ON THE CHOI-LAM ANALOGUE OF HILBERT'S 1888 THEOREM FOR SYMMETRIC FORMS

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Abstract. A famous theorem of Hilbert from 1888 states that a positive semidefinite (psd) real form is a sum of squares (sos) of real forms if and only if $n = 2$ or $d = 1$ or $(n, 2d) = (3, 4)$, where n is the number of variables and $2d$ the degree of the form. In 1976, Choi and Lam proved the analogue of Hilbert's Theorem for symmetric forms by assuming the existence of psd not sos symmetric n-ary quartics for $n \geq 5$. In this paper we complete their proof by constructing explicit psd not sos symmetric n-ary quartics for $n \geq 5$.

1. Introduction

A real form (homogeneous polynomial) f is called *positive semidefinite* (psd) if it takes only non-negative values and it is called a *sum of squares* (sos) if there exist other forms h_j so that $p = h_1^2 + \cdots + h_k^2$. The question whether a real psd form can be written as a sum of squares of real forms has many ramifications and has been studied extensively. Since a psd form always has even degree, it is sufficient to consider this question for even degree forms. We refer to this question as (Q). The first significant result in this direction was given by D. Hilbert [9] in 1888. His celebrated theorem states that a psd form is sos if and only if $n = 2$ or $d = 1$ or $(n, 2d) = (3, 4)$, where n is the number of variables and $2d$ the degree of the form.

The above answer to (Q) can be summarized by the following chart:

deg \ var	2	3	4	5	6	...
2	✓	✓	✓	✓	✓	...
4	✓	✓	×	×	×	...
6	✓	×	×	×	×	...
8	✓	×	×	×	×	...
⋮	⋮	⋮	⋮	⋮	⋮	⋱

where, a tick (✓) denotes a positive answer to (Q), whereas a cross (×) denotes a negative answer to (Q).

Let $\mathcal{P}_{n,2d}$ and $\Sigma_{n,2d}$ denote the cone of psd and sos n-ary $2d$-ic forms (i.e. forms of degree $2d$ in n variables) respectively. Hilbert made a careful study of quaternary quartics and ternary sextics, and demonstrated that $\Sigma_{3,6} \subsetneq \mathcal{P}_{3,6}$ and $\Sigma_{4,4} \subsetneq \mathcal{P}_{4,4}$. Moreover he showed that

$$\text{if } \Sigma_{4,4} \subsetneq \mathcal{P}_{4,4} \text{ and } \Sigma_{3,6} \subsetneq \mathcal{P}_{3,6}, \text{ then}$$

$$\Sigma_{n,2d} \subsetneq \mathcal{P}_{n,2d} \text{ for all } n \geq 3, 2d \geq 4 \text{ and } (n, 2d) \neq (3, 4). \tag{1}$$

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So it is sufficient to produce psd not sos forms in these two crucial cases of quaternary quartics and ternary sextics to get psd not sos forms in all remaining cases as in assertion (1) above. In those two cases Hilbert described a method to produce examples of psd not sos forms, which was "elaborate and unpractical" (see [3, p.387]), so no explicit examples appeared in literature for next 80 years. Explicit examples with $(n, 2d) = (3, 6)$ were found by T. S. Motzkin [12] in 1967 and R. M. Robinson [14] in 1969; Robinson also found an explicit example with $(n, 2d) = (4, 4)$. M. D. Choi and T. Y. Lam [2, 3, 4] produced many more examples in the mid 1970's. More examples were given later by B. Reznick [13] and K. Schmüdgen [15].

In 1976, Choi and Lam [3] considered the question when a psd form is sos for the special case when the form considered is moreover symmetric ($f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = f(x_1, \ldots, x_n)\ \forall\ \sigma \in S_n$). Let $S\mathcal{P}_{n,2d}$ and $S\Sigma_{n,2d}$ denote the set of symmetric psd and symmetric sos n-ary $2d$-ic forms respectively. They demonstrated that it is enough to find symmetric psd not sos forms in the two crucial cases of n-ary quartics for $n \geq 4$ and ternary sextics to obtain symmetric psd not sos n-ary $2d$-ics for all $n \geq 3, 2d \geq 4$ and $(n, 2d) \neq (3, 4)$ (see Proposition 2.2). They showed that the answer to this question is the same as the answer to (Q), by assuming the existence of psd not sos symmetric n-ary quartics for $n \geq 5$. For the convenience of the reader we include the following citation from [3],

> *"the construction of $f_{n,4} \in S\mathcal{P}_{n,4} \setminus S\Sigma_{n,4}$ $(n \geq 4)$ requires considerable effort, so we shall not go into the full details here. Suffice it to record the special form $f_{4,4} = \sum x^2y^2 + \sum x^2yz - 2xyzw$. Here the two summations denote the full symmetric sums (w.r.t. the variables x, y, z, w); hence the summation lengths are respectively 6 and 12".*

We complete their proof by constructing explicit psd not sos symmetric n-ary quartics for $n \geq 5$ (see Theorems 2.8 and 2.9). These theorems will be further used in [7], where we consider the question when an even symmetric psd form is sos.

2. Analogue of Hilbert's 1888 Theorem for Symmetric forms

We revisit the question: for which pairs $(n, 2d)$ will a symmetric psd n-ary $2d$-ic form be sos? We refer to this question as $Q(S)$.

Choi and Lam in [3] claimed that the answer to $Q(S)$, that classifies the pairs $(n, 2d)$ for which a symmetric psd n-ary $2d$-ic form is sos, is:

$$S\mathcal{P}_{n,2d} = S\Sigma_{n,2d} \text{ if and only if } n = 2 \text{ or } d = 1 \text{ or } (n, 2d) = (3, 4). \tag{2}$$

One direction of (2) follows from Hilbert's Theorem. Conversely for proving $S\mathcal{P}_{n,2d} \subseteq S\Sigma_{n,2d}$ only if $n = 2$ or $d = 1$ or $(n, 2d) = (3, 4)$, they showed that it is enough to find $f \in S\mathcal{P}_{n,2d} \setminus S\Sigma_{n,2d}$ for all pairs $(n, 4)$ with $n \geq 4$ and for the pair $(3, 6)$, i.e. they demonstrated that

$$\text{if } S\Sigma_{n,4} \subsetneq S\mathcal{P}_{n,4} \text{ for all } n \geq 4 \text{ and } S\Sigma_{3,6} \subsetneq S\mathcal{P}_{3,6}, \text{ then}$$

$$S\Sigma_{n,2d} \subsetneq S\mathcal{P}_{n,2d} \text{ for all } n \geq 3, 2d \geq 4 \text{ and } (n, 2d) \neq (3, 4). \tag{3}$$

Lemma 2.1. Let $f \in \mathcal{F}_{n,2d}$ be a psd not sos form and p an irreducible indefinite form of degree r in $\mathbb{R}[x_1, \ldots, x_n]$. Then $p^2 f \in \mathcal{F}_{n,2d+2r}$ is also a psd not sos form.

Proof. Clearly $p^2 f$ is psd. If $p^2 f = \sum_k h_k^2$, then for every real tuple $\underline{a}$ with $p(\underline{a}) = 0$, it follows that $(p^2 f)(\underline{a}) = 0$. This implies $h_k^2(\underline{a}) = 0\ \forall\ k$ (since h_k^2 is psd), and so on the real variety $p = 0$, we have $h_k = 0$ as well.

So (using [1, Theorem 4.5.1]), for each k, there exists g_k so that $h_k = pg_k$. This gives $f = \sum_k g_k^2$, which is a contradiction. □

Proposition 2.2. If $S\Sigma_{n,4} \subsetneq S\mathcal{P}_{n,4}$ for all $n \geq 4$ and $S\Sigma_{3,6} \subsetneq S\mathcal{P}_{3,6}$, then $S\Sigma_{n,2d} \subsetneq S\mathcal{P}_{n,2d}$ for all $n \geq 3, d \geq 2$ and $(n, 2d) \neq (3, 4)$.

Proof. Suppose we have forms $f \in S\mathcal{P}_{n,2d} \setminus S\Sigma_{n,2d}$ for all pairs $(n, 4)$ with $n \geq 4$, and for the pair $(3, 6)$. Then we can construct symmetric n-ary forms of higher degree by taking $(x_1+\ldots+x_n)^{2i} f$, which can be seen to be in $S\mathcal{P}_{n,2d+2i} \setminus S\Sigma_{n,2d+2i}\ \forall\ i \geq 0$, by i applications of Lemma 2.1 with $p = x_1 + \ldots + x_n$. □

For the pair $(3, 6)$, Robinson [14] constructed the symmetric ternary sextic form $R(x, y, z) := x^6 + y^6 + z^6 - (x^4y^2 + y^4z^2 + z^4x^2 + x^2y^4 + y^2z^4 + z^2x^4) + 3x^2y^2z^2$ and showed that it is psd but not sos. For the pair $(4, 4)$, Choi and Lam [3] gave the form $f_{4,4} \in S\mathcal{P}_{4,4} \setminus S\Sigma_{4,4}$. So in view of Proposition 2.2, it remains to find psd not sos symmetric n-ary quartics for $n \geq 5$.

We will now construct explicit forms $f \in S\mathcal{P}_{n,4} \setminus S\Sigma_{n,4}$ for $n \geq 4$. For $n \geq 4$, consider the symmetric n-ary quartic (studied in [5])

$$L_n(\underline{x}) := m(n-m)\sum_{i<j}(x_i - x_j)^4 - \Big(\sum_{i<j}(x_i - x_j)^2\Big)^2,$$

where $m = \left\lfloor \frac{n}{2} \right\rfloor$. We shall show that L_n is psd for all n and L_n is not sos for all odd $n \geq 5$.

We need an important result (Theorem 2.3 below) of Choi, Lam and Reznick [5]. The same argument was modified in [8, Theorem 2.3] to treat even symmetric n-ary octics for $n \geq 4$.

Theorem 2.3. A symmetric n-ary quartic f is psd iff $f(\underline{x}) \geq 0$ for every $\underline{x} \in \mathbb{R}^n$ with at most two distinct coordinates (if $n \geq 4$), i.e. $\Lambda_{n,2} = \{\underline{x} \in \mathbb{R}^n \mid x_i \in \{r, s\}; r \neq s\}$ is a test set for symmetric n-ary quartics.

Proof. See [6, Corollary 3.11]. □

Remark 2.4. V. Timofte's half degree principle [16] gives a complete generalisation of above theorem for both symmetric polynomials (i.e. invariant under the action of the group S_n) and even symmetric polynomials (i.e. invariant under the action of the group $S_n \times \mathbb{Z}_2^n$) of degree $2d$ in n variables. See [10] for an application of this principle to elementary symmetric functions.

For $n = 5$, $L_n(\underline{x})$ has been discussed by A. Lax and P. D. Lax. They showed [11, p.72] that

$$A_5(\underline{x}) := \sum_{i=1}^{5}\prod_{j\neq i}(x_i - x_j) = \frac{1}{8}L_5,$$

a psd symmetric quartic in five variables, is not sos.

Proposition 2.5. L_n is psd for all n.

Proof. In view of Theorem 2.3, it is enough to prove that $L_n \geq 0$ on the test set $\Lambda_{n,2} = \{(\underbrace{r,\dots,r}_{k}, \underbrace{s,\dots,s}_{n-k}) \mid r \neq s \in \mathbb{R}; 0 \leq k \leq n\}$.

Now for $\underline{x} \in \Lambda_{n,2}$,

$$x_i - x_j = \begin{cases} \pm(r-s) \neq 0, & \text{for } k(n-k) \text{ terms,} \\ 0 & \text{, otherwise} \end{cases}$$

so L_n takes the value

$$\begin{aligned} L_n(\underline{x}) &= m(n-m)k(n-k)(r-s)^4 - [k(n-k)(r-s)^2]^2 \\ &= k(n-k)(r-s)^4[(m-k)(n-m-k)], \end{aligned}$$

which is non-negative since there is no integer between m and $n-m$. □

Definition 2.6. Let $\{0,1\}^n$ be the set of all n-tuples $\underline{x} = (x_1,\dots,x_n)$ with $x_i \in \{0,1\}$ for all $i = 1,\dots,n$. A subset $S \subset \{0,1\}^n$ is called a **0/1 set** and $\underline{x} \in \{0,1\}^n$ a **0/1 point**.

Lemma 2.7. Suppose $n \geq 4$ and $h(x_1,\dots,x_n)$ is a quadratic form that vanishes on all 0/1 points with m or $(m+1)$ 1's, where $m = \left\lfloor \frac{n}{2} \right\rfloor$, i.e. $h(\underline{x}) = 0$ for all $\underline{x}$ with m or $(m+1)$ 1's and $\begin{cases} (m+1) \text{ or } m \text{ 0's (respectively) for odd } n = 2m+1; \\ m \text{ or } (m-1) \text{ 0's (respectively) for even } n = 2m. \end{cases}$
Then h is identically zero.

Proof. Set $h(x_1,\dots,x_n) = \sum_{i=1}^{n} a_i x_i^2 + \sum_{i<j} a_{ij} x_i x_j$. Fix distinct i, j, k and let S such that $|S| = m-1$, be a set of indices not containing i, j, k. Then $h = 0$ on $\underline{x}$, where the 1's on $\underline{x}$ occur precisely on $S \cup \{i\}$, $S \cup \{i,k\}$, $S \cup \{j\}$, $S \cup \{j,k\}$. So we have:

$$\text{on } S \cup \{i\}: \quad 0 = \sum_{l \in S} a_l + a_i + \sum_{l<l' \in S} a_{ll'} + \sum_{l \in S} a_{il},$$

$$\text{on } S \cup \{i,k\}: \quad 0 = \sum_{l \in S} a_l + a_i + a_k + \sum_{l<l' \in S} a_{ll'} + \sum_{l \in S} a_{il} + \sum_{l \in S} a_{kl} + a_{ik}.$$

Subtracting above two equations gives:

$$a_k + \sum_{l \in S} a_{kl} + a_{ik} = 0. \tag{4}$$

Doing the same with $S \cup \{j\}$ and $S \cup \{j,k\}$ gives:

$$a_k + \sum_{l \in S} a_{kl} + a_{jk} = 0. \tag{5}$$

Thus $a_{ik} = a_{jk}$ (from equations (4) and (5)).

Since i, j, k are arbitrary, $a_{ik} = a_{jk} = a_{jl}$ for any $l \neq i, j, k$. So all the coefficients of $x_i x_j$ (for $i \neq j$) in h are equal, say $a_{ij} = u; i \neq j$.

It follows from equation (4) that $a_k + mu = 0$. So $a_k = -mu \ \forall\, k$, which gives:

$$h(x_1,\dots,x_n) = u\left(-m \sum_{i=1}^{n} x_i^2 + \sum_{i<j} x_i x_j\right).$$

But then $h(\underbrace{1,\dots,1}_{m},0,\dots,0) = 0$ gives $u\left(-m(m) + \dfrac{m(m-1)}{2}\right) = 0$, which implies $u = 0$, which implies $h = 0$. □

Theorem 2.8. If $n \geq 5$ is odd, then L_n is not sos.

Proof. Fix odd $n \geq 5, n = 2m+1$. Then

$$L_{2m+1} = m(m+1)\sum_{i<j}(x_i - x_j)^4 - \Big(\sum_{i<j}(x_i - x_j)^2\Big)^2.$$

If $L_{2m+1} = \sum_t h_t^2$, then $L_{2m+1}(\underline{x}) = 0 \Rightarrow$ each $h_t(\underline{x}) = 0$, for any $\underline{x} \in \mathbb{R}^n$.

In particular, $L_{2m+1}(\underline{x}) = 0$ when $\underline{x}$ has m or $(m+1)$ 1's and $(m+1)$ or m 0's. So, $h_t(\underline{x}) = 0$ for $\underline{x}$ with m or $(m+1)$ 1's and $(m+1)$ or m 0's respectively. Write

$$h_t(\underline{x}) = \sum_{i=1}^{n} a_i x_i^2 + \sum_{i<j}^{n} a_{ij} x_i x_j.$$

Then by Lemma 2.7, we get $h_t = 0$. Hence L_{2m+1} is not sos. □

Now we construct $f \in S\mathcal{P}_{2m,4} \setminus S\Sigma_{2m,4}$ for $m \geq 2$.

Unfortunately,

$$L_{2m}(\underline{x}) = \sum_{i<j}(x_i - x_j)^2\Big(-(x_1 + \ldots + x_{2m}) + m(x_i + x_j)\Big)^2$$

(see [6, Proposition 3.13]) is sos, and so we need a different example in $S\mathcal{P}_{2m,4} \setminus S\Sigma_{2m,4}$. For $2m \geq 4$, let

$$C_{2m}(x_1,\dots,x_{2m}) := L_{2m+1}(x_1,\dots,x_{2m},0).$$

Trivially, C_{2m} is a symmetric $2m$-ary quartic and psd. We shall show it is not sos.

Theorem 2.9. For $m \geq 2$, $C_{2m}(x_1,\dots,x_{2m})$ is not sos.

Proof. If $C_{2m} = \sum_t h_t^2$, then $C_{2m}(\underline{x}) = 0 \Rightarrow$ each $h_t(\underline{x}) = 0$, for any $\underline{x} \in \mathbb{R}^n$. In particular, $C_{2m}(\underline{x}) = 0$ when $\underline{x}$ has m or $(m+1)$ 1's and m or $(m-1)$ 0's. So, $h_t(\underline{x}) = 0$ for $\underline{x}$ with m or $(m+1)$ 1's and m or $(m-1)$ 0's respectively.

Write

$$h_t(\underline{x}) = \sum_{i=1}^{n} a_i x_i^2 + \sum_{i<j}^{n} a_{ij} x_i x_j.$$

Then by Lemma 2.7, we get $h_t = 0$. Hence, C_{2m} is not sos. □

To sum up, the answer to $Q(S)$ can be summarised by the same chart as for Hilbert's Theorem, given in the Introduction.

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