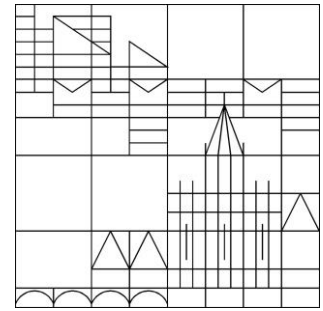


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A SEMIGROUP APPROACH TO NONLINEAR LÉVY PROCESSES

ROBERT DENK, MICHAEL KUPPER, AND MAX NENDEL

ABSTRACT. We study the relation between Lévy processes under nonlinear expectations, nonlinear semigroups and fully nonlinear PDEs. First, we establish a one-to-one relation between nonlinear Lévy processes and nonlinear Markovian convolution semigroups. Second, we provide a condition on a family of infinitesimal generators $(A_\lambda)_{\lambda \in \Lambda}$ of linear Lévy processes which guarantees the existence of a nonlinear Lévy processes such that the corresponding nonlinear Markovian convolution semigroup is a viscosity solution of the fully nonlinear PDE $\partial_t u = \sup_{\lambda \in \Lambda} A_\lambda u$. The results are illustrated with several examples.

Key words: Lévy process, convex expectation space, Markovian convolution semigroup, fully nonlinear PDE, Nisio semigroup

AMS 2010 Subject Classification: 60G51; 49L25; 47H20

1. INTRODUCTION

In this paper we study the relation between Lévy processes under nonlinear expectations, nonlinear semigroups and fully nonlinear PDEs. Let $(X_t)_{t \geq 0}$ be an \mathbb{R}^d -valued Lévy process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then

$$(S(t)f)(x) := \mathbb{E}(f(x + X_t))$$

for $f \in \text{BUC}(\mathbb{R}^d)$, $t \geq 0$ and $x \in \mathbb{R}^d$, defines a strongly continuous semigroup on the space of bounded and uniformly continuous functions $\text{BUC}(\mathbb{R}^d)$ whose infinitesimal generator $A: D(A) \subset \text{BUC}(\mathbb{R}^d) \rightarrow \text{BUC}(\mathbb{R}^d)$ is given by an integro-differential operator which is uniquely determined by a Lévy triplet, see Applebaum [1] or Sato [21]. Moreover, $(S(t)f)_{t \geq 0}$ is the solution of the abstract Cauchy problem

$$\partial_t u(t) = Au(t), \quad t > 0,$$

with $u(0) = f \in \text{BUC}(\mathbb{R}^d)$. For a detailed discussion on operator semigroups we refer to Pazy [16] or Engel and Nagel [8].

We first extend the well-known relation between Lévy processes and Markovian convolution semigroups of probability measures to a nonlinear setting. Nonlinear Lévy processes were introduced in [10] as càdlàg processes with stationary and independent increments under a sublinear expectation. The G-Brownian motion due to [17, 18] is a special case of a nonlinear Lévy process, see also Dolinsky et al. [6] or Denis et al. [4]. For an introduction to nonlinear expectations we refer to [20]. Since we do not require any path regularity we call a process $(X_t)_{t \geq 0}$ an \mathcal{E} -Lévy process with values in an abelian group G , if it has stationary and independent increments and $X_t \rightarrow X_0$ in distribution as $t \searrow 0$ with respect to a convex expectation \mathcal{E} . We then provide a relation between \mathcal{E} -Lévy processes and convex Markovian convolution semigroups, i.e. strongly continuous semigroups $(\mathcal{S}(t))_{t \geq 0}$ on $\text{BUC}(G)$ such that each $\mathcal{S}(t)$ has a translation invariant convex kernel. The proof relies on a version of Kolmogorov's extension theorem for nonlinear expectations elaborated in [5].

Our main focus lies on the construction of \mathcal{E} -Lévy processes with nonlinear generators. We start with an arbitrary family $(A_\lambda)_{\lambda \in \Lambda}$ of generators

$$A_\lambda: D(A_\lambda) \subset \text{BUC}(G) \rightarrow \text{BUC}(G)$$

of Markovian convolution semigroups $S_\lambda = (S_\lambda(t))_{t \geq 0}$ of linear operators. We then construct the smallest Markovian convolution semigroup $(\mathcal{S}(t))_{t \geq 0}$ which dominates each $(S_\lambda(t))_{t \geq 0}$. The corresponding \mathcal{E} -Lévy process can be viewed as a process with independent increments whose distribution is uncertain, i.e. any distribution of the increments associated to $(S_\lambda)_{\lambda \in \Lambda}$ is taken into account. We basically follow an idea by Nisio [15] in order to construct a sublinear Markovian convolution semigroup which results from a given family of linear Markovian convolution semigroups by constant optimization. In [15] Nisio considers strongly continuous semigroups on the space of all bounded measurable functions. However, by a theorem of Lotz (see e.g. [2, Corollary 4.3.19]), any strongly continuous semigroup on the space of bounded measurable functions already has a bounded generator, which is not suitable for most applications. We therefore modify Nisio's construction to the space $\text{BUC}(G)$. Under the condition that the subspace

$$\left\{ f \in \bigcap_{\lambda \in \Lambda} D(A_\lambda) : \{A_\lambda f : \lambda \in \Lambda\} \text{ is bounded and uniformly equicontinuous} \right\} \quad (1.1)$$

is dense in $\text{BUC}(G)$ we construct a (strongly continuous) Markovian convolution semigroup $(\mathcal{S}(t))_{t \geq 0}$ on $\text{BUC}(G)$ with corresponding \mathcal{E} -Lévy process $(X_t)_{t \geq 0}$ on a sublinear expectation space $(\Omega, \mathcal{F}, \mathcal{E})$ such that

$$u(t, x) := (\mathcal{S}(t)f)(x) = \mathcal{E}(f(x + X_t)), \quad t \geq 0, x \in G,$$

is a viscosity solution of the fully nonlinear PDE

$$\partial_t u = \sup_{\lambda \in \Lambda} A_\lambda u \quad \text{on } (0, \infty) \times G$$

with $u(0) = f$ for all $f \in \text{BUC}(G)$. In particular, the generator of the \mathcal{E} -Lévy process $(X_t)_{t \geq 0}$ is given by $\sup_{\lambda \in \Lambda} A_\lambda$. Here, we use a slightly different notion of viscosity solution which fits to the semigroup setting. However, in many cases, particularly for the classical case $G = \mathbb{R}^d$, this leads to the same class or an even larger class of test functions. We refer to Crandall et al. [3] for the classical definition and a detailed discussion of viscosity solutions. Moreover, we give a condition on the generators $(A_\lambda)_{\lambda \in \Lambda}$ which guarantees that the corresponding \mathcal{E} -Lévy process is tight, or equivalently each $\mathcal{S}(t)$ is continuous from above. Throughout, the state space is an abelian group, which gives the opportunity to consider certain classes of cylindrical G -Wiener Processes as an infinite dimensional extension of the G -Brownian motion, or nonlinear Lévy processes on the d -dimensional Torus.

Nonlinear \mathbb{R}^d -valued Lévy processes have first been introduced in Hu and Peng [10], where G -Lévy processes with a decomposition $X = X^c + X^d$ into a continuous and a jump part are considered. Under the assumption that X^c is a G -Brownian motion and $\mathcal{E}(|X_t^d|) \leq ct$ for some constant c , it is shown that $u(t, x) = \mathcal{E}(f(x + X_t))$ is a viscosity solution of $\partial_t u(t, x) - G(u(t, x + \cdot) - u(t, x)) = 0$ and $u(0) = f$, where $G(\varphi(\cdot)) := \lim_{h \downarrow 0} \frac{1}{h} \mathcal{E}(\varphi(X_h))$. The function G is shown to have a Lévy-Khinchine representation in terms of a set Λ of Lévy triplets (b, Σ, μ) satisfying an integrability condition, i.e. $u(t, x)$ is the solution of

$$\partial_t u = \sup_{(b, \Sigma, \mu) \in \Lambda} A_{b, \Sigma, \mu} u, \quad \text{and } u(0) = f, \quad (1.2)$$

where $A_{b,\Sigma,\mu}$ is the generator with the Lévy triplet (b, Σ, μ) , see also Example 3.2. Conversely, starting from the unique solution of (1.2), Hu and Peng [10] give a construction of the respective nonlinear Lévy process. In Nutz and Neufeld [14] the authors consider upper expectations $\mathcal{E}(\cdot) = \sup_{\mathbb{P}} \mathbb{E}_{\mathbb{P}}(\cdot)$ over a class of all semimartingales with given differential characteristics in a set Λ of Lévy triplets, which in [13] is shown to be analytic. This allows them to construct conditional nonlinear expectations and nonlinear Lévy processes with general characteristics whose distributions are defined for all measurable functions. Under the conditions

$$\sup_{(b,\Sigma,\mu) \in \Lambda} \left(|b| + |\Sigma| + \int_{\mathbb{R}^d} |y| \wedge |y|^2 d\mu(y) \right) < \infty \quad \text{and} \quad \lim_{\varepsilon \searrow 0} \sup_{(b,\Sigma,\mu) \in \Lambda} \int_{|z| \leq \varepsilon} |z|^2 d\mu(z) = 0 \quad (1.3)$$

it is shown that $u(t, x) = \mathcal{E}(f(x + X_t))$ is the unique viscosity solution of (1.2). The conditions in (1.3) are weaker than the integrability condition in [10] and allow for instance to consider classes of Lévy processes with infinite variation jumps. Our main condition (1.1) in the context of \mathbb{R}^d -valued processes is guaranteed under

$$\sup_{(b,\Sigma,\mu) \in \Lambda} \left(|b| + |\Sigma| + \int_{\mathbb{R}^d} 1 \wedge |y|^2 d\mu(y) \right) < \infty, \quad (1.4)$$

which does not exclude any Lévy triplet at all. In particular, Lévy processes with non-integrable jumps can be considered, see e.g. Example 3.6, and for finite Λ the condition (1.4) is always satisfied. In order to obtain uniqueness for the viscosity solution of (1.2) one additionally needs the second condition in (1.3) and tightness of the family of Lévy measures $\{\mu : (b, \Sigma, \mu) \in \Lambda\}$, which is due to [10]. A related concept to nonlinear Lévy processes are second order backward stochastic differential equation with jumps, see Kazi-Tani et al. [11], [12] and also Soner et al. [22].

The paper is organized as follows. In Section 2 we introduce the notation and discuss our main results which are illustrated with several examples in Section 3. The relation between \mathcal{E} -Lévy processes and Markovian convolution semigroups is given in Section 4. Finally, in Section 5 we prove the main result by constructing a version of Nisio semigroups on $\text{BUC}(G)$.

2. MAIN RESULTS

We say that $(\Omega, \mathcal{F}, \mathcal{E})$ is a *convex expectation space* if (Ω, \mathcal{F}) is a measurable space and $\mathcal{E} : \mathcal{L}^\infty(\Omega, \mathcal{F}) \rightarrow \mathbb{R}$ is a convex expectation which is continuous from below. Here, as usual $\mathcal{L}^\infty(\Omega, \mathcal{F})$ denotes the space of all bounded measurable functions $\Omega \rightarrow \mathbb{R}$. Recall that a convex expectation on a convex set M with $\mathbb{R} \subset M$ is a functional $\mathcal{E} : M \rightarrow \mathbb{R}$ which satisfies

$$\begin{aligned} \mathcal{E}(X) &\leq \mathcal{E}(Y) \text{ whenever } X \leq Y, \\ \mathcal{E}(\alpha) &= \alpha \text{ for all } \alpha \in \mathbb{R}, \text{ and} \\ \mathcal{E}(\lambda X + (1 - \lambda)Y) &\leq \lambda \mathcal{E}(X) + (1 - \lambda) \mathcal{E}(Y) \text{ for all } \lambda \in [0, 1]. \end{aligned}$$

If in addition \mathcal{E} is positive homogeneous, i.e. $\mathcal{E}(\lambda X) = \lambda \mathcal{E}(X)$ for all $\lambda > 0$, then \mathcal{E} is called a *sublinear* expectation and $(\Omega, \mathcal{F}, \mathcal{E})$ is a *sublinear* expectation space. A convex expectation is said to be *continuous from below* if $\mathcal{E}(X_n) \nearrow \mathcal{E}(X)$ for every increasing sequence (X_n) in $\mathcal{L}^\infty(\Omega, \mathcal{F})$ which converges pointwise to $X \in \mathcal{L}^\infty(\Omega, \mathcal{F})$.

Let $(\Omega, \mathcal{F}, \mathcal{E})$ be a convex expectation space, and let G be an abelian group with a translation invariant metric d such that (G, d) is separable and complete. We denote by $C_b(G)$ and $\text{BUC}(G)$ the spaces of all bounded functions $f : G \rightarrow \mathbb{R}$ which are continuous

and uniformly continuous, respectively. For a random variable $X: \Omega \rightarrow G$ with values in G endowed with the Borel σ -algebra \mathcal{B} , the functional

$$\mathcal{E} \circ X^{-1}: C_b(G) \rightarrow \mathbb{R}, \quad f \mapsto \mathcal{E}(f(X))$$

defines a convex expectation which is called the *distribution* of X under \mathcal{E} . Given another random variables $Y: \Omega \rightarrow S$ with values in a Polish space S , for $f \in C_b(S \times G)$ the function

$$S \rightarrow \mathbb{R}, \quad y \mapsto \mathcal{E}(f(y, X))$$

is bounded and lower semicontinuous (approximate f from below by $g \in \text{BUC}(S \times G)$ for which $y \mapsto \mathcal{E}(g(y, X))$ is continuous, see also [5, Remark 5.5]). Hence, $\mathcal{E}(f(y, X))|_{y=Y}$ is in $\mathcal{L}^\infty(\Omega, \mathcal{F})$, which shows that $\mathcal{E}(\mathcal{E}(f(y, X))|_{y=Y})$ is well-defined. Then, X is called *independent* of Y if

$$\mathcal{E}(f(Y, X)) = \mathcal{E}(\mathcal{E}(f(y, X))|_{y=Y})$$

for all $f \in C_b(S \times G)$.

Definition 2.1. a) We say that $\mathcal{S}: \text{BUC}(G) \rightarrow \text{BUC}(G)$ is a convex *kernel* if $(\mathcal{S} \cdot)(x)$ is a convex expectation on $\text{BUC}(G)$ for all $x \in G$. It is called *continuous from above* if $\mathcal{S}f_n \searrow \mathcal{S}f$ (pointwise convergence) for every decreasing sequence (f_n) in $\text{BUC}(G)$ which converges pointwise to $f \in \text{BUC}(G)$.

b) A convex kernel $\mathcal{S}: \text{BUC}(G) \rightarrow \text{BUC}(G)$ is called a convex *Markovian convolution* if $\mathcal{S}f_n \nearrow \mathcal{S}f$ for every increasing sequence (f_n) in $\text{BUC}(G)$ which converges pointwise to $f \in \text{BUC}(G)$ and

$$(\mathcal{S}f)(x) = (\mathcal{S}f_x)(0)$$

for all $f \in \text{BUC}(G)$ and $x \in G$, where $f_x: G \rightarrow \mathbb{R}$ is given by $y \mapsto f(x + y)$.

c) We say that $(\mathcal{S}(t))_{t \geq 0}$ is a convex *Markovian convolution semigroup* on $\text{BUC}(G)$ if

- (i) $\mathcal{S}(t)$ is a convex Markovian convolution for all $t \geq 0$,
- (ii) $\mathcal{S}(0)f = f$ for all $f \in \text{BUC}(G)$,
- (iii) $\mathcal{S}(s + t) = \mathcal{S}(s)\mathcal{S}(t)$ for all $s, t \geq 0$,
- (iv) $\lim_{t \searrow 0} \|\mathcal{S}(t)f - f\|_\infty = 0$ for all $f \in \text{BUC}(G)$.

In this case, we say that $(\mathcal{S}(t))_{t \geq 0}$ is continuous from above if each $\mathcal{S}(t)$ is so.

d) Let $(\Omega, \mathcal{F}, \mathcal{E})$ be a convex expectation space. Then, $(X_t)_{t \geq 0}$ is called an \mathcal{E} -*Lévy process* if

- (i) $X_t: \Omega \rightarrow G$ is measurable for all $t \geq 0$,
- (ii) $\mathcal{E}(f(X_0)) = f(0)$ for all $f \in C_b(G)$,
- (iii) $\mathcal{E} \circ (X_{s+t} - X_s)^{-1} = \mathcal{E} \circ X_t^{-1}$ for all $s, t \geq 0$,
- (iv) $X_{s+t} - X_s$ is independent of $(X_{t_1}, \dots, X_{t_n})$ for all $s, t \geq 0$, $n \in \mathbb{N}$, $0 \leq t_1 < \dots < t_n \leq s$,
- (v) $\mathcal{E}(f(X_t)) \rightarrow f(0)$ for all $f \in C_b(G)$, i.e. $X_t \rightarrow X_0$ in distribution as $t \searrow 0$.

Remark 2.2. Let $\mathcal{S}: \text{BUC}(G) \rightarrow \text{BUC}(G)$ be a convex kernel which is continuous from above. Then it has a unique extension to a convex kernel $\mathcal{S}: C_b(G) \rightarrow C_b(G)$ which is continuous from above. If \mathcal{S} is in addition a Markovian convolution, then there exists a random variable $X: \Omega \rightarrow G$ on a convex expectation space $(\Omega, \mathcal{F}, \mathcal{E})$ such that

$$(\mathcal{S}f)(x) = \mathcal{E}(f(x + X))$$

for all $f \in C_b(G)$ and $x \in G$. Moreover, for every $\varepsilon > 0$ there exists a compact set $K \subset G$ such that $\mathcal{E}(1_{G \setminus K}(X)) < \varepsilon$, i.e. $\mathcal{E} \circ X^{-1}$ is *tight*. For the details we refer to [5].

Our first result connects convex Markovian convolution semigroups and \mathcal{E} -Lévy processes. The proof is given in Section 4.

Theorem 2.3. *For every convex Markovian convolution semigroup $(\mathcal{S}(t))_{t \geq 0}$ which is continuous from above, there exists a convex expectation space $(\Omega, \mathcal{F}, \mathcal{E})$ and an \mathcal{E} -Lévy process $(X_t)_{t \geq 0}$ such that*

$$(\mathcal{S}(t)f)(x) = \mathcal{E}(f(x + X_t)) \quad (2.1)$$

for every $f \in \text{BUC}(G)$ and $x \in G$.

Conversely, for every \mathcal{E} -Lévy process $(X_t)_{t \geq 0}$ on a convex expectation space $(\Omega, \mathcal{F}, \mathcal{E})$, the family $(\mathcal{S}(t))_{t \geq 0}$ defined by (2.1) is a convex Markovian convolution semigroup.

In this paper, we use the following definition of a viscosity solution.

Definition 2.4. Let $D \subset \text{BUC}(G)$ and $\mathcal{A}: D \subset \text{BUC}(G) \rightarrow \text{BUC}(G)$. Then, we say that $u: [0, \infty) \rightarrow \text{BUC}(G)$ is a D -viscosity subsolution of the PDE

$$u_t = \mathcal{A}u \quad (2.2)$$

if $u: [0, \infty) \rightarrow \text{BUC}(G)$ is continuous and for every $t > 0$ and $x \in G$ we have

$$\partial_t \psi(t, x) \leq (\mathcal{A}\psi(t))(x)$$

for every differentiable $\psi: (0, \infty) \rightarrow \text{BUC}(G)$ which satisfies $(\psi(t))(x) = (u(t))(x)$, $\psi(s) \geq u(s)$ and $\psi(s) \in D$ for all $s > 0$. Here and in the following, we use the notation $\psi(t, x) := (\psi(t))(x)$.

Analogously, u is called a D -viscosity supersolution of (2.2) if $u: [0, \infty) \rightarrow \text{BUC}(G)$ is continuous and for every $t > 0$ and $x \in G$ we have

$$\partial_t \psi(t, x) \geq (\mathcal{A}\psi(t))(x)$$

for every differentiable $\psi: (0, \infty) \rightarrow \text{BUC}(G)$ which satisfies $(\psi(t))(x) = (u(t))(x)$, $\psi(s) \leq u(s)$ and $\psi(s) \in D$ for all $s > 0$.

We say that u is a D -viscosity solution of (2.2) if u is a D -viscosity subsolution and a D -viscosity supersolution.

Now we are ready to state our main result. Given a family $(A_\lambda)_{\lambda \in \Lambda}$ of generators of Lévy processes we provide the existence of an \mathcal{E} -Lévy process on a sublinear expectation space with generator $\sup_{\lambda \in \Lambda} A_\lambda$. The corresponding sublinear Markovian convolution semigroup is a viscosity solution of the fully nonlinear PDE $u_t = \sup_{\lambda \in \Lambda} A_\lambda u$ with $u(0, \cdot) = f \in \text{BUC}(G)$. The proof is postponed to Section 5.

Theorem 2.5. *Let $\Lambda \neq \emptyset$ be an index set. We make the following assumptions:*

(A1) *For each $\lambda \in \Lambda$ let $A_\lambda: D(A_\lambda) \subset \text{BUC}(G) \rightarrow \text{BUC}(G)$ be the generator of a Markovian convolution semigroup $(S_\lambda(t))_{t \geq 0}$ of linear operators.*

(A2) *The subspace*

$$D := \left\{ f \in \bigcap_{\lambda \in \Lambda} D(A_\lambda) : \{A_\lambda f : \lambda \in \Lambda\} \text{ is bounded and uniformly equicontinuous} \right\}$$

is dense in $\text{BUC}(G)$.

Then, there exists a sublinear expectation space $(\Omega, \mathcal{F}, \mathcal{E})$ and an \mathcal{E} -Lévy process $(X_t)_{t \geq 0}$ such that for each $f \in \text{BUC}(G)$ the function

$$(u(t))(x) := \mathcal{E}(f(x + X_t)), \quad t \geq 0, x \in G \quad (2.3)$$

is a D -viscosity solution of the fully nonlinear PDE

$$u_t(t, x) = \sup_{\lambda \in \Lambda} (A_\lambda u(t))(x), \quad (t, x) \in (0, \infty) \times G, \quad (2.4)$$

$$u(0, x) = f(x), \quad x \in G. \quad (2.5)$$

Moreover, there exists a set \mathcal{P} of probability measures on (Ω, \mathcal{F}) such that $\mathcal{E}(Y) = \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}(Y)$ for all $Y \in \mathcal{L}^\infty(\Omega, \mathcal{F})$.

Remark 2.6. In the situation of Theorem 2.5, if each $A_\lambda : \text{BUC}(G) \rightarrow \text{BUC}(G)$ is a bounded operator and $\sup_{\lambda \in \Lambda} \|A_\lambda\| < \infty$, then the mapping

$$\text{BUC}(G) \rightarrow \text{BUC}(G), \quad u \mapsto \sup_{\lambda \in \Lambda} A_\lambda u$$

is Lipschitz continuous. Therefore, by the Picard-Lindelöf theorem, the function u in (2.3) is a classical solution of the fully nonlinear PDE (2.4)-(2.5), which satisfies $u \in C^1([0, \infty); \text{BUC}(G))$.

In most applications, the conditions (A1) and (A2) in Theorem 2.5 can be easily verified as shown in Section 3. For the sake of illustration, we consider the case $G = \mathbb{R}^d$, where the Lévy-Khintchine formula characterizes generators of Markovian convolution semigroups of linear operators by means of so-called Lévy triplets, see e.g. Applebaum [1] or Sato [21]. Given a set Λ of Lévy triplets (b, Σ, μ) , i.e. $b \in \mathbb{R}^d$, $\Sigma \in \mathbb{R}^{d \times d}$ is a symmetric positive semidefinite matrix and μ is a Lévy measure, the condition (1.4) is sufficient to guarantee (A2) with $\text{BUC}^2(\mathbb{R}^d) \subset D$. By Theorem 2.5 there exists an \mathcal{E} -Lévy process $(X_t)_{t \geq 0}$ such that $u(t, x) = \mathcal{E}(f(x + X_t))$ is a D -viscosity solution of the fully nonlinear PDE

$$\partial_t u = \sup_{(b, \Sigma, \mu) \in \Lambda} A_{b, \Sigma, \mu} u, \quad \text{and} \quad u(0) = f \in \text{BUC}(G). \quad (2.6)$$

For more details, we refer to Example 3.2. Assuming in addition to (1.4) that for every $\varepsilon > 0$ there exists some $M > 0$ such that

$$\sup_{(b, \Sigma, \mu) \in \Lambda} \mu(\mathbb{R}^d \setminus B(0, M)) < \varepsilon, \quad (2.7)$$

where $B(0, M) := \{x \in \mathbb{R}^d : |x| \leq M\}$, and the second condition in (1.3) one obtains from [10, Corollary 53] the uniqueness of the viscosity solution of the PDE (2.6).

Remark 2.7. Let $\psi \in C_b^{2,3}((0, \infty) \times \mathbb{R}^d)$, where $C_b^{2,3}((0, \infty) \times \mathbb{R}^d)$ stands for the space of all functions of $(t, x) \in (0, \infty) \times \mathbb{R}^d$ for which all partial derivatives up to order 2 in t and up to order 3 in x exist, are continuous and bounded. Then,

$$\lim_{h \rightarrow 0} \sup_{x \in \mathbb{R}^d} \left| \frac{\psi(t+h, x) - \psi(t, x)}{h} - \partial_t \psi(t, x) \right| = 0$$

for all $t > 0$ and therefore, $\psi : (0, \infty) \rightarrow \text{BUC}(\mathbb{R}^d)$ is differentiable with $\psi(s) \in \text{BUC}^2(\mathbb{R}^d)$ for all $s > 0$ using the identification $(\psi(s))(x) := \psi(s, x)$. Therefore, the class of test functions considered in the framework of $\text{BUC}^2(\mathbb{R}^d)$ -viscosity solutions includes the class $C_b^{2,3}((0, \infty) \times \mathbb{R}^d)$ of test functions, which is often considered in classical viscosity theory, see e.g. Denis et al. [4] or Hu and Peng [10].

Let $C_0(G)$ be the closure of the space $C_c(G)$ of all continuous functions with compact support w.r.t. the supremum norm $\|\cdot\|_\infty$. Note that the existence of a function in $C_0(G) \setminus \{0\}$ already implies that G is locally compact and vice versa since G is a topological abelian group, which is metrizable. The following additional condition on the generators $(A_\lambda)_{\lambda \in \Lambda}$ implies that the related \mathcal{E} -Lévy process is tight.

Proposition 2.8. *In addition to the assumptions in Theorem 2.5, suppose:*

- (A3) For every $\varepsilon > 0$ there exists $\varphi \in \bigcap_{\lambda \in \Lambda} (D(A_\lambda) \cap C_0(G))$ with $0 \leq \varphi \leq 1$, $\varphi(0) = 1$ and $\sup_{\lambda \in \Lambda} \|A_\lambda \varphi\|_\infty \leq \varepsilon$.

Then $\mathcal{E} \circ X_t^{-1}$ is tight for all $t \geq 0$.

The proof is given at the end of Section 4. For instance, in the case $G = \mathbb{R}^d$, where the generators are given by Lévy triplets $(b, \Sigma, \mu) \in \Lambda$, the condition (A3) holds if (1.4) is satisfied and the set of Lévy measures $\{\mu : (b, \Sigma, \mu) \in \Lambda\}$ is tight.

3. EXAMPLES

Let $\text{ca}_+^1(G)$ be the set of all Borel probability measures on G .

Example 3.1 (Compound Poisson processes). For $\lambda \geq 0$ and $\mu \in \text{ca}_+^1(G)$, let

$$(A_{\lambda, \mu} f)(x) := \lambda \int_G f(x+y) - f(x) d\mu(y), \quad f \in \text{BUC}(G), x \in G.$$

Then, $A_{\lambda, \mu}: \text{BUC}(G) \rightarrow \text{BUC}(G)$ is a bounded linear operator which satisfies the strong maximum principle, i.e. for $f \in \text{BUC}(G)$ and $x_0 \in G$ with $f(x_0) = \max_{x \in G} f(x)$ one has $(A_{\lambda, \mu} f)(x_0) \leq 0$. Further, it generates the linear Markovian convolution semigroup $S_{\lambda, \mu}(t) := e^{tA_{\lambda, \mu}}$, which satisfies

$$(S_{\lambda, \mu}(t)f)(x) = \mathbb{E}(f(x + J_t)) = \int_G f(x+y) d(\mathbb{P} \circ J_t^{-1})(y), \quad f \in \text{BUC}(G),$$

where $(J_t)_{t \geq 0}$ is a compound Poisson process with rate λ and jump size distribution μ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Indeed, let $J_t = \sum_{i=1}^{N_t} Y_i$ for an i.i.d. sequence $(Y_i)_{i \in \mathbb{N}}$ of random variables $Y_i: \Omega \rightarrow G$ such that $\mathbb{P} \circ Y_i^{-1} = \mu$, and a random variable $N_t: \Omega \rightarrow \mathbb{N}_0$ which is independent of $(Y_i)_{i \in \mathbb{N}}$ and satisfies $\mathbb{P}(N_t = n) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}$ for all $n \in \mathbb{N}$. Then, for $f \in \text{BUC}(G)$ and $x \in G$ we have

$$\begin{aligned} \mathbb{E}(f(x + J_t)) &= \sum_{n=0}^{\infty} \mathbb{E}(f(x + Y_1 + \dots + Y_n)) e^{-\lambda t} \frac{(\lambda t)^n}{n!} \\ &= e^{-\lambda t} \sum_{n=0}^{\infty} \frac{t^n \lambda^n}{n!} \int_G \dots \int_G f(x + y_1 + \dots + y_n) d\mu(y_1) \dots d\mu(y_n) \\ &= e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(t^n (A_{\lambda, \mu} + \lambda)^n f)(x)}{n!} = e^{-\lambda t} (e^{t(A_{\lambda, \mu} + \lambda)} f)(x) \\ &= (e^{tA_{\lambda, \mu}} f)(x) = (S_{\lambda, \mu}(t)f)(x), \end{aligned}$$

where we used $((A_{\lambda, \mu} + \lambda)f)(x) = \lambda \int_G f(x+y) d\mu(y)$.

Now, assume that $\Lambda \subset [0, \infty)$ is bounded and $\mathcal{Q} \subset \text{ca}_+^1(G)$. Since

$$\left\{ f \in \text{BUC}(G) : \left. \begin{array}{l} \{A_{\lambda, \mu} f : (\lambda, \mu) \in \Lambda \times \mathcal{Q}\} \text{ is bounded} \\ \text{and uniformly equicontinuous} \end{array} \right\} = \text{BUC}(G) \right.$$

the assumptions (A1) and (A2) are satisfied with $D = \text{BUC}(G)$. Hence, by Theorem 2.5 and Remark 2.6 there exists a nonlinear expectation space $(\Omega, \mathcal{F}, \mathcal{E})$ and an \mathcal{E} -Lévy process $(X_t)_{t \geq 0}$ such that for all $f \in \text{BUC}(G)$ the function

$$u(t, x) := (u(t))(x) := \mathcal{E}(f(x + X_t)), \quad t \geq 0, x \in G,$$

is the unique classical solution of the fully nonlinear PIDE

$$\begin{aligned} u_t(t, x) &= \sup_{(\lambda, \mu) \in \Lambda \times \mathcal{Q}} (A_{\lambda, \mu} u(t))(x), \quad (t, x) \in (0, \infty) \times G, \\ u(0, x) &= f(x), \quad x \in G, \end{aligned}$$

with $u \in C^1([0, \infty); \text{BUC}(G))$.

Example 3.2 (General Lévy processes). Let $G = \mathbb{R}^d$ for some $d \in \mathbb{N}$, and consider a set Λ of Lévy triplets (b, Σ, μ) , i.e. $b \in \mathbb{R}^d$, $\Sigma \in \mathbb{R}^{d \times d}$ is a symmetric positive semidefinite matrix and μ is a Lévy measure. For any Lévy triplet (b, Σ, μ) consider

$$(A_{b, \Sigma, \mu} f)(x) := b \cdot \nabla f(x) + \frac{1}{2} \operatorname{tr}(\Sigma \nabla^2 f(x)) + \int_{\mathbb{R}^d} f(x+y) - f(x) - \nabla f(x) \cdot h(y) \, d\mu(y)$$

for $x \in \mathbb{R}^d$ and $f \in D(A_{b, \Sigma, \mu}) := \{g \in \operatorname{BUC}(\mathbb{R}^d) : A_{b, \Sigma, \mu} g \in \operatorname{BUC}(\mathbb{R}^d)\}$, where the function $h: \mathbb{R}^d \rightarrow \mathbb{R}^d$ is defined by $h(y) = y$ for $|y| \leq 1$, and $h(y) = 0$ whenever $|y| > 1$. Then, by the Lévy-Khintchine formula, for each Lévy triplet (b, Σ, μ) the operator $A_{b, \Sigma, \mu}$ generates a Markovian convolution semigroup $(S_{b, \Sigma, \mu}(t))_{t \geq 0}$ of linear operators on $\operatorname{BUC}(\mathbb{R}^d)$. In particular, (A1) is satisfied. Assume that

$$C := \sup_{(b, \Sigma, \mu) \in \Lambda} \left(|b| + |\Sigma| + \int_{\mathbb{R}^d} 1 \wedge |y|^2 \, d\mu(y) \right) < \infty. \quad (3.1)$$

We will verify that (A2) is satisfied under (3.1). Let

$$D := \left\{ f \in \bigcap_{(b, \Sigma, \mu) \in \Lambda} D(A_{b, \Sigma, \mu}) : \begin{array}{l} \{A_{b, \Sigma, \mu} f : (b, \Sigma, \mu) \in \Lambda\} \text{ is bounded} \\ \text{and uniformly equicontinuous} \end{array} \right\}.$$

Since $\operatorname{BUC}^2(\mathbb{R}^d)$ is dense in $\operatorname{BUC}(\mathbb{R}^d)$, it suffices to show that $\operatorname{BUC}^2(\mathbb{R}^d) \subset D$. Let $f \in \operatorname{BUC}^2(\mathbb{R}^d)$. Then, $f \in D(A_{b, \Sigma, \mu})$ for any Lévy triplet $(b, \Sigma, \mu) \in \Lambda$. Moreover, by Taylor's theorem we have

$$\begin{aligned} |(A_{b, \Sigma, \mu} f)(x)| &\leq |b| |\nabla f(x)| + |\Sigma| |\nabla^2 f(x)| + \int_{\mathbb{R}^d} |f(x+y) - f(x) - \nabla f(x) \cdot h(y)| \, dy \\ &\leq |b| \|\nabla f\|_\infty + |\Sigma| \|\nabla^2 f\|_\infty + \max \left\{ 2\|f\|_\infty, \frac{1}{2} \|\nabla^2 f\|_\infty \right\} \int_{\mathbb{R}^d} 1 \wedge |y|^2 \, d\mu(y) \\ &\leq 2C \max \{ \|f\|_\infty, \|\nabla f\|_\infty, \|\nabla^2 f\|_\infty \} \end{aligned}$$

for all $x \in \mathbb{R}^d$ and all $(b, \Sigma, \mu) \in \Lambda$, so that

$$\sup_{(b, \Sigma, \mu) \in \Lambda} \|A_{b, \Sigma, \mu} f\|_\infty \leq 2C \max \{ \|f\|_\infty, \|\nabla f\|_\infty, \|\nabla^2 f\|_\infty \}. \quad (3.2)$$

Let $\varepsilon > 0$. Since $f \in \operatorname{BUC}^2(\mathbb{R}^d)$ there exists $\delta > 0$ such that for all $x, z \in \mathbb{R}^d$ with $|x - z| \leq \delta$ and $\theta \in \{f, \nabla f, \nabla^2 f\}$ it holds

$$2C \sup_{y \in \mathbb{R}^d} |\theta(x+y) - \theta(z+y)| \leq \varepsilon.$$

Let $x, z \in \mathbb{R}^d$ be fixed with $|x - z| \leq \delta$ and $g: \mathbb{R}^d \rightarrow \mathbb{R}$ be defined by

$$g(y) := f(x+y) - f(z+y)$$

for all $y \in \mathbb{R}^d$, so that $g \in \operatorname{BUC}^2(\mathbb{R}^d)$ with $2C \max \{ \|g\|_\infty, \|\nabla g\|_\infty, \|\nabla^2 g\|_\infty \} \leq \varepsilon$. By (3.2) we get

$$|(A_{b, \Sigma, \mu} f)(x) - (A_{b, \Sigma, \mu} f)(z)| = |(A_{b, \Sigma, \mu} g)(0)| \leq 2C \max \{ \|g\|_\infty, \|\nabla g\|_\infty, \|\nabla^2 g\|_\infty \} \leq \varepsilon$$

for all $(b, \Sigma, \mu) \in \Lambda$, which shows that (A2) is satisfied. Therefore, by Theorem 2.5 there exists a sublinear expectation space $(\Omega, \mathcal{F}, \mathcal{E})$ and an \mathcal{E} -Lévy process $(X_t)_{t \geq 0}$ such that for all $f \in \operatorname{BUC}(\mathbb{R}^d)$ the function

$$u(t, x) := (u(t))(x) := \mathcal{E}(f(x + X_t)), \quad t \geq 0, x \in \mathbb{R}^d,$$

is a $\text{BUC}^2(\mathbb{R}^d)$ -viscosity solution of the fully nonlinear PIDE

$$\begin{aligned} u_t(t, x) &= \sup_{(b, \Sigma, \mu) \in \Lambda} (A_{b, \Sigma, \mu} u(t))(x), \quad (t, x) \in (0, \infty) \times \mathbb{R}^d, \\ u(0, x) &= f(x), \quad x \in \mathbb{R}^d. \end{aligned}$$

Example 3.3 (Lévy processes on the d -dimensional Torus \mathbb{T}^d). Let $G = \mathbb{T}^d$, where \mathbb{T}^d denotes the d -dimensional Torus represented by $(-\pi, \pi]^d$. We say that (b, Σ, μ, ν) is a *Lévy quadruple* if $b \in \mathbb{R}^d$, $\Sigma \in \mathbb{R}^{d \times d}$ is a symmetric positive semidefinite matrix, μ is a positive finite measure on \mathbb{T}^d and ν is a positive measure on \mathbb{T}^d with $\nu(\{0\}) = 0$ and

$$\int_{\mathbb{T}^d} |y|^2 d\nu(y) < \infty.$$

For each Lévy quadruple (b, Σ, μ, ν) we consider

$$\begin{aligned} (A_{b, \Sigma, \mu, \nu} f)(x) &:= b \cdot \nabla f(x) + \frac{1}{2} \text{tr}(\Sigma \nabla^2 f(x)) + \int_{\mathbb{T}^d} f(x+y) - f(x) d\mu(y) \\ &\quad + \int_{\mathbb{T}^d} f(x+y) - f(x) - \nabla f(x) \cdot y d\nu(y) \end{aligned}$$

for $x \in \mathbb{T}^d$ and $f \in D(A_{b, \Sigma, \mu, \nu}) := \{g \in C(\mathbb{T}^d) : A_{b, \Sigma, \mu, \nu} g \in C(\mathbb{T}^d)\}$. Recall that $C(\mathbb{T}^d)$ can be identified with the set of all 2π -periodic continuous functions on \mathbb{R}^d .

We start by proving that for any Lévy quadruple (b, Σ, μ, ν) the operator $A_{b, \Sigma, \mu, \nu}$ generates a Markovian convolution semigroup $(S_{b, \Sigma, \mu, \nu}(t))_{t \geq 0}$ of linear operators on \mathbb{T}^d . In order to do so, let (b, Σ, μ, ν) be a Lévy quadruple and define

$$\eta f := \int_{(-\pi, \pi]^d} f(y) d\nu(y) + \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \lambda_k \int_{(-\pi, \pi]^d} f(x + 2\pi k) d\mu(y)$$

for $f \in \mathcal{L}^\infty(\mathbb{R}^d)$, where $\lambda_k \geq 0$ for all $k \in \mathbb{Z}^d \setminus \{0\}$ and $\sum_{k \in \mathbb{Z}^d \setminus \{0\}} \lambda_k = 1$. Then, η is a Lévy measure on \mathbb{R}^d , see e.g. Example 3.2, so that (b, Σ, η) is a Lévy triplet. Hence, there exists a Markovian convolution semigroup $(S_{b, \Sigma, \eta}(t))_{t \geq 0}$ of linear operators on \mathbb{R}^d with generator $A_{b, \Sigma, \eta}$. As the space $C(\mathbb{T}^d)$ of all 2π -periodic continuous functions is a closed subspace of $\text{BUC}(\mathbb{R}^d)$, which is invariant under $S_{b, \Sigma, \eta}(t)$ for all $t \geq 0$, we obtain that

$$S(t) := (S_{b, \Sigma, \eta}(t))|_{C(\mathbb{T}^d)}, \quad t \geq 0$$

defines a Markovian convolution semigroup of linear operators on $C(\mathbb{T}^d)$. Let A denote the generator of the semigroup $(S(t))_{t \geq 0}$. As $C(\mathbb{T}^d)$ is a closed subspace of $\text{BUC}(\mathbb{R}^d)$ and $\sum_{k \in \mathbb{Z}^d \setminus \{0\}} \lambda_k = 1$, we get that

$$Af = A_{b, \Sigma, \eta} f = A_{b, \Sigma, \mu, \nu} f$$

for all $f \in D(A)$. In particular, $A_{b, \Sigma, \mu, \nu}$ is the generator of $(S(t))_{t \geq 0}$.

Now, let Λ be a set of Lévy quadruples with

$$\sup_{(b, \Sigma, \mu, \nu) \in \Lambda} \left(|b| + |\Sigma| + \mu(\mathbb{T}^d) + \int_{\mathbb{T}^d} |y|^2 d\nu(y) \right) < \infty. \quad (3.3)$$

Then, in a similar way as in Example 3.2, one can show that for every $f \in C^2(\mathbb{T}^d) = \text{BUC}^2(\mathbb{T}^d)$, the set $\{A_{b, \Sigma, \mu, \nu} f : (b, \Sigma, \mu, \nu) \in \Lambda\}$ is bounded and equicontinuous. Therefore, the assumptions (A1) and (A2) are satisfied. Hence, there exists a nonlinear

expectation space $(\Omega, \mathcal{F}, \mathcal{E})$ and an \mathcal{E} -Lévy process $(X_t)_{t \geq 0}$ such that for all $f \in C(\mathbb{T}^d)$ the function

$$u(t, x) := (u(t))(x) := \mathcal{E}(f(x + X_t)), \quad t \geq 0, x \in \mathbb{T}^d,$$

is a $C^2(\mathbb{T}^d)$ -viscosity solution of the fully nonlinear PIDE

$$\begin{aligned} u_t(t, x) &= \sup_{(b, \Sigma, \mu, \nu) \in \Lambda} (A_{b, \Sigma, \mu, \nu} u(t))(x), \quad (t, x) \in (0, \infty) \times \mathbb{T}^d, \\ u(0, x) &= f(x), \quad x \in \mathbb{T}^d. \end{aligned}$$

We also refer to Hunt [9] for a Lévy-Khintchine formula on compact Lie groups.

Example 3.4. Let $A: D(A) \subset \text{BUC}(G) \rightarrow \text{BUC}(G)$ be the generator of a Markovian convolution semigroup $(S(t))_{t \geq 0}$ and $\Lambda \subset [0, \infty)$ bounded. For all $\lambda \in \Lambda$ let $A_\lambda := \lambda A$ and $S_\lambda(t) := S(\lambda t)$ for $t \geq 0$. Then, the assumptions (A1) and (A2) are satisfied with $D = D(A)$. Hence, there exists a nonlinear expectation space $(\Omega, \mathcal{F}, \mathcal{E})$ and an \mathcal{E} -Lévy process $(X_t)_{t \geq 0}$ such that for all $u_0 \in \text{BUC}(G)$ the function

$$u(t, x) := \mathcal{E}(f(x + X_t)), \quad t \geq 0, x \in G$$

is a $D(A)$ -viscosity solution of the fully nonlinear PDE

$$\begin{aligned} u_t(t, x) &= \sup_{\lambda \in \Lambda} (\lambda A u(t))(x), \quad (t, x) \in (0, \infty) \times G, \\ u(0, x) &= f(x), \quad x \in G. \end{aligned}$$

For instance, if A is the generator of a cylindrical Wiener process on a separable Hilbert space, one obtains an \mathcal{E} -Lévy process which can be viewed as a cylindrical G -Wiener process.

Example 3.5. For all $h > 0$ let $\mu_h := \frac{1}{h^2} \delta_h$ and consider $\Lambda := \{(0, 0, \mu_h) : h > 0\}$ in Example 3.2 with $d = 1$. Then, we have that

$$\sup_{h > 0} \int_{\mathbb{R}} 1 \wedge |y|^2 d\mu_h(y) = \sup_{h > 0} \frac{1}{h^2} \int_{\mathbb{R}} |y|^2 d\delta_h(y) = 1,$$

so that the assumptions (A1) and (A2) are satisfied. However, the second condition in (1.3) does not hold. By Example 3.2, there exists a sublinear expectation space $(\Omega, \mathcal{F}, \mathcal{E})$ and an \mathcal{E} -Lévy process $(X_t)_{t \geq 0}$ such that for all $f \in \text{BUC}(\mathbb{R})$ the function

$$u(t, x) := \mathcal{E}(f(x + X_t)), \quad t \geq 0, x \in \mathbb{R},$$

is a $\text{BUC}^2(\mathbb{R})$ -viscosity solution of the fully nonlinear PIDE

$$\begin{aligned} u_t(t, x) &= \sup_{h > 0} (A_{0, 0, \mu_h} u(t))(x), \quad (t, x) \in (0, \infty) \times \mathbb{R}, \\ u(0, x) &= f(x), \quad x \in \mathbb{R}. \end{aligned}$$

Note that $\|A_{(0, 0, \mu_h)} f - \frac{1}{2} f''\|_\infty = \sup_{x \in \mathbb{R}} \left| \frac{f(x+h) - f(x) - f'(x)h}{h^2} - \frac{1}{2} f''(x) \right| \rightarrow 0$ as $h \searrow 0$ for all $f \in \text{BUC}^2(\mathbb{R})$.

Example 3.6 (Cauchy distributed jumps). For $\gamma > 0$ let δ_γ be given by

$$\mu_\gamma((-\infty, b)) := \frac{\gamma}{\pi} \int_{-\infty}^b \frac{1}{y^2 + \gamma^2} dy = \frac{1}{2} + \arctan\left(\frac{b}{\gamma}\right)$$

for $b \in \mathbb{R}$. Let $\Gamma \subset (0, \infty)$. Then, we have that

$$\sup_{\gamma \in \Gamma} \int_{\mathbb{R}} 1 \wedge |y|^2 d\mu_\gamma(y) \leq \sup_{\gamma \in \Gamma} \mu_\gamma(\mathbb{R}) = 1,$$

so that (A1) and (A2) are satisfied, but the first condition in (1.3) is violated. By Example 3.1 with $G = \mathbb{R}$ (using the notation from Example 3.2), there exists a sublinear expectation space $(\Omega, \mathcal{F}, \mathcal{E})$ and an \mathcal{E} -Lévy process $(X_t)_{t \geq 0}$ such that for all $f \in \text{BUC}(\mathbb{R})$ the function

$$u(t, x) := \mathcal{E}(f(x + X_t)), \quad t \geq 0, x \in \mathbb{R},$$

is the unique classical solution of the fully nonlinear PIDE

$$\begin{aligned} u_t(t, x) &= \sup_{\gamma \in \Gamma} (A_{0,0,\mu_\gamma} u(t))(x), \quad (t, x) \in (0, \infty) \times \mathbb{R}, \\ u(0, x) &= f(x), \quad x \in \mathbb{R} \end{aligned}$$

with $u \in C^1([0, \infty); \text{BUC}(\mathbb{R}))$.

4. PROOF OF THEOREM 2.3

Proof. Let $(\mathcal{S}(t))_{t \geq 0}$ be a convex Markovian convolution semigroup which is continuous from above. By Remark 2.2, every $\mathcal{S}(t)$ has a unique extension to a convex kernel $\mathcal{S}(t) : C_b(G) \rightarrow C_b(G)$ which is continuous from above. Then the family

$$\mathcal{E}_{s,t}(x, f) := (\mathcal{S}(t-s)f)(x), \quad 0 \leq s < t$$

of convex kernels on $C_b(G)$ is continuous from above and satisfies the Chapman-Kolmogorov equations, i.e.

$$\mathcal{E}_{s,t}(x, \mathcal{E}_{t,u}(\cdot, f)) = \mathcal{E}_{s,u}(x, f)$$

for all $0 \leq s < t < u$, $f \in C_b(G)$ and $x \in G$. Hence, it follows from [5, Theorem 5.6] that there exists a convex expectation space $(\Omega, \mathcal{F}, \mathcal{E})$ and a family of random variables $X_t : \Omega \rightarrow G$, $t \geq 0$, such that

$$\mathcal{E}(f(X_0)) = f(0)$$

for all $f \in C_b(G)$ and

$$\mathcal{E}(f(X_{t_1}, \dots, X_{t_n}, X_t)) = \mathcal{E}(\mathcal{E}_{s,t}(X_s, f(X_{t_1}, \dots, X_{t_n}, \cdot)))$$

for all $0 \leq s < t$, $n \in \mathbb{N}$, $0 \leq t_1 < \dots < t_n \leq s$, and $f \in C_b(G^{n+1})$. Recall that $(x_{t_1}, \dots, x_{t_n}, x_s) \mapsto \mathcal{E}_{s,t}(x_s, f(x_{t_1}, \dots, x_{t_n}, \cdot))$ is continuous, see e.g. [5, Proposition 5.5]. Next, we verify that $(X_t)_{t \geq 0}$ is a \mathcal{E} -Lévy process. For $f \in C_b(G)$ and $s, t \geq 0$ one has

$$\mathcal{E}(f(X_t)) = \mathcal{E}(\mathcal{E}_{0,t}(X_0, f)) = \mathcal{E}((\mathcal{S}(t)f)(X_0)) = (\mathcal{S}(t)f)(0)$$

and

$$\begin{aligned} \mathcal{E}(f(X_{s+t} - X_s)) &= \mathcal{E}(\tilde{f}(X_s, X_{s+t})) = \mathcal{E}(\mathcal{E}_{s,s+t}(X_s, \tilde{f}(X_s, \cdot))) \\ &= \mathcal{E}((\mathcal{S}(t)\tilde{f}(X_s, \cdot))(X_s)) = \mathcal{E}((\mathcal{S}(t)f)(0)) = (\mathcal{S}(t)f)(0), \end{aligned} \quad (4.1)$$

where $\tilde{f}(x, y) := f(y - x)$ and $(\mathcal{S}(t)\tilde{f}(x, \cdot))(x) = (\mathcal{S}(t)f_{-x})(x) = (\mathcal{S}(t)f)(0)$ because $\mathcal{S}(t)$ is a Markovian convolution. This shows that the random variables $X_{s+t} - X_s$ and X_t have the same distribution under \mathcal{E} . Moreover, for $s, t \geq 0$, $0 \leq t_1 < \dots < t_n \leq s$ and $f \in C_b(G^{n+1})$, it follows by (4.1) that

$$(\mathcal{S}(t)f(x_{t_1}, \dots, x_{t_n}, \cdot))(0) = \mathcal{E}(f(x_{t_1}, \dots, x_{t_n}, X_{s+t} - X_s))$$

for all $x_{t_1}, \dots, x_{t_n} \in G$ and therefore,

$$\begin{aligned} \mathcal{E}(f(X_{t_1}, \dots, X_{t_n}, X_{s+t} - X_s)) &= \mathcal{E}(f_{-X_s}(X_{t_1}, \dots, X_{t_n}, X_{s+t})) \\ &= \mathcal{E}(\mathcal{E}_{s,s+t}(X_s, f_{-X_s}(X_{t_1}, \dots, X_{t_n}, \cdot))) \\ &= \mathcal{E}((\mathcal{S}(t)f_{-X_s}(X_{t_1}, \dots, X_{t_n}, \cdot))(X_s)) \\ &= \mathcal{E}((\mathcal{S}(t)f(X_{t_1}, \dots, X_{t_n}, \cdot))(0)) \\ &= \mathcal{E}(\mathcal{E}(f(x_{t_1}, \dots, x_{t_n}, X_{s+t} - X_s))|_{x_{t_1}=X_{t_1}, \dots, x_{t_n}=X_{t_n}}) \end{aligned}$$

which shows that $X_{s+t} - X_s$ is independent of $(X_{t_1}, \dots, X_{t_n})$. It remains to verify that $X_t \rightarrow X_0$ in distribution as $t \searrow 0$. To that end, fix $f \in C_b(G)$ and notice that there exists an increasing sequence (f_n) in $\text{BUC}(G)$ which converges pointwise to f . By Dini's theorem it follows that the continuous functions $t \mapsto (\mathcal{S}(t)f_n)(0)$ converges uniformly on $[0, 1]$ to the continuous function $t \mapsto (\mathcal{S}(t)f)(0)$. Hence, for every $\varepsilon > 0$ there exists some $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} |\mathcal{E}(f(X_t)) - f(0)| &\leq |(\mathcal{S}(t)f)(0) - (\mathcal{S}(t)f_{n_0})(0)| + \|\mathcal{S}(t)f_{n_0} - f_{n_0}\|_\infty \\ &\quad + |f_{n_0}(0) - f(0)| \\ &\leq \varepsilon, \quad \text{as } t \searrow 0. \end{aligned}$$

Conversely, let $(X_t)_{t \geq 0}$ be an \mathcal{E} -Lévy process on a convex expectation space $(\Omega, \mathcal{F}, \mathcal{E})$. Then, the family $(\mathcal{S}(t))_{t \geq 0}$ defined by

$$(\mathcal{S}(t)f)(x) := \mathcal{E}(f(x + X_t))$$

for $f \in \text{BUC}(G)$ and $x \in G$ is a convex Markovian convolution semigroup. Indeed, for $f \in \text{BUC}(G)$, $x \in G$ and $t \geq 0$, it holds that

$$(\mathcal{S}(t)f_x)(0) = \mathcal{E}(f(x + X_t)) = (\mathcal{S}(t)f)(x),$$

so that $\mathcal{S}(t)$ is a convex Markovian convolution which satisfies

$$(\mathcal{S}(0)f)(x) = \mathcal{E}(f(x + X_0)) = \mathcal{E}(f_x(X_0)) = f_x(0) = f(x).$$

Moreover, since $\mathcal{E}(f(x + (X_{t+s} - X_s))) = \mathcal{E}(f(x + X_t)) = (\mathcal{S}(t)f)(x)$ for all $s, t \geq 0$, $f \in \text{BUC}(G)$, $x \in G$, and $X_{t+s} - X_s$ is independent of X_s we obtain

$$\begin{aligned} (\mathcal{S}(t+s)f)(0) &= \mathcal{E}(f(X_{t+s})) = \mathcal{E}(f(X_s + (X_{t+s} - X_s))) \\ &= \mathcal{E}((\mathcal{S}(t)f)(X_s)) = (\mathcal{S}(s)\mathcal{S}(t)f)(0). \end{aligned}$$

Since $\mathcal{S}(s)$ and $\mathcal{S}(t)$ are Markovian convolutions we conclude the semigroup property $\mathcal{S}(t+s) = \mathcal{S}(s)\mathcal{S}(t)$. It remains to show that $\lim_{t \searrow 0} \|\mathcal{S}(t)f - f\|_\infty = 0$ for all $f \in \text{BUC}(G)$. To do so, we first show that for every $c \geq 0$ and $\delta > 0$ we have that

$$\mathcal{E}(c1_{G \setminus B(0, \delta)}(X_t)) \rightarrow 0 \tag{4.2}$$

as $t \searrow 0$. Let $\varphi: G \rightarrow \mathbb{R}$ be defined by

$$\varphi(y) := \frac{c}{\delta}(d(y, 0) \wedge \delta)$$

for $y \in G$. Then, $\varphi \in C_b(G)$ with $\varphi \geq 0$, $\varphi(0) = 0$ and $\varphi(y) = c$ for all $y \in G \setminus B(0, \delta)$, so that

$$0 \leq \mathcal{E}(c1_{G \setminus B(0, \delta)}(X_t)) = \mathcal{E}(\varphi(X_t)1_{G \setminus B(0, \delta)}(X_t)) \leq \mathcal{E}(\varphi(X_t)) \rightarrow 0$$

as $t \searrow 0$. Now, fix $\varepsilon > 0$ and $f \in \text{BUC}(G)$. Then there exists some $\delta > 0$ such that $|f(x+y) - f(x)| < \frac{\varepsilon}{2}$ for all $x, y \in G$ with $d(y, 0) < \delta$. For each $x \in G$ let $g_x \in \mathcal{L}^\infty(G)$

be given by $y \mapsto 1_{B(0,\delta)}(y)(f(x+y) - f(x))$. Then $\|g_x\|_\infty \leq \frac{\varepsilon}{2}$ for all $x \in G$, and by (4.2) there exists some $t_0 > 0$ such that

$$\mathcal{E}(2\|f\|_\infty 1_{G \setminus B(0,\delta)}(X_t)) \leq \frac{\varepsilon}{2}$$

for all $0 < t < t_0$. Since $\mathcal{E}: \mathcal{L}^\infty(\Omega, \mathcal{F}) \rightarrow \mathbb{R}$ is 1-Lipschitz continuous, we get

$$\begin{aligned} |(\mathcal{S}(t)f)(x) - f(x)| &= |\mathcal{E}(f(x+X_t) - f(x))| \leq \mathcal{E}(|f(x+X_t) - f(x)|) \\ &\leq \mathcal{E}(g_x(X_t) + 2\|f\|_\infty 1_{G \setminus B(0,\delta)}(X_t)) \\ &\leq \|g_x\|_\infty + \mathcal{E}(2\|f\|_\infty 1_{G \setminus B(0,\delta)}(X_t)) \leq \varepsilon \end{aligned}$$

for all $0 < t < t_0$, which shows that $\|\mathcal{S}(t)f - f\|_\infty \leq \varepsilon$ for all $0 < t < t_0$. \square

5. RANDOMIZING LINEAR SEMIGROUPS AND THE PROOF OF THEOREM 2.5

Throughout this section all Markovian convolutions are defined on $BUC(G)$. We assume that $\{A_\lambda : \lambda \in \Lambda\}$ is a given non-empty family of operators which satisfies the assumptions (A1) and (A2). Recall that A_λ generates a Markovian convolution semigroup $(S_\lambda(t))_{t \geq 0}$ of linear operators, and the domain

$$D = \left\{ f \in \bigcap_{\lambda \in \Lambda} D(A_\lambda) : \{A_\lambda f : \lambda \in \Lambda\} \text{ is bounded and uniformly equicontinuous} \right\}$$

is dense in $BUC(G)$.

We consider finite partitions $P := \{\pi \subset [0, \infty) : 0 \in \pi, \pi \text{ finite}\}$. For a partition $\pi = \{t_0, t_1, \dots, t_m\} \in P$ with $0 = t_0 < t_1 < \dots < t_m$ we set $|\pi|_\infty := \max_{j=1, \dots, m} (t_j - t_{j-1})$. The set of partitions with end-point t is denoted by P_t , i.e. $P_t := \{\pi \in P : \max \pi = t\}$.

For $f \in BUC(G)$ and $t \geq 0$, we define

$$(J_t f)(x) := \sup_{\lambda \in \Lambda} (S_\lambda(t)f)(x), \quad x \in G,$$

and for a partition $\pi \in P$, we set

$$J_\pi f := J_{t_1-t_0} \cdots J_{t_m-t_{m-1}} f,$$

where we assume that $\pi = \{t_0, t_1, \dots, t_m\}$ with $0 = t_0 < t_1 < \dots < t_m$, and $J_{\{0\}} f := f$. Note that $J_t = J_{\{0,t\}}$ for $t > 0$. Moreover, since $S_\lambda(t)$ is continuous from below for all $\lambda \in \Lambda$ and $t \geq 0$, it follows that J_π is continuous from below for all $\pi \in P$.

Lemma 5.1. a) J_π is a sublinear Markovian convolution for all $\pi \in P$.

b) Let $f \in D$ and $L_f := \sup_{\lambda \in \Lambda} \|A_\lambda f\|_\infty (< \infty)$. Then,

$$\|J_{t_1} f - J_{t_2} f\|_\infty \leq L_f |t_1 - t_2|, \quad t_1, t_2 \geq 0, \quad (5.1)$$

$$\|J_\pi f - f\|_\infty \leq L_f t, \quad \pi \in P_t, t > 0. \quad (5.2)$$

Proof. a) Since $S_\lambda(t)$ is a linear Markovian convolution for all $\lambda \in \Lambda$, J_t is a sublinear Markovian convolution for all $t \geq 0$. As this property is preserved under compositions, the same holds for J_π .

b) Let $f \in D$. For $t_1, t_2 \geq 0$, $x \in G$, and $\lambda_0 \in \Lambda$ we have that

$$\begin{aligned} (S_{\lambda_0}(t_1)f)(x) - (J_{t_2}f)(x) &\leq (S_{\lambda_0}(t_1)f)(x) - (S_{\lambda_0}(t_2)f)(x) \\ &\leq \|S_{\lambda_0}(t_1)f - S_{\lambda_0}(t_2)f\|_\infty \\ &\leq \sup_{\lambda \in \Lambda} \|S_\lambda(t_1)f - S_\lambda(t_2)f\|_\infty \end{aligned}$$

and therefore, taking the supremum over $\lambda_0 \in \Lambda$,

$$(J_{t_1}f)(x) - (J_{t_2}f)(x) \leq \sup_{\lambda \in \Lambda} \|S_\lambda(t_1)f - S_\lambda(t_2)f\|_\infty.$$

By symmetry and taking the supremum over all $x \in G$, we thus get that

$$\|J_{t_1}f - J_{t_2}f\|_\infty \leq \sup_{\lambda \in \Lambda} \|S_\lambda(t_1)f - S_\lambda(t_2)f\|_\infty.$$

Let $\lambda \in \Lambda$ and w.l.o.g. let $t_1 < t_2$. Then, as $f \in D(A_\lambda)$, it follows that (see, e.g. [7, Lemma II.1.3])

$$\begin{aligned} \|S_\lambda(t_1)f - S_\lambda(t_2)f\|_\infty &= \left\| \int_{t_1}^{t_2} S_\lambda(s)A_\lambda f \, ds \right\|_\infty \leq \int_{t_1}^{t_2} \|S_\lambda(s)A_\lambda f\|_\infty \, ds \\ &\leq (t_2 - t_1)\|A_\lambda f\|_\infty \leq L_f|t_1 - t_2| \end{aligned}$$

for all $\lambda \in \Lambda$. Taking the supremum over all $\lambda \in \Lambda$, we obtain (5.1).

To show (5.2), let $\pi = \{0, t_1, \dots, t_{m-1}, t\} \in P_t$ with $0 < t_1 < \dots < t_{m-1} < t$ and $m \in \mathbb{N}$. Note that the case $\pi = \{0\}$ is trivial. For $m = 1$, we have $\pi = \{0, t\}$, and by (5.1),

$$\|J_\pi f - f\|_\infty = \|J_t f - J_0 f\|_\infty \leq L_f t.$$

Now let $m \geq 2$, and set $\pi' := \{0, t_1, \dots, t_{m-1}\}$. By induction, we have $\|J_{\pi'} f - f\|_\infty \leq L_f t_{m-1}$. As $J_{\pi'}$ is sublinear and therefore 1-Lipschitz continuous, we obtain

$$\begin{aligned} \|J_\pi f - f\|_\infty &= \|J_{\pi'} J_{t-t_{m-1}} f - f\|_\infty \\ &\leq \|J_{\pi'} J_{t-t_{m-1}} f - J_{\pi'} f\|_\infty + \|J_{\pi'} f - f\|_\infty \\ &\leq \|J_{t-t_{m-1}} f - f\|_\infty + \|J_{\pi'} f - f\|_\infty \\ &\leq L_f(t - t_{m-1}) + L_f t_{m-1} = L_f t. \end{aligned}$$

□

The following result shows that $J_\pi f$ depends continuously on the partition π .

Lemma 5.2. *Let $m \in \mathbb{N}$, $\pi = \{0, t_1, \dots, t_m\} \in P$ with $0 < t_1 < \dots < t_m$, and for $n \in \mathbb{N}$ let $\pi_n = \{0, t_1^n, \dots, t_m^n\} \in P$ with $\lim_{n \rightarrow \infty} t_j^n = t_j$ for $j = 1, \dots, m$. Then*

$$\|J_\pi f - J_{\pi_n} f\|_\infty \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

for all $f \in BUC(G)$.

Proof. Note that $0 < t_1^n < \dots < t_m^n$ for sufficiently large n . Fix $f \in BUC(G)$. We have to show the continuity of the map $(t_1, \dots, t_m) \mapsto J_{\{0, t_1, \dots, t_m\}} f$. By definition of J_π , it is sufficient to show that the map

$$[0, \infty) \rightarrow BUC(G), \quad t \mapsto J_t f$$

is continuous. Let $\varepsilon > 0$ and $t \geq 0$, and let (t^n) be a sequence in $[0, \infty)$ such that $t^n \rightarrow t$. By assumption (A2) there exists $f_0 \in D$ with $\|f - f_0\|_\infty \leq \frac{\varepsilon}{3}$, and it follows from Lemma 5.1 that

$$\begin{aligned} \|J_t f - J_{t^n} f\|_\infty &\leq \|J_t f - J_t f_0\|_\infty + \|J_t f_0 - J_{t^n} f_0\|_\infty + \|J_{t^n} f - J_{t^n} f_0\|_\infty \\ &\leq \frac{2}{3}\varepsilon + L_{f_0}|t - t^n| < \varepsilon \end{aligned}$$

for sufficiently large $n \in \mathbb{N}$. □

The above results allow to consider the limit of $J_\pi f$ when the mesh size of the partition tends to zero. To that end, we first note that for $s, t \geq 0$, $f \in \text{BUC}(G)$ and $x \in G$, it holds

$$\begin{aligned} (J_{t+s}f)(x) &= \sup_{\lambda \in \Lambda} (S_\lambda(t+s)f)(x) = \sup_{\lambda \in \Lambda} (S_\lambda(t)S_\lambda(s)f)(x) \\ &\leq \sup_{\lambda \in \Lambda} (S_\lambda(t)J_s f)(x) = (J_t J_s f)(x). \end{aligned}$$

From this, we obtain for $\pi_1, \pi_2 \in P_t$ with $\pi_1 \subset \pi_2$ the pointwise inequality

$$J_{\pi_1} f \leq J_{\pi_2} f. \quad (5.3)$$

In particular, for $\pi_1, \pi_2 \in P_t$ we have that

$$(J_{\pi_1} f) \vee (J_{\pi_2} f) \leq J_\pi f, \quad (5.4)$$

where $\pi := \pi_1 \cup \pi_2 \in P_t$. Therefore, we define

$$(\mathcal{S}(t)f)(x) := \sup_{\pi \in P_t} (J_\pi f)(x) \quad (5.5)$$

for all $t \geq 0$, $x \in G$ and $f \in \text{BUC}(G)$. Note that $\mathcal{S}(0)f = f$ for all $f \in \text{BUC}(G)$.

Lemma 5.3. $\mathcal{S}(t)$ is a sublinear Markovian convolution for all $t \geq 0$, which satisfies

$$\|\mathcal{S}(t)f - f\|_\infty \leq L_f t, \quad f \in D.$$

Moreover,

$$\lim_{t \searrow 0} \|\mathcal{S}(t)f - f\|_\infty = 0$$

for all $f \in \text{BUC}(G)$.

Proof. As J_π is a sublinear Markovian convolution for all $\pi \in P_t$, the same holds for $\mathcal{S}(t)$. For $f \in D$, $x \in G$ and $\varepsilon > 0$ there exists $\pi_0 \in P_t$ such that

$$(\mathcal{S}(t)f)(x) - f(x) \leq (J_{\pi_0} f)(x) - f(x) + \varepsilon \leq \sup_{\pi \in P_t} \|J_\pi f - f\|_\infty + \varepsilon$$

and

$$f(x) - (\mathcal{S}(t)f)(x) \leq f(x) - (J_{\pi_0} f)(x) \leq \sup_{\pi \in P_t} \|J_\pi f - f\|_\infty.$$

By Lemma 5.1 it follows that

$$\|\mathcal{S}(t)f - f\|_\infty \leq \sup_{\pi \in P_t} \|J_\pi f - f\|_\infty \leq L_f t.$$

From this, we obtain that $\lim_{t \searrow 0} \|\mathcal{S}(t)f - f\|_\infty = 0$ for $f \in \text{BUC}(G)$ with the same density argument as in the proof of Lemma 5.2. \square

Lemma 5.4. For $t \geq 0$, let $(\pi_n)_{n \in \mathbb{N}}$ be a sequence in P_t such that $\pi_n \subset \pi_{n+1}$ for all $n \in \mathbb{N}$, and $\lim_{n \rightarrow \infty} |\pi_n|_\infty = 0$. Then

$$J_{\pi_n} f \nearrow \mathcal{S}(t)f, \quad \text{as } n \rightarrow \infty,$$

for all $f \in \text{BUC}(G)$.

Proof. Fix $f \in \text{BUC}(G)$. For $t = 0$, the statement is trivial. For $t > 0$ and $x \in G$ we define

$$(J_\infty f)(x) := \sup_{n \in \mathbb{N}} (J_{\pi_n} f)(x)$$

Then, J_∞ is a sublinear Markovian convolution. Since $\pi_n \subset \pi_{n+1}$, it follows from (5.3) that

$$J_{\pi_n} f \nearrow J_\infty f, \quad \text{as } n \rightarrow \infty.$$

By definition of $\mathcal{S}(t)$, it clearly holds $J_\infty f \leq \mathcal{S}(t)f$. As for the other inequality, let $\pi = \{t_0, t_1, \dots, t_m\} \in P_t$ with $m \in \mathbb{N}$ and $0 = t_0 < t_1 < \dots < t_m = t$. Since $|\pi_n|_\infty \searrow 0$ as $n \rightarrow \infty$, we may w.l.o.g. assume that $\#\pi_n \geq m+1$ for all $n \in \mathbb{N}$. Moreover, we can choose $0 = t_0^n < t_1^n < \dots < t_m^n = t$ with $\pi_n' := \{t_0^n, t_1^n, \dots, t_m^n\} \subset \pi_n$ and $\lim_{n \rightarrow \infty} t_i^n = t_i$ for all $i = 1, \dots, m-1$. Then, by Lemma 5.2, we have that

$$\|J_\pi f - J_{\pi_n'} f\|_\infty \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Since

$$J_\infty f \geq J_{\pi_n} f \geq J_{\pi_n'} f \geq J_\pi f - \|J_\pi f - J_{\pi_n'} f\|_\infty$$

we obtain that $J_\infty f \geq J_\pi f$. Taking the supremum over all $\pi \in P_t$, we thus get that $J_\infty f = \mathcal{S}(t)f$. \square

Corollary 5.5. *Let $t \geq 0$. Then, there exists a sequence (π_n) in P_t such that*

$$J_{\pi_n} f \nearrow \mathcal{S}(t)f, \quad \text{as } n \rightarrow \infty.$$

Moreover,

$$\mathcal{S}(t)f = \sup_{n \in \mathbb{N}} (J_{\frac{t}{n}})_n^n f = \sup_{n \in \mathbb{N}} (J_{2^{-n}t})^{2^n} f = \lim_{n \rightarrow \infty} (J_{2^{-n}t})^{2^n} f, \quad (5.6)$$

for all $f \in BUC(G)$, where the supremum is understood pointwise.

Proof. Choose $\pi_n := \{\frac{kt}{2^n} : k \in \{0, \dots, 2^n\}\}$ in Lemma 5.4 to obtain the first statement. For $\tilde{\pi}_n := \{\frac{kt}{n} : k \in \{0, \dots, n\}\}$ it holds $\tilde{\pi}_{2^n} = \pi_n$ for all $n \in \mathbb{N}$, which yields the first equality in (5.6) and therefore the assertion. \square

Theorem 5.6 (Dynamic programming principle). *$(\mathcal{S}(t))_{t \geq 0}$ is a Markovian convolution semigroup of sublinear operators. In particular, for every $s, t \geq 0$ one has*

$$\mathcal{S}(s+t)f = \mathcal{S}(s)\mathcal{S}(t)f \quad (5.7)$$

for every $f \in BUC(G)$.

Proof. Let $\pi_0 \in P_{s+t}$ and $\pi := \pi_0 \cup \{s\}$. Then, $\pi \in P_{s+t}$ with $\pi_0 \subset \pi$, and by (5.3) we get that

$$J_{\pi_0} f \leq J_\pi f.$$

We have already shown all properties of a Markovian convolution semigroup except the semigroup property (5.7), i.e. the dynamic programming principle.

If $s = 0$ or $t = 0$ the statement is trivial. Therefore, let $s, t > 0$. Let $m \in \mathbb{N}$ and $0 = t_0 < t_1 < \dots < t_m = s+t$ with $t_i = s$ for some $i \in \{1, \dots, m\}$, and define $\pi := \{t_0, \dots, t_m\} \in P_{s+t}$. Then, for $\pi_1 := \{t_0, \dots, t_i\} \in P_s$ and $\pi_2 := \{t_i - s, \dots, t_m - s\} \in P_t$ we have

$$J_{\pi_1} = J_{t_1 - t_0} \cdots J_{t_i - t_{i-1}}, \quad \text{and} \quad J_{\pi_2} = J_{t_{i+1} - t_i} \cdots J_{t_m - t_{m-1}},$$

and therefore

$$\begin{aligned} J_\pi f &= J_{t_1 - t_0} \cdots J_{t_m - t_{m-1}} f = (J_{t_1 - t_0} \cdots J_{t_i - t_{i-1}}) (J_{t_{i+1} - t_i} \cdots J_{t_m - t_{m-1}} f) \\ &= J_{\pi_1} J_{\pi_2} f \leq J_{\pi_1} \mathcal{S}(t)f \leq \mathcal{S}(s)\mathcal{S}(t)f. \end{aligned}$$

Taking the supremum over all $\pi \in P_{s+t}$, we get $\mathcal{S}(s+t)f \leq \mathcal{S}(s)\mathcal{S}(t)f$. Conversely, by Corollary 5.5 there exists a sequence (π_n) in P_t such that $J_{\pi_n} f \nearrow \mathcal{S}(t)f$ as $n \rightarrow \infty$. For $\pi_0 \in P_s$ and $\pi_n' := \pi_0 \cup \{r+s : r \in \pi_n\} \in P_{s+t}$ it holds $J_{\pi_n'} = J_{\pi_0} J_{\pi_n}$. Since J_{π_0} is continuous from below we have

$$J_{\pi_0} \mathcal{S}(t)f = \lim_{n \rightarrow \infty} J_{\pi_0} J_{\pi_n} f = \lim_{n \rightarrow \infty} J_{\pi_n'} f \leq \mathcal{S}(s+t)f.$$

Taking the supremum over all $\pi_0 \in P_s$, we get that $\mathcal{S}(s)\mathcal{S}(t)f \leq \mathcal{S}(s+t)f$. \square

Lemma 5.7. *Let $\mathcal{S} : \text{BUC}(G) \rightarrow \text{BUC}(G)$ be convex and continuous. Let $(\Omega, \mathcal{F}, \nu)$ be a finite measure space with $\nu \neq 0$. Further, let $g : \Omega \rightarrow \text{BUC}(G)$ be bounded and \mathcal{F} - $\mathcal{B}(\text{BUC}(G))$ -measurable with separable range $g(\Omega)$, i.e. g is Bochner integrable. Then, $\mathcal{S}g : \Omega \rightarrow \text{BUC}(G)$, $\omega \mapsto \mathcal{S}(g(\omega))$ is again bounded and \mathcal{F} - $\mathcal{B}(\text{BUC}(G))$ -measurable with separable range $(\mathcal{S}g)(\Omega)$. Hence, $\mathcal{S}g$ is Bochner integrable and we have that*

$$\mathcal{S} \left(\frac{1}{\nu(\Omega)} \int_{\Omega} g \, d\nu \right) \leq \frac{1}{\nu(\Omega)} \int_{\Omega} \mathcal{S}g \, d\nu.$$

Proof. Since \mathcal{S} is continuous, we obtain that $\mathcal{S}g : \Omega \rightarrow \text{BUC}(G)$ is \mathcal{F} - $\mathcal{B}(\text{BUC}(G))$ -measurable with separable range $(\mathcal{S}g)(\Omega)$. If g is a simple function, $\mathcal{S}g$ is a simple function and the assertion follows by convexity of \mathcal{S} . Since g is \mathcal{F} - $\mathcal{B}(\text{BUC}(G))$ -measurable with separable range $g(\Omega)$, there exists a sequence of simple functions $(g_n)_{n \in \mathbb{N}}$ with $\lim_{n \rightarrow \infty} \|g_n(\omega) - g(\omega)\|_{\infty} = 0$ for all $\omega \in \Omega$. By continuity of \mathcal{S} , we obtain $\lim_{n \rightarrow \infty} \|\mathcal{S}g_n(\omega) - \mathcal{S}g(\omega)\|_{\infty} = 0$ for all $\omega \in \Omega$. Hence, by definition of Bochner's integral it follows that

$$\begin{aligned} \mathcal{S} \left(\frac{1}{\nu(\Omega)} \int_{\Omega} g \, d\nu \right) &= \lim_{n \rightarrow \infty} \mathcal{S} \left(\frac{1}{\nu(\Omega)} \int_{\Omega} g_n \, d\nu \right) \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{\nu(\Omega)} \int_{\Omega} \mathcal{S}g_n \, d\nu = \frac{1}{\nu(\Omega)} \int_{\Omega} \mathcal{S}g \, d\nu. \end{aligned}$$

\square

Fix $f \in D$. Since $\{A_{\lambda}f : \lambda \in \Lambda\} \subset \text{BUC}(G)$ is bounded and uniformly equicontinuous, it follows that

$$\mathcal{A}f := \sup_{\lambda \in \Lambda} A_{\lambda}f \in \text{BUC}(G),$$

where the supremum is understood pointwise.

Lemma 5.8. *Fix $f \in D$. Then, for $\pi \in P$ and $t > 0$ it holds*

$$J_{\pi}f - f \leq \int_0^{\max \pi} \mathcal{S}(s)\mathcal{A}f \, ds \quad \text{and} \quad \mathcal{S}(t)f - f \leq \int_0^t \mathcal{S}(s)\mathcal{A}f \, ds.$$

Proof. Since $\mathcal{A}f \in \text{BUC}(G)$, the mapping $[0, \infty) \rightarrow \text{BUC}(G)$, $s \mapsto \mathcal{S}(s)\mathcal{A}f$ is continuous. Therefore, the Bochner integrals are well-defined. Then, for all $t, h > 0$ we have

$$\begin{aligned} J_h f - f &= \sup_{\lambda \in \Lambda} S_{\lambda}(h)f - f = \sup_{\lambda \in \Lambda} \int_0^h S_{\lambda}(s)A_{\lambda}f \, ds \\ &\leq \int_0^h \mathcal{S}(s)\mathcal{A}f \, ds = \int_t^{t+h} \mathcal{S}(s-t)\mathcal{A}f \, ds. \end{aligned} \tag{5.8}$$

We prove the first inequality by induction on $m = \#\pi$. If $m = 1$, i.e. if $\pi = \{0\}$, the statement is trivial. Hence, assume that

$$J_{\pi'}f - f \leq \int_0^{\max \pi'} \mathcal{S}(s)\mathcal{A}f \, ds$$

for all $\pi' \in P$ with $\#\pi' = m$ for some $m \in \mathbb{N}$. Let $\pi = \{t_0, t_1, \dots, t_m\}$ with $0 = t_0 < t_1 < \dots < t_m$ and $\pi' := \pi \setminus \{t_m\}$. Then, it follows from (5.8) and Lemma 5.7 that

$$\begin{aligned} J_\pi f - J_{\pi'} f &\leq J_{\pi'}(J_{t_m - t_{m-1}} f - f) \leq J_{\pi'} \left(\int_{t_{m-1}}^{t_m} \mathcal{S}(s - t_{m-1}) \mathcal{A}f \, ds \right) \\ &\leq \mathcal{S}(t_m - t_{m-1}) \left(\int_{t_{m-1}}^{t_m} \mathcal{S}(s - t_{m-1}) \mathcal{A}f \, ds \right) \leq \int_{t_{m-1}}^{t_m} \mathcal{S}(s) \mathcal{A}f \, ds, \end{aligned}$$

where the first inequality follows from the sublinearity of $J_{\pi'}$. By induction, we thus get

$$\begin{aligned} J_\pi f - f &= (J_\pi f - J_{\pi'} f) + (J_{\pi'} f - f) \leq \int_{t_{m-1}}^{t_m} \mathcal{S}(s) \mathcal{A}f \, ds + \int_0^{t_{m-1}} \mathcal{S}(s) \mathcal{A}f \, ds \\ &= \int_0^{\max \pi} \mathcal{S}(s) \mathcal{A}f \, ds. \end{aligned}$$

In particular, for every $\pi \in P_t$ it holds

$$J_\pi f - f \leq \int_0^t \mathcal{S}(s) \mathcal{A}f \, ds.$$

Taking the supremum over all $\pi \in P_t$ yields the second assertion. \square

Lemma 5.9. *Let $M \subset \text{BUC}(G)$ be bounded and uniformly equicontinuous. Then,*

$$\sup_{\lambda \in \Lambda} \sup_{g \in M} \|S_\lambda(t)g - g\|_\infty \rightarrow 0, \quad \text{as } t \searrow 0.$$

Proof. Let $\varepsilon > 0$ and $C := \sup_{g \in M} \|g\|_\infty$. Then, there exists some $\delta > 0$ such that

$$\sup_{g \in M} |g(x) - g(y)| \leq \varepsilon$$

for all $x, y \in G$ with $d(x, y) \leq \delta$. Let $\varphi(y) := \frac{1}{\delta}(d(y, 0) \wedge \delta)$ for all $y \in G$. Then, $\varphi \in \text{BUC}(G)$, $0 \leq \varphi \leq 1$, $\varphi(0) = 0$ and $\varphi(y) = 1$ for all $y \in G \setminus B(0, \delta)$. Fix $\lambda \in \Lambda$, $g \in M$ and $x \in G$. Since $g(x + \cdot) - g(x) \leq \varepsilon + 2C\varphi$, we get

$$|(S_\lambda(t)g)(x) - g(x)| = |[S_\lambda(t)(g(x + \cdot) - g(x))](0)| \leq \varepsilon + 2C(\mathcal{S}(t)\varphi)(0),$$

so that

$$\sup_{\lambda \in \Lambda} \sup_{g \in M} \|S_\lambda(t)g - g\|_\infty \leq \varepsilon + 2C(\mathcal{S}(t)\varphi)(0).$$

Since $(\mathcal{S}(t)\varphi)(0) \rightarrow 0$ as $t \searrow 0$ by Lemma 5.3, we obtain the assertion. \square

Lemma 5.10. *For every $f \in D$ one has*

$$\lim_{h \searrow 0} \left\| \frac{\mathcal{S}(h)f - f}{h} - \mathcal{A}f \right\|_\infty = 0.$$

Proof. Let $\varepsilon > 0$. Then, by Lemma 5.9 and Lemma 5.3 there exists $h_0 > 0$ such that

$$S_\lambda(h)A_\lambda f - A_\lambda f \geq -\varepsilon \quad \text{for all } \lambda \in \Lambda$$

and

$$\mathcal{S}(h)\mathcal{A}f - \mathcal{A}f \leq \varepsilon$$

for all $0 < h \leq h_0$. Then, for every $0 < h \leq h_0$ and $\lambda \in \Lambda$ we get

$$\mathcal{S}(h)f - f \geq S_\lambda(h)f - f = \int_0^h S_\lambda(s)A_\lambda f \, ds \geq (A_\lambda f - \varepsilon)h$$

so that

$$\frac{\mathcal{S}(h)f - f}{h} \geq \mathcal{A}f - \varepsilon. \quad (5.9)$$

On the other hand, it follows from Lemma 5.8 that for $0 < h \leq h_0$

$$\mathcal{S}(h)f - f - h\mathcal{A}f \leq \int_0^h \mathcal{S}(s)\mathcal{A}f \, ds - h\mathcal{A}f = \int_0^h \mathcal{S}(s)\mathcal{A}f - \mathcal{A}f \, ds \leq h\varepsilon,$$

which yields

$$\frac{\mathcal{S}(h)f - f}{h} - \mathcal{A}f \leq \varepsilon.$$

Together with (5.9), we obtain

$$\left\| \frac{\mathcal{S}(h)f - f}{h} - \mathcal{A}f \right\|_{\infty} \leq \varepsilon$$

for all $0 < h \leq h_0$. □

Theorem 5.11. *For $f \in \text{BUC}(G)$ the function*

$$u(t, x) = (u(t))(x) = (\mathcal{S}(t)f)(x), \quad t \geq 0, x \in G,$$

is a D -viscosity solution of the fully nonlinear PDE

$$\begin{aligned} u_t(t, x) &= \sup_{\lambda \in \Lambda} (A_{\lambda}u(t))(x), \quad (t, x) \in (0, \infty) \times G, \\ u(0, x) &= f(x), \quad x \in G. \end{aligned}$$

Proof. Fix $t > 0$ and $x \in G$. We first show that u is a D -viscosity subsolution. Let $\psi: (0, \infty) \rightarrow \text{BUC}(G)$ differentiable with $(\psi(t))(x) = (u(t))(x)$, $\psi(s) \geq u(s)$ and $\psi(s) \in D$ for all $s > 0$. Then, for every $h \in (0, t)$, it follows from Theorem 5.6 that

$$\begin{aligned} 0 &= \frac{\mathcal{S}(h)\mathcal{S}(t-h)f - \mathcal{S}(t)f}{h} = \frac{\mathcal{S}(h)u(t-h) - u(t)}{h} \\ &\leq \frac{\mathcal{S}(h)\psi(t-h) - u(t)}{h} \leq \frac{\mathcal{S}(h)(\psi(t-h) - \psi(t)) + \mathcal{S}(h)\psi(t) - u(t)}{h} \\ &= \mathcal{S}(h) \left(\frac{\psi(t-h) - \psi(t)}{h} \right) + \frac{\mathcal{S}(h)\psi(t) - \psi(t)}{h} + \frac{\psi(t) - u(t)}{h}. \end{aligned}$$

Let $\varepsilon > 0$. Then, by Lemma 5.10 and Lemma 5.3, there exists $0 < h_0 < t$ such that for all $0 < h < h_0$ one has

$$\frac{\mathcal{S}(h)\psi(t) - \psi(t)}{h} \leq \mathcal{A}\psi(t) + \frac{\varepsilon}{3}, \quad \frac{\psi(t-h) - \psi(t)}{h} \leq -\psi_t(t) + \frac{\varepsilon}{3},$$

and

$$\mathcal{S}(h)(-\psi_t(t)) \leq -\psi_t(t) + \frac{\varepsilon}{3}.$$

Hence, we get

$$\begin{aligned} 0 &\leq \mathcal{S}(h)(-\psi_t(t)) + \mathcal{A}\psi(t) + \frac{2\varepsilon}{3} + \frac{\psi(t) - u(t)}{h} \\ &\leq -\psi_t(t) + \mathcal{A}\psi(t) + \varepsilon + \frac{\psi(t) - u(t)}{h} \end{aligned}$$

for all $0 < h < h_0$. Since $(\psi(t))(x) = (u(t))(x)$ we obtain that

$$0 \leq -(\psi_t(t))(x) + (\mathcal{A}\psi(t))(x) + \varepsilon.$$

Letting $\varepsilon \searrow 0$ yields $(\psi_t(t))(x) \leq (\mathcal{A}\psi(t))(x)$.

To show that u is a D -viscosity supersolution, let $\psi: (0, \infty) \rightarrow \text{BUC}(G)$ differentiable with $(\psi(t))(x) = (u(t))(x)$, $\psi(s) \in D$ and $\psi(s) \leq u(s)$ for all $s > 0$. By Theorem 5.6, for all $h > 0$ with $0 < h < t$ we get

$$\begin{aligned} 0 &= \frac{\mathcal{S}(t)f - \mathcal{S}(h)\mathcal{S}(t-h)f}{h} = \frac{u(t) - \mathcal{S}(h)u(t-h)}{h} \leq \frac{u(t) - \mathcal{S}(h)\psi(t-h)}{h} \\ &= \frac{u(t) - \psi(t)}{h} + \frac{\psi(t) - \mathcal{S}(h)\psi(t)}{h} + \frac{\mathcal{S}(h)\psi(t) - \mathcal{S}(h)\psi(t-h)}{h} \\ &\leq \frac{u(t) - \psi(t)}{h} + \frac{\psi(t) - \mathcal{S}(h)\psi(t)}{h} + \mathcal{S}(h) \left(\frac{\psi(t) - \psi(t-h)}{h} \right) \end{aligned}$$

Let $\varepsilon > 0$. Then, by Lemma 5.10 and Lemma 5.3 there exists $0 < h_0 < t$ such that

$$\begin{aligned} \frac{\psi(t) - \mathcal{S}(h)\psi(t)}{h} &\leq -\mathcal{A}\psi(t) + \frac{\varepsilon}{3}, \\ \frac{\psi(t) - \psi(t-h)}{h} &\leq \psi_t(t) + \frac{\varepsilon}{3}, \end{aligned}$$

and

$$\mathcal{S}(h)(\psi_t(t)) \leq \psi_t(t) + \frac{\varepsilon}{3}$$

for all $0 < h < h_0$. We thus get

$$0 \leq \frac{u(t) - \psi(t)}{h} - \mathcal{A}\psi(t) + \mathcal{S}(h)\psi_t(t) + \frac{2\varepsilon}{3} \leq \frac{u(t) - \psi(t)}{h} - \mathcal{A}\psi(t) + \psi_t(t) + \varepsilon$$

for all $0 < h < h_0$. Since $(\psi(t))(x) = (u(t))(x)$ we obtain that

$$0 \leq -(\mathcal{A}\psi(t))(x) + (\psi_t(t))(x) + \varepsilon.$$

Letting $\varepsilon \searrow 0$ yields $(\psi_t(t))(x) \geq (\mathcal{A}\psi(t))(x)$. \square

In order to complete the proof of Theorem 2.5, it remains to show that there exists a set \mathcal{P} of probability measures and a stochastic process $(X_t)_{t \geq 0}$ on measurable space (Ω, \mathcal{F}) such that the viscosity solution in Theorem 5.11 is of the form

$$u(t, x) = \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}(f(x + X_t))$$

for $t \geq 0$ and $x \in G$. This is shown in the following proposition for which the assumption (A2) is not needed.

Proposition 5.12. *There exists a set \mathcal{P} of probability measures on a measurable space (Ω, \mathcal{F}) and an \mathcal{E} -Lévy process $(X_t)_{t \geq 0}$ such that*

$$(\mathcal{S}(t)f)(x) = \mathcal{E}(f(x + X_t)) = \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}(f(x + X_t))$$

for all $f \in \text{BUC}(G)$, $t \geq 0$ and $x \in G$, where

$$\mathcal{E}(Y) = \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}(Y), \quad Y \in \mathcal{L}^\infty(\Omega, \mathcal{F}).$$

Proof. Let $h_1, h_2 > 0$. Then,

$$(J_{h_1} J_{h_2} f)(x) = \sup_{\eta \in \mathcal{Q}} \int_G f(x + \cdot) d\eta,$$

where \mathcal{Q} is the set of probability measures of the form

$$\eta: \mathcal{B} \rightarrow [0, 1], \quad B \mapsto \left[S_{\lambda_0}(h_1) \left(\sum_{k=1}^n 1_{B_k} S_{\lambda_k}(h_2) 1_B \right) \right](0)$$

with $k \in \mathbb{N}$, $\lambda_0, \lambda_1, \dots, \lambda_n \in \Lambda$ and $B_1, \dots, B_n \in \mathcal{B}$. Here, we identify $S_\lambda(h)$ for $\lambda \in \Lambda$ and $h > 0$ with the unique translation invariant kernel associated to it. Inductively, we therefore obtain that J_π admits a dual representation in terms of distributions of stochastic integrals with a finite space-time partition for all $\pi \in P$. As each one of these distributions can be extended to the path space by Kolmogorov's extension theorem, there exists a set $\mathcal{P} \subset \text{ca}_+^1(G^{[0,\infty)}, \mathcal{B}^{[0,\infty)})$ such that

$$(\mathcal{S}(t)f)(0) = \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}(f \circ X_t)$$

for all $t \geq 0$ and $f \in \text{BUC}(G)$, where $(X_t)_{t \geq 0}$ is the canonical process. Then, $(X_t)_{t \geq 0}$ satisfies all properties in the definition of an \mathcal{E} -Lévy process. Note that (iii) and (iv) follow immediately from the dual representation of the constructed expectation $\mathcal{E}(\cdot) = \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}(\cdot)$ and (v) follows as in the proof of Theorem 2.3. \square

We finish with the proof of Proposition 2.8.

Proof. Fix $t > 0$ and $\varepsilon > 0$. Let $\varphi \in \bigcap_{\lambda \in \Lambda} (D(A_\lambda) \cap C_0(G))$ with $0 \leq \varphi \leq 1$, $\varphi(0) = 1$ and $\sup_{\lambda \in \Lambda} \|A_\lambda \varphi\|_\infty \leq \frac{\varepsilon}{2t}$. Since $\varphi(0) = 1$ and $1 - \varphi \in \bigcap_{\lambda \in \Lambda} D(A_\lambda)$ with $A_\lambda(1 - \varphi) = -A_\lambda \varphi$, it follows from Lemma 5.3 that

$$\begin{aligned} |(\mathcal{S}(t)(1 - \varphi))(0)| &\leq \|\mathcal{S}(t)(1 - \varphi) - (1 - \varphi)\|_\infty \\ &\leq t \sup_{\lambda \in \Lambda} \|A_\lambda \varphi\|_\infty \leq \frac{\varepsilon}{2}. \end{aligned}$$

Note that the estimate in Lemma 5.3 holds for all functions in $\bigcap_{\lambda \in \Lambda} D(A_\lambda)$. Since $\varphi \in C_0(G)$ there exists a compact set $K \subset G$ such that

$$1_{G \setminus K}(x) \leq 1 - \varphi(x) + \frac{\varepsilon}{2} \quad \text{for all } x \in G.$$

Hence,

$$\mathcal{E}(1_{G \setminus K}(X_t)) \leq \mathcal{E}(1 - \varphi(X_t)) + \frac{\varepsilon}{2} = |(\mathcal{S}(t)(1 - \varphi))(0)| + \frac{\varepsilon}{2} \leq \varepsilon. \quad \square$$

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