{\mathfrak{a}}\right)\right) & \text{if } \left\langle P_t\left(P_{t-1}\right), \vartheta_t^{\mathfrak{a}}\left(\vartheta_{t-1}^{\mathfrak{a}}\right)\right\rangle \leq 0, \\ -\mathbb{1}_A \infty & \text{else,} \end{cases}
\end{aligned}$$

and

$$\mathbb{1}_A v_{t-1}\left(E_{t-1}^{\mathfrak{a}}, \vartheta_{t-1}^{\mathfrak{a}}\right) \quad := \begin{cases} \mathbb{1}_A u_t\left(E_{t-1}^{\mathfrak{a}}, \vartheta_{t-1}^{\mathfrak{a}}\right) & \text{if } \left\langle P_{t-1}, \vartheta_{t-1}^{\mathfrak{a}}\right\rangle \leq 0, \\ -\mathbb{1}_A \infty & \text{else.} \end{cases} \tag{4.45}$$

All properties on utilities and generator that are applied in the sequel have been proved in Chapters 3 and 4. In fact, the original properties that have been imposed on the utilities directly are replaced by those of the generator.

Now, we try to find $\hat{\vartheta}_{t-1}^{\mathfrak{a}}$ such that

$$v_{t-1}^{\mathfrak{a}}\left(E^{\mathfrak{a}}, \hat{\vartheta}_{t-1}^{\mathfrak{a}}, P_{t-1}\right) = \operatorname*{ess\,sup}_{\vartheta_{t-1}^{\mathfrak{a}} \in \mathcal{C}_{t-1}} v_{t-1}^{\mathfrak{a}}\left(E^{\mathfrak{a}}, \vartheta_{t-1}^{\mathfrak{a}}, P_{t-1}\right). \tag{4.46}$$

In the sequel, we omit the agent in the terms, and to shorten notation, we write $\chi_{\mathcal{C}}(\vartheta)$ which is 1 if $\vartheta \in \mathcal{C}$ and $-\infty$ else, and conclude that

$$\begin{aligned}
&\operatorname*{ess\,sup}_{\vartheta_{t-1}^{\mathfrak{a}} \in \mathcal{C}_{t-1}} v_{t-1}\left(E^{\mathfrak{a}}, \vartheta_{t-1}^{\mathfrak{a}}, P_{t-1}\right) \\
&= \operatorname*{ess\,sup}_{\vartheta_{t-1}^{\mathfrak{a}} \in \mathcal{C}_{t-1}} \chi_{\{\langle P_{t-1}, \vartheta_{t-1}^{\mathfrak{a}}\rangle \leq 0\}}\left(\vartheta_{t-1}^{\mathfrak{a}}\right) u_{t-1}\left(E^{\mathfrak{a}}, \vartheta_{t-1}^{\mathfrak{a}}\right) \\
&= \operatorname*{ess\,sup}_{\vartheta_{t-1}^{\mathfrak{a}} \in \mathcal{C}_{t-1}} \chi_{\{\langle P_{t-1}, \vartheta_{t-1}^{\mathfrak{a}}\rangle \leq 0\}}\left(\vartheta_{t-1}^{\mathfrak{a}}\right) \\
&\qquad \operatorname*{ess\,sup}_{\vartheta_t^{\mathfrak{a}} \in \mathcal{C}_t} G_{t-1}\left(v_t\left(E^{\mathfrak{a}}, E_t^{\mathfrak{a}}\left(E^{\mathfrak{a}}\right), \vartheta_{t-1}^{\mathfrak{a}}, \vartheta_t^{\mathfrak{a}}\left(\vartheta_{t-1}^{\mathfrak{a}}\right), P_{t-1}, P_t\left(P_{t-1}\right)\right), E^{\mathfrak{a}}, E_t^{\mathfrak{a}}\left(E^{\mathfrak{a}}\right)\right) \\
&= \operatorname*{ess\,sup}_{\vartheta_{t-1}^{\mathfrak{a}} \in \mathcal{C}_{t-1}} \chi_{\{\langle P_{t-1}, \vartheta_{t-1}^{\mathfrak{a}}\rangle \leq 0\}}\left(\vartheta_{t-1}^{\mathfrak{a}}\right) \\
&\qquad \operatorname*{ess\,sup}_{\vartheta_t^{\mathfrak{a}} \in \mathcal{C}_t} G_{t-1}(u_t\left(E^{\mathfrak{a}}, E_t^{\mathfrak{a}}\left(E^{\mathfrak{a}}\right), \vartheta_{t-1}^{\mathfrak{a}}, \vartheta_t^{\mathfrak{a}}\left(\vartheta_{t-1}^{\mathfrak{a}}\right)\right) \chi_{\{\langle P_t(P_{t-1}), \vartheta_t^{\mathfrak{a}}\left(\vartheta_{t-1}^{\mathfrak{a}}\right)\rangle \leq 0\}}, E^{\mathfrak{a}}, E_t^{\mathfrak{a}}\left(E^{\mathfrak{a}}\right)).
\end{aligned}$$

By assumption on the equilibrium price $\hat{P}_t$ and 4.33,

$$= \operatorname*{ess\,sup}_{\vartheta^{\mathfrak{a}}_{t-1} \in \mathcal{C}_{t-1}} \chi_{\{\langle P_{t-1}, \vartheta^{\mathfrak{a}}_{t-1}\rangle \leq 0\}} \left(\vartheta^{\mathfrak{a}}_{t-1}\right)$$
$$G_{t-1}\left(u_t\left(E^{\mathfrak{a}}, E^{\mathfrak{a}}_t\left(E^{\mathfrak{a}}\right), \vartheta^{\mathfrak{a}}_{t-1}, \hat{\vartheta}^{\mathfrak{a}}_t\left(\vartheta^{\mathfrak{a}}_{t-1}\right)\right)\chi_{\{\langle \hat{P}_t(P_{t-1}), \hat{\vartheta}^{\mathfrak{a}}_t\left(\vartheta^{\mathfrak{a}}_{t-1}\right)\rangle \leq 0\}}, E^{\mathfrak{a}}, E^{\mathfrak{a}}_t\left(E^{\mathfrak{a}}\right)\right)$$

Since $G_{t-1}$ has the property that if it is $-\infty$ at time $t$, it is $-\infty$ at time $t-1$, this is equal to

$$= \operatorname*{ess\,sup}_{\vartheta^{\mathfrak{a}}_{t-1} \in \mathcal{C}_{t-1}} \chi_{\{\langle P_{t-1}, \vartheta^{\mathfrak{a}}_{t-1}\rangle \leq 0\}} \chi_{\{\langle \hat{P}_t(P_{t-1}), \hat{\vartheta}^{\mathfrak{a}}_t\left(\vartheta^{\mathfrak{a}}_{t-1}\right)\rangle \leq 0\}} \left(\vartheta^{\mathfrak{a}}_{t-1}\right)$$
$$G_{t-1}\left(u_t\left(E^{\mathfrak{a}}, E^{\mathfrak{a}}_t\left(E^{\mathfrak{a}}\right), \vartheta_{t-1}, \hat{\vartheta}_t\left(\vartheta_{t-1}\right)\right), E^{\mathfrak{a}}, E^{\mathfrak{a}}_t\left(E^{\mathfrak{a}}\right)\right)$$

Here, the optimization problem has the same properties as in the induction assumption, with one exception. The half plane condition for the trading strategies and the structure of the sets $\mathcal{C}^{\mathfrak{a}}\left(P_{t-1}\right)$ have changed. It is now, whatever prices are at time $t-1$ and $t$, the trading restrictions are never violated, even if prices in previous periodes have changed. But this assumption is the classical one, also imposed when considering a one step model from initial time to final time. Thus, the claim is proven.

## 4.5 Proofs of the lemmas

We give now the proof of Lemma 4.9.

*Proof.* First, we consider the mapping $F$ on the conditional set $\mathbb{R} \times E\left(\mathcal{F}\right) \times L^0\left(\mathcal{F}\right)$. Let $Y = \sum_{1 \leq k \leq \bar{k}} \mathbb{1}_{A_k} y_k = \sum_{1 \leq k' \leq \bar{k}'} \mathbb{1}_{A_{k'}} y_{k'}$ be two normal representations of $Y \in E\left(\mathcal{F}\right)$. By the defining properties (4.4) and (4.5), $\mathbb{1}_{A_k \cap A_{k'}} F\left(x, Y, \vartheta\right) = \mathbb{1}_{A_k \cap A_{k'}} f\left(x, y_k, \vartheta\right) = \mathbb{1}_{A_k \cap A_{k'}} f\left(x, y_{k'}, \vartheta\right)$ which is independent of the representation of $Y$ since $\mathbb{1}_{A_k \cap A_{k'}} y_k = \mathbb{1}_{A_k \cap A_{k'}} y_{k'} = \mathbb{1}_{A_k \cap A_{k'}} Y$. Next, we prove that $F$ defined on $\mathbb{R} \times E\left(\mathcal{F}\right) \times L^0\left(\mathcal{F}\right)$ is consistent with the definition in (4.6). To that end, let $\left(x, Y, \vartheta\right) \in \mathbb{R} \times E\left(\mathcal{F}\right) \times L^0\left(\mathcal{F}\right)$ with $Y = \sum_{1 \leq k \leq \bar{k}} \mathbb{1}_{A_k} y_k$ and $\left(Y^n\right)_{n \in \mathbb{N}}$ in $E\left(\mathcal{F}\right)$ with $\lim_{n \in \mathbb{N}} Y^n = Y$ and $Y^n =$

$\sum_{1\leq k^n\leq \overline{k}^n} \mathbb{1}_{A^n_{k^n}} y^n_{k^n}$. Then, it holds that

$$\begin{aligned}
Y &= \operatorname*{ess\,lim\,sup}_{K\in\mathbb{N}} \Big( \mathbb{1}_{\{\lim_{n\in\mathbb{N}} Y^n \geq K\}} K + \mathbb{1}_{\{-K\geq \lim_{n\in\mathbb{N}} Y^n\}} (-K) \\
&\qquad + \mathbb{1}_{\{-K\leq \lim_{n\in\mathbb{N}} Y^n \leq K\}} \lim_{n\in\mathbb{N}} Y^n \Big) \\
&= \operatorname*{ess\,lim\,sup}_{K\in\mathbb{N}} \lim_{n\in\mathbb{N}} Y^n \wedge K \vee -K
\end{aligned}$$

since $Y$ is elementary. Now, for any choice of $k, k^n$ with $1 \leq k \leq \overline{k}$ and $1 \leq k^n \leq \overline{k}^n$, we have that

$$\begin{aligned}
\mathbb{1}_{A_k \cap \bigcap_{n\in\mathbb{N}} A^n_{k^n}} \lim_{n\in\mathbb{N}} F(x, Y^n, \vartheta) &= \mathbb{1}_{A_k \cap \bigcap_{n\in\mathbb{N}} A^n_{k^n}} \lim_{n\in\mathbb{N}} f(x, y^n_{k^n}, \vartheta) \\
&= \mathbb{1}_{A_k \cap \bigcap_{n\in\mathbb{N}} A^n_{k^n}} f(x, y, \vartheta) \\
&= \mathbb{1}_{A_k \cap \bigcap_{n\in\mathbb{N}} A^n_{k^n}} F(x, Y, \vartheta).
\end{aligned}$$

by assumption (4.3). Put together, by stability,

$$\begin{aligned}
&\lim_{m\in\mathbb{N}} F(x, Y^m, \vartheta) \\
&= \lim_{m\in\mathbb{N}} \operatorname*{ess\,lim\,sup}_{K\in\mathbb{N}} \operatorname*{ess\,lim\,sup}_{\substack{(Y^n)_{n\in\mathbb{N}}\subset E(\mathcal{F}) \\ \lim_{n\in\mathbb{N}} Y^n = Y^m}} F(x, Y^n \wedge K \vee -K, \vartheta) \\
&= \lim_{m\in\mathbb{N}} \operatorname*{ess\,lim\,sup}_{K\in\mathbb{N}} \operatorname*{ess\,lim\,sup}_{\substack{(Y^n)_{n\in\mathbb{N}}\subset E(\mathcal{F}) \\ \lim_{n\in\mathbb{N}} Y^n = Y^m}} \mathbb{1}_{A^K_n} F(x, K, \vartheta) \\
&\qquad\qquad\qquad\qquad + \mathbb{1}_{A^*_n} F(x, Y^n, \vartheta) + \mathbb{1}_{A^{-K}_n} F(x, -K, \vartheta) \\
&= \lim_{m\in\mathbb{N}} \operatorname*{ess\,lim\,sup}_{K\in\mathbb{N}} \mathbb{1}_{A^K_m} F(x, K, \vartheta) + \mathbb{1}_{A^*_m} F(x, Y^m, \vartheta) + \mathbb{1}_{A^{-K}_m} F(x, -K, \vartheta) \\
&= F(x, Y, \vartheta)
\end{aligned}$$

with $A^K_n := \{Y^n \geq K\}$, $A^{-K}_n := \{-K \geq Y^n\}$ and $A^*_n := \left(A^K_n \cup A^{-K}_n\right)^c$. Thus, $F$ on $\mathbb{R} \times E(\mathcal{F}) \times L^0(\mathcal{F})$ fulfills (4.6).

For the general case, let $\delta \in L^0(\mathcal{F}_t)_{++}$. Assume that $\lim_{n\in\mathbb{N}} x^n = x$ in $\mathbb{R}$, $\lim_{n\in\mathbb{N}} Y^n = Y$ in $L^0(\mathcal{F})$ and $\lim_{n\in\mathbb{N}} \vartheta^n = \vartheta$ in $L^0(\mathcal{F})$. We want to show that here exists $\overline{n} \in \mathbb{N}$ such that $\|F(x^n, Y^n, \vartheta^n) - F(x, Y, \vartheta)\| < \delta$ for all $n \geq \overline{n}$. By definition of $F$, there are sequences $(Y^n_k)_{k\in\mathbb{N}}$ and $(Y_k)_{k\in\mathbb{N}}$ in $E(\mathcal{F})$ with $\lim_{k\in\mathbb{N}} Y^n_k = Y^n$, $\lim_{k\in\mathbb{N}} Y_k = Y$, $\lim F(x^n, Y^n_k, \vartheta^n) = F(x^n, Y^n, \vartheta^n)$ and $\lim F(x, Y_k, \vartheta) = F(x, Y, \vartheta)$.

This implies that there exists $\overline{k} \in \mathbb{N}$, such that, by triangle inequality,

$$\begin{aligned}
&\|F(x^n, Y^n, \vartheta^n) - F(x, Y, \vartheta)\| \\
&\quad \leq \|F(x^n, Y^n, \vartheta^n) - F(x^n, Y_k^n, \vartheta^n)\| + \|F(x^n, Y_k^n, \vartheta^n) - F(x, Y_k, \vartheta)\| \\
&\qquad + \|F(x, Y_k, \vartheta) - F(x, Y, \vartheta)\| \\
&\quad \leq \frac{\delta}{3} + \|F(x^n, Y_k^n, \vartheta^n) - F(x, Y_k, \vartheta)\| + \frac{\delta}{3}
\end{aligned} \tag{4.47}$$

for all $k \geq \overline{k}$. Therefore, it is left to consider the term $\|F(x^n, Y_k^n, \vartheta^n) - F(x, Y_k, \vartheta)\|$. Again, for all $\varepsilon \in L^0(\mathcal{F})_{++}$, there exists $\overline{k}' \in \mathbb{N}$ such that, by triangle inequality,

$$\|Y_k - Y_k^n\| \leq \|Y_k - Y\| + \|Y - Y^n\| + \|Y^n - Y_k^n\| \leq \varepsilon \tag{4.48}$$

for all $k \geq \overline{k}'$. In terms of normal representations, the remaining term in (4.47) and inequality (4.48) are reformulated by

$$\|F(x^n, Y_k^n, \vartheta^n) - F(x, Y_k, \vartheta)\| = \sum_{\substack{m \in \mathbb{N} \\ m' \in \mathbb{N}}} \mathbb{1}_{A_m \cap A_{m'}} \|f\left(x^n, y_m^{n,k}, \vartheta^n\right) - f\left(x, y_{m'}^k, \vartheta\right)\| \tag{4.49}$$

and

$$\sum_{m \in \mathbb{N}, m' \in \mathbb{N}} \mathbb{1}_{A_m \cap A_{m'}} \|y_m^{n,k} - y_{m'}^k\| \leq \varepsilon \tag{4.50}$$

for $Y_k^n = \sum_{m \in \mathbb{N}} \mathbb{1}_{A_m^{n,k}} y_m^{n,k}$ and $Y_k = \sum_{m' \in \mathbb{N}} \mathbb{1}_{A_{m'}^k} y_{m'}^k$. Inequality (4.50) together with the assumptions $\|x^n - x\| < \varepsilon'$ and $\|\vartheta^n - \vartheta\| < \varepsilon''$ of the general case and (4.3) yield that there exists $\overline{n} \in \mathbb{N}$ such that

$$\|f\left(x^n, y_m^{n,k}, \vartheta^n\right) - f\left(x, y_{m'}^k, \vartheta\right)\| < \frac{\delta}{3}$$

for all $n \geq \overline{n}$ by Lemma 4.14 on the compact interval $\left[y_m^{n,k} \wedge y_{m'}^k, y_m^{n,k} \vee y_{m'}^k\right]$. Hence, with inequality (4.47), we conclude that

$$\|F(x^n, Y^n, \vartheta^n) - F(x, Y, \vartheta)\| < \delta$$

for all $n \geq \overline{n}$. For the uniqueness, let $\tilde{F}$ fulfill properties (4.4), (4.5) and (4.7). Then, there exist $A \in \mathcal{F}$, $x \in \mathbb{R}$, $Y \in L^0(\mathcal{F})$ and $\vartheta \in L^0(\mathcal{F})$ such that $F(x, Y, \vartheta) \neq$

$\tilde{F}(x, Y, \vartheta)$ on $A$. Now, there exist $Y_n \in E(\mathcal{F})$ such that $F(x, Y_n, \vartheta) = \tilde{F}(x, Y_n, \vartheta)$ on $A$ and $\lim_{n\in\mathbb{N}} Y_n = Y$. Then, it holds that $F(x, Y, \vartheta) = \lim_{n\in\mathbb{N}} F(x, Y_n, \vartheta) = \lim_{n\in\mathbb{N}} \tilde{F}(x, Y_n, \vartheta) = \tilde{F}(x, Y, \vartheta)$ by (4.7), in contradiction to $F(x, Y, \vartheta) \neq \tilde{F}(x, Y, \vartheta)$ on $A$. Thus, the lemma holds. $\square$

The following lemma is a conditional version of a characterization of $\mathbb{P}$-almost sure uniformly continuity.

**Lemma 4.14.** *Let $f\colon \mathbb{R} \times \mathbb{R} \times L^0(\mathcal{F}) \to L^0(\mathcal{F})$ be a mapping with the following properties.*

$$\begin{aligned} &f\left(x, y, \mathbb{1}_A\vartheta + \mathbb{1}_{A^c}\vartheta'\right) = \mathbb{1}_A f(x, y, \vartheta) + \mathbb{1}_{A^c} f\left(x, y, \vartheta'\right) \\ &\qquad \textit{for all } (x, y) \in \mathbb{R}^2,\ \vartheta, \vartheta' \in L^0(\mathcal{F}) \textit{ and } A \in \mathcal{F}, \end{aligned} \tag{4.51}$$

*that is, $\vartheta \mapsto f(x, y, \vartheta)$ is $\mathcal{F}$-stable for fixed $(x, y) \in \mathbb{R} \times \mathbb{R}$, and*

$$\begin{aligned} &\lim_{n\in\mathbb{N}} f(x^n, y^n, \vartheta^n) = f(x, y, \vartheta) \\ &\qquad \textit{for all } \lim_{n\in\mathbb{N}} x^n = x \textit{ in } \mathbb{R}, \lim_{n\in\mathbb{N}} y^n = y \textit{ in } \mathbb{R} \textit{ and } \lim_{n\in\mathbb{N}} \vartheta^n = \vartheta \textit{ in } L^0(\mathcal{F}). \end{aligned} \tag{4.52}$$

*Then, for a compact interval $I \subset \mathbb{R}$ and a conditionally compact conditional interval $\mathcal{I} \sqsubset L^0(\mathcal{F})$, for all $\varepsilon \in L^0(\mathcal{F})_{++}$ there exist $\delta \in \mathbb{R}_{++}$ and $\overline{\delta} \in L^0(\mathcal{F})_{++}$ such that*

$$\begin{aligned} &\|f(x_1, y_1, \vartheta_1) - f(x_2, y_2, \vartheta_2)\| < \varepsilon \\ &\qquad \textit{if} |x_1 - x_2| < \delta, |y_1 - y_2| < \delta \textit{ and} \|\vartheta_1 - \vartheta_2\| < \overline{\delta} \end{aligned}$$

*for all $x_1, x_2, y_1, y_2 \in I$, $\vartheta_1, \vartheta_2 \in \mathcal{I}$.*

We remark here that the stricter assumption $\|\vartheta_1 - \vartheta_2\| < \overline{\delta}\mathbb{1}_\Omega$ for $\delta \in \mathbb{R}_{++}$ is a uniform assumption, thus, a convergence assumption in $(L^\infty, \|\cdot\|_\infty)$. For our purpose, the latter is too strong and not often fulfilled.

*Proof.* The proof is as in classical analysis. Assume to the contrary, that there exist sequences $(x^n)_{n\in\mathbb{N}}, (\overline{x}^n)_{n\in\mathbb{N}}$ in $I$, $(y^n)_{n\in\mathbb{N}}, (\overline{y}^n)_{n\in\mathbb{N}}$ in $I$ and $(\vartheta^n)_{n\in\mathbb{N}}, \left(\overline{\vartheta}^n\right)_{n\in\mathbb{N}}$ in $\mathcal{I}$ such that $|x^n - \overline{x}^n| < \frac{1}{n}$, $|y^n - \overline{y}^n| < \frac{1}{n}$, $|\vartheta^n - \overline{\vartheta}^n| < \frac{1}{n}$ and $\|f(x^n, y^n, \vartheta^n) - f(x^n, y^n, \vartheta^n)\| \geq \varepsilon$ for all $n \in \mathbb{N}$. On the compact subset $I \subset \mathbb{R}$, there is a cluster point $x$ of the sequence $(x^n)_{n\in\mathbb{N}}$ and a converging subsequence $(x^{n_k})_{k\in\mathbb{N}}$ with limit $x$. Similarly, there is a cluster point $y$ of the sequence $(y^{n_k})_{k\in\mathbb{N}}$ and a converging subsequence $(y^{n_l})_{l\in\mathbb{N}}$ with limit $y$. On the conditionall compact conditional subset $\mathcal{I} \subset L^0(\mathcal{F})$, the sequence $(\vartheta^{n_l})_{l\in\mathbb{N}}$ has a converging conditional subsequence with limit $\vartheta$, in other words, there is a subsequence $(\vartheta^{n_m})_{m\in\mathbb{N}}$ with $\lim_{m\in\mathbb{N}}\vartheta^{n_m}=\vartheta$. By construction, also $\lim_{m\in\mathbb{N}}\overline{x}^{n_m}=x$, $\lim_{m\in\mathbb{N}}\overline{y}^{n_m}=y$ and $\lim_{m\in\mathbb{N}}\overline{\vartheta}^{n_m}=\vartheta$. By assumption (4.52), it holds that $\lim_{m\in\mathbb{N}}f\left(x^{n_m},y^{n_m},\vartheta^{n_m}\right)=f\left(x,y,\vartheta\right)$ and $\lim_{m\in\mathbb{N}}f\left(\overline{x}^{n_m},\overline{y}^{n_m},\overline{\vartheta}^{n_m}\right)=f\left(x,y,\vartheta\right)$. Therefore,

$$\begin{aligned}\varepsilon &\leq \|\lim_{m\in\mathbb{N}}f\left(x^{n_m},y^{n_m},\vartheta^{n_m}\right)-\lim_{m\in\mathbb{N}}f\left(\overline{x}^{n_m},\overline{y}^{n_m},\overline{\vartheta}^{n_m}\right)\|\\ &\leq \|\lim_{m\in\mathbb{N}}f\left(x^{n_m},y^{n_m},\vartheta^{n_m}\right)-f\left(x,y,\vartheta\right)\|\\ &\qquad+\|(f\left(x,y,\vartheta\right)-\lim_{m\in\mathbb{N}}f\left(\overline{x}^{n_m},\overline{y}^{n_m},\overline{\vartheta}^{n_m}\right)\|=0,\end{aligned}$$

which is a contradiction. Thus, the lemma holds. □

We give the proof of Lemma 4.13.

*Proof.* How to deal with the limit for $K\to\infty$ has been shown in the proof of Lemma 4.9. Thus, we may assume that $Y$ is conditionally bounded. Let $\lim_{n\in\mathbb{N}}x^n=x$, $\lim_{n\in\mathbb{N}}Y^n=Y$ and $\lim_{n\in\mathbb{N}}\vartheta^n=\vartheta$. We want to show that

$$\operatorname{ess}\limsup_{n\in\mathbb{N}}F\left(x^n,Y^n,\vartheta^n\right)\leq F\left(x,Y,\vartheta\right).$$

By definition of the essential supremum, there exists a sequence $\left(\overline{x}^n,\overline{Y}^n,\overline{\vartheta}^n\right)_{n\in\mathbb{N}}$ such that

$$\operatorname{ess}\limsup_{n\in\mathbb{N}}F\left(x^n,Y^n,\vartheta^n\right)=\lim_{n\in\mathbb{N}}F\left(\overline{x}^n,\overline{Y}^n,\overline{\vartheta}^n\right).$$

Further, by definition of $F$, there exists a sequence $\left(\overline{Y}^n_{k_n}\right)_{k_n\in\mathbb{N}}$ in $E\left(\mathcal{F}\right)$ such that

$$\operatorname{ess}\limsup_{n\in\mathbb{N}}F\left(x^n,Y^n,\vartheta^n\right)=\lim_{n\in\mathbb{N}}F\left(\overline{x}^n,\overline{Y}^n,\overline{\vartheta}^n\right)=\lim_{n\in\mathbb{N}}\lim_{k_n\in\mathbb{N}}F\left(\overline{x}^n,\overline{Y}^n_{k_n},\overline{\vartheta}^n\right).$$

We make use of the proof of Lemma 4.9. Therefore, we may choose a subsequence $\left(\tilde{x}^n,\tilde{Y}^n,\tilde{\vartheta}^n\right)_{n\in\mathbb{N}}$ of $\left(\overline{x}^n,\overline{Y}^n_{k_n},\overline{\vartheta}^n\right)_{n\in\mathbb{N}}$ such that $\lim_{n\in\mathbb{N}}\tilde{x}^n=x$, $\lim_{n\in\mathbb{N}}\tilde{Y}^n=Y$ for $\tilde{Y}^n\in E\left(\mathcal{F}\right)$, $\lim_{n\in\mathbb{N}}\tilde{\vartheta}^n=\vartheta$, $F\left(\tilde{x}^n,\tilde{Y}^n,\tilde{\vartheta}^n\right)\leq F\left(\tilde{x}^{n'},\tilde{Y}^{n'},\tilde{\vartheta}^{n'}\right)$ for $n\leq n'$ and $\operatorname{ess}\limsup_{n\in\mathbb{N}}F\left(x^n,Y^n,\vartheta^n\right)=\lim_{n\in\mathbb{N}}F\left(\tilde{x}^n,\tilde{Y}^n,\tilde{\vartheta}^n\right)$. Since $\tilde{Y}^n\in E\left(\mathcal{F}\right)$, we write $\tilde{Y}^n=\sum_{m_n\leq\overline{m}_n}\mathbb{1}_{\tilde{A}^n_{m_n}}\tilde{y}^n_{m_n}$ for a partition $\left(\tilde{A}^n_{m_n}\right)_{m_n\leq\overline{m}_n}$ of $\Omega$ and $\tilde{y}^n_{m_n}\in\mathbb{R}$, $m_n\leq\overline{m}_n$. We consider the joint partition $\left(\bigcap_{n\in\mathbb{N}}\tilde{A}^n_{\hat{m}_n}\mid\hat{m}_n\in\{1,\ldots,\overline{m}_n\},\,n\in\mathbb{N}\right)$ on $\Omega$ which is countable by definition and therefore can be denoted by $\left(\tilde{A}_{\tilde{n}}\right)_{\tilde{n}\in\mathbb{N}}$. Since by construction $\lim_{n\in\mathbb{N}} \tilde{Y}^n = Y$ we observe that the sequence $\left(\sum_{\tilde{n}\leq\tilde{n}'} \mathbb{1}_{\tilde{A}_{\tilde{n}}} \tilde{y}_{\tilde{n}}\right)_{\tilde{n}'\in\mathbb{N}}$ is in $E(\mathcal{F})$ and its conditional limit is $Y$ where the indexing for $\tilde{y}$ and $\tilde{A}$ are corresponding, thus, $\tilde{y}_{\tilde{n}} = \tilde{y}^n_{m_n}$ whenever $\tilde{A}_{\tilde{n}} = \tilde{A}^n_{m_n}$.

On $\bigcap_{n\in\mathbb{N}} A^n_{\tilde{m}_n}$ for fixed $\tilde{m}_n \leq \overline{m}_n$, we observe that

$$\begin{aligned}
\mathbb{1}_{\tilde{A}^n_{\tilde{m}_n}} \operatorname{ess}\limsup_{n\in\mathbb{N}} F\left(x^n, Y^n, \vartheta^n\right) &= \mathbb{1}_{\tilde{A}^n_{\tilde{m}_n}} \lim_{n\in\mathbb{N}} F\left(\tilde{x}^n, \tilde{Y}^n, \tilde{\vartheta}^n\right) \\
&= \mathbb{1}_{\tilde{A}^n_{m_n}} \lim_{n\in\mathbb{N}} f\left(\tilde{x}^n_{m_n}, \tilde{y}^n_{m_n}, \tilde{\vartheta}^n\right) \\
&= \mathbb{1}_{\tilde{A}^n_{m_n}} \limsup_{n\in\mathbb{N}} f\left(\tilde{x}^n_{m_n}, \tilde{y}^n_{m_n}, \tilde{\vartheta}^n\right) \\
&\leq \mathbb{1}_{\tilde{A}^n_{m_n}} f\left(x, y^n_{m_n}\right) = F\left(x, \mathbb{1}_{\tilde{A}^n_{m_n}} y^n_{m_n}\right)
\end{aligned} \tag{4.53}$$

since $\left(\tilde{x}^n, \tilde{Y}^n, \tilde{\vartheta}^n\right)_{n\in\mathbb{N}}$ has been chosen such that $F\left(\tilde{x}^n, \tilde{Y}^n, \tilde{\vartheta}^n\right) \leq F\left(\tilde{x}^{n'}, \tilde{Y}^{n'}\tilde{\vartheta}^{n'}\right)$ for $n \leq n'$, by 4.37 and $\lim_{n\in\mathbb{N}} x^n = x$ and for $\lim_{\tilde{n}'\in\mathbb{N}}\left(\sum_{\tilde{n}\leq\tilde{n}'} \mathbb{1}_{\tilde{A}_{\tilde{n}}} \tilde{y}_{\tilde{n}}\right) = Y$ and $\lim_{n\in\mathbb{N}} \vartheta^n = \vartheta$. Thus, since $\left(\tilde{A}_{\tilde{n}}\right)_{\tilde{n}\in\mathbb{N}}$ is a partition of $\Omega$, in (4.53), we consider the conditional union on all $\tilde{A}_{\tilde{n}}$, $\tilde{n} \in \mathbb{N}$. Hence,

$$\begin{aligned}
\operatorname{ess}\limsup_{n\in\mathbb{N}} F\left(x^n, Y^n, \vartheta^n\right) &= \sum_{\tilde{n}\in\mathbb{N}} \mathbb{1}_{\tilde{A}_{\tilde{n}}} \operatorname{ess}\limsup_{n\in\mathbb{N}} F\left(x^n, Y^n, \vartheta^n\right) \\
&\leq \sum_{\tilde{n}\in\mathbb{N}} \mathbb{1}_{\tilde{A}_{\tilde{n}}} F\left(x, \mathbb{1}_{\tilde{A}_{\tilde{n}}} y_{\tilde{n}}, \vartheta\right) \\
&\leq F\left(x, Y, \vartheta\right)
\end{aligned}$$

by the very definition of $F$ which shows the claim. □

## 4.6 Conditional topology conditioned to a sub-$\sigma$-algebra

Finally, we give some results for conditional topological spaces with respect to different sigma-algebras.

Let $\mathbf{X}$ be a conditional set with respect to the $\sigma$-algebra $\mathcal{F}$ and $\mathfrak{T}$ a conditional topology on $\mathbf{X}$. Let $\mathcal{G} \subset \mathcal{F}$ be a sub-$\sigma$-algebra. We want to consider $\mathbf{X}$ as a conditional set with respect to $\mathcal{G}$. Then, the conditional topology $\mathfrak{T}$ on $(\mathbf{X}, \mathcal{F})$ restricted to $\mathcal{G}$ will be a $\mathcal{G}$-conditional topology as a direct consequence of the following definition.

**Definition 4.15.** Let $\mathcal{C} \in \mathfrak{T}$ be conditionally open. It lives on $A_{\mathcal{C}} \in \mathcal{F}$. Define $A^*_{\mathcal{C}} :=$

$\operatorname{ess\,sup}\{A \in \mathcal{G} \mid A \subset A_{\mathcal{C}}\}$. Then, $\mathcal{C}^* := \{X \in \mathcal{C} \mid X$ lives on $A^*_{\mathcal{C}}$ and is $\mathcal{G}$-measurable$\}$ is a conditional subset of $(\mathbf{X}, \mathcal{G})$. Further, the family $\mathfrak{T}|\mathcal{G} := \{\mathcal{C}^* \mid \mathcal{C} \in \mathfrak{T}\}$ is a conditional topology on $(\mathbf{X}, \mathcal{G})$ and called the conditional topology $\mathfrak{T}$ restricted to the sub-$\sigma$-algebra $\mathcal{G}$.

**Example 4.16.** The conditional Euclidean topology on $L^0(\mathcal{F})$ restricted to $\mathcal{G}$ is the $\mathcal{G}$-conditional Euclidean topology by construction.

Now, let $\mathbf{X}$ be a conditional set with respect to the $\sigma$-algebra $\mathcal{F}$. Let $\mathcal{G} \subset \mathcal{F}$ be a sub-$\sigma$-algebra. Let $\mathfrak{T}$ be a conditional topology on the conditional set $\mathbf{X}$ with respect to $\mathcal{G}$.

**Definition 4.17.** The smallest conditional $\sigma$-algebra with respect to $\mathcal{F}$ that contains all $\sigma_{\mathcal{F}}(\mathcal{C})$ for $\mathcal{C} \in \mathfrak{T}$ is a conditional topology on $(\mathbf{X}, \mathcal{F})$ and is called the $\mathcal{F}$-conditional topology generated by $\mathfrak{T}$ and denoted by $\sigma_{\mathcal{F}}(\mathfrak{T})$.

**Example 4.18.** The $\mathcal{G}$-conditional Euclidean topology on $L^0(\mathcal{F})$ generates the $\mathcal{F}$-conditional Euclidean topology on $L^0(\mathcal{F})$ by construction.

**Remark 4.19.** We remark that $(\sigma_{\mathcal{F}}(\mathfrak{T}_{\mathcal{G}}))|\mathcal{G} = \mathfrak{T}_{\mathcal{G}}$ and $\sigma_{\mathcal{F}}(\mathfrak{T}_{\mathcal{F}}|\mathcal{G}) = \mathfrak{T}_{\mathcal{F}}$ for a $\mathcal{F}$-conditional topology $\mathfrak{T}_{\mathcal{F}}$ on $\mathbf{X}$ and $\mathcal{G} \subset \mathcal{F}$. The latter property is the reason why we do not consider conditional expectation instead of a conditional topology restricted to a sub-$\sigma$-algebra since it is not reversible. Indeed, let $Y$ be $\mathcal{F}$-measurable, but not $\mathcal{G}$-measurable. For simplicity, let $\mathcal{G}$ be trivial. The $L^0(\mathcal{F})$-open conditional set $\{Y\}^{\sqsubset}$ is not reproduced when considering the $\mathcal{F}$-conditional Euclidean topology, since the $\sigma_{\mathcal{F}}$-stable hull of $\left\{\mathbb{E}[X \mid \mathcal{G}] \mid X \in \{Y\}^{\sqsubset}\right\} = L^0(\mathcal{G})$ is $L^0(\mathcal{F}) \neq \{Y\}^{\sqsubset}$.

Next, we give an example of a convergence property of conditional sequences with respect to these different $\sigma$-algebras.

**Example 4.20.** Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $\mathcal{F} = 2^{\Omega}$ and $\mathcal{G} = \{\emptyset, \{\omega_1, \omega_2\}, \{\omega_3\}, \Omega\}$. The conditional balls $\mathcal{B}^{\varepsilon}(\mathbb{1}_{\omega_1})$ for all $\varepsilon \in L^0(\mathcal{F})_{++}$ which lives on $A$ with $\varepsilon < 1$ for all $A \in \mathcal{F}$ form an $\mathcal{F}$-conditional topology $\mathfrak{T}$ on $L^0(\mathcal{F})$. Now, by definition, $\mathfrak{T}|\mathcal{G}$ consists of the conditional balls $(-\varepsilon', \varepsilon')$ living on $\{\omega_3\}$ for $\varepsilon \in \mathbb{R}_{++}$. If the restriction $\varepsilon < 1$ is omitted, we obtain the conditional topology $\mathfrak{T}'$ on $L^0(\mathcal{F})$, an enlargement of $\mathfrak{T}$, then $\mathfrak{T}'|\mathcal{G}$ consists of the conditional balls $(-\varepsilon', \varepsilon')\,\mathbb{1}_{\omega_3}$ living on $A$ for $\varepsilon \in \mathbb{R}_{++}$ and $A \in \mathcal{G}$.

Next, the consider an example how convergence with respect to the conditional topologies is transmitted. The conditional sequence $(X_N)_{N \in (\mathbf{N}, \mathcal{F})}$ defined by $X_N := \left(1 - \frac{1}{N}\right)\mathbb{1}_{\omega_1} + \frac{1}{N}\mathbb{1}_{\{\omega_2, \omega_3\}}$ converges to $X = \mathbb{1}_{\omega_1}$ in $\mathfrak{T}$ and $\mathfrak{T}'$. If we consider the $\mathcal{G}$-measurable elements of the conditional sequence $Y_N = \frac{1}{N}\mathbb{1}_{\omega_3}$ we may examine convergence with respect to the conditional topologies $\mathfrak{T}|\mathcal{G}$ with limit $Y = 0$ and $\mathfrak{T}'|\mathcal{G}$ with limit $Y' = 0$ living only on $\{\omega_3\}$.

**Remark 4.21.** Let $\mathcal{G} \subset \mathcal{F}$ be a sub-$\sigma$-algebra and let $(X_N)_{N\in\mathbb{N}(\mathcal{F})}$ be a conditional sequence in $L^0(\mathcal{F})$ that converges to $X \in L^0(\mathcal{F})$ with respect to the Euclidean topology $\mathfrak{T}$. Then, the conditional sequence $(X_N)_{N\in\mathbb{N}(\mathcal{G})}$ converges to $X$ on $A^*$ with respect to $\mathfrak{T}|\mathcal{G}$ on $A^* \in \mathcal{G}$ where $A^* = \operatorname{ess\,sup}\{A \in \mathcal{G} \mid X \text{ lives on } A \text{ and is } \mathcal{G} \cap A\text{-measurable}\}$. If $X_N$ is additionally $\mathcal{G}$-measurable, the conditional sequence converges in $L^0(\mathcal{G})$ with respect to the conditional topology $\mathfrak{T}|\mathcal{G}$, and the limit is $X$ if it is $\mathcal{G}$-measurable, too. Indeed, since the conditional sequence $(X_N)_{N\in\mathbb{N}(\mathcal{F})}$ converges to $X$ in $\mathfrak{T}$, for any $\varepsilon \in L^0(\mathcal{F})_{++}$ there is $\overline{N} \in \mathbb{N}(\mathcal{F})$ such that $\|X_N - X\| < \varepsilon$ for all $N \geq \overline{N}$. Now, by definition of $\mathfrak{T}|\mathcal{G}$, a conditional ball $\mathcal{B}^\varepsilon(Y)$ for $\varepsilon \in L^0(\mathcal{F})_{++}$ and $Y \in L^0(\mathcal{F})$ is mapped to $\mathcal{B}^\varepsilon(Y) \sqsubset L^0(\mathcal{G})$ and lives on $A^*_\varepsilon \cap A^*_Y$. Therefore, we show that $(X_N)_{N\in\mathbb{N}(\mathcal{G})}$ converges to $X$ on $A^*_X$. For any $\varepsilon' \in L^0(\mathcal{G})_{++}$, the conditional ball $\mathcal{B}^{\varepsilon'}(X)$ is mapped to the conditional ball $\mathcal{B}^{\varepsilon'}(X)$ on $A^*_X$, thus, it holds that $X_N \in \mathcal{B}^{\varepsilon'}(X)$ on $A^*_X$ for all $N \geq \overline{N}$ since $X_N$ is $\mathcal{G}$-measurable. That is the claim, and for the second part, we observe that $X$ lives on $A^*_X$ if $X$ is $\mathcal{G}$-measurable.

Conversely, let $(X_N)_{N\in\mathbb{N}(\mathcal{G})}$ be a conditional sequence in $L^0(\mathcal{G})$ that converges to $X$ with respect to the Euclidean topology $\mathfrak{T}$. Then, the conditional sequence $(X_N)_{N\in\mathbb{N}(\mathcal{F})}$ in $L^0(\mathcal{F})$ converges to $X$ with respect to the Euclidean topology $\sigma_{\mathcal{F}}(\mathfrak{T})$ which is just a finer stability property. Indeed, for any conditionally open $\mathcal{O} \sqsubset L^0(\mathcal{F})$ that is the $\mathcal{F}$-$\sigma$-stable hull of some $\mathcal{O}' \in \mathfrak{T}$, all $X_N \in \mathcal{O}'$ with $N \in \mathbb{N}(\mathcal{G})$ belong to $\mathcal{O}$, thus, also for $N \in \mathbb{N}(\mathcal{F})$. Further, also by definition, for $A \in \mathcal{F}$, $X_N \in \mathcal{O}$ on $A$ as well as $X_N \in \mathcal{O} \sqcap \mathcal{O}^*$ for all $N \geq \overline{N} \vee \overline{N}^*$ if $X_N \in \mathcal{O}'$ for $N \geq \overline{N}$ and $X_N \in \mathcal{O}^*$ for $N \geq \overline{N}'$. Thus, the conditional sequence $(X_N)_{N\in\mathbb{N}(\mathcal{F})}$ converges with respect to the generator of $\sigma_{\mathcal{F}}(\mathfrak{T})$.

Summing up, it is important to consider the corresponding conditional topology when considering the limit of a conditional sequence. Thus, we give the following notation.

**Definition 4.22.** Let $f\colon L^0(\mathcal{G}) \to L^0(\mathcal{F})$ be a $\mathcal{G}$-stable conditional function for $\mathcal{G} \subset \mathcal{F}$. Let $(X_N)_{N\in\mathbb{N}(\mathcal{G})}$ be a conditional sequence in $L^0(\mathcal{G})$. Then, the convergence of the $\mathcal{G}$-conditional sequence $f(X_N)_{N\in\mathbb{N}(\mathcal{G})}$ ist considered with respect to the conditional euclidean topology $\mathfrak{T}|\mathcal{G}$ where $\mathfrak{T}$ is the conditional topology on $L^0(\mathcal{F})$. We write $\lim_{n\in\mathbb{N}} f(X_n)$ if the limit exists for the $\mathcal{G}$-conditional sequence $(X_n)_{n\in\mathbb{N}}$ and do not name the conditional topology explicitely if it is clear.

Let $g\colon L^0(\mathcal{F}) \to L^0(\mathcal{G})$ be a $\mathcal{G}$-stable conditional function for $\mathcal{G} \subset \mathcal{F}$. Let $(X_N)_{N\in\mathbb{N}(\mathcal{F})}$ be a conditional sequence in $L^0(\mathcal{F})$. If we want to consider the convergence of the $\mathcal{G}$-conditional sequence $f(X_N)_{N\in\mathbb{N}(\mathcal{G})}$ we only may assume that the $\mathcal{G}$-conditional sequence $(X_N)_{N\in\mathbb{N}(\mathcal{G})}$ converges with respect to the conditional euclidean topology $\mathfrak{T}|\mathcal{G}$ where $\mathfrak{T}$ is the conditional topology on $L^0(\mathcal{F})$.

We remark here, that this definition will often be applied for the identity. Also, any other property in terms of conditionally open conditional sets is handled the same way. If stability in the conditional set is with respect to a different $\sigma$-algebra we always understand the property as given here.

**Example 4.23.** [Conditional expectation with respect to a conditional measure] From Section 1.2, we can define integrals with respect to a conditional measure $\mu\colon \mathfrak{F} \to L^0(\mathcal{F})_+$ where $\mathfrak{F}$ is a conditional $\sigma$-algebra in $L^0(\mathcal{F})$. We extend this definition to the space $L^0(\mathcal{G})$ for $\mathcal{G} \subset \mathcal{F}$, thus, a conditional expectation is defined for an arbitrary sub-conditional-$\sigma$-algebra with respect to an underlying sub-$\sigma$-algebra. For a conditional sub-$\sigma$-algebra with respect to the same $\sigma$-algebra, we refer to Section 1.2. Thus, it suffices to consider the setup as in Definition 4.22. So, let $\mathfrak{G}$ be a conditional $\sigma$-algebra in $L^0(\mathcal{G})$ defined by $\mathfrak{F}|\mathcal{G}$. Then, the conditional measure $\mu$ is projected such that $\mu_{\mathcal{G}}(\mathcal{C}) = \mu(\sigma_{\mathcal{F}}(\mathcal{C}))$ for $\mu_{\mathcal{G}}\colon \mathfrak{G} \to L^0(\mathcal{G})_+$, but, $\mu_{\mathcal{G}}$ is only $\mathcal{G}$-stable, and hence, the conditional integral with respect to $\mu_{\mathcal{G}}$.

Further, for the definition of the integral with respect to $\mu_{\mathcal{G}}$, let $f\colon L^0(\mathcal{F}) \to L^0(\mathcal{F})$ be a $\mu$-integrable conditional function. Then, the $\mathcal{G}$-conditional function $g\colon L^0(\mathcal{G}) \to L^0(\mathcal{G})$ is called the conditional expectation of $f$ with respect to $\mathfrak{G}$ if $\int g\chi_{\mathcal{C}} d\mu_{\mathcal{G}} = \int f\chi_{\sigma_{\mathcal{F}}(\mathcal{C})} d\mu$ for all $\mathcal{C} \in \mathfrak{G}$ and it is denoted by $\mathbb{E}_\mu[f \mid \mathfrak{G}]$. To show that such a mapping $g$ exists remark that by definition of $\mu_{\mathcal{G}}$, for indicator functions, then step functions, and by monotone convergence, the equality holds for fositive conditional functions. Considering positive and negative part yields the claim.

# Concluding remarks and discussion

Conditional theory and namely conditional variational analysis provide a large toolkit of methods for optimization problems. Compared to classical variational analysis with random sets and measurable selection there are the following remarks.

If we fix an underlying probability space for the conditional sets there is no need of topological assumptions such as closed-valued mappings or Polish spaces. Integrability of the utility is also not presumed.

These rather technical assumptions are replaced by stability conditions which usually can be verified easily. Furthermore, conditional variational analysis also works in infinite dimensional spaces, further examples are given in [JKZ18].

Further, the optimization of the utility has mainly been driven by assumptions on the utility coming from the Walras setting in an economy driven by offer and demand. We may apply this methodology to risk averse agents, or, just simple examples such as $u\left(x,Y,t\right)=1-e^{-\alpha_x \mathbb{E}[Y|\mathcal{F}_t]\frac{T-t}{T}}$ where $Y$ is some allocation as in [CHKP16] and $\alpha_x$ is a scalar depending on the history $x$.

The price dependencies in multiple periods in the Walras setting are not clear cut. Other assumptions on the utility such as translation invariance may allow to control the prices. At any case, the path-dependency allows for a wider class of utility functions that takes history in count.

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