Universität Konstanz

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Konstanzer Schriften in Mathematik

Nr. 352, März 2016

ISSN 1430-3558

# Maximal regularity for the thermoelastic plate equations with free boundary conditions

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February 25, 2016

**Abstract**

We consider the linear thermoelastic plate equations with free boundary conditions in the $L_p$ in time and $L_q$ in space setting. We obtain unique solvability with optimal regularity for the inhomogeneous problem in a uniform $C^4$-domain, which includes the cases of a bounded domain and of an exterior domain with $C^4$-boundary. Moreover, we prove uniform a priori-estimates for the solution. The proof is based on the existence of $\mathcal{R}$-bounded solution operators of the corresponding generalized resolvent problem which is shown with the help of an operator-valued Fourier multiplier theorem due to Weis.

## 1 Introduction

Let $\Omega$ be a domain in the $N$-dimensional Euclidean space $\mathbb{R}^N$ with boundary $\Gamma$. In the present paper we consider the linearized thermoelastic plate equations given by

$$\begin{aligned} u_{tt} + \Delta^2 u + \Delta\theta &= f_1 \quad \text{in } (0,\infty)\times\Omega, \\ \theta_t - \Delta\theta - \Delta u_t &= f_2 \quad \text{in } (0,\infty)\times\Omega \end{aligned} \tag{1-1}$$

with initial conditions

$$\begin{aligned} u|_{t=0} &= u_0 \quad \text{in } \Omega, \\ u_t|_{t=0} &= u_1 \quad \text{in } \Omega, \\ \theta|_{t=0} &= \theta_0 \quad \text{in } \Omega. \end{aligned} \tag{1-2}$$

In (1-1)–(1-2), we omit all physical constants for simplicity of presentation. These equations model the behaviour of a thin plate with the elastic properties being influenced by the temperature (see, e.g., [14]). In (1-1), $u = u(t,x)$ stands for the vertical displacement at time $t$ and at the point $x = (x_1,\dots,x_N) \in \Omega$ while $\theta = \theta(t,x)$ describes the temperature relative to a constant reference temperature. For (1-1), several boundary conditions are of interest, see, e.g. [15] for a survey on physically relevant boundary conditions. In the present paper, we consider free boundary conditions which are given by

$$\begin{aligned} \Delta u - (1-\beta)\Delta' u + \theta &= g_1 \quad \text{on } (0,\infty)\times\Gamma, \\ \partial_\nu\big(\Delta u + (1-\beta)\Delta' u + \theta\big) &= g_2 \quad \text{on } (0,\infty)\times\Gamma, \\ \partial_\nu\theta &= g_3 \quad \text{on } (0,\infty)\times\Gamma. \end{aligned} \tag{1-3}$$

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Partially supported by JSPS Grant-in-aid for Scientific Research (S) # 24224004 and Top Global University Project
**Subjectclass(2010)** : 35K35; 35J40; 42B15
**Keywords**: Thermoelastic plate equations; generation of analytic semigroups; maximal $L_p$-$L_q$ regularity; $\mathcal{R}$-bounded solution operator, operator-valued Fourier multipliers

In (1-3), $\beta \in [0, 1)$ is a parameter which is fixed throughout this paper, $\Delta$ and $\Delta'$ stand for the Laplace operator in $\Omega$ and the Laplace-Beltrami operator on $\Gamma$, respectively, and $\partial_\nu$ denotes the derivative in outer normal direction. Note that the term $\theta$ can be omitted in the second line of (1-3) if we replace $g_2$ by $g_2 - g_3$.

There is a rich literature on the thermoelastic plate equations under various kinds of boundary conditions. Exponential stability of the associated semigroup in $L_2$ (in the case of a bounded domain) has been proved by Kim [12], Munos Rivera and Racke [22], Liu and Zheng [21], Lasiecka and Triggiani [15] –[16], and Shibata [26]. For a survey on general von Karman evolution equations, we refer to Chuesov and Lasiecka [2]. It turns out that the generated semigroup is even analytic, see also Liu and Renardy [20], Liu and Liu [18], and Liu and Yong [19] in the $L_2$-setting. This means that the effect from the heat equation in $\theta$ is strong enough to obtain analytic behaviour of the whole system although the first equation in (1-1) is a simply dispersive equation (the product of two Schrödinger equations) with respect to $u$.

Most of the results mentioned above are obtained in an $L_2$-setting, where energy methods are available. However, as the original equations modelling thermoelastic plates are non-linear, an $L_p$-approach is also relevant in order to handle equations with low regularity of the data. Therefore, several results on (1-1) in $L_p$-spaces were obtained. In the whole-space case, analyticity of the generated semigroup in $L_p$ was shown by Denk and Racke [4]. In the case of the half-space and of bounded domains, equations (1-1) with Dirichlet boundary conditions

$$u = \partial_\nu u = \theta = 0 \quad \text{on } (0, \infty) \times \Gamma$$

were studied by Naito and Shibata [24] and by Naito [23]. It was shown that in $L_p$ an analytic $C^0$-semigroup is generated and that even maximal $L_p$-$L_q$-regularity holds which is the key property for the analysis of the non-linear equations. By Denk, Racke and Shibata [5], [6] energy estimates for the generated semigroup in $L_q$ were shown.

The proof of maximal $L_p$-regularity for the linearized system and a rather complete analysis of the non-linear thermoelastic plate equations can be found in a recent paper by Lasiecka and Wilke [17]. In that paper, the boundary conditions

$$u = \Delta u = \theta = 0 \quad \text{on } (0, \infty) \times \Gamma$$

are studied. From a mathematical point of view, these boundary conditions are easier to handle. This is due to the fact that the operator $\Delta^2$ appearing in the first line of (1-1) can then be interpreted as the square of the Dirichlet-Laplace operator, and solvability of (1-1) can be shown by abstract operator-theoretic methods. For the boundary conditions (1-3) studied in the present paper, such an abstract approach seems to be not available, and one needs a thorough analysis of the (localized) solution operators.

The purpose of this paper is to prove maximal $L_p$-$L_q$-regularity of the initial boundary value problem (1-1)-(1-3). In our approach, setting $v = u_t$ we rewrite (1-1) as a first-order system acting on $U = (u, v, \theta)^\top$, where $M^\top$ denotes the transposed of $M$, and being of the form

$$U_t - A(D)U = F \quad \text{in } (0, T) \times \Omega, \quad B(D)U = G \quad \text{on } (0, T) \times \Gamma, \quad U|_{t=0} = U_0 \quad \text{in } \Omega \tag{1-4}$$

with operator-matrices $A(D)$ and $B(D)$ being defined by

$$A(D) := \begin{pmatrix} 0 & 1 & 0 \\ -\Delta^2 & 0 & -\Delta \\ 0 & \Delta & \Delta \end{pmatrix}, \quad B(D) := \begin{pmatrix} \Delta - (1-\beta)\Delta' & 0 & 1 \\ \partial_\nu(\Delta + (1-\beta)\Delta') & 0 & 0 \\ 0 & 0 & \partial_\nu \end{pmatrix}.$$

Setting $F = (0, f_1, f_2)^\top$, $U_0 = (u_0, u_1, \theta_0)^\top$, $G = (g_1, g_2, g_3)^\top$, and $U = (u, u_t, \theta)^\top$ in (1-4) represents the equations (1-1)-(1-3). To prove maximal $L_p$-$L_q$-regularity of problem (1-4), we prove the existence of an $\mathcal{R}$-bounded solution operator of the problem:

$$\lambda U - A(D)U = F \quad \text{in } \Omega, \quad B(D)U = G \quad \text{on } \Gamma \tag{1-5}$$

with $F = (0, f_1, f_2)^\top \in L_q(\Omega)^2$ and $G = (g_1, g_2, g_3)^\top \in H_q^2(\Omega) \times H_q^1(\Omega)^2$, which is the generalized resolvent problem corresponding to problem (1-4).

To state the main results pricesely, at this point we introduce several symbols used throughout the paper. $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of all natural numbers, real numbers, and complex numbers, respectively. Set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For any multi-index $\kappa = (\kappa_1, \ldots, \kappa_N) \in \mathbb{N}_0^N$, we write $|\kappa| = \kappa_1 + \cdots + \kappa_N$ and $\partial_x^\kappa = \partial_1^{\kappa_1} \cdots \partial_N^{\kappa_N}$ with $\partial_i = \partial/\partial x_i$. For any scalar function $f$ and vector-valued function $\mathbf{g} = (g_1, \ldots, g_k)$, let

$$\nabla f = (\partial_1 f, \ldots, \partial_N f), \ \ \nabla^\ell f = (\partial_x^\alpha f \mid |\alpha| = \ell),$$
$$\nabla \mathbf{g} = (\partial_i g_j \mid i = 1, \ldots, N, j = 1, \ldots, k), \ \ \nabla^\ell \mathbf{g} = (\nabla^\ell g_1, \ldots, \nabla^\ell g_k).$$

For any domain $D$, let $L_q(D)$, $H_q^m(D)$ $(m \in \mathbb{N})$ and $B_{q,p}^s(D)$ $(s \in (0, \infty) \setminus \mathbb{N})$ be the Lebesgue space, Sobolev space and Besov space, while $\|\cdot\|_{L_q(D)}$, $\|\cdot\|_{H_q^m(D)}$ and $\|\cdot\|_{B_{q,p}^s(D)}$ denote their norms, respectively. We write $H_q^0(D) = L_q(D)$ and $B_{qq}^s(D) = W_q^s(D)$. Let $X$ and $Y$ be Banach spaces, and let $\mathcal{L}(X, Y)$ be the space of all bounded linear operators from $X$ to $Y$. We use the abbreviation $\mathcal{L}(X) = \mathcal{L}(X, X)$. For an interval $J = (0, T)$ with $T \in (0, \infty]$, $L_p(J, X)$ denotes the $X$-valued Lebesgue space and $H_p^m(J, X)$ $(m \in \mathbb{N})$ the $X$-valued Sobolev space, while $\|\cdot\|_{L_p(J,X)}$ and $\|\cdot\|_{H_p^m(J,X)}$ denote their norms, respectively. For any domain $V$ in $\mathbb{C}$, $\mathcal{C}(V, X)$ denotes the set of all $X$-valued functions $f = f(\lambda)$ defined for $\lambda = \gamma + i\tau \in V$ which are continuously differentiable with respect to $\tau$ when $\lambda \in V$. Let $\Sigma_\vartheta$ and $\Sigma_{\vartheta,\lambda_0}$ be the sets in $\mathbb{C}$ defined by

$$\Sigma_\vartheta := \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg \lambda| < \vartheta\}, \quad \Sigma_{\vartheta,\lambda_0} := \{\lambda \in \Sigma_\vartheta \mid |\lambda| \geq \lambda_0\} \tag{1-6}$$

for any $0 < \vartheta \leq \pi$ and $\lambda_0 > 0$. Let $X^d = \{\mathbf{f} = (f_1, \ldots, f_d)^\top \mid f_i \in X \ \ (i = 1, \ldots, d)\}$, while the norm of $X^d$ is written by $\|\cdot\|_X$ instead of $\|\cdot\|_{X^d}$ for short. In particular, we write

$$\|G\|_{H_q^m(D) \times H_q^\ell(D)^2} = \|g_1\|_{H_q^m(D)} + \|(g_2, g_3)\|_{H_q^\ell(D)} \quad \text{for } G = (g_1, g_2, g_3)^\top \in H_q^m(D) \times H_q^\ell(D)^2.$$

Let

$$\begin{aligned}
&\mathbb{G}_q(D) := \{(F, G) \mid F = (0, f_1, f_2)^\top, \ \ (f_1, f_2) \in L_q(D)^2, \ \ G = (g_1, g_2, g_3)^\top \in H_q^2(D) \times H_q^1(D)^2\},\\
&\|(F, G)\|_{\mathbb{G}_q(D)} := \|(f_1, f_2)\|_{L_q(D)} + \|G\|_{H_q^2(D) \times H_q^1(D)^2},\\
&\mathcal{X}_q(\Omega) := \{\mathcal{H} = (F', G, G', g_1'') \mid F' = (f_1, f_2)^\top \in L_q(\Omega)^2,\\
&\qquad G = (g_1, g_2, g_3)^\top \in H_q^2(\Omega) \times H_q^1(\Omega)^2, G' = (g_1', g_2', g_3')^\top \in H_q^1(\Omega) \times L_q(\Omega)^2, \ \ g_1'' \in L_q(\Omega)\},\\
&\|\mathcal{H}\|_{\mathcal{X}_q(D)} := \|(f_1, f_2)\|_{L_q(D)} + \|G\|_{H_q^2(D) \times H_q^1(D)} + \|G'\|_{H_q^1(D) \times L_q(D)^2} + \|g_1''\|_{L_q(D)}.
\end{aligned} \tag{1-7}$$

For any $\lambda \in \mathbb{C}$ and $(F, G) \in \mathbb{G}_q(D)$, let $H_\lambda(F, G) := (f_1, f_2, G, \lambda^{1/2} G, \lambda g_1)$ with $F = (0, f_1, f_2)^\top$ and $G = (g_1, g_2, g_3)^\top$. In particular, $G'$ and $g_1''$ are the corresponding variables to $\lambda^{1/2} G$ and $\lambda g_1$. For any exponent $q \in (1, \infty)$, let $q' = q/(q-1)$ be the dual exponent of $q$. The letters $C$ and $c$ denote generic positive constants and the constant $C_{a,b,\ldots}$ depends on $a$, $b$, .... The values of the constants $C$, $c$ and $C_{a,b,\ldots}$ may change from line to line.

Next, we introduce two definitions (see, e.g., [3], [13]).

**Definition 1.1.** A family $\mathcal{T} \subset \mathcal{L}(X, Y)$ of operators is called *$\mathcal{R}$-bounded* if for one (and then all) $p \in [1, \infty)$ there exists a constant $C > 0$ such that for all $m \in \mathbb{N}$, $(T_k)_{k=1,\ldots,m} \subset \mathcal{T}$, and $(x_k)_{k=1,\ldots,m} \subset X$ we have

$$\Big\| \sum_{k=1}^m r_k T_k x_k \Big\|_{L_p([0,1],Y)} \leq C \Big\| \sum_{k=1}^m r_k x_k \Big\|_{L_p([0,1],X)}.$$

Here the Rademacher functions $r_k$, $k \in \mathbb{N}$, are given by $r_k \colon [0,1] \to \{-1, 1\}, t \mapsto \operatorname{sign}(\sin(2^k \pi t))$. The smallest such $C$ is called the $\mathcal{R}$-bound of $\mathcal{T}$ on $\mathcal{L}(X, Y)$ which is written by $\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})$ in what follows. Note that we omit the dependence on $p$ in the notation of the $\mathcal{R}$-bound.

**Definition 1.2.** A domain $\Omega$ is called a uniform $C^4$-domain if there exist positive constants $\alpha$, $\beta$ and $K$ such that for any $x_0 \in \Gamma$ there exist a coordinate number $j$ and a $C^4$-function $h(x')$ defined on $B'_\alpha(x'_0)$ such that $\|h\|_{H_\infty^4(B'_\alpha(x'_0))} \leq K$ and

$$\Omega \cap B_\beta(x_0) = \{x \in \mathbb{R}^N \mid x_j > h(x') \ \ (x' \in B'_\alpha(x'_0))\} \cap B_\beta(x_0),$$
$$\Gamma \cap B_\beta(x_0) = \{x \in \mathbb{R}^N \mid x_j = h(x') \ \ (x' \in B'_\alpha(x'_0))\} \cap B_\beta(x_0).$$

Here, $x'$ has been defined by $x' = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_N)$ for $x = (x_1, \ldots, x_N)$,

$$B'_\alpha(x'_0) = \{x' \in \mathbb{R}^{N-1} \mid |x' - x'_0| < \alpha\}, \quad B_\beta(x_0) = \{x \in \mathbb{R}^N \mid |x - x_0| < \beta\}.$$

In what follows, $\Omega$ is assumed to be a uniform $C^4$-domain. Let $\iota : L_{1,\mathrm{loc}}(\Omega) \to L_{1,\mathrm{loc}}(\mathbb{R}^N)$ be an extension operator possessing the following properties:

(e-1) For any $1 < q < \infty$ and $f \in H^i_q(\Omega)$ we have $\iota f \in H^i_q(\mathbb{R}^N)$, $\iota f = f$ in $\Omega$ and $\|\iota f\|_{H^i_q(\mathbb{R}^N)} \leq C_q \|f\|_{H^i_q(\Omega)}$ for $i = 0, , \ldots, 4$.

(e-2) For any $1 < q < \infty$ and $f \in H^1_q(\Omega)$, $\|(1 - \Delta)^{-1/2}\iota(\nabla f)\|_{L_q(\mathbb{R}^N)} \leq C_q \|f\|_{L_q(\Omega)}$.

Here, $(1-\Delta)^{-1/2}$ is the operator defined by $(1-\Delta)^{-1/2} f = \mathscr{F}^{-1}[(1+|\xi|^2)^{-1/2}\mathscr{F} f]$ with the help of the Fourier transform $\mathscr{F}$ and its inverse transform $\mathscr{F}^{-1}$ defined by

$$(\mathscr{F}\varphi)(\xi) = \int_{\mathbb{R}^n} \varphi(x) e^{-ix\xi} dx \quad (\xi \in \mathbb{R}^n), \quad (\mathscr{F}^{-1}\varphi)(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \varphi(\xi) e^{ix\xi} d\xi \quad (x \in \mathbb{R}^n).$$

For the existence of such an extension operator, we refer, e.g., to Schade and Shibata [25, Appendix A]. In what follows, such $\iota$ is fixed. Let $\mathbf{W}^{-1}_q(\Omega)$ be the space defined by

$$\mathbf{W}^{-1}_q(\Omega) = \big\{f \in L_{1,\mathrm{loc}}(\Omega) \mid \|f\|_{\mathbf{W}^{-1}_q(\Omega)} = \|(1-\Delta)^{-1/2}\iota f\|_{L_q(\mathbb{R}^N)} < \infty\big\}.$$

Finally, we state the main results of this paper.

**Theorem 1.3** (Maximal $L_p$-$L_q$-regularity)**.** *Let $T > 0$. Let $1 < p, q < \infty$. Assume that $\Omega$ is a uniform $C^4$-domain in $\mathbb{R}^N$. Then, there exists a number $\lambda_1$ such that for any initial data $U_0 = (u_0, u_1, \theta_0)^\top \in B^{4-2/p}_{q,p}(\Omega) \times B^{2-2/p}_{q,p}(\Omega)^2$, right-hand side $F = (0, f_1, f_2)^\top$ with $(f_1, f_2)^\top \in L_p((0,T), L_q(\Omega)^2)$ and boundary data $G = (g_1, g_2, g_3)^\top$ with*

$$G \in H^1_p((0,T), L_q(\Omega) \times \mathbf{W}^{-1}_q(\Omega)^2) \cap L_p((0,T), H^2_q(\Omega) \times H^1_q(\Omega)^2)$$

*satisfying the compatibility condition: $G|_{t=0} = B(D)U_0$ on $\Omega$, problem* (1-4) *admits a unique solution $U = (u, u_t, \theta)^\top$ with*

$$u \in \bigcap_{\ell=0}^{2} H^\ell_p((0,T), H^{4-2\ell}_q(\Omega)), \quad \theta \in \bigcap_{\ell=0}^{1} H^\ell_p((0,T), H^{2-2\ell}_q(\Omega))$$

*possessing the estimate:*

$$\sum_{\ell=0}^{2} \|\partial_t^\ell u\|_{L_p((0,T),H^{4-2\ell}_q(\Omega))} + \sum_{\ell=0}^{1} \|\partial_t^\ell \theta\|_{L_p((0,T),H^{2-2\ell}_q(\Omega))} \leq e^{\gamma T}\Big\{\|u_0\|_{B^{4-2/p}_{q,p}(\Omega)} + \|\theta_0\|_{B^{2-2/p}_{q,p}(\Omega)}$$
$$+ \|(f_1, f_2)\|_{L_p((0,T),L_q(\Omega)^2)} + \|G\|_{L_p((0,T),H^2_q(\Omega)\times H^1_q(\Omega)^2)} + \|\partial_t G\|_{L_p((0,T),L_q(\Omega)\times \mathbf{W}^{-1}_q(\Omega)^2)}\Big\}$$

*with some positive constant $\gamma > 0$ independent of $T$.*

Concerning the compatibility conditions, we remark that $G|_{t=0}$ and $B(D)U_0$ both belong to the space $B^{2-2/p}_{qp}(\Omega) \times (B^{1-2/p}_{pq}(\Omega))^2$ in the case $p > 2$. For simplicity, we assume that the compatibility conditions holds in $\Omega$. In view of the nonlinear equation, we typically assume $2 < p < \infty$, $N < q < \infty$, and $2/p + N/q < 1$. In this case, the traces of $G|_{t=0}$ and $B(D)U_0$ on the boundary exist, and the compatibility condition can be formulated with respect to these traces.

In this paper, to prove Theorem 1.3, we prove the existence of $\mathcal{R}$-bounded solution operators associated with problem (1-5). Namely, we prove

**Theorem 1.4** (Existence of $\mathcal{R}$-bounded solution operators)**.** *Let $1 < q < \infty$. Assume that $\Omega$ is a uniform $C^4$-domain in $\mathbb{R}^N$. Let $\mathbb{G}_q(\Omega)$ and $\mathcal{X}_q(\Omega)$ be defined as in* (1-7)*. Then, there exist a number $\vartheta > \pi/2$, a positive number $\lambda_0$, and an operator family $\mathcal{B}_i(\lambda)$ $(i = 1, 2)$ with*

$$\mathcal{B}_1(\lambda) \in \mathcal{C}(\Sigma_{\vartheta,\lambda_0}, \mathcal{L}(\mathcal{X}_q(\Omega), H^4_q(\Omega))), \quad \mathcal{B}_2(\lambda) \in \mathcal{C}(\Sigma_{\vartheta,\lambda_0}, \mathcal{L}(\mathcal{X}_q(\Omega), H^2_q(\Omega))),$$

*such that problem* (1-5) *admits a unique solution $U = \mathcal{B}(\lambda)\mathcal{H}_\lambda(F, G)$ with $\mathcal{B}(\lambda) = (\mathcal{B}_1(\lambda), \lambda\mathcal{B}_1(\lambda), \mathcal{B}_2(\lambda))^\top$ for any $(F, G) \in \mathbb{G}_q(\Omega)$ and $\lambda \in \Sigma_{\vartheta,\lambda_0}$, where $\mathcal{H}_\lambda(F, G) = (f_1, f_2, G, \lambda^{1/2}G, \lambda g_1)$ for $F = (0, f_1, f_2)^\top$ and $G = (g_1, g_2, g_3)^\top$, and there hold the estimates:*

$$\begin{aligned} &\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega), H^{4-j}_q(\Omega))}(\{(\tau\partial_\tau)^s(\lambda^{j/2}\mathcal{B}_1(\lambda)) \mid \lambda \in \Sigma_{\vartheta,\lambda_0}\}) \le C \quad (s = 0, 1, \ \ j = 0, 1, 2, 3, 4), \\ &\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega), H^{2-j}_q(\Omega))}(\{(\tau\partial_\tau)^s(\lambda^{j/2}\mathcal{B}_2(\lambda)) \mid \lambda \in \Sigma_{\vartheta,\lambda_0}\}) \le C \quad (s = 0, 1, \ \ j = 0, 1, 2). \end{aligned} \tag{1-8}$$

## 2 Analysis in the whole space

The purpose of this section is to prove the existence of an $\mathcal{R}$-bounded solution operator associated with the resolvent problem:

$$\lambda U - A(D)U = F \quad \text{in } \mathbb{R}^N \tag{2-1}$$

with $F = (0, f_1, f_2)^\top$. One main tool for the proof is the following lemma due to Denk and Schnaubelt [7, Lemma 2.1] and Enomoto and Shibata [9, Theorem 3.3].

**Lemma 2.1.** *Let $1 < q < \infty$ and let $\Lambda$ be a set in $\mathbb{C}$. Let $m = m(\lambda, \xi)$ be a function defined on $\Lambda \times (\mathbb{R}^N \setminus \{0\})$ which is infinitely differentiable with respect to $\xi \in \mathbb{R}^N \setminus \{0\}$ for each $\lambda \in \Lambda$. Assume that for any multi-index $\alpha \in \mathbb{N}_0^N$ there exists a constant $C_\alpha$ depending on $\alpha$ and $\Lambda$ such that*

$$|\partial_\xi^\alpha m(\lambda, \xi)| \le C_\alpha |\xi|^{-|\alpha|} \tag{2-2}$$

*for any $(\lambda, \xi) \in \Lambda \times (\mathbb{R}^N \setminus \{0\})$. Let $K_\lambda$ be an operator defined by $K_\lambda f = \mathscr{F}_\xi^{-1}[m(\lambda, \xi)\mathscr{F} f(\xi)]$. Then, the family of operators $\{K_\lambda \mid \lambda \in \Lambda\}$ is $\mathcal{R}$-bounded on $\mathcal{L}(L_q(\mathbb{R}^N))$ and*

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N))}(\{K_\lambda \mid \lambda \in \Lambda\}) \le C_{q,N} \max_{|\alpha| \le N+1} C_\alpha \tag{2-3}$$

*with some constant $C_{q,N}$ depending only on $q$ and $N$.*

The symbol of the operator matrix $A(D)$ is given by

$$A(\xi) := \begin{pmatrix} 0 & 1 & 0 \\ -|\xi|^4 & 0 & |\xi|^2 \\ 0 & -|\xi|^2 & -|\xi|^2 \end{pmatrix},$$

i.e., we have $A(D) = \mathscr{F}^{-1}A(\cdot)\mathscr{F}$. Thus, by the Fourier transform equation (2-1) is transformed to

$$(\lambda I - A(\xi))\hat{U}(\xi) = \hat{F}(\xi) \tag{2-4}$$

with $\hat{U} = (\mathscr{F} u, \mathscr{F} v, \mathscr{F} \theta)^\top$ and $\hat{F} = (0, \mathscr{F} f_1, \mathscr{F} f_2)^\top$. The analysis of the inverse matrix $(\lambda I - A(\xi))^{-1}$ was essentially done in [24]. As we need some variants of the results in [24], we summarize the main properties of $(\lambda I - A(\xi))^{-1}$ and give a short indication of the proofs.

In the following, define $\gamma_1, \gamma_2, \gamma_3$ by the equality

$$p(t) := t^3 + t^2 + 2t + 1 = (t + \gamma_1)(t + \gamma_2)(t + \gamma_3) \tag{2-5}$$

with $\gamma_1 \in \mathbb{R}$, $\gamma_2 = \bar{\gamma}_3$ and $\operatorname{Im} \gamma_2 > 0$. Then $\gamma_1 \in (0, 1)$, $\operatorname{Re} \gamma_2 = \operatorname{Re} \gamma_3 \in (0, \frac{1}{2})$, and $\det(\lambda - A(\xi)) = \prod_{j=1}^3 (\lambda + \gamma_j |\xi|^2)$ (see [24, Lemma 2.3]). Let $\vartheta_0 > \pi/2$ and $\vartheta_1 > \pi/2$ be chosen in such a way that

$$\lambda \gamma_i^{-1} \in \Sigma_{\vartheta_1} \quad (i = 1, 2, 3) \quad \text{for } \lambda \in \Sigma_{\vartheta_0}. \tag{2-6}$$

Then the inequality

$$|\lambda\gamma_i^{-1} + |\xi|^2| \geq c(|\lambda| + |\xi|^2) \tag{2-7}$$

holds for any $\lambda \in \Sigma_{\vartheta_0}$ and $\xi \in \mathbb{R}^N$ with some positive constant $c$.

It was shown in [24, Section 2] that for all $\lambda \in \Sigma_{\vartheta_0}$ we have

$$(\lambda I - A(\xi))^{-1} = \frac{1}{\det(\lambda I - A(\xi))} \begin{pmatrix} \lambda(\lambda + |\xi|^2) + |\xi|^4 & \lambda + |\xi|^2 & |\xi|^2 \\ -(\lambda + |\xi|^2)|\xi|^4 & \lambda(\lambda + |\xi|^2) & \lambda|\xi|^2 \\ |\xi|^6 & -\lambda|\xi|^2 & \lambda^2 + |\xi|^4 \end{pmatrix}.$$

Since $\gamma_1\gamma_2\gamma_3 = 1$ as follows from Vieta's formula, we have $\det(\lambda I - A(\xi)) = \prod_{i=1}^3(\lambda\gamma_i^{-1} + |\xi|^2)$, and then a solution $U = (u, \lambda u, \theta)^\top$ of problem (2-1) is given by $U = \mathscr{F}^{-1}\big((\lambda I - A(\xi))^{-1}\mathscr{F}f(\xi)\big)$, i.e.,

$$\begin{aligned} u(x) &= \mathscr{F}^{-1}\Big[\frac{\lambda + |\xi|^2}{\prod_{i=1}^3(\lambda\gamma_i^{-1} + |\xi|^2)}\mathscr{F}f_1(\xi)\Big] + \mathscr{F}^{-1}\Big[\frac{|\xi|^2}{\prod_{i=1}^3(\lambda\gamma_i^{-1} + |\xi|^2)}\mathscr{F}f_2(\xi)\Big], \\ \theta(x) &= -\mathscr{F}^{-1}\Big[\frac{\lambda|\xi|^2}{\prod_{i=1}^3(\lambda\gamma_i^{-1} + |\xi|^2)}\mathscr{F}f_1(\xi)\Big] + \mathscr{F}^{-1}\Big[\frac{\lambda^2 + |\xi|^4}{\prod_{i=1}^3(\lambda\gamma_i^{-1} + |\xi|^2)}\mathscr{F}f_2(\xi)\Big]. \end{aligned} \tag{2-8}$$

Let the operator $\mathcal{S}_0(\lambda)$ acting on $f$ be defined by

$$\mathcal{S}_0(\lambda)f := \mathscr{F}^{-1}\Big[\Big(\prod_{i=1}^3(\lambda\gamma_i + |\xi|^2)\Big)^{-1}\mathscr{F}f(\xi)\Big].$$

By Lemma 2.1 and (2-7),

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N))}\Big(\Big\{(\tau\partial_\tau)^s(\lambda^{j/2}\partial_x^\alpha\mathcal{S}_0(\lambda)) \mid \lambda \in \Sigma_{\vartheta_0}\Big\}\Big) \leq C_{j,\alpha} \quad (s = 0, 1) \tag{2-9}$$

for $j \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0^N$ with $j + |\alpha| = 6$. Moreover, by (2-8) $u = \mathcal{S}_1(\lambda)F$ and $\theta = \mathcal{S}_2(\lambda)F$ for $F = (f_1, f_2)^\top$ with

$$\begin{aligned} \mathcal{S}_1(\lambda)F &:= (\lambda - \Delta)\mathcal{S}_0(\lambda)f_1 - \Delta S_0(\lambda)f_2, \\ \mathcal{S}_2(\lambda)F &:= \lambda\Delta\mathcal{S}_0(\lambda)f_1 + (\lambda^2 + \Delta^2)\mathcal{S}_0(\lambda)f_2. \end{aligned}$$

At this point, we introduce some fundamental properties of $\mathcal{R}$-bounded operators and Bourgain's results concerning Fourier multiplier theorems with scalar multiplier.

**Proposition 2.2.** *a) Let $X$ and $Y$ be Banach spaces, and let $\mathcal{T}$ and $\mathcal{S}$ be $\mathcal{R}$-bounded families in $\mathcal{L}(X, Y)$. Then, $\mathcal{T} + \mathcal{S} = \{T + S \mid T \in \mathcal{T},\ S \in \mathcal{S}\}$ is also an $\mathcal{R}$-bounded family in $\mathcal{L}(X, Y)$ and*

$$\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T} + \mathcal{S}) \leq \mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T}) + \mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{S}).$$

*b) Let $X$, $Y$ and $Z$ be Banach spaces, and let $\mathcal{T}$ and $\mathcal{S}$ be $\mathcal{R}$-bounded families in $\mathcal{L}(X, Y)$ and $\mathcal{L}(Y, Z)$, respectively. Then, $\mathcal{S}\mathcal{T} = \{ST \mid T \in \mathcal{T},\ S \in \mathcal{S}\}$ also an $\mathcal{R}$-bounded family in $\mathcal{L}(X, Z)$ and*

$$\mathcal{R}_{\mathcal{L}(X,Z)}(\mathcal{S}\mathcal{T}) \leq \mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})\mathcal{R}_{\mathcal{L}(Y,Z)}(\mathcal{S}).$$

*c) Let $1 < p, q < \infty$ and let $D$ be a domain in $\mathbb{R}^N$. Let $m = m(\lambda)$ be a bounded function defined on a subset $\Lambda$ in $\mathbb{C}$ and let $M_m(\lambda)$ be a map defined by $M_m(\lambda)f = m(\lambda)f$ for any $f \in L_q(D)$. Then, $\mathcal{R}_{\mathcal{L}(L_q(D))}(\{M_m(\lambda) \mid \lambda \in \Lambda\}) \leq C_{N,q,D}\|m\|_{L_\infty(\Lambda)}$.*

*d) Let $n = n(\tau)$ be a $C^1$-function defined on $\mathbb{R} \setminus \{0\}$ that satisfies the conditions $|n(\tau)| \leq \gamma$ and $|\tau n'(\tau)| \leq \gamma$ with some constant $c > 0$ for any $\tau \in \mathbb{R} \setminus \{0\}$. Let $T_n$ be the operator-valued Fourier multiplier defined by $T_n f = \mathscr{F}^{-1}(n\mathscr{F}[f])$ for any $f$ with $\mathscr{F}[f] \in \mathscr{D}(\mathbb{R}, L_q(D))$. Then, $T_n$ is extended to a bounded linear operator from $L_p(\mathbb{R}, L_q(D))$ into itself. Moreover, denoting this extension also by $T_n$, we have*

$$\|T_n\|_{\mathcal{L}(L_p(\mathbb{R}, L_q(D)))} \leq C_{p,q,D}\gamma.$$

*Here, $\mathscr{D}(\mathbb{R}, L_q(D))$ denotes the set of all $L_q(D)$-valued $C^\infty$-functions on $\mathbb{R}$ with compact support.*

*Proof.* The assertions a) and b) follow from [3, p.28, Proposition 3.4], and the assertions c) and d) follow from [3, p.27, Remarks 3.2] (see also Bourgain [1]). □

Since

$$\Big|\partial_\xi^\alpha (\tau\partial_\tau)^s\Big(\frac{\lambda^{j/2}(i\xi)^\beta}{\prod_{i=1}^3(\lambda\gamma_i^{-1}+|\xi|^2)}\Big)\Big| \le C_\alpha|\xi|^{-|\alpha|}$$

for any $s \in \{0,1\}$, $j \in \mathbb{N}_0$ and $\beta \in \mathbb{N}_0^N$ with $j+|\beta| = 6$ and $(\lambda,\xi) \in \Sigma_{\vartheta_0} \times (\mathbb{R}^N \setminus \{0\})$ as follows from (2-7), by Lemma 2.1 and Proposition 2.2 a),

$$\begin{aligned}
&\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N))}(\{(\tau\partial_\tau)^s(\lambda^{j/2}\partial_x^\alpha(\lambda-\Delta)\mathcal{S}_0(\lambda)) \mid \lambda \in \Sigma_{\vartheta_0}\}) \le \gamma_0 \quad (s=0,1,\ \ j=0,1,2,3,4\ \ j+|\alpha|=4),\\
&\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N))}(\{(\tau\partial_\tau)^s(\lambda^{j/2}\partial_x^\alpha\Delta\mathcal{S}_0(\lambda)) \mid \lambda \in \Sigma_{\vartheta_0}\}) \le \gamma_0 \quad (s=0,1,\ \ j=0,1,2,3,4,\ \ j+|\alpha|=4),\\
&\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N))}(\{(\tau\partial_\tau)^s(\lambda^{j/2}\partial_x^\alpha(\lambda\Delta\mathcal{S}_0(\lambda))) \mid \lambda \in \Sigma_{\vartheta_0}\}) \le \gamma_0 \quad (s=0,1,\ \ j=0,1,2,\ \ j+|\alpha|=2),\\
&\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N))}(\{(\tau\partial_\tau)^s(\lambda^{j/2}\partial_x^\alpha(\lambda^2+\Delta^2)\mathcal{S}_0(\lambda)) \mid \lambda \in \Sigma_{\vartheta_0}\}) \le \gamma_0 \quad (s=0,1,\ \ j=0,1,2,\ \ j+|\alpha|=2)
\end{aligned}$$

for some constant $\gamma_0 > 0$. Combined with Proposition 2.2 c), this yields the following result.

**Theorem 2.3.** *Let $1 < q < \infty$ and $\lambda_0 > 0$. Let $\vartheta_0$ be the number given in* (2-6)*. Then, there exist operator families $\mathcal{S}_i(\lambda)$ $(i = 1,2)$ with*

$$\mathcal{S}_1 \in \mathcal{C}(\Sigma_{\vartheta_0,\lambda_0}, \mathcal{L}(L_q(\mathbb{R}^N)^2, H_q^4(\mathbb{R}^N))), \quad \mathcal{S}_2 \in \mathcal{C}(\Sigma_{\vartheta_0,\lambda_0}, \mathcal{L}(L_q(\mathbb{R}^N)^2, H_q^2(\mathbb{R}^N)))$$

*such that problem* (2-1) *admits a unique solution $U = (\mathcal{S}_1(\lambda)F', \lambda\mathcal{S}_1(\lambda)F', \mathcal{S}_2(\lambda)F')^\top$ for any $\lambda \in \Sigma_{\vartheta_0,\lambda_0}$ and $F = (0, f_1, f_2)^\top$ with $F' = (f_1,f_2)^\top \in L_q(\mathbb{R}^N)^2$, and there hold the estimates:*

$$\begin{aligned}
&\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N)^2, H_q^{4-j}(\mathbb{R}^N))}\big(\{(\tau\partial_\tau)^s(\lambda^{j/2}\mathcal{S}_1(\lambda)) \mid \lambda \in \Sigma_{\vartheta_0,\lambda_0}\}\big) \le C_{\lambda_0}\gamma_0 \quad (s=0,1,\ \ j=0,1,2,3,4),\\
&\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N)^2, H_q^{2-j}(\mathbb{R}^N))}\big(\{(\tau\partial_\tau)^s(\lambda^{j/2}\mathcal{S}_2(\lambda)) \mid \lambda \in \Sigma_{\vartheta_0,\lambda_0}\}\big) \le C_{\lambda_0}\gamma_0 \quad (s=0,1,\ \ j=0,1,2),
\end{aligned}$$

*with some constant $C_{\lambda_0} > 0$.*

# 3 Solution operators in the half-space

Let $\mathbb{R}_+^N = \{x = (x_1,\dots,x_N) \in \mathbb{R}^N \mid x_N > 0\}$ and $\mathbb{R}_0^N = \{x = (x_1,\dots,x_N) \in \mathbb{R}^N \mid x_N = 0\}$. The purpose of this section is to prove the existence of $\mathcal{R}$-bounded solution operators of the generalized resolvent problem:

$$\lambda U - A(D)U = F \ \text{ in } \mathbb{R}_+^N, \quad B(D)U = G \ \text{ on } \mathbb{R}_0^N \tag{3-1}$$

in the half-space $\mathbb{R}_+^N$ with $U = (u, \lambda u, \theta)^\top$, $F = (0, f_1, f_2)^\top$ and $G = (g_1, g_2, g_3)^\top$. The boundary condition in (3-1) is represented componentwise by

$$\begin{aligned}
\Delta u - (1-\beta)\Delta' u + \theta &= g_1 \quad \text{on } \mathbb{R}_0^N,\\
\partial_N(\Delta u + (1-\beta)\Delta' u) &= g_2 \quad \text{on } \mathbb{R}_0^N,\\
\partial_N\theta &= g_3 \quad \text{on } \mathbb{R}_0^N.
\end{aligned} \tag{3-2}$$

Here, $\Delta' = \sum_{j=1}^{N-1}\partial_j^2$. Then, the main result of this section is

**Theorem 3.1.** *Let $1 < q < \infty$ and $\lambda_0 > 0$. Then, there exist a number $\vartheta > \pi/2$ and operator families $\mathcal{T}_i(\lambda)$ $(i = 1,2)$ with*

$$\mathcal{T}_1 \in \mathcal{C}(\Sigma_\vartheta, \mathcal{L}(\mathcal{X}_q(\mathbb{R}_+^N), H_q^4(\mathbb{R}_+^N))), \quad \mathcal{T}_2 \in \mathcal{C}(\Sigma_\vartheta, \mathcal{L}(\mathcal{X}_q(\mathbb{R}_+^N), H_q^2(\mathbb{R}_+^N)))$$

*such that problem* (3-1) *admits a unique solution*

$$U = \big(\mathcal{T}_1(\lambda)H_\lambda(F,G), \lambda\mathcal{T}_1(\lambda)H_\lambda(F,G), \mathcal{T}_2(\lambda)H_\lambda(F,G)\big)^\top$$

*for any $(F,G)\in\mathbb{G}_q(\mathbb{R}^N_+)$ and $\lambda\in\Sigma_\vartheta$, and there hold the estimates:*

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\mathbb{R}^N_+),H^{4-j}_q(\mathbb{R}^N_+))}\big(\{(\tau\partial_\tau)^s(\lambda^{j/2}\mathcal{T}_1(\lambda))\mid\lambda\in\Sigma_{\vartheta,\lambda_0}\}\big)\le C_{N,q,\lambda_0}\quad(s=0,1,\ \ j=0,1,2,3,4),$$
$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\mathbb{R}^N_+),H^{2-j}_q(\mathbb{R}^N_+))}\big(\{(\tau\partial_\tau)^s(\lambda^{j/2}\mathcal{T}_2(\lambda))\mid\lambda\in\Sigma_{\vartheta,\lambda_0}\}\big)\le C_{N,q,\lambda_0}\quad(s=0,1,\ \ j=0,1,2).$$

In what follows, we prove Theorem 3.1. Let $\iota_h$ be the Lions extension operator of the form:

$$[\iota_h f](x):=\begin{cases}f(x',x_N) & (x_N>0),\\ \sum_{j=1}^6 a_j f(x',-jx_N) & (x_N<0),\end{cases}\tag{3-3}$$

for any given $f$ on $\mathbb{R}^N_+$, where $x'=(x_1,\dots,x_{N-1})$, and $a_j$ are real numbers satisfying the relations:

$$\sum_{j=1}^6(-j)^k a_j=1\quad\text{for }k=-1,0,1,\dots,4.$$

Let $\mathcal{S}(\lambda)=(\mathcal{S}_1(\lambda),\mathcal{S}_2(\lambda))^\top$ be the solution operator given in Theorem 2.3, and let the operator $\mathcal{S}_+(\lambda)=(\mathcal{S}_{+1}(\lambda),\mathcal{S}_{+2}(\lambda))^\top$ acting on $F'=(f_1,f_2)\in L_q(\mathbb{R}^N_+)^2$ be defined by

$$\mathcal{S}_{+i}(\lambda)F'=\mathcal{S}_i(\lambda)(\iota_h F')\quad(i=1,2).\tag{3-4}$$

Obviously,
$$V=\big(\mathcal{S}_{+1}(\lambda)(\iota_hF'),\lambda\mathcal{S}_{+1}(\lambda)(\iota_hF'),\mathcal{S}_{+2}(\lambda)(\iota_hF')\big)^\top$$
satisfies the equation: $\lambda V-A(D)V=F$ in $\mathbb{R}^N_+$ with $F=(0,f_1,f_2)^\top$ and the estimate:

$$\begin{aligned}&\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N_+)^2,H^{4-j}_q(\mathbb{R}^N_+))}(\{(\tau\partial_\tau)^s(\lambda^{j/2}\mathcal{S}_{+1}(\lambda))\mid\lambda\in\Sigma_{\vartheta_0,\lambda_0}\})\le C_{\lambda_0}\quad(s=0,1,\ \ j=0,1,2,3,4),\\&\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N_+)^2,H^{2-j}_q(\mathbb{R}^N_+))}(\{(\tau\partial_\tau)^s(\lambda^{j/2}\mathcal{S}_{+2}(\lambda))\mid\lambda\in\Sigma_{\vartheta_0,\lambda_0}\})\le C_{\lambda_0}\quad(s=0,1,\ \ j=0,1,2).\end{aligned}\tag{3-5}$$

Setting $U=V+W$ yields that $W$ should solve the equations (3-1) replacing $F$ and $G$ by 0 and $G-B(D)V$. Since the second component of $W$ coincides with $\lambda$ times the first component, in what follows, it suffices to consider the equations:

$$\lambda^2u+\Delta^2u+\Delta\theta=0,\quad\lambda\theta-\Delta\theta-\lambda\Delta u=0\quad\text{in }\mathbb{R}^N_+\tag{3-6}$$

with non-homogeneous boundary condition (3-2).

Applying partial Fourier transform

$$\mathscr{F}'[u](\xi',x_N):=\int_{\mathbb{R}^{N-1}}u(x',x_N)e^{-ix'\xi'}\,dx',$$

where $\xi'=(\xi_1,\dots,\xi_{N-1})$, to (3-6) and (3-2) yields an ordinary differential equation system in $x_N>0$:

$$\begin{aligned}\lambda^2w+(\partial_N^2-|\xi'|^2)^2w+(\partial_N^2-|\xi'|^2)\tau=0\quad(x_N>0),\\ \lambda\tau-(\partial_N^2-|\xi'|^2)\tau-\lambda(\partial_N^2-|\xi'|^2)w=0\quad(x_N>0)\end{aligned}\tag{3-7}$$

with initial conditions

$$\begin{aligned}(\partial_N^2-|\xi'|^2)w(0)+(1-\beta)|\xi'|^2w(0)+\tau(0)&=\mathscr{F}'[g_1](\xi',0),\\ \partial_N\big((\partial_N^2-|\xi'|^2)w(0)-(1-\beta)|\xi'|^2w(0)\big)&=\mathscr{F}'[g_2](\xi',0),\\ \partial_N\tau(0)&=\mathscr{F}'[g_3](\xi',0).\end{aligned}\tag{3-8}$$

Here we have set $w(\xi',x_N)=(\mathscr{F}'u)(\xi',x_N)$ and $\tau(\xi',x_N)=(\mathscr{F}'\theta)(\xi',x_N)$.

We find solutions $w$ and $\tau$ of (3-7)–(3-8). For this, we use the representation formula of $w$ and $\tau$ which was derived in [24, Eq. (3.15)]. There it was shown that every stable solution of (3-7) has the form

$$
\begin{aligned}
w(\xi', x_N, \lambda) &= \sum_{i=1}^{3} P_i \exp(-A_i(\xi', \lambda)x_N), \\
\tau(\xi', x_N, \lambda) &= -\lambda \sum_{i=1}^{3} (\gamma_i^2 + 2) P_i \exp(-A_i(\xi', \lambda)x_N).
\end{aligned}
\tag{3-9}
$$

Here $\gamma_1, \gamma_2, \gamma_3$ are given by (2-5). The numbers $A_i$ appearing in (3-9) are defined by

$$
A_k(\xi', \lambda) = \sqrt{\lambda \gamma_i^{-1} + |\xi'|^2} \quad (k = 1, 2, 3), \tag{3-10}
$$

and $P_1, P_2, P_3$ are constants which will determined later by the boundary conditions. Inserting (3-9) into the boundary conditions (3-8), we get a linear equation system for the coefficients $P_i$:

$$
\begin{aligned}
\sum_{i=1}^{3} (A_i^2 - \beta|\xi'|^2 - \lambda(\gamma_i^2 + 2)) P_i &= \mathscr{F}'[g_1](\xi', 0), \\
\sum_{i=1}^{3} (-A_i^3 + A_i(2 - \beta)|\xi'|^2) P_i &= \mathscr{F}'[g_2](\xi', 0), \\
\sum_{i=1}^{3} \lambda A_i(\gamma_i^2 + 2) P_i &= \mathscr{F}'[g_3](\xi', 0).
\end{aligned}
$$

Noting $A_i^2 = \lambda\gamma_i^{-1} + |\xi'|^2$ and $\gamma_i^{-1} - \gamma_i^2 - 2 = \frac{1}{\gamma_i}(1 - \gamma_i^3 - 2\gamma_i) = -\gamma_i$ by $p(-\gamma_i) = 0$, the linear equations above are re-written of the form:

$$
\Delta(\xi', \lambda)(P_1, P_2, P_3)^\top = (\mathscr{F}'[g_1](\xi', 0), \mathscr{F}'[g_2](\xi', 0), \mathscr{F}'[g_3](\xi', 0))^\top \tag{3-11}
$$

with

$$
\Delta(\xi', \lambda) := \begin{pmatrix} -\gamma_1\lambda + \zeta & -\gamma_2\lambda + \zeta & -\gamma_3\lambda + \zeta \\ A_1(-\frac{\lambda}{\gamma_1} + \zeta) & A_2(-\frac{\lambda}{\gamma_2} + \zeta) & A_3(-\frac{\lambda}{\gamma_3} + \zeta) \\ \lambda A_1(\gamma_1^2 + 2) & \lambda A_2(\gamma_2^2 + 2) & \lambda A_3(\gamma_3^2 + 2) \end{pmatrix}. \tag{3-12}
$$

Here $\zeta := (1 - \beta)|\xi'|^2$. The matrix $\Delta(\xi', \lambda)$ is called the Lopatinskiĭ matrix of (3-6), (3-2).

It is the most important of this paper to analyze the inverse matrix of the Lopatinskiĭ matrix. For this purpose, we introduce some classes of multipliers.

**Definition 3.2.** Let $V$ be a domain in $\mathbb{C}$, let $\Xi \subset (\mathbb{R}^{N-1} \setminus \{0\}) \times V$, and let $m \colon \Xi \to \mathbb{C}$, $(\xi', \lambda) \mapsto m(\xi', \lambda)$ be $C^1$ with respect to $\tau$ (where $\lambda = \gamma + i\tau$) and $C^\infty$ with respect to $\xi'$.

(1) $m(\xi', \lambda)$ is called a multiplier of order $s$ with type 1 on $\Xi$ if there hold the estimates:

$$
|\partial_{\xi'}^{\kappa'} m(\xi', \lambda)| \leq C_{\kappa'}(|\lambda|^{1/2} + |\xi'|)^{s - |\kappa'|}, \quad |\partial_{\xi'}^{\kappa'}(\tau\partial_\tau m(\xi', \lambda))| \leq C_{\kappa'}(|\lambda|^{1/2} + |\xi'|)^{s - |\kappa'|} \tag{3-13}
$$

for any multi-index $\kappa' \in \mathbb{N}_0^{N-1}$ and $(\xi', \lambda) \in \Xi$ with some constant $C_{\kappa'}$ depending solely on $\kappa'$ and $\Xi$.

(2) $m(\xi', \lambda)$ is called a multiplier of order $s$ with type 2 on $\Xi$ if there hold the estimates:

$$
|\partial_{\xi'}^{\kappa'} m(\xi', \lambda)| \leq C_{\kappa'}(|\lambda|^{1/2} + |\xi'|)^{s}|\xi'|^{-|\kappa'|}, \quad |\partial_{\xi'}^{\kappa'}(\tau\partial_\tau m(\xi', \lambda))| \leq C_{\kappa'}(|\lambda|^{1/2} + |\xi'|)^{s}|\xi'|^{-|\kappa'|}. \tag{3-14}
$$

for any multi-index $\kappa' \in \mathbb{N}_0^{N-1}$ and $(\xi', \lambda) \in \Xi$ with some constant $C_{\kappa'}$ depending solely on $\kappa'$ and $\Xi$.

Let $\mathbb{M}_{s,i}(\Xi)$ be the set of all multipliers of order $s$ with type $i$ on $\Xi$ ($i = 1, 2$). In the standard case $\Xi = (\mathbb{R}^{N-1} \setminus \{0\}) \times V$, we write $\mathbb{M}_{s,i}(V)$ instead of $\mathbb{M}_{s,i}(\Xi)$.

Obviously, $\mathbb{M}_{s,i}(\Xi)$ are complex vector spaces. Moreover, the following lemma follows from the inequality $(|\lambda|^{1/2}+|\xi'|)^{-|\alpha'|}\le|\xi'|^{-|\alpha'|}$ and the Leibniz rule immediately.

**Lemma 3.3.** *Let $s_1$, $s_2$ be two real numbers. Then, the following three assertions hold.*
*a) Given $m_i\in\mathbb{M}_{s_i,1}(\Xi)$ $(i=1,2)$, we have $m_1m_2\in\mathbb{M}_{s_1+s_2,1}(\Xi)$.*
*b) Given $\ell_i\in\mathbb{M}_{s_i,i}(\Xi)$ $(i=1,2)$, we have $\ell_1\ell_2\in\mathbb{M}_{s_1+s_2,2}(\Xi)$.*
*c) Given $n_i\in\mathbb{M}_{s_i,2}(\Xi)$ $(i=1,2)$, we have $n_1n_2\in\mathbb{M}_{s_1+s_2,2}(\Xi)$.*

Due to $\partial_{\xi'}^{\alpha'}\xi_\ell=0$ for $|\alpha'|\ge1$, we have $(\xi',\lambda)\mapsto\xi_\ell\in\mathbb{M}_{1,1}(\Sigma_{\vartheta_0})$. Similarly, due to $\partial_{\xi'}^{\alpha'}|\xi'|^2=0$ for $|\alpha'|\ge3$ we obtain for $\zeta:=(1-\beta)|\xi'|^2$ (see (3-12))

$$|\partial_{\xi'}^{\alpha'}\zeta|\le C(|\lambda|^{1/2}+|\xi'|)^{2-|\alpha'|},\tag{3-15}$$

which yields that $\zeta\in\mathbb{M}_{2,1}(\Sigma_{\vartheta_0})$. Here and in the following, $\vartheta_0$ is the number given in (2-6). By (2-7),

$$c(|\lambda|^{1/2}+|\xi'|)\le\operatorname{Re}A_i(\xi',\lambda)\le|A_i(\xi',\lambda)|\le C(|\lambda|^{1/2}+|\xi'|)\tag{3-16}$$

with some positive constants $c$ and $C$, which furnishes that

$$A_i(\xi',\lambda)^s\in\mathbb{M}_{s,1}(\Sigma_{\vartheta_0}),\quad(A_i(\xi',\lambda)+|\xi'|)^{-1}\in\mathbb{M}_{-1,2}(\Sigma_{\vartheta_0}),\tag{3-17}$$

where $s$ is any real number. The property of the Lopatinskiĭ matrix $\Delta$ is given in

**Theorem 3.4.** *There exists a number $\frac{\pi}{2}<\vartheta\le\pi$ such that the inverse matrix $\Delta(\xi',\lambda)^{-1}$ exists for any $\lambda\in\Sigma_\vartheta$ and $\xi'\in\mathbb{R}^{N-1}$. Let*

$$\Delta(\xi',\lambda)^{-1}=(g_{ij}(\xi',\lambda))_{i,j=1,2,3}.$$

*Then,*

$$\lambda g_{i1}\in\mathbb{M}_{0,1}(\Sigma_\vartheta)\ \ (i=1,2,3),\quad\lambda g_{ij}\in\mathbb{M}_{-1,1}(\Sigma_\vartheta)\ \ (i=1,2,3,\ \ j=2,3).\tag{3-18}$$

*Moreover, there exists a positive constant $\sigma_0>0$ such that*

$$\begin{aligned}\lambda g_{i1}&\in\mathbb{M}_{0,2}(\Xi_{\vartheta,\sigma_0})\ \ (i=1,2,3),&\sum_{i=1}^3 g_{i1}&\in\mathbb{M}_{-2,2}(\Xi_{\vartheta,\sigma_0})\ \ (i=1,2,3),\\ \lambda g_{ij}&\in\mathbb{M}_{-1,2}(\Xi_{\vartheta,\sigma_0})\ \ (i=1,2,3,\ \ j=2,3),&\sum_{i=1}^3 g_{ij}&\in\mathbb{M}_{-3,2}(\Xi_{\vartheta,\sigma_0})\ \ (i=1,2,3,j=2,3)\end{aligned}\tag{3-19}$$

*with*

$$\Xi_{\vartheta,\sigma_0}=\{(\xi',\lambda)\in\mathbb{R}^{N-1}\times\Sigma_\vartheta\ \mid\sigma_0|\xi'|\ge|\lambda|^{1/2},\quad\lambda\in\Sigma_\vartheta\}.\tag{3-20}$$

The proof of Theorem 3.4 is the highlight of this paper, But, it is postponed to Section 4 and using Theorem 3.4, we are going to investigate the solution operator of the parameter-dependent system (3-6) and (3-2). By (3-9), (3-11) and Theorem 3.4,

$$\begin{aligned}w(\xi',x_N,\lambda)&=\sum_{i,j=1}^3 e^{-A_i(\xi',\lambda)x_N}g_{ij}(\xi',\lambda)\mathscr{F}'[g_j](\xi',0),\\ \tau(\xi',x_N,\lambda)&=-\lambda\sum_{i,j=1}^3 e^{-A_i(\xi',\lambda)x_N}(\gamma_i^2+2)g_{ij}(\xi',\lambda)\mathscr{F}'[g_j](\xi',0).\end{aligned}$$

Let $\psi$ be a $C^\infty$ function on $\mathbb{R}$ such that $\psi(t)=1$ for $t<1$ and $\psi(t)=0$ for $t>2$, and set $\varphi_0(\xi',\lambda)=\psi(c_0|\xi'|/|\lambda|^{1/2})$ and $\varphi_\infty(\xi',\lambda)=1-\varphi_0(\xi',\lambda)$. Note that

$$\varphi_0(\xi',\lambda)=\begin{cases}1&\text{if }c_0|\xi'|\le|\lambda|^{1/2},\\0&\text{if }c_0|\xi'|\ge2|\lambda|^{1/2},\end{cases}\quad\varphi_\infty(\xi',\lambda)=\begin{cases}0&\text{if }c_0|\xi'|\le|\lambda|^{1/2},\\1&\text{if }c_0|\xi'|\ge2|\lambda|^{1/2}.\end{cases}\tag{3-21}$$

In view of Theorem 3.4, we write

$$
\begin{aligned}
w(\xi', x_N, \lambda) &= \sum_{i,j=1}^{3} e^{-A_i(\xi',\lambda)x_N} g_{ij}(\xi',\lambda)\mathscr{F}'[g_j](\xi',0),\\
&= \sum_{i,j=1}^{3} e^{-A_i(\xi',\lambda)x_N} \varphi_0(\xi',\lambda) g_{ij}(\xi',\lambda)\mathscr{F}'[g_j](\xi',0)\\
&\quad + \sum_{i,j=1}^{3} e^{-A_i(\xi',\lambda)x_N} \varphi_\infty(\xi',\lambda) g_{ij}(\xi',\lambda)\mathscr{F}'[g_j](\xi',0)\\
&= \sum_{i,j=1}^{3} e^{-A_i(\xi',\lambda)x_N} \varphi_0(\xi',\lambda) g_{ij}(\xi',\lambda)\mathscr{F}'[g_j](\xi',0)\\
&\quad + \sum_{i,j=1}^{3} (e^{-A_i(\xi',\lambda)x_N} - e^{-|\xi'|x_N})\varphi_\infty(\xi',\lambda) g_{ij}(\xi',\lambda)\mathscr{F}'[g_j](\xi',0)\\
&\quad + \sum_{j=1}^{3} e^{-|\xi'|(\xi',\lambda)x_N} \Big\{\varphi_\infty(\xi',\lambda)\sum_{i=1}^{3} g_{ij}(\xi',\lambda)\Big\}\mathscr{F}'[g_j](\xi',0).
\end{aligned}
$$

In what follows, let $A = |\xi'|$ for short, and let

$$
\mathcal{M}_k = \mathcal{M}_k(\xi', x_N, \lambda) := \frac{e^{-A_k(\xi',\lambda)x_N} - e^{-Ax_N}}{A_k(\xi',\lambda) - A} \quad (k = 1, 2, 3). \tag{3-22}
$$

We have the identities

$$
\begin{aligned}
\partial_N \mathcal{M}_k(\xi', x_N, \lambda) &= -e^{-A_k(\xi',\lambda)x_N} - A\mathcal{M}_k(\xi', x_N, \lambda),\\
\partial_N^2 \mathcal{M}_k(\xi', x_N, \lambda) &= (A_k(\xi',\lambda) + A)e^{-A_k(\xi',\lambda)x_N} + A^2\mathcal{M}_k(\xi', x_N, \lambda).
\end{aligned} \tag{3-23}
$$

Let $\mathscr{F}'^{-1}[\varphi]$ denote the inverse partial Fourier transform defined by

$$
\mathscr{F}'^{-1}[\varphi](x', x_N) := \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'} \varphi(\xi', x_N)\, d\xi'.
$$

Let

$$
u(x,\lambda) = \mathscr{F}'^{-1}[w(\xi', x_N, \lambda)], \quad \theta(x,\lambda) = \mathscr{F}'^{-1}[\tau(\xi', x_N, \lambda)]
$$

First, we treat $u(x,\lambda)$. Using the Volevich trick and (3-23) and applying the inverse partial Fourier transform, we rewrite $u(x,\lambda) = \sum_{k=1}^{6} T_j(\lambda)g$, $g = (g_1, g_2, g_3)^\top$, with

$$
\begin{aligned}
T_1(\lambda)g(x) &= -\int_0^\infty \mathscr{F}'^{-1}\Big[\sum_{i,j=1}^{3} g_{ij}(\xi',\lambda)\varphi_0(\xi',\lambda)e^{-A_i(\xi',\lambda)(x_N+y_N)}\mathscr{F}'[\partial_N g_j](\xi', y_N)\Big](x')\, dy_N,\\
T_2(\lambda)g(x) &= \int_0^\infty \mathscr{F}'^{-1}\Big[\sum_{i,j=1}^{3} g_{ij}(\xi',\lambda)\varphi_0(\xi',\lambda)e^{-A_i(\xi',\lambda)(x_N+y_N)}A_i(\xi',\lambda)\mathscr{F}'[g_j](\xi', y_N)\Big](x')\, dy_N,\\
T_3(\lambda)g(x) &= -\int_0^\infty \mathscr{F}'^{-1}\Big[\sum_{i,j=1}^{3} g_{ij}(\xi',\lambda)\varphi_\infty(\xi',\lambda)(e^{-A_i(\xi',\lambda)(x_N+y_N)} - e^{-A(x_N+y_N)})\\
&\qquad\qquad \mathscr{F}'[\partial_N g_j](\xi', y_N)\Big](x')\, dy_N,\\
T_4(\lambda)g(x) &= \int_0^\infty \mathscr{F}'^{-1}\Big[\sum_{i,j=1}^{3} g_{ij}(\xi',\lambda)\varphi_\infty(\xi',\lambda)\Big\{(A_i(\xi',\lambda) - A)e^{-A_i(\xi',\lambda)(x_N+y_N)},
\end{aligned}
$$

$$+ A(e^{-A_i(\xi',\lambda)(x_N+y_N)} - e^{-A(x_N+y_N)})\Big\}\mathscr{F}'[g_j](\xi', y_N)\Big](x')\,dy_N$$

$$T_5(\lambda)g(x) = -\int_0^\infty \mathscr{F}'^{-1}\Big[\sum_{j=1}^3\Big(\sum_{i=1}^3 g_{ij}(\xi',\lambda)\Big)\varphi_\infty(\xi',\lambda)e^{-A(x_N+y_N)}\mathscr{F}'[\partial_N g_j](\xi', y_N)\Big](x')\,dy_N,$$

$$T_6(\lambda)g(x) = \int_0^\infty \mathscr{F}'^{-1}\Big[\sum_{j=1}^3\Big(\sum_{i=1}^3 g_{ij}(\xi',\lambda)\Big)\varphi_\infty(\xi',\lambda)Ae^{-A(x_N+y_N)}\mathscr{F}'[g_j](\xi', y_N)\Big](x')\,dy_N.$$

To construct $\mathcal{R}$-bounded solution operators associated with problem (3-1) and (3-2), we use the following lemma due to Shibata and Shimizu [30, Lemma 5.6].

**Lemma 3.5.** *Let $\vartheta > \pi/2$ be the same number as in Theorem 3.4 and let $A_k(\xi',\lambda)$ and $\mathcal{M}_k(\xi', x_N, \lambda)$ $(k = 1, 2, 3)$ be functions given in* (3-10) *and* (3-22)*, respectively. Given $\ell_0(\xi,'\,\lambda) \in \mathbb{M}_{-2,1}(\Sigma_\vartheta)$ and $\ell_1(\xi',\lambda) \in \mathbb{M}_{-2,2}(\Sigma_\vartheta)$, we define the operators $L_j(\lambda)$ $(j = 1, 2, 3, 4)$ by*

$$[L_1(\lambda)h](x) = \int_0^\infty \mathscr{F}'^{-1}[\ell_0(\xi',\lambda)\lambda^{1/2}e^{-A_k(\xi',\lambda)(x_N+y_N)}\mathscr{F}'[h](\xi', y_N)](x')\,dy_N,$$
$$[L_2(\lambda)h](x) = \int_0^\infty \mathscr{F}'^{-1}[\ell_1(\xi,\lambda)Ae^{-A_k(\xi',\lambda)(x_N+y_N)}\mathscr{F}'[h](\xi', y_N)](x')\,dy_N,$$
$$[L_3(\lambda)h](x) = \int_0^\infty \mathscr{F}'^{-1}[\ell_1(\xi',\lambda)Ae^{-A(x_N+y_N)}\mathscr{F}'[h](\xi', y_N)](x')\,dy_N,$$
$$[L_4(\lambda)h](x) = \int_0^\infty \mathscr{F}'^{-1}[\ell_1(\xi',\lambda)A^2\mathcal{M}_k(\xi', x_N+y_N,\lambda)\mathscr{F}'[h](\xi', y_N)](x')\,dy_N.$$

*Then,*

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N_+))}\big(\{(\tau\partial_\tau)^s(\lambda^{j/2}\partial_x^\alpha L_i(\lambda)) \mid \lambda \in \Sigma_\vartheta\}\big) \le C_{N,q} \quad (s = 0, 1,\ \ j + |\alpha| = 2,\ \ i = 1, 2, 3, 4).$$

From Lemma 3.5 and Lemma 3.3 the following lemma follows immediately.

**Lemma 3.6.** *Let $\vartheta > \pi/2$ be the same number as in Theorem 3.4 and let $A_k(\xi',\lambda)$ and $M_k(\xi', x_N, \lambda)$ $(k = 1, 2, 3)$ be functions given in* (3-10) *and* (3-22)*, respectively. Given $m_0(\xi',\lambda) \in \mathbb{M}_{-4,1}(\Sigma_\vartheta)$ and $m_1(\xi',\lambda) \in \mathbb{M}_{-4,2}(\Sigma_\vartheta)$, we define the operators $M_j(\lambda)$ $(j = 1, 2, 3, 4)$ by*

$$[M_1(\lambda)h](x) = \int_0^\infty \mathscr{F}'^{-1}[m_0(\xi',\lambda)\lambda^{1/2}e^{-A_k(\xi',\lambda)(x_N+y_N)}\mathscr{F}'[h](\xi', y_N)](x')\,dy_N,$$
$$[M_2(\lambda)h](x) = \int_0^\infty \mathscr{F}'^{-1}[m_1(\xi',\lambda)Ae^{-A_k(\xi',\lambda)(x_N+y_N)}\mathscr{F}'[h](\xi', y_N)](x')\,dy_N,$$
$$[M_3(\lambda)h](x) = \int_0^\infty \mathscr{F}'^{-1}[m_1(\xi',\lambda)Ae^{-A(x_N+y_N)}\mathscr{F}'[h](\xi', y_N)](x')\,dy_N,$$
$$[M_4(\lambda)h](x) = \int_0^\infty \mathscr{F}'^{-1}[m_1(\xi',\lambda)A^2\mathcal{M}_k(\xi', x_N+y_N,\lambda)\mathscr{F}'[h](\xi', y_N)](x')\,dy_N.$$

*Then,*

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N_+))}(\{(\tau\partial_\tau)^s(\lambda^{j/2}\partial_x^\alpha M_i(\lambda)) \mid \lambda \in \Sigma_\vartheta\}) \le C_{N,q} \quad (s = 0, 1,\ \ j + |\alpha| = 4,\ \ i = 1, 2, 3, 4). \tag{3-24}$$

*Proof.* Since $\lambda m_0(\xi',\lambda) \in \mathbb{M}_{-2,2}(\Sigma_\vartheta)$, it follows from Lemma 3.5 that (3-24) holds for $i = 1$ and $j+|\alpha| = 4$ with $j = 2, 3, 4$. Let $a, b \in \{1, \ldots, N-1\}$. We have

$$\partial_a\partial_b[M_1(\lambda)h](x) = -\int_0^\infty \mathscr{F}'^{-1}[\xi_a\xi_b m_0(\lambda,\xi')\lambda^{1/2}e^{-A_i(\xi',\lambda)(x_N+y_N)}\mathscr{F}'[h](\xi', y_N)](x')\,dy_N,$$
$$\partial_a\partial_N[M_1(\lambda)h](x) = -i\int_0^\infty \mathscr{F}'^{-1}[\xi_a A_i(\xi',\lambda)m_0(\lambda,\xi')\lambda^{1/2}e^{-A_i(\xi',\lambda)(x_N+y_N)}\mathscr{F}'[h](\xi', y_N)](x')\,dy_N,$$

$$\partial_N^2[M_1(\lambda)h](x) = \int_0^\infty \mathscr{F}'^{-1}[A_i(\xi',\lambda)^2 m_0(\lambda,\xi')\lambda^{1/2}e^{-A_i(\xi',\lambda)(x_N+y_N)}\mathscr{F}'[h](\xi',y_N)](x')\,dy_N.$$

Since $\xi_a$ and $A_k(\xi',\lambda) \in \mathbb{M}_{1,1}(\Sigma_\vartheta)$, it follows from Lemma 3.3 that $\xi_a\xi_b m_0(\xi',\lambda)$, $\xi_a A_k(\xi',\lambda)m_0(\xi',\lambda)$, and $A_k(\xi',\lambda)^2 m_0(\xi',\lambda)$ belong to $\mathbb{M}_{-2,1}(\Sigma_\vartheta)$, so that it follows from Lemma 3.5 that (3-24) holds for $i = 1$ and $j + |\alpha| = 4$ with $|\alpha| = 2,3,4$. Since $\lambda m_1(\xi',\lambda) \in \mathbb{M}_{-2,2}(\Sigma_\vartheta)$, it follows from Lemma 3.5 that (3-24) holds for $i = 2,3,4$ and $j + |\alpha| = 4$ with $j = 2,3,4$. By Lemma 3.3, $\xi_a\xi_b m_1(\xi',\lambda)$, $\xi_a A_k(\xi',\lambda)m_1(\xi',\lambda)$, $A_k(\xi',\lambda)^2 m_1(\xi',\lambda)$, $\xi_a A m_1(\xi',\lambda)$, and $A^2 m_1(\xi',\lambda)$ belong to $\mathbb{M}_{-2,2}(\Sigma_\vartheta)$. It follows from Lemma 3.5 that (3-24) holds for $i = 2,3$ and $j + |\alpha| = 4$ with $|\alpha| = 2,3,4$. Since $\xi_a\xi_b m_1(\xi',\lambda)$, $\xi_a A m_1(\xi',\lambda)$, $(A_k(\xi',\lambda) + A)A m_1(\xi',\lambda)$, and $A^2 m_1(\xi',\lambda)$ belong to $\mathbb{M}_{-2,2}(\Sigma_\vartheta)$, it follows from Lemma 3.5 that (3-24) holds for $i = 4$ and $j + |\alpha| = 4$ with $|\alpha| = 2,3,4$, which completes the proof of Lemma 3.6. □

To define the solution operators, we change the formulas $T_j(\lambda)$. First of all, we rewrite $T_1(\lambda)g$ as follows:

$$\begin{aligned} T_1(\lambda)g(x) = -\int_0^\infty \mathscr{F}'^{-1}\Big[\sum_{k=1}^3 \Big(&\lambda^{-1}g_{k1}(\xi',\lambda)\varphi_0(\xi',\lambda)\mathscr{F}'[\lambda^{1/2}\partial_N g_1](\xi',y_N) \\ &+ \sum_{j=2}^3 \lambda^{-1/2}g_{kj}(\xi',\lambda)\varphi_0(\xi',\lambda)\mathscr{F}'[\partial_N g_j](\xi',y_N)\Big)\lambda^{1/2}e^{-A_k(\xi',\lambda)(x_N+y_N)}\Big](x')\,dy_N. \end{aligned}$$

Recall that $G' = (g_1', g_2', g_3')$ and $g_1''$ correspond to $\lambda^{1/2}G = \lambda^{1/2}(g_1,g_2,g_3)$ and $\lambda g_1$, respectively. Let

$$\mathcal{X}_q'(D) = \{(G, G', g_1'') \mid G \in H_q^2(D) \times H_q^1(D)^2,\ \ G' \in H_q^1(D) \times L_q(D)^2, g_1'' \in L_q(D)\},$$

and let

$$\begin{aligned} \mathcal{U}_1(\lambda)(G,G',g_1'')(x) = -\int_0^\infty \mathscr{F}'^{-1}\Big[\sum_{k=1}^3 \Big(&\lambda^{-1}g_{k1}(\xi',\lambda)\varphi_0(\xi',\lambda)\mathscr{F}'[\partial_N g_1'](\xi',y_N) \\ &+ \sum_{j=2}^3 \lambda^{-1/2}g_{kj}(\xi',\lambda)\varphi_0(\xi',\lambda)\mathscr{F}'[\partial_N g_j](\xi',y_N)\Big)\lambda^{1/2}e^{-A_k(\xi',\lambda)(x_N+y_N)}\Big](x')\,dy_N. \end{aligned}$$

Obviously, $T_1(\lambda)g = \mathcal{U}_1(\lambda)(g, \lambda^{1/2}g, \lambda g_1)$. Moreover, it follows from Theorem 3.4 and (3-21) that $\lambda^{-1}g_{k1}(\xi',\lambda)\varphi_0(\xi',\lambda)$ and $\lambda^{-1/2}g_{kj}(\xi',\lambda)\varphi_0(\xi',\lambda)$ $(j = 2,3)$ belong to $\mathbb{M}_{-4,1}(\Sigma_\vartheta)$, which, combined with Lemma 3.6, furnishes that

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_q'(\mathbb{R}_+^N), L_q(\mathbb{R}_+^N))}\big(\{(\tau\partial_\tau)^s(\lambda^{j/2}\partial_x^\alpha \mathcal{U}_1(\lambda)) \mid \lambda \in \Sigma_\vartheta\}\big) \le C_{N,q} \quad (s = 0,1) \tag{3-25}$$

for any $j \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0^N$ with $j + |\alpha| = 4$.

Next, analogously to $T_1(\lambda)g$, we rewrite $T_2(\lambda)g$ as follows:

$$\begin{aligned} T_2(\lambda)g(x) = \int_0^\infty \mathscr{F}'^{-1}\Big[\sum_{k=1}^3 \Big(&g_{k1}(\xi',\lambda)\varphi_0(\xi',\lambda)A_k(\xi',\lambda)\lambda^{-3/2}\mathscr{F}'[\lambda g_1](\xi',y_N) \\ &+ \sum_{j=2}^3 g_{kj}(\xi',\lambda)\varphi_0(\xi',\lambda)A_k(\xi',\lambda)\lambda^{-1}\mathscr{F}'[\lambda^{1/2}g_j](\xi',y_N)\Big)\lambda^{1/2}e^{-A_k(\xi',\lambda)(x_N+y_N)}\Big](x')\,dy_N. \end{aligned}$$

Let

$$\begin{aligned} \mathcal{U}_2(\lambda)(G,G',g_1'')(x) = \int_0^\infty \mathscr{F}'^{-1}\Big[\sum_{k=1}^3 \Big(&g_{k1}(\xi',\lambda)\varphi_0(\xi',\lambda)A_k(\xi',\lambda)\lambda^{-3/2}\mathscr{F}'[g_1''](\xi',y_N) \\ &+ \sum_{j=2}^3 g_{kj}(\xi',\lambda)\varphi_0(\xi',\lambda)A_k(\xi',\lambda)\lambda^{-1}\mathscr{F}'[g_j'](\xi',y_N)\Big)\lambda^{1/2}e^{-A_k(\xi',\lambda)(x_N+y_N)}\Big](x')\,dy_N. \end{aligned}$$

Obviously, $T_2(\lambda)g = \mathcal{U}_2(\lambda)(g, \lambda^{1/2}g, \lambda g_1)$. Moreover, it follows from Theorem 3.4 and (3-21) that $g_{k1}(\xi',\lambda)\varphi_0(\xi',\lambda)A_k(\xi',\lambda)\lambda^{-3/2}$ and $g_{kj}(\xi',\lambda)\varphi_0(\xi',\lambda)A_k(\xi',\lambda)\lambda^{-1}$ $(j=2,3)$ belong to $\mathbb{M}_{-4,1}(\Sigma_\vartheta)$, which, combined with Lemma 3.6, furnishes that

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}'_q(\mathbb{R}^N_+), L_q(\mathbb{R}^N_+))}(\{(\tau\partial_\tau)^s(\lambda^{j/2}\partial_x^\alpha \mathcal{U}_2(\lambda)) \mid \lambda \in \Sigma_\vartheta\}) \leq C_{N,q} \quad (s=0,1) \tag{3-26}$$

for any $j \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0^N$ with $j + |\alpha| = 4$.

Next, we consider $T_3(\lambda)g$. Using the identities

$$\lambda = \gamma_k(A_k(\xi',\lambda)+A)(A_k(\xi',\lambda)-A), \quad 1 = -\sum_{\ell=1}^{N-1}(i\xi_\ell)(i\xi_\ell)A^{-2}, \tag{3-27}$$

we rewrite $T_3(\lambda)g$ as follows:

$$\begin{aligned} T_3(\lambda)g(x) = \gamma_k^{-1}\int_0^\infty \mathscr{F}'^{-1}\Bigg[\sum_{k=1}^3\Big(\sum_{\ell=1}^{N-1}\Big[\lambda g_{k1}(\xi',\lambda)\varphi_\infty(\xi',\lambda)(A_k(\xi',\lambda)+A)^{-1}(i\xi_\ell A^{-4})\mathscr{F}'[\partial_\ell\partial_N g_1(\xi',y_N)]\Big] \\ -\sum_{j=2}^3 \lambda g_{ij}(\xi',\lambda)\varphi_\infty(\xi',\lambda)(A_k(\xi',\lambda)+A)^{-1}A^{-2}\mathscr{F}'[\partial_N g_j](\xi',y_N)\Big)A^2\mathcal{M}_k(\xi',x_N+y_N,\lambda)\Bigg](x')\,dy_N. \end{aligned}$$

Let, $\mathcal{U}_3(\lambda)(G,G',g_1'') = T_3(\lambda)g$. Since the multipliers symbols:

$$\lambda g_{k1}(\xi',\lambda)\varphi_\infty(\xi',\lambda)(A_k(\xi',\lambda)+A)^{-1}(i\xi_\ell A^{-4}),\ \ \lambda g_{ij}(\xi',\lambda)\varphi_\infty(\xi',\lambda)(A_k(\xi',\lambda)+A)^{-1}A^{-2}$$

$(j=2,3)$ belong to $\mathbb{M}_{-4,2}(\Sigma_\vartheta)$ as follows from Theorem 3.4 and (3-21), by Lemma 3.6 we have

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}'_q(\mathbb{R}^N_+), L_q(\mathbb{R}^N_+))}(\{(\tau\partial_\tau)^s(\lambda^{j/2}\partial_x^\alpha \mathcal{U}_3(\lambda)) \mid \lambda \in \Sigma_\vartheta\}) \leq C_{N,q} \quad (s=0,1) \tag{3-28}$$

for any $j \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0^N$ with $j + |\alpha| = 4$.

Using (3-27), we also have

$$\begin{aligned} T_4(\lambda)g(x) = -\int_0^\infty \mathscr{F}'^{-1}\Bigg[\sum_{k=1}^3 \gamma_k^{-1}\Big\{\Big(\lambda g_{k1}(\xi',\lambda)\varphi_\infty(\xi',\lambda)(A_k(\xi',\lambda)+A)^{-1}A^{-3}\mathscr{F}'[\Delta g_1](\xi',y_N) \\ +\sum_{j=2}^3\sum_{\ell=1}^{N-1}\lambda g_{kj}(\xi',\lambda)\varphi_\infty(\xi',\lambda)(A_k(\xi',\lambda)+A)^{-1}A^{-3}(i\xi_\ell)\mathscr{F}'[\partial_\ell g_j](\xi',y_N)\Big) \\ (Ae^{-A_k(\xi',\lambda)(x_N+y_N)} + A^2\mathcal{M}_k(\xi',x_N+y_N,\lambda))\Big\}\Bigg](x')\,dy_N. \end{aligned}$$

Let $\mathcal{U}_4(\lambda)(G,G',g_1'') = T_4(\lambda)g$. Since the multipliers

$$\lambda g_{k1}(\xi',\lambda)\varphi_\infty(\xi',\lambda)(A_k(\xi',\lambda)+A)^{-1}A^{-3},\ \ \lambda g_{kj}(\xi',\lambda)\varphi_\infty(\xi',\lambda)(A_k(\xi',\lambda)+A)^{-1}A^{-3}(i\xi_\ell)$$

$(j=2,3)$ belong to $\mathbb{M}_{-4,2}(\Sigma_\vartheta)$ as follows from Theorem 3.4 and (3-21), by Lemma 3.6 we have

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}'_q(\mathbb{R}^N_+), L_q(\mathbb{R}^N_+))}(\{(\tau\partial_\tau)^s(\lambda^{j/2}\partial_x^\alpha \mathcal{U}_4(\lambda)) \mid \lambda \in \Sigma_\vartheta\}) \leq C_{N,q} \quad (s=0,1) \tag{3-29}$$

for any $j \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0^N$ with $j + |\alpha| = 4$.

Using the second indentity in (3-27), we rewrite $T_5(\lambda)g$ as follows:

$$T_5(\lambda)g(x) = \int_0^\infty \mathscr{F}'^{-1}\Bigg[\Big\{\Big(\sum_{k=1}^3 g_{k1}(\xi',\lambda)\Big)\varphi_\infty(\xi',\lambda)A^{-3}\sum_{\ell=1}^{N-1} i\xi_\ell\mathscr{F}'[\partial_\ell\partial_N g_1](\xi',y_N)$$

$$+\sum_{j=2}^{3}\Big(\sum_{k=1}^{3} g_{kj}(\xi',\lambda)\Big)\varphi_\infty(\xi',\lambda)A^{-1}\mathscr{F}'[\partial_N g_j](\xi',y_N)\Big\}Ae^{-A(x_N+y_N)}\Big](x')\,dy_N.$$

Let $\mathcal{U}_5(\lambda)(G,G',g_1'')=T_5(\lambda)g$. Since the multipliers:

$$\Big(\sum_{k=1}^{3} g_{k1}(\xi',\lambda)\Big)\varphi_\infty(\xi',\lambda)A^{-3}(i\xi_\ell),\quad \Big(\sum_{k=1}^{3} g_{kj}(\xi',\lambda)\Big)\varphi_\infty(\xi',\lambda)A^{-1}$$

$(j=2,3,\ \ell=1,\dots,N-1)$ belong to $\mathbb{M}_{-4,2}(\Sigma_\vartheta)$ as follows from Theorem 3.4 and (3-21), by Lemma 3.6 we have

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_q'(\mathbb{R}^N_+),L_q(\mathbb{R}^N_+))}(\{(\tau\partial_\tau)^s(\lambda^{j/2}\partial_x^\alpha\mathcal{U}_5(\lambda))\mid\lambda\in\Sigma_\vartheta\})\le C_{N,q}\quad(s=0,1)\tag{3-30}$$

for any $j\in\mathbb{N}_0$ and $\alpha\in\mathbb{N}_0^N$ with $j+|\alpha|=4$.

Finally, using the second identity in (3-27), we rewrite $T_6(\lambda)g$ as follows:

$$T_6(\lambda)g(x)=-\int_0^\infty\mathscr{F}'^{-1}\Big[\Big\{\Big(\sum_{k=1}^{3} g_{k1}(\xi',\lambda)\Big)\varphi_\infty(\xi',\lambda)A^{-2}\mathscr{F}'[\Delta g_1](\xi',y_N)$$
$$+\sum_{j=2}^{3}\sum_{\ell=1}^{N-1}\Big(\sum_{k=1}^{3} g_{kj}(\xi',\lambda)\Big)\varphi_\infty(\xi',\lambda)A^{-2}(i\xi_\ell)\mathscr{F}'[\partial_\ell g_j](\xi',y_N)\Big\}Ae^{-A(x_N+y_N)}\Big](x')\,dy_N.$$

Let $\mathcal{U}_6(\lambda)(G,G',g_1'')=T_6(\lambda)g$. Since the multipliers

$$\Big(\sum_{k=1}^{3} g_{k1}(\xi',\lambda)\Big)\varphi_\infty(\xi',\lambda)A^{-2},\quad \Big(\sum_{k=1}^{3} g_{kj}(\xi',\lambda)\Big)\varphi_\infty(\xi',\lambda)A^{-2}(i\xi_\ell)$$

$(j=2,3,\ \ell=1,\dots,N-1)$ belong to $\mathbb{M}_{-4,2}(\Sigma_\vartheta)$ as follows from Theorem 3.4 and (3-21), by Lemma 3.6 we have

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_q'(\mathbb{R}^N_+),L_q(\mathbb{R}^N_+))}(\{(\tau\partial_\tau)^s(\lambda^{j/2}\partial_x^\alpha\mathcal{U}_6(\lambda))\mid\lambda\in\Sigma_\vartheta\})\le C_{N,q}\quad(s=0,1)\tag{3-31}$$

for any $j\in\mathbb{N}_0$ and $\alpha\in\mathbb{N}_0^N$ with $j+|\alpha|=4$. Thus, setting $\mathcal{V}_1(\lambda)(G,G',g_1'')=\sum_{\ell=1}^{6}\mathcal{U}_\ell(\lambda)(G,G',g_1'')$ and using (3-25)–(3-31), we see that $u(x,\lambda)=\mathcal{V}_1(\lambda)(g,\lambda^{1/2}g,\lambda g)$ and that

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_q'(\mathbb{R}^N_+),L_q(\mathbb{R}^N_+))}(\{(\tau\partial_\tau)^s(\lambda^{j/2}\partial_x^\alpha\mathcal{V}_1(\lambda))\mid\lambda\in\Sigma_\vartheta\})\le C_{N,q}\quad(s=0,1)\tag{3-32}$$

for any $j\in\mathbb{N}_0$ and $\alpha\in\mathbb{N}_0^N$ with $j+|\alpha|=4$.

Next, we consider $\theta(x,\lambda)=\mathscr{F}'^{-1}[\tau(\xi',x_N,\lambda)](x')$. Applying the Volevich trick and using the identities (3-27) and $A_k(\xi',\lambda)^4=\gamma_k^{-2}\lambda^2+2\gamma_k^{-1}\lambda A^2+A^4$, we have

$$\theta(x,\lambda)=-\lambda\sum_{k,j=1}^{3}(\gamma_k^2+2)\mathscr{F}'^{-1}[g_{kj}(\xi',\lambda)e^{-A_k(\xi',\lambda)x_N}\mathscr{F}[g_j](\xi',0)](x')$$
$$=\sum_{k,j=1}^{3}(\gamma_k^2+2)\int_0^\infty\mathscr{F}'^{-1}\Big[e^{-A_k(\xi',\lambda)(x_N+y_N)}\lambda g_{kj}(\xi',\lambda)\mathscr{F}[\partial_N g_j](\xi',y_N)\Big](x')\,dy_N$$
$$-\sum_{k,j=1}^{3}(\gamma_k^2+2)\int_0^\infty\mathscr{F}'^{-1}\Big[e^{-A_k(\xi',\lambda)(x_N+y_N)}\lambda A_k(\xi',\lambda)g_{kj}(\xi',\lambda)\mathscr{F}[g_j](\xi',y_N)\Big](x')\,dy_N$$
$$=\sum_{k=1}^{3}(\gamma_k^2+2)\int_0^\infty\mathscr{F}'^{-1}\Big[\lambda g_{k1}(\xi',\lambda)\gamma_k^{-1}A_k(\xi',\lambda)^{-2}\lambda^{\frac12}e^{-A_k(\xi',\lambda)(x_N+y_N)}\mathscr{F}[\lambda^{\frac12}\partial_N g_1](\xi',y_N)\Big](x')\,dy_N$$
$$-\sum_{k=1}^{3}\sum_{\ell=1}^{N-1}(\gamma_k^2+2)\int_0^\infty\mathscr{F}'^{-1}\Big[\lambda g_{k1}(\xi',\lambda)A_k(\xi',\lambda)^{-2}(i\xi_\ell)A^{-1}$$

$$Ae^{-A_k(\xi',\lambda)(x_N+y_N)}\mathscr{F}[\partial_\ell\partial_N g_1](\xi',y_N)\Big](x')\,dy_N$$

$$+\sum_{j=2}^{3}\sum_{k=1}^{3}(\gamma_k^2+2)\int_0^\infty \mathscr{F}'^{-1}\Big[\lambda g_{kj}(\xi',\lambda)\gamma_k^{-1}A_k(\xi',\lambda)^{-2}\lambda^{\frac12}$$

$$\lambda^{\frac12}e^{-A_k(\xi',\lambda)(x_N+y_N)}\mathscr{F}[\partial_N g_j](\xi',y_N)\Big](x')\,dy_N$$

$$+\sum_{j=2}^{3}\sum_{k=1}^{3}(\gamma_k^2+2)\int_0^\infty \mathscr{F}'^{-1}\Big[\lambda g_{kj}(\xi',\lambda)A_k(\xi',\lambda)^{-2}AAe^{-A_k(\xi',\lambda)(x_N+y_N)}\mathscr{F}[\partial_N g_j](\xi',y_N)\Big](x')\,dy_N$$

$$-\sum_{k=1}^{3}(\gamma_k^2+2)\int_0^\infty \mathscr{F}'^{-1}\Big[\lambda g_{k1}(\xi',\lambda)A_k(\xi',\lambda)^{-3}\Big(\gamma_k^{-2}\lambda^{\frac12}\lambda^{\frac12}e^{-A_k(\xi',\lambda)(x_N+y_N)}\mathscr{F}[\lambda g_1](\xi',y_N)$$

$$+2\gamma_k^{-1}AAe^{-A_k(\xi',\lambda)(x_N+y_N)}\mathscr{F}[\lambda g_1]-AAe^{-A_k(\xi',\lambda)(x_N+y_N)}\mathscr{F}[\Delta' g_1](\xi',\lambda)\Big)\Big](x')\,dy_N$$

$$-\sum_{j=2}^{3}\sum_{k=1}^{3}(\gamma_k^2+2)\int_0^\infty \mathscr{F}'^{-1}\Big[\lambda g_{kj}(\xi',\lambda)A_k(\xi',\lambda)^{-1}\Big(\gamma_k^{-1}\lambda^{\frac12}\lambda^{\frac12}e^{-A_k(\xi',\lambda)(x_N+y_N)}\mathscr{F}[\lambda^{\frac12}g_j](\xi',y_N)$$

$$-\sum_{\ell=1}^{N-1}(i\xi_\ell)A^{-1}Ae^{-A_k(\xi',\lambda)(x_N+y_N)}\mathscr{F}'[\partial_\ell g_j](\xi',y_N)\Big)\Big](x')\,dy_N.$$

According to the formula above, we define an operator $\mathcal{V}_2(\lambda)$ acting on $(G,G',g_1'')$ by

$$\mathcal{V}_2(\lambda)(G,G',g_1'')$$

$$=\sum_{k=1}^{3}(\gamma_k^2+2)\int_0^\infty \mathscr{F}'^{-1}\Big[\lambda g_{k1}(\xi',\lambda)\gamma_k^{-1}A_k(\xi',\lambda)^{-2}\lambda^{\frac12}e^{-A_k(\xi',\lambda)(x_N+y_N)}\mathscr{F}[\partial_N g_1'](\xi',y_N)\Big](x')\,dy_N$$

$$-\sum_{k=1}^{3}\sum_{\ell=1}^{N-1}(\gamma_k^2+2)\int_0^\infty \mathscr{F}'^{-1}\Big[\lambda g_{k1}(\xi',\lambda)A_k(\xi',\lambda)^{-2}(i\xi_\ell)A^{-1}$$

$$Ae^{-A_k(\xi',\lambda)(x_N+y_N)}\mathscr{F}[\partial_\ell\partial_N g_1](\xi',y_N)\Big](x')\,dy_N$$

$$+\sum_{j=2}^{3}\sum_{k=1}^{3}(\gamma_k^2+2)\int_0^\infty \mathscr{F}'^{-1}\Big[\lambda g_{kj}(\xi',\lambda)\gamma_k^{-1}A_k(\xi',\lambda)^{-2}\lambda^{\frac12}$$

$$\lambda^{\frac12}e^{-A_k(\xi',\lambda)(x_N+y_N)}\mathscr{F}[\partial_N g_j](\xi',y_N)\Big](x')\,dy_N$$

$$+\sum_{j=2}^{3}\sum_{k=1}^{3}(\gamma_k^2+2)\int_0^\infty \mathscr{F}'^{-1}\Big[\lambda g_{kj}(\xi',\lambda)A_k(\xi',\lambda)^{-2}AAe^{-A_k(\xi',\lambda)(x_N+y_N)}\mathscr{F}[\partial_N g_j](\xi',y_N)\Big](x')\,dy_N$$

$$-\sum_{k=1}^{3}(\gamma_k^2+2)\int_0^\infty \mathscr{F}'^{-1}\Big[\lambda g_{k1}(\xi',\lambda)A_k(\xi',\lambda)^{-3}\Big(\gamma_k^{-2}\lambda^{\frac12}\lambda^{\frac12}e^{-A_k(\xi',\lambda)(x_N+y_N)}\mathscr{F}[g_1''](\xi',y_N)$$

$$+2\gamma_k^{-1}AAe^{-A_k(\xi',\lambda)(x_N+y_N)}\mathscr{F}[g_1'']-AAe^{-A_k(\xi',\lambda)(x_N+y_N)}\mathscr{F}[\Delta' g_1](\xi',\lambda)\Big)\Big](x')\,dy_N$$

$$-\sum_{j=2}^{3}\sum_{k=1}^{3}(\gamma_k^2+2)\int_0^\infty \mathscr{F}'^{-1}\Big[\lambda g_{kj}(\xi',\lambda)A_k(\xi',\lambda)^{-1}\Big(\gamma_k^{-1}\lambda^{\frac12}\lambda^{\frac12}e^{-A_k(\xi',\lambda)(x_N+y_N)}\mathscr{F}[g_j'](\xi',y_N)$$

$$-\sum_{\ell=1}^{N-1}(i\xi_\ell)A^{-1}Ae^{-A_k(\xi',\lambda)(x_N+y_N)}\mathscr{F}'[\partial_\ell g_j](\xi',y_N)\Big)\Big](x')\,dy_N.$$

With this definition, we obtain $\mathcal{V}_2(\lambda)(g, \lambda^{1/2}g, \lambda g_1) = \theta$. Moreover, by Theorem 3.4 we see that the multipliers

$$\lambda g_{k1}(\xi', \lambda)A_k(\xi', \lambda)^{-2}, \ \ \lambda g_{kj}(\xi', \lambda)A_k(\xi', \lambda)^{-2}\lambda^{1/2}, \ \ \lambda g_{k1}(\xi', \lambda)A_k(\xi', \lambda)^{-3}\lambda^{1/2},$$

belong to $\mathbb{M}_{-2,1}(\Sigma_\vartheta)$ and that the multipliers

$$\begin{aligned}&\lambda g_{k1}(\xi', \lambda)A_k(\xi', \lambda)^{-2}(i\xi_\ell)A^{-1}, \ \ \lambda g_{kj}(\xi', \lambda)A_k(\xi', \lambda)^{-2}A,\\ &\lambda g_{k1}(\xi', \lambda)A_k(\xi', \lambda)^{-3}A, \ \ \lambda g_{kj}(\xi', \lambda)A_k(\xi', \lambda)^{-1}(i\xi_\ell)A^{-1}\end{aligned}$$

belong to $\mathbb{M}_{-2,2}(\Sigma_\vartheta)$. Thus, by Lemma 3.5, we have

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}'_q(\mathbb{R}^N_+), L_q(\mathbb{R}^N_+))}(\{(\tau\partial_\tau)^s(\lambda^{j/2}\partial_x^\alpha \mathcal{V}_2(\lambda)) \mid \lambda \in \Sigma_\vartheta\}) \leq C_{N,q} \quad (s = 0, 1) \tag{3-33}$$

for any $j \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0^N$ with $j + |\alpha| = 2$.

Let $\mathcal{V}_1(\lambda) = \sum_{\ell=1}^6 \mathcal{U}_\ell(\lambda)$ and $\mathcal{V}(\lambda) = (\mathcal{V}_1(\lambda), \mathcal{V}_2(\lambda))^\top$. Then

$$U = (\mathcal{V}_1(\lambda)(g, \lambda^{1/2}g, \lambda g_1), \lambda\mathcal{V}_1(\lambda)(g, \lambda^{1/2}g, \lambda g_1), \mathcal{V}_2(\lambda)(g, \lambda^{1/2}g, \lambda g_1))^\top$$

satisfies the equations (3-1) with $F = 0$. And, by Proposition 2.2 c), (3-32), and (3-33), we have

$$\begin{aligned}&\mathcal{R}_{\mathcal{L}(\mathcal{X}'_q(\mathbb{R}^N_+), H^{4-j}_q(\mathbb{R}^N_+))}(\{(\tau\partial_\tau)^s(\lambda^{j/2}\mathcal{V}_1(\lambda)) \mid \lambda \in \Sigma_{\vartheta,\lambda_0}\}) \leq C_{N,q,\lambda_0} \quad (j = 0, 1, 2, 3, 4),\\ &\mathcal{R}_{\mathcal{L}(\mathcal{X}'_q(\mathbb{R}^N_+), H^{2-j}_q(\mathbb{R}^N_+))}(\{(\tau\partial_\tau)^s(\lambda^{j/2}\mathcal{V}_k(\lambda)) \mid \lambda \in \Sigma_{\vartheta,\lambda_0}\}) \leq C_{N,q,\lambda_0} \quad (j = 0, 1, 2, \ \ k = 2, 3),\end{aligned} \tag{3-34}$$

for any $\lambda_0 > 0$ and $s = 0, 1$.

To define $\mathcal{T}(\lambda)$ required in Theorem 3.1, we want to apply $\mathcal{V}(\lambda)$ to $g - B(D)\mathcal{S}_+(\lambda)F'$ with $\mathcal{S}_+(\lambda) = (\mathcal{S}_{+1}(\lambda), \lambda\mathcal{S}_{+1}(\lambda), \mathcal{S}_{+2}(\lambda))^\top$ where $\mathcal{S}_{+1}(\lambda)$ and $\mathcal{S}_{+2}(\lambda)$ are defined in (3-4). Observe that

$$B(D)\tilde{\mathcal{S}}_+(\lambda)F' = (\Delta - (1-\beta)\Delta')\mathcal{S}_{+1}(\lambda)F' + \mathcal{S}_{+2}(\lambda)F', \partial_N(\Delta + (1-\beta)\Delta')\mathcal{S}_{+1}(\lambda)F', \partial_N\mathcal{S}_{+2}(\lambda)F')^\top.$$

Thus, we define $\mathcal{T}(\lambda)$ acting on $(F', G, G', g_1'')$ by

$$\mathcal{T}(\lambda)(F', G, G', g_1'') = \mathcal{S}_+(\lambda)F' + \mathcal{V}(\lambda)(G, G', g_1'') - \mathcal{V}(\lambda)(B(D)\mathcal{S}_+(\lambda)F', \lambda^{1/2}B(D)\mathcal{S}_+(\lambda)F', \mathcal{G}_3)$$

with $\mathcal{G}_3 = (\Delta - (1-\beta)\Delta')\mathcal{S}_{+1}(\lambda)F' + \mathcal{S}_{+2}(\lambda)F'$. By (3-5),

$$\begin{aligned}&\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N_+)^2, H^{4-j}(\mathbb{R}^N_+))}\{(\tau\partial_t)^s(\lambda^{j/2}\mathcal{S}_{+1}(\lambda)) \mid \lambda \in \Sigma_{\vartheta_0,\lambda_0}\}) \leq C_{\lambda_0} \quad (s = 0, 1, \ \ j = 0, 1, 2, 3, 4),\\ &\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N_+)^2, H^{2-j}(\mathbb{R}^N_+))}\{(\tau\partial_t)^s(\lambda^{j/2}\mathcal{S}_{+2}(\lambda)) \mid \lambda \in \Sigma_{\vartheta_0,\lambda_0}\}) \leq C_{\lambda_0} \quad (s = 0, 1, \ \ j = 0, 1, 2),\end{aligned} \tag{3-35}$$

and therefore, by (3-34), (3-35) and Proposition 2.2, we obtain Theorem 3.1.

# 4 Proof of Theorem 3.4

The purpose of this section is to prove Theorem 3.4. For this, we start with the uniqueness of the solution of the system of ordinary differential equations:

$$\begin{aligned}\lambda^2 w + (\partial_{\scriptscriptstyle N}^2 - |\xi'|^2)^2 w + (\partial_{\scriptscriptstyle N}^2 - |\xi'|^2)\tau = 0 \quad (x_{\scriptscriptstyle N} > 0),\\ \lambda\tau - (\partial_{\scriptscriptstyle N}^2 - |\xi'|^2)\tau - \lambda(\partial_{\scriptscriptstyle N}^2 - |\xi'|^2)w = 0 \quad (x_{\scriptscriptstyle N} > 0)\end{aligned} \tag{4-1}$$

with boundary conditions

$$\begin{aligned}(\partial_{\scriptscriptstyle N}^2 - |\xi'|^2)w(0) + (1-\beta)|\xi'|^2 w(0) + \tau(0) = 0,\\ \partial_{\scriptscriptstyle N}\big((\partial_{\scriptscriptstyle N}^2 - |\xi'|^2)w(0) - (1-\beta)|\xi'|^2 w(0)\big) = 0,\\ \partial_{\scriptscriptstyle N}\tau(0) = 0.\end{aligned} \tag{4-2}$$

Let $w$ and $\tau$ be $C^\infty$-functions defined on $(0, \infty)$. If $w$ and $\tau$ satisfy equations (4-1) and (4-2) and the condition: $|w(x_{\scriptscriptstyle N})| + |\tau(x_{\scriptscriptstyle N}| \leq Ce^{-cx_{\scriptscriptstyle N}}$ with some positive constants $C$ and $c$, then $w$ and $\tau$ are called stable solutions.

**Lemma 4.1.** *Let $\lambda \in \mathbb{C} \setminus \{0\}$ with $\operatorname{Re}\lambda \geq 0$ and $\xi' \in \mathbb{R}^{N-1}$. If $w$ and $\tau$ be stable solutions of problem* (4-1) *and* (4-2)*, then $w = \tau = 0$.*

*Proof.* Following [24, proof of Lemma 4.1], we multiply the first equation in (4-1) by $\bar{\lambda}\bar{w}$, the second by $\bar{\tau}$, take the sum and integrate over $(0, \infty)$. We get

$$\begin{aligned} S &:= \lambda|\lambda|^2\|w\|^2 + \bar{\lambda}\langle(\partial_N^2 - |\xi'|^2)^2 w, w\rangle + \bar{\lambda}\langle(\partial_N^2 - |\xi'|^2)\tau, w\rangle + \lambda\|\tau\|^2 \\ &\quad - \langle(\partial_N^2 - |\xi'|^2)\tau, \tau\rangle - \lambda\langle(\partial_N^2 - |\xi'|^2)w, \tau\rangle = 0. \end{aligned}$$

Here $\langle\cdot,\cdot\rangle$ is the scalar product in $L_2((0,\infty))$ and $\|\cdot\|$ the norm in $L_2((0,\infty))$. First we show that for all $w$ satisfying the boundary conditions (4-2), we have

$$\langle(\partial_N^2 - |\xi'|^2)^2 w, w\rangle = s - \tau(0)\partial_N \bar{w}(0)$$

with a real and non-negative number $s$. For this we write by integration by parts

$$\begin{aligned} \langle(\partial_N^2 - |\xi'|^2)^2 w, w\rangle &= \|\partial_N^2 w\|^2 + 2|\xi'|^2\|\partial_N w\|^2 + |\xi'|^4\|w\|^2 \\ &\quad - \partial_N^3 w(0)\bar{w}(0) + \partial_N^2 w(0)\partial_N\bar{w}(0) + 2|\xi'|^2\partial_N w(0)\bar{w}(0). \end{aligned}$$

Inserting the boundary conditions (4-2) yields that

$$\begin{aligned} \langle(\partial_N^2 - |\xi'|^2)^2 w, w\rangle &= \|\partial_N^2 w\|^2 + 2|\xi'|^2\|\partial_N w\|^2 + |\xi'|^4\|w\|^2 \\ &\quad - (2-\beta)|\xi'|^2\partial_N w(0)\bar{w}(0) + (\beta|\xi'|^2 w(0) - \tau(0))\partial_N\bar{w}(0) + 2|\xi'|^2\partial_N w(0)\bar{w}(0) \\ &= s - \tau(0)\partial_N\bar{w}(0) \end{aligned}$$

with $s := \|\partial_N^2 w\|^2 + 2|\xi'|^2\|\partial_N w\|^2 + |\xi'|^4\|w\|^2 + 2\beta|\xi'|^2\operatorname{Re}\big(w(0)\partial_N\bar{w}(0)\big)$. Since

$$\begin{aligned} \big|\operatorname{Re}\big(w(0)\partial_N\bar{w}(0)\big)\big| &\leq \big|w(0)\partial_N\bar{w}(0)\big| = \Big|\int_0^\infty \partial_N\big[w(x_N)\partial_N\bar{w}(x_N)\big]dx_N\Big| \\ &\leq \int_0^\infty \Big[|\partial_N w(x_N)|^2 + |w(x_N)\partial_N^2\bar{w}(x_N)|\Big]dx_N \leq \|\partial_N w\|^2 + \|w\|\cdot\|\partial_N^2 w\|, \end{aligned}$$

noting that $\beta < 1$, we have

$$s \geq \|\partial_N^2 w\|^2 + |\xi'|^4\|w\|^2 - 2|\xi'|^2\|w\|\,\|\partial_N^2 w\| \geq 0.$$

With integration by parts again, we have

$$\begin{aligned} S &= \lambda|\lambda|^2\|w\|^2 + \bar{\lambda}s - \bar{\lambda}\tau(0)\partial_N\bar{w}(0) \\ &\quad - \bar{\lambda}\langle\partial_N\tau, \partial_N w\rangle - \bar{\lambda}\partial_N\tau(0)\bar{w}(0) - \bar{\lambda}|\xi'|^2\langle\tau, w\rangle + \lambda\|\tau\|^2 + \|\partial_N\tau\|^2 + \partial_N\tau(0)\bar{\tau}(0) \\ &\quad + |\xi'|^2\|\tau\|^2 + \lambda\langle\partial_N w, \partial_N\tau\rangle + \lambda\partial_N w(0)\bar{\tau}(0) + \lambda|\xi'|^2\langle w, \tau\rangle = 0. \end{aligned}$$

Taking the real part and noting that $\partial_N\tau(0) = 0$ yield that

$$0 = \operatorname{Re} S = \operatorname{Re}\lambda\big[|\lambda|^2\|w\|^2 + s + \|\tau\|^2\big] + \|\partial_N\tau\|^2 + |\xi'|^2\|\tau\|^2.$$

As $\operatorname{Re}\lambda \geq 0$ and $s \geq 0$, we obtain $\|\partial_N\tau\| = 0$. Therefore, $\tau$ is a constant and from $\tau(x_N) \to 0 \quad (x_N \to \infty)$ we get $\tau = 0$. Inserting this into the second equation in (4-1) and noting $\lambda \neq 0$ yield that $(\partial_N^2 - |\xi'|^2)w = 0 \quad (x_N > 0)$. From the first equation in (4-1) we now get $\lambda^2 w = 0$ and thus $w = 0$. □

In what follows, let $\Delta(\xi', \lambda)$ be the Lopatinskiĭ matrix defined in (3-12). To prove Theorem 3.4, one of the tasks is to analyze the determinant of $\Delta(\xi', \lambda)$. The determinant is given by

$$\begin{aligned} \det\Delta(\xi', \lambda) &= \lambda(-\gamma_1\lambda + \zeta)A_2A_3\Big[(-\tfrac{\lambda}{\gamma_2} + \zeta)(\gamma_3^2 + 2) - (-\tfrac{\lambda}{\gamma_3} + \zeta)(\gamma_2^2 + 2)\Big] \\ &\quad + \lambda(-\gamma_2\lambda + \zeta)A_3A_1\Big[(-\tfrac{\lambda}{\gamma_3} + \zeta)(\gamma_1^2 + 2) - (-\tfrac{\lambda}{\gamma_1} + \zeta)(\gamma_3^2 + 2)\Big] \end{aligned}$$

$$+\lambda(-\gamma_3\lambda+\zeta)A_1A_2\Big[(-\tfrac{\lambda}{\gamma_1}+\zeta)(\gamma_2^2+2)-(-\tfrac{\lambda}{\gamma_2}+\zeta)(\gamma_1^2+2)\Big]$$

with $\zeta := (1-\beta)|\xi'|^2$. A simple calculation shows that for $i \neq j$ we have

$$(-\tfrac{\lambda}{\gamma_i}+\zeta)(\gamma_j^2+2)-(-\tfrac{\lambda}{\gamma_j}+\zeta)(\gamma_i^2+2)=(\gamma_i^2-\gamma_j^2)(\tfrac{\lambda}{\gamma_i\gamma_j}-\zeta).$$

Moreover, recalling that $\gamma_1\gamma_2\gamma_3=1$ we have $\frac{\lambda}{\gamma_2\gamma_3}-\zeta=\gamma_1\lambda-\zeta$. Therefore, we get

$$\det\Delta(\xi',\lambda)=\lambda\Big\{(\gamma_3^2-\gamma_2^2)A_2A_3(\gamma_1\lambda-\zeta)^2+(\gamma_1^2-\gamma_3^2)A_3A_1(\gamma_2\lambda-\zeta)^2+(\gamma_2^2-\gamma_1^2)A_1A_2(\gamma_3\lambda-\zeta)^2\Big\}.$$

**(i)** Firstly, we estimate $\det\Delta(\xi',\lambda)$ for $(\xi',\lambda)\in\mathbb{R}^{N-1}\times\Sigma_{\vartheta_0}$ with $|\lambda|\leq\sigma_0|\xi'|^2$, where $\vartheta_0$ is the number given in (2-6) and $\sigma_0$ is a small positive number chosen later. We derive an asymptotic expansion of $\det\Delta(\xi',\lambda)$ in $t:=\frac{\lambda}{|\xi|^2}$ for $t\to 0$. In what follows, the order $O(t^\ell)$ ($\ell=1,2$) means that there exists a $C^\infty$-function $g(t)$ defined on $|t|\leq\sigma_0$ such that $O(t^\ell)$ is represented by $O(t^\ell)=t^\ell g(t)$. Let $f(t)=t^\ell g(t)$, and then by the Bell formula we have

$$|\partial_{\xi'}^{\alpha'}f(\lambda/|\xi'|^2)|\leq C_{f,\alpha'}(|\lambda|/|\xi'|^2)^\ell|\xi'|^{-|\alpha'|} \tag{4-3}$$

for any $\alpha'\in\mathbb{N}_0^{N-1}$ whenever $|\lambda|^{1/2}/|\xi'|\leq\sigma_0$. In fact, we have

$$\begin{aligned}|\partial_{\xi'}^{\alpha'}f(\lambda/|\xi'|^2)|&\leq C_{\alpha'}\sum_{m=1}^{|\alpha'|}|f^{(m)}(\lambda/|\xi'|^2)|\sum_{\substack{\alpha'_1+\cdots+\alpha'_m=\alpha'\\|\alpha'_k|\geq1}}|\partial_{\xi'}^{\alpha'_1}(\lambda/|\xi'|^2)|\cdots|\partial_{\xi'}^{\alpha'_m}(\lambda/|\xi'|^2)|\\&\leq C_{f,\alpha'}(|\lambda/|\xi'|^2)^\ell|\xi'|^{-|\alpha'|}\end{aligned}$$

provided that $|\lambda|^{1/2}/|\xi'|\leq\sigma_0(\leq 1)$.

Since $\sigma_0$ will be chosen as a small positive number eventually, we may assume that $\sigma_0|\gamma_j|^{-1}\leq 1/2$ for $j=1,2,3$. Let $t=\lambda/|\xi'|^2$ and $|t|\leq\sigma_0/2$. By the Taylor formula, we have $A_j=\sqrt{\gamma_j^{-1}\lambda+|\xi'|^2}=|\xi'|(1+(2\gamma_j)^{-1}t+O(t^2))$. Since

$$\gamma_j\lambda-\zeta=\gamma_j\lambda-(1-\beta)|\xi'|^2=|\xi'|^2(\gamma_jt-(1-\beta)),$$

we obtain

$$\begin{aligned}\det\Delta(\xi',\lambda)=\lambda|\xi'|^6\Big\{&(\gamma_3^2-\gamma_2^2)\big(1+\tfrac{t}{2\gamma_2}+O(t^2)\big)\big(1+\tfrac{t}{2\gamma_3}+O(t^2)\big)(\gamma_1t-(1-\beta))^2\\&+(\gamma_1^2-\gamma_3^2)+O(t^2)\big(1+\tfrac{t}{2\gamma_3}+O(t^2)\big)\big(1+\tfrac{t}{2\gamma_1}+O(t^2)\big)(\gamma_2t-(1-\beta))^2\\&+(\gamma_2^2-\gamma_1^2)\big(1+\tfrac{t}{2\gamma_1}+O(t^2)\big)\big(1+\tfrac{t}{2\gamma_2}+O(t^2)\big)(\gamma_3t-(1-\beta))^2\Big\}\\=\lambda|\xi'|^6&\big(c_0+c_1t+O(t^2)\big),\quad t\to0.\end{aligned}$$

Here

$$c_0=(1-\beta)^2\big((\gamma_3^2-\gamma_2^2)+(\gamma_1^2-\gamma_3^2)+(\gamma_2^2-\gamma_1^2)\big)=0$$

and

$$c_1=\frac{1}{2}(1-\beta)^2c_{11}-2(1-\beta)c_{12}$$

with

$$\begin{aligned}c_{11}&=(\gamma_3^2-\gamma_2^2)\big(\tfrac{1}{\gamma_2}+\tfrac{1}{\gamma_3}\big)+(\gamma_1^2-\gamma_3^2)\big(\tfrac{1}{\gamma_3}+\tfrac{1}{\gamma_1}\big)+(\gamma_2^2-\gamma_1^2)\big(\tfrac{1}{\gamma_1}+\tfrac{1}{\gamma_2}\big),\\c_{12}&=(\gamma_3^2-\gamma_2^2)\gamma_1+(\gamma_1^2-\gamma_3^2)\gamma_2+(\gamma_2^2-\gamma_1^2)\gamma_3.\end{aligned}$$

Making use of $\gamma_j^3=\gamma_j^2-2\gamma_j+1$ and $\gamma_1\gamma_2\gamma_3=1$, straightforward calculation shows

$$c_{11}=c_{12}=(\gamma_1-\gamma_2)(\gamma_1-\gamma_3)(\gamma_2-\gamma_3).$$

Consequently,

$$c_1 = (\gamma_1 - \gamma_2)(\gamma_1 - \gamma_2)(\gamma_2 - \gamma_3)(1-\beta)\Big(-\frac{3}{2} - \frac{\beta}{2}\Big) \neq 0.$$

We obtain

$$\det \Delta(\xi', \lambda) = |\xi'|^4 \lambda^2 \big(c_1 + O(t)\big) \quad (t = \frac{\lambda}{|\xi'|^2}) \tag{4-4}$$

for $|t| \le \sigma_0$ with sufficiently small $\sigma_0 > 0$.

**(ii)** Secondly, we estimate $\Delta(\xi', \lambda)$ for $(\xi', \lambda) \in \mathbb{R}^{N-1} \times \mathbb{C}$ with $\sigma_0|\xi'|^2 \le |\lambda|$. Let $\tilde\lambda = \lambda r^{-2}$ and $\tilde\xi = \xi r^{-1}$ with $r = (|\lambda| + |\xi'|^2)^{1/2}$, and then, $|\tilde\lambda| + |\tilde\xi'|^2 = 1$. Note that

$$\det \Delta(r\xi', r^2\lambda) = r^8 \det \Delta(\tilde\xi', \tilde\lambda) \quad (r > 0).$$

Let

$$\Xi = \{(\tilde\xi', \tilde\lambda) \in \mathbb{R}^{N-1} \times \mathbb{C} \mid |\tilde\lambda| + |\tilde\xi'|^2 = 1, \ |\tilde\lambda| \ge \frac{\sigma_0}{1+\sigma_0}, \quad |\tilde\xi'|^2 \le \frac{1}{1+\sigma_0}\}.$$

Note that $(\tilde\xi', \tilde\lambda) \in \Xi$ for $(\xi', \lambda) \in \mathbb{R}^{N-1} \times \mathbb{C}$ with $\sigma_0|\xi'|^2 \le |\lambda|$. By Lemma 4.1, we know that $\det \Delta(\tilde\xi', \tilde\lambda) \neq 0$ for $(\tilde\xi', \tilde\lambda) \in \Xi$ with $\operatorname{Re}\tilde\lambda \ge 0$. Since $\det \Delta(\tilde\xi', \tilde\lambda)$ is a continuous function, there exists a $\vartheta > \pi/2$ such that $\det \Delta(\tilde\xi', \tilde\lambda) \neq 0$ for $(\tilde\xi', \tilde\lambda) \in \Xi$ with $\tilde\lambda \in \Sigma_\vartheta$, which furnishes that

$$|\det \Delta(\xi', \lambda)| \ge c_2 (|\lambda|^{1/2} + |\xi|)^8 \tag{4-5}$$

with some $c_2 > 0$ provided that $(\xi', \lambda) \in \mathbb{R}^{N-1} \times \Sigma_\vartheta$ and $\sigma_0|\xi'|^2 \le |\lambda|$.

Next, we consider $\Delta(\xi', \lambda)^{-1} = (g_{ij}(\xi', \lambda))$. Let

$$\begin{aligned}
\Xi_{\vartheta,\sigma_0} &= \{(\xi', \lambda) \in \mathbb{R}^{N-1} \times \Sigma_\vartheta \ \mid \sigma_0|\xi'| \ge |\lambda|^{1/2}, \quad \lambda \in \Sigma_\vartheta\}, \\
\Xi'_{\vartheta,\sigma_0} &= \{(\xi', \lambda) \in \mathbb{R}^{N-1} \times \Sigma_\vartheta \ \mid \sigma_0|\xi'| \le |\lambda|^{1/2}, \quad \lambda \in \Sigma_\vartheta\}.
\end{aligned}$$

**(iii)** Firstly, we consider $g_{i1}(\xi', \lambda)$. The coefficient of $g_{11}$ is given by

$$\begin{aligned}
g_{11}(\xi', \lambda) &= \frac{\lambda A_2 A_3}{\det \Delta(\xi', \lambda)} \Big[\Big(-\frac{\lambda}{\gamma_2} + \zeta\Big)(\gamma_3^2 + 2) - \Big(-\frac{\lambda}{\gamma_3} + \zeta\Big)(\gamma_2^2 + 2)\Big] \\
&= \frac{\lambda A_2 A_3}{\det \Delta(\xi', \lambda)} \Big[\lambda\Big(\frac{\gamma_2^2}{\gamma_3} - \frac{\gamma_3^2}{\gamma_2} + \frac{2}{\gamma_3} - \frac{2}{\gamma_2}\Big) + \zeta(\gamma_3^2 - \gamma_2^2)\Big] \\
&= \frac{\lambda A_2 A_3}{\det \Delta(\xi', \lambda)} \Big[\lambda \ \frac{\gamma_2^3 - \gamma_3^3 + 2\gamma_2 - 2\gamma_3}{\gamma_2\gamma_3} + \zeta(\gamma_3^2 - \gamma_2^2)\Big] \\
&= \frac{\lambda A_2 A_3}{\det \Delta(\xi', \lambda)} (\gamma_2^2 - \gamma_3^2)(\gamma_1 \lambda - \zeta)
\end{aligned}$$

where we again used $\gamma_j^3 + 2\gamma_j = \gamma_j^2 + 1$ and $\gamma_1\gamma_2\gamma_3 = 1$. By (4-4) and the Bell formula, we have

$$|\partial_{\xi'}^{\alpha'} (\det \Delta(\xi', \lambda))^{-1}| \le C_{\alpha'} (|\lambda|^{1/2} + |\xi'|)^{-8-|\alpha'|} \tag{4-6}$$

for any $\alpha' \in \mathbb{N}_0^{N-1}$ and $(\xi', \lambda) \in \Xi'_{\vartheta,\sigma_0}$, so that by the Leibniz rule, (3-15), (3-17) with $s = 1$, and the permutation of indices, we have

$$|\partial_{\xi'}^{\alpha'} g_{i1}(\xi', \lambda)| \le C_{\alpha'} (|\lambda|^{1/2} + |\xi'|)^{-2-|\alpha'|} \quad (i = 1, 2, 3) \tag{4-7}$$

for any $\alpha' \in \mathbb{N}_0^{N-1}$ and $(\xi', \lambda) \in \Xi'_{\vartheta,\sigma_0}$.

Moreover, by (4-4)

$$\begin{aligned}
g_{11}(\xi', \lambda) &= \frac{1}{\lambda} \ \frac{1}{c_1 + O(t)} \Big[\Big(1 + \frac{t}{2\gamma_2}\Big)\Big(1 + \frac{t}{2\gamma_2}\Big)(\gamma_2^2 - \gamma_3^2)(\gamma_1 t - (1-\beta)) + O(t^2)\Big] \\
&= \frac{1}{\lambda}(\mu_{11} + O(t)) \quad (t = \frac{\lambda}{|\xi'|^2})
\end{aligned}$$

with $\mu_{11} = -\frac{(\gamma_2^2-\gamma_3^2)(1-\beta)}{c_1}$ for any $(\xi', \lambda) \in \Xi_{\vartheta,\sigma_0}$. By permutation of indices, we get the same result for $g_{21}, g_{31}$ with

$$\mu_{21} = -\frac{(\gamma_3^2 - \gamma_1^2)(1-\beta)}{c_1}, \quad \mu_{31} = -\frac{(\gamma_1^2 - \gamma_2^2)(1-\beta)}{c_1},$$

so that we have

$$|\partial_{\xi'}^{\alpha'}(\lambda g_{i1}(\xi', \lambda) - \mu_{i1})| \leq C_{\alpha'}|\lambda||\xi'|^{-2-|\alpha'|} \quad (i = 1, 2, 3) \tag{4-8}$$

for any $\alpha' \in \mathbb{N}_0^{N-1}$ and $(\xi', \lambda) \in \Xi_{\vartheta,\sigma_0}$, which, combined with (4-7), furnishes that

$$|\partial_{\xi'}^{\alpha'}\lambda g_{i1}(\xi', \lambda)| \leq C_{\alpha'}(|\lambda|^{-1/2} + |\xi'|)^{-|\alpha'|} \quad (i = 1, 2, 3)$$

for any $\alpha' \in \mathbb{N}_0^{N-1}$ and any $(\xi', \lambda) \in \mathbb{R}^{N-1} \times \Sigma_\vartheta$. Here, we have used the fact: $|\xi'|^{-1} \leq C(|\lambda|^{1/2} + |\xi'|)^{-1}$ for $(\xi', \lambda) \in \mathbb{R}^{N-1} \times \Sigma_\vartheta$ with $|\lambda| \leq c_0|\xi'|^2$.

Moreover, since $\mu_{11} + \mu_{21} + \mu_{31} = 0$, by (4-8) we have

$$|\partial_{\xi'}^{\alpha'}(\sum_{i=1}^{3} g_{i1}(\xi', \lambda))| \leq C_{\alpha'}|\xi'|^{-2-|\alpha'|} \tag{4-9}$$

for any $\alpha' \in \mathbb{N}_0^{N-1}$ and $(\xi', \lambda) \in \Xi_{\vartheta,\sigma_0}$

**(iv)** Secondly, we estimate $g_{i2}$. The coefficient of $g_{12}$ is given by

$$g_{12}(\xi', \lambda) = \frac{\lambda}{\det \Delta(\xi', \lambda)}\Big[(-\gamma_3\lambda + \zeta)A_2(\gamma_2^2 + 2) - (-\gamma_2\lambda + \zeta)A_3(\gamma_3^2 + 2)\Big].$$

By (4-6), Leibniz rule, (3-15), (3-16) with $s = 1$, and the permutation of indices, we have

$$|\partial_{\xi'}^{\alpha'} g_{i2}(\xi', \lambda)| \leq C_{\alpha'}(|\lambda|^{1/2} + |\xi'|)^{-3-|\alpha'|} \quad (i = 1, 2, 3) \tag{4-10}$$

for any $\alpha' \in \mathbb{N}_0^{N-1}$ and $(\xi', \lambda) \in \Xi'_{\vartheta,\sigma_0}$.

The asymptotic expansion for $t = \frac{\lambda}{|\xi'|^2} \to 0$ is given by

$$g_{12}(\xi', \lambda) = \frac{1}{\lambda|\xi'|}\Big(\frac{(1-\beta)(\gamma_2^2 - \gamma_3^2)}{c_1} + O(t)\Big)$$

for any $(\xi', \lambda) \in \Xi_{\vartheta,\sigma_0}$. By permutation of indices, we get

$$\mu_{12} = \frac{1}{c_1}(1-\beta)(\gamma_2^2 - \gamma_3^2), \quad \mu_{22} = \frac{1}{c_1}(1-\beta)(\gamma_3^2 - \gamma_1^2), \quad \mu_{32} = \frac{1}{c_1}(1-\beta)(\gamma_1^2 - \gamma_2^2).$$

Thus, noting that $\mu_{12} + \mu_{22} + \mu_{23} = 0$, we have

$$|\partial_{\xi'}^{\alpha'}\lambda g_{i2}(\xi', \lambda)| \leq C_{\alpha'}|\xi'|^{-1-|\alpha'|}, \quad |\partial_{\xi'}^{\alpha'}(\sum_{i=1}^{3} g_{i2}(\xi', \lambda))| \leq C_{\alpha'}|\xi'|^{-3-|\alpha'|}$$

for any $\alpha' \in \mathbb{N}_0^{N-1}$ and $(\xi', \lambda) \in \Xi_{\vartheta,\sigma_0}$. Moreover, using (4-10), we have

$$|\partial_{\xi'}^{\alpha'}\lambda g_{i2}(\xi', \lambda)| \leq C_{\alpha'}(|\lambda|^{-1/2} + |\xi'|)^{-1-|\alpha'|} \quad (i = 1, 2, 3)$$

for any $\alpha' \in \mathbb{N}_0^{N-1}$ and any $(\xi', \lambda) \in \mathbb{R}^{N-1} \times \Sigma_\vartheta$.

**(v)** Thirdly, we estimate $g_{i3}$. The coefficient of $g_{13}$ is given by

$$g_{13}(\xi', \lambda) = \frac{1}{\det \Delta(\xi', \lambda)}\Big[(-\gamma_2\lambda + \zeta)A_3\Big(-\frac{\lambda}{\gamma_3} + \zeta\Big) - (-\gamma_3\lambda + \zeta)A_2\Big(-\frac{\lambda}{\gamma_2} + \zeta\Big)\Big].$$

By (4-6), Leibniz rule, (3-15), (3-16) with $s = 1$, and the permutation of indices, we have

$$|\partial_{\xi'}^{\alpha'} g_{i3}(\xi', \lambda)| \leq C_{\alpha'}(|\lambda|^{1/2} + |\xi'|)^{-3-|\alpha'|} \quad (i = 1, 2, 3) \tag{4-11}$$

for any $\alpha' \in \mathbb{N}_0^{N-1}$ and $(\xi', \lambda) \in \Xi'_{\vartheta,\sigma_0}$.

The asymptotic expansion for $t = \frac{\lambda}{|\xi'|^2} \to 0$ is given by

$$\begin{aligned} g_{13}(\xi', \lambda) &= \frac{|\xi'|^5}{\lambda^2 |\xi'|^4 (c_1 + O(t))} \Big[ ((1-\beta) - \gamma_2 t)\Big(1 + \frac{t}{2\gamma_3}\Big)\Big((1-\beta) - \frac{t}{\gamma_3}\Big) \\ &\quad - ((1-\beta) - \gamma_3 t)\Big(1 + \frac{t}{2\gamma_2}\Big)\Big((1-\beta) - \frac{t}{\gamma_2}\Big) + O(t^2)\Big] \\ &= \frac{|\xi'|}{\lambda^2 (c_1 + O(t))}\Big[d_1 t + O(t^2)\Big] = \frac{1}{\lambda|\xi'|}\Big[\frac{d_1}{c_1} + O(t)\Big] \end{aligned}$$

for any $(\xi', \lambda) \in \Xi_{\vartheta,\sigma_0}$, where

$$\begin{aligned} d_1 &= \Big( -\gamma_2(1-\beta) + \frac{1}{2\gamma_3}(1-\beta)^2 - \frac{1}{\gamma_3}(1-\beta)\Big) \\ &\quad - \Big( -\gamma_3(1-\beta) + \frac{1}{2\gamma_2}(1-\beta)^2 - \frac{1}{\gamma_2}(1-\beta)\Big) \\ &= (1-\beta)(\gamma_3 - \gamma_2) - (1-\beta)\Big(\frac{1}{2} + \frac{\beta}{2}\Big)\Big(\frac{1}{\gamma_3} - \frac{1}{\gamma_2}\Big). \end{aligned}$$

It is easily seen that $d_1 \neq 0$, and we get $g_{13}(\xi', \lambda) = \dfrac{1}{\lambda|\xi'|}(\mu_{31} + O(t))$ with

$$\mu_{31} = \frac{1}{c_1}(1-\beta)\Big(\gamma_3 - \gamma_2 - \Big(\frac{1}{2} + \frac{\beta}{2}\Big)\Big(\frac{1}{\gamma_3} - \frac{1}{\gamma_2}\Big)\Big)$$

for $(\xi', \lambda) \in \Xi_{\vartheta,\sigma_0}$. For $g_{23}$ and $g_{33}$ we obtain the same result with

$$\begin{aligned} \mu_{32} &= \frac{1}{c_1}(1-\beta)\Big(\gamma_1 - \gamma_3 - \Big(\frac{1}{2} + \frac{\beta}{2}\Big)\Big(\frac{1}{\gamma_1} - \frac{1}{\gamma_3}\Big)\Big), \\ \mu_{33} &= \frac{1}{c_1}(1-\beta)\Big(\gamma_2 - \gamma_1 - \Big(\frac{1}{2} + \frac{\beta}{2}\Big)\Big(\frac{1}{\gamma_2} - \frac{1}{\gamma_1}\Big)\Big), \end{aligned}$$

Thus, noting that $\mu_{31} + \mu_{32} + \mu_{33} = 0$, we have

$$|\partial_{\xi'}^{\alpha'} \lambda g_{i3}(\xi', \lambda)| \le C_{\alpha'}|\xi'|^{-1-|\alpha'|}, \quad |\partial_{\xi'}^{\alpha'}(\sum_{i=1}^{3} g_{i3}(\xi', \lambda))| \le C_{\alpha'}|\xi'|^{-3-|\alpha'|}$$

for any $\alpha' \in \mathbb{N}_0^{N-1}$ and $(\xi', \lambda) \in \Xi_{\vartheta,\sigma_0}$. Moreover, using (4-11), we have

$$|\partial_{\xi'}^{\alpha'} \lambda g_{i3}(\xi', \lambda)| \le C_{\alpha'}(|\lambda|^{-1/2} + |\xi'|)^{-1-|\alpha'|} \quad (i = 1, 2, 3)$$

for any $\alpha' \in \mathbb{N}_0^{N-1}$ and any $(\xi', \lambda) \in \mathbb{R}^{N-1} \times \Sigma_\vartheta$.

Analogously, we can treat $\tau\partial_\tau g_{ij}(\xi', \lambda)$, so that we have proved Theorem 3.4.

# 5 Problem in a bent half-space

Let $\Phi$ be a diffeomorphism of class $H^4_\infty$ from $\mathbb{R}^N$ onto itself and let $\Phi^{-1}$ be its inverse operator. Let $\nabla\Phi(x) = \mathcal{A} + B(x)$ and $\nabla\Phi^{-1}(y) = \mathcal{A}_{-1} + B_{-1}(y)$, where we assume that $\mathcal{A}$ and $\mathcal{A}_{-1}$ are orthonormal matrices with constant coefficients and $B(x)$ and $B_{-1}(y)$ are matrices of $H^3_\infty(\mathbb{R}^N)$-functions which satisfy the conditions:

$$\|(B, B_{-1})\|_{L_\infty(\mathbb{R}^N)} \le M_1, \quad \|\nabla(B, B_{-1})\|_{H^2_\infty(\mathbb{R}^N)} \le M_2. \tag{5-1}$$

We choose $M_1$ small enough eventually, so that we may assume that $0 < M_1 \le 1 \le M_2$ without loss of generality. Let $\Omega_+ = \Phi(\mathbb{R}^N_+)$ and $\Gamma_+ = \Phi(\mathbb{R}^N_0)$ and let $\nu_+$ be the unit outer normal to $\Gamma_+$.

In this section, we consider the equations:

$$\lambda U - A(D)U = F \quad \text{in } \Omega_+, \quad B(D)U = G \quad \text{on } \Gamma_+ \tag{5-2}$$

with $F = (0, f_1, f_2)^\top$ and $G = (g_1, g_2, g_3)^\top$. The purpose of this section is to prove

**Theorem 5.1.** *Let $1 < q < \infty$ and let $\vartheta$ be the same number as in Theorem 3.1. Let $M_1$ in* (5-1) *be sufficiently small. Then, there exist a constant $\lambda_1 > 0$ and operator families $\mathcal{T}_{+i}(\lambda)$ $(i = 1, 2)$ with*

$$\mathcal{T}_{+1}(\lambda) \in \mathcal{C}(\Sigma_{\vartheta,\lambda_1}, \mathcal{L}(\mathcal{X}_q(\Omega_+), H^4_q(\Omega_+))), \quad \mathcal{T}_{+2}(\lambda) \in \mathcal{C}(\Sigma_{\vartheta,\lambda_1}, \mathcal{L}(\mathcal{X}_q(\Omega_+), H^2_q(\Omega_+)))$$

*such that problem* (5-2) *admits a unique solution*

$$U = (\mathcal{T}_{+1}(\lambda)H_\lambda(F, G), \lambda\mathcal{T}_{+1}(\lambda)H_\lambda(F, G), \mathcal{T}_{+2}(\lambda)H_\lambda(F, G))^\top$$

*for any $(F, G) \in \mathbb{G}_q(\Omega_+)$ and $\lambda \in \Sigma_{\vartheta,\lambda_1}$, and there hold the estimates:*

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega_+),H^{4-j}_q(\Omega_+))}(\{(\tau\partial_\tau)^s(\lambda^{j/2}\mathcal{T}_{+1}(\lambda)) \mid \lambda \in \Sigma_{\vartheta,\lambda_0}\}) \le C_{N,q,\lambda_0} \quad (s = 0, 1, \ \ j = 0, 1, 2, 3, 4),$$
$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega_+),H^{2-j}_q(\Omega_+))}(\{(\tau\partial_\tau)^s(\lambda^{j/2}\mathcal{T}_{+2}(\lambda)) \mid \lambda \in \Sigma_{\vartheta,\lambda_0}\}) \le C_{N,q,\lambda_0} \quad (s = 0, 1, \ \ j = 0, 1, 2).$$

In what follows, we prove Theorem 5.1. We use the change of variables: $y = \Phi(x)$ to transfer problem (5-2) to the half space case. Let

$$\frac{\partial x_j}{\partial y_k}(\Phi(x)) = A_{jk} + B_{jk}(x), \tag{5-3}$$

and then, by (5-1)

$$\sum_{j=1}^N A_{jk}A_{j\ell} = \sum_{j=1}^N A_{kj}A_{\ell j} = \delta_{k\ell}, \quad \|B_{jk}\|_{L_\infty(\mathbb{R}^N)} \le M_1, \quad \|\nabla B_{jk}\|_{H^2_\infty(\mathbb{R}^N)} \le M_2. \tag{5-4}$$

Since $\Gamma_+$ is represented by $x_N = \Phi_{-,N}(y) = 0$ with $\Phi^{-1} = (\Phi_{-,1}, \dots, \Phi_{-,N})$, we have

$$\nu_+(y) = -d^{-1}\Big(\frac{\partial x_N}{\partial y_1}, \dots, \frac{\partial x_N}{\partial y_N}\Big)(y) \quad (y \in \Gamma_+) \tag{5-5}$$

with $d = \sqrt{\sum_{j=1}^N(\partial x_N/\partial y_j)^2}$. By (5-5), we may assume that $\nu_+$ is defined in $\mathbb{R}^N$. Moreover, choosing $M_1 > 0$ sufficiently small, by (5-3) and (5-4) we have

$$\nu_+ = -(A_{N1}, \dots, A_{NN}) + \tilde\nu_+ \tag{5-6}$$

with some vector of functions $\tilde\nu_+$ defined on $\mathbb{R}^N$ satisying the estimate:

$$\|\tilde\nu_+\|_{L_\infty(\mathbb{R}^N)} \le CM_1, \quad \|\nabla\tilde\nu_+\|_{H^2_\infty(\mathbb{R}^N)} \le C_{M_2}, \tag{5-7}$$

where $C_{M_2}$ is some constant depending on $M_2$. Using the relation

$$\frac{\partial}{\partial y_j} = \sum_{k=1}^N \frac{\partial x_k}{\partial y_j}\frac{\partial}{\partial x_k} = \sum_{k=1}^N (A_{kj} + B_{kj}(x))\frac{\partial}{\partial x_k}, \tag{5-8}$$

by (5-4) we have

$$\Delta_y = \Delta_x + E^1(D), \quad \Delta_y^2 = \Delta_x^2 + E^2(D), \tag{5-9}$$

where $E^1(D)$ and $E^2(D)$ are some partial differential operators of the form:

$$E^1(D) = \sum_{1\le|\alpha|\le2} e^1_\alpha(x)\partial_x^\alpha, \quad E^2(D) = \sum_{1\le|\alpha|\le4} e^2_\alpha(x)\partial_x^\alpha,$$

with

$$\begin{aligned} &\|e^1_\alpha\|_{L_\infty(\mathbb{R}^N)} \le CM_1 \ \ (|\alpha| = 2), \quad \|e^2_\alpha\|_{L_\infty(\mathbb{R}^N)} \le CM_1 \ \ (|\alpha| = 4), \\ &\|\nabla e^1_\alpha\|_{H^2_\infty(\mathbb{R}^N)} \le C_{M_2} \ \ (|\alpha| = 2), \quad \|\nabla e^2_\alpha\|_{H^2_\infty(\mathbb{R}^N)} \le C_{M_2} \ \ (|\alpha| = 4), \\ &\|e^1_\alpha\|_{H^2_\infty(\mathbb{R}^N)} \le C_{M_2} \ \ (|\alpha| = 1), \quad \|e^2_\alpha\|_{H^{|\alpha|-1}_\infty(\mathbb{R}^N)} \le C_{M_2} \ \ (1 \le |\alpha| \le 3). \end{aligned} \tag{5-10}$$

Moreover, by (5-1), (5-4), (5-6) and (5-7),

$$\partial_{\nu_+} \circ \Phi = -\partial_N + E^3(D), \quad \Delta' = \Delta_{x'} + E^4(D) \tag{5-11}$$

where $\Delta_{x'} = \sum_{j=1}^{N-1} \partial_j^2$, and $E^3(D)$ and $E^4(D)$ are some partial differential operators of the form:

$$E^3(D) = \sum_{|\alpha|=1} e^3_\alpha \partial^\alpha_x, \quad E^4(D) = \sum_{j,k=1}^{N-1} e^4_{jk}\partial_j\partial_k + \sum_{j=1}^{N-1} e^4_j \partial_j.$$

with

$$\begin{aligned} \|e^3_\alpha\|_{L_\infty(\mathbb{R}^N)} \le CM_1, \quad \|\nabla e^3_\alpha\|_{H^2_\infty(\mathbb{R}^N)} \le C_{M_2}, \\ \|e^4_{jk}\|_{L_\infty(\mathbb{R}^N)} \le CM_1, \quad \|(\nabla e^4_{jk}, e^4_j)\|_{H^2_\infty(\mathbb{R}^N)} \le C_{M_2}. \end{aligned} \tag{5-12}$$

Thus, problem (5-2) is transformed to

$$\lambda U - A(D)U - P(D)U = \tilde F \quad \text{in } \mathbb{R}^N_+, \quad B'(D)U - Q(D)U = \tilde G \quad \text{on } \mathbb{R}^N_0 \tag{5-13}$$

with $\tilde F = F\circ\Phi$, $\tilde G = (g_1\circ\Phi, -g_2\circ\Phi, -g_3\circ\Phi)^\top$. For simplicity, we continue to write $F$ and $G$ instead of $\tilde F$ and $\tilde G$, respectively.

$$\begin{aligned} &P(D) = \begin{pmatrix} 0 & 0 & 0 \\ -E^2(D) & 0 & -E^1(D) \\ 0 & E^1(D) & E^1(D) \end{pmatrix} \quad B'(D) = \begin{pmatrix} \Delta - (1-\beta)\Delta'_{x'} & 0 & 1 \\ \partial_N(\Delta + (1-\beta)\Delta_{x'}) & 0 & 0 \\ 0 & 0 & \partial_N \end{pmatrix}, \\ &Q(D) = -\begin{pmatrix} E^1(D) - (1-\beta)E^4(D) & 0 & 0 \\ Q_{21}(D) & 0 & 0 \\ 0 & 0 & E^3(D) \end{pmatrix}, \\ &Q_{21}(D) = E^3(D)(\Delta + (1-\beta)\Delta_{x'}) + (\partial_N + E^3(D))(E^1(D) + (1-\beta)E^4(D)). \end{aligned}$$

Recall that

$$H_\lambda(F,G) = (f_1, f_2, G, \lambda^{1/2}G, \lambda g_1) \quad \text{for } (F,G) \in \mathbb{G}_q(\mathbb{R}^N_+) \text{ with } F = (0, f_1, f_2)^\top \text{ and } G = (g_1, g_2, g_3)^\top.$$

Let $\mathcal{T}_i(\lambda)$ $(i = 1, 2)$ be the operators given in Theorem 3.1. Setting $\mathcal{T}(\lambda) := (\mathcal{T}_1(\lambda), \lambda\mathcal{T}_1(\lambda), \mathcal{T}_2(\lambda))^\top$ and $\tilde U := \mathcal{T}(\lambda)H_\lambda(F,G)$, we have

$$\begin{cases} \lambda\tilde U - A(D)\tilde U - P(D)\tilde U = F - P(D)\mathcal{T}(\lambda)H_\lambda(F,G) & \text{in } \mathbb{R}^N_+, \\ \quad B'(D)\tilde U - Q(D)\tilde U = G - Q(D)\mathcal{T}(\lambda)H_\lambda(F,G) & \text{on } \mathbb{R}^N_0 \end{cases} \tag{5-14}$$

with $F = (0, f_1, f_2)^\top$. Let

$$\begin{aligned} &L_\lambda(D) = (\lambda\mathbf{I} - A(D) - P(D), \quad B'(D) - Q(D)), \\ &\mathcal{U}(\lambda)(F', G, G', g_1'') = (P(D)\mathcal{T}(\lambda)(F', G, G', g_1''), Q(D)\mathcal{T}(\lambda)(F', G, G', g_1'')), \end{aligned}$$

where $\mathbf{I}$ is the $3\times 3$ identity matrix. Then, problem (5-14) can be written in the form

$$L_\lambda(D)\mathcal{T}(\lambda)H_\lambda(F,G) = (F,G) - \mathcal{U}(\lambda)H_\lambda(F,G) = \big(I - \mathcal{U}(\lambda)H_\lambda\big)(F,G) \quad \text{on } \mathbb{R}^N_+ \times \mathbb{R}^N_0 \tag{5-15}$$

with $F = (0, f_1, f_2)^\top$ and $G = (g_1, g_2, g_3)^\top$. In the following, we will show that $I - H_\lambda\mathcal{U}(\lambda)$ is invertible.

Noting that

$$P(D)\mathcal{T}(\lambda)(F', G, G', g_1'') = \begin{pmatrix} 0 \\ -E^2(D)\mathcal{T}_1(\lambda)(F', G, G', g_1'') - E^1(D)\mathcal{T}_2(\lambda)(F', G, G', g_1'') \\ E^1(D)\lambda\mathcal{T}_1(\lambda)(F', G, G', g_1'') + E^1(D)\mathcal{T}_2(\lambda)(F', G, G', g_1'') \end{pmatrix},$$

we have

$$H_\lambda \mathcal{U}(\lambda)(F', G, G', g_1'') = \begin{pmatrix} -E^2(D)\mathcal{T}_1(\lambda)(F', G, G', g_1'') - E^1(D)\mathcal{T}_2(\lambda)(F', G, G', g_1'') \\ E^1(D)\lambda\mathcal{T}_1(\lambda)(F', G, G', g_1'') + E^1(D)\mathcal{T}_2(\lambda)(F', G, G', g_1'') \\ Q(D)\mathcal{T}(\lambda)(F', G, G', g_1'') \\ \lambda^{1/2}Q(D)\mathcal{T}(\lambda)(F', G, G', g_1'') \\ \lambda(E^1(D)\mathcal{T}_1(\lambda) - (1-\beta)E^4(D))\mathcal{T}_1(\lambda)(F', G, G', g_1'') \end{pmatrix}.$$

Recall the definition of $\mathcal{X}_q(D)$ given in (1-7). By Theorem 3.1, Proposition 2.2, (5-10) and (5-12), we will show that

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\mathbb{R}^N_+))}(\{(\tau\partial_\tau)^s(H_\lambda \mathcal{U}(\lambda)) \mid \lambda \in \Sigma_{\vartheta,\lambda_1}\}) \leq CM_1 + C_{M_2}\lambda_1^{-1/2} \quad (s = 0, 1) \tag{5-16}$$

holds for any $\lambda_1 \geq \lambda_0$. Here, $\lambda_0$ is the same number as in Theorem 3.1, and $C$ and $C_{M_2}$ are constants, where $C$ is independent of $M_1$, $M_2$ and $\lambda_1$ and $C_{M_2}$ independent of $M_1$ and $\lambda_1$. In fact,

$$\begin{aligned}
&\int_0^1 \Big\| \sum_{k=1}^n r_k(\omega) P(D)\mathcal{T}(\lambda_k)(F_k', G_k, G_k', g_{1k}'') \Big\|^q_{L_q(\mathbb{R}^N_+)^3} \, d\omega \\
&\leq C \int_0^1 \Big\{ M_1 \Big\| \nabla^4 (\sum_{k=1}^n r_k(\omega)\mathcal{T}_1(\lambda_k)(F_k', G_k, G_k', g_{1k}'')) \Big\|^q_{L_q(\mathbb{R}^N_+)} \\
&+ C_{M_2} \Big\| \sum_{k=1}^n r_k(\omega)\mathcal{T}_1(\lambda_k)(F_k', G_k, G_k', g_{1k}'') \Big\|^q_{H^3_q(\mathbb{R}^N_+)} \\
&+ M_1 \Big\| \nabla^2 (\sum_{k=1}^n r_k(\omega)\lambda_k\mathcal{T}_1(\lambda_k)(F_k', G_k, G_k', g_{1k}'')) \Big\|^q_{L_q(\mathbb{R}^N_+)} \\
&+ C_{M_2} \| \sum_{k=1}^n r_k(\omega)\lambda_k\mathcal{T}_1(\lambda_k)(F_k', G_k, G_k', g_{1k}'') \Big\|^q_{H^1_q(\mathbb{R}^N_+)} \\
&+ M_1 \Big\| \nabla^2 (\sum_{k=1}^n r_k(\omega)\mathcal{T}_2(\lambda_k)(F_k', G_k, G_k', g_{1k}'')) \Big\|^q_{L_q(\mathbb{R}^N_+)} \\
&+ C_{M_2} \Big\| \sum_{k=1}^n r_k(\omega)\mathcal{T}_2(\lambda_k)(F_k', G_k, G_k', g_{1k}'') \Big\|^q_{H^1_q(\mathbb{R}^N_+)} \Big\} \, d\omega \\
&\leq (CM_1 + C_{M_2}\lambda_1^{-1/2}) \int_0^1 \Big\| \sum_{k=1}^n r_k(\omega)(F_k', G_k, G_k', g_{1k}'') \Big\|^q_{\mathcal{X}_q(\mathbb{R}^N_+)} \, d\omega,
\end{aligned}$$

which, combined with Theorem 3.1, furnishes that

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\mathbb{R}^N_+), Lq(\mathbb{R}^N_+)^3)}(\{P(D)\mathcal{T}(\lambda) \mid \lambda \in \Sigma_{\vartheta,\lambda_1}\}) \leq CM_1 + C_{M_2}\lambda_1^{-1/2}$$

for any $\lambda_1 \geq \lambda_0$. By Theorem 3.1, Proposition 2.2, (5-10), and (5-12),

$$\begin{aligned}
&\int_0^1 \Big\| \sum_{k=1}^n r_k(\omega) Q(D)\mathcal{T}(\lambda_k)(F_k', G_k, G_k', g_{1k}'') \Big\|^q_{H^2_q(\mathbb{R}^N_+) \times H^1_q(\mathbb{R}^N_+)^2} \, d\omega \\
&\leq C \int_0^1 \Big\{ M_1 \Big\| \nabla^4 (\sum_{k=1}^n r_k(\omega)\mathcal{T}_1(\lambda_k)(F_k', G_k, G_k', g_{1k}'')) \Big\|^q_{L_q(\mathbb{R}^N_+)} \\
&+ C_{M_2} \Big\| \sum_{k=1}^n r_k(\omega)\mathcal{T}_1(\lambda_k)(F_k', G_k, G_k', g_{1k}'') \Big\|^q_{H^3_q(\mathbb{R}^N_+)} \\
&+ M_1 \Big\| \nabla^2 (\sum_{k=1}^n r_k(\omega)\mathcal{T}_2(\lambda_k)(F_k', G_k, G_k', g_{1k}'')) \Big\|^q_{L_q(\mathbb{R}^N_+)}
\end{aligned}$$

$$
\begin{aligned}
&+ C_{M_2}\Big\| \sum_{k=1}^n r_k(\omega)\mathcal{T}_2(\lambda_k)(F_k', G_k, G_k', g_{1k}'')\Big\|_{H^1_q(\mathbb{R}^N_+)}^q \Bigg\} \, d\omega \\
&\le (CM_1 + C_{M_2}\lambda_1^{-1/2}) \int_0^1 \Big\| \sum_{k=1}^n r_k(\omega)(F_k', G_k, G_k', g_{1k}'')\Big\|_{\mathcal{X}_q(\mathbb{R}^N_+)}^q \, d\omega;
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^1 \Big\| \sum_{k=1}^n r_k(\omega)\lambda_k^{1/2} Q(D)\mathcal{T}(\lambda_k)(F_k', G_k, G_k', g_{1k}'')\Big\|_{H^1_q(\mathbb{R}^N_+)\times L_q(\mathbb{R}^N_+)^2}^q \, d\omega \\
&\quad \le C\int_0^1 \Bigg\{ M_1 \Big\| \nabla^3 (\sum_{k=1}^n r_k(\omega)\lambda_k^{1/2}\mathcal{T}_1(\lambda_k)(F_k', G_k, G_k', g_{1k}''))\Big\|_{L_q(\mathbb{R}^N_+)}^q \\
&\quad + C_{M_2}\Big\| \sum_{k=1}^n r_k(\omega)\lambda_k^{1/2}\mathcal{T}_1(\lambda_k)(F_k', G_k, G_k', g_{1k}'')\Big\|_{H^2_q(\mathbb{R}^N_+)}^q \\
&\quad + M_1 \Big\| \nabla (\sum_{k=1}^n r_k(\omega)\lambda_k^{1/2}\mathcal{T}_2(\lambda_k)(F_k', G_k, G_k', g_{1k}''))\Big\|_{L_q(\mathbb{R}^N_+)}^q \\
&\quad + C_{M_2}\Big\| \sum_{k=1}^n r_k(\omega)\lambda_k^{1/2}\mathcal{T}_2(\lambda_k)(F_k', G_k, G_k', g_{1k}'')\Big\|_{L_q(\mathbb{R}^N_+)}^q \Bigg\} \, d\omega \\
&\le (CM_1 + C_{M_2}\lambda_1^{-1/2}) \Big\| \sum_{k=1}^n r_k(\omega)(F_k', G_k, G_k', g_{1k}'')\Big\|_{\mathcal{X}_q(\mathbb{R}^N_+)}^q \, d\omega.
\end{aligned}
$$

Moreover, setting $A^1(\lambda) = (E^1(D) - (1-\beta)E^4(D))\mathcal{T}_1(\lambda)$, which is the first component of $Q(D)\mathcal{T}(\lambda)$, we have

$$
\begin{aligned}
&\int_0^1 \| \sum_{k=1}^n r_k(\omega)\lambda_k A^1(\lambda_k)(F_k', G_k, G_k', g_{1k}'')\|_{L_q(\mathbb{R}^N_+)}^q \, d\omega \\
&\quad \le C\int_0^1 \Bigg\{ M_1 \Big\| \nabla^2 (\sum_{k=1}^n r_k(\omega)\lambda_k\mathcal{T}_1(\lambda_k)(F_k', G_k, G_k', g_{1k}''))\Big\|_{L_q(\mathbb{R}^N_+)}^q \\
&\quad + C_{M_2}\Big\| \sum_{k=1}^n r_k(\omega)\lambda_k\mathcal{T}_1(\lambda_k)(F_k', G_k, G_k', g_{1k}'')\Big\|_{H^1_q(\mathbb{R}^N_+)}^q \Bigg\} \, d\omega \\
&\le (CM_1 + C_{M_2})\lambda_1^{-1/2} \int_0^1 \Big\| \sum_{k=1}^n r_k(\omega)(F_k', G_k, G_k', g_{1k}'')\Big\|_{\mathcal{X}_q(\mathbb{R}^N_+)}^q \, d\omega.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\mathbb{R}^N_+), H^2_q(\mathbb{R}^N_+)\times H^1_q(\mathbb{R}^N_+)^2)}(\{Q(D)\mathcal{T}(\lambda) \mid \lambda \in \Sigma_{\vartheta,\lambda_1}\}) &\le CM_1 + C_{M_2}\lambda_1^{-1/2}, \\
\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\mathbb{R}^N_+), H^1_q(\mathbb{R}^N_+)\times L_q(\mathbb{R}^N_+)^2)}(\{\lambda^{1/2}Q(D)\mathcal{T}(\lambda) \mid \lambda \in \Sigma_{\vartheta,\lambda_1}\}) &\le CM_1 + C_{M_2}\lambda_1^{-1/2}, \\
\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\mathbb{R}^N_+), L_q(\mathbb{R}^N_+))}(\{\lambda A_1(\lambda) \mid \lambda \in \Sigma_{\vartheta,\lambda_1}\}) &\le CM_1 + C_{M_2}\lambda_1^{-1/2}.
\end{aligned}
$$

Summing up, we have proved that

$$
\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\mathbb{R}^N_+))}(\{H_\lambda \mathcal{U}(\lambda) \mid \lambda \in \Sigma_{\vartheta,\lambda_1}\}) \le CM_1 + C_{M_2}\lambda_1^{-1/2}.
$$

Analogously, we have

$$
\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\mathbb{R}^N_+))}(\{\tau\partial_\tau (H_\lambda \mathcal{U}(\lambda)) \mid \lambda \in \Sigma_{\vartheta,\lambda_1}\}) \le CM_1 + C_{M_2}\lambda_1^{-1/2}.
$$

Therefore, we have obtained (5-16).

Choosing $M_1 > 0$ so small that $CM_1 < 1/4$ and $\lambda_1 \geq \lambda_0$ so large that $C_{M_2}\lambda_1^{-1/2} \leq 1/4$, respectively, we have

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\mathbb{R}^N_+))}(\{(\tau\partial_\tau)^s(H_\lambda\mathcal{U}(\lambda)) \mid \lambda \in \Sigma_{\vartheta,\lambda_1}\}) \leq 1/2 \quad (s=0,1). \tag{5-17}$$

Thus, by Proposition 2.2, $(I - H_\lambda\mathcal{U}(\lambda))^{-1} = I + \sum_{j=1}^\infty (H_\lambda\mathcal{U}(\lambda))^j$ exists in $\mathcal{L}(\mathcal{X}_q(\mathbb{R}^N_+))$ and satisfies the estimate:

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\mathbb{R}^N_+))}(\{(\tau\partial_\tau)^s(I - H_\lambda\mathcal{U}(\lambda))^{-1}) \mid \lambda \in \Sigma_{\vartheta,\lambda_1}\}) \leq 2 \quad (s=0,1). \tag{5-18}$$

Recall the definition of space $\mathbb{G}_q(\mathbb{R}^N_+)$ and its norm $\|\cdot\|_{\mathbb{G}_q(\mathbb{R}^N_+)}$ given in (1-7) with $D = \mathbb{R}^N_+$. For any $\lambda \neq 0$, there exists a positive constant $c_\lambda$ depending on $\lambda$ such that

$$c_\lambda^{-1}\|H_\lambda(F,G)\|_{\mathcal{X}_q(\mathbb{R}^N_+)} \leq \|(F,G)\|_{\mathbb{G}_q(\mathbb{R}^N_+)} \leq c_\lambda\|H_\lambda(F,G)\|_{\mathcal{X}_q(\mathbb{R}^N_+)}, \tag{5-19}$$

i.e., $\|H_\lambda(F,G)\|_{\mathcal{X}_q(\mathbb{R}^N_+)}$ is an equivalent norm of $\mathbb{G}_q(\mathbb{R}^N_+)$. By (5-17),

$$\|H_\lambda\mathcal{U}(\lambda)H_\lambda(F,G)\|_{\mathcal{X}_q(\mathbb{R}^N_+)} \leq \tfrac{1}{2}\|H_\lambda(F,G)\|_{\mathcal{X}_q(\mathbb{R}^N_+)},$$

which yields that $(I - \mathcal{U}(\lambda)H_\lambda)^{-1} = I + \sum_{j=1}^\infty(\mathcal{U}(\lambda)H_\lambda)^j$ exists in $\mathcal{L}(\mathbb{G}_q(\mathbb{R}^N_+))$ for any $\lambda \in \Sigma_{\vartheta,\lambda_1}$. In view of (5-15), $U := \mathcal{T}(\lambda)H_\lambda(I - \mathcal{U}(\lambda)H_\lambda)^{-1}(F,G)$ satisfies the equation

$$L_\lambda(D)U = (I - \mathcal{U}(\lambda)H_\lambda)(I - \mathcal{U}(\lambda)H_\lambda)^{-1}(F,G) = (F,G),$$

that is, $U$ satisfies equations (5-13). Thus, we define the operator $\mathcal{V}(\lambda)$ by

$$\mathcal{V}(\lambda) = \mathcal{T}(\lambda)(I - H_\lambda\mathcal{U}(\lambda))^{-1},$$

we obtain $U = \mathcal{V}(\lambda)H_\lambda(F,G)$. In fact, noting that

$$(I - H_\lambda\mathcal{U}(\lambda))^{-1}H_\lambda = H_\lambda + \sum_{j=1}^\infty (H_\lambda\mathcal{U}(\lambda))^j H_\lambda = H_\lambda(I + \sum_{j=1}^\infty(\mathcal{U}(\lambda)H_\lambda)^j) = H_\lambda(I - \mathcal{U}(\lambda)H_\lambda)^{-1},$$

we see that $\mathcal{V}(\lambda)H_\lambda(F,G) = \mathcal{T}(\lambda)H_\lambda(I - \mathcal{U}(\lambda)H_\lambda)^{-1}(F,G) = U$. Therefore, $U = \mathcal{V}(\lambda)H_\lambda(F,G)$ is the unique solution of the equations (5-13) for any $\lambda \in \Sigma_{\vartheta,\lambda_1}$ and $(F,G) \in \mathbb{G}_q(\mathbb{R}^N_+)$. Here, the uniqueness of the solution follows from the existence of solutions of the dual problem. (For this, see also Shibata and Shimizu [29, Proof of Theorem 4.3] for a similar argument.)

Setting $\mathcal{V}(\lambda) = (\mathcal{V}_1(\lambda), \lambda\mathcal{V}_1(\lambda), \mathcal{V}_2(\lambda))^\top$, by (5-18), Theorem 3.1 and Proposition 2.2, we have

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\mathbb{R}^N_+), H^{4-j}_q(\mathbb{R}^N_+))}(\{(\tau\partial_\tau)^s(\lambda^{j/2}\mathcal{V}_1(\lambda)) \mid \lambda \in \Sigma_{\vartheta,\lambda_1}\}) \leq C_{N,q,\lambda_1} \quad (s=0,1, \;\; j=0,1,2,3,4),$$
$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\mathbb{R}^N_+), H^{2-j}_q(\mathbb{R}^N_+))}(\{(\tau\partial_\tau)^s(\lambda^{j/2}\mathcal{V}_2(\lambda)) \mid \lambda \in \Sigma_{\vartheta,\lambda_1}\}) \leq C_{N,q,\lambda_1} \quad (s=0,1, \;\; j=0,1,2).$$

Finally, defining $\mathcal{T}_{+i}(\lambda)$ $(i=1,2)$ by

$$\mathcal{T}_{+i}(\lambda)(F', G, G', g_1'') = (\mathcal{V}_i(\lambda)[(F', G, G', g_1'') \circ \Phi]) \circ \Phi^{-1}, ,$$

we see that $\mathcal{T}_{+i}(\lambda)$ $(i=1,2)$ are the operators satisfying the required properties in Theorem 5.1, which completes the proof of Theorem 5.1.

# 6 Proof of Theorem 1.4

First of all, we state several properties of a uniform $C^4$-domain.

**Proposition 6.1.** *Let $\Omega$ be a uniform $C^4$-domain in $\mathbb{R}^N$ with boundary $\Gamma$. Then, for any positive constant $M_1$, there exist constants $M_2 > 0$, $d \in (0,1)$, an open set $U$, at most countably many functions $\Phi_j \in H^4_\infty(\mathbb{R}^N)^N \cap C^4(\mathbb{R}^N, \mathbb{R}^N)$ and points $x_j \in \Gamma$ such that the following assertions hold:*

(i) *For every $j \in \mathbb{N}$, the map $\mathbb{R}^N \ni x \mapsto \Phi_j(x) \in \mathbb{R}^N$ is bijective.*

(ii) $\Omega = U \cup \bigcup_{j=1}^\infty (\Phi_j(\mathbb{R}^N_+) \cap B_d(x_j))$, $\Phi_j(\mathbb{R}^N_+) \cap B_d(x_j) = \Omega \cap B_d(x_j)$, *and* $\Gamma \cap B_d(x_j) = \Phi_j(\mathbb{R}^N_0) \cap B_d(x_j)$.

(iii) *There exist* $C^\infty$*-functions* $\zeta_j$ *and* $\tilde\zeta_j$ $(j = 0, 1, 2, \ldots)$ *such that*

$$\operatorname{supp}\zeta_0, \operatorname{supp}\tilde\zeta_0 \subset G, \ \operatorname{supp}\zeta_j, \operatorname{supp}\tilde\zeta_j \subset B_d(x_j), \ \|\zeta_j\|_{H^4_\infty(\mathbb{R}^N)} \le c_0, \ \|\tilde\zeta_j\|_{H^4_\infty(\mathbb{R}^N)} \le c_0,$$
$$\tilde\zeta_j = 1 \ \text{on} \operatorname{supp}\zeta_j, \ \sum_{j=0}^\infty \zeta_j = 1 \ \text{on} \ \overline{\Omega}, \ \sum_{j=1}^\infty \zeta_j = 1 \ \text{on} \ \Gamma.$$

*Here,* $c_0$ *is a constant which depends on* $M_2$*,* $N$*,* $q$*,* $q'$ *and* $r$*, but is independent of* $j = 0, 1, 2, \ldots$*.*

(iv) $\nabla\Phi_j = \mathcal{R}_j + R_j$, $\nabla(\Phi_j)^{-1} = \mathcal{R}_{-j} + R_{-j}$, *where* $\mathcal{R}_j$ *and* $\mathcal{R}_{-j}$ *are* $N \times N$ *constant orthogonal matrices, and* $R_j$ *and* $R_{-j}$ *are* $N \times N$ *matrices of* $H^3_\infty$*-functions defined on* $\mathbb{R}^N$ *which satisfy the conditions:* $\|R_j\|_{L_\infty(\mathbb{R}^N)} \le M_1$*,* $\|R_{-j}\|_{L_\infty(\mathbb{R}^N)} \le M_1$*,* $\|\nabla R_j\|_{H^2_\infty(\mathbb{R}^N)} \le M_2$ *and* $\|\nabla R_{-j}\|_{H^2_\infty(\mathbb{R}^N)} \le M_2$ *for any* $j = 1, 2, \ldots$*.*

(v) *There exists a natural number* $L \ge 2$ *such that any* $L+1$ *distinct sets of* $\{U_j \mid j = 0, 1, 2, \ldots\}$ *with* $U_0 = U$ *and* $U_j = \Phi_j(B_d(0))$ $(j = 1, 2, 3, \ldots)$ *have an empty intersection.*

In what follows, we write $\Omega_\ell = \Phi_\ell(\mathbb{R}^N_+)$, and $\Gamma_\ell = \Phi_\ell(\mathbb{R}^N_0)$. For $f \in L_q(\Omega)$, we have

$$\sum_{\ell=1}^\infty \|\tilde\zeta_\ell f\|^q_{L_q(\Omega_\ell)} + \|\tilde\zeta_0 f\|^q_{L_q(\mathbb{R}^N)} \le C_q \|f\|^q_{L_q(\Omega)}. \tag{6-1}$$

For $(F, G) \in \mathbb{G}_q(\Omega)$, let $U_0$ and $U_\ell$ be solutions of the equations:

$$\lambda U_0 - A(D)U_0 = \tilde\zeta_0 F \quad \text{in } \mathbb{R}^N, \tag{6-2}$$
$$\lambda U_\ell - A(D)U_\ell = \tilde\zeta_\ell F \quad \text{in } \Omega_\ell, \quad B(D)U_\ell = \tilde\zeta_\ell G \quad \text{on } \Gamma_\ell. \tag{6-3}$$

By Theorem 2.3 and Theorem 5.1, there exist operator families $\mathcal{S}_{01}(\lambda)$, $\mathcal{S}_{02}(\lambda)$, $\mathcal{S}_{\ell 1}(\lambda)$ and $\mathcal{S}_{\ell 2}(\lambda)$ with

$$\mathcal{S}_{01}(\lambda) \in \mathcal{C}(\Sigma_{\vartheta,\lambda_0}, \mathcal{L}(L_q(\mathbb{R}^N)^2, H^4_q(\mathbb{R}^N))), \quad \mathcal{S}_{02}(\lambda) \in \mathcal{C}(\Sigma_{\vartheta,\lambda_0}, \mathcal{L}(L_q(\mathbb{R}^N)^2, H^2_q(\mathbb{R}^N))), \tag{6-4}$$
$$\mathcal{S}_{\ell 1}(\lambda) \in \mathcal{C}(\Sigma_{\vartheta,\lambda_0}, \mathcal{L}(L_q(\Omega_\ell)^2, H^4_q(\Omega_\ell))), \quad \mathcal{S}_{\ell 2}(\lambda) \in \mathcal{C}(\Sigma_{\vartheta,\lambda_0}, \mathcal{L}(L_q(\Omega_\ell)^2, H^2_q(\Omega_\ell))) \tag{6-5}$$

such that

$$\begin{aligned} U_0 &= (\mathcal{S}_{01}(\lambda)\tilde\zeta_0 F', \lambda\mathcal{S}_{01}(\lambda)\tilde\zeta_0 F', \mathcal{S}_{02}(\lambda)\tilde\zeta_0 F')^\top, \\ U_\ell &= (\mathcal{S}_{\ell 1}(\lambda)H_\lambda(\tilde\zeta_\ell F, \tilde\zeta_\ell G), \lambda\mathcal{S}_{\ell 1}(\lambda)H_\lambda(\tilde\zeta_\ell F, \tilde\zeta_\ell G), \mathcal{S}_{\ell 2}(\lambda)H_\lambda(\tilde\zeta_\ell F, \tilde\zeta_\ell G))^\top \end{aligned} \tag{6-6}$$

uniquely solve (6-2) with $\lambda \in \Sigma_{\vartheta,\lambda_0}$ and (6-3) with $\lambda \in \Sigma_{\vartheta,\lambda_1}$, respectively. Here and hereafter, $H_\lambda$ is an operator acting on $(F, G)$ defined by $H_\lambda(F, G) = (f_1, f_2, G, \lambda^{1/2}G, \lambda g_1)$, where $F = (0, f_1, f_2)$ and $G = (g_1, g_2, g_3)$. Moreover, we have

$$\begin{aligned} &\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N)^2, H^{4-j}_q(\mathbb{R}^N))}(\{(\tau\partial_\tau)^s(\lambda^{j/2}\mathcal{S}_{01}(\lambda)) \mid \lambda \in \Sigma_{\vartheta,\lambda_0}\}) \le \kappa_1 \quad (j = 0, 1, 2, 3, 4); \\ &\mathcal{R}_{L_q(\mathbb{R}^N)^2, H^{2-j}_q(\mathbb{R}^N))}(\{(\tau\partial_\tau)^s(\lambda^{j/2}\mathcal{S}_{02}(\lambda)) \mid \lambda \in \Sigma_{\vartheta,\lambda_0}\}) \le \kappa_1 \quad (j = 0, 1, 2); \\ &\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega_\ell), H^{4-j}_q(\Omega_\ell))}(\{(\tau\partial_\tau)^s(\lambda^{j/2}\mathcal{S}_{\ell 1}(\lambda)) \mid \lambda \in \Sigma_{\vartheta,\lambda_1}\}) \le \kappa_1 \quad (j = 0, 1, 2, 3, 4), \\ &\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega_\ell), H^{2-j}_q(\Omega_\ell))}(\{(\tau\partial_\tau)^s(\lambda^{j/2}\mathcal{S}_{\ell 2}(\lambda)) \mid \lambda \in \Sigma_{\vartheta,\lambda_1}\}) \le \kappa_1 \quad (j = 0, 1, 2), \end{aligned} \tag{6-7}$$

for $s = 0, 1$ with some constant $\kappa_1$ independent of $\ell$. We may assume that $0 \le \lambda_0 \le \lambda_1$. By (6-7),

$$\begin{aligned} &\sum_{j=0}^4 \|\lambda^{j/2}u_0\|_{H^{4-j}_q(\mathbb{R}^N)} + \sum_{j=0}^2 \|\lambda^{j/2}(v_0, \theta_0)\|_{H^{2-j}_q(\mathbb{R}^N)} \le \kappa_1 \|(\tilde\zeta_0 f_1, \tilde\zeta_0 f_2)\|_{L_q(\mathbb{R}^N)}, \\ &\sum_{j=0}^4 \|\lambda^{j/2}u_\ell\|_{H^{4-j}_q(\Omega_\ell)} + \sum_{j=0}^2 \|\lambda^{j/2}(v_\ell, \theta_\ell)\|_{H^{2-j}_q(\Omega_\ell)} \le \kappa_1 \|(\tilde\zeta_\ell f_1, \tilde\zeta_\ell f_2, \tilde\zeta_\ell G, \lambda^{1/2}\tilde\zeta_\ell G, \lambda\tilde\zeta_\ell g_1)\|_{\mathcal{X}_q(\Omega_\ell)} \end{aligned} \tag{6-8}$$

where we have set $U_0 = (u_0, v_0, \theta_0)^\top$ and $U_\ell = (u_\ell, v_\ell, \theta_\ell)^\top$. For any $(F', G, G', g_1'') \in \mathcal{X}_q(\Omega)$ and $\lambda \in \Sigma_{\vartheta,\lambda_1}$, we define an operator $\mathcal{A}_i(\lambda)$ acting on $(F', G, G', g_1'')$ by

$$\mathcal{A}_i(\lambda)(F,G,G',g_1'') = \zeta_0 \mathcal{S}_{0i}(\lambda)(\tilde{\zeta}_0 F') + \sum_{\ell=1}^{\infty} \zeta_\ell \mathcal{S}_{\ell i}(\lambda)(\tilde{\zeta}_\ell F', \tilde{\zeta}_\ell G, \tilde{\zeta}_\ell G', \tilde{\zeta}_\ell g_1'') \quad (i = 1,2). \tag{6-9}$$

By (6-1) and (6-7),

$$\begin{aligned} &\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega), H_q^{4-j}(\Omega))}(\{(\tau\partial_\tau)^s(\lambda^{j/2}\mathcal{A}_1(\lambda)) \mid \lambda \in \Sigma_{\vartheta,\lambda_1}\}) \le C_q\kappa_1 \quad (s = 0,1, \ \ j = 0,1,2,3,4), \\ &\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega), H_q^{2-j}(\Omega))}(\{(\tau\partial_\tau)^s(\lambda^{j/2}\mathcal{A}_2(\lambda)) \mid \lambda \in \Sigma_{\vartheta,\lambda_1}\}) \le C_q\kappa_1 \quad (s = 0,1, \ \ j = 0,1,2). \end{aligned} \tag{6-10}$$

For any $C^\infty$-function $\zeta$, we define operators $A_R(\zeta, D)$ and $B_R(\zeta, D)$ acting on $U$ by

$$A_R(\zeta,D)U := A(D)(\zeta U) - \zeta A(D)U, \quad B_R(\zeta,D)U := B(D)(\zeta U) - \zeta B(D)U,$$

and then we have

$$A_R(\zeta,D) = (0, A_{R2}(\zeta,D)U, A_{R3}(\zeta,D)U)^\top, \quad B_R(\zeta,D)U = (B_{R1}(\zeta,D)U, B_{R2}(\zeta,D)U, B_{R3}(\zeta,D)U)^\top$$

with

$$\begin{aligned} &A_{R2}(\zeta,D)U = -\{\Delta((\Delta\zeta)u + 2(\nabla\zeta)\cdot(\nabla u)) + (\Delta\zeta)(\Delta u) + 2(\nabla\zeta)\cdot(\nabla\Delta u) + (\Delta\zeta)\theta + 2(\nabla\zeta)\cdot(\nabla\theta)\}, \\ &A_{R3}(\zeta,D)U = (\Delta\zeta)v + 2(\nabla\zeta)\cdot(\nabla v) + (\Delta\zeta)\theta + 2(\nabla\zeta)\cdot(\nabla\theta), \\ &B_{R1}(\zeta,D)U = 2(\nabla\zeta)\cdot(\nabla u) + (\Delta\zeta)u - (1-\beta)(2(\nabla'\zeta)\cdot(\nabla' u) + (\Delta'\zeta)u), \\ &B_{R2}(\zeta,D)U = \partial_\nu\{2(\nabla\zeta)\cdot(\nabla u) + (\Delta\zeta)u + (1-\beta)(2(\nabla'\zeta)\cdot(\nabla' u) + (\Delta'\zeta)u)\} + (\partial_\nu\zeta)(\Delta + (1-\beta)\Delta')u, \\ &B_{R3}(\zeta,D)U = (\partial_\nu\zeta)\theta, \end{aligned}$$

where $U = (u, v, \theta)^\top$. Let $\mathcal{A}(\lambda) = (\mathcal{A}_1(\lambda), \lambda\mathcal{A}_1(\lambda), \mathcal{A}_2(\lambda))^\top$ and then by (6-2), (6-3) and (6-6), we see that $U = \mathcal{A}(\lambda)H_\lambda(F,G)$ satisfies the equation:

$$\lambda U - A(D)U = F - \mathbb{U}_1(\lambda)(F,G) \quad \text{in } \Omega, \quad B(D)U = G - \mathbb{U}_2(\lambda)(F,G) \quad \text{on } \Gamma, \tag{6-11}$$

where

$$\begin{aligned} &\mathbb{U}_1(\lambda)(F,G) = A_R(\zeta_0, D)\mathcal{S}_0(\lambda)(\tilde{\zeta}_0 F') + \sum_{\ell=1}^{\infty} A_R(\zeta_\ell, D)\mathcal{S}_\ell(\lambda)H_\lambda(\tilde{\zeta}_\ell F, \tilde{\zeta}_\ell G), \\ &\mathbb{U}_2(\lambda)(F,G) = \sum_{\ell=1}^{\infty} B_R(\zeta_\ell, D)\mathcal{S}_\ell(\lambda)H_\lambda(\tilde{\zeta}_\ell F, \tilde{\zeta}_\ell G). \end{aligned}$$

Setting $L_\lambda(D) = (\lambda I - A(D), B(D))$ and $\mathbb{U}(\lambda)(F,G) = (\mathbb{U}_1(\lambda)(F,G), \mathbb{U}_2(\lambda)(F,G))$, we have

$$L_\lambda(D)\mathcal{A}(\lambda)H_\lambda(F,G) = (I - \mathbb{U}(\lambda))(F,G) \quad \text{on } \Omega \times \Gamma. \tag{6-12}$$

By (6-2) and (6-3), we have

$$H_\lambda \mathbb{U}(\lambda)(F,G) = \begin{pmatrix} A_R(\zeta_0,D)\mathcal{S}_0(\lambda)(\tilde{\zeta}_0 F') \\ 0 \\ 0 \\ 0 \end{pmatrix} + \sum_{\ell=1}^{\infty} \begin{pmatrix} A_R(\zeta_\ell,D)\mathcal{S}_\ell(\lambda)H_\lambda(\tilde{\zeta}_\ell F, \tilde{\zeta}_\ell G) \\ B_R(\zeta_\ell,D)\mathcal{S}_\ell(\lambda)H_\lambda(\tilde{\zeta}_\ell F, \tilde{\zeta}_\ell G) \\ \lambda^{1/2} B_R(\zeta_\ell,D)\mathcal{S}_\ell(\lambda)H_\lambda(\tilde{\zeta}_\ell F, \tilde{\zeta}_\ell G) \\ \lambda B_{R1}(\zeta_\ell,D)(\mathcal{S}_{\ell 1}(\lambda)H_\lambda(\tilde{\zeta}_\ell F, \tilde{\zeta}_\ell G) \end{pmatrix},$$

where $F = (0, f_1, f_2)^\top$ and $F' = (f_1, f_2)^\top$. We have

$$H_\lambda \mathbb{U}(\lambda)(F,G) = \mathcal{U}(\lambda)H_\lambda(F,G) \tag{6-13}$$

with

$$\mathcal{U}(\lambda)(F', G, G', g_1'') = \begin{pmatrix} A_R(\zeta_0, D)\mathcal{S}_0(\lambda)(\tilde\zeta_0 F') \\ 0 \\ 0 \\ 0 \end{pmatrix} + \sum_{\ell=1}^\infty \begin{pmatrix} A_R(\zeta_\ell, D)\mathcal{S}_\ell(\lambda)(\tilde\zeta_\ell F', \tilde\zeta_\ell G, \tilde\zeta_\ell G', \tilde\zeta_\ell g_1'') \\ B_R(\zeta_\ell, D)\mathcal{S}_\ell(\lambda)(\tilde\zeta_\ell F', \tilde\zeta_\ell G, \tilde\zeta_\ell G', \tilde\zeta_\ell g_1'') \\ \lambda^{1/2} B_R(\zeta_\ell, D)\mathcal{S}_\ell(\lambda)(\tilde\zeta_\ell F', \tilde\zeta_\ell G, \tilde\zeta_\ell G', \tilde\zeta_\ell g_1'') \\ \lambda B_{R1}(\zeta_\ell, D)(\mathcal{S}_{\ell 1}(\lambda)(\tilde\zeta_\ell F', \tilde\zeta_\ell G, \tilde\zeta_\ell G', \tilde\zeta_\ell g_1'') \end{pmatrix}.$$

By (6-1), (6-7) and Proposition 2.2, we have

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega))}(\{(\tau\partial_\tau)^s \mathcal{U}(\lambda) \mid \lambda \in \Sigma_{\vartheta,\lambda_2}\}) \le C_q \kappa_1 \lambda_2^{-1/2} \quad (s = 0, 1) \tag{6-14}$$

for any $\lambda_2 \ge \lambda_1$. We choose $\lambda_2$ so large that

$$C_q \kappa_1 \lambda_2^{-1/2} \le 1/2. \tag{6-15}$$

By (6-13), (6-14) and (6-15), we have

$$\|H_\lambda \mathbb{U}(\lambda)(F, G)\|_{\mathcal{X}_q(\Omega)} \le \tfrac{1}{2}\|H_\lambda(F, G)\|_{\mathcal{X}_q(\Omega)}. \tag{6-16}$$

By the equivalence of the norms $\|H_\lambda(\cdot)\|_{\mathcal{X}_q(\Omega)}$ and $\|\cdot\|_{\mathbb{G}_q(\Omega)}$ (cf. (5-19)), the inverse $(I - \mathbb{U}(\lambda))^{-1} = I + \sum_{n=1}^\infty \mathbb{U}(\lambda)^n$ exists in $\mathcal{L}(\mathbb{G}_q(\Omega))$. Let $V = \mathcal{A}(\lambda)H_\lambda(I - \mathbb{U}(\lambda))^{-1}(F, G)$, and then by (6-12) $V$ solves the equation:

$$L_\lambda(D)V = (F, G) \quad \text{on } \Omega \times \Gamma. \tag{6-17}$$

The uniqueness of solutions follows from the existence of solutions of the dual problem, so that $V$ is the unique solution of the equation (6-17).

On the other hand, by (6-14) and (6-15), $I - \mathcal{U}(\lambda) = I + \sum_{n=1}^\infty \mathcal{U}(\lambda)^n$ exists in $\mathcal{C}(\Sigma_{\vartheta,\lambda_2}, \mathcal{L}(\mathcal{X}_q(\Omega)))$ and satisfies the estimate:

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega))}(\{(\tau\partial_\tau)^s (I - \mathcal{U}(\lambda))^{-1} \mid \lambda \in \Sigma_{\vartheta,\lambda_2}\}) \le 2 \quad (s = 0, 1). \tag{6-18}$$

Moreover, by (6-13),

$$H_\lambda(I - \mathbb{U}(\lambda))^{-1} = H_\lambda + \sum_{n=1}^\infty H_\lambda \mathbb{U}(\lambda)^n = (I + \sum_{n=1}^\infty \mathcal{U}(\lambda)^n)H_\lambda = (I - \mathcal{U}(\lambda))^{-1}H_\lambda. \tag{6-19}$$

Let $\mathcal{B}_i(\lambda) = \mathcal{A}_i(\lambda)(I - \mathcal{U}(\lambda))^{-1}$ $(i = 1, 2)$. Then, by (6-9), (6-18) and Proposition 2.2,

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega), H_q^{4-j}(\Omega))}(\{(\tau\partial_\tau)^s (\lambda^{j/2}\mathcal{B}_1(\lambda)) \mid \lambda \in \Sigma_{\vartheta,\lambda_1}\}) \le C_q \kappa_1 \quad (s = 0, 1, \; j = 0, 1, 2, 3, 4),$$
$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega), H_q^{2-j}(\Omega))}(\{(\tau\partial_\tau)^s (\lambda^{j/2}\mathcal{B}_2(\lambda)) \mid \lambda \in \Sigma_{\vartheta,\lambda_1}\}) \le C_q \kappa_1 \quad (s = 0, 1, \; j = 0, 1, 2).$$

Moreover, setting $\mathcal{B}(\lambda) = (\mathcal{B}_1(\lambda), \lambda\mathcal{B}_1(\lambda), \mathcal{B}_2(\lambda))^\top$, by (6-19) we see that

$$\mathcal{B}(\lambda)H_\lambda(F, G) = \mathcal{A}(\lambda)H_\lambda(I - \mathbb{U}(\lambda))^{-1}(F, G),$$

and therefore $V = \mathcal{B}(\lambda)H_\lambda(F, G)$ is the unique solution of the equation (1-5) (cf. (6-17)). This completes the proof of Theorem 1.4 replacing $\lambda_0$ by $\lambda_2$.

# 7 Proof of Theorem 1.3

To prove Theorem 1.3, we start with

**Lemma 7.1.** *Let $1 < p, q < \infty$. For any $\theta_0 \in B^{2-2/p}_{q,p}(\Omega)$, $u_0 \in B^{4-2/p}_{q,p}(\Omega)$, and $u_1 \in B^{2-2/p}_{q,p}(\Omega)$, there exist $\omega$ and $w$ such that $\omega|_{t=0} = \theta_0$, $w|_{t=0} = u_0$, and $\partial_t w|_{t=0} = u_1$ in $\Omega$,*

$$\omega \in \bigcap_{\ell=0}^{1} H^\ell_p((0, \infty), H^{2-2\ell}_{q,p}(\Omega)), \quad w \in \bigcap_{\ell=0}^{2} H^\ell_p((0, \infty), H^{4-2\ell}_q(\Omega),$$

*and*

$$\sum_{\ell=0}^1 \|\partial_t^\ell \omega\|_{L_p((0,\infty),H_q^{2-2\ell}(\Omega))} \leq C\|\theta_0\|_{B^{2-2/p}_{q,p}(\Omega)},$$
$$\sum_{\ell=0}^2 \|\partial_t^\ell w\|_{L_p((0,\infty),H_{q,p}^{4-2\ell}(\Omega))} \leq C(\|u_0\|_{B_q^{4-2/p}(\Omega)} + \|u_1\|_{B^{2-2/p}_{q,p}(\Omega)}).$$

*with some positive constant $C$.*

*Proof.* To prove the lemma, we consider the shifted heat equation:

$$\partial_t v + v - \Delta v = f \quad \text{in } (0,\infty)\times\mathbb{R}^N, \quad v|_{t=0} = v_0. \tag{7-1}$$

Employing the same argument as in the previous sections, it is easy to prove that for any $f \in L_p((0,\infty), L_q(\mathbb{R}^N))$ and $v_0 \in B^{2-2/p}_{q,p}(\mathbb{R}^N)$, problem (7-1) admits a unique solution $v \in \bigcap_{\ell=0}^1 H_p^\ell((0,\infty), H_q^{2-2\ell}(\mathbb{R}^N))$ possessing the estimate:

$$\sum_{\ell=0}^1 \|\partial_t^\ell v\|_{L_p((0,\infty),H_q^{2-2\ell}(\mathbb{R}^N))} \leq C(\|v_0\|_{B^{2-2/p}_{q,p}(\mathbb{R}^N)} + \|f\|_{L_p((0,\infty),L_q(\Omega))}). \tag{7-2}$$

Assume that $v_0 \in B^{4-2/p}_{q,p}(\mathbb{R}^N)$ and $f \in \bigcap_{\ell=0}^1 H_p^\ell((0,\infty), H_q^{2-2\ell}(\mathbb{R}^N))$. Then, for any multi-index $\alpha \in \mathbb{N}_0^N$ with $|\alpha| \leq 2$ we have

$$\partial_t\partial_x^\alpha v + \partial_x^\alpha v - \Delta\partial_x^\alpha v = \partial_x^\alpha f \quad \text{in } (0,\infty)\times\mathbb{R}^N, \quad \partial_x^\alpha v|_{t=0} = \partial_x^\alpha v_0.$$

Thus, the unique existence of solutions of (7-1) yields that

$$\begin{aligned} &v \in \bigcap_{\ell=0}^1 H_p^\ell((0,\infty), H_q^{4-2\ell}(\mathbb{R}^N)),\\ &\sum_{\ell=0}^1 \|\partial_t^\ell v\|_{L_p((0,\infty),H_q^{4-2\ell}(\mathbb{R}^N))} \leq C(\|v_0\|_{B^{4-2/p}_{q,p}(\mathbb{R}^N)} + \|f\|_{L_p((0,\infty),H_q^2(\mathbb{R}^N))}). \end{aligned} \tag{7-3}$$

Moreover, the relation: $\partial_t^2 v = -\partial_t v + \Delta\partial_t v + \partial_t f$ yields that $\partial_t^2 v \in L_p((0,\infty), L_q(\mathbb{R}^N))$ and

$$\|\partial_t^2 v\|_{L_p((0,\infty),L_q(\mathbb{R}^N))} \leq C(\|v_0\|_{B^{4-2/p}_{q,p}(\mathbb{R}^N)} + \sum_{\ell=0}^1 \|\partial_t^\ell f\|_{L_p((0,\infty),H_q^{2-2\ell}(\mathbb{R}^N))}). \tag{7-4}$$

Let $\iota_h$ be an extension map as given in Introduction satisfying property (e-1). Let $\omega$ be a solution of the shifted heat equation:

$$\partial_t\omega + \omega - \Delta\omega = 0 \quad \text{in } (0,\infty)\times\mathbb{R}^N, \quad \omega|_{t=0} = \iota_h\theta_0.$$

Since $\iota_h\theta_0 \in B^{2-2/p}_{q,p}(\mathbb{R}^N)$, by (7-2) with $f = 0$ there exists a unique $\omega \in \bigcap_{\ell=0}^1 H_p^\ell((0,\infty), H_q^{2-2\ell}(\mathbb{R}^N))$ satisfying the estimate

$$\sum_{\ell=0}^1 \|\partial_t^\ell\omega\|_{L_p((0,\infty),H_q^{2-2\ell}(\mathbb{R}^N))} \leq C\|\iota_h\theta_0\|_{B^{2-2/p}_{q,p}(\mathbb{R}^N)}.$$

Since $\iota_h\theta_0 = \theta_0$ on $\Omega$ and $\|\iota_h\theta_0\|_{B^{2-2/p}_{q,p}(\mathbb{R}^N)} \leq C\|\theta_0\|_{B^{2-2/p}_{q,p}(\Omega)}$ with some constant $C > 0$, the restriction of $\omega$ on $\Omega$ is the function satisfying the required properties.

Next, let $f$ be a solution of the shifted heat equation:

$$\partial_t f + f - \Delta f = 0 \quad \text{in } (0,\infty)\times\mathbb{R}^N, \quad f|_{t=0} = v_0$$

with $v_0 = \iota_h u_1 + \iota_h u_0 - \iota_h \Delta u_0$. Since $v_0 \in B^{2-2/p}_{q,p}(\mathbb{R}^N)$ and

$$\|v_0\|_{B^{2-2/p}_{q,p}(\mathbb{R}^N)} \leq C(\|u_1\|_{B^{2-2/p}_{q,p}(\Omega)} + \|u_0\|_{B^{4-2/p}_{q,p}(\Omega)}),$$

there exists a unique $f \in \bigcap_{\ell=0}^1 H^\ell_p((0,\infty), H^{2-2\ell}_q(\mathbb{R}^N))$ satisfying the estimate:

$$\sum_{\ell=0}^1 \|\partial_t^\ell f\|_{L_p((0,\infty),H^{2-2\ell}_q(\mathbb{R}^N))} \leq C(\|u_1\|_{B^{2-2/p}_{q,p}(\Omega)} + \|u_0\|_{B^{4-2/p}_{q,p}(\Omega)}). \tag{7-5}$$

Let $w$ be a solution of the shifted heat equation:

$$\partial_t w + w - \Delta w = f \quad \text{in } (0,\infty)\times\mathbb{R}^N, \quad w|_{t=0} = \iota u_0. \tag{7-6}$$

Then, by (7-3), (7-4) and (7-5), there exists a unique $w \in \bigcap_{\ell=0}^2 H^\ell_p((0,\infty), H^{4-2\ell}_q(\mathbb{R}^N))$ satisfying the estimate:

$$\sum_{\ell=0}^2 \|\partial_t^\ell w\|_{L_p((0,\infty),H^{4-2\ell}_q(\mathbb{R}^N))} \leq C(\|u_1\|_{B^{2-2/p}_{q,p}(\Omega)} + \|u_0\|_{B^{4-2/p}_{q,p}(\Omega)}).$$

Moreover, by (7-6), $\partial_t w|_{t=0} = v_0 - \iota_h u_0 + \Delta \iota_h u_0 = \iota_h u_1$, so that the restriciton of $w$ on $\Omega$ satisfies the required properties, which completes the proof of Lemma 7.1. □

In view of Lemma 7.1, to prove Theorem 1.3, it suffices to consider the equations (1-4) with $U_0 = 0$. Let $F = (0, f_1, f_2)^\top$ and $G = (g_1, g_2, g_3)$ satisfy the regularity condition:

$$(f_1, f_2) \in L_p((0,T), L_q(\Omega)^2), \quad G \in H^1_p((0,T), L_q(\Omega)\times \mathbf{W}^{-1}_q(\Omega)^2) \cap L_p((0,T), H^2_q(\Omega)\times H^1_q(\Omega)^2)$$

and the compatibility condition: $G|_{t=0} = 0$. In the following, given $f(t,\cdot)$ defined for $t \in [0,T]$, $f_0(t,\cdot)$ denotes the zero extension of $f$ to $t < 0$, that is, $f_0(t,\cdot) = f(t,\cdot)$ for $0 < t < T$, and $f_0(t,\cdot) = 0$ for $t < 0$, and $E_T f$ denotes the extention of $f$ to $\mathbb{R}$ defined by

$$[E_T f](t,\cdot) = \begin{cases} f_0(t,\cdot) & \text{for } t < T, \\ f_0(2T-t,\cdot) & \text{for } t \geq T. \end{cases} \tag{7-7}$$

Note that $E_T f$ vanishes for $t \notin [0, 2T]$ and moreover, if $f|_{t=0} = 0$, then

$$\partial_t[E_T f](t,\cdot) = \begin{cases} \partial_t f(t,\cdot) & \text{for } t \leq T, \\ -(\partial_t f)(2T-t,\cdot) & \text{for } t \geq T, \\ 0 & \text{for } t \notin [0,2T]. \end{cases} \tag{7-8}$$

Since $E_T f = f$ for $0 \leq t \leq T$, instead of the equations (1-4) with $U_0 = 0$, we consider the equations:

$$U_t - A(D)U = E_T F \quad \text{in } \mathbb{R}\times\Omega, \quad B(D)U = E_T G \quad \text{on } \mathbb{R}\times\Gamma. \tag{7-9}$$

Let $\mathcal{L}$ be the Laplace transform with respect to time variable $t$ and let $\mathcal{L}^{-1}$ be its inverse transform, which are defined by

$$\mathcal{L}[f](\lambda,\cdot) = \int_{-\infty}^\infty e^{-\lambda t} f(t,\cdot)\,dt = \int_{\mathbb{R}} e^{-i\tau t} e^{-\gamma t} f(t,\cdot)\,dt,$$
$$\mathcal{L}^{-1}[f](t,\cdot) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t} f(\lambda,\cdot)\,d\lambda = \frac{e^{\gamma t}}{2\pi}\int_{-\infty}^\infty f(\gamma+i\tau,\cdot)\,d\tau,$$

where $\lambda = \gamma + i\tau \in \mathbb{C}$. Let $\mathcal{L}[E_T F](\lambda,\cdot) = J_\lambda$ and $\mathcal{L}[E_T G](\lambda,\cdot) = K_\lambda$. Since $E_T F$ and $E_T G$ vanish for $t \notin [0, 2T]$, $J_\lambda$ and $K_\lambda$ are entire functions with respect to $\lambda \in \mathbb{C}$. Applying the Laplace transform to (7-9), we have

$$\lambda V_\lambda - A(D)V_\lambda = J_\lambda \quad \text{in } \Omega, \quad B(D)V_\lambda = K_\lambda \quad \text{on } \Gamma, \tag{7-10}$$

where we have set $V_\lambda = \mathcal{L}[U](\lambda, \cdot)$. Now, $V_\lambda = \mathcal{B}(\lambda)H_\lambda(J_\lambda, K_\lambda)$ is the unique solution of (7-10), so that the uniqueness of the solution yields that $V_\lambda$ is holomorphic for $\lambda \in \Sigma_{\vartheta,\lambda_0}$. Let $\mathcal{B}_i(\lambda)$ $(i = 1, 2)$ be the operators given in Theorem 1.4 and set $\mathcal{B}(\lambda) = (\mathcal{B}_1(\lambda), \lambda\mathcal{B}_1(\lambda), \mathcal{B}_2(\lambda))^\top$. Then, the unique solution $U$ of (7-9) is given by

$$U(t, \cdot) = \mathcal{L}^{-1}[\mathcal{B}(\lambda)H_\lambda(J_\lambda, K_\lambda)] = e^{\gamma t}\mathcal{F}^{-1}[\mathcal{B}(\lambda)H_\lambda(\mathcal{F}[e^{-\gamma\cdot}E_T F], \mathcal{F}[e^{-\gamma\cdot}E_T G])](t, \cdot).$$

Let $\Lambda^{1/2}$ be the operator defined by

$$[\Lambda^{1/2}f](t, \cdot) = \mathcal{L}^{-1}[\lambda^{1/2}\mathcal{L}[f](\lambda, \cdot)](t),$$

and let

$$\|e^{-\gamma t}f\|_{L_p(\mathbb{R},X)} = \Big\{\int_{-\infty}^{\infty} e^{-\gamma t}\|f(t, \cdot)\|_X)^p\,dt\Big\}^{1/p}.$$

Since $|\lambda|^{1/2} \leq |\lambda|(1 + |\xi|^2)^{-1/2}$ when $|\lambda| \geq 1 + |\xi|^2$ and $|\lambda|^{1/2} \leq (1 + |\xi|^2)^{1/2}$ when $|\lambda| \leq 1 + |\xi|^2$, by the Bourgain theorem (cf. Proposition 2.2) we have

$$\|e^{\gamma t}\Lambda^{1/2}f\|_{L_p(\mathbb{R},L_q(\mathbb{R}^N))} \leq C(\|e^{\gamma t}\partial_t(1-\Delta)^{-1/2}f\|_{L_p(\mathbb{R},L_q(\mathbb{R}^N))} + \|e^{\gamma t}(1-\Delta)^{1/2}f\|_{L_p(\mathbb{R},L_q(\mathbb{R}^N))}),$$

so that by using property (e-2) of the extension map $\iota_h$ given in the introduction we have

$$\|e^{\gamma t}\Lambda^{1/2}f\|_{L_p(\mathbb{R},L_q(\Omega))} \leq C(\|e^{\gamma t}\partial_t f\|_{L_p(\mathbb{R},\mathbf{W}_q^{-1}(\Omega))} + \|e^{\gamma t}f\|_{L_p(\mathbb{R},H_q^1(\Omega))}). \tag{7-11}$$

And also, using the extension map $\iota_h$ and Proposition 2.2, we have

$$\begin{aligned}
&\|e^{-\gamma t}\partial_t f\|_{L_p(\mathbb{R},H_q^m(\Omega))} + \|e^{-\gamma t}f\|_{L_p(\mathbb{R},H_q^{m+2}(\Omega))} \\
&\quad\leq C_{m,q}(\|e^{-\gamma t}\partial_t f\|_{L_p(\mathbb{R},H_q^m(\Omega))} + \|e^{-\gamma t}\Lambda^{1/2}\nabla f\|_{L_p(\mathbb{R},H_q^m(\Omega))} + \|e^{-\gamma t}\nabla^2 f\|_{L_p(\mathbb{R},H_q^m(\Omega))})
\end{aligned} \tag{7-12}$$

for any $m \in \mathbb{N} \cup \{0\}$. Therefore, using (7-11) and (7-12) and applying the Weis operator valued Fourier multiplier theorem with the help of Theorem 1.4, we have

$$\begin{aligned}
&\sum_{\ell=0}^{2}\|e^{-\gamma t}\partial_t^\ell u\|_{L_p(\mathbb{R},H_q^{4-2\ell}(\Omega))} + \sum_{\ell=0}^{1}\|e^{-\gamma t}\partial_t^\ell\theta\|_{L_p(\mathbb{R},H_q^{2-2\ell}(\Omega))} \\
&\quad\leq C(\|e^{-\gamma t}E_T(f_1, f_2)\|_{L_p(\mathbb{R},L_q(\Omega))} + \|e^{-\gamma t}E_T G\|_{L_p(\mathbb{R},H_q^2(\Omega)\times H_q^1(\Omega)^2)} \\
&\qquad + \|e^{-\gamma t}\Lambda^{1/2}E_T G\|_{L_p(\mathbb{R},H_q^1(\Omega)\times L_q(\Omega)^2)} + \|e^{-\gamma t}\partial_t E_T g_1\|_{L_p(\mathbb{R},L_q(\Omega))}) \\
&\quad\leq C(\|e^{-\gamma t}E_T(f_1, f_2)\|_{L_p(\mathbb{R},L_q(\Omega))} + \|e^{-\gamma t}E_T G\|_{L_p(\mathbb{R},H_q^2(\Omega)\times H_q^1(\Omega)^2)} \\
&\qquad + \|e^{-\gamma t}\partial_t E_T G\|_{L_p(\mathbb{R},L_q(\Omega)\times\mathbf{W}_q^{-1}(\Omega)^2)})
\end{aligned} \tag{7-13}$$

for any $\gamma \geq \lambda_0$ with constant $C$ independent of $\gamma$, where we have set $U = (u, \partial_t u, \theta)$. Using (7-7) and (7-8) and noting that $G|_{t=0} = 0$, we have

$$\begin{aligned}
\|e^{-\gamma t}E_T(f_1, f_2)\|_{L_p(\mathbb{R},L_q(\Omega))} &\leq C\|(f_1, f_2)\|_{L_p((0,T),L_q(\Omega))}, \\
\|e^{-\gamma t}E_T G\|_{L_p(\mathbb{R},H_q^2(\Omega)\times H_q^1(\Omega)^2)} &\leq C\|G\|_{L_p((0,T),H_q^2(\Omega)\times H_q^1(\Omega)^2)}, \\
\|e^{-\gamma t}\partial_t E_T G\|_{L_p(\mathbb{R},L_q(\Omega)\times\mathbf{W}_q^{-1}(\Omega)^2)} &\leq C\|\partial_t G\|_{L_p((0,T),L_q(\Omega)\times\mathbf{W}_q^{-1}(\Omega)^2)},
\end{aligned}$$

which, combined with (7-13), furnishes that

$$\sum_{\ell=0}^{2}\|e^{-\gamma t}\partial_t^\ell u\|_{L_p(\mathbb{R},H_q^{4-2\ell}(\Omega))} + \sum_{\ell=0}^{1}\|e^{-\gamma t}\theta\|_{L_p(\mathbb{R},H_q^{2-2\ell}(\Omega))} \leq CI_T \tag{7-14}$$

with

$$I_T = \|e^{-\gamma t}(f_1, f_2)\|_{L_p((0,T),L_q(\Omega))} + \|G\|_{L_p(\mathbb{R},H_q^2(\Omega)\times H_q^1(\Omega)^2)} + \|\partial_t G\|_{L_p(\mathbb{R},L_q(\Omega)\times\mathbf{W}_q^{-1}(\Omega)^2)}.$$

Especially, for any $t < 0$, we have

$$\begin{aligned} & e^{\gamma|t|}\sum_{\ell=0}^{2}\|\partial_t^\ell u\|_{L_p((-\infty,t),H^{4-2\ell}_q(\Omega))} + \sum_{\ell=0}^{1}\|\theta\|_{L_p((-\infty,t),H^{2-2\ell}_q(\Omega))} \\ & \leq \sum_{\ell=0}^{2}\|e^{-\gamma t}\partial_t^\ell u\|_{L_p(\mathbb{R},H^{4-2\ell}_q(\Omega))} + \sum_{\ell=0}^{1}\|e^{-\gamma t}\theta\|_{L_p(\mathbb{R},H^{2-2\ell}_q(\Omega))} \leq CI_T \end{aligned}$$

for any $\gamma \geq \lambda_0$, so that letting $\gamma \to \infty$, we have

$$e^{\gamma|t|}\sum_{\ell=0}^{2}\|\partial_t^\ell u\|_{L_p((-\infty,t),H^{4-2\ell}_q(\Omega))} + \sum_{\ell=0}^{1}\|\theta\|_{L_p((-\infty,t),H^{2-2\ell}_q(\Omega))} = 0$$

for any $t < 0$, which implies that $(u, \theta) = 0$ for $t < 0$.

Summing up, we have proved that $U = (u, \partial_t u, \theta)^\top$ satisifes the equations:

$$U_t - A(D)U = F \quad \text{in } \Omega\times(0,T), \quad B(D)U = G \quad \text{on } \Gamma\times(0,T), \quad U|_{t=0} = 0 \quad \text{in } \Omega$$

and the estimate:

$$\sum_{\ell=0}^{2}\|e^{-\gamma t}\partial_t^\ell u\|_{L_p((0,T),H^{4-2\ell}_q(\Omega))} + \sum_{\ell=0}^{1}\|e^{-\gamma t}\theta\|_{L_p((0,T),H^{2-2\ell}_q(\Omega))} \leq CI_T.$$

The uniqueness of the solutions follwos from the existence of solutions of the dual problem, which completes the proof of Theorem 1.3.

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