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Toric completions and bounded functions on real algebraic varieties

Daniel Plaumann and Claus Scheiderer

Abstract

Given a semi-algebraic set S, we study compactifications of S that arise from embeddings into complete toric varieties. This makes it possible to describe the asymptotic growth of polynomial functions on S in terms of combinatorial data. We extend our earlier work in Plaumann and Scheiderer ['The ring of bounded polynomials on a semi-algebraic set', *Trans. Amer. Math. Soc.* 364 (2012) 4663–4682] to compute the ring of bounded functions in this setting, and discuss applications to positive polynomials and the moment problem. Complete results are obtained in special cases, like sets defined by binomial inequalities. We also show that the wild behaviour of certain examples constructed by Krug ['Geometric interpretations of a counterexample to Hilbert's 14th problem, and rings of bounded polynomials on semialgebraic sets', Preprint, 2011, arXiv:1105.2029] and Mondal-Netzer ['How fast do polynomials grow on semialgebraic sets?', *J. Algebra* 413 (2014) 320–344] cannot occur in a toric setting.

Introduction

The simplest measure for the asymptotic growth of a real polynomial in n variables on $\mathbb{R}^n$ is its total degree. However, when we pass from $\mathbb{R}^n$ to an unbounded semi-algebraic subset $S \subseteq \mathbb{R}^n$, the total degree of a polynomial may not reflect the growth of the restriction $f|_S$ any more.

The degree of a polynomial can be understood as its pole order along the hyperplane at infinity when $\mathbb{R}^n$ is embedded into projective space in the usual way. How this relates to the growth of $f|_S$ depends on how the closure of S in $\mathbb{P}^n(\mathbb{R})$ meets the hyperplane at infinity. Unless this intersection is empty (which would mean that S is bounded in $\mathbb{R}^n$) or of maximal dimension, the total degree alone will usually not suffice to understand the growth of polynomials on S. We may however hope to improve control of the growth by suitable blow-ups at infinity.

To make this idea more precise, we consider the following setup. Suppose that V is an affine real variety and S is the closure of an open semi-algebraic subset of $V(\mathbb{R})$. An open dense embedding of V into a complete variety X is called *compatible with* S if the geometry of S at infinity is regular in the following sense: if Z is any hypersurface at infinity, that is, any irreducible component of the complement of V in X, the closure $\overline{S}$ of S in $X(\mathbb{R})$ meets Z either in a Zariski-dense subset of Z or not at all. Under this condition, the pole orders of a regular function f on V along the hypersurfaces at infinity intersecting $\overline{S}$ accurately reflects the qualitative growth of f on S.

Compatible completions were introduced by the authors in [11], as well as in the dissertation of the first author, motivated by earlier work of Powers and Scheiderer [12]. An S-compatible completion of V yields in particular a description of

$$B_V(S) = \{f \in \mathbb{R}[V] : \exists \lambda \in \mathbb{R}\ |f| \leqslant \lambda \text{ on S}\},$$

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the ring of regular functions on V that are bounded on S. If $V \hookrightarrow X$ is such an S-compatible completion and Y is the union of those irreducible components of $X \setminus V$ that are disjoint from $\overline{S}$, then $B_V(S)$ is naturally identified with $\mathscr{O}_X(X \setminus Y)$, the ring of regular functions of the variety $X \setminus Y$.

The main goal of this paper is to improve on the results in [**11**] and make them more explicit in the controlled setting of toric varieties. Specifically, we study the following questions:

(1) One of the main results of [**11**] is the existence of regular completions in the case $\dim V \leqslant 2$. The higher-dimensional case remains open and hinges on the existence of a certain type of embedded resolution of singularities. In the toric setting, we introduce a stronger, purely combinatorial compatibility condition in the spirit of toric geometry (Section 2). We show that this condition can be satisfied if S is defined by binomial inequalities (Corollary 2.16) or if S is what we call a tentacle (Corollary 2.20), generalizing a concept introduced by Netzer [**10**].

Since the compatible completions in dimension 2 constructed in [**11**] are built from an embedded resolution of singularities, they are typically quite hard to compute explicitly. In contrast, our results in the toric setting only require the usual arithmetic of semigroups derived from rational polyhedral cones.

(2) The transcendence degree of the ring of regular functions $\mathscr{O}(X \setminus Y)$ of the complement of a divisor Y in a complete variety X is called the *Iitaka dimension* of Y. It is a natural generalization of the Kodaira dimension studied extensively in complex algebraic geometry. Thus, in the case of a compatible completion, when $B_V(S)$ is identified with $\mathscr{O}(X \setminus Y)$, the Iitaka dimension measures in how many independent directions the set S is bounded.

In dimension 2, the Iitaka dimension is strongly related to the signature of the intersection matrix A_Y of the divisor Y. However, the correspondence is not perfect if A_Y is singular. Specifically, if A_Y is negative semidefinite, but not definite, Iitaka's criterion (Proposition 3.4) does not give anything. In the toric setting, on the other hand, we show that the signature of A_Y is sufficient to determine the Iitaka dimension (Proposition 3.13). It seems plausible that this has been observed before, but we were unable to find any trace in the literature. We exploit the result in an application to positive polynomials explained below.

(3) The existence of an S-compatible completion X of V yields a good description of the ring of bounded functions $B_V(S)$. However, it does not imply that $B_V(S)$ is a finitely generated $\mathbb{R}$-algebra. This was discussed in [**11**] and much further explored by Krug [**5**]. When a toric S-compatible completion exists, $B_V(S)$ is always finitely generated (Proposition 2.10).

Beyond bounded polynomials, an S-compatible completion also provides control over the asymptotic growth of arbitrary polynomials, as indicated in the beginning. Let Y' be the union of all irreducible components of $X \setminus V$ that intersect the closure of S in X. In Section 4, we study the linear subspaces

$$L_{X,m}(S) = \{f \in \mathbb{R}[V] : \text{all poles of } f \text{ along } Y' \text{ have order at most } m\},$$

which consist of functions of bounded growth on S. Assume that $B_V(S) = \mathbb{R}$. In analogy with the case $S = \mathbb{R}^n$, one might expect that the spaces $L_{X,m}(S)$ ($m \in \mathbb{N}$) are finite-dimensional. If so, the filtration $L_{X,0}(S) \subseteq L_{X,1}(S) \subseteq L_{X,2}(S) \subseteq \cdots$ of $\mathbb{R}[V]$ behaves much like the filtration of the polynomial ring by total degree. The properties of filtrations obtained in this way and further generalizations have also been studied in complex algebraic geometry (see [**8**]). For us, this question is particularly relevant in the context of positive polynomials and the moment problem, as it concerns possible degree cancellations in sums of positive polynomials, as explained in Section 5. However, a subtle example due to Mondal and Netzer [**9**] (see Example 4.4) implies that the $L_{X,m}(S)$ may have infinite dimension. This construction is complemented by Theorem 5.5, which combines with the results of [**13**] to show that if S is basic open of dimension at least 2 and admits an S-compatible toric completion, but no non-constant bounded function, then the spaces $L_{X,m}(S)$ are finite-dimensional and, consequently, the moment problem for S is not finitely solvable. This comprises the results of Netzer [**10**] for tentacles and of Powers and Scheiderer [12]. It is also related to a theorem of Vinzant [14], which constructs a certain kind of toric compatible completion under an algebraic assumption on the description of S and the ideal of V, as explained in the last section of [14].

1. *Compatible completions of semi-algebraic sets*

We briefly summarize some of the definitions and results in [11].

Definition 1.1. Let V be a normal affine $\mathbb{R}$-variety and let S be a semi-algebraic subset of $V(\mathbb{R})$. By a *completion* of V, we mean an open dense embedding $V \hookrightarrow X$ into a normal complete $\mathbb{R}$-variety. The completion X is said to be *compatible with* S (or S*-compatible*) if, for every irreducible component Z of $X \smallsetminus V$, the following condition holds: The set $Z(\mathbb{R}) \cap \overline{S}$ is either empty or Zariski-dense in Z.

Here, when taking the closure $\overline{S}$ of a semi-algebraic subset S of $X(\mathbb{R})$, we refer to the Euclidean topology on $X(\mathbb{R})$, rather than the Zariski topology. Note that every irreducible component of $X \smallsetminus V$ is a divisor on X, that is, has codimension 1 [3, p. 66].

Theorem 1.2 ([11, Theorem 3.8]). *Let V be a normal affine $\mathbb{R}$-variety, let $S \subseteq V(\mathbb{R})$ be a semi-algebraic subset and assume that the completion $V \hookrightarrow X$ of V is compatible with S. Let Y denote the union of those irreducible components Z of $X \smallsetminus V$ for which $\overline{S} \cap Z(\mathbb{R}) = \varnothing$, and put $U = X \smallsetminus Y$. Then the inclusion $V \subseteq U$ induces an isomorphism of $\mathbb{R}$-algebras*

$$\mathscr{O}_X(U) \cong B_V(S).$$

A semi-algebraic set is called *regular* if its closure coincides with the closure of its interior. It is called *regular at infinity* if it is the union of a regular and a relatively compact semi-algebraic set. One of the main results of [11] is the existence of compatible completions for 2-dimensional semi-algebraic sets regular at infinity.

Theorem 1.3 ([11, Theorem 4.5]). *Let V be a normal quasi-projective surface over $\mathbb{R}$, and let S be a semi-algebraic subset of $V(\mathbb{R})$ that is regular at infinity. Then V has an S-compatible projective completion. If V is non-singular, then the completion can be chosen to be non-singular as well.*

1.4. The proof of Theorem 1.3 is essentially constructive and relies on embedded resolution of singularities. We summarise the procedure for our present purposes. Let $V \hookrightarrow X$ be any open dense embedding of V into a normal projective surface. Let C_∂ be the Zariski closure of the boundary of S in $X(\mathbb{R})$ and let $C_\infty = X \smallsetminus V$. Put $C = C_\partial \cup C_\infty$, a reduced curve in X. We write

$$\partial_X^\infty S = \overline{S} \cap C_\infty(\mathbb{R})$$

for the set of *boundary points of S at infinity* in X. A sufficient condition for X to be an S-compatible completion of V is that C has *only normal crossings* in $\partial_X^\infty S$. Explicitly, this means the following. If $P \in \partial_X^\infty S$, then

(1) P is a non-singular point of all irreducible components of C that contain it.
(2) P is contained in exactly one component C_0 of C_∂ and one component C_1 of C_∞, and $C_0(\mathbb{R})$ and $C_1(\mathbb{R})$ have independent tangents in P. (Equivalently, the local equations for C_0 and C_1 generate the maximal ideal of the local ring $\mathscr{O}_{X,P}$.)

By blowing-up any points in $\partial_X^\infty S$ in which conditions (1) or (2) are violated and proceeding inductively, we can produce a completion $\widetilde{X}$ and a corresponding curve $\widetilde{C} = \widetilde{C}_\partial \cup \widetilde{C}_\infty$, defined as before, such that all points of $\partial^{\infty}_{\widetilde{X}} S$ are normal crossings of $\widetilde{C}$, which is therefore an S-compatible completion of V. Note that blowing-up increases the number of irreducible components in C_∞, since the exceptional divisor is added. In the resulting S-compatible completion X, the divisor Y of Theorem 1.2 consists of those irreducible components of C_∞ that are disjoint from $\overline{S}$.

Explicit computation of the ring of bounded polynomials following the above procedure is possible, but can quickly turn into a cumbersome task. We give the following simple example as an illustration. A much more interesting example will be discussed in Section 4.

EXAMPLE 1.5. Let $S = \{(x, y) \in \mathbb{R}^2 : -1 \leqslant x \leqslant 1\}$ be a strip in the affine plane $V = \mathbb{A}^2_{\mathbb{R}}$ and consider the embedding $V \hookrightarrow \mathbb{P}^2_{\mathbb{R}}$ into the projective plane given by $(u, v) \mapsto (u : v : 1)$. Then $C_\infty = \mathbb{P}^2 \smallsetminus V$ is the line at infinity and C_∂ is the Zariski closure of the two lines $\mathscr{V}(x-1)$ and $\mathscr{V}(x+1)$ in V. The set $\partial S^{\infty}_{\mathbb{P}^2}$ is the point $P = (0 : 1 : 0)$, which is also the unique intersection point of C_∂ and C_∞. In local coordinates $r = x/y$ and $s = 1/y$ of $\mathbb{P}^2_{\mathbb{R}}$ centred around P, we have $C_\infty = \mathscr{V}(s)$ and $C_\partial = \mathscr{V}((r-s)(r+s))$. Since all three components of $C = C_\infty \cup C_\partial$ pass through P, C does not have normal crossings in P. Indeed, the completion of S is not S-compatible, since $\overline{S} \cap C_\infty(\mathbb{R}) = \{P\}$ is not Zariski-dense in C_∞.

Let $\widetilde{X}$ be the blow-up of $\mathbb{P}^2_{\mathbb{R}}$ in P. It is given in local coordinates by the quadratic transformation $r = r_1,\ s = r_1 s_1$. In the new coordinates r_1, s_1, the exceptional divisor is $E = \mathscr{V}(r_1)$. The strict transforms of the components of C in X are $C_\infty = \mathscr{V}(s_1)$ and $C_\partial = \mathscr{V}((s_1-1)(s_1+1))$. Now $\widetilde{C}_\infty = \widetilde{X} \smallsetminus V$ has the two components C_∞ and E. Since C_∂ meets E in the points $(0, 1)$ and $(0, -1)$, but does not meet C_∞, we see that $\widetilde{X}$ is an S-compatible completion of V and $Y = C_\infty$ is the component of $\widetilde{C}_\infty$ that is disjoint from $\overline{S}_{\widetilde{X}(\mathbb{R})}$. To compute $\mathscr{O}(\widetilde{X} \smallsetminus Y)$, write $f \in \mathbb{R}[x, y]$ as $f = \sum_{i,j} a_{ij} x^i y^j = \sum_{i,j} a_{ij} r_1^{-j} s_1^{-i-j}$, so that f lies in $\mathscr{O}(\widetilde{X} \smallsetminus Y)$ if and only if $j = 0$. Thus $B(S) = \mathscr{O}(\widetilde{X} \smallsetminus Y) = \mathbb{R}[x]$.

In dimensions $\geqslant 3$, it is not even guaranteed that the ring $B(S)$ is finitely generated (see [**11**, Section 5]).

2. *Toric completions*

Let V be an affine toric variety. By a *toric completion* of V, we mean an open embedding of V into a complete toric variety X which is compatible with the torus actions. Let $S \subseteq V(\mathbb{R})$ be a semi-algebraic subset. We are going to work out conditions on S ensuring that V has a toric completion $V \subseteq X$ that is compatible with S. The existence of such a completion allows us to make the ring of bounded polynomial functions on S completely explicit. It also prevents several pathologies that can occur in more general cases.

We start by reviewing some general notions on toric varieties. An excellent reference is the book of Cox *et al.* [**2**].

2.1. Let T be an n-dimensional split $\mathbb{R}$-torus and let $T(\mathbb{R}) \cong (\mathbb{R}^*)^n$ be the group of $\mathbb{R}$-points. All toric varieties will be T-varieties. Let $M = \mathit{Hom}(T, \mathbb{G}_m)$ (respectively, $N = \mathit{Hom}(\mathbb{G}_m, T)$), the group of characters (respectively, of co-characters) of T. Both are free abelian groups of rank n, each being the natural dual of the other. We write both groups additively and denote the character corresponding to $\alpha \in M$ by $\mathtt{x}^\alpha$, the co-character corresponding to $v \in N$ by λ_v. The pairing between M and N will be denoted by $\langle \alpha, v \rangle$.

2.2. Let $M_{\mathbb{R}} = M \otimes \mathbb{R}$, $N_{\mathbb{R}} = N \otimes \mathbb{R}$. By a cone $\sigma \subseteq N_{\mathbb{R}}$ we always mean a finitely generated rational convex cone. Let $\sigma^* \subseteq M_{\mathbb{R}}$ denote the dual cone of σ, let $H_\sigma = M \cap \sigma^*$, and write $\mathbb{R}[H_\sigma]$ for the semigroup algebra of H_σ. Then $U_\sigma = Spec\mathbb{R}[H_\sigma]$ is an affine toric variety that contains a unique closed T-orbit, denoted O_σ.

Assume that the cone $\sigma \subseteq N_\mathbb{R}$ is pointed. Then the dense T-orbit U_0 in U_σ is isomorphic to T, and we may use any fixed $\xi_0 \in U_0(\mathbb{R})$ to equivariantly identify U_0 with T. Let $v \in N \cap \mathrm{relint}(\sigma)$. For any $\xi \in U_0(\mathbb{R})$, the limit

$$L_v(\xi) := \lim_{s\to 0}(\lambda_v(s)\cdot\xi)$$

exists in $U_\sigma(\mathbb{R})$ and lies in $O_\sigma(\mathbb{R})$. Clearly, the map $L_v : U_0(\mathbb{R}) \to O_\sigma(\mathbb{R})$ is equivariant under the $T(\mathbb{R})$-action. In particular, L_v is an open map.

2.3. Fixing $v \in N$, we consider the v-grading of $\mathbb{R}[T]$, which is the grading that makes the character $\mathtt{x}^\alpha$ homogeneous of degree $\langle \alpha, v\rangle$ for every $\alpha \in M$. We say that $f \in \mathbb{R}[T]$ is v-homogeneous if f is homogeneous in the v-grading. For $0 \neq f \in \mathbb{R}[T]$, let $\mathrm{in}_v(f) \in \mathbb{R}[T]$ denote the leading component of f in the v-grading, that is, the non-zero v-homogeneous component of f of *smallest* v-degree. Two vectors v, $v' \in N$ satisfy $\mathrm{in}_v(f) = \mathrm{in}_{v'}(f)$ if and only if v and v' lie in the relative interior of the same cone of the normal fan of the Newton polytope of f. (Note that since we define the leading form $\mathrm{in}_v(f)$ to be the homogeneous component of smallest v-degree, we are using inward, rather than outward, normal cones here.)

2.4. A fan is a finite non-empty set Σ of closed pointed rational cones in $N_\mathbb{R}$, which is closed under taking faces and such that the intersection of any two elements of Σ is a face of both. The union of all cones in Σ is called the support of Σ, denoted by $|\Sigma|$; if $|\Sigma| = N_\mathbb{R}$, then Σ is called complete. The fan Σ gives rise to a toric variety X_Σ, obtained by glueing the affine toric varieties U_σ, $\sigma \in \Sigma$. The variety X_Σ is complete if and only if the fan Σ is complete. In general, the ring of global regular functions $\mathscr{O}(X_\Sigma)$ is the semigroup algebra $\mathbb{R}[H]$, where $H = M \cap |\Sigma|^*$. By Dickson's lemma, this is a finitely generated $\mathbb{R}$-algebra.

Let U_σ be an affine toric variety and let $S \subseteq U_\sigma(\mathbb{R})$ be a semi-algebraic set. We are going to study conditions under which there exists a toric completion of U_σ that is compatible with S, and which therefore allows the explicit computation of the ring $B_{U_\sigma}(S)$ of polynomials bounded on S. We first propose an abstract framework; see Proposition 2.10. After this, we will exhibit concrete situations to which the abstract framework applies.

We will always assume that the semi-algebraic set S is open and contained in the dense torus orbit in U_σ.

2.5. Let $S \subseteq T(\mathbb{R})$ be an open semi-algebraic subset. Given $v \in N$, put

$$S(v) := \{\xi \in T(\mathbb{R}) : \forall\, 0 < s \ll 1\ \lambda_v(s)\xi \in S\}.$$

It is easily seen that $(S_1 \cup S_2)(v) = S_1(v) \cup S_2(v)$ and $(S_1 \cap S_2)(v) = S_1(v) \cap S_2(v)$ hold for all $v \in N$ and all open semi-algebraic sets $S_1,\ S_2 \subseteq T(\mathbb{R})$. Further let

$$K(S) := \{v \in N : S(v) \neq \varnothing\}, \quad K_0(S) := \{v \in N : \mathrm{int}(S(v)) \neq \varnothing\}.$$

Then $K(S_1 \cup S_2) = K(S_1) \cup K(S_2)$ and $K_0(S_1 \cup S_2) = K_0(S_1) \cup K_0(S_2)$ hold.

LEMMA 2.6. *Given any open semi-algebraic set $S \subseteq T(\mathbb{R})$, there exists a fan Σ in $N_\mathbb{R}$ such that*

$$K(S) = N \cap \bigcup_{\sigma\in E} \mathrm{relint}(\sigma), \quad K_0(S) = N \cap \bigcup_{\sigma\in E_0} \mathrm{relint}(\sigma)$$

hold for suitable subsets E_0, E of Σ. Any such fan Σ is said to be adapted to S.

Proof. We may assume that $S = \{\xi \in T(\mathbb{R}) : f_i(\xi) > 0 \ (i = 1, \ldots, r)\}$ is basic open, with $f_1, \ldots, f_r \in \mathbb{R}[T]$. Given $f \in \mathbb{R}[T]$ and $v \in N$, let $f_{v,d} \in \mathbb{R}[T]$ be the v-homogeneous component of f of degree d. Thus

$$f(\lambda_v(s)\xi) = \sum_{d\in\mathbb{Z}} f_{v,d}(\xi) \cdot s^d$$

for $s \in \mathbb{R}$ and $\xi \in T(\mathbb{R})$. So $\xi \in S(v)$ holds if and only if, for every $i = 1, \ldots, r$, there exists $d_i \in \mathbb{Z}$ with $(f_i)_{v,d_i}(\xi) > 0$ and with $(f_i)_{v,d'}(\xi) = 0$ for all $d' < d_i$. Let $\Lambda(f, v)$ denote the sequence of non-zero v-homogeneous components of f, ordered by increasing degree. Then, if v, $v' \in N$ satisfy $\Lambda(f_i, v) = \Lambda(f_i, v')$ for $i = 1, \ldots, r$, it follows that $S(v) = S(v')$. It is clear that there is a fan Σ such that any two vectors v, v' in the relative interior of the same cone of Σ satisfy this condition. Such Σ satisfies the condition of the lemma. □

REMARK 2.7. If S is a subset of the positive orthant in $\mathbb{R}^n$, the set $K(S)$ is closely related to the tropicalization of S constructed by Alessandrini [1].

LEMMA 2.8. *Let $S \subseteq T(\mathbb{R})$ be an open semi-algebraic set and let $\rho \subseteq N_\mathbb{R}$ be a pointed cone:*

(a) *If $K(S) \cap \operatorname{relint}(\rho) \neq \varnothing$, then $\overline{S} \cap O_\rho(\mathbb{R}) \neq \varnothing$.*
(b) *If $K_0(S) \cap \operatorname{relint}(\rho) \neq \varnothing$, then $\overline{S} \cap O_\rho(\mathbb{R})$ is Zariski-dense in O_ρ.*

Here we fix an equivariant identification $T = U_0$. The closures are taken inside the affine toric variety U_ρ and with respect to the Euclidean topology. Recall that U_ρ contains $U_0 = T$ (respectively, O_ρ) as an open dense (respectively, as a closed) T-orbit.

Proof. Given $v \in N \cap \operatorname{relint}(\rho)$, the map $L_v : T(\mathbb{R}) \to O_\rho(\mathbb{R})$ (see 2.2) is open and maps $S(v)$ into $\overline{S} \cap O_\rho(\mathbb{R})$. The hypothesis $v \in K(S)$ (respectively, $v \in K_0(S)$) means $S(v) \neq \varnothing$ (respectively, $\operatorname{int}(S(v)) \neq \varnothing$). This proves the lemma. □

2.9. Now let Σ be a complete fan in $N_\mathbb{R}$ and let X_Σ be the associated complete toric variety. We write $\Sigma(d)$ for the set of d-dimensional cones in Σ. For $\tau \in \Sigma$ let Y_τ be the Zariski closure of O_τ in X_Σ. In particular, Y_τ is a prime Weil divisor on X_Σ when $\tau \in \Sigma(1)$. We fix a cone $\sigma \in \Sigma$ and consider X_Σ as a toric completion of the affine toric variety U_σ.

Let $S \subseteq T(\mathbb{R}) = U_0(\mathbb{R})$ be an open semi-algebraic set. We will require the following *toric compatibility* assumption:

(TC) For any $\tau \in \Sigma(1)$ with $\tau \nsubseteq \sigma$, either $\overline{S} \cap Y_\tau(\mathbb{R})$ is empty or $K_0(S) \cap \operatorname{relint}(\tau)$ is non-empty.

(The two cases are mutually exclusive by Lemma 2.8.) We define the subfan F_S of Σ by

$$F_S := \left\{ \rho \in \Sigma : \begin{array}{l} \text{Every } 1-\textit{dimensional} \text{ face } \tau \text{ of } \rho \text{ satisfies} \\ \tau \subseteq \sigma \text{ or } K_0(S) \cap \operatorname{relint}(\tau) \neq \varnothing \end{array} \right\}.$$

PROPOSITION 2.10. *With the above notation, assume that the toric compatibility condition (TC) holds. Then the toric variety X_Σ is an S-compatible completion of U_σ. In particular, let $B_{U_\sigma}(S)$ be the subring of $\mathbb{R}[U_\sigma]$ consisting of the regular functions that are bounded on S. Then*

$$B_{U_\sigma}(S) = \mathscr{O}(X_{F_S}) = \mathbb{R}[H]$$

with $H = M \cap |F_S|^*$. In particular, the $\mathbb{R}$-algebra $B_{U_\sigma}(S)$ is finitely generated.

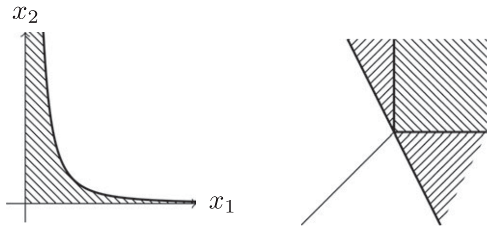

FIGURE 1. *The semi-algebraic set from Example 2.11(2) and its corresponding fan.*

Proof. Write $X = X_\Sigma$, a normal and complete toric variety containing U_σ as an open affine toric subvariety. The irreducible components of $X \smallsetminus U_\sigma$ are the Y_τ, where $\tau \in \Sigma(1)$ and $\tau \nsubseteq \sigma$. Given such τ with $\overline{S} \cap Y_\tau(\mathbb{R}) \neq \varnothing$, we know that $\overline{S} \cap Y_\tau(\mathbb{R})$ is Zariski-dense in Y_τ, by condition (TC) and Lemma 2.8(b). So the completion X of U_σ is compatible with the semi-algebraic set $S \subseteq U_\sigma(\mathbb{R})$, in the sense of Definition 1.1. By Theorem 1.2, we therefore have $B_{U_\sigma}(S) = \mathscr{O}(X \smallsetminus Y)$, where Y is the union of those irreducible components Y_τ of $X \smallsetminus U_\sigma$ for which $\overline{S} \cap Y_\tau(\mathbb{R}) = \varnothing$. By condition (TC), the latter means $\tau \in \Sigma(1)$, $\tau \nsubseteq \sigma$ and $\operatorname{relint}(\tau) \cap K_0(S) = \varnothing$. So $X \smallsetminus Y$ is precisely the toric variety associated to the subfan F_S of Σ defined above. $\square$

EXAMPLE 2.11. Let $n = 2$. We compatibly identify $M = \mathbb{Z}^2$, $N = \mathbb{Z}^2$ and $T(\mathbb{R}) = (\mathbb{R}^*)^2$. We denote by (e_1, e_2) the standard basis of $N_\mathbb{R} = \mathbb{R}^2$ and by (e_1^*, e_2^*) the dual basis of $M_\mathbb{R}$. Let $\sigma = \operatorname{cone}(e_1, e_2)$ be the positive quadrant in $N_\mathbb{R}$, so that $\mathbb{R}[U_\sigma] = \mathbb{R}[x_1, x_2]$ and $U_\sigma = \mathbb{A}^2$. Let Σ_0 be the standard fan of $\mathbb{P}^2$ with ray generators e_1, e_2, and $-(e_1 + e_2)$. We use homogeneous coordinates $(u_0 : u_1 : u_2)$ on $\mathbb{P}^2$ with $x_i = \frac{u_i}{u_0}$ $(i = 1, 2)$.

(1) Consider the set

$$S := \{(\xi_1, \xi_2) \in (\mathbb{R}^*)^2 : -1 < \xi_1 < 1\}.$$

It is easily seen that $K(S) = K_0(S) = \{(v_1, v_2) \in N : v_1 \geqslant 0\}$. Let Σ be the refinement of Σ_0 generated by the additional ray generator $-e_2$. Then Σ is adapted to S; cf. Lemma 2.6. The toric variety X_Σ is the blow-up of $\mathbb{P}^2$ in the point (0:0:1), which is exactly the compatible completion of the strip $\overline{S} \subseteq \mathbb{R}^2$ we considered in Example 1.5. By definition, $|F_S| = \{(v_1, v_2) \in N_\mathbb{R} : v_1 \geqslant 0\}$, so that $M \cap |F_S|^* = \{(k, 0) \in M : k \geqslant 0\}$, whence $\mathscr{O}(X_{F_S}) = \mathbb{R}[x_1]$. It is not hard to check that condition (TC) is met in this example, so that Proposition 2.10 yields $B_{\mathbb{A}^2}(S) = \mathbb{R}[x_1]$.

(2) Let $k \geqslant 1$ and let

$$S := \{(\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_1^k \xi_2 < 1,\ \xi_1 > 0,\ \xi_2 > 0\}$$

(see Figure 1 for $k = 2$). Here we find that $K(S) = K_0(S)$ is the half-space $kv_1 + v_2 \geqslant 0$ in N. We again refine Σ_0 by adding ray generators $\pm(e_1 - ke_2)$ to Σ_0, and obtain the fan Σ shown on the right of Figure 1.

By construction, Σ is adapted to S, and $|F_S| = \{v \in N_\mathbb{R} : kv_1 + v_2 \geqslant 0\}$. We check that condition (TC) is satisfied. This amounts to showing, for $\tau = \operatorname{cone}(-e_1 - e_2)$, that $Y_\tau(\mathbb{R}) \cap \overline{S} = \varnothing$. Indeed, let $\rho = \operatorname{cone}(-e_1 - e_2, -e_1 + ke_2)$. Then ρ^* is generated by $e_2^* - e_1^*$ and $-(ke_1^* + e_2^*)$, so that $\mathbb{R}[U_\rho] = \mathbb{R}[H]$, where H is the saturated semigroup generated by $y_1 = x_1^{-k} x_2^{-1}$, $y_2 = x_1^{-1}$, and $y_3 = x_1^{-1} x_2$, so that $\mathbb{R}[U_\rho] \cong \mathbb{R}[y_1, y_2, y_3]/(y_1 y_3 - y_2^{k+1})$. Under this identification, we find $y_1 = 0$ on $Y_\tau \cap U_\rho$ while $y_1 > 1$ on $S \cap U_\rho(\mathbb{R})$. So $\overline{S} \cap (U_\rho \cap Y_\tau)(\mathbb{R}) = \varnothing$. Essentially the same computation applies to $\rho' = \operatorname{cone}(-e_1 - e_2, e_1 - ke_2)$. Hence we conclude $B_{\mathbb{A}^2}(S) = \mathscr{O}(X_{F_S}) = \mathbb{R}[x_1^k x_2]$. This will be discussed in general below (cf. Corollary 2.16).

(3) Let

$$S := \{(\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_1(\xi_1 - \xi_2) + 1 > 0,\ \xi_2(\xi_2 - \xi_1) + 1 > 0,\ \xi_1 > 0,\ \xi_2 > 0\}$$

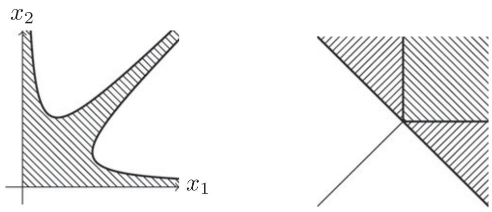

FIGURE 2. *The semi-algebraic set from Example 2.11(3) and its corresponding fan.*

(see Figure 2). In this example, we have $K_0(S) = \{v \in N : v_1 + v_2 \geqslant 0\}$, while $K(S)$ consists of $K_0(S)$ and the half-line τ generated by $-(e_1 + e_2)$. Let Σ be the complete fan with ray generators e_1, e_2, $\pm(e_1 - e_2)$ and $-(e_1 + e_2)$ (see Figure 2). Then Σ is adapted to S and $|F_S|$ is the half-plane $v_1 + v_2 \geqslant 0$. But condition (TC) is not satisfied. If it were, we could conclude $B_{\mathbb{A}^2}(S) = \mathbb{R}[x_1 x_2]$, which is clearly not true since $x_1 x_2$ is unbounded on S. Indeed, it follows from Lemma 2.8 that $Y_\tau(\mathbb{R}) \cap \overline{S} \neq \varnothing$. (It is not hard to check directly that it is a single point).

(4) The key property for Proposition 2.10 to apply is condition (TC) from 2.9. For a general open semi-algebraic set, this condition cannot be satisfied by any choice of a fan Σ in $N_\mathbb{R}$, as is demonstrated by the following simple example: for the open set $S = \{\xi \in (\mathbb{R}_+^*)^2 : \xi_1, \xi_2 > 1$ and $1 < \xi_1 - \xi_2 < 2\}$ in the 2-dimensional torus, we have $K(S) = \mathbb{Z}_+(-e_1 - e_2)$ and $K_0(S) = \{0\}$. So, at least with respect to $\sigma = \{0\}$, condition (TC) cannot hold for any complete fan Σ.

REMARK 2.12. Example (4) can still be saved by making a linear change of coordinates. However, it is clear that more complicated examples of open sets may be constructed for which no linear coordinate change allows to apply condition (TC). There is also an indirect way to see this. Whenever condition (TC) applies, we see from Proposition 2.10 that the ring $B_{U_\sigma}(S)$ of bounded polynomials on U_σ is finitely generated as an $\mathbb{R}$-algebra. On the other hand, it is known that there exist open semi-algebraic subsets S of $(\mathbb{R}^*)^n$ for $n \geqslant 3$ for which the $\mathbb{R}$-algebra $B_{\mathbb{A}^n}(S)$ fails to be finitely generated (see [**5**]).

Condition (TC) can be rather cumbersome to check, as the above examples show. We therefore seek favourable situations in which this condition can be guaranteed, and therefore allows a purely combinatorial computation of the ring of bounded functions. We discuss two classes of sets where this approach is successful, namely, binomially defined sets and the so-called 'tentacles' considered by Netzer [**10**].

2.13. Let $Q := (\mathbb{R}_+^*)^n \subseteq T(\mathbb{R})$, and let

$$S = \{\xi \in Q : a_i \xi^{\alpha_i} < b_i \xi^{\beta_i} \ (i = 1, \ldots, r)\}$$

be a non-empty basic open set in Q defined by binomial inequalities, where $0 \neq a_i,\ b_i \in \mathbb{R}$, and $\alpha_i,\ \beta_i \in M = \mathbb{Z}^n$ $(i = 1, \ldots, r)$. An easy argument shows that the inequalities can be rewritten with $a_i = 1$ and $\beta_i = 0$ for all i. For the following discussion we will therefore assume

$$S = \{\xi \in Q : \xi^{\gamma_i} < c_i \ (i = 1, \ldots, r)\},$$

where $\gamma_i \in M$ and $c_i > 0$ $(i = 1, \ldots, r)$.

We use the notation introduced in 2.5. Let $v \in N$. If $\langle \gamma_i, v \rangle > 0$ for all i, then $S(v) = Q$. If $\langle \gamma_i, v \rangle \geqslant 0$ for all i, then $S \subseteq S(v)$. If $\langle \gamma_i, v \rangle < 0$ for some i, then $S(v) = \varnothing$. So we see that

$$K(S) = K_0(S) = C_S^* \cap N,$$

where $C_S := \operatorname{cone}(\gamma_1, \ldots, \gamma_r) \subseteq M_\mathbb{R}$ and $C_S^* \subseteq N_\mathbb{R}$ is the dual cone of C_S.

The next lemma contains the reason why condition (TC) can be met.

LEMMA 2.14. *Let $\rho \subseteq N_{\mathbb{R}}$ be a pointed cone satisfying $\overline{S} \cap O_\rho(\mathbb{R}) \neq \varnothing$. Then $C_S^* \cap \operatorname{relint}(\rho) \neq \varnothing$.*

Proof. We may work in the toric affine variety $U_\rho = Spec\mathbb{R}[H_\rho]$ with $H_\rho = M \cap \rho^*$. Any point $\xi \in O_\rho(\mathbb{R})$ satisfies $\xi^\gamma = 0$ for all $\gamma \in H_\rho \smallsetminus (-H_\rho)$. Let us write $\tau := -C_S^*$, so that $H_\tau := M \cap \tau^*$ is the saturation inside M of the semi-group generated by $-\gamma_1, \dots, -\gamma_r$. Any $\beta \in H_\tau$ can be written in the form $\beta = -\sum_{i=1}^r b_i \gamma_i$ with rational numbers $b_i \geqslant 0$. Therefore, there exists $c > 0$ with $\xi^\beta > c$ for all $\xi \in S$. Hence we have $\xi^\beta \geqslant c > 0$ for any $\xi \in \overline{S}$, which implies $\beta \notin H_\rho \smallsetminus (-H_\rho)$. Thus $H_\tau \cap H_\rho \subseteq -H_\rho$, or equivalently, by dualizing, $-\rho \subseteq \rho + \tau$. Choose any $u \in \operatorname{relint}(\rho)$. There exists $v \in \rho$ with $-u \in v + \tau$, that is, with $u + v \in -\tau = C_S^*$. This proves the lemma since $u + v \in \operatorname{relint}(\rho)$. □

COROLLARY 2.15. *Let Σ be a complete fan in $N_{\mathbb{R}}$ which is adapted to S. Then condition (TC) from 2.9 is satisfied.*

Proof. Adapted simply means here that C_S^* is a union of cones from Σ. The claim is clear from Lemma 2.14: if $\tau \in \Sigma(1)$ satisfies $\overline{S} \cap Y_\tau(\mathbb{R}) \neq \varnothing$, then $\overline{S} \cap O_\rho(\mathbb{R}) \neq \varnothing$ for some $\rho \in \Sigma$ containing τ. By Lemma 2.14, this implies $C_S^* \cap \operatorname{relint}(\rho) \neq \varnothing$. By adaptedness, this implies $\tau \subseteq C_S^*$. □

We conclude that an S-compatible toric completion exists whenever S is defined by binomial inequalities.

COROLLARY 2.16. *Let σ be a pointed cone in $N_{\mathbb{R}}$, and let $S = \{\xi \in Q : \xi^{\gamma_i} < c_i \ (i = 1, \dots, r)\}$ as before, considered as a subset of $U_\sigma(\mathbb{R})$. The ring of polynomials on U_σ that are bounded on S is given by*

$$B_{U_\sigma}(S) = \mathbb{R}[H],$$

where $H = M \cap \sigma^* \cap C_S$.

2.17. A polynomial function $f \in \mathbb{R}[U_\sigma]$ is therefore bounded on S if and only if, for every monomial m occurring in f, some power of m is a product of $\mathtt{x}^{\gamma_1}, \dots, \mathtt{x}^{\gamma_r}$. It is obvious that such f is bounded on S; the content of Corollary 2.16 is that no other f is bounded on S. In particular, we see that $B_{U_\sigma}(S) = \mathbb{R}$ if and only if $\sigma + C_S^* = N_{\mathbb{R}}$.

2.18. For a second class of examples, let U be a non-empty open semi-algebraic subset of $Q = \{\xi \in T(\mathbb{R}) = (\mathbb{R}^*)^n : \xi_i > 0 \ (i = 1, \dots, n)\}$, and let $v \in N$. We consider the open set

$$S := S_v(U) := \{\lambda_v(s)\xi : \xi \in U, \ 0 < s \leqslant 1\}$$

in Q, which we may call a v-tentacle, following Netzer [**10**]. Multiplying v by a positive integer does not change S, therefore we may assume that v is a primitive element of N.

LEMMA 2.19. *Assume that U is relatively compact in Q. Let $S = S_v(U)$ be the associated v-tentacle as above.*

(a) $K(S) = K_0(S) = \mathbb{Z}_+ v$.
(b) *If $\{0\} \neq \rho \subseteq N_{\mathbb{R}}$ is a pointed cone with $\overline{S} \cap O_\rho(\mathbb{R}) \neq \varnothing$, then $v \in \operatorname{relint}(\rho)$.*

Proof. (a) We obviously have $U \subseteq S(v)$, and therefore $v \in K_0(S)$. Conversely let $u \in N$ with $S(u) \neq \varnothing$. So there is $\xi \in Q$ such that $\lambda_u(s)\xi \in S$ for all sufficiently small real $s > 0$.

Thus, for any small $s > 0$ there exist $0 < t \leqslant 1$ and $\eta \in U$ with $\lambda_u(s)\xi = \lambda_v(t)\eta$. Assume $u \notin \mathbb{R}_+ v$. Then there exists $\alpha \in M$ with $\langle \alpha, u\rangle > 0 > \langle \alpha, v\rangle$. Evaluating the character $\mathtt{x}^\alpha$, we get $s^{\langle \alpha,u\rangle}\xi^\alpha = t^{\langle \alpha,v\rangle}\eta^\alpha \geqslant \eta^\alpha$. The right-hand side is positive and bounded away from zero, since $\mathtt{x}^\alpha$ does not approach zero on U. On the other hand, the left-hand side tends to zero for $s \to 0$. This contradiction proves the claim.

(b) The proof is similar to that of Lemma 2.14. Again we may work in the affine toric variety U_ρ. Let $\gamma \in H_\rho \smallsetminus (-H_\rho)$. For any $\xi \in U_\rho(\mathbb{R})$ we have $\xi^\gamma = 0$. Since U is relatively compact, there exists $c > 1$ with $c^{-1} \leqslant \xi^\gamma \leqslant c$ for all $\xi \in U$. We have $\mathtt{x}^\gamma(\lambda_v(s)\xi) = s^{\langle \alpha,v\rangle} \cdot \xi^\gamma$ for $s > 0$, and we conclude $\langle \alpha, v\rangle > 0$. Thus $\langle \alpha, v\rangle > 0$ holds for every $\gamma \in H_\rho \smallsetminus (-H_\rho)$. This means $M \cap (-\mathbb{R}_+ v)^* \cap \rho^* \subseteq -\rho^*$, or $-\rho \subseteq \rho - \mathbb{R}_+ v$ after dualizing. As before, this implies $v \in \mathrm{relint}(\rho)$. □

Similarly to Proposition 2.16, we deduce the following corollary.

Corollary 2.20. *Let $U \neq \varnothing$ be an open and relatively compact subset of Q, and let $S = S_v(U)$ be the associated v-tentacle. Let σ be a pointed cone in $N_\mathbb{R}$. The ring of polynomials on U_σ that are bounded on S is $B_{U_\sigma}(S) = \mathbb{R}[H]$, where $H = M \cap \sigma^* \cap (\mathbb{R}_+ v)^*$.*

2.21. Thus a polynomial function $f \in \mathbb{R}[U_\sigma]$ is bounded on S if and only if every monomial $\mathtt{x}^\alpha$ occurring in f satisfies $\langle \alpha, v\rangle \geqslant 0$. In particular, $B_{U_\sigma}(S) = \mathbb{R}$ is equivalent to $\sigma + \mathbb{R}_+ v = N_\mathbb{R}$.

3. *Iitaka dimension on toric surfaces*

Let X be a non-singular projective surface over a field k. We always assume that X is absolutely irreducible. We first discuss how the intersection matrix A_D of an effective divisor D on X relates to the Iitaka dimension $\kappa(D)$ of D. Since $\kappa(D)$ is the transcendence degree of $\mathscr{O}(X \smallsetminus D)$, these facts have implications for rings of bounded polynomials on 2-dimensional semi-algebraic sets, by Theorem 1.3. In general, the intersection matrix does not uniquely determine $\kappa(D)$. However, when X is a toric surface and the divisor D is toric, we show that $\kappa(D)$ can be read off from A_D in a simple manner (Proposition 3.13).

3.1. Given two divisors D, D' on X, we denote by $D\,.\,D'$ the intersection number of D and D'. The intersection pairing is invariant under linear equivalence and therefore induces a bilinear pairing on the divisor class group $Pic(X)$. As usual, we write $D^2 := D\,.\,D$ for the self-intersection number of D.

Definition 3.2. Let D be an effective (not necessarily reduced) divisor on X whose distinct irreducible components are $C_1, \ldots, C_r$. We define the *intersection matrix* of D to be the symmetric $r \times r$ matrix with integer entries $C_i\,.\,C_j$ $(i,\, j = 1, \ldots, r)$; cf. [4, 8.3]. It will be denoted by A_D.

3.3. Let D be an effective divisor on X. For $m \geqslant 1$ let $\phi_m : X \dashrightarrow |mD|$ be the rational map associated with the complete linear series $|mD|$. The *Iitaka dimension* of D is defined to be

$$\kappa(X, D) := \max_{m \geqslant 1} \dim \phi_m(X);$$

see [4, Section 10.1] or [6, 2.1.3]. It is well known that $\kappa(D, X)$ is equal to the transcendence degree of $\mathscr{O}(X \smallsetminus D)$, the ring of regular functions on the open subvariety $X \smallsetminus D := X \setminus \mathrm{supp}(D)$ of X (see [4, Proposition 10.1]). The Iitaka dimension of D is closely related to the intersection matrix A_D.

Proposition 3.4. (a) *If A_D is negative definite, then $\kappa(X,D)=0$, that is, $\mathscr{O}(X\smallsetminus D)=k$.*
(b) *If $D^2>0$, then $\kappa(X,D)=2$.*

Proof. (a) is [4, Proposition 8.5]; (b) is Lemma 8.5. Assertion (a) is also a consequence of Proposition 3.6. □

Corollary 3.5. *If $\mathscr{O}(X\smallsetminus D)$ has transcendence degree $\leqslant 1$, then A_D is negative semidefinite.*

In particular, $\kappa(X,D)$ is determined by the Sylvester signature of A_D whenever A_D is nonsingular.

Proof. Let $C_1,\dots,C_r$ be the irreducible components of D. The intersection matrix A_D has non-negative off-diagonal entries. Therefore, if A_D has a positive eigenvalue, there exist integers $m_i\geqslant 0$ with $(\sum_i m_iC_i)^2>0$. By Proposition 3.4(b), this implies $trdeg\mathscr{O}(X\smallsetminus D)=2$. □

Part (a) of 3.4 can be generalized as follows.

Proposition 3.6. *Let $D\subseteq X$ be an effective divisor whose intersection matrix A_D is negative definite. Then, for any line bundle L on X, the space $H^0(X\smallsetminus D,\,\mathscr{O}(L))$ is a finite-dimensional k-vector space.*

For the proof we need two lemmas.

Lemma 3.7. *Let D be an effective divisor with irreducible components $C_1,\dots,C_r$, and assume $C_i\,.\,D<0$ for $i=1,\dots,s$. Then, for any divisor E there exists an integer $n_0=n_0(E)$ such that*

$$|E+nD|=(n-n_0)D+|E+n_0D|$$

holds for all $n\geqslant n_0$.

Proof. Say $D=\sum_{i=1}^r m_iC_i$, with $m_i\geqslant 1$. Choose an integer n such that the inequality

$$n(C_i\,.\,D)<-C_i\,.\,(E+a_1C_1+\dots+a_sC_s) \tag{3.1}$$

holds for $i=1,\dots,r$ and every tuple $(a_1,\dots,a_r)$ with $0\leqslant a_j\leqslant m_j$ $(j=1,\dots,r)$. Then we claim

$$|E+(n+1)D|=|E+nD|+D.$$

Indeed, if $a_1,\dots,a_r$ are integers with $0\leqslant a_j\leqslant m_j$ $(j=1,\dots,r)$, we show

$$\left|E+nD+\sum_j a_jC_j\right|=|E+nD|+\sum_j a_jC_j$$

by induction on $\sum_j a_j$. The assertion is trivial for $\sum_j a_j=0$. If $(a_1,\dots,a_s)\neq(0,\dots,0)$ is a tuple with $0\leqslant a_j\leqslant m_j$, and if i is an index with $a_i\geqslant 1$, we have

$$C_i\,.\left(E+nD+\sum_j a_jC_j\right)<0$$

by (3.1). Any effective divisor linearly equivalent to $E + nD + \sum_j a_j C_j$ must therefore contain C_i, which implies

$$\left|E + nD + \sum_j a_j C_j\right| = \left|-C_i + E + nD + \sum_j a_j C_j\right| + C_i,$$

and so

$$\left|E + nD + \sum_j a_j C_j\right| = |E + nD| + \sum_j a_j C_j$$

by the inductive hypothesis. □

LEMMA 3.8. *If $x_1, \dots, x_r$ is a linear basis of $\mathbb{R}^r$, there exist integers $m_1, \dots, m_r \geqslant 1$ such that $x = \sum_i m_i x_i$ satisfies $\langle x, x_i \rangle > 0$ for all $i = 1, \dots, r$.*

Proof. Let K be the convex cone spanned by $x_1, \dots, x_r$, and let $K^* = \{y \in \mathbb{R}^r : \langle x, y \rangle \geqslant 0\}$ be the dual cone. Since $x_1, \dots, x_r$ are a basis, both K and K^* have non-empty interior. We have to show that the interiors intersect. Assuming $\operatorname{int}(K) \cap \operatorname{int}(K^*) = \varnothing$, there exists $0 \neq z \in \mathbb{R}^r$ with $\langle x, z \rangle \geqslant 0$ for all $x \in K$ and $\langle y, z \rangle \leqslant 0$ for all $y \in K^*$. Hence $z \in K^* \cap (-K^{**}) = (K^*) \cap (-K)$, which implies $\langle z, z \rangle \leqslant 0$, whence $z = 0$, which is a contradiction. Now interior$(K) \cap$ interior(K^*) is a non-empty open cone, hence it contains integer points with respect to the basis $x_1, \dots, x_r$. □

Proof of Proposition 3.6. Let $C_1, \dots, C_r$ be the irreducible components of D, let $U := X \setminus D$ and let E be a divisor on X such that $L \cong \mathscr{O}_X(E)$. Every section in $\Gamma(U, L)$ is a meromorphic section of L on X, which means that

$$\Gamma(U, L) = \bigcup_{n \geqslant 1} \Gamma(X, E + nD)$$

(ascending union). Since A_D is negative definite, we find integers $m_1, \dots, m_r \geqslant 1$ such that $D := \sum_{i=1}^r m_i C_i$ satisfies $C_i \,.\, D < 0$ $(i = 1, \dots, r)$, using Lemma 3.8. By Lemma 3.7, there exists $n_0 \geqslant 1$ such that $|E + nD| = (n - n_0)D + |E + n_0 D|$ for all $n \geqslant n_0$, which means $\Gamma(X, E + nD) = \Gamma(X, E + n_0 D)$. Hence $\Gamma(U, L) = \Gamma(X, E + n_0 D)$, and so this space has finite dimension. □

REMARK 3.9. The hypothesis that A_D is negative definite in Proposition 3.6 entails $\mathscr{O}(X \setminus D) = k$. One may wonder whether 3.6 remains true if only $\mathscr{O}(X \setminus D) = k$ is assumed. An example due to Mondal and Netzer [**9**] shows that this usually fails. We will revisit their construction in Example 4.4. On the other hand, we will see in 3.13 that such problems do not occur in a toric setting.

3.10. Let Σ be the fan of a non-singular projective toric surface X. For $\rho \in \Sigma(1)$ let $Y_\rho = \overline{O_\rho}$. Let $\rho_0, \dots, \rho_{m-1}, \rho_m = \rho_0$ be the elements of $\Sigma(1)$, written in cyclic order, so that ρ_{i-1} and ρ_i bound a cone from $\Sigma(2)$ for $i = 1, \dots, m$. Let v_i be the primitive generator of ρ_i. The divisor class group of X is generated by the $Y_i = Y_{\rho_i}$, and the intersection form on X has the following description (see [**2**, Section 10.4]): given $1 \leqslant i < m$, there is an integer b_i such that $b_i v_i = v_{i-1} + v_{i+1}$. Then we have $Y_i^2 = -b_i$, $Y_i \,.\, Y_j = 1$ if $j - i = \pm 1$, and $Y_i \,.\, Y_j = 0$ otherwise. Similarly for $i = 0$.

LEMMA 3.11. *Let $n \geqslant 1$ and let $\rho_0, \dots, \rho_{n+1}$ be a sequence of pairwise different cones in $\Sigma(1)$ such that ρ_{i-1} and ρ_i bound a cone from $\Sigma(2)$ for $i = 1, \dots, n+1$. Let $l = \rho_0 \cup (-\rho_0)$ and let A be the intersection matrix of the divisor $\sum_{i=1}^n Y_{\rho_i}$ on X. Then*:

(a) $\det(A) = 0 \;\Leftrightarrow\; \rho_{n+1} = -\rho_0$;
(b) $A \prec 0 \;\Leftrightarrow\; (-1)^n \det(A) > 0 \;\Leftrightarrow\; \rho_1$ *and* ρ_{n+1} *lie (strictly) on the same side of the line* l;
(c) A *is indefinite* $\Leftrightarrow\; (-1)^n \det(A) < 0 \;\Leftrightarrow\; \rho_1$ *and* ρ_{n+1} *lie (strictly) on opposite sides of* l.

In case (a) *we have $A \preceq 0$ and $rk(A) = n-1$. In case* (c) *the matrix A has a unique positive eigenvalue.*

The hypothesis indicates that $\rho_0, \dots, \rho_{n+1}$ are given in cyclic order and that there exists no further cone from $\Sigma(1)$ in between them. Since $\rho_{n+1} \neq \rho_0$, there exists at least one cone in $\Sigma(2)$ that is not of the form $\text{cone}(\rho_{i-1}, \rho_i)$ with $1 \leqslant i \leqslant n+1$.

Proof. Let $v_i \in N$ be the primitive vector generating ρ_i for $i = 0, \dots, n+1$. Let $b_1, \dots, b_n \in \mathbb{Z}$ be defined by $v_{i+1} + v_{i-1} = b_i v_i$ $(i = 1, \dots, n)$. Then

$$A = \begin{pmatrix} -b_1 & 1 & & & \\ 1 & -b_2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -b_{n-1} & 1 \\ & & & 1 & -b_n \end{pmatrix}.$$

Let $\delta(b_1, \dots, b_i)$ be the upper left $i \times i$ principal minor of A $(i = 1, \dots, n)$. Then

$$v_{i+1} = (-1)^i\, \delta(b_1, \dots, b_i)\, v_1 + (-1)^i\, \delta(b_2, \dots, b_i)\, v_0$$

holds for $i = 1, \dots, n$. In particular,

$$v_{n+1} = (-1)^n \det(A)\, v_1 + (-1)^n\, \delta(b_2, \dots, b_n)\, v_0.$$

Since v_0, v_1 are linearly independent and $v_0 \neq v_{n+1}$, the lemma follows easily from these identities. □

3.12. We keep the previous hypotheses. Let T be a subset of $\Sigma(1)$, let $T' = \Sigma(1) \smallsetminus T$ and let

$$U = X \smallsetminus \bigcup_{\tau \in T} Y_\tau,$$

an open toric subvariety of X. Let $C = \text{cone}(\tau' : \tau' \in T') \subseteq N_{\mathbb{R}}$; then $\mathscr{O}(U) = k[M \cap C^*]$, the semi-group algebra of $M \cap C^*$. The following list exhausts all possible cases:

(1) $C = N_{\mathbb{R}}$. Then $|T'| \geqslant 3$ and $\mathscr{O}(U) = k$.
(2) C is a half-plane. Then $|T'| \geqslant 3$ and $\mathscr{O}(U) \cong k[u]$ (polynomial ring in one variable).
(3) C is a line. Then $|T'| = 2$ and $\mathscr{O}(U) \cong k[u, u^{-1}]$ (ring of Laurent polynomials in one variable).
(4) $C \smallsetminus \{0\}$ is contained in an open half-plane. Then we have $trdeg\mathscr{O}(U) = 2$.

The following result shows that, for toric divisors on non-singular toric surfaces, the Iitaka dimension is characterized by the signature of the intersection matrix.

PROPOSITION 3.13. *Let X be a non-singular toric projective surface with fan Σ. Let $T \subseteq \Sigma(1)$ with $T \neq \varnothing$, let $U = X \smallsetminus \bigcup_{\tau \in T} Y_\tau$ and let A be the intersection matrix of T.*

(a) $A \prec 0 \;\Leftrightarrow\; \mathscr{O}(U) = k$;
(b) $A \preceq 0$ *and* $\det(A) = 0 \;\Leftrightarrow\; trdeg\mathscr{O}(U) = 1$;
(c) A *has a positive eigenvalue* $\Leftrightarrow\; trdeg\mathscr{O}(U) = 2$.

Moreover, in case (c) *we have* $\det(A) = 0$ *if and only if* $|\Sigma(1) \smallsetminus T| \leqslant 1$.

Proof. Write $T' = \Sigma(1) \smallsetminus T$ as before. The implications from the left to the right in (a) (respectively, in (c)) hold for general reasons (and could be easily reproved using 3.12); see Proposition 3.6 and Corollary 3.5. The group $Pic(X)$ is free abelian of rank $|\Sigma(1)| - 2$. The intersection form on $Pic(X)$ is non-degenerate, and all of its eigenvalues are negative except for one, by the Hodge index theorem. Therefore it is clear for $|T'| \leqslant 1$ that A is singular and has a unique positive eigenvalue. Also, $trdeg\mathscr{O}(U) = 2$ is clear for $|T'| \leqslant 1$; see 3.12.

So we assume $|T'| \geqslant 2$ for the rest of the proof. The matrix A is a block diagonal sum of matrices $A_1, \ldots, A_r$. For each $i = 1, \ldots, r$ there is a sequence $\rho_0, \ldots, \rho_{n+1}$ as in Lemma 3.11 with ρ_0, $\rho_{n+1} \in T'$ and $\rho_1, \ldots, \rho_n \in T$, such that A_i is the intersection matrix of $\rho_1, \ldots, \rho_n$. By 3.12 we have $trdeg\mathscr{O}(U) = 2$ if and only if all $\tau' \in T'$ are contained in a common open half-plane (together with 0). By 3.11 it is equivalent that A_i is indefinite for one index $i \in \{1, \ldots, r\}$. Since $|T'| \geqslant 2$, it is equivalent that A is indefinite and $\det(A) \neq 0$. On the other hand, $\mathscr{O}(U) = k$ is by 3.12 equivalent to the condition that T' is not contained in a half-plane. By 3.11, this in turn is equivalent to $A_i \prec 0$ for every $i = 1, \ldots, r$, and hence to $A \prec 0$. This proves (a) and (c) together with the last statement, and so it also implies (b). □

Example 3.14. Recalling the setup in 2.9, we let $k = \mathbb{R}$, fix a pointed cone $\sigma \in \Sigma$, and regard X_Σ as a completion of the affine toric variety U_σ. If $S \subseteq U_\sigma$ is an open semi-algebraic set and the toric compatibility condition (TC) is satisfied, the subset $T \subseteq \Sigma(1)$ in 3.12 consists of those $\tau \in \Sigma(1)$ with $\tau \nsubseteq \sigma$ and $K_0(S) \cap \operatorname{relint}(\tau) = \varnothing$.

All of the four different cases in 3.12 can occur in this situation. Cases (1) and (4) may arise in a trivial way: for example, let Σ be the standard fan of $\mathbb{P}^2$ and $\sigma = \{\operatorname{cone}(e_1, e_2)\}$, so that $U_\sigma \hookrightarrow X_\Sigma$ is the usual embedding of the affine into the projective plane. Then if $S = (\mathbb{R}^*)^2$, we find $T = \varnothing$ and $T' = \Sigma(1)$, hence $|T'| = 3$ and $U = \mathbb{P}^2$, so that $\mathscr{O}(U) = B_{U_\sigma}(S) = \mathbb{R}$. The same will hold whenever $S \subseteq (\mathbb{R}^*)^2$ contains a non-empty open cone. On the other hand, if $S \subseteq (\mathbb{R}^*)^2$ is bounded, we have $T = \{\operatorname{cone}(-e_1 - e_2)\}$ and $T' = \{\operatorname{cone}(e_1), \operatorname{cone}(e_2)\}$, so that $U = U_\sigma$ and $\mathscr{O}(U) = B_{U_\sigma} = \mathbb{R}[x_1, x_2]$ (for a more interesting example leading to case (4), see also [**11**, Example 3.10]).

In Example 2.11(2), the fan Σ is the refinement of the standard fan of $\mathbb{P}^2$ in which $\Sigma(1) = \{\operatorname{cone}(e_1), \operatorname{cone}(e_2), \operatorname{cone}(-e_1 - e_2), \pm\operatorname{cone}(e_1 - ke_2)\}$. Here, $T = \{\operatorname{cone}(-e_1 - e_2)\}$ and $C = \operatorname{cone}(\tau' : \tau' \in T') = \operatorname{cone}(e_1, e_2, \pm(e_1 - ke_2))$ is a half-plane, so we are in case (2). Indeed we found $\mathscr{O}(U) = B_{U_\sigma}(S) = \mathbb{R}[x_1^k x_2]$.

Case (3) obviously will not come up if we start from $\sigma = \operatorname{cone}(e_1, e_2)$, since $\mathbb{R}[x_1, x_2]$ cannot contain a ring of Laurent polynomials. But if we consider, for example, $S = \{(\xi_1, \xi_2) \in (\mathbb{R}^*)^2 : 1 < \xi_1 < 2\}$ and Σ as above with $k = 0$, we have the same completion as in Example 2.11(1), but with $\sigma = \{0\}$. Here, we find $T' = \{\pm\operatorname{cone}(e_2)\}$ and $\mathscr{O}(U) = B_{U_\sigma}(S) = \mathbb{R}[x_1, x_1^{-1}]$.

4. *Filtration by degree of boundedness*

4.1. Let $S \subseteq \mathbb{R}^d$ be a semi-algebraic set. In [**9**], Mondal and Netzer studied the following filtration on the polynomial ring. For $n \geqslant 0$ let

$$B_n(S) = \{f \in \mathbb{R}[x] : \exists\, g \in \mathbb{R}[x] \text{ with } \deg(g) \leqslant 2n \text{ and } f^2 \leqslant g \text{ on } S\}.$$

Then the $B_n(S)$ form an ascending filtration on $\mathbb{R}[x] = \mathbb{R}[x_1, \dots, x_d]$ by linear subspaces, satisfying $B_m(S)\, B_n(S) \subseteq B_{m+n}(S)$ for all $m, n \geqslant 0$. Clearly $B_0(S) = B(S)$, the ring of polynomials bounded on S.

4.2. We propose to generalize the construction from 4.1. Let V be a normal affine $\mathbb{R}$-variety, let $S \subseteq V(\mathbb{R})$ be a semi-algebraic set, and let $V \subseteq X$ be an open dense embedding into a normal and complete variety X. We assume that the completion is compatible with S, in the sense of 1.1. Let Y (respectively, Y') be the union of those irreducible components Z of $X \smallsetminus V$ for which $\overline{S} \cap Z(\mathbb{R})$ is empty (respectively, non-empty), and put $U = X \smallsetminus Y$. Then $V = X \smallsetminus (Y \cup Y')$. For $n \geqslant 0$ let

$$L_{X,n}(S) = \Gamma(U, \mathscr{O}_X(nY')).$$

Since Y' is disjoint from $V \subseteq U$, we may consider $L_{X,n}(S)$ as a subspace of $\mathscr{O}_X(V) = \mathbb{R}[V]$, namely,

$$L_{X,n}(S) = \{f \in \mathbb{R}[V] : \mathrm{ord}_Z(f) \geqslant -n \text{ for all components } Z \text{ of } Y'\}.$$

The $L_{X,n}(S)$ $(n \geqslant 0)$ define an ascending and exhaustive filtration of $\mathbb{R}[V]$ by linear subspaces, satisfying

$$L_{X,m}(S)\, L_{X,n}(S) \subseteq L_{X,m+n}(S)$$

for $m, n \geqslant 0$. Moreover, $L_{X,0}(S) = B_V(S)$ by Theorem 1.2. In particular, the $L_{X,n}(S)$ are modules over the ring $B_V(S)$.

For $V = \mathbb{A}^d$ the affine space, the two filtrations $\{B_n(S)\}$ and $\{L_{X,n}(S)\}$ on $\mathbb{R}[\mathtt{x}]$ are compatible in the following sense.

PROPOSITION 4.3. *With notation as above, fix* $m \geqslant 0$.

(a) *There exists* $n \geqslant 0$ *such that* $B_m(S) \subseteq L_{X,n}(S)$.
(b) *There exists* $n \geqslant 0$ *such that* $L_{X,m}(S) \subseteq B_n(S)$.

Proof. (a) Choose n such that $\mathbb{R}[x]_{2m} \subseteq L_{X,2n}(S)$ (the existence of such n is clear since $\mathbb{R}[x]_{2m}$ is finite-dimensional). Now given $f \in B_m(S)$, choose $g \in \mathbb{R}[x]_{2m}$ such that $f^2 \leqslant g$ on S. The rational function $f^2/(g+1)$ is defined and bounded on S. We apply Theorem 1.2 to the S-compatible completion $(V \cap \mathrm{dom}(g+1)) \subseteq X$ and conclude that $\mathrm{ord}_Z(f^2/(g+1)) \geqslant 0$ for all components Z of Y'. Then $f \in L_{X,n}(S)$.

(b) Choose $g \in \mathbb{R}[x]$ with $\mathrm{ord}_Z(g) \leqslant -m$ for all components Z of Y'. Let $f \in L_{X,m}(S)$. By Theorem 1.2, the rational function $f^2/(g^2+1)$ is defined and bounded on S. Thus, if $n = \deg(g)$, then $L_{X,m} \subseteq B_n(S)$. □

EXAMPLE 4.4. The following example is due to Mondal and Netzer [9]. Let $V = \mathbb{A}^2$, $\mathbb{R}[V] = \mathbb{R}[x, y]$, put

$$f_1 = x^3y + x^6 - x, \quad S_1 = \{(a,b) \in \mathbb{R}^2 : 2 \geqslant f_1(a,b) \geqslant 1,\ a \geqslant 1\},$$
$$f_2 = x^3y - x^6 - x, \quad S_2 = \{(a,b) \in \mathbb{R}^2 : 2 \geqslant f_2(a,b) \geqslant 1,\ a \geqslant 1\},$$

and let $S = S_1 \cup S_2$. Applying the procedure of 1.4 to S_1, starting with $V \subseteq \mathbb{P}^2$, requires a sequence of nine blow-ups. In the resulting completion $V \subseteq X_1$, the complement $C_\infty = X_1 \smallsetminus V$ has ten irreducible components $E_0, \dots, E_9$, which are the strict transforms of the line $\mathbb{P}^2 \smallsetminus V$ and the exceptional divisors of the nine blow-ups. Only E_9 meets $\overline{S}$, so we need to consider the divisor $Y_1 = \sum_{i=0}^{8} E_i$. The configuration of the irreducible components of Y_1 is shown in its intersection graph

$$\begin{array}{cccccccccccc}
E_0 & & & & & & & & & & \\
| & & & & & & & & & & \\
E_2 & & & & & & & & & & \\
| & & & & & & & & & & \\
E_3 & — & E_4 & — & E_5 & — & E_6 & — & E_7 & — & E_8 \\
| & & & & & & & & & & \\
E_1 & & & & & & & & & &
\end{array}$$

and the full intersection matrix is

$$M_{Y_1} = \begin{bmatrix}
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2
\end{bmatrix}.$$

This matrix has a positive eigenvalue, which shows that $B_V(S_1)$ has transcendence degree 2. The compatible completion for S_2 is isomorphic to that of S_1.

When resolving the union $S = S_1 \cup S_2$, the first three blow-ups are the same as for S_1. The blow-ups for S_1 and S_2 in the fourth step are centred at two different points of E_3. In the resulting S-compatible completion X of V, the divisor Y has intersection graph

$$\begin{array}{ccccccccccccccccccccc}
 & & & & & & & & & & E_0 & & & & & & & & & & \\
 & & & & & & & & & & | & & & & & & & & & & \\
 & & & & & & & & & & E_2 & & & & & & & & & & \\
 & & & & & & & & & & | & & & & & & & & & & \\
E_8' & — & E_7' & — & E_6' & — & E_5' & — & E_4' & — & E_3 & — & E_4 & — & E_5 & — & E_6 & — & E_7 & — & E_8 \\
 & & & & & & & & & & | & & & & & & & & & & \\
 & & & & & & & & & & E_1 & & & & & & & & & &
\end{array}$$

and intersection matrix

$$M_Y = \begin{bmatrix}
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & -3 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2
\end{bmatrix},$$

which is negative semi-definite of corank 1. Mondal and Netzer show through direct computation that $B(S) = \mathbb{R}$, and at the same time that $B_1(S)$ has infinite dimension over $\mathbb{R}$. We thus conclude from Proposition 4.3 that there exists $n \geqslant 0$ such that $\dim H^0(X \smallsetminus Y, \mathscr{O}(n(E_9 + E_9'))) = \infty$, even though $\mathscr{O}_X(X \smallsetminus Y) = B(S) = \mathbb{R}$.

We now show that the phenomenon in the above example does not occur for semi-algebraic sets inside a compatible toric completion. It follows in particular that the set constructed by Mondal and Netzer does not admit such a completion.

Proposition 4.5. *Let U be a toric variety. For any Weil divisor D on U, the space $H^0(U, \mathscr{O}_U(D))$ is finitely generated as a module over $\mathscr{O}(U) = H^0(U, \mathscr{O}_U)$.*

Proof. Let Σ be the fan associated to U. For every $\rho \in \Sigma(1)$ let $u_\rho \in N$ be the primitive generator of ρ. We can assume that the Weil divisor D is torus invariant ([**2**] 4.1.3). So $D = \sum_{\rho\in\Sigma(1)} m_\rho Y_\rho$ with $m_\rho \in \mathbb{Z}$. By [**2**] 4.1.2 and 4.3.2, the space $H^0(U, \mathscr{O}(D))$ is linearly spanned by the characters $\mathtt{x}^\beta$ with β in

$$B = \Big\{\beta \in M : \langle\beta, u_\rho\rangle \geqslant -m_\rho \text{ for all } \rho \in \Sigma(1)\Big\}.$$

On the other hand, the ring $\mathscr{O}(U)$ is linearly spanned by the characters $\mathtt{x}^\alpha$ with α in

$$A = \Big\{\alpha \in M : \langle\beta, u_\rho\rangle \geqslant 0 \text{ for all } \rho \in \Sigma(1)\Big\}.$$

From Dickson's Lemma it follows that there exists a finite subset B_0 of B such that $B = B_0 + A$. Hence the $\mathscr{O}(U)$-module $\mathscr{O}(D)$ is generated by the characters $\mathtt{x}^\beta$ with $\beta \in B_0$. □

Corollary 4.6. *Let V be an affine toric $\mathbb{R}$-variety and let $S \subseteq V(\mathbb{R})$ be a semi-algebraic set. Assume that there exists a toric completion $V \subseteq X$ of V which is compatible with S. Then $L_{X,n}(S)$ is finitely generated as a module over $B_V(S)$ for every $n \geqslant 0$.*

Note that Proposition 2.10 provides a sufficient condition for the existence of a completion X as required. In particular, such X exists if S is defined by binomial inequalities (2.16).

Proof. By assumption, X is a complete toric variety, and every irreducible component of $X \smallsetminus V$ is torus invariant. Let Y (respectively, Y') be the union of those irreducible components of $X \smallsetminus V$ for which $\overline{S} \cap Y(\mathbb{R}) = \varnothing$ (respectively, $\overline{S} \cap Y(\mathbb{R}) \neq \varnothing$), and write $U = X \smallsetminus Y$. Then U is a toric variety. By definition, $L_{X,n}(S) = H^0(U, \mathscr{O}(-nY'))$ for $n \geqslant 0$. By Proposition 4.5, $L_{X,n}(S)$ is finitely generated as a module over $\mathscr{O}(U)$, and $\mathscr{O}(U) = B_V(S)$ by Theorem 1.2. □

In particular, $B_V(S) = \mathbb{R}$ implies that the spaces $L_{X,n}(S)$ are all finite-dimensional. If $V = \mathbb{A}^d$, this also implies that the spaces $B_n(S)$ of Mondal–Netzer are all finite-dimensional, using Proposition 4.3.

5. *Positive polynomials and stability*

Let V be an irreducible affine $\mathbb{R}$-variety, $S \subseteq V(\mathbb{R})$ be a closed semi-algebraic set and

$$\mathscr{P}_V(S) = \{f \in \mathbb{R}[V] : f|_S \geqslant 0\},$$

the cone of non-negative regular functions on S. We will always assume that S is the closure of its interior in $V(\mathbb{R})$. (This property is sometimes referred to by saying that S is 'regular'.)

Definition 5.1. We say that $\mathscr{P}_V(S)$ is *totally stable* if the following holds: for every finite-dimensional subspace U of $\mathbb{R}[V]$ there exists a finite-dimensional subspace W of $\mathbb{R}[V]$

such that, for all $r \geqslant 2$ and $f_1, \ldots, f_r \in \mathscr{P}_V(S)$,

$$f_1 + \cdots + f_r \in U \implies f_1, \ldots, f_r \in W.$$

Consider the case $V = \mathbb{A}^n_{\mathbb{R}}$, $\mathbb{R}[V] = \mathbb{R}[x]$ with $x = (x_1, \ldots, x_n)$. Then $\mathscr{P}(S)$ is totally stable if and only if, for every $d \geqslant 0$, there exists $e \geqslant d$ such that whenever $f_1, \ldots, f_r \in \mathscr{P}(S)$ are such that $\deg(\sum_{i=1}^r f_i) \leqslant d$, it follows that $\deg(f_i) \leqslant e$ for all i. Note that if $S = \mathbb{R}^n$, the leading forms of two non-negative polynomials cannot cancel, so we may take $e = d$.

The property of total stability has consequences for the existence of degree bounds for representations of positive polynomials by weighted sums of squares, and for the moment problem in polynomial optimisation and functional analysis. These questions have been a major motivation for the study of bounded polynomials. The precise statement is as follows: the moment problem for S is said to be finitely solvable if $\mathscr{P}_V(S)$ contains a dense finitely generated quadratic module M (where dense means that no element of $\mathscr{P}_V(S)$ can be strictly separated from M by any linear functional on $\mathbb{R}[V]$). The main result of [**13**] implies the following theorem.

THEOREM 5.2. *If $S \subseteq V(\mathbb{R})$ has dimension at least 2 and $\mathscr{P}_V(S)$ is totally stable, then the moment problem for S is not finitely solvable.*

See [**7**, **12**, **13**] for a fuller discussion and further references.

We now examine total stability for open semi-algebraic sets that admit a compatible completion. First note the following simple observation that holds without any additional assumptions.

PROPOSITION 5.3. *If $\mathscr{P}_V(S)$ is totally stable, then $B_V(S) = \mathbb{R}$.*

Proof. Let $f \in B_V(S)$, say $|f| \leqslant \lambda$ on S for some $\lambda \in \mathbb{R}$. Then $f^{2n}, \lambda^{2n} - f^{2n} \in \mathscr{P}_V(S)$ for all $n \geqslant 1$. Since the sum of these two elements is constant, total stability implies the existence of a finite-dimensional subspace W of $\mathbb{R}[V]$ containing f^{2n} for all $n \geqslant 1$. Thus f is algebraic over $\mathbb{R}$ and therefore constant. □

We are interested in the converse. Suppose that V is normal and admits an S-compatible completion $V \hookrightarrow X$. Following the notation in 4.2, let Y be the union of all irreducible components Z of $X \setminus V$ for which $\overline{S} \cap Z(\mathbb{R})$ is empty and let Y' be the union of those for which $\overline{S} \cap Z(\mathbb{R})$ is dense in Z. Consider the filtration

$$L_{X,1}(S) \subseteq L_{X,2}(S) \subseteq \cdots$$

of $\mathbb{R}[V]$ by (not necessarily finite-dimensional) subspaces, defined in 4.2. Given $f_1, \ldots, f_r \in \mathscr{P}_V(S)$, we have

$$\mathrm{ord}_Z(f_1 + \cdots + f_r) = \min\{\mathrm{ord}_Z(f_1), \ldots, \mathrm{ord}_Z(f_r)\}$$

for every component Z of Y', by [**11**, Lemma 3.4]. Thus if $f_1 + \cdots + f_r \in L_{X,n}(S)$, then already $f_1, \ldots, f_r \in L_{X,n}(S)$. It follows that if the spaces $L_{X,n}(S)$ are finite-dimensional, then $\mathscr{P}_V(S)$ is totally stable.

In the 2-dimensional case, we have a sufficient condition in terms of the intersection matrix, as seen in Section 3.

PROPOSITION 5.4. *Let V be a non-singular real affine surface and $S \subseteq V(\mathbb{R})$ be a closed semi-algebraic set that is the closure of its interior. Assume that V admits a non-singular*

S-compatible completion $V \hookrightarrow X$ such that the intersection matrix of the divisor Y defined as above is negative definite. Then $\mathscr{P}_V(S)$ is totally stable.

Proof. In this case, the spaces $L_{X,n}(S)$ are finite-dimensional by Proposition 3.6, so that $\mathscr{P}_V(S)$ is totally stable by the preceding discussion. □

In view of Proposition 5.3, we are also led to the question of whether $B_V(S) = \mathbb{R}$ implies that the spaces $L_{X,n}(S)$ are finite-dimensional over $\mathbb{R}$ for all $n \geqslant 0$. The example of Mondal and Netzer discussed in 4.1 shows this to be false in general. On the other hand, it is true in the toric setting, as we saw in the preceding section.

THEOREM 5.5. *Let S be a closed semi-algebraic set in an affine toric variety V that is the closure of its interior. Assume that $B_V(S) = \mathbb{R}$ and that V admits a toric S-compatible completion $V \hookrightarrow X$. Then $\mathscr{P}_V(S)$ is totally stable.*

Proof. By Corollary 4.6, the spaces $L_{X,n}(S)$ constructed from the completion X are finite-dimensional over $B_V(S) = \mathbb{R}$. Hence $\mathscr{P}_V(S)$ is totally stable by the argument above. □

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Daniel Plaumann
Fakultät für Mathematik
Technische Universität Dortmund
Vogelpothsweg 87
44227 Dortmund
Germany

Daniel.Plaumann@math.tu-dortmund.de

Claus Scheiderer
Fachbereich Mathematik und Statistik
Universität Konstanz
78457 Konstanz
Germany

Claus.Scheiderer@uni-konstanz.de