Essays on Multidimensional BSDEs and FBSDEs

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Presented by
PENG LUO

at the

Universität Konstanz

Faculty of Sciences
Department of Mathematics and Statistics

Accepted on the recommendation of
Prof. Dr. Michael Kupper, examiner
Prof. Dr. Ying Hu, co-examiner
Prof. Dr. Stefan Volkwein, co-examiner

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Abstract

This thesis elaborates on several topics on multidimensional BSDEs and FBSDEs.

In the first part, we consider multidimensional quadratic BSDEs with generators which can be separated into a coupled and an uncoupled part allowing to analyse the degree of coupling of the system in terms of the growth coefficients. We provide conditions on the relation between the size of the terminal condition and the degree of coupling which guarantee existence and uniqueness of solutions.

In the second part, we derive two existence and uniqueness results for multidimensional and coupled systems of forward-backward SDEs when the generator of the backward equation may have quadratic growth in the control variable and the parameters of the forward equation are Lipschitz continuous. In the Markovian setting, we show existence and uniqueness in the superquadratic case for unbounded terminal conditions. Furthermore, the Markovian setting can be dropped if the generator can be separated into a quadratic and a subquadratic part, and the terminal condition is bounded. In this case the solution exists on a small time interval.

In the last part, we consider a BSDE with a generator that can be subjected to delay, in the sense that its current value depends on the weighted past values of the solutions, for instance a distorted recent average. Existence and uniqueness results are provided in the case of possibly infinite time horizon for equations with, and without reflection. Furthermore, we show that when the delay vanishes, the solutions of the delayed equations converge to the solution of the equation without delay. We argue that these equations are naturally linked to forward backward systems, and we exemplify a situation where this observation allows to derive results for quadratic delayed equations with non-bounded terminal conditions in multidimension.

Zusammenfassung

Diese Arbeit beschäftigt sich mit mehreren Themen auf dem Gebiet von mehrdimensionale BSDEs und FBSDEs.

Im ersten Teil betrachten wir mehrdimensionale quadratische BSDEs mit Generatoren, die eine Zerlegung in einen zusammenhängenden und einen nicht zusammenhängenden Teil erlauben. Damit kann der Kopplungsgrad des Systems in Bezug auf den Wachstumskoeffizienten analysiert werden. Wir liefern Bedingungen an die Relation zwischen der Endbedingung und dem Kopplungsgrad, welche die Existenz und Eindeutigkeit von Lösungen sichern.

Im zweiten Teil zeigen wir zwei Resultate zur Existenz und Eindeutigkeit für mehrdimensionale und zusammenhängende Systeme von Forward-Backward-SDEs, wobei der Generator der Backward Gleichung quadratischen Wachstum in dem Control Prozess haben darf und die Parameter der Forward Gleichung Lipschitz-stetig sind. Im Markov Fall zeigen wir Existenz und Eindeutigkeit für den superquadratischen Fall bei unbeschränkten Endbedingungen. Weiterhin kann die Markov Bedinung aufgehoben werden, sofern der Generator in einen quadratischen und einen subquadratischen Teil getrennt werden kann, und die Endbedingung beschränkt ist. In diesem Fall existiert die Lösung auf einem kleinen Zeitintervall.

Im letzten Teil betrachten wir eine BSDE mit einem verzögerten Generator, in dem Sinne, dass der aktuelle Wert von den gewichteten vergangenen Werten abhängt, z.B. ein verzerrter Mittelwert. Existenz und Eindeutigkeit der Lösung werden im Fall von möglicherweise unendlichen Zeithorizont für Gleichungen mit und ohne Reflexion bewiesen. Darüber hinaus zeigen wir die Konvergenz der Lösungen der verzögerten Gleichungen gegen die Lösung der Gleichung ohne Verzögerung, sofern die Verzögerung verschwindet. Wir argumentieren noch, dass diese Gleichungen mit Forward-Backward-SDEs verbunden sind, und damit veranschaulichen wir eine Situation, wo die Ergebnisse der quadratischen verzögerten Gleichungen aus FBSDEs mit unbeschränkte Endbedingungen ableitbar sind.

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Dedications

To my family

Contents

Chapter 1

Introduction

Backward stochastic differential equations (BSDEs) were first introduced by Bismut [9] as adjoint equations in stochastic optimization problems. On a filtered probability space, a BSDE usually takes the form:

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \tag{1.0.1}$$

where W is a standard Brownian motion, ξ is called the terminal condition and g the generator. A solution is a pair of predictable processes (Y, Z) such that (1.0.1) holds, Y is called the value process and Z the control process. The first general solvability result is due to Pardoux and Peng [63] for square integrable terminal conditions and Lipschitz continuous generators. Since then, BSDEs have been intensively studied and used as a powerful tool in applied and theoretical areas, particularly in mathematical finance. In their survey paper, El Karoui et al. [28] presented possible applications of BSDEs in stochastic control theory and financial modeling. For instance, BSDEs naturally appear in the theory of contingent claim valuation in complete market. It is pointed out that the works by Black and Scholes [10], Duffie [27], Harrison and Kreps [34], Harrison and Pliska [35], Karatzas [45] and Merton [59] can be expressed as BSDEs. BSDE also connects to the theory of risk measure. Peng [67] defined g-expectation and conditional g-expectation through the solution of a BSDE with g as the generator. Rosazza Gianin [70] showed that g-expectation corresponds to a coherent (resp. convex) risk measure if g is sublinear (resp. convex). She suggested a conditional g-expectation as a dynamic risk measure and proved that a dynamic coherent or convex risk measure can be represented as a conditional g-expectation under strictly monotone and dominated conditions. Delbaen et al. [21] represented the penalty term of general dynamic concave utilities (hence of dynamic convex risk measures) in the context of a Brownian filtration and with a fixed finite time horizon. Their approach relies on the theory of g-expectation.

Moreover, considerable works have been done to weaken the assumptions on the terminal conditions and the generators. Among them, Kobylanski [48] obtained the existence and uniqueness of the solution of a BSDE when the generator can grow quadratically in the control process and the terminal condition is bounded for the 1-dimensional case. The main technique is to use an exponential change of variable. She also proved the comparison theorem and established the stability result and its relation with PDEs. By an approximation procedure based on Malliavin calculus, Briand and Elie [13] provided a simple approach to construct the solution to a quadratic BSDE with bounded terminal condition. This method allows them to consider a delayed quadratic BSDE whose generator depends on the recent past of the value process. Briand and Hu [14, 15] obtained the existence of the solution by relaxing the boundedness on the terminal conditions to the existence of exponential moments. By additionally assuming the generator to be convex in the control process, the uniqueness holds. Barrieu and El Karoui [7] studied the stability and convergence of some general quadratic semimartingales. They proved the existence of solutions of general quadratic BSDEs under minimal exponential integrability assumptions relying on their convergence result. When the generator has superquadratic growth in the control process. The first result is due to Delbaen et al. [22] who consider a generator which is convex in z and bounded terminal conditions. They showed that there exists a bounded terminal condition such that the associated BSDE does not admit any bounded solution and if the BSDE has a bounded solution, there exist infinitely many of them. When the terminal condition and the generator are deterministic functions of a forward SDE, they obtain an existence result. Richou [69] studied the existence and uniqueness of solutions to quadratic and superquadratic Markovian BSDEs with unbounded terminal conditions. Based on a priori estimate on the control process, he proved the existence of a viscosity solution to a semilinear parabolic PDE having quadratic or superquadratic growth in the gradient of the solution and gave explicit convergence rates for time approximation of quadratic or superquadratic Markovian BSDEs. His solvability result was later extended by Masiero and Richou [58] where the regularity assumption on the terminal condition is weakened. Cheridito and Nam [16] obtained the existence and uniqueness of solutions of BSDEs when the generator can grow arbitrarily fast in z and the terminal condition has bounded Malliavin derivative.

A predominant area of applications of quadratic BSDEs is utility maximization and indifference pricing. In a financial market with constrains on the portfolios, Rouge and El Karoui [71] characterized the price for a claim as a quadratic BSDE. Sekine [72] studied the maximization problem for the exponential and power utility functions based on a duality result obtained by Cvitanic and Karatzas [19]. He derived a quadratic BSDE as a necessary and sufficient condition for optimality via a variational method and dynamic programming. Their results are extended by Hu et al. [42] by applying the theory of BMO martingales. They obtained appropriate quadratic BSDEs for the value processes of several constrained utility maximization problems. Morlais [60] studied the existence and uniqueness of solutions for a kind of quadratic BSDEs driven by a continuous martingale and gave applications to the utility maximization problem. Building on the work by Mania and Tevzadze [56, 57], Nutz [62] investigated the respective BSDE for a power utility function in a more general setting. He established a one-to-one correspondence between solutions to BSDEs and solutions to the so-called primal and dual problems of utility maximization. Mania and Schweizer [55] studied the dynamics of the exponential utility indifference valuation. They obtained that the indifference value process is the unique solution of a quadratic BSDE and provided BMO estimates for the components of this solution. Becherer [8] considered the same problem in a discontinuous setting. Bordigoni et al. [12] studied a stochastic control problem arising in the context of utility maximization under model uncertainty. They characterized the dynamic value process as the unique solution of a generalized quadratic BSDE. Their approach is extended to an infinite time horizon in Hu and Schweizer [40]. Heyne et al. [36] studied the utility maximization problem of an agent with non-trivial endowment and whose preferences are modeled by the maximal subsolution of a BSDE. They proved that the utility maximization problem can be seen as a robust control problem admitting a saddle point if the generator of the BSDE is convex and satisfies a quadratic growth condition.

Multidimensional quadratic BSDEs naturally arise in equilibrium pricing models in financial mathematics. Cheridito et al. [18] solved a problem of valuing a derivative in an incomplete market in a discrete setting. They closed their work by considering the continuous case which leads to a fully coupled multidimensional quadratic BSDE whose solvability is unknown. Kramkov and Pulido [49] considered a financial model where the prices of risky assets are quoted by a representative market maker who takes into account an exogenous demand. These prices can be characterized as a system of quadratic BSDEs. They obtained a unique solution of this system for bounded terminal condition when the market maker's risk aversion is sufficiently small. They also proved that the established equilibrium is unique in the global sense. Kardaras et al. [46] studied existence an uniqueness of continuous time stochastic Radner equilibria in an incomplete markets model. This problem is equivalent to solving a fully coupled system of quadratic BSDEs. By introducing the notion of distance to Pareto optimality, they proved the existence and uniqueness of the equilibrium when the distance is small enough.

However, a general existence theory does not exist for multidimensional quadratic BSDEs. Frei and dos Reis [30] and Frei [29] provided counterexamples which show that multidimensional quadratic BSDEs may fail to have a global solution. The main difficulty is that the comparison theorem may fail to hold for BSDE systems (see [39]). Tevzadze [73] proved that when the terminal condition is small enough, one has a unique solution for multidimensional quadratic BSDE. The main idea is to construct a contraction mapping on $\mathcal{S}^\infty \times BMO$. Cheridito and Nam [17] and Hu and Tang [41] obtained local solvability on $[T-\varepsilon, T]$ for some $\varepsilon > 0$ of systems of BSDEs with subquadratic generators and diagonally quadratic generators respectively, which under additional assumptions on the generator can be extended to global solutions. Cheridito and Nam [17] provided solvability for Markovian quadratic BSDEs and projectable quadratic BSDEs. Frei [29] introduced the notion of split solution and studied the existence of multidimensional quadratic BSDEs by considering a special kind of terminal condition. In Bahlali et al. [6] existence is shown when the generator $g(s, y, z)$ is strictly subquadratic in z and satisfies some monotonicity condition. In this thesis, we study multidimensional quadratic BSDEs with separated generators. Sufficient conditions are provided which guarantee the existence and uniqueness of solutions.

Similar to stochastic differential equations (SDEs), BSDEs are related to partial differential equations (PDEs). Peng [66] showed that the solution of a BSDE provides a probabilistic interpretation of a solution for a quasilinear PDE in the spirit of the well-known Feynman-Kac formula when the BSDE is Markovian, i.e., the randomness of the terminal condition and the generator comes form a forward SDE. We usually call this system decoupled forward-backward stochastic differential equation (FBSDE). When the drift and diffusion coefficients in the forward SDE depend on the solution of the BSDE, we call this system coupled FBSDE. Antonelli [4] obtained the first solvability result of a coupled FBSDE over a small time horizon. He also constructed a counterexample which shows that for coupled FBSDEs, large time horizon may lead to non-solvability. This method is later detailed by Pardoux and Tang [64]. They studied the existence and uniqueness of the solution for a coupled FBSDE. Continuous dependence of the solution on a parameter is obtained. They also provided the connection between FBSDEs and quasilinear parabolic PDES. Ma et al. [53] studied Markovian FBSDEs by using the so-called "Four Step Scheme". By requiring the non-degeneracy of the forward diffusion coefficient and non-randomness of the coefficients, they proved that the backward component of the solution are determined explicitly by the forward component via a quasilinear PDE. This method works for arbitrarily large time horizon. Another method is the "Method of Continuation" initialed by Hu and Peng [38], Peng and Wu [65], developed by Yong [74, 75]. Under monotonicity conditions on the coefficients, they obtained solvability for non-Markovian FBSDEs with arbitrary duration. In a Markovian setting with forward diffusion process being uniformly non-degenerate, Delarue [20] obtained the existence and uniqueness of the solution of an FBSDE over arbitrary time horizon by combining contraction mapping method and the "Four Step Scheme" method and some delicate PDE arguments. This result was later extended to non-Markovian case by Zhang [76]. Recently, Ma et al. [54] established a unified approach to study the wellposedness of general non-Markovian FBSDEs. They introduced the concept of "Decoupling Fields". They provided sufficient conditions under which the associated characteristic BSDE is wellposed which leads to the existence of decoupling fields, and ultimately to the solvability of FBSDE. This method is significantly refined and extended to multi-dimensional systems by Fromm and Imkeller [32]. The above mentioned results on coupled FBSDEs assume Lipschitz continuity of the generator g. However, FBSDEs appearing in the study of stochastic control problems are typically of quadratic growth in Z. For instance, this class of systems are shown to characterize solutions of utility maximization problems with non-trivial terminal endowment, see Horst et al. [37]. In this thesis, we consider the existence and uniqueness of solutions of coupled FBSDEs, with quadratic or even superquadratic growth and in the multi-dimensional case.

BSDEs with time-delayed generators were introduced in Delong and Imkeller [24]. In this type of equation, the generator may depend on the path of the value and control processes with some weighted measures. They obtained existence and uniqueness of a solution for a sufficiently small time horizon or for a sufficiently small Lipschitz constant of the generator. For some special classes of generators, they obtained that the existence and uniqueness may still hold for arbitrarily large time horizon and Lipschitz coefficient. They also showed that solutions of BSDEs with time-delayed generators do not in general inherit the boundedness and BMO properties. Delong and Imkeller [25] investigated BSDEs with time delayed generators diven by Brownian motions and Poisson random measures. The existence and uniquess of solutions were obtained when the time horizon or the Lipschitz coefficient is sufficiently small. They also studied differentiability in the variational or Malliavin sense and derived equations that are satisfied by the Malliavin gradient processes. This class of equations turned out to have natural applications in pricing and hedging of insurance contracts, see Delong [23]. dos Reis et al. [26] provided sufficient conditions for the solution of a BSDE with time delayed generator to exist in L^p. They also considered the decoupled systems of SDEs and BSDEs with time delayed generators. Sufficient conditions for their variational differentiability were provided. By usual representation formulas, variational derivatives and the Malliavin derivatives are connected. Some path regularity results are obtained. Zhou and Ren [77] established the existence and uniqueness of the solution for a reflected BSDE with time delayed generator for a sufficiently small Lipschitz coefficient of the generator and a continuous barrier process. In this thesis, we consider BSDEs with time delayed generators on finite and infinite time horizon. Moreover, we study reflected BSDE with time delayed generator and a RCLL (right continuous with left limits) barrier process. We also study quadratic and superquadratic BSDEs with delay only in the value process from the connection between BSDEs with time delayed generators and FBSDEs.

Structure and Main Results of the Thesis: This thesis consists of three main chapters which have resulted in three preprints: Jamneshan et al. [44], Luo and Tangpi [51] and Luo and Tangpi [52].

In chapter 2, we will study the existence and uniqueness of solutions of multidimensional quadratic BSDEs. We start with coupled system with partial dependence where the generator is sum of squares of the control processes. We will study the interplay between terminal conditions and coefficients which guarantees solvability of this system. The idea is that we first solve a family of 1-dimensional parameterized BSDEs by using Pardoux and Peng [63] or an extension of Lemma 2.5 in [41]. We provide conditions such that we can define a functional which maps a bounded subset of $\mathcal{S}^\infty \times BMO$ into itself. Later, under some additional conditions, we can obtain a unique solution by applying Banach fixed point theorem. When the generator only has coupledness in the value process, i.e., the i-th component of the generator only depends on the i-th component of the control process, we obtain the existence and uniqueness of the solution for arbitrarily large terminal condition and time horizon. The main technique is to obtain a contraction mapping on S^∞ when time duration is small by using Girsanov's theorem. We then obtain the solvability for arbitrarily large time horizon by a pasting technique. For the general case, we consider generators which can be separated into a coupled part and an uncoupled part. Two kinds of sufficient conditions are provided which yield the existence and uniqueness of solutions of multidimensional quadratic BSDEs with separated generators.

In chapter 3, we will study multi-dimensional and coupled systems of forward-backward SDEs when the generator of the backward equation may have quadratic growth in the control variable and the parameters of the forward equation are Lipschitz continuous. In the Markovian setting, we consider superquadratic generator and unbounded terminal condition. The generator is assumed to have only coupledness in the value process. The drift coefficient of the forward part does not depend on the control process, and the diffusion coefficient is assumed to be bounded and Borel measurable. We first assume that all the coefficients are continuously differentiable. Given $(X^0, Y^0, Z^0) = (0, 0, 0)$, we can obtain a sequence of solutions (X^n, Y^n, Z^n) of a family of decoupled FBSDEs. The main technique is that we first show that X^1 is Malliavin differentiable and then we solve the backward part by using an extension of the existence result of Cheridito and Nam [16] to multidimension. Moreover, we have Z^1 is bounded. By induction, we obtain (X^n, Y^n, Z^n) such that Z^n is uniformly bounded from which we show that (X^n, Y^n, Z^n) is a Cauchy sequence in $\mathcal{S}^2 \times \mathcal{S}^2 \times \mathcal{H}^2$ whenever the time horizon is sufficiently small. For the general case, by considering a sequence of nonnegative C^∞ operators, we can obtain a sequence of continuously differentiable coefficients by defining the convolution with these operators. From the first step, we have a sequence of solutions (X^n, Y^n, Z^n). We show that (X^n, Y^n, Z^n) converges to (X, Y, Z) which is the unique solution of our original FBSDE. Under additional growth conditions on the coefficients and the strictly positive definiteness of the diffusion coefficient, we can extend the solvability result to arbitrarily large time horizon. Since Z is uniformly bounded, by a transformation, we are actually considering coupled FBSDEs with Lipschitz generators. By the uniqueness of solution and a pasting technique, the result follows. For the non-Markovian case, we consider generators which can be separated into a quadratic and a subquadratic part, and bounded terminal conditions. The diffusion coefficient is assumed to be a given process in $\mathcal{H}^2$. For any $(y, z \cdot W) \in \mathcal{S}^\infty \times BMO$, we solve a decoupled FBSDE. By using the results in Hu and Tang [41] or chapter 2, we obtain a contraction mapping on a bounded subset of $\mathcal{S}^\infty \times BMO$ if the time horizon is sufficiently small. The existence and uniqueness of the solution follows from Banach fixed point theorem. Moreover, the continuity and differentiability of the solution with respect to the initial value are presented.

In chapter 4, we will investigate a new kind of BSDEs with time-delayed generators. Except the weighted measures, we also consider the existence of weighting functions in the delay. We assume the generator to satisfy the standard Lipschitz condition and we allow the time horizon to be infinity. If the Lipschitz coefficient or the mass of weighted measures or the norm of the weighting functions is sufficiently small, we obtain the existence and uniqueness of the solution to BSDE with time-delayed generator. The result holds similarly for BSDEs with time-delayed generators and constrained above a RCLL barrier. When the time horizon is finite and the generator has no delay in the control process, we establish the link between FBSDEs and BSDEs with time-delayed generators from which we obtain some solvability results for BSDE with quadratic and superquadratic growth and with delay only in the value process by using the results in chapter 3. Compared to Briand and Elie [13], we consider multidimensional case and a different kind of delay. Moreover, our argument allows to consider a superquadratic generator.

Chapter 2

Multidimensional Quadratic BSDEs with Separated Generators

2.1 Introduction

Backward stochastic differential equations (BSDEs) were introduced by Bismut [9]. A BSDE is an equation of the form

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \quad t \in [0, T], \tag{2.1.1}$$

where W is a d-dimensional Brownian motion, the terminal condition ξ is an n-dimensional random variable, and $g : \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \to \mathbb{R}^n$ is the generator. A solution consists of a pair of predictable processes (Y, Z) with values in $\mathbb{R}^n$ and $\mathbb{R}^{n \times d}$, called the value and control process, respectively. The first existence and uniqueness result for BSDEs with an L^2-terminal condition and a generator satisfying a Lipschitz growth condition is due to Pardoux and Peng [63]. In case that the generator satisfies a quadratic growth condition in the control z, the situation is more involved and a general existence theory does not exist. Frei and dos Reis [30] and Frei [29] provided counterexamples which show that multidimensional quadratic BSDEs may fail to have a global solution. In the one-dimensional case the existence of quadratic BSDE was shown by Kobylanski [48] for bounded terminal conditions, and by Briand and Hu [14, 15] for unbounded terminal conditions. Solvability results for superquadratic BSDEs are discussed in Delbaen et al. [22], see also Masiero and Richou [58], Richou [69] and Cheridito and Nam [16].

The focus of the present work lies on multidimensional quadratic BSDEs, which naturally arise in equilibrium pricing models in financial mathematics. In case that the terminal condition is small enough the existence and uniqueness of a solution was first shown by Tevzadze [73]. Cheridito and Nam [17] and Hu and Tang [41] obtained local solvability on $[T-\varepsilon, T]$ for some $\varepsilon > 0$ of systems of BSDEs with subquadratic generators and diagonally quadratic generators respectively, which under additional assumptions on the generator can be extended to global solutions. Cheridito and Nam [17] provided solvability for Markovian quadratic BSDEs and projectable quadratic BSDEs. Frei [29] introduced the notion of split solution and studied the existence of multidimensional quadratic BSDEs by considering a special kind of terminal condition. In Bahlali et al. [6] existence is shown when the generator $g(s, y, z)$ is strictly subquadratic in z and satisfies some monotonicity condition.

For the sake of illustration of our results we consider the following system of quadratic BSDEs:

$$\begin{aligned} Y_t^1 &= \xi^1 + \int_t^T \theta_1 |Z_s^1|^2 + \vartheta_1 |Z_s^2|^2 ds - \int_t^T Z_s^1 dW_s, \\ Y_t^2 &= \xi^2 + \int_t^T \vartheta_2 |Z_s^1|^2 + \theta_2 |Z_s^2|^2 ds - \int_t^T Z_s^2 dW_s, \end{aligned} \tag{2.1.2}$$

where $t \in [0, T]$, $\xi^i \in L^\infty$ and $\theta_i, \vartheta_i \in \mathbb{R}$, $i = 1, 2$. In the case that $\vartheta_1 = \vartheta_2 = 0$, the system (2.1.2) reduces to decoupled one-dimensional quadratic BSDEs, which by Kobylanski [48] have solutions for every terminal conditions $\xi^i \in L^\infty$ and $\theta_i \in \mathbb{R}$, $i = 1, 2$. Moreover, by Tevzadze [73] the system (2.1.2) has a solution whenever the terminal conditions ξ^1 and ξ^2 are small enough. In the present work we give two different sets of conditions on the interplay between the parameters θ^i, ϑ^i and the terminal conditions ξ^i, $i = 1, 2$, which guarantee the solvability of system (2.1.2) in Section 2.2. For instance, given θ^i and ξ^i the system (2.1.2) has a solution if $|\vartheta^i|$ is small enough for $i = 1, 2$. To the best of our knowledge there is no literature which can answer this question.

The general case is treated in Section 2.3. We consider generators which can be separated into two parts: the coupled and the uncoupled part. We use the growth coefficients of the coupled part to characterize the degree of the coupling. In the first step of the construction of the solution we view the coupled part as a parameter and solve in Lemma 2.A.1 a 1-dimensional quadratic BSDE by means of Theorem 2 in [14]. This leads to a bounded set of candidate solutions where the value process is uniformly bounded and the control process is bounded in BMO. These bounds in combination with our conditions on the interplay between the parameters allow in a second step to apply Banach's fixed point theorem. If the generator is independent of the value process, the method allows to consider unbounded terminal conditions.

This chapter is organized as follows. In Section 2.2, we state our setting and main results for coupled systems with partial dependence. Section 2.3 is devoted to fully coupled systems. We present an auxiliary result for 1-dimensional quadratic BSDEs in Appendix 2.A.

2.2 Coupled systems with partial dependence

Fix a real number $T > 0$, and let $(W_t)_{t\geq 0}$ be a d-dimensional Brownian motion on a complete probability space $(\Omega, \mathcal{F}, P)$. Let $(\mathcal{F}_t)_{t\geq 0}$ be the augmented filtration generated by the Brownian motion W. For two real numbers $a, b \geq 0$, the minimum and maximum of a and b are denoted by $a \wedge b$ and $a \vee b$ respectively. The Euclidean norm is denoted by $|\cdot|$ and we denote by $\|\cdot\|_\infty$ the L^∞-norm. We assume that $\mathcal{F}_T = \mathcal{F}$ and denote by $\mathcal{P}$ the predictable σ-algebra on $\Omega \times [0, T]$. Inequalities and equalities between random variables and processes are understood in the P-almost sure and $P \otimes dt$-almost sure sense respectively. For $p \in [1, \infty)$ and $m, n \in \mathbb{N} = \{1, 2, \ldots\}$, we denote by $\mathcal{S}^p(\mathbb{R}^m)$ the space of $\mathbb{R}^m$-valued predictable and continuous processes X with $\|X\|^p_{\mathcal{S}^p} := E[(\sup_{t\in[0,T]} |X_t|)^p] < \infty$, and by $\mathcal{H}^p(\mathbb{R}^n)$ the space of $\mathbb{R}^{n\times d}$-valued predictable processes Z with $\|Z\|^p_{\mathcal{H}^p} := E[(\int_0^T |Z_u|^2 \, du)^{p/2}] < \infty$. For a suitable integrand Z, we denote by $Z \cdot W$ the stochastic integral $(\int_0^t Z_u \, dW_u)_{t\in[0,T]}$ of Z with respect to W. Let $\mathcal{S}^\infty(\mathbb{R}^n)$ denote the space of all n-dimensional continuous adapted processes such that

$$\|Y\|_\infty := \| \sup_{0\leq t\leq T} |Y_t| \|_\infty < \infty.$$

Let $\mathcal{T}$ be the set of stopping times with values in $[0, T]$.

By a solution we mean a pair of predictable processes (Y, Z) such that (2.1.1) holds and $t \mapsto Y_t$ is continuous, $t \mapsto Z_t$ belongs to $L^2([0, T])$ and $t \mapsto g(t, Y_t, Z_t)$ belongs to $L^1([0, T])$ P-a.s..

In the following, we give two existence results for the system (2.1.2) under two different conditions on the interplay between terminal conditions and coefficients. We assume $\vartheta_1 \neq 0,\ \vartheta_2 \neq 0$.

Theorem 2.2.1. *If $\theta_1 = \theta_2 = 0$ and suppose that*

(i) $8|\vartheta_2|\|\xi^1\|^2_\infty \leq \|\xi^2\|_\infty$, $8|\vartheta_1|\|\xi^2\|^2_\infty \leq \|\xi^1\|_\infty$,

(ii) $16|\vartheta_1|\|\xi^2\|_\infty \leq 1$, $16|\vartheta_2|\|\xi^1\|_\infty \leq 1$,

then the system (2.1.2) *admits a unique solution* (Y, Z) *such that* Y *is bounded and* $\|Z^1 \cdot W\|_{BMO} \leq 2\|\xi^1\|_\infty$ *and* $\|Z^2 \cdot W\|_{BMO} \leq 2\|\xi^2\|_\infty$.

Proof. For any $z \cdot W \in BMO$, it holds $\int_0^T |z_s|^2 ds \in L^2$. Fix $i = 1, 2$. By [63, Lemma 2.1], the BSDE

$$Y^i_t = \xi^i + \int_t^T \vartheta_i |z_s|^2 ds - \int_t^T Z^i_s dW_s,\ t \in [0, T], \tag{2.2.1}$$

admits a unique solution $(Y^i, Z^i) \in \mathcal{S}^2(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R}^d)$. For $t \in [0, T]$, taking conditional expectation with respect to $\mathcal{F}_t$, one obtains

$$Y^i_t = E\left[\xi^i + \int_t^T \vartheta_i |z_s|^2 ds \middle| \mathcal{F}_t \right],$$

and thus

$$|Y_t^i| \leq \|\xi^i\|_\infty + |\vartheta_i| \|z \cdot W\|_{BMO}^2.$$

By Itô's formula, it holds

$$|Y_t^i|^2 + \int_t^T |Z_s^i|^2 ds = |\xi^i|^2 + 2\vartheta_i \int_t^T Y_s^i |z_s|^2 ds - 2 \int_t^T Y_s^i Z_s^i dW_s.$$

Taking conditional expectation with respect to $\mathcal{F}_t$ yields

$$\begin{aligned}
&E\left[\int_t^T |Z_s^i|^2 \middle| \mathcal{F}_t\right] \\
&\leq E\left[|\xi^i|^2 + 2\vartheta_i \int_t^T Y_s^i |z_s|^2 ds \middle| \mathcal{F}_t\right] \\
&\leq \|\xi^i\|_\infty^2 + 2|\vartheta_i| \left(\|\xi^i\|_\infty + |\vartheta_i| \|z \cdot W\|_{BMO}^2\right) E\left[\int_t^T |z_s|^2 ds \middle| \mathcal{F}_t\right].
\end{aligned}$$

Hence

$$\|Z^i \cdot W\|_{BMO}^2 \leq \|\xi^i\|_\infty^2 + 2|\vartheta_i| \|\xi^i\|_\infty \|z \cdot W\|_{BMO}^2 + 2|\vartheta_i|^2 \|z \cdot W\|_{BMO}^4.$$

Let $M = \left\{(z^1, z^2) : \|z^1 \cdot W\|_{BMO} \leq 2\|\xi^1\|_\infty, \|z^2 \cdot W\|_{BMO} \leq 2\|\xi^2\|_\infty\right\}$. For $(z^1, z^2) \in M$, define $I(z^1, z^2) = (Z^1, Z^2)$, where Z^i is the second component of the solution of (2.2.1) when z is replaced by z^2 for $i = 1$, and z^1 for $i = 2$. By assumption (i), it is easy to check that I maps M into itself.

For $(z^1, z^2), (\bar{z}^1, \bar{z}^2) \in M$, let $(Z^1, Z^2) = I(z^1, z^2), (\bar{Z}^1, \bar{Z}^2) = I(\bar{z}^1, \bar{z}^2)$. Denote $\delta Z^i = Z^i - \bar{Z}^i$, $\delta z^i = z^i - \bar{z}^i$, $\delta Y^i = Y^i - \bar{Y}^i$ for $i = 1, 2$. Since

$$\delta Y_t^1 = \int_t^T \vartheta_1 \left(z_s^2 + \bar{z}_s^2\right) \delta z_s^2 ds - \int_t^T \delta Z_s^1 dW_s,$$

it follows from Itô's formula that

$$|\delta Y_t^1|^2 + \int_t^T |\delta Z_s^1|^2 ds = 2\vartheta_1 \int_t^T \delta Y_s^1 \left(z_s^2 + \bar{z}_s^2\right) \delta z_s^2 ds - 2 \int_t^T Y_s^1 \delta Z_s^1 dW_s.$$

Taking conditional expectation with respect to $\mathcal{F}_t$ and using $2ab \leq \frac{1}{4}a^2 + 4b^2$, one has

$$\begin{aligned}
&|\delta Y_t^1|^2 + E\left[\int_t^T |\delta Z_s^1|^2 ds \middle| \mathcal{F}_t\right] \\
&\leq \frac{1}{4}\|\delta Y^1\|_\infty^2 + 4|\vartheta_1|^2 E^2\left[\int_t^T \left(|z_s^2| + |\bar{z}_s^2|\right)|\delta z_s^2| ds \middle| \mathcal{F}_t\right].
\end{aligned}$$

Noting that

$$\begin{aligned}
\frac{1}{2}\left(\|\delta Y^1\|_\infty^2 + \|\delta Z^1\cdot W\|_{BMO}^2\right) &\leq \|\delta Y^1\|_\infty^2 \vee \|\delta Z^1\cdot W\|_{BMO}^2 \\
&\leq \operatorname*{ess\,sup}_{\tau\in\mathcal{T}}\left\{|\delta Y_\tau^1|^2 + E\left[\int_\tau^T |\delta Z^1|^2 ds|\mathcal{F}_\tau\right]\right\},
\end{aligned}$$

it follows from Hölder's inequality and $2ab \leq a^2 + b^2$ that

$$\begin{aligned}
\|\delta Z^1\cdot W\|_{BMO}^2 &\leq 8|\vartheta_1|^2 \operatorname*{ess\,sup}_{\tau\in\mathcal{T}} E^2\left[\int_t^T \left(|z_s^2| + |\bar{z}_s^2|\right)|\delta z_s^2| ds \middle| \mathcal{F}_t\right] \\
&\leq 16|\vartheta_1|^2\left(\|z^2\cdot W\|_{BMO}^2 + \|\bar{z}^2\cdot W\|_{BMO}^2\right)\|\delta z^2\cdot W\|_{BMO}^2 \\
&\leq 128|\vartheta_1|^2\|\xi^2\|_\infty^2\|\delta z^2\cdot W\|_{BMO}^2.
\end{aligned}$$

Similarly, one obtains

$$\|\delta Z^2\cdot W\|_{BMO}^2 \leq 128|\vartheta_2|^2\|\xi^1\|_\infty^2\|\delta z^1\cdot W\|_{BMO}^2.$$

By assumption (ii), I is a contraction. □

Theorem 2.2.2. *If $\theta_1 > 0$ and $\theta_2 > 0$ and suppose that*

(i) $4\theta_1|\vartheta_1|e^{2\theta_2\|\xi^2\|_\infty} \leq \theta_2^2$, $4\theta_2|\vartheta_2|e^{2\theta_1\|\xi^1\|_\infty} \leq \theta_1^2$,

(ii) $8L_4^4c_2^2|\vartheta_1|^2e^{2\theta_2\|\xi^2\|_\infty^2} \leq c_1\theta_2^2$ *and* $8L_4^4\bar{c}_2^2|\vartheta_2|^2e^{2\theta_1\|\xi^1\|_\infty^2} \leq \bar{c}_1\theta_1^2$,

where L_4 is given by Lemma A.1.4, c_1, c_2 (resp. $\bar{c}_1, \bar{c}_2$) are given by Lemma A.1.3 for K equals to $2e^{\theta_1\|\xi_1\|_\infty}$ (resp. $2e^{\theta_2\|\xi_2\|_\infty}$).

Then the system (2.1.2) *admits a unique solution (Y, Z) such that Y is bounded and $\|Z^1\cdot W\|_{BMO} \leq \frac{e^{\theta_1\|\xi^1\|_\infty}}{\theta_1}$ and $\|Z^2\cdot W\|_{BMO} \leq \frac{e^{\theta_2\|\xi^2\|_\infty}}{\theta_2}$.*

Proof. For $i = 1, 2$ and $z\cdot W \in BMO$ with $\|z\cdot W\|_{BMO}^2 \leq \frac{1}{4\theta_i|\vartheta_i|}$, from Lemma 2.A.1, the following BSDE

$$Y_t^i = \xi^i + \int_t^T \theta_i|Z_s^i|^2 ds + \int_t^T \vartheta_i|z_s|^2 ds - \int_t^T Z_s^i dW_s \tag{2.2.2}$$

admits a unique solution (Y^i, Z^i) such that $(Y^i, Z^i \cdot W) \in \mathcal{S}^\infty(\mathbb{R}) \times BMO$ and $\|Z^i \cdot W\|_{BMO} \leq \frac{e^{\theta_i \|\xi^i\|_\infty}}{\theta_i}$. Let

$$M = \left\{ (z^1, z^2) : \|z^1 \cdot W\|_{BMO} \leq \frac{e^{\theta_1 \|\xi^1\|_\infty}}{\theta_1}, \|z^2 \cdot W\|_{BMO} \leq \frac{e^{\theta_2 \|\xi^2\|_\infty}}{\theta_2} \right\}.$$

For $(z^1, z^2) \in M$, define $I(z^1, z^2) = (Z^1, Z^2)$, where Z^i is the second component of solution of equation (2.2.2) when z is replaced by z^2 for $i = 1$, and z^1 for $i = 2$. By assumption (i), it is easy to check that I maps M to itself.

For $(z^1, z^2), (\bar{z}^1, \bar{z}^2) \in M$, let $(Z^1, Z^2) = I(z^1, z^2)$ and $(\bar{Z}^1, \bar{Z}^2) = I(\bar{z}^1, \bar{z}^2)$. Denote $\delta Z^i = Z^i - \bar{Z}^i$, $\delta z^i = z^i - \bar{z}^i$, $\delta Y^i = Y^i - \bar{Y}^i$ for $i = 1, 2$. One has

$$\begin{aligned}
\delta Y_t^1 &= \int_t^T \theta_1 \left(Z_s^1 + \bar{Z}_s^1 \right) \delta Z_s^1 ds + \int_t^T \vartheta_1 \left(z_s^2 + \bar{z}_s^2 \right) \delta z_s^2 ds - \int_t^T \delta Z_s^1 dW_s \\
&= \int_t^T \vartheta_1 \left(z_s^2 + \bar{z}_s^2 \right) \delta z_s^2 ds - \int_t^T \delta Z_s^1 d\tilde{W}_s,
\end{aligned}$$

where $\tilde{W}_t := W_t - \int_0^t \theta_1 \left(Z_s^1 + \bar{Z}_s^1 \right) ds$ is a Brownian motion under the equivalent probability measure $\frac{d\tilde{P}}{dP} = \mathcal{E}_T(\theta_1 \left(Z_s^1 + \bar{Z}_s^1 \right) \cdot W)$. Putting the second term on the right hand to the left hand, taking square and conditional expectation with respect to $\mathcal{F}_t$ and $\tilde{P}$ and using Hölder's inequality, one obtains

$$\begin{aligned}
&\|\delta Y_t^1\| + \tilde{E} \left[\int_t^T |\delta Z_s^1|^2 ds \middle| \mathcal{F}_t \right] \\
&\leq \tilde{E} \left[\left(\int_t^T \vartheta_1 \left(z_s^2 + \bar{z}_s^2 \right) \delta z_s^2 ds \right)^2 \middle| \mathcal{F}_t \right] \\
&\leq |\vartheta_1|^2 \tilde{E} \left[\left(\int_t^T \left(z_s^2 + \bar{z}_s^2 \right)^2 ds \right)^2 \middle| \mathcal{F}_t \right]^{\frac{1}{2}} \tilde{E} \left[\left(\int_t^T |\delta z_s^2|^2 ds \right)^2 \middle| \mathcal{F}_t \right]^{\frac{1}{2}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\|\delta Z^1 \cdot \tilde{W}\|^2_{BMO(\tilde{P})} \\
&\leq 2|\vartheta_1|^2 \left(\|z_s^2 \cdot \tilde{W}\|^2_{BMO_4(\tilde{P})} + \|\bar{z}_s^2 \cdot \tilde{W}\|^2_{BMO_4(\tilde{P})} \right) \|\delta z_s^2 \cdot \tilde{W}\|^2_{BMO_4(\tilde{P})}.
\end{aligned}$$

Lemma A.1.4 implies

$$\begin{aligned}
&\|\delta Z^1 \cdot \tilde{W}\|^2_{BMO(\tilde{P})} \\
&\leq 2L_4^4|\vartheta_1|^2 \left(\|z_s^2 \cdot \tilde{W}\|^2_{BMO(\tilde{P})} + \|\bar{z}_s^2 \cdot \tilde{W}\|^2_{BMO(\tilde{P})} \right) \|\delta z_s^2 \cdot \tilde{W}\|^2_{BMO(\tilde{P})}.
\end{aligned}$$

Hence there exist constants $c_1 > 0$ and $c_2 > 0$ given by Lemma A.1.3 with $K = 2e^{\theta_1 \|\xi^1\|_\infty}$ such that

$$c_1 \|\delta Z^1 \cdot W\|_{BMO}^2 \leq 2L_4^4 c_2^2 |\vartheta_1|^2 \left(\|z_s^2 \cdot W\|_{BMO}^2 + \|\bar{z}_s^2 \cdot W\|_{BMO}^2 \right) \|\delta z_s^2 \cdot W\|_{BMO}^2$$
$$\leq \frac{4L_4^4 c_2^2 |\vartheta_1|^2 e^{2\theta_2 \|\xi^2\|_\infty^2}}{\theta_2^2} \|\delta z_s^2 \cdot W\|_{BMO}^2.$$

Similarly there are $\bar{c}_1 > 0$ and $\bar{c}_2 > 0$ for $K = 2e^{\theta_2 \|\xi^2\|_\infty}$ s.t.

$$\bar{c}_1 \|\delta Z^2 \cdot W\|_{BMO}^2 \leq \frac{4L_4^4 \bar{c}_2^2 |\vartheta_2|^2 e^{2\theta_1 \|\xi^1\|_\infty^2}}{\theta_1^2} \|\delta z_s^1 \cdot W\|_{BMO}^2.$$

Assumption (ii) implies that I is a contraction mapping. □

We finally state an existence result for the BSDE (4.2.1) where the coupling is only in the value process. We make the following assumptions:

(H5) $g : \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \to \mathbb{R}^n$ is predictable and $g^i(t, y, z) = g^i(t, y, z^i)$, $i = 1, \ldots, n$, for any $y \in \mathbb{R}^n$, $z \in \mathbb{R}^{n \times d}$. There exist constants $C \geq 0, \theta > 0$ and $\beta > 0$ such that

$$|g(t, 0, 0)| \leq C,$$
$$|g(t, y, z) - g(t, y', z')| \leq \beta |y - y'| + \theta(1 + |z| + |z'|)|z - z'|,$$

for all $t \in [0, T]$, $y, y' \in \mathbb{R}^n$ and $z, z' \in \mathbb{R}^{n \times d}$.

(H6) $\xi \in L^\infty(\mathcal{F}_T)$.

Theorem 2.2.3. *If (H5) and (H6) are satisfied, then the BSDE* (4.2.1) *admits a unique solution* (Y, Z) *such that* Y *is bounded and* $Z \cdot W \in BMO$.

Proof. For any $y \in \mathcal{S}^\infty(\mathbb{R}^n)$, it follows from Kobylanski [48] that for any $i = 1, \ldots, n$, the following BSDE

$$Y_t^i = \xi^i + \int_t^T g^i(s, y_s, Z_s^i) ds - \int_t^T Z_s^i dW_s \tag{2.2.3}$$

admits a unique solution (Y^i, Z^i) such that Y^i is bounded and $Z^i \cdot W \in BMO$. Hence by defining $I(y) = Y$, where the i-th component of Y is given by the first component of solution of BSDE (2.2.3), I maps $\mathcal{S}^\infty(\mathbb{R}^n)$ to itself. For $y, \bar{y} \in \mathcal{S}^\infty(\mathbb{R}^n)$, let $Y = I(y)$ and $\bar{Y} = I(\bar{y})$. Denote $\delta Z^i = Z^i - \bar{Z}^i$, $\delta y = y - \bar{y}$,

$\delta Y = Y - \bar{Y}$ $\delta Y^i = Y^i - \bar{Y}^i$ for $i = 1, 2$. One has

$$
\begin{aligned}
&\delta Y_t^i \\
&= \int_t^T g^i(s, y_s, Z_s^i) - g^i(s, \bar{y}_s, \bar{Z}_s^i) ds - \int_t^T \delta Z_s^i dW_s \\
&= \int_t^T g^i(s, y_s, Z_s^i) - g^i(s, y_s, \bar{Z}_s^i) + g^i(s, y_s, \bar{Z}_s^i) - g^i(s, \bar{y}_s, \bar{Z}_s^i) ds - \int_t^T \delta Z_s^i dW_s \\
&= \int_t^T b_s \delta Z_s^i + g^i(s, y_s, \bar{Z}_s^i) - g^i(s, \bar{y}_s, \bar{Z}_s^i) ds - \int_t^T \delta Z_s^i dW_s,
\end{aligned}
$$

where $|b_s| \leq \theta(1 + |Z_s^i| + |\bar{Z}_s^i|)$ implies $b \cdot W$ is a BMO martingale. By Girsanov's theorem, $\tilde{W}_t := W_t - \int_0^t b_s ds$ is a Brownian motion under the equivalent probability measure $\frac{d\tilde{P}}{dP} = \mathcal{E}_T(b \cdot W)$. Hence

$$
\delta Y_t^i = \int_t^T g^i(s, y_s, \bar{Z}_s^i) - g^i(s, \bar{y}_s, \bar{Z}_s^i) ds - \int_t^T \delta Z_s^i d\tilde{W}_s.
$$

Taking conditional expectation with respect to $\mathcal{F}_t$ and $\tilde{P}$ and using condition (H5), one obtains

$$
|\delta Y_t^i| \leq (T - t)\beta \|\delta y\|_{\infty,[T-t,T]},
$$

where $\|\delta y\|_{\infty,[T-t,T]} := \| \sup_{T-t \leq r \leq T} |\delta y_r| \|_\infty$. By setting $\lambda = \frac{1}{2\beta n}$, we have on $[T - \lambda, T]$,

$$
\|\delta Y\|_{\infty,[T-\lambda,T]} \leq \frac{1}{2} \|\delta y\|_{\infty,[T-\lambda,T]}.
$$

Thus I defines a contraction on $[T - \lambda, T]$. Then BSDE (4.2.1) has a unique solution on $[T - \lambda, T]$ such that Y is bounded. Similarly, with $T - \lambda$ as terminal time and $Y_{T-\lambda}$ as terminal condition, BSDE (4.2.1) has a unique solution on $[T - 2\lambda, T - \lambda]$ such that Y is bounded. By pasting, we obtain a unique solution of BSDE (4.2.1) on $[T - 2\lambda, T]$ such that Y is bounded. Since λ is a fixed constant, we can extend (Y, Z) to the whole interval $[0, T]$ in finitely many steps. Noting that for any $i = 1, \ldots, n$ and $t \in [0, T]$,

$$
g^i(t, y, z) = g^i(t, y, z^i) = g^i(t, 0, z^i) + g^i(t, y, z^i) - g^i(t, 0, z^i),
$$

with $Y \in \mathcal{S}^\infty(\mathbb{R}^n)$, one has $Z \cdot W$ is a BMO martingale by using a similar argument as in Lemma 2.A.1. Hence, for any $i = 1, \ldots, n$ and $t \in [0, T]$

$$
\begin{aligned}
Y_t^i &= \xi^i + \int_t^T g^i(s, Y_s, Z_s^i)ds - \int_t^T Z_s^i dW_s \\
&= \xi^i + \int_t^T g^i(s, Y_s, Z_s^i) - g^i(s, Y_s, 0) + g^i(s, Y_s, 0) - g^i(s, 0, 0) + g^i(s, 0, 0)ds \\
&\quad - \int_t^T Z_s^i dW_s \\
&= \xi^i + \int_t^T \eta_s Z_s^i + g^i(s, Y_s, 0) - g^i(s, 0, 0) + g^i(s, 0, 0)ds - \int_t^T Z_s^i dW_s,
\end{aligned}
$$

where $|\eta_s| \leq \theta(1 + |Z_s^i|)$ implies that $\eta \cdot W$ is a BMO martingale. By Girsanov's theorem, $\bar{W}_t := W_t - \int_0^t \eta_s ds$ is a Brownian motion under the equivalent probability measure $\frac{d\bar{P}}{dP} = \mathcal{E}_T(b \cdot W)$. Taking conditional expectation with respect to $\mathcal{F}_t$ and $\bar{P}$ and using condition (H5), it holds

$$
\begin{aligned}
|Y_t^i| &\leq \|\xi^i\|_\infty + \bar{E}\left[\int_t^T |g^i(s, 0, 0)| + |g^i(s, Y_s, 0) - g^i(s, 0, 0)|ds \middle| \mathcal{F}_t\right] \\
&\leq \|\xi^i\|_\infty + \bar{E}\left[\int_t^T C + \beta |Y_s| ds \middle| \mathcal{F}_t\right].
\end{aligned}
$$

Thus $|Y_t^i| \leq u_t$, where u_t is the solution of the following ODE

$$
u_t = \sum_{i=1}^n \|\xi^i\|_\infty + nCT + \int_t^T n\beta u_s ds.
$$

It is easy to check that the unique solution to the preceding ODE is given by

$$
u_t = \left(\sum_{i=1}^n \|\xi^i\|_\infty + nCT\right) e^{n\beta(T-t)}.
$$

Therefore

$$
|Y_t^i| \leq \left(\sum_{i=1}^n \|\xi^i\|_\infty + nCT\right) e^{n\beta(T-t)}. \qquad \square
$$

Remark 2.2.4. Since Y is uniformly bounded, the previous result follows from the arguments in Hu and Tang [41]. For completeness, we give a detailed proof and state the bound for Y explicitly. ♦

2.3 Fully coupled systems

In this section, we consider the general case where we have coupledness both in the value process and control process. For the sake of simplicity, we consider only 2-dimensional quadratic BSDEs, an extension to n-dimensions is straightforward. The two dimensional system of BSDEs is given by:

$$Y_t^i = \xi^i + \int_t^T g^i(s, Y_s^1, Y_s^2, Z_s^1, Z_s^2)ds - \int_t^T Z_s^i dW_s, \quad t \in [0,T],\ i = 1,2, \quad (2.3.1)$$

where $g^i : \Omega \times [0,T] \times \mathbb{R}^2 \times \mathbb{R}^{2\times d} \to \mathbb{R}$ is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^2) \otimes \mathcal{B}(\mathbb{R}^{2\times d})$-measurable by $\mathcal{B}(\mathbb{R}^2)$ and $\mathcal{B}(\mathbb{R}^{2\times d})$ denote the Borel sigma-algebra of $\mathbb{R}^2$ and $\mathbb{R}^{2\times d}$ respectively and where the generator is of the form

$$g^i(s, y^1, y^2, z^1, z^2) = f^i(s, z^i) + h^i(s, y^1, y^2, z^1, z^2), \quad i = 1, 2.$$

We consider the following conditions. If there is no risk of confusion we write y and z for the vectors (y^1, y^2) and (z^1, z^2) respectively. For each $i = 1, 2$, there are constants $C, \gamma_i, \eta_i, \theta_i, \vartheta_i > 0$ and $\alpha_i, \beta_i \geq 0$ such that[1]

(B1) $\xi^i \in L^\infty(\mathcal{F}_T)$,

(B2) $|f^i(t, z^i)| \leq C + \gamma_i |z^i|^2$,

(B3) $|f^i(t, z^i) - f^i(t, \bar{z}^i)| \leq \theta_i (1 + |z^i| + |\bar{z}^i|)|z^i - \bar{z}^i|$,

(B4) $|h^i(t, y^1, y^2, z^1, z^2) - h^i(t, \bar{y}^1, \bar{y}^2, \bar{z}^1, \bar{z}^2)| \leq \alpha_i |y - \bar{y}| + \vartheta_i (1 + |z| + |\bar{z}|)|z - \bar{z}|$,

(B5) $|h^i(t, y^1, y^2, z^1, z^2)| \leq C + \beta_i |y| + \eta_i (|z^1|^2 + |z^2|^2)$.

Theorem 2.3.1. *Assume (B1)-(B5) and*

(i) $(\beta_1 + \beta_2)T < 1$.

(ii) There exists $\delta \in (0,1)$ *such that*

$$D_1 + D_2 \leq \frac{\delta}{2\gamma_1 \eta_1} \wedge \frac{\delta}{2\eta_2 \gamma_2},$$
$$\gamma_1 \vee \gamma_2 < \frac{1}{2A},$$
$$\theta_1^2 (T + 2D_1) \vee \theta_2^2 (T + 2D_2) < \frac{1}{72},$$

[1] When the growth of the generator is purely quadratic, C is allowed to be 0.

$$48\alpha_1^2T^2 + 48\alpha_2^2T^2 + \frac{24\alpha_1^2T^2}{1-72\theta_1^2(T+2D_1)} + \frac{24\alpha_2^2T^2}{1-72\theta_2^2(T+2D_2)} < 1,$$

$$144\vartheta_1^2(T+2(D_1+D_2)) + 144\vartheta_2^2(T+2(D_1+D_2)) + \frac{72\vartheta_1^2(T+2(D_1+D_2))}{1-72\theta_1^2(T+2D_1)} + \frac{72\vartheta_2^2(T+2(D_1+D_2))}{1-72\theta_2^2(T+2D_2)} < 1,$$

where

$$D_1 := \frac{\|\xi^1\|_\infty^2 + 4CTA + 2\beta_1TA^2 + \frac{\delta A}{\gamma_1}}{1-2\gamma_1A},$$
$$D_2 := \frac{\|\xi^2\|_\infty^2 + 4CTA + 2\beta_2TA^2 + \frac{\delta A}{\gamma_2}}{1-2\gamma_2A},$$
$$A := \frac{\|\xi^1\|_\infty + \|\xi^2\|_\infty + 4CT + \frac{\ln\frac{1}{1-\delta}}{2\gamma_1} + \frac{\ln\frac{1}{1-\delta}}{2\gamma_2}}{1-(\beta_1+\beta_2)T}.$$

Then the system of BSDEs (2.3.1) *has a unique solution* (Y,Z) *such that*

$$\|Y\|_\infty \le A \quad \textit{and} \quad \|Z\cdot W\|_{BMO} \le \sqrt{D_1+D_2}.$$

Proof. Fix $y \in \mathcal{S}^\infty(\mathbb{R}^2)$ and let $z\cdot W$ be a BMO martingale with

$$\|z\cdot W\|_{BMO} \le \frac{\sqrt{\delta}}{\sqrt{2\gamma_1\eta_1}\vee\sqrt{2\eta_2\gamma_2}}.$$

Define the function I mapping (y,z) to (Y,Z) where for each $i=1,2$, (Y^i,Z^i) is the solution of

$$Y_t^i = \xi^i + \int_t^T f^i(s,Z_s^i) + h^i(s,y_s,z_s)ds - \int_t^T Z_s^i dW_s.$$

By Lemma 2.A.1, the 1-dimensional equation (2.3.1) admits a unique solution (Y^i,Z^i) such that

$$|Y_t^i| \le \|\xi^i\|_\infty + 2C(T-t) + \beta_i(T-t)\|y\|_\infty + \frac{1}{2\gamma_i}\ln\frac{1}{1-\delta}.$$

By Itô's formula,

$$|Y_t^i|^2 = |\xi^i|^2 + \int_t^T \left(2Y_s^i(f^i(s,Z_s^i) + h^i(s,y^1,y^2,z_s^1,z_s^2)) - |Z_s^i|^2\right)ds - \int_t^T 2Y_s^iZ_s^i dW_s.$$

By (B2) and (B4) and since y and ξ^i are bounded,

$$
\begin{aligned}
|Y_t^i|^2 &\le \|\xi^i\|_\infty^2 + 4CT\|Y^i\|_\infty + 2\beta_i T\|y\|_\infty \\
&\quad + \int_t^T \left(2\|Y^i\|_\infty(\gamma_i|Z_s^i|^2 + \eta_i|z_s|^2) - |Z_s^i|^2\right) ds - \int_t^T 2Y_s^i Z_s^i dW_s.
\end{aligned}
$$

By taking conditional expectation with respect to $\mathcal{F}_t$ on both sides of the previous inequality and using the BMO-norm of $z \cdot W$,

$$
\|Z^i \cdot W\|_{BMO}^2 \le \frac{\|\xi^i\|_\infty^2 + 4CT\|Y^i\|_\infty + 2\beta_i T\|Y^i\|_\infty\|y\|_\infty + \frac{\delta\|Y^i\|_\infty}{\gamma_i}}{1 - 2\gamma_i A}.
$$

Thus the set of candidate solutions is given by

$$
M := \left\{(Y, Z) : \|Y\|_\infty \le A \text{ and } \|Z \cdot W\|_{BMO} \le \sqrt{D_1 + D_2}\right\}.
$$

Next we show that $I : M \to M$ mapping $(y, z) \mapsto (Y, Z)$ is a contraction. Let $(Y, Z) = I(y, z)$ and $(\bar{Y}, \bar{Z}) = I(\bar{y}, \bar{z})$. Then by Itô's formula,

$$
\begin{aligned}
|Y_t^i - \bar{Y}_t^i|^2 &= \int_t^T 2(Y_s^i - \bar{Y}_s^i)(f^i(s, Z_s^i) + h^i(s, y_s, z_s) - f^i(s, \bar{Z}_s^i) - h^i(s, \bar{y}_s, \bar{z}_s))ds \\
&\quad - \int_t^T (Z_s^i - \bar{Z}_s^i)^2 ds - \int_t^T 2(Y_s^i - \bar{Y}_s^i)(Z_s^i - \bar{Z}_s^i)dW_s.
\end{aligned}
$$

If we take conditional expectation with respect to $\mathcal{F}_t$ on both sides of the previous equation, the last term of the right hand side vanishes. If in the first term on the right hand side we extract the uniform norm of $(Y - \bar{Y})$ to the outside of the expectation and then by Young's inequality $2ab \le \frac{1}{4}a^2 + 4b^2$, we obtain

$$
\begin{aligned}
|Y_t^i - \bar{Y}_t^i|^2 + E\left[\int_t^T (Z_s^i - \bar{Z}_s^i)^2 ds \middle| \mathcal{F}_t\right] &\le \frac{1}{4}\|Y^i - \bar{Y}^i\|_\infty^2 \\
+ 4E^2\left[\int_t^T |f^i(s, Z_s^i) + h^i(s, y_s, z_s) - f^i(s, \bar{Z}_s^i) - h^i(s, \bar{y}_s, \bar{z}_s)| ds \middle| \mathcal{F}_t\right].
\end{aligned}
$$

Since

$$
\begin{aligned}
&\frac{1}{2}\left(\|Y^i - \bar{Y}^i\|_\infty^2 + \|(Z^i - \bar{Z}^i) \cdot W\|_{BMO}^2\right) \\
&\le \|Y^i - \bar{Y}^i\|_\infty^2 \vee \|(Z^i - \bar{Z}^i) \cdot W\|_{BMO}^2 \\
&\le \operatorname*{ess\,sup}_{\tau \in \mathcal{T}} \left\{ |Y_\tau^i - \bar{Y}_\tau^i|^2 + E\left[\int_\tau^T (Z_s^i - \bar{Z}_s^i)^2 ds \middle| \mathcal{F}_\tau\right] \right\},
\end{aligned}
$$

it follows from Hölder's inequality and $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$ that

$$\begin{aligned}
&\frac{1}{2}\|Y^i - \bar{Y}^i\|_\infty^2 + \|(Z^i - \bar{Z}^i)\cdot W\|_{BMO}^2 \\
&\leq 8 \operatorname*{ess\,sup}_{\tau\in\mathcal{T}} E^2\left[\int_\tau^T |f^i(s,Z_s^i) + h^i(s,y_s,z_s) - f^i(s,\bar{Z}_s^i) - h^i(s,\bar{y}_s,\bar{z}_s)|ds\middle|\mathcal{F}_\tau\right] \\
&\leq 24 \operatorname*{ess\,sup}_{\tau\in\mathcal{T}}\left\{E^2\left[\int_\tau^T \theta_i(1+|Z_s^i|+|\bar{Z}_s^i|)|Z_s^i - \bar{Z}_s^i|ds\middle|\mathcal{F}_\tau\right]\right. \\
&\quad\left. + E^2\left[\int_\tau^T \alpha_i|y_s-\bar{y}_s|ds\middle|\mathcal{F}_\tau\right] + E^2\left[\int_\tau^T \vartheta_i(1+|z_s|+|\bar{z}_s|)|z_s-\bar{z}_s|ds\middle|\mathcal{F}_\tau\right]\right\} \\
&\leq 24\alpha_i^2T^2\|y-\bar{y}\|_\infty^2 + 72\theta_i^2(T+2D_i)\|(Z^i-\bar{Z}^i)\cdot W\|_{BMO}^2 \\
&\quad + 72\vartheta_i^2(T+2(D_1+D_2))\|(z-\bar{z})\cdot W\|_{BMO}^2.
\end{aligned}$$

By assumption (ii), I is a contraction. □

In the following theorem we obtain solvability of the system (2.3.1) under a slightly different set of conditions.

Theorem 2.3.2. *Assume (B1)-(B5) and*

(i) $(\beta_1+\beta_2)T < 1$.

(ii)

$$\begin{aligned}
&D_1 + D_2 \leq \frac{1}{4\gamma_1\eta_1} \wedge \frac{1}{4\eta_2\gamma_2}, \\
&\alpha_1^2T^2\left(1+\frac{1}{c_1}\right) + \alpha_2^2T^2\left(1+\frac{1}{\bar{c}_1}\right) < \frac{1}{4}, \\
&\vartheta_1^2c_2L_4^2(T+2c_2L_4^2(D_1+D_2))\left(1+\frac{1}{c_1}\right) \\
&\qquad + \vartheta_2^2\bar{c}_2L_4^2(T+2\bar{c}_2L_4^2(D_1+D_2))\left(1+\frac{1}{\bar{c}_1}\right) < \frac{1}{12\sqrt{3}},
\end{aligned}$$

where

$$\begin{aligned}
D_1 &:= \frac{e^{2\gamma_1\|\xi^1\|_\infty}}{2\gamma_1^2}\left(1 + e^{4\gamma_1 CT + 2\gamma_1\beta_1 TA}(8\gamma_1 CT + 4\gamma_1\beta_1 TA + 1)\right), \\
D_2 &:= \frac{e^{2\gamma_2\|\xi^2\|_\infty}}{2\gamma_2^2}\left(1 + e^{4\gamma_2 CT + 2\gamma_2\beta_2 TA}(8\gamma_2 CT + 4\gamma_2\beta_2 TA + 1)\right), \\
A &:= \frac{\|\xi^1\|_\infty + \|\xi^2\|_\infty + 4CT + \frac{\ln 2}{2\gamma_1} + \frac{\ln 2}{2\gamma_2}}{1-(\beta_1+\beta_2)T},
\end{aligned}$$

where c_1, c_2 (resp. $\bar{c}_1, \bar{c}_2$) are given by Lemma A.1.3 with K equals to $2\theta_1 D_1$ (resp. $2\theta_2 D_2$), and L_4 is given by Lemma A.1.4.

Then the system of BSDEs (2.3.1) *has a unique solution* (Y, Z) *such that*

$$\|Y\|_\infty \leq A \quad \text{and} \quad \|Z \cdot W\|_{BMO} \leq \sqrt{D_1 + D_2}.$$

Proof. In order to obtain the set of candidate solutions

$$M := \Big\{(Y, Z) : \|Y\|_\infty \leq A \text{ and } \|Z \cdot W\|_{BMO} \leq \sqrt{D_1 + D_2}\Big\}$$

we can argue analogously to the proof of Theorem 2.3.1 by applying Itô's formula to the function $u(x) = \frac{1}{(2\gamma)^2}(e^{2\gamma x} - 1 - 2\gamma x)$ by letting $\gamma = \gamma_i$ for each $i = 1, 2$ as in Lemma 2.A.1. Let $I : M \to M$ map (y, z) to (Y, Z) where (Y^i, Z^i) is the solution of

$$Y_t^i = \xi^i + \int_t^T f^i(s, Z_s^i) + h^i(s, y_s, z_s)ds - \int_t^T Z_s^i dW_s.$$

For the contraction argument we need to proceed differently. Let $(Y, Z) = I(y, z)$ and $(\bar{Y}, \bar{Z}) = I(\bar{y}, \bar{z})$. Defining $\Delta Y = Y^1 - \bar{Y}^1$ and $\Delta Z = Z^1 - \bar{Z}^1$, we obtain

$$\begin{aligned}
\Delta Y &= \int_t^T \big(f^1(s, Z_s^1) + h^1(s, y_s, z_s) - f^1(s, \bar{Z}_s^1) - h^1(s, \bar{y}_s, \bar{z}_s)\big)\, ds - \int_t^T \Delta Z_s dW_s \\
&= \int_t^T \big(b_s \Delta Z_s + h^1(s, y_s, z_s) - h^1(s, \bar{y}_s, \bar{z}_s)\big)\, ds - \int_t^T \Delta Z_s dW_s,
\end{aligned}$$

where $|b_s| \leq \theta_1(1 + |Z_s^1| + |\bar{Z}_s^1|)$ which implies already that $b \cdot W$ is a BMO martingale. By Girsanov's theorem $\tilde{W}_t := W_t - \int_0^t b_s ds$ is a Brownian motion under the equivalent probability measure $\frac{d\tilde{P}}{dP} = \mathcal{E}_T(b \cdot W)$. Hence

$$\Delta Y = \int_t^T \big(h^1(s, y_s, z_s) - h^1(s, \bar{y}_s, \bar{z}_s)\big)\, ds - \int_t^T \Delta Z_s d\tilde{W}_s.$$

First taking square, second conditional expectation with respect to $\mathcal{F}_t$ and $\tilde{P}$ on both sides of the previous equality, and third by Hölder's inequality and $2ab \leq a^2 + b^2$,

$$\begin{aligned}
&|\Delta Y_t|^2 + \tilde{E}\left[\int_t^T |\Delta Z_s|^2 ds \,\middle|\, \mathcal{F}_t\right] \\
&= \tilde{E}\left[\left(\int_t^T (h^1(s, y_s, z_s) - h^1(s, \bar{y}_s, \bar{z}_s))ds\right)^2 \,\middle|\, \mathcal{F}_t\right]
\end{aligned}$$

$$\le 2\alpha_1^2 \tilde{E}\left[\left(\int_t^T |y_s-\bar{y}_s|ds\right)^2 \middle| \mathcal{F}_t\right]$$

$$+ 2\vartheta_1^2 \tilde{E}\left[\left(\int_t^T (1+|z_s|+|\bar{z}_s|)|z_s-\bar{z}_s|ds\right)^2 \middle| \mathcal{F}_t\right]$$

$$\le 2\alpha_1^2(T-t)^2\|y-\bar{y}\|_\infty^2 + 2\vartheta_1^2 \tilde{E}\left[\int_t^T (1+|z_s|+|\bar{z}_s|)^2 ds \int_t^T |z_s-\bar{z}_s|^2 ds \middle| \mathcal{F}_t\right]$$

$$\le 2\alpha_1^2(T-t)^2\|y-\bar{y}\|_\infty^2$$

$$+ 2\vartheta_1^2 \tilde{E}\left[\left(\int_t^T (1+|z_s|+|\bar{z}_s|)^2 ds\right)^2 \middle| \mathcal{F}_t\right]^{\frac{1}{2}} \tilde{E}\left[\left(\int_t^T |z_s-\bar{z}_s|^2 ds\right)^2 \middle| \mathcal{F}_t\right]^{\frac{1}{2}}$$

By Lemma A.1.4 and $(a+b+c)^2 \le 3(a^2+b^2+c^2)$, it holds

$$\tilde{E}\left[\left(\int_t^T (1+|z_s|+|\bar{z}_s|)^2 ds\right)^2 \middle| \mathcal{F}_t\right]^{\frac{1}{2}}$$

$$\le 3\sqrt{3}\tilde{E}\left[(T-t)^2 + \left(\int_t^T |z_s|^2 ds\right)^2 + \left(\int_t^T |\bar{z}_s|^2 ds\right)^2 \middle| \mathcal{F}_t\right]^{\frac{1}{2}}$$

$$\le 3\sqrt{3}\tilde{E}\left[(T-t)^2 \middle| \mathcal{F}_t\right]^{\frac{1}{2}} + 3\sqrt{3}\tilde{E}\left[\left(\int_t^T |z_s|^2 ds\right)^2 \middle| \mathcal{F}_t\right]^{\frac{1}{2}}$$

$$+ 3\sqrt{3}\tilde{E}\left[\left(\int_t^T |\bar{z}_s|^2 ds\right)^2 \middle| \mathcal{F}_t\right]^{\frac{1}{2}}$$

$$\le 3\sqrt{3}\left(T + \|z_s\cdot\tilde{W}\|_{BMO_4(\tilde{P})}^2 + \|\bar{z}_s\cdot\tilde{W}\|_{BMO_4(\tilde{P})}^2\right)$$

$$\le 3\sqrt{3}\left(T + L_4^2\|z_s\cdot\tilde{W}\|_{BMO(\tilde{P})}^2 + L_4^2\|\bar{z}_s\cdot\tilde{W}\|_{BMO(\tilde{P})}^2\right),$$

and

$$\tilde{E}\left[\left(\int_t^T |z_s-\bar{z}_s|^2 ds\right)^2 \middle| \mathcal{F}_t\right]^{\frac{1}{2}} \le \|(z_s-\bar{z}_s)\cdot\tilde{W}\|_{BMO_4(\tilde{P})}^2$$

$$\le L_4^2\|(z_s-\bar{z}_s)\cdot\tilde{W}\|_{BMO(\tilde{P})}^2.$$

Therefore

$$\begin{aligned}&\|\Delta Y\|_\infty^2 + \|\Delta Z \cdot \tilde{W}\|^2_{BMO(\tilde{P})}\\ &\leq 4\alpha_1^2 T^2 \|y-\bar{y}\|_\infty^2 + 12\sqrt{3}\vartheta_1^2 L_4^2 \Big(T + L_4^2 \|z_s \cdot \tilde{W}\|^2_{BMO(\tilde{P})}\\ &\quad + L_4^2 \|\bar{z}_s \cdot \tilde{W}\|^2_{BMO(\tilde{P})}\Big) \|(z_s - \bar{z}_s)\cdot \tilde{W}\|^2_{BMO(\tilde{P})}.\end{aligned}$$

For two constants $c_1 > 0$ and $c_2 > 0$ given by Lemma A.1.3 with $K = 2\theta_1 D_1$, we obtain

$$\begin{aligned}&\|Y^1 - \bar{Y}^1\|_\infty^2 + c_1 \|(Z^1 - \bar{Z}^1)\cdot W\|^2_{BMO} \leq 4\alpha_1^2 T^2 \|y - \bar{y}\|_\infty^2\\ &\quad + 12\sqrt{3}\vartheta_1^2 c_2 L_4^2 (T + 2c_2 L_4^2 (D_1 + D_2)) \|(z - \bar{z})\cdot W\|^2_{BMO}.\end{aligned}$$

By a similar argument, we obtain

$$\begin{aligned}&\|Y^2 - \bar{Y}^2\|_\infty^2 + \bar{c}_1 \|(Z^2 - \bar{Z}^2)\cdot W\|^2_{BMO} \leq 4\alpha_2^2 T^2 \|y - \bar{y}\|_\infty^2\\ &\quad + 12\sqrt{3}\vartheta_2^2 \bar{c}_2 L_4^2 (T + 2\bar{c}_2 L_4^2 (D_1 + D_2)) \|(z - \bar{z})\cdot W\|^2_{BMO}.\end{aligned}$$

By assumption (ii), I is a contraction. □

Remark 2.3.3. When the generator is independent of the value process, we can consider unbounded terminal condition as in (iii) of Lemma 2.A.1. By the martingale representation theorem it holds that

$$\xi = E[\xi] + \int_0^T v_s dW_s.$$

Thus $\hat{Y}_t := Y_t - E[\xi|\mathcal{F}_t]$ is bounded and

$$\hat{Y}_t = \int_t^T g(s, Z_s)ds - \int_t^T (Z_s - v_s)dW_s, \quad t \in [0,T].$$

Applying Itô's formula to $\varphi(|x|) = |x|^2$ and $\varphi(|x|) = \frac{1}{(4\gamma)^2}(e^{4\gamma|x|} - 1 - 4\gamma|x|)$ and by arguing similarly as in the Theorems 2.3.1 and 2.3.2, respectively. We obtain similar results as the ones we obtained for the bounded case. ♦

2.A Auxiliary result for the one-dimensional BSDE

In this section, we present an extension of Lemma 2.5 in [41]. We consider the following 1-dimensional BSDE

$$Y_t = \xi + \int_t^T [f(s, Z_s) + g_s]ds - \int_t^T Z_s dW_s, \quad t \in [0,T], \tag{2.A.1}$$

where $f : \Omega \times [0,T] \times \mathbb{R}^d \to \mathbb{R}$ is a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable function and $g : \Omega \times [0,T] \to \mathbb{R}$ is $\mathcal{P}$-measurable. We assume that the function $f(\omega, t, z)$ is continuous in z for $P \otimes dt$-almost all $(\omega, t) \in \Omega \times [0,T]$ and that there exist constants $C \geq 0$ and $\gamma > 0$ such that

$$|f(\cdot, z)| \leq C + \gamma |z|^2, \quad \text{for all } z \in \mathbb{R}^d.$$

Moreover, we consider the following conditions:

(A1) There exists a constant $\theta > 0$ such that

$$|f(\cdot, z) - f(\cdot, \bar{z})| \leq \theta(1 + |z| + |\bar{z}|)|z - \bar{z}|, \quad \text{for all } z, \bar{z} \in \mathbb{R}^d.$$

(A2) $E[e^{2\gamma|\xi + \int_0^T g_s ds|}] < \infty$.

(A3) $\xi \in L^\infty(\mathcal{F}_T)$ and $|g| \leq |z|^2$ where $z \cdot W$ is a BMO martingale with $\|z \cdot W\|_{BMO} < \frac{1}{\sqrt{2\gamma}}$.

(A4) $E[\xi|\mathcal{F}_\cdot] - E[\xi]$ is a BMO martingale with $\|E[\xi|\mathcal{F}_\cdot] - E[\xi]\|_{BMO_1} < \frac{1}{16\gamma}$ and $|g| \leq |z|^2$ where $z \cdot W$ is a BMO martingale with $\|z \cdot W\|_{BMO} < \frac{1}{\sqrt{4\gamma}}$.

Lemma 2.A.1.

(i) Assume that (A2) holds, then the BSDE (2.A.1) *has at least a solution* (Y, Z) *such that*

$$-\frac{1}{2\gamma} \ln E\left[e^{-2\gamma\xi + 2\gamma C(T-t) - 2\gamma \int_t^T g_s ds} \big| \mathcal{F}_t\right] \leq Y_t \leq \frac{1}{2\gamma} \ln E\left[e^{2\gamma\xi + 2\gamma C(T-t) + 2\gamma \int_t^T g_s ds} \big| \mathcal{F}_t\right], \tag{2.A.2}$$

for all $t \in [0,T]$.

(ii) Assume that (A3) holds, then the BSDE (2.A.1) *has a solution* (Y, Z) *such that* Y *is bounded and* $Z \cdot W$ *is a BMO martingale with*

$$\|Z \cdot W\|_{BMO} \leq \frac{e^{\gamma\|\xi\|_\infty}}{\sqrt{2\gamma}} \sqrt{1 + e^{2\gamma CT}\left(\frac{2\gamma CT + 2\gamma\|z \cdot W\|_{BMO}^2}{1 - 2\gamma\|z \cdot W\|_{BMO}^2}\right)}. \tag{2.A.3}$$

(iii) Assume that (A4) holds, then the BSDE (2.A.1) *has a solution* (Y, Z) *such that* $Z \cdot W$ *is a BMO martingale.*

Moreover, suppose that (A1) holds, let $(\tilde{Y}, \tilde{Z})$ *solve the BSDE* (2.A.1) *where* $\tilde{\xi}$ *and* $\tilde{f}$ *satisfy the set of conditions as in (ii) or (iii), and* $\tilde{\xi} \geq \xi$ *and* $\tilde{f} \geq f$*. Then it holds that* $\tilde{Y}_t \geq Y_t$*. In particular, under (ii) and (iii) the solution is unique.*

Proof. (i) By Theorem 2 in [14],

$$\bar{Y}_t = \left(\xi + \int_0^T g_s ds\right) + \int_t^T f(s, \bar{Z}_s) ds - \int_t^T \bar{Z}_s dW_s, \quad t \in [0, T],$$

admits at least a solution $(\bar{Y}, \bar{Z})$ such that

$$-\frac{1}{2\gamma} \ln E\left[e^{-2\gamma\xi + 2\gamma C(T-t) - 2\gamma \int_0^T g_s ds} \big| \mathcal{F}_t\right] \leq \bar{Y}_t \\ \leq \frac{1}{2\gamma} \ln E\left[e^{2\gamma\xi + 2\gamma C(T-t) + 2\gamma \int_0^T g_s ds} \big| \mathcal{F}_t\right].$$

Defining $Y_t := \bar{Y}_t - \int_0^t g_s ds$, the pairing $(Y, \bar{Z})$ satisfies the BSDE (2.A.1) and Y satisfies (2.A.2).

(ii) Since $E[\exp(2\gamma \int_t^T z_s^2 ds)|\mathcal{F}_t] < \infty$ for all $t \in [0, T]$, by Lemma A.1.2 and $\xi \in L^\infty(\mathcal{F}_T)$ it follows from (2.A.2) that Y is bounded by

$$|Y_t| \leq \|\xi\|_\infty + C(T-t) + \frac{1}{2\gamma} \ln \frac{1}{1 - 2\gamma \|z \cdot W\|_{BMO}^2}. \tag{2.A.4}$$

For $n \geq 1$ let τ_n be the stopping time

$$\tau_n := \inf\left\{t \geq 0 : \int_0^t e^{4\gamma|Y_s|} |Z_s|^2 ds \geq n\right\} \wedge T.$$

Let $u : \mathbb{R}_+ \to \mathbb{R}_+$ be given by

$$u(x) = \frac{1}{(2\gamma)^2}(e^{2\gamma x} - 1 - 2\gamma x).$$

Then $u(|\cdot|)$ is a $\mathcal{C}^2$-function on $\mathbb{R}$, and by Itô's formula

$$u(|Y_{t\wedge\tau_n}|) = u(|Y_{\tau_n}|) \\ + \int_{t\wedge\tau_n}^{\tau_n} \left(u'(|Y_s|)\text{sgn}(Y_s)(f(s, Z_s) + g_s) - \frac{1}{2}u''(|Y_s|)|Z_s|^2\right) ds \\ - \int_{t\wedge\tau_n}^{\tau_n} u'(|Y_s|)\text{sgn}(Y_s) Z_s dW_s,$$

where $\text{sgn}(x) = -\mathbf{1}_{\{x\leq 0\}} + \mathbf{1}_{\{x>0\}}$. Since $u'(x) = \frac{1}{2\gamma}(e^{2\gamma x} - 1) \geq 0$ and $(u'' - 2\gamma u')(x) = 1$ for all $x \geq 0$ and by the growth condition on f and (A3) it holds that

$$
\begin{aligned}
u(|Y_{t\wedge\tau_n}|) \leq u(|Y_{\tau_n}|) \\
&+ \int_{t\wedge\tau_n}^{\tau_n} u'(|Y_s|)(C+|z_s|^2)ds - \frac{1}{2}\int_{t\wedge\tau_n}^{\tau_n} |Z_s|^2 ds \\
&- \int_{t\wedge\tau_n}^{\tau_n} u'(|Y_s|)\text{sgn}(Y_s)Z_s dW_s.
\end{aligned}
$$

By taking conditional expectation with respect to $\mathcal{F}_t$ on both sides of the previous inequality, the last term vanishes whenever $\tau_n \leq t$. On the complement $t < \tau_n$ it is a martingale since it is a local martingale the quadratic variation process of which is bounded by definition. Hence we obtain a uniform norm of the left hand term for each n by (2.A.4) and (A3) and thus the dominated convergence theorem yields

$$
\begin{aligned}
\frac{1}{2}E\left[\int_t^T |Z_s|^2 ds \big| \mathcal{F}_t\right] &\leq u\left(\|\xi\|_\infty\right) \\
&+ u'\left(\|\xi\|_\infty + CT + \frac{1}{2\gamma}\ln\frac{1}{1-2\gamma\|z\cdot W\|^2_{BMO}}\right)(CT + \|z\cdot W\|^2_{BMO}).
\end{aligned}
$$

By taking the essential supremum over all stopping times in $\mathcal{T}$ to the left hand side of the previous inequality, we get the the following BMO-bound on $Z \cdot W$:

$$
\|Z\cdot W\|_{BMO} \leq \frac{e^{\gamma\|\xi\|_\infty}}{\sqrt{2}\gamma}\sqrt{1 + e^{2\gamma CT}\left(\frac{2\gamma CT + 2\gamma\|z\cdot W\|^2_{BMO}}{1-2\gamma\|z\cdot W\|^2_{BMO}}\right)}.
$$

(iii) By the martingale representation theorem,

$$
\xi = E[\xi] + \int_0^T v_s dW_s
$$

for some $v \in \mathcal{H}^2$. By Lemma A.1.2, it follows from $\|E[\xi \mid \mathcal{F}.] - E[\xi]\|_{BMO_1} < \frac{1}{16\gamma}$ that

$$
E[e^{4\gamma|\xi|}] < \frac{e^{4\gamma E[|\xi|]}}{1 - 16\gamma\|E[\xi|\mathcal{F}.]\|_{BMO_1}}
$$

Combining the previous estimate with (2.A.2) we conclude that $\hat{Y}_t := Y_t - E[\xi|\mathcal{F}_t] \in \mathcal{S}^\infty(\mathbb{R})$. Moreover, $\hat{Y}$ satisfies the following BSDE

$$
\hat{Y}_t = \int_t^T (f(s,Z_s) + g_s)ds - \int_t^T (Z_s - v_s)dW_s, \quad t\in[0,T].
$$

Applying Itô's formula to $\varphi(|x|) = \frac{1}{(4\gamma)^2}(e^{4\gamma|x|} - 1 - 4\gamma|x|)$ and arguing as in (ii) (by using additionally the inequality $(a-b)^2 \geq \frac{1}{2}b^2 - a^2)^2$ we obtain for $0 \leq s \leq t \leq T$

$$
\begin{aligned}
\varphi(|Y_s|) = \varphi(|Y_t|) - \int_s^t \varphi'(|Y_s|)\mathrm{sgn}(Y_s)(Z_s - v_s)dW_s \\
+ \int_s^t \left(\varphi'(|Y_s|)\mathrm{sgn}(Y_s)(f(s,Z_s)+g_s) - \frac{1}{2}\varphi''(|Y_s|)|Z_s - v_s|^2\right)ds \\
\leq \varphi(|Y_t|) - \int_s^t \varphi'(|Y_s|)\mathrm{sgn}(Y_s)(Z_s - v_s)dW_s \\
+ \int_s^t \left(\varphi'(|Y_s|)\mathrm{sgn}(Y_s)(f(s,Z_s)+g_s) + \frac{1}{2}\varphi''(|Y_s|)|v_s|^2 \right. \\
\left. - \frac{1}{4}\varphi''(|Y_s|)|Z_s|^2\right)ds.
\end{aligned}
$$

By a similar argument as in (ii), it holds that $Z \cdot W$ is a BMO martingale.

(iv) Let $\Delta Y := \tilde{Y} - Y$ and $\Delta Z := \tilde{Z} - Z$. Then

$$
\Delta Y_t = \tilde{\xi} - \xi + \int_t^T (\tilde{f}(s,\tilde{Z}_s) - \tilde{f}(s,Z_s) + \tilde{f}(s,Z_s) - f(s,Z_s))ds - \int_t^T \Delta Z_s dW_s.
$$

We can find a predictable process b such that $|b_s| \leq \theta(1+|Z_s|+|\tilde{Z}_s|)$ which already implies that $b \cdot W$ is a BMO martingale satisfying the following equation

$$
\Delta Y_t = \tilde{\xi} - \xi + \int_t^T (b_s \Delta Z_s + \tilde{f}(s,Z_s) - f(s,Z_s))ds - \int_t^T \Delta Z_s dW_s.
$$

Let $\tilde{W}_t := W_t - \int_0^t b_s ds$ and define $\frac{d\tilde{P}}{dP} := \mathcal{E}_T(b \cdot W)$. Then

$$
\Delta Y_t = \tilde{\xi} - \xi + \int_t^T (\tilde{f}(s,Z_s) - f(s,Z_s))ds - \int_t^T \Delta Z_s d\tilde{W}_s, \quad t \in [0,T].
$$

By taking conditional expectation with respect to $\tilde{P}$ and $\mathcal{F}_t$ on both sides of the previous equation it follows that $\Delta Y \geq 0$. □

[2] Indeed, $(a-b)^2 - \frac{1}{2}b^2 + a^2 = 2a^2 - 2ab + \frac{1}{2}b^2 = 2(a - \frac{1}{2}b)^2 \geq 0$.

Chapter 3

Solvability of Coupled FBSDEs with Quadratic and Superquadratic Growth

3.1 Introduction

Nonlinear FBSDEs are systems of forward and backward stochastic differential equations. They generally take the form

$$\begin{cases} X_t = x + \int_0^t b_u(X_u, Y_u, Z_u)\, du + \int_0^t \sigma_u(X_u, Y_u, Z_u)\, dW_u \\ Y_t = h(X_T) + \int_t^T g_u(X_u, Y_u, Z_u)\, du - \int_t^T Z_u\, dW_u \end{cases}$$

for a given initial value x and a multi-dimensional Brownian motion W. These systems naturally appear in numerous areas of applied mathematics including stochastic control and mathematical finance. Moreover, they provide solutions or viscosity solutions to various types of parabolic partial differential equations, and as shown recently by Fromm et al. [33], they can be used in the study of the Skorokhod embedding problem.

In the Markovian setting, coupled FBSDEs are linked to parabolic PDEs, the solutions of which provide existence for the FBSDE, see Ma et al. [53]. For non-Markovian systems, existence for sufficiently small time horizons has been obtained by Delarue [20] using a contraction method. Well-posedness of the system has been investigated by Ma et al. [54] using the so-called decoupling field method, a technique that is significantly refined and extended to multi-dimensional systems by Fromm and Imkeller [32]. The above mentioned results on coupled FBSDEs assume Lipschitz continuity of the generator g. However, FBSDEs appearing in the study of stochastic control problems are typically of quadratic growth in Z. For instance, this class of systems are shown to characterize solutions of utility maximization problems with non-trivial terminal endowment, see Horst et al. [37]. The present chapter is concerned with existence and uniqueness of solutions of such coupled systems, with quadratic or even superquadratic growth and in the multi-dimensional case.

If the system is decoupled, then the forward stochastic differential equation (SDE) and the backward stochastic differential equation (BSDE) can be studied independently. SDEs with Lipschitz continuous coefficients are well understood, see for instance Protter [68]. In case that the terminal condition $\xi = h(X_T)$ is square integrable and the generator Lipschitz continuous, existence and uniqueness of the solutions of the BSDEs has been proved by Pardoux and Peng [63]. If Y is one-dimensional and g is allowed to have quadratic growth in the control process Z, BSDEs' solutions have been obtained by Kobylanski [48] for the case of bounded terminal conditions. In the superquadratic growth case Delbaen et al. [22] showed that BSDEs with bounded terminal conditions are typically ill-posed. For a generator allowed to grow arbitrarily fast, existence of maximal subsolutions of decoupled FBSDEs was studied by Heyne et al. [36] under convexity assumptions. BSDEs' solutions both for linear growth and quadratic growth generators have many desirable features, for instance they are Malliavin differentiable and the trace of the Malliavin's derivative of the value process Y is a version of the control process, see El Karoui et al. [28] and Ankirchner et al. [2]. Based on this observation, Cheridito and Nam [16] showed that boundedness of the Malliavin's derivative of the terminal condition of a Lipschitz BSDE ensures boundedness of the control process, this enabled them to solve BSDEs when g can grow arbitrarily fast in Z by a truncation and approximation procedure on the generator. Boundedness of the process Z derived through the Malliavin's derivative of the value process constitutes a key argument in our study of coupled FBSDEs with superquadratic growth.

We first consider a Markovian system the generator of which is Lipschitz continuous in X and Y and can have arbitrary growth in Z, with non-necessarily bounded terminal condition h. Based on an extension of the existence result of Cheridito and Nam [16] to multi-dimensional BSDEs, we propose a Picard iteration scheme for the coupled system. This iterative sequence can be proved to be a Cauchy sequence in an appropriate Banach space under uniform boundedness of the control processes derived using Malliavin calculus arguments, and for small enough time horizon. If a stronger growth condition on the generator and non-degeneracy of the volatility σ are assumed, solvability can be extended to any finite time horizon by a truncation of the generator and an iterative pasting of local solutions. We further show that in the non-Markovian setting existence and uniqueness can be obtained under a uniform boundedness assumption on the Malliavin's derivative of the generator and the terminal condition.

Existence of quadratic BSDEs in the multi-dimensional case is being the subject of intensive research. Recent contributions have been made for instance by Cheridito and Nam [17] and Hu and Tang [41]. In [41], it was proved using BMO-martingale estimates that if the terminal condition is bounded and the generator can be decomposed into the sum of a quadratic function of Z and a function that has linear growth in Y and subquadratic growth in Z, then the equation admits a solution for sufficiently small time horizons. BMO-martingale estimates also play a central role in our investigation of coupled FBSDEs with quadratic growth.

Our second main result focuses on a non-Markovian setting, where we consider an FBSDE with bounded terminal condition and a generator that does not grow faster than the quadratic function. In this setting, we show that the stochastic integral of the candidate control process is a BMO-martingale so that its stochastic exponential defines an equivalent probability measure. Thus, the Banach fixed point theorem can be applied using a change of measures and properties of BMO-martingales to prove existence and uniqueness. We further show using similar estimates that the solution (X, Y, Z) is continuous and differentiable with respect to the initial value x.

To the best of our knowledge, the only works considering existence of coupled FBSDEs with quadratic growth are the article of Antonelli and Hamadène [5] and the Ph.D. thesis of Fromm [31]. In [5] the focus is on global solvability. The authors consider a one-dimensional equation with one dimensional Brownian motion and impose monotonicity conditions on the coefficient so that comparison principles for SDEs and BSDEs can be applied. A (non-necessarily unique) solution is then obtained by monotone convergence of an iterative scheme. In [31, Chapter 3], a fully coupled Markovian FBSDE is considered with one-dimensional forward and value processes and locally Lipschitz generator in (Y, Z) and a existence of a unique global solution is obtained using the technique of decoupling fields. See also [31, Chapter 4] for an extension of this result to multi-dimension and locally Lipschitz generator in Z, for small time horizons.

The structure of the rest of this chapter is the following: In the next section we make precise the probabilistic setting, introduce some notations and state our two main existence and uniqueness results. Section 3.3 and Section 3.4 are dedicated to the study of coupled FBSDE with superquadratic and quadratic growth, respectively. We present an extended result of Cheridito and Nam [16] in Appendix 3.A. We provide some results for multidimensional BSDEs with superquadratic growth in Appendix 3.B.

3.2 Preliminaries and main results

We work on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ with $T \in (0, \infty)$. We assume that the filtration is generated by a d-dimensional Brownian motion W and it is complete and right continuous. Let us also assume that $\mathcal{F} = \mathcal{F}_T$. We endow $\Omega \times [0, T]$ with the predictable σ-algebra and $\mathbb{R}^k$ with its Borel σ-algebra. Unless otherwise stated, all equalities and inequalities between random variables and processes will be understood in the P-a.s. and $P \otimes dt$-a.e. sense, respectively. For $p \in [1, \infty)$ and $m, m' \in \mathbb{N}$, we denote by $\mathcal{S}^p(\mathbb{R}^m)$ the space of predictable and continuous processes X valued in $\mathbb{R}^m$ such that $\|X\|_{\mathcal{S}^p}^p := E[(\sup_{t\in[0,T]} |X_t|)^p] < \infty$ and by $\mathcal{H}^p(\mathbb{R}^{m'\times d})$ the space of predictable processes Z valued in $\mathbb{R}^{m'\times d}$ such that $\|Z\|_{\mathcal{H}^p}^p := E[(\int_0^T |Z_u|^2\, du)^{p/2}] < \infty$. For a suitable integrand Z, we denote by $Z \cdot W$ the stochastic integral $(\int_0^t Z_u\, dW_u)_{t\in[0,T]}$ of Z with respect to W. From

Protter [68], $Z \cdot W$ defines a continuous martingale for any $Z \in \mathcal{H}^p$. Let us further define by BMO the martingales $Z \cdot W$ valued in $\mathbb{R}^{m'}$ such that

$$\|Z \cdot W\|_{\text{BMO}} := \sup_{\tau} \left\| E\left[\int_{\tau}^{T} Z_u^2 \, du \,|\, \mathcal{F}_\tau \right] \right\|_{\infty} < \infty,$$

where the supremum is taken over all stopping times valued in $[0, T]$. We are interested in studying existence and uniqueness of predictable solutions (X, Y, Z) of a coupled system of the form

$$\begin{cases} X_t = x + \int_0^t b_u(X_u, Y_u, Z_u)\, du + \int_0^t \sigma_u(X_u, Y_u, Z_u)\, dW_u \\ Y_t = h(X_T) + \int_t^T g_u(X_u, Y_u, Z_u)\, du - \int_t^T Z_u\, dW_u \end{cases} \tag{3.2.1}$$

in the case where the generator g has at least quadratic growth in the control variable Z.

Let $\mathcal{M}$ be the class of smooth random variables of the form

$$\xi = F\left(\int_0^T h_s^1 \, dW_s, \ldots, \int_0^T h_s^m \, dW_s \right)$$

where $F \in C_p^\infty(\mathbb{R}^{m\times d})$, the space of infinitely continuously differentiable functions whose partial derivatives have polynomial growth, and $h^1, \ldots, h^m \in L^2([0,T]; \mathbb{R}^d)$. For any $\xi \in \mathcal{M}$, consider the operator $D = (D^1, \ldots, D^d) : \mathcal{M} \to L^2(\Omega \times [0,T])$ given by

$$D_t^i \xi := \sum_{j=1}^m \frac{\partial F}{\partial x_{i,j}} \left(\int_0^T h_s^1 \, dW_s, \ldots, \int_0^T h_s^m \, dW_s \right) h_t^{i,j}, \quad 0 \le t \le T, \; 1 \le i \le d$$

and the norm $\|\xi\|_{1,2} := (E[|\xi|^2 + \int_0^T |D_t \xi|^2 \, dt])^{1/2}$. As shown in Nualart [61], the operator D extends to the closure $\mathcal{D}^{1,2}$ of the set $\mathcal{M}$ with respect to the norm $\|\cdot\|_{1,2}$. A random variable ξ will be said to be Malliavin differentiable if $\xi \in \mathcal{D}^{1,2}$ and we will denote by $D_t \xi$ its Malliavin derivative. Note that if ξ is $\mathcal{F}_t$ measurable, then $D_u \xi = 0$ for all $u \in (t, T]$. By $\mathcal{L}_a^{1,2}(\mathbb{R}^{m'})$, we denote the space of processes $X \in \mathcal{H}^2(\mathbb{R}^{m'})$ such that $X_t \in (\mathcal{D}^{1,2})^{m'}$ for all $t \in [0,T]$, the process $DX_t(\omega)$ admits a square integrable progressively measurable version and

$$\|X\|^2_{\mathcal{L}_a^{1,2}} := \|X\|_{\mathcal{H}^2} + \left\| \left(\int_0^T \int_0^T |D_r X_t|^2 \, dr \, dt \right)^{1/2} \right\|_{L^2} < \infty.$$

We refer to Nualart [61] for a thorough treatment of the theory of Malliavin calculus.

A crucial observation in BSDEs has been that assumptions on the derivatives of the parameters of the equation allow to give bounds for the control process Z

and thereby solvability in the local Lipschitz case. See for instance Cheridito and Nam [16] and Richou [69], where BSDEs and decoupled Markovian FBSDEs are studied in such frameworks. Now consider the conditions

(A1) $b : [0,T] \times \mathbb{R}^m \times \mathbb{R}^{m'} \to \mathbb{R}^m$ is a continuous function such that there exist $k_1, k_2, \lambda_1 \geq 0$ such that

$$\left|b_t(x,y) - b_t(x',y')\right| \leq k_1 \left|x - x'\right| + k_2 \left|y - y'\right| \text{ and}$$
$$\left|b_t(x,y)\right| \leq \lambda_1(1 + |x| + |y|)$$

for all $x, x' \in \mathbb{R}^m$ and $y, y' \in \mathbb{R}^{m'}$.

(A2) $\sigma : [0,T] \to \mathbb{R}^{m \times d}$ is a Borel measurable function such that there exists $\lambda_2 \geq 0$ such that $|\sigma(t)| \leq \lambda_2$ for all $t \in [0,T]$.

(A3) $g : [0,T] \times \mathbb{R}^m \times \mathbb{R}^{m'} \times \mathbb{R}^{m' \times d} \to \mathbb{R}^{m'}$ is a continuous function such that $g_t(0,0,0) \in L^2(dt)$, $g^i(x,y,z) = g^i(x,y,z^i)$ and there exist $k_3, k_4 \geq 0$ as well as a nondecreasing function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\begin{aligned}\left|g_t(x,y,z) - g_t(x',y',z')\right| \leq\ & k_3 \left|x - x'\right| + k_4 \left|y - y'\right| \\ & + \rho\left(|z| \vee \left|z'\right|\right) \left|z - z'\right|\end{aligned}$$

for all $x, x' \in \mathbb{R}^m$, $y, y' \in \mathbb{R}^{m'}$ and $z, z' \in \mathbb{R}^{m' \times d}$.

(A4) There exists a constant $K \geq 0$ such that

$$\begin{aligned}&\left|g_t(x,y,z) - g_t(x',y,z) - g_t(x,y',z') + g_t(x',y',z')\right| \\ &\qquad \leq K \left|x - x'\right| \left(\left|y - y'\right| + \left|z - z'\right|\right)\end{aligned}$$

for all $t \in [0,T]$, $x, x' \in \mathbb{R}^m$, $y, y' \in \mathbb{R}^{m'}$ and $z, z' \in \mathbb{R}^{m' \times d}$.

(A5) $h : \mathbb{R}^m \to \mathbb{R}^{m'}$ is a continuous function such that there exists $k_5 \geq 0$ such that

$$\left|h(x) - h(x')\right| \leq k_5 \left|x - x'\right|$$

for all $x, x' \in \mathbb{R}^m$.

Under these assumptions, we obtain an existence and uniqueness result for fully coupled FBSDEs with generators of superquadratic growth.

Theorem 3.2.1. *If* (A1) *-* (A5) *hold, then there exists a constant $C_{k,\lambda,m',d}$ only depending on k_i, λ_2, m', d, $i = 1, \ldots, 5$, such that if $T \leq C_{k,\lambda,m',d}$, then the FBSDE*

$$\begin{cases} X_t = x + \int_0^t b_u(X_u, Y_u)\, du + \int_0^t \sigma_u \, dW_u \\ Y_t = h(X_T) + \int_t^T g_u(X_u, Y_u, Z_u)\, du - \int_t^T Z_u \, dW_u \end{cases} \tag{3.2.2}$$

has a unique solution $(X, Y, Z) \in \mathcal{S}^2(\mathbb{R}^m) \times \mathcal{S}^2(\mathbb{R}^{m'}) \times \mathcal{S}^\infty(\mathbb{R}^{m' \times d})$ *such that*

$$|Z_t^{ij}| \leq 2\lambda_2 m' e^{k_1 T + m' k_4 T} \left(k_5 + k_3 T\right) \quad P \otimes dt\text{-a.e.} \tag{3.2.3}$$

If in addition there exist $\lambda_1, \lambda_3, \lambda_4 \geq 0$ *and* $\lambda_5 > 0$ *as well as a nondecreasing function* $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$\begin{cases} |b_t(x,y)| & \leq \lambda_1(1 + |y|), \quad |h(x)| \leq \lambda_4 \\ |g_t(x,y,z)| & \leq \lambda_3\big(1 + |y| + \rho(|z|)\,|z|\big) \\ \langle x, \sigma_t \sigma_t^* x\rangle & \geq \lambda_5 |x|^2 \end{cases} \tag{3.2.4}$$

for all $x \in \mathbb{R}^m$*,* $y \in \mathbb{R}^{m'}$ *and* $z \in \mathbb{R}^{m' \times d}$*, then the solution* (X, Y, Z) *exists for any* $T \in (0, \infty)$*.*

The following counter example shows that the condition (3.2.4) cannot be dropped without violating global solvability. Consider the FBSDE

$$\begin{cases} X_t & = \int_0^t Y_u \, du \\ Y_t & = \int_t^T k X_u \, du - \int_t^T Z_u \, dW_u. \end{cases}$$

This equation can be rewritten as

$$Y_t = \int_t^T \int_0^s k Y_u \, du \, ds - \int_t^T Z_u \, dW_u. \tag{3.2.5}$$

It has been shown in [24, Example 3.2] that if $T\sqrt{k} < \frac{\pi}{2}$ then the time-delayed BSDE (3.2.5) has a unique solution whereas if $T\sqrt{k} = \frac{\pi}{2}$, (3.2.5) may not have any solutions and if it does have one, there are infinitely many.

We will also show that in the non-Markovian case the fully coupled system (3.2.1) can be solved under boundedness conditions on the Malliavin's derivative of the generator and the terminal condition. This is Theorem 3.3.1 below. Moreover, still in the non-Markovian setting, such boundedness conditions are not needed for existence of (3.2.2) provided that the generator has at most quadratic growth and the time horizon is sufficiently small. In fact, consider the conditions

(B1) $b : \Omega \times [0, T] \times \mathbb{R}^m \times \mathbb{R}^{m'} \to \mathbb{R}^m$ is measurable and there exist $k_1, k_2, \lambda_1 \geq 0$ such that

$$\left|b_t(x,y) - b_t(x', y')\right| \leq k_1 \left|x - x'\right| + k_2 \left|y - y'\right| \quad \text{and}$$
$$|b_t(x,y)| \leq \lambda_1(1 + |x| + |y|)$$

for all $x, x' \in \mathbb{R}^m$ and $y, y' \in \mathbb{R}^{m'}$.

(B2) $\sigma : \Omega \times [0, T] \to \mathbb{R}^{m \times d}$ is a predictable process such that $\sigma \in \mathcal{H}^2(\mathbb{R}^{m \times d})$.

(B3) $g : \Omega \times [0,T] \times \mathbb{R}^m \times \mathbb{R}^{m'} \times \mathbb{R}^{m' \times d} \to \mathbb{R}^{m'}$ is measurable, $g_t(x,y,z) = f_t(z) + l_t(x,y,z)$ where f and l are measurable functions with $f_t^i(z) = f_t^i(z^i),\ i = 1,\dots,m'$ and there exists $k_3, k_4, k_5, k_6, \lambda_2, \lambda_3, \lambda_4 \geq 0$ such that

$$\begin{aligned}
&|f_t(z) - f_t(z')| \leq k_3(1 + |z| + |z'|)|z - z'|,\\
&|l_t(x,y,z) - l_t(x',y',z')| \leq k_4|x - x'| + k_5|y - y'|\\
&\qquad\qquad\qquad\qquad + k_6(1 + |z|^{\varepsilon} + |z'|^{\varepsilon})|z - z'|,\\
&|f_t(z)| \leq \lambda_2(1 + |z|^2),\\
&|l_t(x,y,z)| \leq \lambda_3(1 + |z|^{1+\varepsilon}) + \lambda_4|y|
\end{aligned}$$

for some $0 \leq \varepsilon < 1$ and for all $x, x' \in \mathbb{R}^m$, $y, y' \in \mathbb{R}^{m'}$ and $z, z' \in \mathbb{R}^{m' \times d}$.

(B4) $h : \Omega \times \mathbb{R}^m \to \mathbb{R}^{m'}$ is measurable and there exist $k_7, \lambda_5 \geq 0$ such that

$$\left|h(x) - h(x')\right| \leq k_7 \left|x - x'\right| \quad \text{and} \quad |h(x)| \leq \lambda_5$$

for all $x, x' \in \mathbb{R}^m$.

The second main result of this work is the following:

Theorem 3.2.2. *If (*B1*) - (*B4*) hold, then there exists a constant $C_{k,\lambda}$ depending only on the coefficients k_i and λ_i such that if $T \leq C_{k,\lambda}$, then there exist two constants C_1 and C_2 such that FBSDE* (3.2.2) *has a unique solution (X,Y,Z) such that $(X, Y, Z \cdot W) \in \mathcal{S}^2(\mathbb{R}^m) \times \mathcal{S}^\infty(\mathbb{R}^{m'}) \times$ BMO and $\|Y\|_{\mathcal{S}^\infty(\mathbb{R}^{m'})} \leq C_1$, and $\|Z \cdot W\|_{BMO} \leq C_2$.*

3.3 FBSDEs with superquadratic growth

3.3.1 Proof of Theorem 3.2.1

Step 1: We first assume that h, b, g are continuously differentiable in all variables. We will show that the sequence (X^n, Y^n, Z^n) given by $X^0 = 0$, $Y^0 = 0$, $Z^0 = 0$ and

$$\begin{cases} X_t^{n+1} &= x + \int_0^t b(X_u^{n+1}, Y_u^n)\, du + \int_0^t \sigma_u \, dW_u \\ Y_t^{n+1} &= h(X_T^{n+1}) + \int_t^T g_u(X_u^{n+1}, Y_u^{n+1}, Z_u^{n+1})\, du - \int_t^T Z_u^{n+1}\, dW_u, \quad n \geq 1 \end{cases}$$

is well defined and that there exists a constant $C > 0$ which does not depend on n such that $|Z^n| < C$ for all n.

By [68] and [61] the process X^1 is well defined, belongs to $\mathcal{D}^{1,2}(\mathbb{R}^m)$ and its Malliavin's derivative satisfies

$$D_t X_r^1 = 0,\ 0 \leq r < t \leq T,$$

$$D_t X_r^1 = \int_t^r (\partial_x b D_t X_u^1 + \partial_y b D_t Y_u^0)\, du + D_t \left(\int_t^r \sigma_u \, dW_u \right),\ 0 \leq t \leq r \leq T,$$

with $D_t(\int_t^r \sigma_u \, dW_u) = \sigma 1_{[t,r]}$, see [61, Theorem 2.2.1]. Hence, since b is Lipschitz continuous, by Gronwall's inequality we have

$$\left| D_t X_r^1 \right| \leq e^{Tk_1} \lambda_2.$$

Moreover, by the chain rule, see [61, Proposition 1.2.4] it follows that $h(X_T^1) \in \mathcal{D}^{1,2}(\mathbb{R}^{m'})$ and $D(h(X_T^1)) = \partial_x h(X_T^1) D X_T^1$. Therefore, $h(X_T^1)$ has bounded Malliavin derivative since $\partial_x h$ is bounded. We then deduce from Theorem 3.A.2 and its proof that (Y^1, Z^1) exists, $(Y^1, Z^1) \in \mathcal{D}^{1,2}(\mathbb{R}^{m'}) \times \mathcal{D}^{1,2}(\mathbb{R}^{m' \times d})$, DY^1 is bounded and $Z_t^1 = D_t Y_t^1$. Now let $n \in \mathbb{N}$, assume that $(X^n, Y^n, Z^n) \in \mathcal{D}^{1,2}(\mathbb{R}^m) \times \mathcal{D}^{1,2}(\mathbb{R}^{m'}) \times \mathcal{D}^{1,2}(\mathbb{R}^{m' \times d})$, DX^n, DY^n are bounded and $Z_t^n = D_t Y_t^n$. The process X^{n+1} is well defined, belongs to $\mathcal{D}^{1,2}(\mathbb{R}^m)$ and its Malliavin derivative satisfies

$$\begin{aligned}
&D_t X_r^{n+1} = 0, \; 0 \leq r < t \leq T, \\
&D_t X_r^{n+1} = \sigma 1_{[t,r]} + \int_t^r (\partial_x b D_t X_u^{n+1} + \partial_y b D_t Y_u^n) \, du, \; 0 \leq t \leq r \leq T.
\end{aligned}$$

Since $\partial_x b$, $\partial_y b$ and σ are bounded by k_1, k_2 and λ_2 respectively, it follows from Gronwall's inequality that

$$\left| D_t X_r^{n+1} \right| \leq e^{Tk_1} \left(\lambda_2 + k_2 \int_0^T \left| D_t Y_u^n \right| \, du \right).$$

Hence,

$$\left\| D_t X^{n+1} \right\|_{\mathcal{S}^\infty} \leq e^{Tk_1} \left(\lambda_2 + k_2 T \left\| D_t Y^n \right\|_{\mathcal{S}^\infty} \right) < \infty. \tag{3.3.1}$$

By the chain rule, $D(h(X_T^{n+1}))$ exists and is bounded. It then follows again from Theorem 3.A.2 and its proof that (Y^{n+1}, Z^{n+1}) exists and Z^{n+1} is bounded. In addition, (Y^{n+1}, Z^{n+1}) are Malliavin differentiable and the derivatives satisfy, for $j = 1, \ldots, d$,

$$\begin{aligned}
&D_t^j Y_r^{n+1} = 0, \; D_t^j Z_r^{n+1} = 0, \quad 0 \leq r < t \leq T, \\
&D_t^j Y_r^{n+1} = \partial_x h(X_T^{n+1}) D_t^j X_T^{n+1} + \int_r^T \Big(\partial_x g D_t^j X_u^{n+1} + \partial_y g D_t^j Y_u^{n+1} \\
&\qquad\qquad + \partial_z g D_t^j Z_u^{n+1} \Big) \, du - \int_r^T D_t^j Z_u^{n+1} \, dW_u, \quad 0 \leq t \leq r \leq T.
\end{aligned}$$

By (A3)-(A5) and the boundedness of Z^{n+1} and DX^{n+1}, it follows from the same procedure of the proof of Lemma 3.A.1 that for $i = 1, \dots, m'; j = 1, \dots, d$.

$$\begin{aligned}
&|D_t^j Y_r^{i,n+1}| \\
&\le m' \left(k_5 \left\| D_t X^{n+1} \right\|_{\mathcal{S}^\infty} + k_3 \int_t^T \left\| D_t X^{n+1} \right\|_{\mathcal{S}^\infty} e^{-m' k_4 (T-s)} ds \right) e^{m' k_4 (T-t)}.
\end{aligned}$$

Hence

$$\left\| D_t Y^{n+1} \right\|_{\mathcal{S}^\infty} \le e^{m' k_4 T} m' \sqrt{m' d} \left(k_5 + k_3 T \right) \left\| D_t X^{n+1} \right\|_{\mathcal{S}^\infty}.$$

Plugging the above estimate in (3.3.1), we obtain

$$\left\| D_t X^{n+1} \right\|_{\mathcal{S}^\infty} \le M + M' \left\| D_t X^n \right\|_{\mathcal{S}^\infty}$$

with $M := \lambda_2 e^{Tk_1}$ and $M' := k_2 T m' \sqrt{m'd} e^{Tk_1 + m'Tk_4}(k_5 + k_3 T)$. Choosing T small enough so that $M' \le 1/2$, we have

$$\left\| D_t X^{n+1} \right\|_{\mathcal{S}^\infty} \le 2M \quad \text{and} \quad \left| Z_t^{ij,n} \right| = \left| D_t^j Y_t^{i,n} \right| \le 2M m' e^{m' k_4 T} \left(k_5 + k_3 T \right).$$

Hence $|Z^n| \le Q$, where $Q = 2Mm'e^{m'k_4 T}\left(k_5 + k_3 T\right)\sqrt{m'd}$.

Step 2: Now we show that (X^n, Y^n, Z^n) is a Cauchy sequence in $\mathcal{S}^2(\mathbb{R}^m) \times \mathcal{S}^2(\mathbb{R}^{m'}) \times \mathcal{H}^2(\mathbb{R}^{m' \times d})$. Indeed, using (A1) we can estimate the norm of the difference $X_t^{n+1} - X_t^n$ as

$$|X_t^{n+1} - X_t^n|^2 \le 2 \left(\int_0^t k_1 |X_s^{n+1} - X_s^n| ds \right)^2 + 2 \left(\int_0^t k_2 |Y_s^n - Y_s^{n-1}| ds \right)^2.$$

Thus

$$\sup_{0 \le t \le T} |X_t^{n+1} - X_t^n|^2 \le 2 \left(\int_0^T k_1 |X_s^{n+1} - X_s^n| ds \right)^2 + 2 \left(\int_0^T k_2 |Y_s^n - Y_s^{n-1}| ds \right)^2.$$

Taking expectation on both sides and using Cauchy-Schwarz' inequality, we have

$$\begin{aligned}
&E\left[\sup_{0 \le t \le T} |X_t^{n+1} - X_t^n|^2 \right] \\
&\le 2Tk_1^2 E\left[\int_0^T |X_s^{n+1} - X_s^n|^2 ds \right] + 2Tk_2^2 E\left[\int_0^T |Y_s^n - Y_s^{n-1}|^2 ds \right] \\
&\le 2T^2 k_1^2 E\left[\sup_{0 \le t \le T} |X_t^{n+1} - X_t^n|^2 \right] + 2T^2 k_2^2 E\left[\sup_{0 \le t \le T} |Y_t^n - Y_t^{n-1}|^2 \right].
\end{aligned}$$

Choosing T to be small enough so that $2T^2k_1^2 \leq \frac{1}{2}$, it follows

$$E\left[\sup_{0\leq t\leq T}|X_t^{n+1}-X_t^n|^2\right] \leq 4T^2k_2^2 E\left[\sup_{0\leq t\leq T}|Y_t^n-Y_t^{n-1}|^2\right]. \tag{3.3.2}$$

On the other hand, applying Itô's formula to $e^{\beta t}|Y_t^{n+1}-Y_t^n|^2$, $\beta \geq 0$, we have

$$\begin{aligned}
&e^{\beta t}|Y_t^{n+1}-Y_t^n|^2 \\
&= e^{\beta T}|h(X_T^{n+1})-h(X_T^n)|^2 - 2\int_t^T e^{\beta s}(Y_s^{n+1}-Y_s^n)(Z_s^{n+1}-Z_s^n)dW_s \\
&\quad - \int_t^T e^{\beta s}(Z_s^{n+1}-Z_s^n)^2 ds - \int_t^T \beta e^{\beta s}(Y_s^{n+1}-Y_s^n)^2 ds \\
&\quad + 2\int_t^T e^{\beta s}(Y_s^{n+1}-Y_s^n)\left[g_s(X_s^{n+1},Y_s^{n+1},Z_s^{n+1}) - g_s(X_s^n,Y_s^n,Z_s^n)\right] ds.
\end{aligned}$$

Hence, due to the condition (A3) and the boundedness of (Z^n), it holds

$$\begin{aligned}
&e^{\beta t}|Y_t^{n+1}-Y_t^n|^2 + \int_t^T e^{\beta s}(Z_s^{n+1}-Z_s^n)^2 ds \\
&\leq e^{\beta T}\left|h(X_T^{n+1})-h(X_T^n)\right|^2 - 2\int_t^T e^{\beta s}(Y_s^{n+1}-Y_s^n)(Z_s^{n+1}-Z_s^n)dW_s \\
&\quad - \int_t^T \beta e^{\beta s}(Y_s^{n+1}-Y_s^n)^2 ds + 2\int_t^T e^{\beta s}\rho(Q)\left|Y_s^{n+1}-Y_s^n\right|\left|Z_s^{n+1}-Z_s^n\right| ds \\
&\quad + 2\int_t^T e^{\beta s}k_7\left|Y_s^{n+1}-Y_s^n\right|\left|X_s^{n+1}-X_s^n\right| ds + 2\int_t^T e^{\beta s}k_4\left|Y_s^{n+1}-Y_s^n\right|^2 ds.
\end{aligned}$$

With some positive constants α_1, α_2, it follows from (A5) and Young's inequality that

$$
\begin{aligned}
&e^{\beta t}|Y_t^{n+1} - Y_t^n|^2 + \int_t^T e^{\beta s}(Z_s^{n+1} - Z_s^n)^2 ds \le e^{\beta T} k_5^2 |X_T^{n+1} - X_T^n|^2 \\
&\quad - 2\int_t^T e^{\beta s}(Y_s^{n+1} - Y_s^n)(Z_s^{n+1} - Z_s^n) dW_s + \alpha_2 \int_t^T e^{\beta s}|X_s^{n+1} - X_s^n|^2 ds \\
&\quad + \left(\frac{(\rho(Q))^2}{\alpha_1} + \frac{k_3^2}{\alpha_2} + 2k_4 - \beta \right) \int_t^T e^{\beta s}(Y_s^{n+1} - Y_s^n)^2 ds \\
&\quad + \alpha_1 \int_t^T e^{\beta s}|Z_s^{n+1} - Z_s^n|^2 ds. \qquad (3.3.3)
\end{aligned}
$$

Letting $\beta = \frac{(\rho(Q))^2}{\alpha_1} + \frac{k_7^2}{\alpha_2} + 2k_8$ and taking expectation on both sides above, we have

$$
\begin{aligned}
&E\left[e^{\beta t}|Y_t^{n+1} - Y_t^n|^2\right] + E\left[\int_t^T e^{\beta s}(Z_s^{n+1} - Z_s^n)^2 ds\right] \le e^{\beta T} k_5^2 E\left[|X_T^{n+1} - X_T^n|^2\right] \\
&\quad + \alpha_1 E\left[\int_t^T e^{\beta s}|Z_s^{n+1} - Z_s^n|^2 ds\right] + \alpha_2 E\left[\int_t^T e^{\beta s}|X_s^{n+1} - X_s^n|^2 ds\right].
\end{aligned}
$$

Putting $\alpha_1 = \frac{1}{2}$ and $\alpha_2 = 1$, the previous estimate yields

$$
\begin{aligned}
&E\left[\int_0^T e^{\beta s}(Z_s^{n+1} - Z_s^n)^2 ds\right] \\
&\quad \le 2e^{\beta T} k_5^2 E\left[|X_T^{n+1} - X_T^n|^2\right] + 2E\left[\int_0^T e^{\beta s}|X_s^{n+1} - X_s^n|^2 ds\right].
\end{aligned}
$$

Next, taking conditional expectation with respect to $\mathcal{F}_t$ in (3.3.3),

$$
\begin{aligned}
&e^{\beta t}|Y_t^{n+1} - Y_t^n|^2 + E\left[\int_t^T e^{\beta s}(Z_s^{n+1} - Z_s^n)^2 ds \middle| \mathcal{F}_t\right] \le e^{\beta T} k_5^2 E\left[|X_T^{n+1} - X_T^n|^2 | \mathcal{F}_t\right] \\
&+ \alpha_1 E\left[\int_t^T e^{\beta s}|Z_s^{n+1} - Z_s^n|^2 ds \middle| \mathcal{F}_t\right] + \alpha_2 E\left[\int_t^T e^{\beta s}|X_s^{n+1} - X_s^n|^2 ds \middle| \mathcal{F}_t\right].
\end{aligned}
$$

Thus, by Burkholder-Davis-Gundy's inequality, with a positive constant c_1 and $\alpha_1 = \frac{1}{2}$, $\alpha_2 = 1$, we have

$$E\left[\sup_{0\le t\le T} e^{\beta t}|Y_t^{n+1} - Y_t^n|^2\right] \le c_1 e^{\beta T} k_5^2 E\left[|X_T^{n+1} - X_T^n|^2\right] + c_1\frac{1}{2}E\left[\int_0^T e^{\beta s}|Z_s^{n+1} - Z_s^n|^2 ds\right] + c_1 E\left[\int_0^T e^{\beta s}|X_s^{n+1} - X_s^n|^2 ds\right]$$

$$\le 2c_1 e^{\beta T} k_5^2 E\left[|X_T^{n+1} - X_T^n|^2\right] + 2c_1 E\left[\int_0^T e^{\beta s}|X_s^{n+1} - X_s^n|^2 ds\right].$$

It now follows from (3.3.2) that

$$E\left[\sup_{0\le t\le T}|Y_t^{n+1} - Y_t^n|^2\right] + E\left[\int_0^T (Z_s^{n+1} - Z_s^n)^2 ds\right] \le 8(c_1+1)e^{\beta T}(k_5^2 + T)T^2 k_2^2 E\left[\sup_{0\le t\le T}|Y_t^n - Y_t^{n-1}|^2\right].$$

Taking T small enough so that

$$8(c_1+1)e^{\beta T}(k_5^2 + T)T^2 k_2^2 \le \frac{1}{2},$$

we obtain that (X^n, Y^n, Z^n) is a Cauchy sequence in $\mathcal{S}^2(\mathbb{R}^m) \times \mathcal{S}^2(\mathbb{R}^{m'}) \times \mathcal{H}^2(\mathbb{R}^{m'\times d})$. By continuity of b, g and h we have the existence of a solution (X, Y, Z) in $\mathcal{S}^2(\mathbb{R}^m) \times \mathcal{S}^2(\mathbb{R}^{m'}) \times \mathcal{H}^2(\mathbb{R}^{m'\times d})$ of FBSDE (3.2.1) and it follows from the boundedness of (Z^n) that Z satisfies (3.2.3). The uniqueness follows from the boundedness of Z and by repeating the above arguments on the difference of two solutions.

Step 3: Let us now turn to the general case. For $n \in \mathbb{N}$, let β_n^1, β_n^2 and β_n^3 be nonnegative C^∞ functions with support on $\{x \in \mathbb{R}^m : |x| \le \frac{1}{n}\}$, $\{x \in \mathbb{R}^{m+m'} : |x| \le \frac{1}{n}\}$ and $\{x \in \mathbb{R}^{m+m'+m'\times d} : |x| \le \frac{1}{n}\}$ respectively, and satisfying $\int_{\mathbb{R}^m} \beta_n^1(r)dr = 1$, $\int_{\mathbb{R}^{m+m'}} \beta_n^2(r)dr = 1$ and $\int_{\mathbb{R}^{m+m'+m'\times d}} \beta_n^3(r)dr = 1$. We define the convolutions

$$h^n(x) := \int_{\mathbb{R}^m} h(x')\beta_n^1(x' - x)dx',$$

$$b_t^n(x, y) := \int_{\mathbb{R}^{m+m'}} b_t(x', y')\beta_n^2(x' - x, y' - y)dx'dy',$$

$$g^n(u, x, y, z) := \int_{\mathbb{R}^{m+m'+m'\times d}} g(u, x', y', z')\beta_n^3(x' - x, y' - y, z' - z)dx'dy'dz'.$$

It is easy to check that b^n satisfies (A1) with the constants k_1, k_2 and $2\lambda_1$ and that g^n and h^n satisfy (A3) - (A4) and (A5), respectively, with the same constants. From the above argument, there exists $\bar{C}_{k,\lambda,m',d}$ independent of n such that if $T \leq \bar{C}_{k,\lambda,m',d}$, FBSDE (3.2.1) with parameters (b^n, h^n, g^n) admits a unique solution $(X^n, Y^n, Z^n) \in \mathcal{S}^2(\mathbb{R}^m) \times \mathcal{S}^2(\mathbb{R}^{m'}) \times \mathcal{S}^\infty(\mathbb{R}^{m'\times d})$ and

$$|Z_t^{ij,n}| \leq 2\lambda_2 m' e^{k_1T + m'k_4T}\,(k_5 + k_3T) \quad P\otimes dt\text{-a.e.}$$

By the Lipschitz continuity conditions on b and h and the locally Lipschitz condition of g, the sequences (b^n) and (h^n) converge uniformly to b and h on $\mathbb{R}^{m+m'}$ and $\mathbb{R}^m$, respectively, and (g^n) converges to g uniformly on $\mathbb{R}^{m+m'} \times \Lambda$ for any compact subset Λ of $\mathbb{R}^{m'\times d}$. Combining these uniform convergences with the boundedness of Z^n, similar to above, we can show that there exists a constant $\tilde{C}_{k,\lambda,m',d}$ depending only on $k_1, k_2, k_3, k_4, k_5, \lambda_2, m', d$ such that if $T \leq \tilde{C}_{k,\lambda,m',d}$, (X^n, Y^n, Z^n) is a Cauchy sequence in $\mathcal{S}^2(\mathbb{R}^m) \times \mathcal{S}^2(\mathbb{R}^{m'}) \times \mathcal{H}^2(\mathbb{R}^{m'\times d})$. Hence with $C_{k,\lambda,m',d} = \bar{C}_{k,\lambda,m',d} \wedge \tilde{C}_{k,\lambda,m',d}$, for any $T \leq C_{k,\lambda,m',d}$, the FBSDE (3.2.2) admits a solution $(X, Y, Z) \in \mathcal{S}^2(\mathbb{R}^m) \times \mathcal{S}^2(\mathbb{R}^{m'}) \times \mathcal{S}^\infty(\mathbb{R}^{m'\times d})$ and $|Z_t^{ij}| \leq 2\lambda_2 m' e^{k_1T+m'k_4T}\,(k_5 + k_3T)$. The uniqueness follows from similar arguments.

Step 4: Now, assume $T > C_{k,\lambda,m',d}$ and the additional growth conditions on b, g and h given by (3.2.4) hold. Let $\tilde{h}_Q : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function whose derivative is bounded by 1 and such that $\tilde{h}'_Q(a) = 1$ for all $-Q \leq a \leq Q$ and

$$\tilde{h}_Q(a) = \begin{cases} (Q+1) & \text{if} \quad a > Q+2 \\ a & \text{if} \quad |a| \leq Q \\ -(Q+1) & \text{if} \quad a < -(Q+2). \end{cases}$$

An example of such a function is given by

$$\tilde{h}_Q(a) = \begin{cases} \left(-Q^2 + 2Qa - a(a-4)\right)/4 & \text{if} \quad a \in [Q, Q+2] \\ \left(Q^2 + 2Qa + a(a+4)\right)/4 & \text{if} \quad [-(Q+2), -Q], \end{cases}$$

see [43]. By the assumptions (A3) the function $\tilde{g} : [0,T]\times\mathbb{R}^m\times\mathbb{R}^{m'}\times\mathbb{R}^{m'\times d} \to \mathbb{R}$ defined by

$$\tilde{g}_t(x,y,z) := g_t(x,y,h_Q(z)) \tag{3.3.4}$$

with $h_Q(z) := (\tilde{h}_Q(z^{ij}))_{ij}$ is Lipschitz continuous in all variables. Thus, it follows from [20, Theorem 2.6] that the equation

$$\begin{cases} \tilde{X}_t = x + \int_0^t b_u(\tilde{X}_u, \tilde{Y}_u)\,du + \int_0^t \sigma_u\,dW_u \\ \tilde{Y}_t = h(\tilde{X}_T) + \int_t^T \tilde{g}_u(\tilde{X}_u, \tilde{Y}_u, \tilde{Z}_u)\,du - \int_t^T \tilde{Z}_u\,dW_u, \quad t \in [0,T] \end{cases} \tag{3.3.5}$$

admits a unique solution $(\tilde{X}, \tilde{Y}, \tilde{Z}) \in \mathcal{S}^2(\mathbb{R}^m)\times\mathcal{S}^\infty(\mathbb{R}^{m'})\times\mathcal{H}^\infty(\mathbb{R}^{m'\times d})$. Moreover, there exists a bounded function $\theta : [0,T]\times\mathbb{R}^m \to \mathbb{R}^{m'}$ which is Lipschitz continuous in x such that $\tilde{Y}_t = \theta(t, \tilde{X}_t)$ for all $t \in [0,T]$. Put $N = [T/C_{k,A,q,m',d}]$, where $[a]$ denotes the integer part of a, and $t_i := iC_{k,A,q,m',d}$, $i = 0, \dots, N$ and $t_{N+1} = T$. Since $t_1 \le C_{k,\tilde{A},q,m',d}$, by the first part of the proof the FBSDE

$$\begin{cases} X_t = x + \int_0^t b_u(X_u, Y_u)\, du + \int_0^t \sigma_u\, dW_u \\ Y_t = \tilde{Y}_{t_1} + \int_t^{t_1} g_u(X_u, Y_u, Z_u)\, du - \int_t^{t_1} Z_u\, dW_u, \quad t \in [0, t_1] \end{cases}$$

admits a unique solution (X^1, Y^1, Z^1) such that $\left|Z_t^1\right| \le Q$ for all $t \in [0, t_1]$. Therefore, $(X^1, Y^1, Z^1)1_{[0,t_1]} = (\tilde{X}, \tilde{Y}, \tilde{Z})1_{[0,t_1]}$. Similarly, we obtain a family (X^i, Y^i, Z^i) of solutions of the FBSDEs

$$\begin{cases} X_t = \tilde{X}_{t_{i-1}} + \int_{t_{i-1}}^t b_u(X_u, Y_u)\, du + \int_{t_{i-1}}^t \sigma_u\, dW_u \\ Y_t = \tilde{Y}_{t_i} + \int_t^{t_i} g_u(X_u, Y_u, Z_u)\, du - \int_t^{t_i} Z_u\, dW_u, \quad t \in [t_{i-1}, t_i] \end{cases}$$

such that $(X^i, Y^i, Z^i)1_{[t_{i-1},t_i]} = (\tilde{X}, \tilde{Y}, \tilde{Z})1_{[t_{i-1},t_i]}$, $i = 1, \dots, N+1$. Define

$$X := \sum_{i=1}^{N+1} X^i 1_{[t_{i-1},t_i]}; \quad Y := \sum_{i=1}^{N+1} Y^i 1_{[t_{i-1},t_i]} \quad \text{and } Z := \sum_{i=1}^{N+1} Z^i 1_{[t_{i-1},t_i]}.$$

Then, $(X, Y, Z) \in \mathcal{S}^2(\mathbb{R}^m) \times \mathcal{S}^\infty(\mathbb{R}^{m'}) \times \mathcal{S}^\infty(\mathbb{R}^{m' \times d})$ is the unique solution of the FBSDE (3.3.6) satisfying $|Z_t| \le Q$ for all $t \in [0, T]$. In fact, it is clear that $(X, Y, Z) \in \mathcal{S}^2(\mathbb{R}^m) \times \mathcal{S}^\infty(\mathbb{R}^{m'}) \times \mathcal{S}^\infty(\mathbb{R}^{m' \times d})$ as a finite sum of elements of the same space. Let $t \in [0, T]$ and $i = 1, \dots, N+1$ such that $t \in [t_{i-1}, t_i]$. We have

$$\begin{aligned} x + \int_0^t b_u(X_u, Y_u)\, du + \int_0^t \sigma_u\, du &= x + \sum_{j=1}^{i} \left(\int_{t_{j-1}}^{t_j \wedge t} b_u(X_u^j, Y_u^j)\, du + \int_{t_{j-1}}^{t_j \wedge t} \sigma_u\, dW_u \right) \\ &= X_t^i = X_t \end{aligned}$$

and

$$\begin{aligned} &h(X_T) + \int_t^T g_u(X_u, Y_u, Z_u)\, du - \int_t^T Z_u\, dW_u \\ &\quad = h(X_T^{N+1}) + \sum_{j=i}^{N+1} \left(\int_{t_{j-1} \vee t}^{t_j} g_u(X_u^j, Y_u^j, Z_u^j)\, du - \int_{t_{j-1} \vee t}^{t_j} Z_u^j\, dW_u \right) \\ &\quad = Y_t^i = Y_t. \end{aligned}$$

That is, (X, Y, Z) satisfies Equation (3.3.6). Uniqueness follows from [20, Theorem 2.6]. This concludes the proof.

3.3.2 Fully coupled systems

In order to consider the fully coupled forward-backward system, i.e., we allow the dependence in (x, y, z) of b and σ, we assume boundedness conditions on the

Malliavin derivatives of the generator and the terminal condition. Under this assumption, we can obtain solvability on any time interval $[0,T]$, $T \in (0,\infty)$ for the Markovian case. Now, consider the following conditions

(A1') $b : \Omega\times[0,T]\times\mathbb{R}^m\times\mathbb{R}^{m'}\times\mathbb{R}^{m'\times d} \to \mathbb{R}^m$ is a continuous and measurable function such that there exist $k_1, k_2, k_3, \lambda_1 \geq 0$ such that

$$\begin{aligned}\left|b_t(x,y,z) - b_t(x',y',z')\right| &\leq k_1\left|x-x'\right| + k_2\left|y-y'\right| + k_3\left|z-z'\right| \\ \text{and} \quad \left|b_t(x,y,z)\right| &\leq \lambda_1(1+|x|+|y|+|z|)\end{aligned}$$

for all $x, x' \in \mathbb{R}^m$, $y, y' \in \mathbb{R}^{m'}$ and $z, z' \in \mathbb{R}^{m'\times d}$.

(A2') $\sigma : \Omega \times [0,T] \times \mathbb{R}^m \times \mathbb{R}^{m'} \times \mathbb{R}^{m'\times d} \to \mathbb{R}^{m\times d}$ is a continuous and measurable function such that there exist $k_4, k_5, k_6, \lambda_2 \geq 0$ such that

$$\begin{aligned}\left|\sigma_t(x,y,z) - \sigma_t(x',y',z')\right| &\leq k_4\left|x-x'\right| + k_5\left|y-y'\right| + k_6\left|z-z'\right| \\ \text{and} \quad \left|\sigma_t(x,y,z)\right| &\leq \lambda_2(1+|x|+|y|+|z|)\end{aligned}$$

for all $x, x' \in \mathbb{R}^m$, $y, y' \in \mathbb{R}^{m'}$ and $z, z' \in \mathbb{R}^{m'\times d}$.

(A3') $g : \Omega\times[0,T]\times\mathbb{R}^m\times\mathbb{R}^{m'}\times\mathbb{R}^{m'\times d} \to \mathbb{R}^{m'}$ is a continuous and measurable function such that $g_t^i(x,y,z) = g_t^i(x,y,z^i)$ for $i = 1,\dots,m'$ and there exist $k_7, k_8, \lambda_3 \geq 0$ as well as a nondecreasing function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\begin{aligned}\left|g_t(x,y,z) - g_t(x',y',z')\right| \leq\ & k_7\left|x-x'\right| + k_8\left|y-y'\right| \\ &+ \rho\left(|z| \vee \left|z'\right|\right)\left|z-z'\right|\end{aligned}$$

for all $x, x' \in \mathbb{R}^m$, $y, y' \in \mathbb{R}^{m'}$ and $z, z' \in \mathbb{R}^{m'\times d}$.

(A4') For every $X \in \mathcal{S}^2$, we have $g_\cdot(X_\cdot,0,0) \in \mathcal{H}^4$ and there exist Borel-measurable functions $q_{ij} : [0,T] \to \mathbb{R}_+$ satisfying $\int_0^T q_{ij}^2(t)dt < \infty$ such that for every pair $(y,z) \in \mathbb{R}^{m'} \times \mathbb{R}^{m'\times d}$ with

$$|z| \leq Q := \sqrt{m'\sum_{j=1}^{d}\left(\sum_{i=1}^{m'}|A_{ij}| + \sum_{i=1}^{m'}\int_0^T |q_{ij}(t)|e^{-m'k_8(T-t)}dt\right)^2 e^{m'k_8T}},$$

one has $g_\cdot(X_\cdot,y,z) \in \mathcal{L}_a^{1,2}(\mathbb{R}^{m'})$ and $\left|D_u^j g_t^i(X_t,y,z)\right| \leq q_{ij}(t)$, $i = 1,\dots,m'; j = 1,\dots,d$ and $u \in [0,T]$,

$$\left|D_u g_t(X_t,y,z) - D_u g_t(X_t,y',z')\right| \leq K_u\left(\left|y-y'\right| + \left|z-z'\right|\right)$$

for some $\mathbb{R}_+$-valued adapted process $(K_u(t))_{t\in[0,T]}$ such that $\int_0^T \|K_u\|_{\mathcal{H}^4}^4\, du < \infty$.

(A5') $h : \Omega \times \mathbb{R}^m \to \mathbb{R}^{m'}$ is continuous and $\mathcal{F}_T$-measurable such that $h(X_T) \in \mathcal{D}^{1,2}(\mathbb{R}^{m'})$ for any $X_T \in L^2(\mathcal{F}_T)$ and there exist constants $k_9 \geq 0$ and $A_{ij} \geq 0$, such that

$$\left|D_t^j h^i(X_T)\right| \leq A_{ij}, \; i = 1, \ldots, m', \; j = 1, \ldots, d, \text{ and}$$
$$|h(x) - h(x')| \leq k_9 |x - x'|,$$

for all $x, x' \in \mathbb{R}^m$.

Theorem 3.3.1. *If* (A1') *-* (A5') *hold, then there exist two constants* $C_{k,A,q,m',d}$ *and* $\varepsilon_{k,A,q,m',d}$ *depending only on* $k_1, k_2, k_3, k_4, k_5, k_7, k_8, k_9, A, q, m', d$ *such that if* $T \leq C_{k,A,q,m',d}$ *and* $k_6 k_9 \leq \varepsilon_{k,A,q,m',d}$, *then the FBSDE* (3.2.1) *has a unique solution* $(X, Y, Z) \in \mathcal{S}^2(\mathbb{R}^m) \times \mathcal{S}^2(\mathbb{R}^{m'}) \times \mathcal{S}^\infty(\mathbb{R}^{m' \times d})$ *such that*

$$|Z_t^{ij}| \leq \left(\sum_{i=1}^{m'} A_{ij} + \sum_{i=1}^{m'} \int_t^T q_{ij}(s) e^{-m' k_8 (T-s)} ds \right) e^{m' k_8 (T-t)}, \quad P \otimes dt\text{-a.e.}$$

Proof. Letting $X^0 = 0, Y^0 = 0, Z^0 = 0$, we consider the sequence (X^n, Y^n, Z^n), solution of the FBSDE

$$\begin{cases} X_t^{n+1} &= x + \int_0^t b_s(X_s^{n+1}, Y_s^n, Z_s^n) + \int_0^t \sigma_s(X_s^{n+1}, Y_s^n, Z_s^n) dW_s \\ Y_t^{n+1} &= h(X_T^{n+1}) + \int_t^T g_s(X_s^{n+1}, Y_s^{n+1}, Z_s^{n+1}) - \int_t^T Z_s^{n+1} dW_s. \end{cases}$$

Under (A1')-(A5'), it follows from [68] and Theorem 3.A.2 that (X^n, Y^n, Z^n) is well defined in $\mathcal{S}^2(\mathbb{R}^m) \times \mathcal{S}^2(\mathbb{R}^{m'}) \times \mathcal{H}^2(\mathbb{R}^{m' \times d})$ and

$$|(Z_t^n)^{ij}| \leq \left(\sum_{i=1}^{m'} A_{ij} + \sum_{i=1}^{m'} \int_t^T q_{ij}(s) e^{-m' k_8 (T-s)} ds \right) e^{m' k_8 (T-t)}, \quad P \otimes dt\text{-a.e.}$$

For simplicity, we give only the estimation for $|X^{n+1} - X^n|$, as that of $|Y^{n+1} - Y^n|$ and $|Z^{n+1} - Z^n|$ follows from exactly the same procedure as in the the proof of Theorem 3.2.1. Indeed, we have

$$\begin{aligned} |X_t^{n+1} - X_t^n|^2 \leq\ & 6 \left(\int_0^t k_1 |X_s^{n+1} - X_s^n| ds \right)^2 + 6 \left(\int_0^t k_2 |Y_s^n - Y_s^{n-1}| ds \right)^2 \\ & + 6 \left(\int_0^t k_3 |Z_s^n - Z_s^{n-1}| ds \right)^2 \\ & + 2 \left(\int_0^t \left[\sigma(s, X_s^{n+1}, Y_s^n, Z_s^n) - \sigma(s, X_s^n, Y_s^{n-1}, Z_s^{n-1}) \right] dW_s \right)^2. \end{aligned}$$

Taking supremum with respect to t, then expectation to both sides and using Cauchy-Schwarz' and Burkholder-Davis-Gundy's inequalities, we have

$$\begin{aligned} E\left[\sup_{0\le t\le T}|X_t^{n+1}-X_t^n|^2\right] &\le \left(6T^2k_1^2+24Tk_4^2\right)E\left[\sup_{0\le t\le T}|X_t^{n+1}-X_t^n|^2\right] \\ &\quad+\left(6T^2k_2^2+24Tk_5^2\right)E\left[\sup_{0\le t\le T}|Y_t^n-Y_t^{n-1}|^2\right] \\ &\quad+\left(6Tk_3^2+24k_6^2\right)E\left[\int_0^T|Z_t^n-Z_t^{n-1}|^2dt\right]. \end{aligned}$$

Choosing T to be small enough so that $\left(6T^2k_1^2+24Tk_4^2\right)\le\frac{1}{2}$, we have

$$\begin{aligned} E\left[\sup_{0\le t\le T}|X_t^{n+1}-X_t^n|^2\right] &\le \left(12T^2k_2^2+48Tk_5^2\right)E\left[\sup_{0\le t\le T}|Y_t^n-Y_t^{n-1}|^2\right] \\ &\quad+\left(12Tk_3^2+48k_6^2\right)E\left[\int_0^T|Z_t^n-Z_t^{n-1}|^2dt\right]. \end{aligned}$$

Hence the result follows directly from the arguments in the proof of Theorem 3.2.1. □

Consider the conditions

(A1") $b:[0,T]\times\mathbb{R}^m\times\mathbb{R}^{m'}\times\mathbb{R}^{m'\times d}\to\mathbb{R}^m$ is continuous and there exist $k_1,k_2,k_3,\lambda_1\ge 0$ such that

$$\begin{aligned} \left|b_t(x,y,z)-b_t(x',y',z')\right| &\le k_1\left|x-x'\right|+k_2\left|y-y'\right|+k_3\left|z-z'\right| \\ \text{and}\quad |b_t(x,y,z)| &\le \lambda_1(1+|y|+|z|) \end{aligned}$$

for all $x,x'\in\mathbb{R}^m$, $y,y'\in\mathbb{R}^{m'}$ and $z,z'\in\mathbb{R}^{m'\times d}$.

(A2") $\sigma:[0,T]\times\mathbb{R}^m\times\mathbb{R}^{m'}\to\mathbb{R}^{m\times d}$ is continuous and there exist $k_4,k_5,\lambda_2\ge 0$, $\lambda_5>0$ such that

$$\begin{aligned} \left|\sigma_t(x,y)-\sigma_t(x',y')\right| &\le k_4\left|x-x'\right|+k_5\left|y-y'\right| \\ |\sigma_t(x,y)| &\le \lambda_2(1+|y|)\quad\text{and} \\ \langle x',\sigma_t(x,y)\sigma_t^*(x,y)x'\rangle &\ge \lambda_5|x'|^2 \end{aligned}$$

for all $x,x'\in\mathbb{R}^m$ and $y,y'\in\mathbb{R}^{m'}$.

(A3") $g:[0,T]\times\mathbb{R}^m\times\mathbb{R}^{m'}\times\mathbb{R}^{m'\times d}\to\mathbb{R}^{m'}$ is continuous and continuously differentiable in y and z, and is such that $g_t^i(x,y,z)=g_t^i(x,y,z^i)$

$i = 1, \dots, m'$ and there exist $k_7, k_8, \lambda_3 \geq 0$ as well as a nondecreasing function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\begin{aligned} \left|g_t(x,y,z) - g_t(x',y',z')\right| &\leq k_7 \left|x - x'\right| + k_8 \left|y - y'\right| \\ &\quad + \rho\left(|z| \vee \left|z'\right|\right)\left|z - z'\right| \\ \text{and} \quad |g_t(x,y,z)| &\leq \lambda_3(1 + |y| + \rho(|z|)\,|z|) \end{aligned}$$

for all $x, x' \in \mathbb{R}^m$, $y, y' \in \mathbb{R}^{m'}$ and $z, z' \in \mathbb{R}^{m' \times d}$.

(A4") For every $X \in \mathcal{S}^2$, we have $g_\cdot(X_\cdot, 0, 0) \in \mathcal{H}^4$ and there exist Borel-measurable functions $q_{ij} : [0,T] \to \mathbb{R}_+$ satisfying $\int_0^T q_{ij}^2(t) ds < \infty$ such that for every pair $(y,z) \in \mathbb{R}^{m'} \times \mathbb{R}^{m' \times d}$ one has $g_\cdot(X_\cdot, y, z) \in \mathcal{L}_a^{1,2}(\mathbb{R}^{m'})$ and $\left|D_u^j g_t^i(X_t, y, z)\right| \leq q_{ij}(t)$, $i = 1, \dots, m'; j = 1, \dots, d$ and, for every $u \in [0,T]$,

$$\left|D_u g_t(X_t, y, z) - D_u g_t(X_t, y', z')\right| \leq K_u\left(\left|y - y'\right| + \left|z - z'\right|\right)$$

for some $\mathbb{R}_+$-valued adapted process $(K_u(t))_{t \in [0,T]}$ such that $\int_0^T \|K_u\|_{\mathcal{H}^4}^4 \, du < \infty$.

(A5") $h : \mathbb{R}^m \to \mathbb{R}^{m'}$ is continuously differentiable and such that $h(X_T) \in \mathcal{D}^{1,2}$ for any $X_T \in L^2(\mathcal{F}_T)$ and there exist constants $k_9, \lambda_4, A_{ij} \geq 0$ such that

$$\begin{aligned} \left|D_t^j h^i(X_T)\right| &\leq A_{ij}, \; i = 1, \dots, m', \; j = 1, \dots, d, \\ |h(x) - h(x')| &\leq k_9 |x - x'|, \end{aligned}$$

and $|h(x)| \leq \lambda_4$ for all $x, x' \in \mathbb{R}^m$.

Theorem 3.3.2. *If* (A1") *-* (A5") *hold, then the FBSDE*

$$\begin{cases} X_t = x + \int_0^t b_u(X_u, Y_u, Z_u)\, du + \int_0^t \sigma_u(X_u, Y_u)\, dW_u \\ Y_t = h(X_T) + \int_t^T g_u(X_u, Y_u, Z_u)\, du - \int_t^T Z_u\, dW_u, \quad t \in [0,T] \end{cases} \tag{3.3.6}$$

has a unique solution $(X, Y, Z) \in \mathcal{S}^2(\mathbb{R}^m) \times \mathcal{S}^2(\mathbb{R}^{m'}) \times \mathcal{S}^\infty(\mathbb{R}^{m' \times d})$ *satisfying*

$$|Z_t| \leq Q := \sqrt{m' \sum_{j=1}^d \left(\sum_{i=1}^{m'} |\tilde{A}_{ij}| + \sum_{i=1}^{m'} \int_0^T |q_{ij}(t)| e^{-m' k_8 (T-t)} dt \right)^2} e^{m' k_8 T}$$

where

$$\tilde{A}_{ij} = \left(\sum_{i=1}^{m'} A_{ij} + \sum_{i=1}^{m'} \int_0^T q_{ij}(t) e^{-m' k_8 (T-t)} dt \right) e^{m' k_8 T}.$$

Proof. Consider the constant $C_{k,A,q,m',d}$ introduced in Theorem 3.3.1. If $T \leq C_{k,A,q,m',d}$, the result follows from Theorem 3.3.1.

In the rest of the proof let us assume that $T > C_{k,A,q,m',d}$. The function $\tilde{g}$ defined by (3.3.4) is Lipschitz continuous and differentiable in (y, z), and satisfies (A4"). Hence, by [20, Theorem 2.6] the FBSDE

$$\begin{cases} \tilde{X}_t = x + \int_0^t b_u(\tilde{X}_u, \tilde{Y}_u, \tilde{Z}_u)\, du + \int_0^t \sigma_u(\tilde{X}_u, \tilde{Y}_u)\, dW_u \\ \tilde{Y}_t = h(\tilde{X}_T) + \int_t^T \tilde{g}_u(\tilde{X}_u, \tilde{Y}_u, \tilde{Z}_u)\, du - \int_t^T \tilde{Z}_u\, dW_u, \quad t \in [0, T] \end{cases} \tag{3.3.7}$$

admits a unique solution $(\tilde{X}, \tilde{Y}, \tilde{Z}) \in \mathcal{S}^2(\mathbb{R}^m) \times \mathcal{S}^\infty(\mathbb{R}^{m'}) \times \mathcal{H}^\infty(\mathbb{R}^{m' \times d})$. Moreover, by [28] the processes $\tilde{Y}$ and $\tilde{Z}$ are Malliavin differentiable and we have for $j = 1, \ldots, d$,

$$\begin{aligned} & D_r^j \tilde{Y}_t = 0, \quad D_r^j \tilde{Z}_t = 0, \quad 0 \leq t < r < T, \\ & D_r^j \tilde{Y}_t = D_r^j h(\tilde{X}_T) + \int_t^T \partial_y \tilde{g}_u D_r^j \tilde{Y}_u + \partial_z \tilde{g}_u D_r^j \tilde{Z}_u + D_r^j \tilde{g}_u(\tilde{X}_y, \tilde{Y}_u, \tilde{Z}_u)\, du \\ & \qquad\qquad - \int_t^T D_r^j \tilde{Z}_u\, dW_u, \quad 0 \leq r \leq t \leq T. \end{aligned}$$

Since $\tilde{g}$ is Lipschitz in z, $\partial_z \tilde{g}(\tilde{X}_y, \tilde{Y}_u, \tilde{Z}_u)$ is bounded. By (A4") and (A5"), it follows from the same procedure as in the proof of Lemma 3.A.1 that

$$|D_r^j \tilde{Y}_t^i| \leq \left(\sum_{i=1}^{m'} A_{ij} + \sum_{i=1}^{m'} \int_t^T q_{ij}(s) e^{-m' k_8 (T-s)} ds \right) e^{m' k_8 (T-t)}, \quad P \otimes dt\text{-a.e.},$$

$i = 1, \ldots, m'; j = 1, \ldots, d$. Let $C_{k,\tilde{A},q,m',d}$ be the constant given by Theorem 3.3.1 replacing A_{ij} by $\tilde{A}_{ij}$. One can easily check that $C_{k,\tilde{A},q,m',d} \leq C_{k,A,q,m',d}$ since $A_{ij} \leq \tilde{A}_{ij}$. Considering a sequence $0 = t_0 \leq t_1 \leq \cdots \leq t_N = T$ with $\max_{1 \leq i \leq N} |t_i - t_{i-1}| \leq C_{k,\tilde{A},q,m',d}$ similar to the last part of the proof of Theorem 3.2.1. Since $D_r \tilde{Y}_{t_i} \in L^\infty$ for all $r \in [t_{i-1}, t_i]$ we can get that for $i = 1, \ldots, N$ that

$$\begin{cases} X_t = \tilde{X}_{t_{i-1}} + \int_{t_{i-1}}^t b_u(X_u, Y_u, Z_u)\, du + \int_{t_{i-1}}^t \sigma_u(X_u, Y_u)\, dW_u \\ Y_t = \tilde{Y}_{t_i} + \int_t^{t_i} g_u(X_u, Y_u, Z_u)\, du - \int_t^{t_i} Z_u\, dW_u, \quad t \in [t_{i-1}, t_i] \end{cases}$$

has a unique solution $(X^i, Y^i, Z^i) \in \mathcal{S}^2(\mathbb{R}^m) \times \mathcal{S}^2(\mathbb{R}^{m'}) \times \mathcal{S}^\infty(\mathbb{R}^{m' \times d})$ and $\left|Z_t^i\right| \leq Q$ for all $t \in [t_{i-1}, t_i]$. By the uniqueness of FBSDE (3.3.7), we have $(X^i, Y^i, Z^i) 1_{[t_{i-1}, t_i]} = (\tilde{X}, \tilde{Y}, \tilde{Z}) 1_{[t_{i-1}, t_i]}$. The result follows from a recursion and pasting procedure as above. □

3.4 FBSDEs with quadratic growth

3.4.1 Proof of Theorem 3.2.2

Consider the function Ψ mapping any processes (y, z) such that $(y, z \cdot W) \in \mathcal{S}^\infty \times$ BMO to the solution (Y, Z) of the following decoupled FBSDE:

$$\begin{cases} X_t &= x + \int_0^t b_s(X_s, y_s)ds + \int_0^t \sigma_s dW_s, \\ Y_t &= h(X_T) + \int_t^T f_s(Z_s) + l_s(X_s, y_s, z_s)ds - \int_t^T Z_s dW_s. \end{cases} \tag{3.4.1}$$

By (B1) and (B2), the process X exists and is unique, see for instance [68]. The backward equation in (3.4.1) is composed of m' times 1-dimensional quadratic BSDEs. Due to (B3) and (B4), it admits a unique solution, see [41, Lemma 2.5]. Thus, Ψ is well defined. Furthermore, for T small enough there exist two positive constants C_1 and C_2 depending only on T and $\lambda_i,\ i = 2, \dots, 5$, such that Ψ maps the set

$$\mathcal{B} := \{(y, z) : \|y\|_{\mathcal{S}^\infty} \le C_1;\ \|z \cdot W\|_{BMO} \le C_2\}$$

to itself, see [41] or chapter 2. Let $(y, z), (\bar{y}, \bar{z}) \in \mathcal{B}$. Put $\Psi(y, z) = (Y, Z)$ and $\Psi(\bar{y}, \bar{z}) = (\bar{Y}, \bar{Z})$ and let X and $\bar{X}$ be the solution of the forward equation in (3.4.1) associated to (y, z) and $(\bar{y}, \bar{z})$, respectively. By the Lipschitz continuity property of b, we have

$$\begin{aligned} \left|X_t - \bar{X}_t\right| &\le \int_0^t \left|b_s(X_s, y_s) - b_s(\bar{X}_s, \bar{y}_s)\right| ds \\ &\le k_1 \int_0^t \left|X_s - \bar{X}_s\right| ds + k_2 \int_0^t |y_s - \bar{y}_s| ds. \end{aligned}$$

Hence Gronwall's inequality yields

$$|X_t - \bar{X}_t| \le k_2 e^{k_1 t} \int_0^t |y_s - \bar{y}_s| ds,$$

thus

$$\|X - \bar{X}\|_{\mathcal{S}^\infty} \le k_2 T e^{k_1 T} \|y - \bar{y}\|_{\mathcal{S}^\infty}.$$

On the other hand, for every $i = 1, \dots, m'$,

$$\begin{aligned} &Y_t^i - \bar{Y}_t^i \\ &= h^i(X_T) - h^i(\bar{X}_T) \\ &\quad + \int_t^T f_s^i(Z_s^i) - f_s^i(\bar{Z}_s^i) + l_s^i(X_s, y_s, z_s) - l_s^i(\bar{X}_s, \bar{y}_s, \bar{z}_s)ds - \int_t^T Z_s^i - \bar{Z}_s^i dW_s \end{aligned}$$

$$= h^i(X_T) - h^i(\bar{X}_T) \\ + \int_t^T \theta_s^i(Z_s^i - \bar{Z}_s^i) + l_s^i(X_s, y_s, z_s) - l_s^i(\bar{X}_s, \bar{y}_s, \bar{z}_s)ds - \int_t^T Z_s^i - \bar{Z}_s^i dW_s$$

where $|\theta_s^i| \le k_3(1 + |Z_s^i| + |\bar{Z}_s^i|)$ which implies that $\theta^i \cdot W$ is a BMO-martingale. By Girsanov's theorem, $\tilde{W}_t^i := W_t - \int_0^t \theta_s^i ds$ is a Brownian motion under the equivalent probability measure given by $\frac{d\tilde{P}^i}{dP} = \mathcal{E}(\theta^i \cdot W)_T$. Hence

$$Y_t^i - \bar{Y}_t^i + \int_t^T Z_s^i - \bar{Z}_s^i d\tilde{W}_s^i = h^i(X_T) - h^i(\bar{X}_T) + \int_t^T l_s^i(X_s, y_s, z_s) - l_s^i(\bar{X}_s, \bar{y}_s, \bar{z}_s)ds.$$

Let us denote by $\delta Y^i := Y^i - \bar{Y}^i$, $\delta Z^i := Z^i - \bar{Z}^i$, $\delta X := X - \bar{X}$, $\delta y := y - \bar{y}$ and $\delta z := z - \bar{z}$. Taking the square and the conditional expectation with respect to $\mathcal{F}_t$ and $\tilde{P}^i$ on both sides of the previous equality, we have

$$|\delta Y_t^i|^2 + \tilde{E}^i \left[\int_t^T |\delta Z_s^i|^2 ds \middle| \mathcal{F}_t \right] \\ = \tilde{E}^i \left[\left(h^i(X_T) - h^i(\bar{X}_T) + \int_t^T l_s^i(X_s, y_s, z_s) - l_s^i(\bar{X}_s, \bar{y}_s, \bar{z}_s)ds \right)^2 \middle| \mathcal{F}_t \right] \\ \le 4k_7^2 \tilde{E}^i \left[|\delta X_T|^2 \middle| \mathcal{F}_t \right] + 4k_4^2 \tilde{E}^i \left[\left(\int_t^T |\delta X_s| ds \right)^2 \middle| \mathcal{F}_t \right] \\ + 4k_5^2 \tilde{E}^i \left[\left(\int_t^T |\delta y_s| ds \right)^2 \middle| \mathcal{F}_t \right] \\ + 4k_6^2 \tilde{E}^i \left[\left(\int_t^T (1 + |z_s|^\varepsilon + |\bar{z}_s|^\varepsilon) |\delta z_s| ds \right)^2 \middle| \mathcal{F}_t \right],$$

where we used the Lipschitz and local Lipschitz continuity properties of h and l, and $2ab \le a^2 + b^2$. By Hölder's inequality and $2ab \le a^2 + b^2$ again, it holds

$$|\delta Y_t^i|^2 + \tilde{E}^i \left[\int_t^T |\delta Z_s^i|^2 ds \middle| \mathcal{F}_t \right] \\ \le 4k_7^2 \tilde{E}^i \left[|\delta X_T|^2 \middle| \mathcal{F}_t \right] + 4k_4^2 (T-t)^2 \tilde{E}^i \left[\sup_{t \le s \le T} |\delta X_s|^2 ds \middle| \mathcal{F}_t \right] + 4k_5^2 (T-t)^2 \|\delta y\|_\infty^2$$

$$
\begin{aligned}
&+ 4k_6^2 \tilde{E}^i \left[\int_t^T (1 + |z_s|^\varepsilon + |\bar{z}_s|^\varepsilon)^2 ds \int_t^T |\delta z_s|^2 ds \middle| \mathcal{F}_t \right] \\
&\leq 4k_7^2 \tilde{E}^i \left[|\delta X_T|^2 \middle| \mathcal{F}_t \right] + 4k_4^2 (T-t)^2 \tilde{E}^i \left[\sup_{t \leq s \leq T} |\delta X_s|^2 ds \middle| \mathcal{F}_t \right] + 4k_5^2 (T-t)^2 \|\delta y\|_\infty^2 \\
&\quad + 4k_6^2 \tilde{E}^i \left[\left(\int_t^T (1 + |z_s|^\varepsilon + |\bar{z}_s|^\varepsilon)^2 ds \right)^2 \middle| \mathcal{F}_t \right]^{\frac{1}{2}} \tilde{E}^i \left[\left(\int_t^T |\delta z_s|^2 ds \right)^2 \middle| \mathcal{F}_t \right]^{\frac{1}{2}} \\
&\leq 4k_7^2 \tilde{E}^i \left[|\delta X_T|^2 \middle| \mathcal{F}_t \right] + 4k_4^2 (T-t)^2 \tilde{E}^i \left[\sup_{t \leq s \leq T} |\delta X_s|^2 ds \middle| \mathcal{F}_t \right] + 4k_5^2 (T-t)^2 \|\delta y\|_\infty^2 \\
&\quad + 12k_6^2 \tilde{E}^i \left[\left(\int_t^T (1 + |z_s|^{2\varepsilon} + |\bar{z}_s|^{2\varepsilon}) ds \right)^2 \middle| \mathcal{F}_t \right]^{\frac{1}{2}} \tilde{E}^i \left[\left(\int_t^T |\delta z_s|^2 ds \right)^2 \middle| \mathcal{F}_t \right]^{\frac{1}{2}} .
\end{aligned}
\tag{3.4.2}
$$

Now, we can further estimate the last term of the right hand side above as follows:

$$
\begin{aligned}
&\tilde{E}^i \left[\left(\int_t^T (1 + |z_s|^{2\varepsilon} + |\bar{z}_s|^{2\varepsilon}) ds \right)^2 \middle| \mathcal{F}_t \right]^{\frac{1}{2}} \tilde{E}^i \left[\left(\int_t^T |\delta z_s|^2 ds \right)^2 \middle| \mathcal{F}_t \right]^{\frac{1}{2}} \\
&\leq \tilde{E}^i \left[\left(T - t + (T-t)^{1-\varepsilon} \left(\int_t^T |z_s|^2 ds \right)^\varepsilon \right. \right. \\
&\quad \left. \left. + \ (T-t)^{1-\varepsilon} \left(\int_0^T |\bar{z}_s|^2 \, ds \right)^\varepsilon \right)^2 \middle| \mathcal{F}_t \right]^{\frac{1}{2}} \tilde{E}^i \left[\left(\int_t^T |\delta z_s|^2 ds \right)^2 \middle| \mathcal{F}_t \right]^{\frac{1}{2}} \\
&\leq \sqrt{3} (T-t)^{1-\varepsilon} \left(T^\varepsilon + 2 + \varepsilon \tilde{E}^i \left[\left(\int_t^T |z_s|^2 ds \right)^2 \middle| \mathcal{F}_t \right]^{\frac{1}{2}} \right. \\
&\quad \left. + \ \varepsilon \tilde{E}^i \left[\left(\int_t^T |\bar{z}_s|^2 ds \right)^2 \middle| \mathcal{F}_t \right]^{\frac{1}{2}} \right) \tilde{E}^i \left[\left(\int_t^T |\delta z_s|^2 ds \right)^2 \middle| \mathcal{F}_t \right]^{\frac{1}{2}} .
\end{aligned}
$$

Therefore, (3.4.2) and Lemma A.1.4 yield

$$
\begin{aligned}
&\|\delta Y^i\|_{\mathcal{S}^\infty}^2 + \|\delta Z^i \cdot \tilde{W}^i\|_{\mathrm{BMO}(\tilde{P}^i)}^2 \\
&\leq 8k_7^2 \|\delta X_T\|_{L^\infty}^2 + 8k_4^2 T^2 \|\delta X\|_{\mathcal{S}^\infty}^2 + 8k_5^2 T^2 \|\delta y\|_{\mathcal{S}^\infty}^2
\end{aligned}
$$

$$
\begin{aligned}
&+ 24\sqrt{3}k_6^2 L_4^2 T^{1-\varepsilon}\left(T^\varepsilon + 2 + \varepsilon L_4^2 \|z\cdot \tilde{W}^i\|^2_{\mathrm{BMO}(\tilde{P}^i)}\right.\\
&\left.+\varepsilon L_4^2 \|\bar{z}\cdot \tilde{W}^i\|^2_{\mathrm{BMO}(\tilde{P}^i)}\right)\|\delta z\cdot \tilde{W}^i\|^2_{\mathrm{BMO}(\tilde{P}^i)}\\
\le{}& \left(8k_2^2T^2e^{2k_1T}\left(k_4^2+k_7^2\right)+8k_5^2T^2\right)\|\delta y\|^2_{\mathcal{S}^\infty}\\
&+ 24\sqrt{3}k_6^2 L_4^2 T^{1-\varepsilon}\left(T^\varepsilon + 2 + \varepsilon L_4^2 \|z\cdot \tilde{W}^i\|^2_{\mathrm{BMO}(\tilde{P}^i)}\right.\\
&\left.+\varepsilon L_4^2 \|\bar{z}\cdot \tilde{W}^i\|^2_{\mathrm{BMO}(\tilde{P}^i)}\right)\|\delta z\cdot \tilde{W}^i\|^2_{\mathrm{BMO}(\tilde{P}^i)}.
\end{aligned}
$$

With the strictly positive constants c_1, c_2 depending only on k_3 and C_2 from Lemma A.1.3,

$$
\begin{aligned}
&\|\delta Y\|^2_{\mathcal{S}^\infty} + c_1\|\delta Z\cdot W\|^2_{\mathrm{BMO}(P)}\\
&\le m'\left(8k_2^2T^2e^{2k_1T}\left(k_4^2+k_7^2\right)+8k_5^2T^2\right)\|\delta y\|^2_{\mathcal{S}^\infty}\\
&\quad+ 24\sqrt{3}k_6^2L_4^2T^{1-\varepsilon}c_2m'\left(T^\varepsilon+2+2\varepsilon L_4^2c_2C_2^2\right)\|\delta z\cdot W\|^2_{\mathrm{BMO}(P)}.
\end{aligned}
$$

Letting T be small enough so that

$$
\begin{cases}
m'\left(1+\frac{1}{c_1}\right)\left(8k_2^2T^2e^{2k_1T}\left(k_4^2+k_7^2\right)+8k_5^2T^2\right) & \le \frac{1}{2}\\
24\sqrt{3}k_6^2L_4^2T^{1-\varepsilon}c_2m'\left(1+\frac{1}{c_1}\right)\left(T^\varepsilon+2+2\varepsilon L_4^2c_2C_2^2\right) & \le \frac{1}{2},
\end{cases}
\tag{3.4.3}
$$

it follows that Ψ defines a contraction mapping. Then, there exists a fixed point $(Y, Z) \in \mathcal{B}$. Hence there exists a constant $C_{k,\lambda}$ which depends only on k_i, λ_i such that when $T \le C_{k,\lambda}$, FBSDE (3.2.2) admits a unique solution (X, Y, Z) such that $(X, Y, Z\cdot W)$ belongs to $\mathcal{S}^2(\mathbb{R}^m)\times\mathcal{S}^\infty(\mathbb{R}^{m'})\times$ BMO and $\|Y\|_{\mathcal{S}^\infty(\mathbb{R}^{m'})} \le C_1$, $\|Z\cdot W\|_{\mathrm{BMO}} \le C_2$.

3.4.2 Regularity of solutions

For any initial value $x \in \mathbb{R}^m$, we denote by (X^x, Y^x, Z^x) the unique solution of the FBSDE (3.2.2). The following two results provide regularity of the solution upon the parameter x.

Theorem 3.4.1 (Continuity). *Assume (*B1*) - (*B4*). With the same constant $C_{k,\lambda}$ as in Theorem 3.2.2, if $T \le C_{k,\lambda}$, the function $x \mapsto (X^x, Y^x, Z^x)$ is continuous.*

Proof. Let $T \le C_{k,\lambda}$ and (X^x, Y^x, Z^x) be the solution of the FBSDE (3.2.2) for any $x \in \mathbb{R}$. Notice that $X^x - X^{x'}$ is bounded. In fact, using the Lipschitz continuity condition on b, we have

$$
\begin{aligned}
|X_t^x - X_t^{x'}| &\le |x-x'| + k_1\int_0^t |X_u^x - X_u^{x'}|\,du + k_2\int_0^t |Y_u^x - Y_u^{x'}|\,du\\
&\le |x-x'| + k_2T\|Y^x - Y^{x'}\|_{\mathcal{S}^\infty} + k_1\int_0^t |X_u^x - X_u^{x'}|\,du
\end{aligned}
$$

$$\leq \left(|x-x'| + k_2 T \|Y^x - Y^{x'}\|_{\mathcal{S}^\infty}\right) e^{k_1 t},$$

by Gronwall's lemma. Thus

$$\|X^x - X^{x'}\|_{\mathcal{S}^\infty} \leq \left(|x-x'| + k_2 T \|Y^x - Y^{x'}\|_{\mathcal{S}^\infty}\right) e^{k_1 T}. \tag{3.4.4}$$

On the other hand, arguing such as in the proof of Theorem 3.2.2, we have, for each $i = 1, \dots, m'$,

$$\begin{aligned}
&Y_t^{i,x} - Y_t^{i,x'} + \int_t^T Z_u^{i,x} - Z_u^{i,x'}\, d\tilde{W}_u^i \\
&= h^i(X_T^x) - h^i(X_T^{x'}) + \int_t^T l_u^i(X_u^x, Y_u^x, Z_u^x) - l_u^i(X_u^{x'}, Y_u^{x'}, Z_u^{x'})\, du
\end{aligned}$$

where $\tilde{W}^i = W - \int_0^\cdot \eta_s^i\, ds$ with $|\eta_s^i| \leq k_3(1+|Z_s^{i,x}|+|Z_s^{i,x'}|)$ is a Brownian motion under the equivalent measure $\tilde{P}^i = \mathcal{E}(\eta^i \cdot W)_T \cdot P$. Hence, similar to Theorem 3.2.2, with the same constants c_1, c_2 and C_2,

$$\begin{aligned}
&\|Y^x - Y^{x'}\|_\infty^2 + c_1 \|(Z^x - Z^{x'}) \cdot W\|_{\text{BMO}}^2 \\
&\leq m' \left(16 k_2^2 T^2 e^{2k_1 T}\left(k_4^2 + k_7^2\right) + 8 k_5^2 T^2\right) \|Y^x - Y^{x'}\|_\infty^2 \\
&\quad + 16 m' e^{2k_1 T}\left(k_4^2 + k_7^2\right) |x-x'|^2 \\
&\quad + 24\sqrt{3} k_6^2 L_4^2 T^{1-\varepsilon} c_2 m' \left(T^\varepsilon + 2 + 2\varepsilon L_4^2 c_2 C_2^2\right) \|(Z^x - Z^{x'}) \cdot W\|_{\text{BMO}}^2.
\end{aligned}$$

Therefore, it follows from (3.4.3) that

$$\|Y^x - Y^{x'}\|_{\mathcal{S}^\infty}^2 \leq \frac{16 m' e^{2k_1 T}\left(k_4^2 + k_7^2\right)}{1 - m'\left(16 k_2^2 T^2 e^{2k_1 T}\left(k_4^2 + k_7^2\right) + 8 k_5^2 T^2\right)} |x-x'|^2. \tag{3.4.5}$$

and

$$c_1 \|Z^x - Z^{x'}\|_{\text{BMO}}^2 \leq 32 m' e^{2k_1 T}\left(k_4^2 + k_7^2\right) |x-x'|^2. \tag{3.4.6}$$

Combining with (3.4.4)

$$\begin{aligned}
&\|X^x - X^{x'}\|_{\mathcal{S}^\infty} \\
&\leq \left(1 + k_2 T \sqrt{\frac{16 m' e^{2k_1 T}\left(k_4^2 + k_7^2\right)}{1 - m'\left(16 k_2^2 T^2 e^{2k_1 T}\left(k_4^2 + k_7^2\right) + 8 k_5^2 T^2\right)}}\right) e^{k_1 T} |x-x'|.
\end{aligned} \tag{3.4.7}$$

This proves continuity of the solution. □

(B5) The functions b; h; f and l are continuously differentiable.

(B6) The functions h'; $\partial_x b$; $\partial_y b$; $\partial_x l$; $\partial_y l$; $\partial_z l$ and f' are Lipschitz continuous in all variables with Lipschitz constant K.

Theorem 3.4.2 (Differentiability). *Assume (B1) - (B6). With the same constant $C_{k,\lambda}$ as in Theorem 3.2.2, if $T \le C_{k,\lambda}$, the function $x \mapsto (X^x, Y^x, Z^x)$ is differentiable.*

Proof. Let $T \le C_{k,\lambda}$, $x, x' \in \mathbb{R}^m$ and $\lambda, \lambda' > 0$. Let $e_j = (0, \dots, 1, \dots, 0)$ be the unit vector in $\mathbb{R}^m$ the jth component of which is 1 and all the others 0. Given $(X^{x+\lambda e_i}, Y^{x+\lambda e_j}, Z^{x+\lambda e_j})$, $(X^{x'+\lambda' e_j}, Y^{x'+\lambda' e_j}, Z^{x'+\lambda' e_j})$, (X^x, Y^x, Z^x) and $(X^{x'}, Y^{x'}, Z^{x'})$ solutions of the FBSDE (3.2.2), we define the processes $N^{x,\lambda} := (X^{x+\lambda e_j} - X^x)/\lambda$; $N^{x',\lambda'} := (X^{x'+\lambda' e_j} - X^{x'})/\lambda'$; $U^{x,\lambda} := (Y^{x+\lambda e_j} - Y^x)/\lambda$; $U^{x',\lambda'} := (Y^{x'+\lambda' e_j} - Y^{x'})/\lambda'$ and $V^{x,\lambda} := (Z^{x+\lambda e_j} - Z^x)/\lambda$; $V^{x',\lambda'} := (Z^{x'+\lambda' e_j} - Z^{x'})/\lambda'$. Furthermore, for $\theta \in [0,1], \lambda > 0, x \in \mathbb{R}^m$, we define the processes $\Lambda^{x,\lambda} := X^x + \theta\lambda N^{x,\lambda}$, $\Gamma^{x,\lambda} := Y^x + \theta\lambda U^{x,\lambda}$ and $\Delta^{x,\lambda} := Z^x + \theta\lambda V^{x,\lambda}$. Let $N^{i,x,\lambda}, U^{i,x,\lambda}, V^{i,x,\lambda}, \Lambda^{i,x,\lambda}, \Gamma^{i,x,\lambda}$ and $\Delta^{i,x,\lambda}$ be the ith component of $N^{x,\lambda}, U^{x,\lambda}$, $V^{x,\lambda}, \Lambda^{x,\lambda}, \Gamma^{x,\lambda}$ and $\Delta^{x,\lambda}$, respectively for each $i = 1, \dots, m'$. Let us first show that there exists a constant C independent of x and λ such that

$$\|N^\lambda\|^2_{\mathcal{S}^\infty} + \|U^\lambda\|^2_{\mathcal{S}^\infty} + \|V^\lambda \cdot W\|^2_{\text{BMO}} \le C. \tag{3.4.8}$$

Since

$$N^{x,\lambda}_t = e_i + \int_0^t \int_0^1 \partial_x b_u(X^x_u + \theta(X^{x+\lambda e_i}_u - X^x_u), Y^x_u + \theta(Y^{x+\lambda e_i}_u - Y^x_u))N^{x,\lambda}_u \, d\theta \, du$$
$$+ \int_0^t \int_0^1 \partial_y b_u(X^x_u + \theta(X^{x+\lambda e_i}_u - X^x_u), Y^x_u + \theta(Y^{x+\lambda e_i}_u - Y^x_u))U^{x,\lambda}_u \, d\theta \, du,$$

and $\partial_x b$ and $\partial_y b$ are bounded, it follows from Gronwall's inequality that

$$|N^{x,\lambda}_t| \le e^{k_1 t}\left(1 + k_2 T\|U^{x,\lambda}\|_{\mathcal{S}^\infty}\right). \tag{3.4.9}$$

We have

$$U^{i,x,\lambda}_t = \int_0^1 \partial_x h^i(\Lambda^{x,\lambda}_T)N^{x,\lambda}_T \, d\theta + \int_t^T \int_0^1 \partial_z f^i_u(\Delta^{i,x,\lambda}_u)V^{i,x,\lambda}_u$$
$$+ \partial_x l^i_u(\Lambda^{x,\lambda}_u, \Gamma^{x,\lambda}_u, \Delta^{x,\lambda}_u)N^{x,\lambda}_u + \partial_y l^i_u(\Lambda^{x,\lambda}_u, \Gamma^{x,\lambda}_u, \Delta^{x,\lambda}_u)U^{x,\lambda}_u$$
$$+ \partial_z l^i_u(\Lambda^{x,\lambda}_u, \Gamma^{x,\lambda}_u, \Delta^{x,\lambda}_u)V^{x,\lambda}_u \, d\theta \, du - \int_t^T V^{i,x,\lambda}_u dW_u.$$

Hence, similar to the proof of Theorem 3.2.2, we have

$$
\begin{aligned}
&U_t^{i,x,\lambda} + \int_t^T V_u^{i,x,\lambda} d\tilde{W}_u^i \\
&= \int_0^1 \partial_x h^i(\Lambda_T^{x,\lambda}) N_T^{x,\lambda}\, d\theta + \int_t^T \int_0^1 \partial_x l_u^i(\Lambda_u^{x,\lambda}, \Gamma_u^{x,\lambda}, \Delta_u^{x,\lambda}) N_u^{x,\lambda} \\
&\quad + \partial_y l_u^i(\Lambda_u^{x,\lambda}, \Gamma_u^{x,\lambda}, \Delta_u^{x,\lambda}) U_u^{x,\lambda} + \partial_z l_u^i(\Lambda_u^{x,\lambda}, \Gamma_u^{x,\lambda}, \Delta_u^{x,\lambda}) V_u^{x,\lambda}\, d\theta\, du,
\end{aligned}
$$

where $\tilde{W}^i = W - \int_0^\cdot \zeta_s^i\, ds$ with $|\zeta_s^i| \leq k_3(1 + 2|(1-\bar{\theta}_s) Z_s^{i,x} + \bar{\theta}_s Z_s^{i,x+\lambda e_j}|)$ for some predictable process $\bar{\theta}_s \in [0,1]$ is a Brownian motion under the equivalent measure $\tilde{P}^i = \mathcal{E}(\zeta^i \cdot W)_T \cdot P$. Therefore similar to Theorem 3.2.2, with the same constants c_1, c_2 and C_2,

$$
\begin{aligned}
&\|U^{x,\lambda}\|_\infty^2 + c_1 \|V^{x,\lambda} \cdot W\|_{\text{BMO}}^2 \\
&\leq m' \left(16 k_2^2 T^2 e^{2k_1 T}\left(k_4^2 + k_7^2\right) + 8 k_5^2 T^2\right) \|U^{x,\lambda}\|_\infty^2 + 16 m' e^{2k_1 T}\left(k_4^2 + k_7^2\right) \\
&\quad + 24\sqrt{3} k_6^2 L_4^2 T^{1-\varepsilon} c_2 m' \left(T^\varepsilon + 2 + 2\varepsilon L_4^2 c_2 C_2^2\right) \|V^{x,\lambda} \cdot W\|_{\text{BMO}}^2.
\end{aligned}
$$

Therefore, it follows from (3.4.3) that

$$
\|U^{x,\lambda}\|_{\mathcal{S}^\infty}^2 \leq \frac{16 m' e^{2k_1 T}\left(k_4^2 + k_7^2\right)}{1 - m'\left(16 k_2^2 T^2 e^{2k_1 T}\left(k_4^2 + k_7^2\right) + 8 k_5^2 T^2\right)}. \tag{3.4.10}
$$

and

$$
c_1 \|V^{x,\lambda}\|_{\text{BMO}}^2 \leq 32 m' e^{2k_1 T}\left(k_4^2 + k_7^2\right). \tag{3.4.11}
$$

Combining with (3.4.9),

$$
\|N^{x,\lambda}\|_{\mathcal{S}^\infty} \leq \left(1 + k_2 T \sqrt{\frac{16 m' e^{2k_1 T}\left(k_4^2 + k_7^2\right)}{1 - m'\left(16 k_2^2 T^2 e^{2k_1 T}\left(k_4^2 + k_7^2\right) + 8 k_5^2 T^2\right)}}\right) e^{k_1 T}. \tag{3.4.12}
$$

Now, estimating the difference gives

$$
\begin{aligned}
|N_t^{x,\lambda} - N_t^{x',\lambda'}| &= \Big| \int_0^t \int_0^1 \partial_x b_u(\Lambda_u^{x,\lambda}, \Gamma_u^{x,\lambda}) N_u^{x,\lambda} + \partial_y b_u(\Lambda_u^{x,\lambda}, \Gamma_u^{x,\lambda}) U_u^{x,\lambda} \\
&\quad - \partial_x b_u(\Lambda_u^{x',\lambda'}, \Gamma_u^{x',\lambda'}) N_u^{x',\lambda'} - \partial_y b_u(\Lambda_u^{x',\lambda'}, \Gamma_u^{x',\lambda'}) U_u^{x',\lambda'}\, d\theta du \Big| \\
&\leq \int_0^t \int_0^1 |\partial_x b_u(\Lambda_u^{x,\lambda}, \Gamma_u^{x,\lambda})| |N_u^{x,\lambda} - N_u^{x',\lambda'}| \\
&\quad + |\partial_x b_u(\Lambda_u^{x,\lambda}, \Gamma_u^{x,\lambda}) - \partial_x b_u(\Lambda_u^{x',\lambda'}, \Gamma_u^{x',\lambda'})| |N_u^{x',\lambda'}| \\
&\quad + |\partial_y b_u(\Lambda_u^{x,\lambda}, \Gamma_u^{x,\lambda})| |U_u^{x,\lambda} - U_u^{x',\lambda'}| \\
&\quad + |\partial_y b_u(\Lambda_u^{x,\lambda}, \Gamma_u^{x,\lambda}) - \partial_y b_u(\Lambda_u^{x',\lambda'}, \Gamma_u^{x',\lambda'})| |U_u^{x',\lambda'}| d\theta du.
\end{aligned} \tag{3.4.13}
$$

Then, using (B1) and (B6) and applying Gronwall's lemma, we have

$$
\begin{aligned}
&\|N^{x,\lambda} - N^{x',\lambda'}\|_{\mathcal{S}^\infty} \\
&\le e^{k_1 T} \Bigg(k_2 T \|U^{x,\lambda} - U^{x',\lambda'}\|_{\mathcal{S}^\infty} + \frac{K(\|N^{x',\lambda'}\|_{\mathcal{S}^\infty} + \|U^{x',\lambda'}\|_{\mathcal{S}^\infty})}{2} \|X^x - X^{x'}\|_{\mathcal{S}^\infty} \\
&\quad + \frac{K(\|N^{x',\lambda'}\|_{\mathcal{S}^\infty} + \|U^{x',\lambda'}\|_{\mathcal{S}^\infty})}{2} \|X^{x+\lambda e_j} - X^{x'+\lambda' e_j}\|_{\mathcal{S}^\infty} \\
&\quad + \frac{K(\|N^{x',\lambda'}\|_{\mathcal{S}^\infty} + \|U^{x',\lambda'}\|_{\mathcal{S}^\infty})}{2} \|Y^x - Y^{x'}\|_{\mathcal{S}^\infty} \\
&\quad + \frac{K(\|N^{x',\lambda'}\|_{\mathcal{S}^\infty} + \|U^{x',\lambda'}\|_{\mathcal{S}^\infty})}{2} \|Y^{x+\lambda e_j} - Y^{x'+\lambda' e_j}\|_{\mathcal{S}^\infty} \Bigg).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&U_t^{i,x,\lambda} - U_t^{i,x',\lambda'} + \int_t^T (V_u^{i,x,\lambda} - V_u^{i,x',\lambda'}) d\tilde{W}_u^i \\
&= \int_0^1 \partial_x h^i(\Lambda_T^{x,\lambda}) N_T^{x,\lambda} - \partial_x h^i(X\Lambda_T^{x',\lambda'}) N_T^{x',\lambda'}\, d\theta \\
&\quad + \int_t^T \int_0^1 (\partial_z f_u^i(\Delta_u^{i,x,\lambda}) - \partial_z f_u^i(\Delta_u^{i,x',\lambda'})) V_u^{i,x',\lambda'}\, d\theta\, du \\
&\quad + \int_t^T \int_0^1 \partial_x l_u^i(\Lambda_u^{x,\lambda}, \Gamma_u^{x,\lambda}, \Delta_u^{x,\lambda}) N_u^{x,\lambda} + \partial_y l_u^i(\Lambda_u^{x,\lambda}, \Gamma_u^{x,\lambda}, \Delta_u^{x,\lambda}) U_u^{x,\lambda} \\
&\quad + \partial_z l_u^i(\Lambda_u^{x,\lambda}, \Gamma_u^{x,\lambda}, \Delta_u^{x,\lambda}) V_u^{x,\lambda} - \partial_x l_u^i(\Lambda_u^{x',\lambda'}, \Gamma_u^{x',\lambda'}, \Delta_u^{x',\lambda'}) N_u^{x',\lambda'} \\
&\quad - \partial_y l_u^i(\Lambda_u^{x',\lambda'}, \Gamma_u^{x',\lambda'}, \Delta_u^{x',\lambda'}) U_u^{x',\lambda'} - \partial_z l_u^i(\Lambda_u^{x',\lambda'}, \Gamma_u^{x',\lambda'}, \Delta_u^{x',\lambda'}) V_u^{x',\lambda'}\, d\theta\, du,
\end{aligned}
$$

where $\tilde{W}^i = W - \int_0^\cdot \zeta_s^i\, ds$ is defined as above. Rearranging the terms on the right hand side such as in (3.4.13) using successively (B3), (B4), (B6) and using Cauchy-Schwarz' inequality, similar to Theorem 3.2.2, with the same constants c_1, c_2, C_2, we have

$$
\begin{aligned}
&\|U^{x,\lambda} - U^{x',\lambda'}\|_\infty^2 + c_1 \|(V^{x,\lambda} - V^{x',\lambda'}) \cdot W\|_{\mathrm{BMO}}^2 \\
&\le m' \left(16 k_2^2 T^2 e^{2k_1 T} \left(k_4^2 + k_7^2\right) + 16 k_5^2 T^2\right) \|U^{x,\lambda} - U^{x',\lambda'}\|_\infty^2 + I_1 + I_2 \\
&\quad + 24\sqrt{3} k_6^2 L_4^2 T^{1-\varepsilon} c_2 m' \left(T^\varepsilon + 2 + 2\varepsilon L_4^2 c_2 C_2^2\right) \|(V^{x,\lambda} - V^{x',\lambda'}) \cdot W\|_{\mathrm{BMO}}^2,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &= 4K^2 m' e^{2k_1 T} (k_4^2 + k_7^2) \left(\|N^{x',\lambda'}\|_{\mathcal{S}^\infty} + \|U^{x',\lambda'}\|_{\mathcal{S}^\infty}\right)^2 \Big(\|X^x - X^{x'}\|_{\mathcal{S}^\infty} \\
&\quad + \|X^{x+\lambda e_j} - X^{x'+\lambda' e_j}\|_{\mathcal{S}^\infty} + \|Y^x - Y^{x'}\|_{\mathcal{S}^\infty} + \|Y^{x+\lambda e_j} - Y^{x'+\lambda' e_j}\|_{\mathcal{S}^\infty}\Big)^2,
\end{aligned}
$$

$$
\begin{aligned}
I_2 = 24m'K^2 \Bigg(& \|N_T^{x',\lambda'}\|_{\mathcal{S}^\infty}^2 \left(\|X_T^x - X_T^{x'}\|_{\mathcal{S}^\infty} + \|X_T^{x+\lambda e_j} - X_T^{x'+\lambda' e_j}\|_{\mathcal{S}^\infty} \right)^2 \\
& + \left(4c_2^2 L_4^2 \|V^{x',\lambda'} \cdot W\|_{\text{BMO}}^2 + 2Tc_2 \left(\|N_T^{x',\lambda'}\|_{\mathcal{S}^\infty} + \|U_T^{x',\lambda'}\|_{\mathcal{S}^\infty} \right)^2 \right) \\
\cdot & \left(\|(Z^x - Z^{x'}) \cdot W\|_{\text{BMO}}^2 + \|(Z^{x+\lambda e_j} - Z^{x'+\lambda' e_j}) \cdot W\|_{\text{BMO}}^2 \right) \\
& + \left(Tc_2 \|V^{x',\lambda'} \cdot W\|_{\text{BMO}}^2 + T^2 \left(\|N_T^{x',\lambda'}\|_{\mathcal{S}^\infty} + \|U_T^{x',\lambda'}\|_{\mathcal{S}^\infty} \right)^2 \right) \\
\cdot & \Big(\|X^x - X^{x'}\|_{\mathcal{S}^\infty} + \|X^{x+\lambda e_j} - X^{x'+\lambda' e_j}\|_{\mathcal{S}^\infty} \\
& + \|Y^x - Y^{x'}\|_{\mathcal{S}^\infty} + \|Y^{x+\lambda e_j} - Y^{x'+\lambda' e_j}\|_{\mathcal{S}^\infty} \Big)^2 \Bigg).
\end{aligned}
$$

Hence, it follows from the Equations (3.4.3), (3.4.5), (3.4.6), (3.4.7) and (3.4.8) that there exists a constant $\tilde{C} > 0$ which does not depend on x, x' and λ, λ' such that

$$
\begin{aligned}
& \|N^{x,\lambda} - N^{x',\lambda'}\|_{\mathcal{S}^\infty}^2 + \|U^{x,\lambda} - U^{x',\lambda'}\|_{\mathcal{S}^\infty}^2 + \|(V^{x,\lambda} - V^{x',\lambda'}) \cdot W\|_{\text{BMO}}^2 \\
& \leq \tilde{C} \left(|x - x'| + |\lambda - \lambda'| \right).
\end{aligned}
$$

This proves the differentiability of $x \mapsto (X^x, Y^x, Z^x)$. □

3.A Multidimensional BSDEs with terminal condition of bounded Malliavin derivative

In this section, we extend the existence result of Cheridito and Nam [16] to the multidimensional case where the i^{th} component of the generator depends only on (y, z^i). For simplicity, we prove the crucial boundedness of Z in this setting and leave out the existence since it follows as in [16, Theorem 2.2]. We consider the BSDE

$$
Y_t = \xi + \int_t^T g_u(Y_u, Z_u) du - \int_t^T Z_u dW_u. \tag{3.A.1}
$$

We make the following assumptions:

(D1) $g : \Omega \times [0,T] \times \mathbb{R}^{m'} \times \mathbb{R}^{m' \times d} \to \mathbb{R}^{m'}$ is a continuous and measurable function such that $g_t^i(y,z) = g_t^i(y,z^i)$, $i = 1, \ldots, m'$ and there exists a constant $B \in \mathbb{R}_+$ and a nondecreasing function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$
\left| g_t(y,z) - g_t(y',z') \right| \leq B \left| y - y' \right| + \rho \left(|z| \vee \left| z' \right| \right) \left| z - z' \right|
$$

for all $t \in [0,T]$, $y, y' \in \mathbb{R}^{m'}$ and $z, z' \in \mathbb{R}^{m' \times d}$.

(D2) $g_\cdot(0,0) \in \mathcal{H}^4$ and there exist Borel-measurable functions $q_{ij} : [0,T] \to \mathbb{R}_+$ satisfying $\int_0^T q_{ij}^2(t)dt < \infty$ such that for every pair $(y,z) \in \mathbb{R}^{m'} \times \mathbb{R}^{m' \times d}$ with

$$|z| \leq Q := \sqrt{m' \sum_{j=1}^{d} \left(\sum_{i=1}^{m'} |A_{ij}| + \sum_{i=1}^{m'} \int_0^T |q_{ij}(t)| e^{-m'B(T-t)} dt \right)^2 e^{m'BT}},$$

one has $g_\cdot(y,z) \in \mathcal{L}_a^{1,2}(\mathbb{R}^{m'})$ and $\left| D_u^j g_t^i(y,z) \right| \leq q_{ij}(t)$, $i = 1, \ldots, m'; j = 1, \ldots, d$ and, for every $u \in [0,T]$,

$$\left| D_u g_t(y,z) - D_u g_t(y',z') \right| \leq K_u \left(|y - y'| + |z - z'| \right)$$

for some $\mathbb{R}_+$-valued adapted process $(K_u(t))_{t \in [0,T]}$ such that $\int_0^T \|K_u\|_{\mathcal{H}^4}^4 \, du < \infty$.

(D3) The terminal condition $\xi \in \mathcal{D}^{1,2}(\mathbb{R}^{m'})$ and there exist constants $A_{ij} \geq 0$ such that $\left| D_t^j \xi^i \right| \leq A_{ij}$ for all $i = 1, \ldots, m'; j = 1, \ldots, d$.

We first prove a useful lemma under the following stronger conditions:

(D1') g is continuously differentiable in (y,z) is such that $g_t^i(y,z) = g_t^i(y,z^i)$, $i = 1, \ldots, m'$ and there exist constants $B \in \mathbb{R}_+, \rho \in \mathbb{R}_+$ such that

$$|\partial_y g_t(y,z)| \leq B, \quad |\partial_z g_t(y,z)| \leq \rho,$$

for all $t \in [0,T]$, $y, y' \in \mathbb{R}^{m'}$ and $z, z' \in \mathbb{R}^{m' \times d}$.

(D2') Condition (D2) holds for all $(y,z) \in \mathbb{R}^{m'} \times \mathbb{R}^{m' \times d}$.

Lemma 3.A.1. *If* (D1')*,* (D2') *and* (D3) *hold, then the BSDE* (3.A.1) *admits a unique solution* $(Y,Z) \in \mathcal{S}^4(\mathbb{R}^{m'}) \times \mathcal{H}^4(\mathbb{R}^{m' \times d})$*, and*

$$|Z_t^{ij}| \leq \left(\sum_{i=1}^{m'} |A_{ij}| + \sum_{i=1}^{m'} \int_t^T |q_{ij}(s)| e^{-m'B(T-s)} ds \right) e^{m'B(T-t)}, \quad P \otimes dt\text{-a.e.}$$

Proof. By [16, Lemma 2.5], condition (D3) implies $E|\xi|^p < +\infty$, for all $p \in [1,\infty)$. It follows from [28, Theorem 5.1 and Proposition 5.3] that the BSDE (3.A.1) has a unique solution $(Y,Z) \in \mathcal{S}^4(\mathbb{R}^{m'}) \times \mathcal{H}^4(\mathbb{R}^{m' \times d})$. Moreover, $(Y,Z) \in \mathcal{L}_a^{1,2}(\mathbb{R}^{m'+m' \times d})$ for $i = 1, \ldots, m';\ j = 1, \ldots, d$,

$$(D_r^j Y_t^i, D_r^j Z_t^i) = (U_t^{ij,r}, V_t^{ij,r}) \quad P \otimes dt \otimes dr\text{-a.e. and } Z_t^{ij} = U_t^{ij,t} \quad P \otimes dt\text{-a.e.},$$

where

$$U_t^{ij,r} = 0, \quad V_t^{ij,r} = 0, \quad \text{for} \quad 0 \leq t < r \leq T,$$

and for each fixed r, denoting $(U_t^{j,r}, V_t^{j,r}) = (D_r^j Y_t, D_r^j Z_t)$, then $(U^{j,r}, V^{j,r})$ is the unique solution in $\mathcal{S}^2(\mathbb{R}^{m'}) \times \mathcal{H}^2(\mathbb{R}^{m' \times d})$ of the BSDE

$$U_t^{j,r} = D_r^j \xi + \int_t^T \partial_y g_s(Y_s, Z_s) U_s^{j,r} + \partial_z g_s(Y_s, Z_s) V_s^{j,r} + D_r^j g_s(Y_s, Z_s) ds - \int_t^T V_s^{j,r} dW_s.$$

Using the conditions (D1'), we have

$$\begin{aligned} U_t^{ij,r} &= D_r^j \xi^i + \int_t^T \partial_y g_s^i(Y_s, Z_s) U_s^{j,r} + \partial_{z^i} g_s^i(Y_s, Z_s) V_s^{ij,r} + D_r^j g_s^i(Y_s, Z_s) ds \\ &\quad - \int_t^T V_s^{ij,r} dW_s \\ &= D_r^j \xi^i + \int_t^T \partial_y g_s^i(Y_s, Z_s) U_s^{j,r} + D_r^j g_s^i(Y_s, Z_s) ds - \int_t^T V_s^{ij,r} d\tilde{W}_s^i, \end{aligned}$$

where $\tilde{W}_t^i = W_t - \int_0^t \partial_{z^i} g_s^i(Y_s, Z_s) ds$ is a Brownian motion under the probability measure $\tilde{P}^i := \mathcal{E}(\partial_{z^i} g_\cdot^i(Y_\cdot, Z_\cdot) \cdot W)_T \cdot P$. Taking conditional expectation with respect to $\mathcal{F}_t$ and $\tilde{P}^i$, using condition (D1') and (D3)

$$|U_t^{ij,r}| \leq \tilde{E}^i \left[A_{ij} + \int_t^T B|U_s^{j,r}| + q_{ij}(s) ds \middle| \mathcal{F}_t \right].$$

Hence, $|U_t^{ij,r}| \leq u_t^j$, where u_t^j is the solution of the following ODE

$$u_t^j = \sum_{i=1}^{m'} A_{ij} + \int_t^T m' B u_s^j + \sum_{i=1}^{m'} q_{ij}(s) ds.$$

It is easy to see that the unique solution of the above ODE is given by

$$u_t^j = \left(\sum_{i=1}^{m'} A_{ij} + \sum_{i=1}^{m'} \int_t^T q_{ij}(s) e^{-m'B(T-s)} ds \right) e^{m'B(T-t)}.$$

Hence

$$|U_t^{ij,r}| \leq \left(\sum_{i=1}^{m'} A_{ij} + \sum_{i=1}^{m'} \int_t^T q_{ij}(s) e^{-m'B(T-s)} ds \right) e^{m'B(T-t)}, \quad P \otimes dt\text{-a.e.} \ \square$$

Theorem 3.A.2. *If* (D1) - (D3) *hold, then the BSDE* (3.A.1) *has a unique solution in* $\mathcal{S}^4(\mathbb{R}^{m'}) \times \mathcal{H}^\infty(\mathbb{R}^{m' \times d})$ *and*

$$|Z_t^{ij}| \leq \left(\sum_{i=1}^{m'} A_{ij} + \sum_{i=1}^{m'} \int_t^T q_{ij}(s) e^{-m'B(T-s)} ds \right) e^{m'B(T-t)}, \quad P \otimes dt\text{-a.e.}$$

Proof. Using Lemma 3.A.1, following the same procedure for each $g^i,\ i = 1, \dots, m'$ as in the proof of [16, Theorem 2.2] and in combination with [28, Proposition 5.1] the result follows. □

3.B Multidimensional BSDEs with superquadratic growth

In this section, we will drop the assumption that the i^{th} component of the generator depends only on (y, z^i). We obtain solvability for small time horizon. Under an additional condition on the growth function ρ, the existence result hold for arbitrarily large time horizon. We consider the BSDE

$$Y_t = \xi + \int_t^T g_u(Y_u, Z_u) du - \int_t^T Z_u dW_u. \tag{3.B.1}$$

We make the following assumptions:

(H1) $g : \Omega \times [0, T] \times \mathbb{R}^{m'} \times \mathbb{R}^{m' \times d} \to \mathbb{R}^{m'}$ is a continuous and measurable function such that there exists a constant $B \in \mathbb{R}_+$ and a nondecreasing function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\left|g_t(y, z) - g_t(y', z')\right| \leq B \left|y - y'\right| + \rho\left(|z| \vee \left|z'\right|\right) \left|z - z'\right|$$

for all $t \in [0, T]$, $y, y' \in \mathbb{R}^{m'}$ and $z, z' \in \mathbb{R}^{m' \times d}$.

(H2) $g_\cdot(0, 0) \in \mathcal{H}^4$ and there exist Borel-measurable functions $q_{ij} : [0, T] \to \mathbb{R}_+$ satisfying $\int_0^T q_{ij}^2(t) dt < \infty$ such that for every pair $(y, z) \in \mathbb{R}^{m'} \times \mathbb{R}^{m' \times d}$ with

$$|z| \leq Q := \sqrt{2 \sum_{j=1}^{d} \sum_{i=1}^{m'} \left(|A_{ij}|^2 + \int_0^T |q_{ij}(t)|^2 dt \right)},$$

one has $g_\cdot(y, z) \in \mathcal{L}_a^{1,2}(\mathbb{R}^{m'})$ and $\left|D_u^j g_t^i(y, z)\right| \leq q_{ij}(t), i = 1, \dots, m'; j = 1, \dots, d$ and, for every $u \in [0, T]$,

$$\left|D_u g_t(y, z) - D_u g_t(y', z')\right| \leq K_u \left(\left|y - y'\right| + \left|z - z'\right|\right)$$

for some $\mathbb{R}_+$-valued adapted process $(K_u(t))_{t \in [0,T]}$ such that $\int_0^T \|K_u\|_{\mathcal{H}^4}^4 \, du < \infty$.

(H3) The terminal condition $\xi \in \mathcal{D}^{1,2}(\mathbb{R}^{m'})$ and there exist constants $A_{ij} \geq 0$ such that $\left|D_t^j \xi^i\right| \leq A_{ij}$ for all $i = 1, \ldots, m'; j = 1, \ldots, d$.

We first prove a useful lemma under the following stronger conditions:

(H1') g is continuously differentiable in (y, z) is such that there exist constants $B \in \mathbb{R}_+, \rho \in \mathbb{R}_+$ such that

$$|\partial_y g_t(y, z)| \leq B, \quad |\partial_z g_t(y, z)| \leq \rho,$$

for all $t \in [0, T]$, $y, y' \in \mathbb{R}^{m'}$ and $z, z' \in \mathbb{R}^{m' \times d}$.

(H2') Condition (D2) holds for all $(y, z) \in \mathbb{R}^{m'} \times \mathbb{R}^{m' \times d}$.

Lemma 3.B.1. *If* (H1')*,* (H2') *and* (H3) *hold, then the BSDE* (3.B.1) *admits a unique solution* $(Y, Z) \in \mathcal{S}^4(\mathbb{R}^{m'}) \times \mathcal{H}^4(\mathbb{R}^{m' \times d})$*, and*

$$|Z_t^j|^2 \leq \sum_{i=1}^{m'} \left(A_{ij}^2 + \int_t^T q_{ij}^2(s) e^{-\left(2B+\rho^2+1\right)(T-s)} ds \right) e^{\left(2B+\rho^2+1\right)(T-t)}, \ P \otimes dt\text{-a.e.}$$

Proof. By [16, Lemma 2.5], condition (H3) implies $E|\xi|^p < +\infty$, for all $p \in [1, \infty)$. It follows from [28, Theorem 5.1 and Proposition 5.3] that the BSDE (3.B.1) has a unique solution $(Y, Z) \in \mathcal{S}^4(\mathbb{R}^{m'}) \times \mathcal{H}^4(\mathbb{R}^{m' \times d})$. Moreover, $(Y, Z) \in \mathcal{L}_a^{1,2}(\mathbb{R}^{m' + m' \times d})$ for $i = 1, \ldots, m';\ j = 1, \ldots, d$,

$$(D_r^j Y_t^i, D_r^j Z_t^i) = (U_t^{ij,r}, V_t^{ij,r}) \quad P \otimes dt \otimes dr\text{-a.e. and } Z_t^{ij} = U_t^{ij,t} \quad P \otimes dt\text{-a.e.},$$

where

$$U_t^{ij,r} = 0, \quad V_t^{ij,r} = 0, \quad \text{for} \quad 0 \leq t < r \leq T,$$

and for each fixed r, denoting $(U_t^{j,r}, V_t^{j,r}) = (D_r^j Y_t, D_r^j Z_t)$, then $(U^{j,r}, V^{j,r})$ is the unique solution in $\mathcal{S}^2(\mathbb{R}^{m'}) \times \mathcal{H}^2(\mathbb{R}^{m' \times d})$ of the BSDE

$$U_t^{j,r} = D_r^j \xi + \int_t^T \partial_y g_s(Y_s, Z_s) U_s^{j,r} + \partial_z g_s(Y_s, Z_s) V_s^{j,r} + D_r^j g_s(Y_s, Z_s) ds - \int_t^T V_s^{j,r} dW_s.$$

Applying Itô's formula to $|U_t^{j,r}|^2$ yields

$$\begin{aligned}|U_t^{j,r}|^2 &= |D_r^j\xi|^2 - \int_t^T 2U_s^{j,r}V_s^{j,r}dW_s + \int_t^T 2U_s^{j,r}\partial_y g_s(Y_s,Z_s)U_s^{j,r} \\ &\quad + 2U_s^{j,r}\partial_z g_s(Y_s,Z_s)V_s^{j,r} + 2U_s^{j,r}D_r^j g_s(Y_s,Z_s) - |V_s^{j,r}|^2 ds \\ &\le |D_r^j\xi|^2 - \int_t^T 2U_s^{j,r}V_s^{j,r}dW_s \\ &\quad + \int_t^T 2B|U_s^{j,r}|^2 + 2\rho|U_s^{j,r}||V_s^{j,r}| + 2\sqrt{\sum_{i=1}^{m'} q_{ij}^2(s)}|U_s^{j,r}| - |V_s^{j,r}|^2 ds \\ &\le |D_r^j\xi|^2 - \int_t^T 2U_s^{j,r}V_s^{j,r}dW_s + \int_t^T \left(2B+\rho^2+1\right)|U_s^{j,r}|^2 + \sum_{i=1}^{m'} q_{ij}^2(s)ds.\end{aligned}$$

Taking conditional expectation with respect to $\mathcal{F}_t$ and P, using condition (H3)

$$|U_t^{j,r}|^2 \le E\left[\sum_{i=1}^{m'} A_{ij}^2 + \int_t^T \left(2B+\rho^2+1\right)|U_s^{j,r}|^2 + \sum_{i=1}^{m'} q_{ij}^2(s)ds \middle| \mathcal{F}_t\right].$$

Gronwall's inequality implies that

$$|U_t^{j,r}|^2 \le \sum_{i=1}^{m'}\left(A_{ij}^2 + \int_t^T q_{ij}^2(s)e^{-\left(2B+\rho^2+1\right)(T-s)}ds\right)e^{\left(2B+\rho^2+1\right)(T-t)},\ P\otimes dt\text{-a.e.}$$

□

Theorem 3.B.2. *If* (H1) *-* (H3) *hold and* $T \le \frac{\log 2}{2B+\rho^2(Q)+1}$*, then the BSDE* (3.B.1) *has a unique solution in* $\mathcal{S}^4(\mathbb{R}^{m'}) \times \mathcal{H}^\infty(\mathbb{R}^{m'\times d})$ *and*

$$|Z_t^j|^2 \le 2\sum_{i=1}^{m'}\left(A_{ij}^2 + \int_0^T q_{ij}^2(s)ds\right), \quad P\otimes dt\text{-a.e.}$$

Proof. Define

$$\tilde{g}_t(y,z) = \begin{cases} g_t(y,z) & \text{if} \quad |z| \le Q, \\ g_t(y, Qz/|z|) & \text{if} \quad |z| > Q.\end{cases}$$

Denote $x = (y,z) \in \mathbb{R}^{m'+m'\times d}$ and let $\beta \in C^\infty(\mathbb{R}^{m'+m'\times d})$ be the mollifier

$$\beta(x) := \begin{cases} \lambda\exp\left(-\frac{1}{1-|x|^2}\right) & \text{if} \quad |x| < 1, \\ 0 & \text{otherwise,}\end{cases}$$

where the constant $\lambda \in \mathbb{R}_+$ is chosen such that $\int_{\mathbb{R}^{m'+m'\times d}} \beta(x)dx = 1$. Set $\beta^n(x) := n^{m'+m'\times d}\beta(nx)$, $n \in \mathbb{N} \setminus \{0\}$, and define

$$g_t^n(\omega, x) := \int\limits_{\mathbb{R}^{m'+m'\times d}} \tilde{g}_t(\omega, x')\beta^n(x - x')dx'.$$

Then all g^n satisfy (H1') and (H2'). Therefore by Lemma 3.B.1 there exist unique solutions $(Y^n, Z^n) \in \mathcal{S}^4(\mathbb{R}^{m'}) \times \mathcal{H}^4(\mathbb{R}^{m'\times d})$ to the BSDEs corresponding to (g^n, ξ), and

$$\begin{aligned}
|Z_t^{n,j}|^2 &\leq \sum_{i=1}^{m'} \left(A_{ij}^2 + \int\limits_t^T q_{ij}^2(s) e^{-\left(2B+\rho^2(Q)+1\right)(T-s)} ds \right) e^{\left(2B+\rho^2(Q)+1\right)(T-t)} \\
&\leq \sum_{i=1}^{m'} \left(A_{ij}^2 + \int\limits_0^T q_{ij}^2(s) ds \right) e^{\left(2B+\rho^2(Q)+1\right)T}.
\end{aligned}$$

Since $T \leq \frac{\log 2}{2B+\rho^2(Q)+1}$, we obtain

$$|Z_t^{n,j}|^2 \leq 2 \left(\sum_{i=1}^{m'} A_{ij}^2 + \sum_{i=1}^{m'} \int\limits_0^T q_{ij}^2(s) ds \right).$$

Thus, following the same procedure as in the proof of [16, Theorem 2.2] and in combination with [28, Proposition 5.1] the result follows. □

Theorem 3.B.3. *If* (H1)*,* (H3) *hold and* ρ *is such that* $\sum_{n=0}^{\infty} \frac{\log 2}{2B+\rho^2(2^n Q)+1} > T$*,* (H2) *holds for all* $(y, z) \in \mathbb{R}^{m'} \times \mathbb{R}^{m'\times d}$ *such that* $|z| \leq 2^N Q$ *where* N *is the smallest integer such that* $\sum_{n=0}^{N} \frac{\log 2}{2B+\rho^2(2^n Q)+1} \geq T$*. Then the BSDE* (3.B.1) *has a unique solution in* $\mathcal{S}^4(\mathbb{R}^{m'}) \times \mathcal{H}^\infty(\mathbb{R}^{m'\times d})$ *and*

$$|Z_t| \leq 2^N Q, \quad P \otimes dt\text{-a.e.}$$

Proof. From Theorem 3.B.2, the BSDE (3.B.1) has a unique solution in $\mathcal{S}^4(\mathbb{R}^{m'}) \times \mathcal{H}^\infty(\mathbb{R}^{m'\times d})$ and $|Z_t| \leq Q$ on $[T - \frac{\log 2}{2B+\rho^2(Q)+1}, T]$. By Lemma 3.B.1, we have

$$|D_r^j Y_{T-\frac{\log 2}{2B+\rho^2(Q)+1}}|^2 \leq \sum_{i=1}^{m'} 2|A_{ij}|^2 + \sum_{i=1}^{m'} \int\limits_0^T 2|q_{ij}(t)|^2 dt.$$

By similar arguments as in Lemma 3.B.1 and Theorem 3.B.2, the BSDE (3.B.1) has a unique solution in $\mathcal{S}^4(\mathbb{R}^{m'}) \times \mathcal{H}^\infty(\mathbb{R}^{m'\times d})$ on $[T - \frac{\log 2}{2B+\rho^2(Q)+1} - \frac{\log 2}{2B+\rho^2(2Q)+1}, T -$

$\frac{\log 2}{2B+\rho^2(Q)+1}]$ with terminal condition $Y_{T-\frac{\log 2}{2B+\rho^2(Q)+1}}$, and

$$|D_r^j Y_{T-\frac{\log 2}{2B+\rho^2(Q)+1}-\frac{\log 2}{2B+\rho^2(2Q)+1}}|^2 \leq \sum_{i=1}^{m'} 2^2|A_{ij}|^2 + \sum_{i=1}^{m'} \int_0^T (2^2+2)|q_{ij}(t)|^2 dt,$$

$$|Z_t| \leq 2Q,\ t \in [T - \frac{\log 2}{2B+\rho^2(Q)+1} - \frac{\log 2}{2B+\rho^2(2Q)+1}, T - \frac{\log 2}{2B+\rho^2(Q)+1}].$$

By recurrence, for $m \geq 2$, the BSDE (3.B.1) has a unique solution in $\mathcal{S}^4(\mathbb{R}^{m'}) \times \mathcal{H}^\infty(\mathbb{R}^{m'\times d})$ on $[T-\sum_{n=0}^{m} \frac{\log 2}{2B+\rho^2(2^nQ)+1}, T-\sum_{n=0}^{m-1} \frac{\log 2}{2B+\rho^2(2^nQ)+1}]$ with terminal condition $Y_{T-\sum_{n=0}^{m-1} \frac{\log 2}{2B+\rho^2(2^nQ)+1}}$, and

$$|D_r^j Y_{T-\sum_{n=0}^{m} \frac{\log 2}{2B+\rho^2(2^nQ)+1}}|^2 \leq \sum_{i=1}^{m'} 2^{m+1}|A_{ij}|^2 + \sum_{i=1}^{m'} \int_0^T (\sum_{k=1}^{m+1} 2^k)|q_{ij}(t)|^2 dt,$$

$$|Z_t| \leq 2^m Q, \quad t \in [T - \sum_{n=0}^{m} \frac{\log 2}{2B+\rho^2(2^nQ)+1}, T - \sum_{n=0}^{m-1} \frac{\log 2}{2B+\rho^2(2^nQ)+1}].$$

Hence the existence follows from a pasting argument. The uniqueness follows from a similar argument as in the proof of [16, Theorem 2.2]. □

Remark 3.B.4. If $\rho(x) = \sqrt{\log(1+x)}$ for $x \geq 0$, then $\sum_{n=0}^{\infty} \frac{\log 2}{2B+\rho^2(2^nQ)+1} = \infty$. Indeed, since $\rho(2^nQ) \leq \sqrt{\log(2^n(1+Q))}$, we have

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{\log 2}{2B+\rho^2(2^nQ)+1} &\geq \sum_{n=0}^{\infty} \frac{\log 2}{2B+\log(2^n(1+Q))+1} \\ &= \sum_{n=0}^{\infty} \frac{\log 2}{2B+\log(1+Q)+n\log 2+1} = \infty.\end{aligned}$$ ♦

Chapter 4

BSDEs on Finite and Infinite Horizon with Time-delayed Generators

4.1 Introduction

In Delong and Imkeller [24, 25], the theory of backward stochastic differential equations (BSDEs) was extended to BSDEs with time delay generators (delay BSDEs). These are non-Markovian BSDEs in which the generator at each positive time t may depend on the past values of the solutions. This class of equations turned out to have natural applications in pricing and hedging of insurance contracts, see Delong [23].

The existence result of Delong and Imkeller [24], proved for standard Lipschitz generators and small time horizon, has been refined by dos Reis et al. [26] who derived additional properties of delay BSDEs such as path regularity and existence of decoupled systems. Furthermore, existence of delay BSDE constrained above a given continuous barrier has been established by Zhou and Ren [77] in a similar setup. More recently, Briand and Elie [13] proposed a framework in which quadratic BSDEs with sufficiently small time delay in the value process can be solved.

In addition to the inherent non-Markovian structure of delay BSDEs, the difficulty in studying these equations comes from that the inter-temporal changes of the value and control processes always depend on their entire past, hence making it hard to obtain boundedness of solutions or even BMO-martingale properties of the stochastic integral of the control process. This suggests that delay BSDEs can actually be solved forward and backward in time and in this regard, share similarities with forward backward stochastic differential equations (FBSDEs), see Section 4.4 for a more detailed discussion.

The object of the present chapter is to study delay BSDEs in the case where the past values of the solutions are weighted with respect to some scaling function.

In economic applications, these weighting functions can be viewed as representing the perception of the past of an agent. For multidimentional BSDEs with possibly infinite time horizon, we derive existence, uniqueness and stability of delay BSDE in this weighting-function setting. In particular, we show that when the delay vanishes, the solutions of the delay BSDEs converge to the solution of the BSDE with no delay, hence recovering a result obtained by Briand and Elie [13] for different types of delay. Moreover, we prove that in our setting existence and uniqueness also hold in the case of reflexion on a càdlàg barrier. We observe a link between delay BSDEs and coupled FBSDE and, based on the findings in chapter 3, we derive existence of delay quadratic BSDEs in the case where only the value process is subjected to delay. We refer to Briand and Elie [13] for a similar result, again for a different type of delay and in the one-dimensional case.

In the next section, we specify our probabilistic structure and the form of the equation, then present existence, uniqueness and stability results. Sections 4.3 and 4.4 are dedicated to the study of reflected delay BSDEs and quadratic and superquadratic BSDEs with delay in value process, respectively.

4.2 BSDEs with time delayed generators

We work on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ with $T \in (0, \infty]$. We assume that the filtration is generated by a d-dimensional Brownian motion W and it is complete and right continuous. Let us also assume that $\mathcal{F} = \mathcal{F}_T$. We endow $\Omega \times [0, T]$ with the predictable σ-algebra and $\mathbb{R}^k$ with its Borel σ-algebra. Unless otherwise stated, all equalities and inequalities between random variables and stochastic processes will be understood in the P-a.s. and $P \otimes dt$-a.e. sense, respectively. For $p \in [1, \infty)$ and $m \in \mathbb{N}$, we denote by $\mathcal{S}^p(\mathbb{R}^m)$ the space of predictable and continuous processes X valued in $\mathbb{R}^m$ such that $\|X\|^p_{\mathcal{S}^p} := E[(\sup_{t\in[0,T]} |X_t|)^p] < \infty$ and by $\mathcal{H}^p(\mathbb{R}^m)$ the space of predictable processes Z valued in $\mathbb{R}^{m\times d}$ such that $\|Z\|^p_{\mathcal{H}^p} := E[(\int_0^T |Z_u|^2\, du)^{p/2}] < \infty$. For a suitable integrand Z, we denote by $Z \cdot W$ the stochastic integral $(\int_0^t Z_u\, dW_u)_{t\in[0,T]}$ of Z with respect to W. From Protter [68], $Z \cdot W$ defines a continuous martingale for every $Z \in \mathcal{H}^p(\mathbb{R}^m)$. Processes $(\phi_t)_{t\in[0,T]}$ will always be extended to $[-T, 0)$ by setting $\phi_t = 0$ for $t \in [-T, 0)$. We equip $\overline{\mathbb{R}}$ with the σ-algebra $\mathcal{B}(\overline{\mathbb{R}})$ consisting of Borel sets of the usual real line with possible addition of the points $-\infty, +\infty$, see Bogachev [11].

Let ξ be an $\mathcal{F}_T$-measurable terminal condition and g an $\mathbb{R}^m$-valued function. Given two measures α_1 and α_2 on $[-\infty, \infty]$, and two weighting functions $u, v : [0, T] \to \mathbb{R}$, we study the existence of the BSDE

$$Y_t = \xi + \int_t^T g(s, \Gamma(s))ds - \int_t^T Z_s dW_s, \quad t \in [0, T], \tag{4.2.1}$$

where

$$\Gamma(s) := \left(\int_{-T}^{0} u(s+r) Y_{s+r} \alpha_1(dr), \int_{-T}^{0} v(s+r) Z_{s+r} \alpha_2(dr) \right). \tag{4.2.2}$$

Example 4.2.1. 1. *BSDE with infinite horizon:* If $u = v = 1$ and $\alpha_1 = \alpha_2 = \delta_0$ the Dirac measure at 0, then Equation (4.2.1) reduces to the classical BSDE with infinite time horizon and stantard Lipschitz generator.
2. *Pricing of insurance contracts:* Let us consider the pricing problem of an insurance contract ξ written on a weather derivative. It is well know, see for instance [3] that such contracts can be priced by investing in a highly correlated, but tradable derivative. In the Merton model, assuming that the latter asset has dynamics

$$dS_t = S_t(\mu_t dt + \sigma_t dW_t),$$

then the insurer chooses a number π_t of shares of S to buy at time t and fixes a cost c_t to be paid by the client. Hence, he seeks to find the price V_0 such that

$$dV_t = c_t\, dt + \pi_t \sigma_t (dW_t + \theta_t\, dt)$$

with $\theta_t = \sigma_t'(\sigma_t \sigma_t)^{-1} \mu_t$. It is natural to demand the cost c_t at time t to depend on the past values of the insurance premium V_t, for instance to account for historical weather data. A possible cost criteria is

$$c_t := M_t \int_{-T}^{0} \cos(\frac{2\pi}{P}(t+s)) V_{t+s}\, ds$$

where P accounts for the weather periodicity and M is a scaling parameter. Thus, the insurance premium satisfies the delay BSDE

$$V_t = \xi + \int_t^T \left(\int_{-T}^{0} M_u \cos(\frac{2\pi}{P}(u+s)) V_{u+s}\, ds + Z_u \sigma_u \theta_u \right) du - \int_t^T Z_u\, dW_u. \ \diamondsuit$$

4.2.1 Existence

Our existence result for the BSDE (4.2.1) is obtained under the following assumptions:

(A1) α_1, α_2 are two deterministic, finite valued measures supported on $[-T, 0]$.

(A2) $u, v : [0, T] \to \mathbb{R}$ are Borel measurable functions such that $u \in L^1(dt)$ and $v \in L^2(dt)$.

(A3) $g : \Omega\times[0,T]\times\mathbb{R}^m\times\mathbb{R}^{m\times d} \to \mathbb{R}^m$ is measurable, such that $\int_0^T g(s,0,0)\,ds \in L^2(\mathbb{R}^m)$ and satisfies the standard Lipschitz condition: there exists a constant $K > 0$ such that

$$|g(t,y,z) - g(t,y',z')| \leq K(|y-y'| + |z-z'|)$$

for every $y, y' \in \mathbb{R}^m$ and $z, z' \in \mathbb{R}^{m\times d}$.

(A4) $\xi \in L^2(\mathbb{R}^m)$ and is $\mathcal{F}_T$-measurable.

Theorem 4.2.2. *Assume* (A1)*-*(A4)*. If*

$$\begin{cases} K^2\alpha_1^2([-T,0])\,\|u\|^2_{L^1(dt)} \leq \frac{1}{25}, \\ K^2\alpha_2^2([-T,0])\,\|v\|^2_{L^2(dt)} \leq \frac{1}{25}, \end{cases} \tag{4.2.3}$$

then BSDE (4.2.1) *admits a unique solution* $(Y,Z) \in \mathcal{S}^2(\mathbb{R}^m)\times\mathcal{H}^2(\mathbb{R}^{m\times d})$.

For the proof we need the following lemma on *a priori* estimates of solutions of (4.2.1).

Lemma 4.2.3 (A priori estimation). *Assume* (A1)*-*(A3)*. For every* $\xi, \bar{\xi} \in L^2(\mathbb{R}^m)$*,* $(y,z),(\bar{y},\bar{z}) \in \mathcal{S}^2(\mathbb{R}^m)\times\mathcal{H}^2(\mathbb{R}^{m\times d})$ *and* $(Y,Z),(\bar{Y},\bar{Z}) \in \mathcal{S}^2(\mathbb{R}^m)\times\mathcal{H}^2(\mathbb{R}^{m\times d})$ *satisfying*

$$\begin{cases} Y_t = \xi + \int_t^T g(s,\gamma(s))ds - \int_t^T Z_s dW_s \\ \bar{Y}_t = \bar{\xi} + \int_t^T g(s,\bar{\gamma}(s))ds - \int_t^T \bar{Z}_s dW_s, \quad t \in [0,T] \end{cases}$$

with

$$\begin{cases} \gamma(s) = \left(\int_{-T}^0 u(s+r)y_{s+r}\alpha_1(dr), \int_{-T}^0 v(s+r)z_{s+r}\alpha_2(dr)\right) \\ \bar{\gamma}(s) = \left(\int_{-T}^0 u(s+r)\bar{y}_{s+r}\alpha_1(dr), \int_{-T}^0 v(s+r)\bar{z}_{s+r}\alpha_2(dr)\right). \end{cases}$$

Then, one has

$$\begin{aligned} &\|Y-\bar{Y}\|^2_{\mathcal{S}^2(\mathbb{R}^m)} + \|Z-\bar{Z}\|^2_{\mathcal{H}^2(\mathbb{R}^{m\times d})} \\ &\leq 20K^2\alpha_1^2([-T,0])\,\|u\|^2_{L^1(dt)}\,\|y-\bar{y}\|^2_{\mathcal{S}^2(\mathbb{R}^m)} \\ &\quad + 10\left\|\xi-\bar{\xi}\right\|^2_{L^2(\mathbb{R}^m)} + 20K^2\alpha_2^2([-T,0])\,\|v\|^2_{L^2(dt)}\,\|z-\bar{z}\|^2_{\mathcal{H}^2(\mathbb{R}^{m\times d})}. \end{aligned}$$

Proof. Let $(y,z) \in \mathcal{S}^2(\mathbb{R}^m)\times\mathcal{H}^2(\mathbb{R}^{m\times d})$, by assumptions (A1) and (A3), using

$2ab \le a^2 + b^2$ and [26, Lemma 1.1], we have

$$E\left(\int_0^T g(s,\gamma(s))ds\right)^2$$

$$\le E\left(\int_0^T |g(s,0,0)|ds + K\int_0^T\int_{-T}^0 |u(s+r)||y_{s+r}|\alpha_1(dr)ds\right.$$

$$\left. + K\int_0^T\int_{-T}^0 |v(s+r)||z_{s+r}|\alpha_2(dr)ds\right)^2$$

$$\le 3E\left[\left(\int_0^T |g(s,0,0)|ds\right)^2 + K^2\left(\int_0^T\int_{-T}^0 |u(s+r)||y_{s+r}|\alpha_1(dr)ds\right)^2\right.$$

$$\left. + K^2\left(\int_0^T\int_{-T}^0 |v(s+r)||z_{s+r}|\alpha_2(dr)ds\right)^2\right]$$

$$\le 3E\left[\left(\int_0^T |g(s,0,0)|ds\right)^2 + K^2\left(\int_0^T \alpha_1([s-T,0])|u(s)||y_s|ds\right)^2\right.$$

$$\left. + K^2\left(\int_0^T \alpha_2([s-T,0])|v(s)||z_s|ds\right)^2\right]$$

$$\le 3E\left(\int_0^T |g(s,0,0)|ds\right)^2 + 3K^2\alpha_1^2([-T,0])\left(\int_0^T |u(s)|ds\right)^2 E\left[\sup_{0\le t\le T}|y_t|^2\right]$$

$$+ 3K^2\alpha_2^2([-T,0])\left(\int_0^T |v(s)|^2ds\right) E\left[\int_0^T |z_s|^2ds\right].$$

Hence, it holds $\int_0^T g(s,\gamma(s))\,ds \in L^2$.

Now, for $t \in [0,T]$, we have

$$Y_t - \bar{Y}_t = \xi - \bar{\xi} + \int_t^T g(s,\gamma(s)) - g(s,\bar{\gamma}(s))ds - \int_t^T Z_s - \bar{Z}_s dW_s \qquad (4.2.4)$$

and taking conditional expectation with respect to $\mathcal{F}_t$ yields

$$Y_t - \bar{Y}_t = E\left[\xi - \bar{\xi} + \int_t^T g(s,\gamma(s)) - g(s,\bar{\gamma}(s))ds\middle|\mathcal{F}_t\right].$$

By Doob's maximal inequality and $2ab \leq a^2 + b^2$, we obtain

$$\begin{aligned}
&E\left[\sup_{0\leq t\leq T} |Y_t - \bar{Y}_t|^2\right] \\
&= E\left(\sup_{0\leq t\leq T}\left|E\left[\xi - \bar{\xi} + \int_t^T g(s,\gamma(s)) - g(s,\bar{\gamma}(s))ds \middle| \mathcal{F}_t\right]\right|\right)^2 \\
&\leq E\left(\sup_{0\leq t\leq T} E\left[|\xi - \bar{\xi}| + \int_0^T |g(s,\gamma(s)) - g(s,\bar{\gamma}(s))|ds \middle| \mathcal{F}_t\right]\right)^2 \\
&\leq 8E\left[|\xi - \bar{\xi}|^2 + \left(\int_0^T |g(s,\gamma(s)) - g(s,\bar{\gamma}(s))|\, ds\right)^2\right].
\end{aligned}$$

On the other hand, for $t = 0$ in (4.2.4), bringing $\int_0^T Z_s - \bar{Z}_s dW_s$ to the left hand side, taking square and expectation to both sides and $2ab \leq a^2 + b^2$, we have

$$\begin{aligned}
E\left[\int_0^T |Z_t - \bar{Z}_t|^2 dt\right] &= E\left(\xi - \bar{\xi} + \int_0^T g(s,\gamma(s)) - g(s,\bar{\gamma}(s))ds\right)^2 - |Y_0 - \bar{Y}_0|^2 \\
&\leq E\left(\xi - \bar{\xi} + \int_0^T g(s,\gamma(s)) - g(s,\bar{\gamma}(s))ds\right)^2 \\
&\leq 2E\left[|\xi - \bar{\xi}|^2 + \left(\int_0^T |g(s,\gamma(s)) - g(s,\bar{\gamma}(s))|ds\right)^2\right].
\end{aligned}$$

By assumption (A3), using [26, Lemma 1.1] and the inequality $2ab \leq a^2 + b^2$, we have

$$\begin{aligned}
&E\left(\int_0^T |g(s,\gamma(s)) - g(s,\bar{\gamma}(s))|ds\right)^2 \\
&\leq K^2 E\left(\int_0^T \int_{-T}^0 |u(s+r)||y_{s+r} - \bar{y}_{s+r}|\alpha_1(dr)ds\right. \\
&\quad \left. + \int_0^T \int_{-T}^0 |v(s+r)||z_{s+r} - \bar{z}_{s+r}|\alpha_2(dr)ds\right)^2 \\
&= K^2 E\left(\int_0^T \alpha_1([s-T,0])|u(s)||y_s - \bar{y}_s|ds + \int_0^T \alpha_2([s-T,0])|v(s)||z_s - \bar{z}_s|ds\right)^2
\end{aligned}$$

$$\leq 2K^2\alpha_1^2([-T,0])\,\|u\|^2_{L^1(dt)}\,\|y-\bar{y}\|^2_{\mathcal{S}^2} + 2K^2\alpha_2^2([-T,0])\,\|v\|^2_{L^2(dt)}\,\|z-\bar{z}\|^2_{\mathcal{H}^2}.$$

Hence,

$$\|Y-\bar{Y}\|^2_{\mathcal{S}^2(\mathbb{R}^m)} + \|Z-\bar{Z}\|^2_{\mathcal{H}^2(\mathbb{R}^{m\times d})} \leq 20K^2\alpha_1^2([-T,0])\,\|u\|^2_{L^1(dt)}\,\|y-\bar{y}\|^2_{\mathcal{S}^2(\mathbb{R}^m)}$$
$$10E\left[|\xi-\bar{\xi}|^2\right] + 20K^2\alpha_2^2([-T,0])\,\|v\|^2_{L^2(dt)}\,\|z-\bar{z}\|^2_{\mathcal{H}^2(\mathbb{R}^{m\times d})}.$$

This concludes the proof. □

Proof (of Theorem 4.2.2). Let $(y,z)\in\mathcal{S}^2(\mathbb{R}^m)\times\mathcal{H}^2(\mathbb{R}^{m\times d})$ and define the process $\gamma(s) := \left(\int_{-T}^0 u(s+r)y_{s+r}\alpha_1(dr), \int_{-T}^0 v(s+r)z_{s+r}\alpha_2(dr)\right)$. Similar to Lemma 4.2.3, it follows from (A1)-(A4) that

$$E\left(\xi + \int_0^T g(s,\gamma(s))ds\right)^2 < \infty.$$

According to the martingale representation theorem, there exists a unique $Z\in\mathcal{H}^2(\mathbb{R}^{m\times d})$ such that for all $t\in[0,T]$,

$$E\left[\xi + \int_0^T g(s,\gamma(s))ds \middle| \mathcal{F}_t\right] = E\left[\xi + \int_0^T g(s,\gamma(s))ds\right] + \int_0^t Z_s dW_s.$$

Putting

$$Y_t := E\left[\xi + \int_t^T g(s,\gamma(s))ds \mid \mathcal{F}_t\right], \quad 0\leq t\leq T,$$

the pair (Y,Z) belongs to $\mathcal{S}^2(\mathbb{R}^m)\times\mathcal{H}^2(\mathbb{R}^{m\times d})$ and satisfies

$$Y_t = \xi + \int_t^T g(s,\gamma(s))ds - \int_t^T Z_s dW_s, \quad 0\leq t\leq T.$$

Thus we have constructed a mapping Φ from $\mathcal{S}^2(\mathbb{R}^m)\times\mathcal{H}^2(\mathbb{R}^{m\times d})$ to itself such that $\Phi(y,z)=(Y,Z)$. Let $(y,z),(\bar{y},\bar{z})\in\mathcal{S}^2(\mathbb{R}^m)\times\mathcal{H}^2(\mathbb{R}^{m\times d})$, and $(Y,Z)=\Phi(y,z)$, $(\bar{Y},\bar{Z})=\Phi(\bar{y},\bar{z})$. By Lemma 4.2.3, we have

$$\|Y-\bar{Y}\|^2_{\mathcal{S}^2(\mathbb{R}^m)} + \|Z-\bar{Z}\|^2_{\mathcal{H}^2(\mathbb{R}^{m\times d})} \leq 10K^2\alpha_1^2([-T,0])\,\|u\|^2_{L^1(dt)}\,\|y-\bar{y}\|^2_{\mathcal{S}^2(\mathbb{R}^m)}$$
$$+\,10K^2\alpha_2^2([-T,0])\,\|v\|^2_{L^2(dt)}\,\|z-\bar{z}\|^2_{\mathcal{H}^2(\mathbb{R}^{m\times d})}$$

so that if condition (4.2.3) is satisfied, Φ is a contraction mapping which therefore admits a unique fixed point on the Banach space $\mathcal{S}^2(\mathbb{R}^m)\times\mathcal{H}^2(\mathbb{R}^{m\times d})$. This completes the proof. □

4.2.2 Stability

In this subsection, we study stability of the BSDE (4.2.1) with respect to the delay measures. In particular, in Corollary 4.2.5 below we give conditions under which a sequence of solutions of BSDEs with time delayed generator converges to the solution of a standard BSDE with no delay. Given two measures α and β, we write $\alpha \leq \beta$ if $\alpha(A) \leq \beta(A)$ for every measurable set A.

Theorem 4.2.4. *Assume* (A2)*-*(A4)*. For $i = 1, 2$ and $n \in \mathbb{N}$, let α_i^n, α_i be measures satisfying (*A1*); with α_i^n satisfying* (4.2.3) *in Theorem 4.2.2 and such that $\alpha_i^n([-T, 0])$ converges to $\alpha_i([-T, 0])$. If $\alpha_1^n \leq \alpha_1$ (or $\alpha_1 \leq \alpha_1^n$) and $\alpha_2^n \leq \alpha_2$ (or $\alpha_2 \leq \alpha_2^n$), then $\|Y^n - Y\|_{\mathcal{S}^2(\mathbb{R}^m)} \to 0$ and $\|Z^n - Z\|_{\mathcal{H}^2(\mathbb{R}^{m\times d})} \to 0$, where (Y^n, Z^n) and (Y, Z) are solutions of the BSDE* (4.2.1) *with delay given by the measures (α_1^n, α_2^n) and (α_2, α_2), respectively.*

Proof. From Theorem 4.2.2, for every n, there exists a unique solution (Y^n, Z^n) to the BSDE (4.2.1) with delay given by the measures (α_1^n, α_2^n). Since α_i^n, $i = 1, 2$ satisfy (4.2.3) in Theorem 4.2.2 and $\alpha_i^n([-T, 0])$ converges to $\alpha_i([-T, 0])$, it follows that α_i satisfy (4.2.3) and by Theorem 4.2.2 there exists a unique solution (Y, Z) to the BSDE with delay given by (α_1, α_2). Using

$$Y_t^n - Y_t = \int_t^T g(s, \Gamma^n(s)) - g(s, \Gamma(s))ds - \int_t^T Z_s^n - Z_s dW_s,$$

it follows similar to the proof of Lemma 4.2.3 that

$$E\left[\sup_{0\leq t\leq T} |Y_t^n - Y_t|^2\right] \leq 4E\left[\left(\int_0^T |g(s, \Gamma^n(s)) - g(s, \Gamma(s))|ds\right)^2\right],$$

and

$$E\left[\int_0^T |Z_t^n - Z_t|^2\right] \leq E\left[\left(\int_0^T |g(s, \Gamma^n(s)) - g(s, \Gamma(s))|ds\right)^2\right].$$

On the other hand, using $2ab \leq a^2 + b^2$, we get

$$E\left[\left(\int_0^T |g(s, \Gamma^n(s)) - g(s, \Gamma(s))|ds\right)^2\right]$$
$$\leq 2K^2 E\left[\left(\int_0^T \left|\int_{-T}^0 u(s+r)Y_{s+r}^n \alpha_1^n(dr) - \int_{-T}^0 u(s+r)Y_{s+r}\alpha_1(dr)\right|ds\right)^2\right]$$

$$+ 2K^2 E\left[\left(\int_0^T \left|\int_{-T}^0 v(s+r) Z^n_{s+r}\alpha^n_2(dr) - \int_{-T}^0 v(s+r) Z_{s+r}\alpha_2(dr)\right| ds\right)^2\right].$$

Without loss of generality, we assume $\alpha_1 \leq \alpha_1^n$ and $\alpha_2 \leq \alpha_2^n$. Hence $\alpha_i^n - \alpha_i$, $i = 1, 2$, define positive measures satisfying (A1). Therefore,

$$E\left[\left(\int_0^T \left|\int_{-T}^0 u(s+r) Y^n_{s+r}\alpha^n_1(dr) - \int_{-T}^0 u(s+r) Y_{s+r}\alpha_1(dr)\right| ds\right)^2\right]$$
$$\leq 2E\left[\left(\int_0^T\int_{-T}^0 |u(s+r)| \left|Y^n_{s+r} - Y_{s+r}\right| \alpha^n_1(dr)ds\right)^2\right]$$
$$+ 2E\left[\left(\int_0^T\int_{-T}^0 |u(s+r)| \left|Y_{s+r}\right| (\alpha^n_1 - \alpha_1)(dr)ds\right)^2\right].$$

Using [26, Lemma 1.1], we obtain

$$E\left[\left(\int_0^T\int_{-T}^0 |u(s+r)| \left|Y^n_{s+r} - Y_{s+r}\right| \alpha^n_1(dr)ds\right)^2\right]$$
$$+ E\left[\left(\int_0^T\int_{-T}^0 |u(s+r)| \left|Y_{s+r}\right| (\alpha^n_1 - \alpha_1)(dr)ds\right)^2\right]$$
$$\leq E\left[\left(\int_0^T \alpha^n_1([s-T,0]) |u(s)| \left|Y^n_s - Y_s\right| ds\right)^2\right]$$
$$+ E\left[\left(\int_0^T (\alpha^n_1 - \alpha_1)([s-T,0]) |u(s)| \left|Y_s\right| ds\right)^2\right]$$
$$\leq \left(\alpha^n_1([-T,0])\right)^2 \|u\|^2_{L^1(dt)} \|Y^n - Y\|^2_{\mathcal{S}^2(\mathbb{R}^m)}$$
$$+ \left((\alpha^n_1 - \alpha_1)([-T,0])\right)^2 \|u\|^2_{L^1(dt)} \|Y\|^2_{\mathcal{S}^2(\mathbb{R}^m)}.$$

Similarly, for the control processes we have

$$E\left[\left(\int_0^T \left|\int_{-T}^0 v(s+r) Z^n_{s+r}\alpha^n_2(dr) - \int_{-T}^0 v(s+r) Z_{s+r}\alpha_2(dr)\right| ds\right)^2\right]$$
$$\leq 2\left(\alpha^n_2([-T,0])\right)^2 \|v\|^2_{L^2(dt)} \|Z^n - Z\|^2_{\mathcal{H}^2(\mathbb{R}^{m\times d})}$$
$$+ 2\left((\alpha^n_2 - \alpha_2)([-T,0])\right)^2 \|v\|^2_{L^2(dt)} \|Z\|^2_{\mathcal{H}^2(\mathbb{R}^{m\times d})}.$$

Hence

$$
\begin{aligned}
&\|Y^n - Y\|^2_{\mathcal{S}^2(\mathbb{R}^m)} + \|Z^n - Z\|^2_{\mathcal{H}^2(\mathbb{R}^{m\times d})} \\
&\le 20K^2 \left(\alpha_1^n([-T,0])\right)^2 \|u\|^2_{L^1(dt)} \|Y^n - Y\|^2_{\mathcal{S}^2(\mathbb{R}^m)} \\
&\quad + 20K^2 \left((\alpha_1^n - \alpha_1)([-T,0])\right)^2 \|u\|^2_{L^1(dt)} \|Y\|^2_{\mathcal{S}^2(\mathbb{R}^m)} \\
&\quad + 20K^2 \left(\alpha_2^n([-T,0])\right)^2 \|v\|^2_{L^2(dt)} \|Z^n - Z\|^2_{\mathcal{H}^2(\mathbb{R}^{m\times d})} \\
&\quad + 20K^2 \left((\alpha_2^n - \alpha_2)([-T,0])\right)^2 \|v\|^2_{L^2(dt)} \|Z\|^2_{\mathcal{H}^2(\mathbb{R}^{m\times d})} \\
&\le \frac{4}{5}\|Y^n - Y\|^2_{\mathcal{S}^2(\mathbb{R}^m)} + \frac{4}{5}\|Z^n - Z\|^2_{\mathcal{H}^2(\mathbb{R}^{m\times d})} \\
&\quad + 20K^2 \left((\alpha_1^n - \alpha_1)([-T,0])\right)^2 \|u\|^2_{L^1(dt)} \|Y\|^2_{\mathcal{S}^2(\mathbb{R}^m)} \\
&\quad + 20K^2 \left((\alpha_2^n - \alpha_2)([-T,0])\right)^2 \|v\|^2_{L^2(dt)} \|Z\|^2_{\mathcal{H}^2(\mathbb{R}^{m\times d})}.
\end{aligned}
$$

Therefore, the result follows from the convergence of $\alpha_i^n([-T,0])$, $i = 1,2$. □

The following is a direct consequence of the above stability result. We denote by δ_0 the Dirac measure at 0.

Corollary 4.2.5. *Assume* (A2)*-*(A4)*. For $i = 1,2$ and $n \in \mathbb{N}$ let α_i^n be measures satisfying* (A1) *and* (4.2.3) *in Theorem 4.2.2 and such that $\alpha_i^n([-T,0])$ converges to* 1*. If $\delta_0 \le \alpha_1^n$ (or $\alpha_1^n \le \delta_0$) and $\delta_0 \le \alpha_2^n$ (or $\alpha_2^n \le \delta_0$), then $\|Y^n - Y\|_{\mathcal{S}^2(\mathbb{R}^m)} \to 0$ and $\|Z^n - Z\|_{\mathcal{H}^2(\mathbb{R}^{m\times d})} \to 0$, where (Y^n, Z^n) is the solution of the BSDE* (4.2.1) *with delay given by (α_1^n, α_2^n) and (Y,Z) is the solution of BSDE without delay.*

We conclude this section with the following counterexample which shows that the condition $\alpha_1 \le \alpha_1^n$ (or $\alpha_1^n \le \alpha_1$) and $\alpha_2 \le \alpha_2^n$ (or $\alpha_2^n \le \alpha_2$) is needed in the above theorem.

Example 4.2.6. Assume that $m = d = 1$. We denote by δ_0 and δ_{-1} the Dirac measures at 0 and -1, respectively. It is clear that $\delta_0([-1,0]) = \delta_{-1}([-1,0])$. Consider the delay BSDEs

$$
Y_t = 1 + \int_t^1 1/5 \left(\int_{-1}^0 Y_{s+r} + Z_{s+r} \right) \delta_0(dr)ds - \int_0^1 Z_s\, dW_s \tag{4.2.5}
$$

and

$$
\bar{Y}_t = 1 + \int_t^1 1/5 \left(\int_{-1}^0 \bar{Y}_{s+r} + \bar{Z}_{s+r} \right) \delta_{-1}(dr)ds - \int_0^1 \bar{Z}_s\, dW_s. \tag{4.2.6}
$$

Since BSDE (4.2.6) takes the form $\bar{Y}_t = 1 - \int_t^1 \bar{Z}_u\, dW_s$, it follows that $\bar{Y}_t = 1$ for all $t \in [0,1]$. On the other hand, (4.2.5) is the standard BSDE without delay, its solution can be written as $Y_t = E[H_1^t \mid \mathcal{F}_t]$, where the deflator $(H_s^t)_{s\ge t}$ at time t is given by $dH_s^t = -\frac{H_s^t}{5}(ds + dW_s)$. Thus, $Y_t = \exp(-1/5(1-t))$ and for $t \in [0,1)$, $Y_t < \bar{Y}_t$. ♢

4.3 Reflected BSDEs with time-delayed generators

The probabilistic setting and the notation of the previous section carries over to the present one. In particular, we fix a time horizon $T \in (0, \infty]$ and we assume $m = 1$. For $p \in [1, \infty)$, we further introduce the space $\mathcal{M}^p(\mathbb{R})$ of adapted càdlàg processes X valued in $\mathbb{R}$ such that $\|X\|^p_{\mathcal{M}^p} := E[(\sup_{t\in[0,T]} |X_t|)^p] < \infty$ and by $\mathcal{A}^p(\mathbb{R})$, we denote the subspace of elements of $\mathcal{M}^p(\mathbb{R})$ which are increasing processes starting at 0. Let $(S_t)_{t\in[0,T]}$ be a càdlàg adapted real-valued process. In this section, we study existence of solutions (Y, Z, K) of BSDEs reflected on the càdlàg barrier S and with time-delayed generators. That is, processes satisfying

$$\begin{aligned} Y_t &= \xi + \int_t^T g(s, \Gamma(s))ds + K_T - K_t - \int_t^T Z_s dW_s, \quad t \in [0, T] && (4.3.1) \\ Y &\geq S && (4.3.2) \\ \int_0^T & (Y_{t-} - S_{t-})dK_t = 0 && (4.3.3) \end{aligned}$$

with Γ defined by (4.2.2). Consider the condition

(A5) $E\left[\sup_{0\leq t\leq T}(S_t^+)^2\right] < \infty$ and $S_T \leq \xi$.

Theorem 4.3.1. *Assume* (A1)*-*(A5)*. If*

$$\begin{cases} K^2\alpha_1^2([-T,0])\,\|u\|^2_{L^1(dt)} \leq \frac{1}{36}, \\ K^2\alpha_2^2([-T,0])\,\|v\|^2_{L^2(dt)} \leq \frac{1}{36}, \end{cases} \tag{4.3.4}$$

then RBSDE (4.3.1) *admits a unique solution* $(Y, Z, K) \in \mathcal{M}^2(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R}^d) \times \mathcal{A}^2(\mathbb{R})$ *satisfying*

$$Y_t = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_t} E\left[\int_t^\tau g(s, \Gamma(s))ds + S_\tau \mathbf{1}_{\{\tau<T\}} + \xi\mathbf{1}_{\{\tau=T\}}\middle|\mathcal{F}_t\right],$$

where $\mathcal{T}$ *is the set of all stopping times taking values in* $[0, T]$ *and* $\mathcal{T}_t = \{\tau \in \mathcal{T} : \tau \geq t\}$.

Proof. For any given $(y, z) \in \mathcal{M}^2(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R}^d)$, similar to the proof of Lemma 4.2.3, we have

$$E\left(\xi + \int_0^T g(s, \gamma(s))ds\right)^2 < \infty$$

with γ defined as in Lemma 4.2.3. Hence, from [50, Theorem 3.3] for $T < \infty$ and [1, Theorem 3.1] for $T = \infty$ the reflected BSDE

$$Y_t = \xi + \int_t^T g(s, \gamma(s))ds + K_T - K_t - \int_t^T Z_s dW_s$$

with barrier S admits a unique solution (Y, Z, K) such that $(Y, Z) \in \mathcal{B}$, the space of processes $(Y, Z) \in \mathcal{M}^2(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R}^d)$ such that $Y \geq S$, and $K \in \mathcal{A}^2(\mathbb{R})$. Moreover, Y admits the representation

$$Y_t = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} E\left[\int_t^\tau g(s, \gamma(s))ds + S_\tau \mathbf{1}_{\{\tau < T\}} + \xi \mathbf{1}_{\{\tau = T\}} \middle| \mathcal{F}_t\right] \quad t \in [0, T].$$

Hence we can define a mapping Φ from $\mathcal{B}$ to $\mathcal{B}$ by setting $\Phi(y, z) := (Y, Z)$. Let $(y, z), (\bar{y}, \bar{z}) \in \mathcal{B}$ and $(Y, Z) = \Phi(y, z)$, $(\bar{Y}, \bar{Z}) = \Phi(\bar{y}, \bar{z})$. From the representation, we deduce

$$\begin{aligned}
&|Y_t - \bar{Y}_t| \\
&\leq \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} E\left[\int_t^\tau |g(s, \gamma(s)) - g(s, \bar{\gamma}(s))| ds \middle| \mathcal{F}_t\right] \\
&\leq E\left[\int_0^T |g(s, \gamma(s)) - g(s, \bar{\gamma}(s))| ds \middle| \mathcal{F}_t\right].
\end{aligned}$$

Doob's maximal inequality implies that

$$E\left[\sup_{0 \leq t \leq T} |Y_t - \bar{Y}_t|^2\right] \leq 4E\left[\left(\int_0^T |g(s, \gamma(s)) - g(s, \bar{\gamma}(s))| ds\right)^2\right].$$

Applying Itô's formula to $|Y_t - \bar{Y}_t|^2$, we obtain

$$\begin{aligned}
|Y_t - \bar{Y}_t|^2 + \int_t^T |Z_s - \bar{Z}_s|^2 ds &= 2\int_t^T (Y_s - \bar{Y}_s)(g(s, \gamma(s)) - g(s, \bar{\gamma}(s)))ds \\
&+ 2\int_t^T (Y_{s-} - \bar{Y}_{s-})d(K_s - \bar{K}_s) - 2\int_t^T (Y_s - \bar{Y}_s)(Z_s - \bar{Z}_s)dW_s \\
= 2\int_t^T (Y_s - \bar{Y}_s)(g(s, \gamma(s)) &- g(s, \bar{\gamma}(s)))ds - 2\int_t^T (Y_s - \bar{Y}_s)(Z_s - \bar{Z}_s)dW_s \\
&+ 2\int_t^T (Y_{s-} - S_{s-})dK_s - 2\int_t^T (Y_{s-} - S_{s-})d\bar{K}_s - 2\int_t^T (\bar{Y}_{s-} - S_{s-})dK_s \\
&+ 2\int_t^T (\bar{Y}_{s-} - S_{s-})d\bar{K}_s.
\end{aligned}$$

Since (Y, K) and $(\bar{Y}, \bar{K})$ satisfy (4.3.2) and (4.3.3), we have

$$|Y_t - \bar{Y}_t|^2 + \int_t^T |Z_s - \bar{Z}_s|^2 ds \leq 2 \int_t^T (Y_s - \bar{Y}_s)(g(s, \gamma(s)) - g(s, \bar{\gamma}(s)))ds \\ - 2 \int_t^T (Y_s - \bar{Y}_s)(Z_s - \bar{Z}_s) dW_s.$$

Hence

$$E\left[\int_0^T |Z_s - \bar{Z}_s|^2 ds\right] \\ \leq E\left[\sup_{0 \leq t \leq T} |Y_t - \bar{Y}_t|^2\right] + E\left[\left(\int_0^T |g(s, \gamma(s)) - g(s, \bar{\gamma}(s))| ds\right)^2\right].$$

In view of the proof of Lemma 4.2.3, we deduce

$$\|Y - \bar{Y}\|^2_{\mathcal{M}^2(\mathbb{R})} + \|Z - \bar{Z}\|^2_{\mathcal{H}^2(\mathbb{R}^d)} \leq 9E\left[\left(\int_0^T |g(s, \gamma(s)) - g(s, \bar{\gamma}(s))| ds\right)^2\right] \\ \leq 18K^2 \alpha_1^2([-T, 0]) \|u\|^2_{L^1(dt)} \|y - \bar{y}\|^2_{\mathcal{M}^2(\mathbb{R})} \\ + 18K^2 \alpha_2^2([-T, 0]) \|v\|^2_{L^2(dt)} \|z - \bar{z}\|^2_{\mathcal{H}^2(\mathbb{R}^d)}.$$

By condition (4.3.4), Φ is a contraction mapping and therefore it admits a unique fixed point which combined with the associated process K is the unique solution of the RBSDE (4.3.1). □

4.4 Quadratic and superquadratic BSDEs with delay in value process

In this section, we study quadratic and superquadratic BSDEs with delay in value process through the connection between BSDEs with time-delayed generators and FBSDEs. We work in the probabilistic setting and with the notation of Section 4.2.

Standard methods to solve BSDEs with quadratic growth in the control variable often rely either on boundedness of the control process, see for instance [69] and [16], or on BMO estimates for the stochastic integral of the control process, see for instance [73]. However, as shown in [24], solutions of BSDEs with time-delayed generators do not, in general, satisfy boundedness and BMO properties so that new methods are required to solve quadratic BSDE with time-delayed generators. Recently, [13] obtained existence and uniqueness of solution for a quadratic BSDE with delay only in the value process. We show below that using FBSDE theory, it is possible to generalize their results to multidimension and considering a different kind of delay. Moreover, our argument allows to solve equations with generators of superquadratic growth.

Let α_1 be the uniform measure on $[-T, 0]$, α_2 the Dirac measure at 0. Put $u(s) = v(s) = 1$, for $s \in [0, T]$. We are considering the following BSDE with time delay only in the value process:

$$Y_t = \xi + \int_t^T g(s, \int_0^s Y_r dr, Z_s) ds - \int_t^T Z_s dW_s, \quad t \in [0, T]. \tag{4.4.1}$$

We denote by $\mathcal{D}^{1,2}$ the space of all Malliavin differentiable random variables and for $\xi \in \mathcal{D}^{1,2}$ denote by $D_t\xi$ its Malliavin derivative. We refer to Nualart [61] for a thorough treatment of the theory of Malliavin calculus, whereas the definition and properties of the BMO-space and norm can be found in [47]. We make the following assumptions:

(B1) $g : [0, T] \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \to \mathbb{R}^m$ is a continuous function such that $g^i(y, z) = g^i(y, z^i)$ and there exists a constant $K > 0$ as well as a nondecreasing function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|g(s, y, z) - g(s, y', z')| \leq K|y - y'| + \rho(|z| \vee |z'|)|z - z'|,$$
$$|g(s, y, z) - g(s, y', z) - g(s, y, z') + g(s, y', z')| \leq K(|y - y'| + |z - z'|)$$

for all $s \in [0, T]$, $y, y' \in \mathbb{R}^m$ and $z, z' \in \mathbb{R}^{m\times d}$.

(B2) ξ is $\mathcal{F}_T$-measurable such that $\xi \in \mathcal{D}^{1,2}(\mathbb{R}^m)$ and there exist constants $A_{ij} \geq 0$ such that

$$|D_t^j \xi^i| \leq A_{ij}, \quad i = 1, \ldots, m;\ j = 1, \ldots, d,$$

for all $t \in [0, T]$.

(B3) $g : \Omega \times [0, T] \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$ is measurable, $g(s, y, z) = f(s, z) + l(s, y, z)$ where f and l are measurable functions with $f^i(s, z) = f^i(s, z^i)$, $i = 1, \ldots, m$ and there exists a constant $K \geq 0$ such that

$$|f(s, z) - f(s, z')| \leq K(1 + |z| + |z'|)|z - z'|,$$
$$|l(s, y, z) - l(s, y', z')| \leq K|y - y'| + K(1 + |z|^\epsilon + |z'|^\epsilon)|z - z'|,$$
$$|f(s, z)| \leq K(1 + |z|^2),$$
$$|l(s, y, z)| \leq K(1 + |z|^{1+\epsilon}),$$

for some $0 \leq \epsilon < 1$ and for all $s \in [0, T]$, $y, y' \in \mathbb{R}^m$ and $z, z' \in \mathbb{R}^{m\times d}$.

(B4) ξ is $\mathcal{F}_T$-measurable such that there exist a constant $K \geq 0$ such that $|\xi| \leq K$.

(B5) $g : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is progressively measurable, continuous process for any choice of the spatial variables and for each fixed $(s,\omega) \in [0,T] \times \Omega$, $g(s,\omega,\cdot)$ is continuous. g is increasing in y and for some constant $K \geq 0$ such that

$$|g(s,y,z)| \leq K(1+|z|),$$

for all $s \in [0,T]$, $y \in \mathbb{R}$ and $z \in \mathbb{R}^d$.

(B6) ξ is $\mathcal{F}_T$-measurable such that $\xi \in L^2$.

(B7) $g : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is progressively measurable, continuous process for any choice of the spatial variables and for each fixed $(s,\omega) \in [0,T] \times \Omega$, $g(s,\omega,\cdot)$ is continuous. g is increasing in y and for some constant $K \geq 0$ such that

$$|g(s,y,z)| \leq K(1+|z|^2),$$

for all $s \in [0,T]$, $y \in \mathbb{R}$ and $z \in \mathbb{R}^d$.

Proposition 4.4.1. *Assume $T \in (0,\infty)$.*

1. *If* (B1)*-*(B2) *are satisfied, then there exists a constant $C \geq 0$ such that for sufficiently small T, BSDE* (4.4.1) *admits a unique solution $(Y,Z) \in \mathcal{S}^2(\mathbb{R}^m) \times \mathcal{H}^2(\mathbb{R}^{m\times d})$ such that $|Z| \leq C$.*

2. *If* (B3)*-*(B4) *are satisfied, then there exist constants $C_1, C_2 \geq 0$ such that for sufficiently small T, BSDE* (4.4.1) *admits a unique solution $(Y,Z) \in \mathcal{S}^2(\mathbb{R}^m) \times \mathcal{H}^2(\mathbb{R}^{m\times d})$ such that $|Y| \leq C_1$ and $\|Z \cdot dW\|_{BMO} \leq C_2$.*

3. *If $m = d = 1$ and* (B5)*-*(B6) *are satisfied, then BSDE* (4.4.1) *admits at least a solution $(Y,Z) \in \mathcal{S}^2(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R}^d)$.*

4. *If $m = d = 1$ and* (B4) *and* (B7) *are satisfied, then BSDE* (4.4.1) *admits at least a solution $(Y,Z) \in \mathcal{S}^2(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R}^d)$ such that Y is bounded and $Z \cdot W$ is a BMO martingale.*

Proof. Define the function $b : \mathbb{R}^m \to \mathbb{R}^m$ by setting for $y \in \mathbb{R}^m$, $b^i(y) = y^i$, $i = 1,\ldots,m$. For $t \in [0,T]$, put

$$X_t = \int_0^t b(Y_s)ds.$$

Thus BSDE (4.4.1) can be written as the coupled FBSDE

$$\begin{cases} X_t = \int_0^t b(Y_s)ds, \\ Y_t = \xi + \int_t^T g(s, X_s, Z_s)ds - \int_t^T Z_s dW_s \end{cases} \tag{4.4.2}$$

so that 1. and 2. follow from chapter 3, and 3. and 4. from [5]. □

The above theorem provides an explanation why it is not enough to solve a time-delayed BSDE backward in time, one actually needs to consider both the forward and backward parts of the solution due to the delay.

Appendix A

Appendix

A.1 BMO martingales

We recall some results and properties of BMO martingales, for a thorough treatment, we refer to Kazamaki [47]. For any uniformly integrable martingale M with $M_0 = 0$ and $p \in [1, \infty)$, define

$$\|M\|_{BMO_p} := \sup_{\tau \in \mathcal{T}} \|E[\langle M \rangle_T - \langle M \rangle_\tau |^{\frac{p}{2}} \mathcal{F}_\tau]^{\frac{1}{p}}\|_\infty.$$

We will use $BMO_p(P)$ when it is necessary to indicate the underlying probability measure, and just write BMO when $p = 2$. We recall the following results from the literature.

Lemma A.1.1. *Let M be a BMO martingale. Then we have:*

(1) The stochastic exponential $\mathcal{E}(M)$ is uniformly integrable.

(2) There exists a number $r > 1$ such that $\mathcal{E}(M)_T \in L^r$. This property follows from the Reverse Hölder inequality. The maximal r with this property can be expressed explicitly in terms of the BMO norm of M. There exists as well an upper bound for $\|\mathcal{E}(M)_T\|_{L^r}^r$ depending only on T, r and the BMO norm of M.

(3) For probability measures P and Q satisfying $dQ = \mathcal{E}(M)_T dP$ for $M \in BMO(P)$, the process $\hat{M} = M - \langle M \rangle$ is a $BMO(Q)$ martingale.

(4) Energy inequalities imply the inclusion $BMO \subset \mathcal{H}^p$ for all $p \geq 1$. More precisely, for $M = \int \alpha dW$ with BMO norm C, the following estimate holds

$$E\left[\left(\int_0^T |\alpha_s|^2 ds\right)^p\right] \leq 2p!(4C^2)^p.$$

Lemma A.1.2 (John-Nierenberg inequalities). *Let M be a local martingale such that $M_0 = 0$.*

(i) If $\|M\|_{BMO_1} < 1/4$, then for any stopping time $\tau \in \mathcal{T}$

$$E\left[\exp\left(|M_T - M_\tau|\right)\middle|\mathcal{F}_\tau\right] \leq \frac{1}{1 - 4\|M\|_{BMO_1}}.$$

(ii) If $\|M\|_{BMO} < 1$, then for any stopping time $\tau \in \mathcal{T}$

$$E\left[\exp\left(\langle M\rangle_T - \langle M\rangle_\tau\right)\middle|\mathcal{F}_\tau\right] \leq \frac{1}{1 - \|M\|^2_{BMO}}.$$

Lemma A.1.3. *For $K > 0$, there are constants $c_1 > 0$ and $c_2 > 0$ such that for any BMO martingale M, we have for any BMO martingale N such that $\|N\|_{BMO(P)} \leq K$,*

$$c_1\|M\|^2_{BMO(P)} \leq \|\tilde{M}\|^2_{BMO(\tilde{P})} \leq c_2\|M\|^2_{BMO(P)}$$

where $\tilde{M} := M - \langle M, N\rangle$ and $\frac{d\tilde{P}}{dP} := \mathcal{E}_T(N)$.

Define

$$\Phi(x) := \left\{1 + \frac{1}{x^2}\log\frac{2x-1}{2(x-1)}\right\}^{\frac{1}{2}} - 1, \quad x > 1.$$

By Lemma 2.4 in [41], the constants in the previous lemma are given by

$$c_1 = \frac{1}{L^4_{2\bar{q}}C_{\bar{p}}^{\frac{2}{\bar{p}}}},$$
$$c_2 = L^4_{2q}C_p^{\frac{2}{p}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $\frac{1}{\bar{p}} + \frac{1}{\bar{q}} = 1$, C_p and $C_{\bar{p}}$ are given by Lemma A.1.5, L_{2q} and $L_{2\bar{q}}$ are given by Lemma A.1.4, and $p, \bar{p}$ are constants such that $\Phi(p) > K$ and $\Phi(\bar{p}) > \bar{K}$, where $\bar{K} = \sqrt{2(q-1)\log(C_p + 1)}$.

Lemma A.1.4. *Let $1 < p < \infty$. There is a positive constant L_p such that for any uniformly integrable martingale M*

$$\|M\|_{BMO_1} \leq \|M\|_{BMO_p} \leq L_p\|M\|_{BMO_1}.$$

If $p \in \mathbb{N}$, L_p is given by $8 \cdot 2^{1/p}(p!)^{1/p}$.

Lemma A.1.5. *Let $1 < p < \infty$. If $\|M\|_{BMO_2} < \Phi(p)$, then*

$$E\left[\mathcal{E}_T(M)^p\middle|\mathcal{F}_\tau\right] \leq C_p\mathcal{E}_\tau(M)^p$$

for any stopping time $\tau \in \mathcal{T}$ with a constant C_p depending only on p. Indeed, $C_p = \frac{2}{1-2(p-1)(2p-1)^{-1}\exp\{p^2 n(M)\}}$ with $n(M) = 2\|M\|_{BMO_1} + \|M\|^2_{BMO_2}$.

A.2 Malliavin Calculus

We briefly recall some definitions and results in the theory of Malliavin calculus. We refer to Nualart [61] for a thorough treatment. Let $\mathcal{S}$ be the class of smooth random variables of the form

$$\xi = F\left(\int_0^T h_s^1\,dW_s,\dots,\int_0^T h_s^m\,dW_s\right)$$

where $F \in C_p^\infty(\mathbb{R}^{m\times d})$, the space of infinitely continuously differentiable functions whose partial derivatives have polynomial growth, and $h^1,\dots,h^m \in L^2([0,T];\mathbb{R}^d)$. For any $\xi \in \mathcal{S}$, consider the operator $D = (D^1,\dots,D^d) : \mathcal{S} \to L^2(\Omega\times[0,T])$ given by

$$D_t^i\xi := \sum_{j=1}^m \frac{\partial F}{\partial x_{i,j}}\left(\int_0^T h_s^1\,dW_s,\dots,\int_0^T h_s^m\,dW_s\right)h_t^{i,j}, \quad 0\le t\le T,\ 1\le i\le d$$

and the norm $\|\xi\|_{1,2} := (E[|\xi|^2 + \int_0^T |D_t\xi|^2\,dt])^{1/2}$. As shown in Nualart [61], the operator D extends to the closure $\mathcal{D}^{1,2}$ of the set $\mathcal{S}$ with respect to the norm $\|\cdot\|_{1,2}$. A random variable ξ will be said to be Malliavin differentiable if $\xi \in \mathcal{D}^{1,2}$ and we will denote by $D_t\xi$ its Malliavin derivative. Note that if ξ is $\mathcal{F}_t$ measurable, then $D_u\xi = 0$ for all $u \in (t,T]$.

The following result is the chain rule ([61, Proposition 1.2.4]).

Proposition A.2.1. *Let $\varphi : \mathbb{R}^m \to \mathbb{R}$ be a function such that*

$$|\varphi(x)-\varphi(y)| \le K|x-y|$$

for any $x, y \in \mathbb{R}^m$. Suppose that $F = (F^1,\dots,F^m)$ is a random vector whose components belong to the space $\mathcal{D}^{1,2}$. Then $\varphi(F) \in \mathcal{D}^{1,2}$, and there exists a random vector $G = (G_1,\dots,G_m)$ bounded by K such that

$$D\left(\varphi(F)\right) = \sum_{i=1}^m G_i DF^i.$$

By $\mathcal{L}_a^{1,2}(\mathbb{R}^{m'})$, we denote the space of processes $X \in \mathcal{H}^2(\mathbb{R}^{m'})$ such that $X_t \in (\mathcal{D}^{1,2})^{m'}$ for all $t \in [0,T]$, the process $DX_t(\omega)$ admits a square integrable progressively measurable version and

$$\|X\|^2_{\mathcal{L}_a^{1,2}} := \|X\|_{\mathcal{H}^2} + \left\|\left(\int_0^T\int_0^T |D_rX_t|^2\,dr\,dt\right)^{1/2}\right\|_{L^2} < \infty.$$

Let $\{X_t, t \in [0,T]\}$ be the solution of the following SDE

$$X_t = x + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, \quad x \in \mathbb{R}^m.$$

Then we have the following result.

Proposition A.2.2. *Suppose that b, σ are globally Lipschitz continuous functions with linear growth and continuously differentiable. Then $X_t \in \left(\mathcal{D}^{1,2}\right)^m$ for any $t \in [0,T]$ and the derivative $D_r X_t$ satisfies for $0 \leq 0 \leq t \leq T$ the SDE*

$$D_r X_t = \sigma(r, X_r) + \int_r^t \nabla_x b(s, X_s) D_r X_s ds + \int_r^t \nabla_x \sigma(s, X_s) D_r X_s dW_s.$$

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