Dual Representation of Convex Increasing Functionals with Applications to Finance

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Presented by
LUDOVIC TANGPI NDOUNKEU

at the

Universität Konstanz

Faculty of Sciences
Department of Mathematics and Statistics

Accepted on the recommendation of
Prof. Dr. Michael Kupper, examinar
Prof. Dr. Halil Mete Soner, co-examinar
Prof. Dr. Robert Denk, co-examinar

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Abstract

This thesis deals with the dual representation of various nonlinear functionals and provides applications to financial mathematics under model uncertainty.

In the first part of the thesis, we begin by assuming that a fixed reference probability measure is given, and we work on a Brownian filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$. In this setting, our study of dual representation focuses on minimal supersolutions of backward stochastic differential equations (BSDEs) with convex generators. These are convex increasing functionals on a space of non-bounded, but integrable random variables. We derive a dual representation under weak requirements on the generator of the equation. On the other hand, we show that any dynamic risk measure satisfying such a representation stems from a BSDE. As an application, we study the utility maximization problem of an agent with non-zero endowment, and whose preferences are modeled by the maximal subsolution of a BSDE. We prove existence of an optimal trading strategy and relate our existence result to the existence of a maximal subsolution to a controlled decoupled FB-SDE. Using BSDE duality, we show that the utility maximization problem can be seen as a robust control problem admitting a saddle point if the generator of the BSDE additionally satisfies a quadratic growth condition. It is then shown that any saddle point of the robust control problem agrees with a primal and a dual optimizer of the utility maximization problem, and can be characterized in terms of the solution of a BSDE.

In the second part of the thesis, we drop the assumption of existence of a reference measure, and work on a topological space Ω which is not assumed to be compact. We give two sorts of conditions guaranteeing the dual representation of convex increasing functionals defined on a space of random variables with respect to countably additive measures. The first conditions, which can be viewed as sequential upper semicontinuity assumptions ensure a max-representation on a Stone vector lattice of continuous random variables. The second condition, which can be viewed as sequential lower semicontinuity assumptions yield a sup-representation on the set of bounded upper semicontinuous random variables; and we characterize functionals admitting a representation on the space of bounded measurable random variables. As applications, we derive a version of the fundamental theorem of asset pricing in continuous and discrete time, and for a market allowing static investments in infinitely many options. We introduce a market efficiency condition stronger than "No Free Lunch With Vanishing Risk" which ensures existence of martingale or local martingale measures for continuous or even càdlàg price processes. On the other hand, we allow trading only in the so-called simple strategies.

Zusammenfassung

Diese Doktorarbeit behandelt die duale Repräsentation von gewissen nichtlinearen Erwartungen und Anwendungen in der Finanzmathematik unter modelunsicherheit.

Im ersten Teil der Arbeit nehmen wir an, dass ein Referenzwahrscheinlichkeitsmaß gegeben ist, und arbeiten auf einem Wahrscheinlichkeitsraum $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ mit einem Brownschen filtration. In diesem Rahmen studieren wir die duale Repräsentation von minimalen Superlösungen von stochastischen Rückwärtsdifferentialgleichungen (BSDEs) mit konvexen Generatoren. Diese sind konvexe und wachsende Funktionale auf dem Raum der unbeschränkten integrierbaren Zufallsvariablen. Andererseits beweisen wir, dass jedes dynamische Risikomaß, welches eine solche Repräsentation erlaubt, einer BSDE entstammt. Diese Ergebnisse wenden wir an, um die Nutzenmaximierung eines Agenten mit nichttrivialer Ausstattung zu studieren, dessen Präferenz durch die maximale Sublösung einer BSDE modelliert ist. Wir beweisen die Existenz einer optimalen Handelsstrategie und stellen den Zusammenhang zur Existenz einer maximalen Sublösung einer kontrolliert entkoppelten FBSDE her. Mithilfe der BSDE-Dualität zeigen wir, dass das Nutzenmaximierungsproblem als ein robustes Kontrollproblem aufgefasst werden kann, das einen Sattelpunkt besitzt, sofern der Generator der BSDE zusätzlich eine quadratische Wachstumsbedingung erfüllt. Es wird gezeigt, dass jeder Sattelpunkt des robusten Kontrollproblems mit dem primalen und dualen Optimierer des Nutzenmaximierungsproblems übereinstimmt und durch die Lösung einer BSDE charakterisiert werden kann.

Im zweiten Teil die Dissertation lassen wir die Annahme der Existenz einer Referenzwahrscheinlichkeit fallen und arbeiten auf einem nichtkompakten topologischen Raum Ω. Wir präsentieren zwei Arten von Bedingungen, die es erlauben, die duale Repräsentation von konvexen wachsenden Funktionalen über einem Raum von Zufallsvariablen durch sigma-additive Maße zu erhalten. Die Bedingungen erster Art, die als Folgenoberhalbstetigkeitsannahme gesehen werden können, garantieren die Max-Repräsentation auf einem Stoneschen Vektorverband von stetigen Zufallsvariablen. Die Bedingungen zweiter Art, die als Folgenunterhalbstetigkeitsannahme gesehen werden können, sichern die Sup-Repräsentation auf einer Menge von beschränkten messbaren Zufallsvariablen. Wir wenden diese Repräsentation an, um eine Version des FTAP in stetiger Zeit und für ein Markt, der statische Investments in unendlich vielen Optionen erlaubt, herzuleiten. Wir führen eine Markteffizienzbedingung ein, die stärker als "No Free Lunch With Vanishing Risk" ist und welchem die Existenz eines Martingalßes oder lokalen Martingalmaßes für stetige oder càdlàg Preisprozesse garantiert. Dabei lassen wir nur das Handeln in sogenannten einfachen Strategien zu.

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Dedications

To my caring parents

Contents

Chapter 1

Introduction

A two-fold character or nature of concepts, people or structures is often referred to as dual. The term is also used to describe two paired items that mirror one another. Duality –or the property of being dual– is frequent around us and is studied in several scientific areas ranging from philosophy, sociology to logic or physics, one of the most prominent examples here probably being the wave-particle duality. It is in the context of projective geometry that duality first appeared in mathematics, in the 1822 treatise "*Traité des proprietés projective des figures*" of Poncelet. Although duality is now widely studied in almost every area of mathematics, there is not a clear mathematical statement defining the concept. Very often, to a given mathematical object (primal), one can associate a related object (dual) sometimes linked by a pairing. The strength of duality theory lies in the fact that the dual object helps to better understand the primal. This form of duality appears, for instance, in Riesz representation theory; topological vector space; distribution theory (very linked to wave-particle duality) and convex analysis.

In this Ph.D. thesis we focus on the duality of solutions of backward stochastic differential equations (BSDEs) and general convex increasing functionals, with applications to financial modeling.

One important question in financial modeling concerns optimal decision making in a random environment. At the beginning of the twentieth century, Frank Knight introduced the concept of model uncertainty, which represents the fact that one may have only knowledge of implausible events, but not of the actual probabilistic model describing the random environment. It is this situation that concerns us in our study of financial mathematics. Here, two further distinctions need to be made: whether the implausible events coincide with the null sets of a single probability measure sometimes called the reference measure (the set of possible models is dominated) or not (the set of possible models is non-dominated). This distinction between the dominated and the non-dominated case will play a key role in our investigations on duality and the applications thereof.

On a probability space, that is when a reference probability measure is known, a BSDEs usually takes the form

$$Y_t = X + \int_t^T g(Y_u, Z_u)\,du - \int_t^T Z_u\,dW_u, \quad 0 \le t \le T,$$

for a given random variable X and function g, respectively called the terminal condition and the generator of the equation, where the equality should hold in the almost sure sense for a given probability measure. The process W governs the stochasticity of the dynamics and the processes (Y, Z) with the right measurability and integrability constitute, when they exist, the solution of the equation. BSDEs were first introduced for linear generators by Bismut [13] as adjoint equations in stochastic optimization problems. They were later generalized to non-linear BSDEs by Pardoux and Peng [81] and Kobylanski [73]. BSDEs have proven to be a powerful tool to solve and model a number of economical problems. For instance, in Hu et al. [68] they are used to solve expected utility maximization problems with a terminal random endowment, in Duffie and Epstein [50] and El Karoui et al. [56] they are used to model agent preferences, in Cheridito et al. [20] they are used to solve an equilibrium pricing problem. These numerous applications have triggered a strong interest in the topic on the theoretical side. A crucial question concerns the existence of solutions of BSDEs. As shown in Delbaen et al. [39], BSDEs are typically ill-posed beyond the quadratic-growth generators investigated by Kobylanski [73] and Briand and Hu [17]. This motivated the study of a weaker form of solutions, namely supersolutions by Drapeau et al. [47]. These can be thought of as superhedging prices and superhedging strategies with non-linear transaction cost of a contingent claim, see also El Karoui et al. [55]. It has been shown that for convex generators that are bounded from below, the existence of a supersolution is sufficient for the existence of a minimal supersolution, see Drapeau et al. [47] and Heyne et al. [65] where the result is extended to non-convex generators. Furthermore, the operator mapping a given terminal condition to the value of the minimal supersolution of a BSDE can be seen as a non-linear expection, similarly to Peng [83]'s g -expectation which maps the terminal condition to the value at a given time of the process Y in the solution. In this thesis we study the duality theory for BSDEs, we investigate which mathematical objects can be seen to be in duality with the processes Y and Z, respectively. Moreover, we study conditions under which separation-type theorems for minimal supersolutions can be obtained.

Under specific conditions on the generator, the g -expectation can be seen as a monetary utility function in the sense of Delbaen [33], see also Rosazza Gianin [94]. That is, a concave, increasing and cash-additive functional on random outcomes. Hence, it is suited to represent the preferences of economic agents. von Neumann and Morgenstern [98] stated a set of rules followed by a *reasonable* person when making a decision; and by the work of Savage [95], under these axioms, preferences can be modeled by expected utility. Since the axioms of von Neumann and Morgenstern have been much criticized by empirical studies, a persistent question in financial mathematics and economic theory has been to find a consistent way to numerically model decision making beyond expected utility. Some alternatives proposed by the academic community include weighted expected utility, Choquet expectation and more recently recursive utility and stochastic differential utility, see Duffie and Epstein [50]. In the present work, we argue that a weaker form of BSDE solutions, namely the maximal subsolution of a BSDE can be used to model preferences of an agent. We study the question of optimal investment of an agent with a concave utility, and investigate the link with the optimization of stochastic differential utilities under (dominated) model uncertainty through duality of BSDEs.

Decision making can also be made by measuring the risk associated to a position on a random outcome. Risk measures are quantitative tools with specific economical features which assign to a future random position a number. These mappings are used, for instance, for hedging purposes and decision making and therefore, are of crucial importance for financial institutions and financial regulatory agencies. Real valued monotone and translation invariant functionals are known as niveloid, see Dolecki and Greco [42]. Risk measures were first introduced by Artzner et al. [3] as minimal capital requirement for regulatory agencies in the form of coherent risk measures. These are convex and positive homogeneous niveloids on the set $L^\infty(P)$ of essentially bounded random variable with respect to the measure P. A robust representation of such risk measures was obtained by Delbaen [32]. Dropping the positive homogeneity axiom, the more general concept of convex risk measures with their representation was introduced by Föllmer and Schied [59] and Frittelli and Rosazza Gianin [61]. Due to the fact that, when computing the risk of a future random position convex risk measures do not encompass the risk incurred by the uncertainty on the time value of money, [54] introduced cash subadditive convex risk measures. When the probabilistic model is known, research on risk measures has been extended in several directions. These include risk measures for processes, see Kupper [75]; dynamic risk measures and time consistency, see Cheridito et al. [19]; law invariant risk measures, see Jouini et al. [69]; quasi convex risk measures, see Drapeau [45].

The study of robust representation of risk measures under model uncertainty, and more precisely when scenarios can be singular measures, is still in its infancy. This constitutes the main theme of the second part of the thesis where, we study representation of general convex increasing functionals. We provide, in addition, applications in financial modeling.

The representation of a positive linear functional ϕ with countable additive measures is a well studied question in analysis. For instance, in the case where the sample space Ω is a compact topological space, the representation on the set of continuous random variables follows from the Riesz representation theorem. When Ω is not a topological space, the representation can be obtained on any Stone vector lattice of random variables from the theorem of Daniell-Stone. These representation results will constitute the main building blocks in our study of representation of general convex increasing functionals. Note that in both cases, ϕ is required to satisfy some continuity condition. To the best of our knowledge, the earliest treatment of the question of robust representation of real valued convex increasing functionals is due to Föllmer and Schied [58]. They introduced a tightness condition on the risk measure that is equivalent to the so called continuity from below, and under which the robust representation holds on the set of bounded continuous random variables on a separable metric space. They represent the risk measure as the worst penalized expected value of the position over every σ-additive probability measures. For subadditive risk measures defined on the set of random variables that are continuous on a finite dimensional Euclidean space, the representation was given by Peng [84]. These results can be seen as nonlinear extensions of the Daniell-Stone theorem. See also Vioglio et al. [97] for a similar representation result on the set of bounded measurable functions and under a sequential continuity condition. In the quasi-sure consideration, the null sets are modeled by a non-dominated set of probability measures. Nonlinear expectations, which differ from coherent risk measures only by a sign, have been studied in this framework. A particularly relevant example constitutes the G -expectation of Shige Peng introduced in Peng [84] and which has given rise to a form of stochastic analysis under model uncertainty known as G -framework and are closely related to the modeling of volatility uncertainty in finance. Using capacity-theoretic techniques, Bion-Nadal and Kervarec [12] obtained a representation of convex risk measures on the closure of the set of bounded continuous random variables with respect to the topology induced by the semi-norm capacity. Nutz and Soner [78] have extended G -expectations to dynamic non-linear expectation, and analogous to the standard case where a probabilistic model is given, they showed that dynamic risk measures can be linked to solutions of second order BSDEs, a natural generalization of BSDEs under model uncertainty.

With regard to the duality of solutions of BSDE in robust non-dominated models, let us focus on zero-generator BSDEs. In this case, the minimal supersolution of the BSDE can be seen as the the minimal superhedging price of a contingent claim under model uncertainty. The dual representation of the minimal superhedging price provides a foundation for the pricing and hedging of claims and allows for numerical approximations. A typical example of such financial model is that of a stock price process modeled with uncertain volatility. One usually takes as set of valid models a set of measures under which the price process is a martingale and its quadratic variation belongs to a fixed interval, this yields a non-dominated set of models. In this situation, a pioneering result is due to Denis and Martini [40], where the canonical process on the canonical space is considered as stock prices and the minimal superhedging price of claims belonging to the set of continuous bounded random variables is represented as the worst expectation of the claim under martingale measures. This result was later extended to the more general class of bounded measurable claims notably by Neufeld and Nutz [77] and Possamaï et al. [86]. In financial applications, one can take advantage of the knowledge of the price of highly liquid derivative securities to gain more insight on the probabilistic structure of the market. This is done by investing statically in the derivative market and dynamically in the stock market. In fact it has been shown that these semi-static hedging strategies are in strong duality with martingale measures consistent with the prices of vanilla options. Since it follows from a result of Breeden and Litzenberger [16] that the knowledge of the prices of vanilla options at all strikes and at a fixed maturity gives the marginal distribution of the underlying stock, consistent martingale measures can be seen as measures with known marginal distribution. The duality in this framework was first established in discrete time by Beiglböck et al. [7] who introduced the martingale transport problem with finitely many given marginal constraints and then extended to continuous time by Dolinsky and Soner [43] and Dolinsky and Soner [44]. Using the fact that martingales can be represented as time-changed Brownian motions this problem has also been studied using various solutions of the Skorokhod Embedding Problem, see for instance the surveys Obłój [79] and Hobson [66] where model-independent arbitrage-free bounds of some exotic options are computed.

The existence of these arbitrage-free prices, induced by martingale measures for the stock price process, are not a priori guaranteed. This existence pertains to a sense of fairness of the market. Defining a meaningful concept of fairness and the characterization thereof, also known as the fundamental theorem of asset pricing (FTAP) is a central question in mathematical finance. We refer to Delbaen and Schachermayer [36] for the state-of-the-art results on the topic when the market is governed by a fixed measure. Under Knightian uncertainty, the discrete-time case has been investigated by Acciaio et al. [1] who introduced the strong notion of model-independent arbitrage and proved a FTAP allowing semi-static trading in infinitely many options. See also Davis and Hobson [30] and Cox and Obłój [24] for related results on path-wise finance. In the quasi-sure analysis, Bouchard and Nutz [15] have shown that markets satisfying a quasi-sure non-arbitrage condition admits a set of martingale measures which have the same polar (quasi-sure null) sets as the set of measures modeling the market.

Our objective when studying duality (and mathematical finance) under model uncertainty is twofold. First, we aim at deriving dual representations of convex increasing functional (and hence of risk measures) without any tightness assumptions neither on the risk measure nor on the set of models, and this for a class of risky positions not only restricted to bounded continuous random variables. Finally, we study market efficiency in a robust non-dominated model and in continuous time settings.

Structure and Main Results of the Thesis: In the first part of the thesis we focus on the case where a reference probabilistic model is known. We study duality of minimal supersolutions of BSDEs and apply our duality results to portfolio optimization under model uncertainty.

In the first chapter, we derive the dual representation of minimal supersolutions of BSDEs on the set of bounded terminal conditions and without continuity or growth condition on the generator, and extend the representations to the non-bounded case. We give conditions stemming from duality theory under which BSDEs admit a solution. These results underlines the importance of convexity in BSDE theory, and strengthen the link between risk measures and BSDE, since we also prove that any risk measure satisfying the representation derived ought to be the minimal supersolution of a BSDE. This chapter is essentially Drapeau et al. [48].

The second chapter is concerned with portfolio optimization. The maximal subsolution of a BSDE can be seen as a non-monetary concave utility. We prove existence of optimal trading strategies when the preferences of the agent are modeled by the maximal subsolution of a BSDE with convex generator. This existence result enables us to obtain the maximal subsolution of a decoupled system of forward-backward SDEs. As an application to our duality results, we show that this portfolio maximization problem can be seen as a maximization of some stochastic differential utilities or penalized expected utilities under (dominated) model uncertainty. Putting additional growth conditions on the generator enables us to prove existence of an optimal model and an optimal discounting factor. We then show that the optimal portfolio and the optimal dual elements constitute a saddle point, that we can further characterize as BSDE solutions. This chapter is essentially Heyne et al. [63].

In the second part of the thesis we focus on the case of non-dominated model uncertainty, and assume that if a set of reference measures exists, it may not be dominated by a single one. We study dual representations of nonlinear monotone functionals and apply our representation results to derive fundamental theorems of asset pricing to different non-dominated financial models.

In the third chapter, we study dual representations of non-linear functionals without fixing a reference measure. Rather simple arguments show that convex increasing functionals on the space of bounded random variables can be represented with respect to finitely additive measures. But since in practice it is not convenient to work with such measures, we study conditions under which convex increasing functionals can be represented with σ -additive measures. If the convex increasing functional satisfies a continuity from above condition, the required representation holds on bounded random variables defined on a general measurable space. This is a strong continuity condition which not easily verified on practical examples. The main results of this chapter give weak conditions under which representations with σ -additive measures can be derived. More precisely, given a topological space equipped we introduce sequential lower and upper semicontinuity conditions ensuring representations of convex increasing functionals on some spaces of random variables. Theses representation results are purely functional analytic. However, we also derive versions of our representation results in the probabilistic setting. This chapter is essentially Cheridito et al. [21].

The fourth chapter focuses on deriving a version of the fundamental theorem of asset pricing in continuous and discrete time. Given a set of reference measures, we introduce the notion of free lunch with disappearing risk (FLDR) which can be thought of as a generalization of Delbaen and Schachermayer's free lunch with vanishing risk. We show, for various continuous and discrete time models, that the absence of such FLDR is equivalent to the existence of martingale (or local martingale) measures for the underlying stock price process. In addition, we prove that these martingale measures are consistent with the prices of some fixed static claims, and have the same polar sets as the set of reference measures. These results are based on the representation theorems of convex increasing functionals of Chapter 4. Regarding the superhedging duality, we obtain a representation of the lower semicontinuous regularization of the superhedging functional. This chapter is essentially Cheridito et al. [22].

Part I

Dominated Case

Chapter 2

Dual Representation of Minimal Supersolutions of Convex BSDEs

2.1 Introduction

Since their introduction by Pardoux and Peng [81], nonlinear Backward Stochastic Differential Equations (BSDEs) have found numerous applications in mathematical finance. For instance, they are used to constructively describe the optimal solution of some utility maximization problems, see Hu et al. [68]. Through the g-expectations of Peng [83], BSDEs offer a framework to study nonlinear expectations and time consistent dynamic risk measures as described by Rosazza Gianin [94] and Delbaen et al. [38]. Mainly driven by its financial applications, the study of BSDEs has been extended in various ways beyond the question of existence and uniqueness of solutions. Many authors have been interested in questions such as numerical approximation, structural and path properties of BSDE solutions, see for instance the survey of El Karoui et al. [55] for an overview. The subject of this chapter is to study BSDEs by convex duality theory.

Deviating from the usual quadratic growth or Lipschitz assumptions on the generator of the BSDE, Drapeau et al. [47] show existence of the minimal supersolution of a BSDE. They study the properties of minimal supersolutions and give the link to cash-subadditive risk measures of El Karoui and Ravanelli [54]. Our main objectives are, on the one hand, to derive a dual representation of minimal supersolutions of BSDEs, and, on the other hand, to study conditions under which an operator satisfying such a representation is the minimal supersolution or a solution of a BSDE.

Dual representation of solutions of BSDE with quadratic growth in the control variable, linear growth in the value process and bounded terminal condition are by now well understood, see for instance Barrieu and El Karoui [5] and El Karoui and Ravanelli [54].

In this first chapter of the thesis, we give the dual representation of the minimal supersolution functional of a BSDE in the framework of Drapeau et al. [47]. The $\mathcal{H}^1$-L^∞ duality turns out to be the right candidate to constitute the basis of our representation. As a starting point, we consider the set of essentially bounded terminal conditions. In this case, we obtain a dual representation of the minimal supersolution at time 0 and a pointwise robust representation in the dynamic case. We show that when the generator of the equation is decreasing in the value process, the minimal supersolution defines a time consistent cash-subadditive risk measure. It allows for a dual representation on the space of essentially bounded random variables, which agrees with the representation of El Karoui and Ravanelli [54] obtained for BSDE solutions. Our dual representation is obtained by showing that the representation of El Karoui and Ravanelli [54] can be restricted on a smaller set. Then we can use truncation and approximation arguments to obtain the representation in the general case, due to monotone stability of minimal supersolutions. A direct consequence of our representation is the identification of BSDEs solution and minimal supersolution in the case of linear growth generators. Note that our truncation technique appears already in the work of Delbaen et al. [38] where it is used to construct a sequence of μ-dominated risk measures. Furthermore, prior to us Barrieu and El Karoui [5] and Bion-Nadal [11] already used the BMO-martingale theory in the study of financial risk measures, but in different settings from ours. Using standard convex duality arguments such as the Fenchel-Moreau theorem and the properties of the Fenchel-Legendre transform of a convex functional, we extend our dual representation to the set of random variables that can be identified to $\mathcal{H}^1$-martingales. Notice that this representation is obtained in the static case.

Our representation results can be seen as extensions of the dual representation of the minimal super-replicating cost of El Karoui and Quenez [53] to the case where we allow for a nonlinear cost function in the dynamics of the wealth process.

The second theme of this work is to give conditions based on convex duality under which a dynamic cash-subadditive risk measure with a given representation can be seen as the solution, or the minimal supersolution of a BSDE. The cash-additive case has been studied by Delbaen et al. [39]. Their results are based on m-stability of the dual space, some supermartingale property and Dood-Meyer decomposition of the risk measure. We shall show that in the cash-subadditive case, discounting the risk measure yields similar results, hence showing an equivalent relationship between existence of the minimal supersolution and the dual representation.

The rest of the chapter is structured as follows: The next section is dedicated to the setting of the probabilistic framework of our study. We also introduce the notation and gather some results on minimal supersolution of BSDEs. Our representation results are stated and proved in Section 2.3. The question of deriving a BSDE from the representation is dealt with in the last section.

2.2 Minimal Supersolution of Convex BSDEs

Given a fixed time horizon $T > 0$, let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ be a filtrated probability space. We assume that the filtration $(\mathcal{F}_t)$ is generated by a d-dimensional Brownian motion W and it satisfies the usual conditions. We further assume that $\mathcal{F}_T = \mathcal{F}$. The set of $\mathcal{F}_t$ measurable random variables is denoted by L^0_t where random variables are identified in the P-almost sure sense. For $1 \leq p < \infty$, we denote by L^p_t the set of random variables in L^0_t which are p-integrable and set $L^p = L^p_T$, and L^∞ is the set of essentially bounded random variables in L^0_T. Statements concerning random variables or processes like inequalities and equalities are to be understood in the P-almost sure or $P \otimes dt$-almost sure sense, respectively. The set of stopping times with values in $[0, T]$ is denoted by $\mathcal{T}$. We consider the sets of processes

$$\mathcal{S} := \{Y : \Omega \times [0,T] \to \mathbb{R}; Y \text{ is adapted and càdlàg}\};$$

$$\mathcal{L} := \left\{ Z : \Omega \times [0,T] \to \mathbb{R}^d; Z \text{ is predictable, and } \int_0^T \|Z_s\|^2 \, ds < +\infty \right\};$$

$$\mathcal{H}^p := \left\{ X \in \mathcal{S} : X \text{ is a continuous martingale with } \sup_{t\in[0,T]} |X_t| \in L^p \right\};$$

$$BMO := \left\{ M : M \in \mathcal{H}^1 \text{ such that } \|M\|_{BMO} < \infty \right\},$$

where $\|M\|_{BMO} := \sup_{\tau\in\mathcal{T}} \|E\left[\langle M\rangle_T - \langle M\rangle_\tau \,|\, \mathcal{F}_\tau\right]^{\frac{1}{2}} \|_\infty$. The set $\mathcal{H}^1_+$ denotes the set of non-negative martingales in $\mathcal{H}^1$. Further, let L^∞_+ and L^∞_{++} be the sets of non-negative and strictly positive random variables in L^∞, respectively. Notice that $X_t = E[X_T \mid \mathcal{F}_t]$ for all $0 \le t \le T$ and every $X \in \mathcal{H}^1$. Therefore, $\mathcal{H}^1$ will be identified with the set of random variables $X \in L^1$, satisfying $\sup_{t\in[0,T]} |E[X \mid \mathcal{F}_t]| \in L^1$. The dual of the Banach space $\mathcal{H}^1$ can be identified with BMO, see [72, Theorem 2.6].

We further consider the sets

$$\mathcal{Q} := \left\{ q \in \mathcal{L} : \exp\left(\int_0^T q_u dW_u - \frac{1}{2} \int_0^T \|q_u\|^2 \, du \right) \in L^\infty \right\},$$

$$\mathcal{D} := \left\{ \beta : \Omega \times [0,T] \to \mathbb{R}; \beta \text{ predictable}, \int_0^T \beta_u^- \, du \in L^\infty \text{ and } \int_0^T \beta_u^+ \, du < \infty \right\}.$$

In our setting, the dual variables will appear to be closely related to the sets $\mathcal{D}$ and $\mathcal{Q}$. The idea of defining the set $\mathcal{Q}$ with stochastic exponentials in L^∞ is motivated by the fact that the representation will rely on the $\mathcal{H}^1$-L^∞ duality. For $q \in \mathcal{Q}$, we denote by Q^q the probability measure whose density process is given by the stochastic exponential $M^q := \exp(\int q_u dW_u - \frac{1}{2}\int q_u^2 du)$ and for $\beta \in \mathcal{D}$ we denote by $D^\beta_{s,t} := \exp(-\int_s^t \beta_u du)$, $0 \le s \le t \le T$ the discounting factors with respect to β. In the case where $\beta \in \mathcal{D}_+ := \{\beta \in \mathcal{D} : \beta \ge 0\}$, the measures with density $M^q_t D^\beta_{0,t}$ was referred to by [54] as subprobability measures.

A generator is a jointly measurable function $g : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to (-\infty, +\infty]$ where $\Omega \times [0,T]$ is endowed with the predictable σ-field, and such that $(y,z) \mapsto g_t(\omega, y, z)$ is $P \otimes dt$-almost surely lower semicontinuous. We denote by g^* the pointwise Fenchel-Legendre transform of g, that is

$$g^*_t(\omega, \beta, q) = \sup_{(y,z)\in\mathbb{R}\times\mathbb{R}^d} \left\{ -y\beta + qz - g_t(\omega, y, z) \right\}, \quad (\beta, q) \in \mathbb{R} \times \mathbb{R}^d,$$

where the scalar product between two vectors $q, z \in \mathbb{R}^d$ is denoted by $qz := q \cdot z$. For any $(\beta, q) \in \mathbb{R} \times \mathbb{R}^d$, the process $g^*(\beta, q)$ is predictable, see [93, Proposition 14.40].

Following [47], a supersolution of the BSDE with terminal condition $X \in L^0$ and driver g is defined as a couple $(Y, Z) \in \mathcal{S} \times \mathcal{L}$ such that

$$\begin{cases} Y_s - \int_s^t g_u(Y_u, Z_u) du + \int_s^t Z_u dW_u \ge Y_t, & \text{for every} \quad 0 \le s \le t \le T \\ Y_T \ge X. \end{cases} \tag{2.1}$$

The following equivalent formulation of (2.1) will sometimes be useful: a pair (Y, Z) is a supersolution if and only if there exists a càdlàg, increasing and adapted process K with $K_0 = 0$ such that

$$Y_t = X + \int_t^T g_u(Y_u, Z_u) du + (K_T - K_t) - \int_t^T Z_u dW_u, \quad \text{for every} \quad 0 \le t \le T. \tag{2.2}$$

The control process Z of a supersolution (Y, Z) is said to be admissible if the continuous local martingale $\int Z \, dW$ is a supermartingale. Given a driver g we define

$$\mathcal{A}(X) := \{(Y, Z) \in \mathcal{S} \times \mathcal{L} : (Y, Z) \text{ fulfills (2.1) and } Z \text{ is admissible}\}, \quad X \in L^0.$$

A supersolution $(\bar{Y}, \bar{Z}) \in \mathcal{A}(X)$ is said to be minimal if $\bar{Y} \le Y$ for every $(Y, Z) \in \mathcal{A}(X)$. A generator g is said to be

(POS) positive, if $g \geq 0$;

(DEC) decreasing, if $g(y,z) \leq g(y',z)$ whenever $y \geq y'$;

(CONV) convex, if $(y,z) \mapsto g(y,z)$ is convex;

(LSC) lower semicontinuous, if $(y,z) \mapsto g(y,z)$ is lower semicontinuous.

Next, we recall the proofs of the existence, uniqueness and monotone stability of the minimal supersolution with respect to the generator. These results were already obtained in [47]. Here we argue that their proofs are also valid, up to a slight change, if we replace the assumption (DEC) by (CONV) on the generator. Recall also that for $X \in \mathcal{X} := \{X \in L^0 : X^- \in L^1\}$, the condition (POS) ensures that the value process Y of a supersolution $(Y,Z) \in \mathcal{A}(X)$ is a supermartingale such that

$$Y_t \geq -E[X^- \mid \mathcal{F}_t] \quad \text{for all } t \in [0,T], \tag{2.3}$$

see [47, Lemma 3.3].

Theorem 2.2.1. *Let g be a driver satisfying (CONV), (LSC) and (POS). For any $X \in \mathcal{X} := \{X \in L^0 : X^- \in L^1\}$ such that $\mathcal{A}(X) \neq \emptyset$, there exists a unique minimal supersolution $(\bar{Y},\bar{Z}) \in \mathcal{A}(X)$ which satisfies*

$$\bar{Y}_t = \operatorname{ess\,inf}\{Y_t : (Y,Z) \in \mathcal{A}(X)\} \quad \textit{for all } t \in [0,T].$$

Sketch of the Proof. The uniqueness of $\bar{Z}$ follows by the supermartingale property of $\bar{Y}$ and the martingale representation theorem. The existence is proved by constructing, through concatenations, a sequence of supersolutions (Y^n, Z^n) whose value processes (Y^n) decrease to the process $\operatorname{ess\,inf}\{Y_t : (Y,Z) \in \mathcal{A}(X)\}$. By a compactness argument, a subsequence in the asymptotic convex hull of (Z^n) which converges strongly to a process $\bar{Z}$ can be selected. The proof is completed by showing that there is a modification $\bar{Y}$ of $\operatorname{ess\,inf}\{Y_t : (Y,Z) \in \mathcal{A}(X)\}$ such the candidate $(\bar{Y},\bar{Z})$ is actually an admissible supersolution. In the case where g does not satisfy (DEC) but (CONV), this is done as in the proof of Theorem 2.2.3 below. □

For a generator g which satisfies (CONV), (LSC) and (POS) we define the operator $\mathcal{E} : \mathcal{X} \to \mathcal{S} \cup \{\infty\}$ as

$$\mathcal{E} : X \longmapsto \begin{cases} \bar{Y} & \text{if } \mathcal{A}(X) \neq \emptyset \\ +\infty & \text{else,} \end{cases}$$

where $\bar{Y}$ is defined in Theorem 2.2.1 and depends on X. We conclude this section by the following structural properties and stability results for $\mathcal{E}$.

Proposition 2.2.2. *Let g satisfying (CONV), (LSC) and (POS), let $X, X' \in L^0$ and $m \in \mathbb{R}$. It holds*

(i) Monotonicity: *if $X' \leq X$ then $\mathcal{E}(X') \leq \mathcal{E}(X)$;*

(ii) Convexity: *$\mathcal{E}_0(\lambda X + (1-\lambda)X') \leq \lambda\mathcal{E}_0(X) + (1-\lambda)\mathcal{E}_0(X')$, for all $\lambda \in (0,1)$;*

(iii) Cash-subadditivity: *if g is (DEC) and $m \geq 0$, then $\mathcal{E}_0(X+m) \leq \mathcal{E}_0(X) + m$;*

(iv) Cash-additivity: *if $g : (y,z) \mapsto g(z)$, then: $\mathcal{E}_0(X+m) = \mathcal{E}_0(X) + m$;*

(v) Normalization: *for every $y \in \mathbb{R}$ such that $g(y,0) = 0$ it holds $\mathcal{E}_0(y) = y$.*

Furthermore, for any sequence of random variables $(X_n) \subseteq L^0$ *such that* $\inf_n X_n \in L^1$*, it holds*

(vi) Monotone convergence: $\lim \mathcal{E}_0(X_n) = \mathcal{E}_0(X)$ *whenever* (X_n) *is increasing and converges* P*-a.s. to* $X \in L^0$*;*

(vii) Fatou: $\mathcal{E}_0(\liminf X_n) \leq \liminf \mathcal{E}_0(X_n)$.

As a restriction on L^1 *the operator* $\mathcal{E}_0$ *is* L^1*-lower semicontinuous.*

Proof. See [47, Proposition 3.2 and Theorems 4.9 and 4.12], but for the sake of readability we give the details for the points *(iii)*, *(iv)* and *(v)*.

As for *(iii)*, let $m \in \mathbb{R}$ with $m \geq 0$ and $X \in \mathcal{X}$. Since $X + m \geq X$, if $\mathcal{A}(X) = \emptyset$ then $\mathcal{A}(X+m) = \emptyset$. In that case $\mathcal{E}_0(X+m) = \infty = \mathcal{E}_0(X)$. If $\mathcal{A}(X) \neq \emptyset$, let $(Y, Z) \in \mathcal{A}(X)$. For all $0 \leq s \leq t \leq T$, since g fulfills (DEC), we have

$$
\begin{aligned}
Y_s + m - \int_s^t g_u(Y_u + m, Z_u)\,du + \int_s^t Z_u\,dW_u \\
&\geq m + Y_s - \int_s^t g_u(Y_u, Z_u)\,du + \int_s^t Z_u\,dW_u \\
&\geq m + Y_t.
\end{aligned}
$$

Thus, $(Y+m, Z) \in \mathcal{A}(X+m)$, which implies $\mathcal{E}_0(X+m) \leq Y_0 + m$. Taking $Y = \mathcal{E}(X)$, we have $\mathcal{E}_0(X+m) \leq \mathcal{E}_0(X) + m$ showing the cash-subadditivity.

As for *(iv)*, if g does not depend on y, one can show that $\mathcal{E}_0$ is additionally cash-superadditive, that is, $\mathcal{E}_0(X+m) \geq \mathcal{E}_0(X) + m$ for $m \geq 0$. Indeed, using the same argument we have $\mathcal{A}(X) \neq \emptyset$ implies $\mathcal{A}(X+m) \neq \emptyset$ and $(Y - m, Z) \in \mathcal{A}(X)$ for all $(Y, Z) \in \mathcal{A}(X+m)$. Then, if g does not depend on y, it follows that $\mathcal{E}_0(X+m) = \mathcal{E}_0(X) + m$ for all $m \in \mathbb{R}_+$. Thus, $\mathcal{E}_0(X) + m = \mathcal{E}_0(X) + m^+ - m^- = \mathcal{E}_0(X + m^+) - m^- = \mathcal{E}_0(X + m + m^-) - m^- = \mathcal{E}_0(X+m)$ for all $m \in \mathbb{R}$.

As for *(v)*, if $g(y, 0) = 0$, we have $(y, 0) \in \mathcal{A}(y)$, and therefore $\mathcal{E}_0(y) \leq y$. If g is (POS), for all $(Y, Z) \in \mathcal{A}(y)$, the supermartingale property of Y and the terminal condition yield $Y_0 \geq E[Y_T] \geq y$. Hence, $\mathcal{E}_0(y) \geq y$. □

Theorem 2.2.3. *Let* $X \in \mathcal{X}$ *be a terminal condition, and let* (g^n) *be an increasing sequence of generators, which converge pointwise to a generator* g*. Suppose that each generator is defined on* $\mathbb{R} \times \mathbb{R}^d$ *and fulfills* (CONV)*,* (LSC) *and* (POS) *and denote by* $\bar{Y}^n$ *the value process of the minimal supersolution of the BSDE with generator* g^n*. Then* $\lim_{n\to\infty} \bar{Y}_0^n = \mathcal{E}_0(X)$*. If, in addition,* $\lim_{n\to\infty} \bar{Y}_0^n < \infty$*, then for all* $t \in [0, T]$ *the set* $\mathcal{A}(X)$ *is nonempty and* $(\bar{Y}_t^n)$ *converges* P*-a.s. to* $\mathcal{E}_t(X)$*.*

Proof. By monotonicity, see Proposition 2.2.2, the sequence $(\bar{Y}_0^n)$ is increasing. Set $Y_0 = \lim_{n\to\infty} \bar{Y}_0^n$, if $Y_0 = \infty$ there is nothing to prove. Else, we put $Y_t := \lim_n \bar{Y}_t^n$, $t \in [0, T]$. It follows from the supermartingale property of $\bar{Y}^n$ and the monotone convergence theorem that Y is a càdlàg supermartingale. Using the arguments of the proof of Theorem 2.2.1, we construct a candidate control Z as pointwise limit of convex combinations $(\tilde{Z}^n)$ of $(\bar{Z}^n)$, where $(\bar{Y}^n, \bar{Z}^n)$ is the minimal supersolution of the BSDE with generator g^n. It remains to verify that $(Y, Z) \in \mathcal{A}(X)$. Fatou's lemma gives

$$
Y_s - \int_s^t g_u(Y_u, Z_u)\,du + \int_s^t Z_u\,dW_u \geq \limsup_{k\to\infty} \left(Y_s - \int_s^t g_u^k(Y_u, Z_u)\,du + \int_s^t Z_u\,dW_u \right).
$$

And for every $k \leq n$, denoting by λ_i^n the convex weights of the convex combination $\tilde{Z}^n$, using (CONV) we have

$$
\begin{aligned}
Y_s - \int_s^t g_u^k(Y_u, Z_u)\,du + \int_s^t Z_u\,dW_u
&\geq \limsup_n \left(\tilde{Y}_s^n - \int_s^t g_u^k(\tilde{Y}_u^n, \tilde{Z}_u^n)\,du - \int_s^t \tilde{Z}_u^n\,dW_u \right) \\
&\geq \limsup_n \sum_{i=n}^{M^n} \lambda_i^n \left(Y_s^i - \int_s^t g^k(Y_u^i, Z_u^i)\,du + \int_s^t Z_u^i\,dW_u \right) \\
&\geq \limsup_n \sum_{i=n}^{M^n} \lambda_i^n \left(Y_s^i - \int_s^t g^i(Y_u^i, Z_u^i)\,du + \int_s^t Z_u^i\,dW_u \right) \\
&\geq Y_t. \qquad (2.4)
\end{aligned}
$$

As to the admissibility of Z, by means of Equations (2.3) and (2.4), we have

$$
\int_0^t Z_u\,dW_u \geq -E[X^- \mid \mathcal{F}_t] - Y_0
$$

so that $\int Z\,dW$ is a supermartingale as a local martingale bounded from below by a martingale. Thus, Z is admissible. □

2.3 Dual Representation

2.3.1 The Bounded Case

To put in perspective some of the difficulties arising from the dependence of the generator in the value process Y, let us begin the study of duality by the case where the generator depends only on Z.

Theorem 2.3.1. *Assume that the generator g does not depend on y, and fulfils (CON), (LSC) and (POS), then $\mathcal{E}_0$ satisfies*

$$
\mathcal{E}_0(X) = \sup_{q \in \mathcal{Q}} E_{Q^q}\left[X - \int_0^T g_u^*(q_u)\,du \right], \quad X \in L^\infty. \qquad (2.5)
$$

Proof. By Proposition 2.2.2 and the Fenchel-Moreau theorem, for every $X \in L^\infty$, we have

$$
\mathcal{E}_0(X) = \sup_{M \in L^\infty} \{E[MX] - \mathcal{E}^*(M)\}, \qquad (2.6)
$$

where $\mathcal{E}^*$ is the convex conjugate of $\mathcal{E}_0$. Since g does not depend on y, $\mathcal{E}_0$ is cash additive. Therefore, the supremum in (2.6) can be taken over those $M \in L^\infty$ such that $M > 0$ and $E[M] = 1$. In fact, by Lemma 2.3.9 below, it holds

$$
\mathcal{E}_0(X) = \sup_{M \in L^\infty, M > 0} \{E[MX] - \mathcal{E}^*(M)\}.
$$

Moreover, let $M \in L^\infty$ be such that $E[M] \neq 1$, for any $X \in L^\infty$ we have

$$\mathcal{E}^*(M) \geq \sup_{n \in \mathbb{Z}} \{E[MX] - \mathcal{E}_0(X) + n(E[M]-1)\} = +\infty.$$

Thus,

$$\mathcal{E}_0(X) = \sup_{M \in L^\infty, M>0, E[M]=1} \{E[MX] - \mathcal{E}^*(M)\} \quad \text{for all } X \in L^\infty. \tag{2.7}$$

Let $M \in L^\infty$ such that $M > 0$ and $E[M] = 1$. By the martingale representation theorem and [88, Proposition VIII.1.6], there is a one-to-one correspondence between $q \in \mathcal{Q}$ and strictly positive continuous martingales $M_t = E[M \mid \mathcal{F}_t]$ such that $E[M_T] = 1$. Hence, Equation (2.7) can be written as

$$\mathcal{E}_0(X) = \sup_{q \in \mathcal{Q}} \{E_{Q^q}[X] - \mathcal{E}^*(q)\}. \tag{2.8}$$

Next, we derive a representation of the penalty term $\mathcal{E}^*$ with respect to the generator.

Fo every $q \in \mathcal{Q}$, it holds

$$\mathcal{E}^*(q) = \sup_{X \in S^1} \{E_{Q^q}[X] - \mathcal{E}_0(X)\}, \tag{2.9}$$

with

$$S^1 := \Big\{X \in L^\infty : \text{there exists } (Y,Z) \in \mathcal{A}(X);\ Y_T = X \text{ and}$$
$$Y_t = \mathcal{E}_0(X) - \int_0^t g_u(Z_u)\,du + \int_0^t Z_u\,dW_u\Big\}.$$

In fact, by definition of the convex conjugate, since $S^1 \subseteq L^\infty$, it is clear that

$$\mathcal{E}^*(q) \geq \sup_{X \in S^1} \{E_{Q^q}[X] - \mathcal{E}_0(X)\} \quad \text{for all } q \in \mathcal{Q}.$$

Let $q \in \mathcal{Q}$ and $X \in L^\infty$. Let (Y, Z) be the minimal supersolution of the BSDE with generator g and terminal condition X. Define

$$\tilde{Y}_t := \mathcal{E}_0(X) - \int_0^t g_u(Z_u)\,du + \int_0^t Z_u\,dW_u \quad t \in [0,T].$$

Then, it holds $(\tilde{Y}, Z) \in \mathcal{A}(\tilde{Y}_T)$; that is, $\tilde{Y}_T \in S^1$. In addition $\tilde{Y}_T \geq X$, which implies $\mathcal{E}_0(\tilde{Y}_T) \geq \mathcal{E}_0(X)$, and since $(\tilde{Y}, Z) \in \mathcal{A}(\tilde{Y}_T)$, we have $\mathcal{E}_0(\tilde{Y}_T) \leq \tilde{Y}_0 = \mathcal{E}_0(X)$. Thus, $\mathcal{E}_0(X) = \mathcal{E}_0(\tilde{Y}_T)$. Therefore, we have

$$E_{Q^q}[X] - \mathcal{E}_0(X) \leq E_{Q^q}[\tilde{Y}_T] - \mathcal{E}_0(\tilde{Y}_T) \leq \sup_{X \in S^1} \{E_{Q^q}[X] - \mathcal{E}_0(X)\},$$

which proves Equation (2.9).

Consider the set

$$S^2 := \left\{X \in L^\infty : X \in S^1 \text{ and } \int Z\,dW \in \mathcal{H}^\infty \text{ for some } (Y,Z) \in \mathcal{A}(X)\right\}.$$

Let us show that

$$\mathcal{E}^*(q) = \sup_{X \in S^2} \{E_{Q^q}[X] - \mathcal{E}_0(X)\} \quad \text{for all } q \in \mathcal{Q}. \tag{2.10}$$

Let $X \in S^1$ with associated supersolution (Y, Z). That is, $(Y, Z) \in \mathcal{A}(X)$ is such that $Y_t = \mathcal{E}_0(X) - \int_0^t g_u(Z_u)\,du + \int_0^t Z\,dW_u$ and $Y_T = X$. For all $n \in \mathbb{N}$, let us define the stopping times

$$\tau_n := \inf\left\{ t > 0 : \left| \int_0^t Z_u dW_u \right| \geq n \right\} \wedge T,$$

and the processes

$$Y_t^n := \mathcal{E}_0(X) - \int_0^{t\wedge\tau_n} g_u(Z_u)\,du + \int_0^{t\wedge\tau_n} Z_u\,dW_u, \quad t \in [0,T],\ n \in \mathbb{N}.$$

Since g satisfies (NOR), we have

$$Y_t^n = \mathcal{E}_0(X) - \int_0^t g_u\left(Z_u 1_{[0,\tau_n]}(u)\right)du + \int_0^t Z_u 1_{[0,\tau_n]}(u)dW_u \quad t \in [0,T].$$

Notice that for every $n \in \mathbb{N}$ we have $Y_t^n = Y_{t\wedge\tau_n}$. In particular, $Y_T^n \geq \mathcal{E}_{\tau_n}(X)$. For every $n \in \mathbb{N}$, $(Y^n, Z1_{[0,\tau_n]}) \in \mathcal{A}(Y_T^n)$ and $\int Z1_{[0,\tau_n]}\,dW \in \mathcal{H}^\infty$. Hence, by minimality,

$$\mathcal{E}_0(Y_T^n) \leq Y_0^n = \mathcal{E}_0(X).$$

For every $n \in \mathbb{N}$ put $X^n := E[X \,|\, \mathcal{F}_{\tau_n}]$. Then, by the martingale convergence theorem the sequence $(X^n)_n$ converges to X P-a.s. and in L^1, since (X^n) is uniformly integrable. Moreover, since dQ^q/dP is bounded, the sequence (X^n) also converges in $L^1(Q^q)$. By the supermartingale property of $\mathcal{E}(X)$ follows $Y_T^n \geq \mathcal{E}_{\tau_n}(X) \geq E[X \,|\, \mathcal{F}_{\tau_n}] = X^n$, thus

$$E_{Q^q}[Y_T^n] \geq E_{Q^q}[X^n]$$

and by $L^1(Q^q)$ convergence, we have

$$\liminf_{n\to\infty} E_{Q^q}[Y_T^n] \geq \liminf_{n\to\infty} E_{Q^q}[X^n] = E_{Q^q}[X].$$

Therefore,

$$E_{Q^q}[X] - \mathcal{E}_0(X) \leq \liminf_{n\to\infty} E_{Q^q}[Y_T^n] - \mathcal{E}_0(Y_T^n). \tag{2.11}$$

Now, let us show that the random variable Y_T^n is in L^∞. Recall that the supersolution $(Y^n, Z1_{[0,\tau_n]}) \in \mathcal{A}(Y_T^n)$ is such that $Y_T^n \geq E[X \,|\, \mathcal{F}_{\tau_n}]$, with $X \in L^\infty$. The positive random variable $\int_0^T g_u(Z_u 1_{[0,\tau_n]}(u))\,du$ can be dominated by an element of L^∞ as follows:

$$\begin{aligned}\int_0^T g_u(Z_u 1_{[0,\tau_n]}(u))\,du &= \int_0^{T\wedge\tau_n} g_u(Z_u)\,du = \mathcal{E}_0(X) - Y_T^n + \int_0^{T\wedge\tau_n} Z_u\,dW_u \\ &\leq \mathcal{E}_0(X) - E\left[X \,|\, \mathcal{F}_{\tau_n}\right] + \int_0^{T\wedge\tau_n} Z_u\,dW_u.\end{aligned}$$

Thus, since

$$Y_T^n = \mathcal{E}_0(X) - \int_0^T g_u(Z_u 1_{[0,\tau_n]}(u))\,du + \int_0^T Z_u 1_{[0,\tau_n]}(u)\,dW_u,$$

we have $Y_T^n \in L^\infty$ and therefore, from Equation (2.11) we have

$$E_{Q^q}[X] - \mathcal{E}_0(X) \le \sup_{X \in S^2} \{E_{Q^q}[X] - \mathcal{E}_0(X)\}.$$

Hence,

$$\mathcal{E}^*(q) \le \sup_{X \in S^2} \{E_{Q^q}[X] - \mathcal{E}_0(X)\}$$

so that Equation (2.10) follows by sets inclusion.

Now, let $q \in \mathcal{Q}$ and let $X \in S^2$ and $(Y,Z) \in \mathcal{A}(X)$ satisfying the equality $X = \mathcal{E}_0(X) - \int_0^T g_u(Z_u)\,du + \int_0^T Z\,dW_u$ and $\int Z\,dW \in \mathcal{H}^\infty$. We have

$$X - \mathcal{E}_0(X) = \int_0^T (q_u Z_u - g_u(Z_u))\,du + \int_0^T Z_u\,dW_u - \int_0^T q_u Z_u\,du.$$

Taking the expectation with respect to Q^q on both sides, and since by Girsanov theorem $\int Z_u\,dW_u - \int q_u Z_u\,du$ is a Q^q-martingale, we are led to

$$E_{Q^q}[X] - \mathcal{E}_0(X) = E_{Q^q}\left[\int_0^T (q_u Z_u - g_u(Z_u))\,du\right].$$

Hence

$$\mathcal{E}^*(q) = \sup_{Z \in \mathcal{L}^\infty} E_{Q^q}\left[\int_0^T (q_u Z_u - g_u(Z_u))\,du\right]. \tag{2.12}$$

Since the set $\mathcal{L}^\infty$ is decomposable in the sense of [93, Definition 14.59], by [93, Theorem 14.60] we have

$$\mathcal{E}^*(q) = E_{Q^q}\left[\int_0^T \sup_{\gamma \in \mathbb{R}^d} \{q_u\gamma - g_u(\gamma)\}\,du\right] = E_{Q^q}\left[\int_0^T g_u^*(q_u)\,du\right]$$

which, in view of Equation (2.8) yields (2.5) □

Remark 2.3.2. *We cannot guaranty that the set $\mathcal{S}$ of admissible value processes is decomposable. Thus, the link between the optimization over processes and the point-wise optimization provided by [93, Theorem 14.60] can no longer be used in the case where g also depends on y.*

For the rest of the chapter we concentrate on the case where the generator is also allowed to depend on y. The following proposition provides the dual representation of g-expectations, see also [55, Proposition 3.3]. Note that such a representation was already obtained in [54] in the more general quadratic case, where the value function of the BSDE was written as a supremum over a set of measures with uniformly integrable densities. Here, we show that under the linear growth assumption the representation can be restricted to a set of measures with densities in L^∞.

Proposition 2.3.3. *Let $X \in L^\infty$ and f be a driver satisfying (*Conv*), (*Lsc*) and (*Pos*), as well as the linear growth condition*

$$f(y,z) \le a + b\,|y| + c\,\|z\|\,, \quad a,b,c > 0.$$

Then the solution (Y,Z) of the BSDE

$$Y_t = X + \int_t^T f_u(Y_u, Z_u)\,du - \int_t^T Z_u\,dW_u, \quad t \in [0,T] \tag{2.13}$$

admits the dual representation

$$Y_t = \operatorname*{ess\,sup}_{(\beta,q)\in\mathcal{D}\times\mathcal{Q}} E_{Q^q}\left[D^{\beta}_{t,T}X - \int_t^T D^{\beta}_{t,u} f^*_u(\beta_u, q_u)\,du \,\middle|\, \mathcal{F}_t\right], \quad t\in[0,T]. \tag{2.14}$$

Before going through the proof, let us provide the following well known lemma, see [54].

Lemma 2.3.4. *Let $f:\mathbb{R}\times\mathbb{R}^d\to(-\infty,\infty]$ be a function satisfying (*LSC*), (*CONV*) as well as*

$$|f(y,z)|\le a+b\,|y|+c\,\|z\|\,,\quad (y,z)\in\mathbb{R}\times\mathbb{R}^d$$

for some positive constants a,b and c. Then, f admits for all $(y,z)\in\mathbb{R}\times\mathbb{R}^d$ the dual representation

$$f(y,z)=\max_{\beta\in\mathbb{R},q\in\mathbb{R}^d}\{-\beta y+qz-f^*(\beta,q)\}=-\bar\beta y+\bar q z-f^*(\bar\beta,\bar q) \tag{2.15}$$

for some $\left|\bar\beta\right|\le b$ and $\|\bar q\|\le c$.

Proof. We shortly present the argument. First, the dual representation of f is a consequence of the Fenchel-Moreau theorem, since the growth condition implies that f is proper. Second, the growth condition on f implies $f^*(\beta,q)\ge-\beta y+qz-f(y,z)\ge -a-\beta y+qz-b\,|y|-c\,\|z\|$ for all $(y,z)\in\mathbb{R}\times\mathbb{R}^d$. In particular $f^*(\beta,q)\ge -a+m\,|\beta|\,(|\beta|-b)+n\,\|q\|\,(\|q\|-c)$ for every $n,m\in\mathbb{N}$, showing that $f^*(\beta,q)=\infty$ for all $b<|\beta|$ or $c<\|q\|$. Hence, the supremum in (2.15) can be restricted to $|\beta|\le b$ and $\|q\|\le c$. Finally, f^* being lower semicontinuous and having a domain contained in a compact set, the supremum is therefore a maximum. □

Proof of Proposition 2.3.3. First notice that by Lemma 2.3.4, f is globally Lipschitz, due to the boundedness of $\bar q$ and $\bar\beta$, which ensures existence and uniqueness of a strong solution for the BSDE with bounded terminal condition, see [81]. Let $(\beta,q)\in\mathcal{D}\times\mathcal{Q}$. With the same arguments as in [54, 55], using Itô's formula applied to $D^{\beta}_{t,u}Y_u$ between t and T where (Y,Z) is the solution of the Lipschitz BSDE with bounded terminal condition (2.13), it holds

$$\begin{aligned}Y_t&=D^{\beta}_{t,T}X-\int_t^T D^{\beta}_{t,u}\left(-\beta_uY_u+q_uZ_u-f_u\left(Y_u,Z_u\right)\right)\,du-\int_t^T D^{\beta}_{t,u}Z_u\,dW^{Q^q}_u\\&=E_{Q^q}\left[D^{\beta}_{t,T}X-\int_t^T D^{\beta}_{t,u}\left(-\beta_uY_u+q_uZ_u-f_u\left(Y_u,Z_u\right)\right)\,du\,\middle|\,\mathcal{F}_t\right]\end{aligned}$$

for all $(\beta,q)\in\mathcal{D}\times\mathcal{Q}$. since $-\beta_uY_u+q_uZ_u-f_u(Y_u,Z_u)\le f^*_u(\beta_u,q_u)$, it follows

$$Y_t\ge\operatorname*{ess\,sup}_{(\beta,q)\in\mathcal{D}\times\mathcal{Q}} E_{Q^q}\left[D^{\beta}_{t,T}X-\int_t^T D^{\beta}_{t,u}f^*_u(\beta_u,q_u)\,du\,\middle|\,\mathcal{F}_t\right]. \tag{2.16}$$

For the other inequality, since f satisfies the conditions of Lemma 2.3.4, for all $(\omega,t)\in\Omega\times[0,T]$ the subgradients $\partial f(\omega,t,Y_t,Z_t)$ with respect to (Y_t,Z_t) are non-empty for all $(\omega,t)\in\Omega\times[0,T]$. Therefore, by means of [93, Theorem 14.56], we can apply a measurable selection theorem, see for instance [92, Corollary 1C], to assert the existence of a predictable $\mathbb{R}\times\mathbb{R}^d$-valued process $(\bar\beta,\bar q)$ such that

$$f(Y,Z)=-\bar\beta Y+\bar qZ-f^*(\bar\beta,\bar q),\quad P\otimes dt\text{-a.s.}, \tag{2.17}$$

and $|\bar{\beta}| \le b$ and $\|\bar{q}\| \le c$. Hence,

$$Y_t = E_{Q^{\bar{q}}}\left[D^{\bar{\beta}}_{t,T} X - \int_t^T D^{\bar{\beta}}_{t,u} f^*_u(\bar{\beta}_u, \bar{q}_u)\, du \,\middle|\, \mathcal{F}_t \right]. \tag{2.18}$$

But even though $\bar{q}$ is bounded it is not guaranteed that the density of $Q^{\bar{q}}$ belongs to L^∞. Thus, we introduce the following localization by defining

$$\sigma^n := \inf\left\{ s > 0 : \left| \int_0^s \bar{q}_u\, dW_u \right| \ge n \right\} \wedge T, \quad n \in \mathbb{N},$$

and put $\bar{q}^n := \bar{q}1_{[0,\sigma^n]} \in \mathcal{Q}$ and $\bar{\beta}^n := \bar{\beta}1_{[0,\sigma^n]} \in \mathcal{D}$. Then, since $\|\bar{q}_u\| \le c$, the density process of $Q^{\bar{q}^n}$ is bounded and the sequence of positive random variables $(D^{\bar{\beta}^n}_{0,T} dQ^{\bar{q}^n}/dP)$ converges P-almost surely to $D^{\bar{\beta}}_{0,T} dQ^{\bar{q}}/dP$. Furthermore, for any $p > 1$ it holds

$$\begin{aligned} E\left[\left| \frac{dQ^{\bar{q}^n}}{dP} \right|^p \right] &= E\left[\exp\left(\int_0^T p\bar{q}^n_u dW_u - \frac{1}{2} \int_0^T \|p\bar{q}^n_u\|^2\, du + \frac{p(p-1)}{2} \int_0^T \|\bar{q}^n_u\|^2\, du \right) \right] \\ &\le \exp\left(\frac{p(p-1)}{2} c^2 T \right). \end{aligned} \tag{2.19}$$

Hence $(D^{\bar{\beta}^n}_{0,T} dQ^{\bar{q}^n}/dP)$ is uniformly integrable. Therefore, since X is bounded it holds

$$\lim_{n\to\infty} E_{Q^{\bar{q}^n}}\left[D^{\bar{\beta}^n}_{t,T} X \mid \mathcal{F}_t \right] = E_{Q^{\bar{q}}}\left[D^{\bar{\beta}}_{t,T} X \mid \mathcal{F}_t \right].$$

Let us show that

$$\lim_{n\to\infty} E_{Q^{\bar{q}^n}}\left[\int_t^T D^{\bar{\beta}^n}_{t,u} f^*_u(\bar{\beta}^n_u, \bar{q}^n_u)\, du \,\middle|\, \mathcal{F}_t \right] = E_{Q^{\bar{q}}}\left[\int_t^T D^{\bar{\beta}}_{t,u} f^*_u(\bar{\beta}_u, \bar{q}_u)\, du \,\middle|\, \mathcal{F}_t \right]. \tag{2.20}$$

For almost all $\omega \in \Omega$ and $t \le u \le T$, by definition of $\bar{\beta}^n$ and $\bar{q}^n$, it holds $(\bar{\beta}^n_u(\omega), \bar{q}^n_u(\omega)) = (\bar{\beta}_u(\omega), \bar{q}_u(\omega))$ for n large enough. Hence, the sequence $(D^{\bar{\beta}^n}_{t,u} f^*_u(\bar{\beta}^n_u, \bar{q}^n_u))$ converges $P \otimes dt$-almost surely to $D^{\bar{\beta}}_{t,u} f^*_u(\bar{\beta}_u, \bar{q}_u)$. Since the processes $\bar{\beta}$ and $\bar{q}$ are bounded, by Equation (2.17) and the linear growth assumption on f, we can find two positive numbers C_1 and C_2 such that

$$\int_0^T \left| f^*_u(\bar{\beta}_u, \bar{q}_u) \right| du \le C_1 \int_0^T |Y_u|\, du + C_2 \int_0^T \|Z_u\|\, du. \tag{2.21}$$

It is known that if X is bounded and f is Lipschitz, then the solution (Y, Z) of the BSDE is such that Y is bounded and $\int Z\, dW$ is in BMO, see for instance [73] and [5, Proposition 7.3][1]. Equation (2.21) and $BMO \subseteq \mathcal{H}^p$ for all $1 \le p < \infty$, see [72], together with

[1]Notice that in [5] , the generator does not depend on y, but the same proof carries over to the general case as mentioned in [54].

Hölder's inequality imply

$$E\left[\left(\frac{dQ^{\bar{q}^n}}{dP}\int_0^T \left|f_u^*(\bar{\beta}_u^n,\bar{q}_u^n)\right|\,du\right)^2\right]$$
$$\leq \tilde{C}_1 E\left[\left(\frac{dQ^{\bar{q}^n}}{dP}\right)^2\right]+\tilde{C}_2 E\left[\left(\frac{dQ^{\bar{q}^n}}{dP}\right)^4\right]^{1/2} E\left[\left(\int_0^T \|Z_u\|^2\,du\right)^2\right]^{1/2} \leq C,$$

where C is a positive real number independent of n. Recalling that D^{β^n} is bounded, we get the required uniform integrability to derive (2.20). Now, from Equation (2.18) and since f^* is positive, we obtain

$$\begin{aligned} Y_t &= E_{Q^{\bar{q}}}\left[D_{t,T}^{\bar{\beta}}X-\int_t^T D_{t,u}^{\bar{\beta}}f_u^*(\bar{\beta}_u,\bar{q}_u)\,du \;\Big|\; \mathcal{F}_t\right] \\ &= \lim_{n\to\infty} E_{Q^{\bar{q}^n}}\left[D_{t,T}^{\bar{\beta}^n}X-\int_t^T D_{t,u}^{\bar{\beta}^n}f_u^*(\bar{\beta}_u^n,\bar{q}_u^n)\,du \;\Big|\; \mathcal{F}_t\right] \\ &\leq \operatorname*{ess\,sup}_{(\beta,q)\in\mathcal{D}\times\mathcal{Q}} E_{Q^q}\left[D_{t,T}^{\beta}X-\int_t^T D_{t,u}^{\beta}f_u^*(\beta_u,q_u)\,du \;\Big|\; \mathcal{F}_t\right]. \end{aligned}$$

Together with Equation (2.16), this concludes the proof. □

Remark 2.3.5. *Equation* (2.18) *enables us already to obtain the representation of the g-expectation with respect to measures with square integrable densities. This is a well-known result. The role of the subsequent localization procedure is to prove that the representation can, in fact, be written with respect to measures with bounded densities. This turns out to be important for the representation in the non-bounded case, since we work on the $\mathcal{H}^1$-L^∞ duality.*

Considering a more general driver, we can build on the result above to represent the minimal supersolution functional defined on the set of essentially bounded random variables.

Theorem 2.3.6. *Let g be a driver satisfying* (CONV)*,* (LSC) *and* (POS)*. Then, the operator $\mathcal{E}_0 : L^\infty \to \mathbb{R}\cup\{\infty\}$ admits the dual representation*

$$\mathcal{E}_0(X) = \sup_{(\beta,q)\in\mathcal{D}\times\mathcal{Q}}\left\{E_{Q^q}\left[D_{0,T}^{\beta}X\right]-\alpha_{0,T}(\beta,q)\right\}, \tag{2.22}$$

where the penalty function α is given by

$$\alpha_{t,s}(\beta,q) := E_{Q^q}\left[\int_t^s D_{t,u}^{\beta}g_u^*(\beta_u,q_u)\,du \;\Big|\; \mathcal{F}_t\right], \quad (\beta,q)\in\mathcal{D}\times\mathcal{Q} \tag{2.23}$$

for every $0\leq t\leq s\leq T$.

In addition, for any $t\in[0,T]$, and $X\in L^\infty$ such that $\mathcal{E}_0(X)<\infty$,

$$\mathcal{E}_t(X) = \operatorname*{ess\,sup}_{(\beta,q)\in\mathcal{D}\times\mathcal{Q}}\left\{E_{Q^q}\left[D_{t,T}^{\beta}X \;\Big|\; \mathcal{F}_t\right]-\alpha_{t,T}(\beta,q)\right\}. \tag{2.24}$$

Proof. First inequality: Let X be a bounded terminal condition. If $\mathcal{A}(X) \neq \emptyset$, then we fix a supersolution $(Y, Z) \in \mathcal{A}(X)$. Let $t \in [0, T]$ and $(\beta, q) \in \mathcal{D} \times \mathcal{Q}$. Let us define the localizing sequence of stopping times (τ_n) by

$$\tau_n := \inf \left\{ s > t : \left| \int_t^s Z_u \, dW_u \right| > n \right\} \wedge T, \quad n \in \mathbb{N}.$$

We apply Itô's formula to $\bar{Y}_u = D^{\beta}_{t,u} Y_u$ for $u \geq t$. Since (Y, Z) satisfies the equivalent formulation (2.2), there exists a nondecreasing process K such that

$$d\bar{Y}_u = -\beta_u D^{\beta}_{t,u} Y_u \, du + D^{\beta}_{t,u} \left(Z_u \, dW_u - g_u(Y_u, Z_u) \, du - dK_u \right).$$

Hence, K being nondecreasing, it follows

$$\bar{Y}_{\tau_n} - \bar{Y}_t \leq \int_t^{\tau_n} D^{\beta}_{t,u} \left(-\beta_u Y_u + q_u Z_u - g(Y_u, Z_u) \right) du + \int_t^{\tau_n} D^{\beta}_{t,u} Z_u \, dW^{Q^q}_u.$$

Applying Girsanov's theorem, it follows that $\int_t^{\cdot \wedge \tau_n} D^{\beta}_{t,u} Z_u \, dW^{Q^q}_u$ is a Q^q-martingale between t and T. Taking conditional expectation on both sides, using the definition of g^*, the facts that $Y_{\tau_n} \geq E\left[X \mid \mathcal{F}_{\tau_n}\right]$ and $g \geq 0$, we are led to

$$Y_t \geq E_{Q^q} \left[D^{\beta}_{t,\tau_n} E\left[X \mid \mathcal{F}_{\tau_n}\right] - \int_t^{\tau_n} D^{\beta}_{t,u} g^*_u(\beta_u, q_u) \, du \;\middle|\; \mathcal{F}_t \right].$$

Since X is bounded, taking the limit on the right hand side we obtain by dominated convergence

$$Y_t \geq E_{Q^q} \left[D^{\beta}_{t,T} X - \int_t^{T} D^{\beta}_{t,u} g^*_u(\beta_u, q_u) \, du \;\middle|\; \mathcal{F}_t \right],$$

so that taking the supremum with respect to β and q and by the fact that Y was chosen arbitrary, we have

$$\mathcal{E}_t(X) \geq \operatorname*{ess\,sup}_{(\beta,q) \in \mathcal{D} \times \mathcal{Q}} E_{Q^q} \left[D^{\beta}_{t,T} X - \int_t^{T} D^{\beta}_{t,u} g^*_u(\beta_u, q_u) \, du \;\middle|\; \mathcal{F}_t \right]. \tag{2.25}$$

If $\mathcal{A}(X) = \emptyset$, then Equation (2.25) is obvious.

Second inequality: Let $n \in \mathbb{N}$, and define

$$g^n(y, z) := \sup_{\{|\beta| \leq n; \|q\| \leq n\}} \left\{ -\beta y + qz - g^*(\beta, q) \right\}.$$

For every $n \in \mathbb{N}$, the function g^n satisfies the assumptions of Proposition 2.3.3. Namely, g^n is proper, has linear growth in y and z and satisfies (CONV), (LSC) and (POS). Moreover, the sequence (g^n) is nondereasing and by the Fenchel-Moreau theorem, it converges pointwise to g. By Proposition 2.3.3, the solution (Y^n, Z^n) of the BSDE with generator g^n and terminal condition X has the dual representation

$$Y^n_t = \operatorname*{ess\,sup}_{(\beta,q) \in \mathcal{D} \times \mathcal{Q}} E_{Q^q} \left[D^{\beta}_{t,T} X - \int_t^{T} D^{\beta}_{t,u} g^{n,*}_u(\beta_u, q_u) \, du \;\middle|\; \mathcal{F}_t \right].$$

Let us denote by $(\bar{Y}^n, \bar{Z}^n)$ the minimal supersolution[2] of the BSDE with driver g^n and terminal condition X. Since for every $n \in \mathbb{N}$ we have $g^n \leq g$, it holds $g^{n,*} \geq g^*$, and, by minimality of $\bar{Y}^n$ we have $\bar{Y}^n_t \leq Y^n_t$. Thus, for all $n \in \mathbb{N}$

$$\bar{Y}^n_t \leq \operatorname*{ess\,sup}_{(\beta,q)\in\mathcal{D}\times\mathcal{Q}} E_{Q^q}\left[D^\beta_{t,T}X - \int_t^T D^\beta_{t,u} g^*_u(\beta_u, q_u)\,du \;\Big|\; \mathcal{F}_t\right]. \tag{2.26}$$

If $t = 0$, taking the limit as n goes to infinity and using the monotone stability of minimal supersolutions of BSDEs, see Theorem 2.2.3, we obtain

$$\mathcal{E}_0(X) \leq \sup_{(\beta,q)\in\mathcal{D}\times\mathcal{Q}} E_{Q^q}\left[D^\beta_{0,T}X - \int_0^T D^\beta_{0,u} g^*_u(\beta_u, q_u)\,du\right].$$

Therefore Equation (2.22) holds true. If $t \in [0, T]$ and $\mathcal{E}_0(X) < \infty$, then it holds, by monotonicity, $\lim_n \bar{Y}^n_0 < \infty$. Hence, taking the limit in Equation (2.26), by Theorem 2.2.3 we have

$$\mathcal{E}_t(X) \leq \operatorname*{ess\,sup}_{(\beta,q)\in\mathcal{D}\times\mathcal{Q}} E_{Q^q}\left[D^\beta_{t,T}X - \int_t^T D^\beta_{t,u} g^*_u(\beta_u, q_u)\,du \;\Big|\; \mathcal{F}_t\right],$$

which ends the proof.

□

In the next corollary, we extend the result of Theorem 2.3.6 by giving conditions under which the representation is valid on the whole space L^∞ even in the dynamic case.

Corollary 2.3.7. *Let g be a driver satisfying (*CONV*), (*DEC*), (*LSC*) and (*POS*). Then either $\mathcal{E}_t(X) \equiv +\infty$ for all $X \in L^\infty$, $t \in [0, T]$, or $\mathcal{E} : L^\infty \to \mathcal{S}$ admits the dual representation*

$$\mathcal{E}_t(X) = \sup_{(\beta,q)\in\mathcal{D}_+\times\mathcal{Q}} \left\{E_{Q^q}\left[D^\beta_{t,T}X \mid \mathcal{F}_t\right] - \alpha_{t,T}(\beta, q)\right\}, \quad X \in L^\infty,\ t \in [0, T], \tag{2.27}$$

where the penalty function α is defined in Theorem 2.3.6.

Proof. If for every $X \in L^\infty$ the set $\mathcal{A}(X)$ is empty, then the domain of $\mathcal{E}$ is empty. On the other hand, if there exists $\xi \in L^\infty$ such that $\mathcal{A}(\xi) \neq \emptyset$, then $\mathcal{A}(X) \neq \emptyset$ for all $X \in L^\infty$. In fact, using $-\|\xi\|_\infty \leq \xi$ we have $\mathcal{A}(-\|\xi\|_\infty) \neq \emptyset$ and by (DEC), see the arguments of the proof of Proposition 2.2.2, we have $\mathcal{A}(-\|\xi\|_\infty + c) \neq \emptyset$ for all $c \geq 0$. Hence $\mathcal{A}(X) \neq \emptyset$ for all $X \in L^\infty$, since $X \leq \|X\|_\infty$ and $\mathcal{A}(\|X\|_\infty) \neq \emptyset$ for all $X \in L^\infty$.

The rest of the proof is similar to that of Theorem 2.3.6. Because g satisfies (DEC), the domain of g^* is concentrated on $\mathbb{R}_+ \times \mathbb{R}^d$, so that the representation can be restricted to $\mathcal{D}_+ \times \mathcal{Q}$. □

Remark 2.3.8. *For a given BSDE, it is not a priori clear that the minimal supersolution solution and the solution agree, since the measure induced by the process K appearing in the definition of the minimal supersolution can be singular to the Lebesgue measure. Proposition 2.3.3 and Corollary 2.3.7 show that if the terminal condition is bounded and the generator is of linear growth both in y and z, then the minimal supersolution of a BSDE coincides with its solution. In particular, $\mathcal{E}(X)$ is a continuous process, compare [47, Proposition 4.4].*

[2] As explained in Remark 2.3.8 we cannot ensure at this point that $Y^n = \bar{Y}^n$.

2.3.2 The Extension to $\mathcal{H}^1$

The goal of this section is to extend the dual representation of $\mathcal{E}_0$ to the space $\mathcal{H}^1$. We define

$$\mathcal{SQ} = \left\{ M \in L^\infty_{++} : E[M] \leq 1 \right\}.$$

We denote by $\mathcal{E}_0^*$ the convex conjugate of $\mathcal{E}_0$, defined as

$$\mathcal{E}_0^*(M) := \sup_{X \in \mathcal{H}^1} \left\{ E\left[MX\right] - \mathcal{E}_0(X) \right\}, \quad M \in L^\infty.$$

The following lemma is a consequence of the Fenchel-Moreau theorem and the structural properties of $\mathcal{E}_0$.

Lemma 2.3.9. *Let g be a driver satisfying* (CONV)*,* (DEC)*,* (LSC)*,* (POS) *and such that $\mathcal{E}_0$ is proper.*[3] *Then, the operator $\mathcal{E}_0 : \mathcal{H}^1 \to]-\infty, \infty]$ is $\sigma(\mathcal{H}^1, L^\infty)$-lower semicontinuous, and admits the dual representation*

$$\mathcal{E}_0(X) = \sup_{M \in \mathcal{SQ}} \left\{ E[MX] - \mathcal{E}_0^*(M) \right\}, \quad X \in \mathcal{H}^1. \tag{2.28}$$

Proof. $\mathcal{E}_0$ is proper, convex since g fulfills (CONV), and $\sigma(L^1, L^\infty)$-lower semicontinuous by [47, Theorem 4.9] and therefore, since $\mathcal{H}^1 \subseteq L^1$, it is $\sigma(\mathcal{H}^1, L^\infty)$-lower semicontinuous. By the Fenchel-Moreau theorem, it follows

$$\mathcal{E}_0(X) = \sup_{M \in L^\infty} \left\{ E\left[MX\right] - \mathcal{E}_0^*(M) \right\}, \quad X \in \mathcal{H}^1. \tag{2.29}$$

A standard argument shows that we can restrict the previous supremum from L^∞ to $\mathcal{SQ}$. On the one hand, let $M \in L^\infty$ with $E[M] > 1$ and $\xi_0 \in \mathcal{H}^1$ such that $\mathcal{E}_0(\xi_0) < \infty$. By cash-subadditivity, see Proposition 2.2.2, it holds

$$\begin{aligned}
\mathcal{E}_0^*(M) &\geq \sup_{n \in \mathbb{N}} \left\{ E[M(\xi_0 + n)] - \mathcal{E}_0(\xi_0 + n) \right\} \\
&\geq \sup_{n \in \mathbb{N}} \left\{ n\left(E[M] - 1\right) \right\} + E[M\xi_0] - \mathcal{E}_0(\xi_0) = +\infty.
\end{aligned}$$

On the other hand, let $M \in L^\infty \setminus L^\infty_+$ and $\xi_0 \in \mathcal{H}^1$ such that $\mathcal{E}_0(\xi_0) < \infty$. There is $\bar{X} \in \mathcal{H}^1_+$ such that $E\left[M\bar{X}\right] < 0$ since L^∞_+ is the polar cone of $\mathcal{H}^1_+$. By monotonicity of $\mathcal{E}_0$, we have $\mathcal{E}_0(-n\bar{X} + \xi_0) \leq \mathcal{E}_0(\xi_0)$. Hence,

$$\begin{aligned}
\mathcal{E}_0^*(M) &\geq \sup_{n \in \mathbb{N}} \left\{ nE\left[-M\bar{X}\right] + E[M\xi_0] - \mathcal{E}_0(-n\bar{X} + \xi_0) \right\} \\
&\geq \sup_{n \in \mathbb{N}} \left\{ nE\left[-M\bar{X}\right] \right\} + E[M\xi_0] - \mathcal{E}_0(\xi_0) = +\infty.
\end{aligned}$$

Therefore, we have

$$\mathcal{E}_0(X) = \sup_{M \in L^\infty_+ :\, E[M] \leq 1} \left\{ E\left[MX\right] - \mathcal{E}_0^*(M) \right\}.$$

Now, let $M \in L^\infty_+$ such that $E[M] \leq 1$, and for all $\lambda \in (0,1)$, we put $M^\lambda = (1-\lambda)M + \lambda$. Then, $M^\lambda \in \mathcal{SQ}$. Since for any $X \in \mathcal{H}^1$ we have $\mathcal{E}_0(X) \geq E[X]$, it follows from the definition of $\mathcal{E}_0^*$ that $\mathcal{E}_0^*(1) \leq 0$ so that by convexity, it holds $\limsup_{\lambda \to 0} \mathcal{E}_0^*(M^\lambda) \leq \mathcal{E}_0^*(M)$. Let $X \in \mathcal{H}^1$, applying dominated convergence theorem to $(M^\lambda X)$ implies

$$E[MX] - \mathcal{E}_0^*(M) \leq \liminf_{\lambda \to 0} \left\{ E\left[M^\lambda X\right] - \mathcal{E}_0^*(M^\lambda) \right\}.$$

Hence, $\mathcal{E}_0(X) \leq \sup_{M \in \mathcal{SQ}} \{E[MX] - \mathcal{E}_0^*(M)\}$. The other inequality follows by sets inclusion. Thus, Equation (2.28) holds true. □

[3] $\mathcal{E}_0$ is proper for instance if there exists $y_0 \in \mathbb{R}$ with $g(y_0, 0) = 0$. In fact, in that case, the pair $(y_0, 0)$ is in $\mathcal{A}(y_0)$ and therefore $\mathcal{E}_0(y_0) \leq y_0 < \infty$. And by (POS), $\mathcal{E}_0(X) \geq E[X] > -\infty$ for all $X \in \mathcal{H}^1$.

We observe that there is a relationship between the sets $\mathcal{SQ}$ and $\mathcal{D}_+ \times \mathcal{Q}$, and the dual representation of $\mathcal{E}_0$.

Remark 2.3.10. *Any element of $\mathcal{SQ}$ may be parametrized by elements of $\mathcal{D}_+ \times \mathcal{Q}$ and vice versa. Indeed, for every $M \in \mathcal{SQ}$, since $M/E[M]$ is a strictly positive random variable with expectation 1, there exists a unique process $q \in \mathcal{Q}$ such that $M_T^q = M/E[M]$, with $M_t^q = \exp(\int_0^t q_u \, dW_u - \frac{1}{2}\int_0^t \|q\|_u^2 \, du)$ and taking $\beta \in \mathcal{D}_+$ such that $\exp(-\int_0^T \beta_s ds) = E[M] \in (0,1]$, we have $M = \exp(-\int_0^T \beta_s ds) M_T^q$. Conversely, given $(\beta, q) \in \mathcal{D}_+ \times \mathcal{Q}$, it holds $\exp(-\int_0^T \beta_s ds)\exp(\int_0^T q_u \, dW_u - \frac{1}{2}\int_0^T \|q\|_u^2 \, du) \in \mathcal{SQ}$. This underlines the importance of working with probability measures with bounded densities in the previous section.*

Remark 2.3.11. *To every $M \in \mathcal{SQ}$ corresponds a unique $q \in \mathcal{Q}$. Hence, for all $X \in L^\infty$, Corollary 2.3.7 yields*

$$\begin{aligned}
\mathcal{E}_0(X) &= \sup_{(\beta,q)\in\mathcal{D}_+\times\mathcal{Q}} \left\{ E\left[\frac{dQ^q}{dP} D_{0,T}^\beta X\right] - E_{Q^q}\left[\int_0^T D_{0,u}^\beta g_u^*(\beta_u, q_u)\, du\right]\right\} \\
&= \sup_{M\in\mathcal{SQ}} \sup_{\{\beta\in\mathcal{D}_+ :\ D_{0,T}^\beta = E[M]\}} \left\{ E[MX] - E_{Q^q}\left[\int_0^T D_{0,u}^\beta g_u^*(\beta_u, q_u)\, du\right]\right\} \\
&= \sup_{M\in\mathcal{SQ}} \left\{ E[MX] - \alpha_{min}(M)\right\},
\end{aligned}$$

for the penalty function

$$\alpha_{min}(M) := \inf_{\{\beta\in\mathcal{D}_+ :\ D_{0,T}^\beta = E[M]\}} E_{Q^q}\left[\int_0^T D_{0,u}^\beta g_u^*(\beta_u, q_u)\, du\right] \tag{2.30}$$

defined on $\mathcal{SQ}$.

We may now present the main result of this section, the extension to $\mathcal{H}^1$ of the dual representation Theorem 2.3.7.

Theorem 2.3.12. *Let g be a driver satisfying* (CONV)*,* (DEC)*,* (LSC) *and* (POS) *and such that $\mathcal{E}_0$ is proper. Then the operator $\mathcal{E}_0 : \mathcal{H}^1 \to]-\infty, +\infty]$ admits the dual representation*

$$\mathcal{E}_0(X) = \sup_{(\beta,q)\in\mathcal{D}_+\times\mathcal{Q}} \left\{ E_{Q^q}\left[D_{0,T}^\beta X\right] - \alpha_0(\beta, q)\right\}, \quad X \in \mathcal{H}^1, \tag{2.31}$$

where

$$\alpha_0(\beta, q) := E_{Q^q}\left[\int_0^T D_{0,u}^\beta g_u^*(\beta_u, q_u)\, du\right], \quad (\beta, q) \in \mathcal{D}_+ \times \mathcal{Q}. \tag{2.32}$$

Proof. Due to Lemma 2.3.9 and Remark 2.3.11, it suffices to show that $\mathcal{E}_0^* = \alpha_{min}$ on $\mathcal{SQ}$, where α_{min} is the penalty function defined by Equation (2.30).

First inequality. For all $X \in \mathcal{H}^1$, it holds

$$\mathcal{E}_0(X) \geq \sup_{(\beta,q)\in\mathcal{D}_+\times\mathcal{Q}} E_{Q^q}\left[D_{0,T}^\beta X - \int_0^T D_{0,u}^\beta g_u^*(\beta_u, q_u)\, du\right]. \tag{2.33}$$

In fact, let $X \in \mathcal{H}^1$. If $\mathcal{A}(X) = \emptyset$, then the result is trivial. Suppose that $\mathcal{A}(X) \neq \emptyset$, and take $(Y, Z) \in \mathcal{A}(X)$. Let $(\beta, q) \in \mathcal{D}_+ \times \mathcal{Q}$, arguing exactly like in the first part of the proof of Theorem 2.3.6 we obtain a localizing sequence of stopping times (τ_n) such that

$$Y_0 \geq E_{Q^q}\left[D^\beta_{0,\tau_n} E[X \mid \mathcal{F}_{\tau_n}] - \int_0^{\tau_n} D^\beta_{0,u} g^*_u(\beta_u, q_u)\, du \right] \quad \text{for all } n \in \mathbb{N}. \tag{2.34}$$

Since $X \in \mathcal{H}^1$, the sequence of martingales (N^n) given by $N^n_t := E[E[X \mid \mathcal{F}_{\tau_n}] \mid \mathcal{F}_t] = E[X \mid \mathcal{F}_{\tau_n \wedge t}]$ is in $\mathcal{H}^1$, and is such that $\left(\sup_{t\in[0,T]} |N^n_t|\right)_n$ is uniformly integrable. Therefore, by [35, Theorem 4.9], see also [72, Lemma 2.5], (N^n) admits a subsequence again denoted by (N^n) which converges weakly in $\mathcal{H}^1$. Thus, the sequence of products $(D^\beta_{0,\tau^n} N^n_T)$ converges weakly in $\mathcal{H}^1$ to $D^\beta_{0,T} X$, since (D^β_{0,τ^n}) is bounded by 1. Now, as a consequence of the boundedness of the martingale $M^q_t = E[dQ^q/dP \mid \mathcal{F}_t]$, the function $X \mapsto E[M^q_T X]$ from $\mathcal{H}^1$ to $\mathbb{R}$ is linear and continuous, and therefore $\sigma(\mathcal{H}^1, BMO)$-continuous. Hence, taking the limit on both sides of Equation (2.34) leads to

$$Y_0 \geq E_{Q^q}\left[D^\beta_{0,T} X - \int_0^{T} D^\beta_{0,u} g^*_u(\beta_u, q_u)\, du \right].$$

This implies, by means of Remark 2.3.11, that

$$\mathcal{E}_0(X) \geq \sup_{M \in \mathcal{SQ}} \left\{ E\left[MX\right] - \alpha_{min}(M) \right\},$$

that is, for every $M \in \mathcal{SQ}$ we have $\alpha_{min}(M) \geq E\left[MX\right] - \mathcal{E}_0(X)$ so that taking the supremum with respect to $X \in \mathcal{H}^1$, we obtain by definition of $\mathcal{E}^*$

$$\alpha_{min}(M) \geq \mathcal{E}^*_0(M).$$

Second inequality. The main argument for the second inequality is to show that the penalty function α_{min} defined by Equation (2.30) is minimal, that is,

$$\mathcal{E}^*_{L^\infty}(M) := \sup_{X \in L^\infty} \left\{ E\left[MX\right] - \mathcal{E}_0(X) \right\} = \alpha_{min}(M), \quad M \in \mathcal{SQ}.$$

In fact, that would imply

$$\mathcal{E}^*_0(M) \geq \mathcal{E}^*_{L^\infty}(M) = \alpha_{min}(M),$$

where the first inequality is obtained by sets inclusion. To that end, it suffices to show that for every $c \geq 0$ the set $\{M \in \mathcal{SQ} : \alpha_{min}(M) \leq c\}$ is convex and closed in L^1, since by convexity, it would then be $\sigma(L^1, L^\infty)$-closed and therefore $\sigma(L^\infty, L^\infty)$-closed.

Convexity: Let $\lambda \in [0, 1]$, $M^1, M^2 \in \mathcal{SQ}$ and $q^i \in \mathcal{Q}$ such that $M^{q^i}_T = M^i/E[M^i]$, $i = 1, 2$. Put $M^\lambda = \lambda M^1 + (1 - \lambda)M^2$. For a given $\varepsilon > 0$, there exists $\beta^i \in \mathcal{D}_+$ such that $D^{\beta^i}_{0,T} = E[M^i]$ and

$$\varepsilon + \alpha_{min}(M^i) \geq E_{Q^{q^i}}\left[\int_0^T D^{\beta^i}_{0,u} g^*(\beta^i_u, q^i_u)\, du \right].$$

Applying Itô's formula to $\log\left(\lambda M^{q^1}_t D^{\beta^1}_t + (1 - \lambda) M^{q^2}_t D^{\beta^2}_t\right)$ such as in the proof of [39, Lemma 2.1] we have

$$\begin{aligned}\lambda M^{q^1}_t D^{\beta^1}_{0,t} + (1 - \lambda) M^{q^2}_t D^{\beta^2}_{0,t} &= \exp\left(\int_0^t q^\lambda_u\, dW_u - \frac{1}{2}\int_0^t \left\| q^\lambda_u \right\|^2 du - \int_0^t \beta^\lambda_u\, du \right) \\ &= M^{q^\lambda}_t D^{\beta^\lambda}_{0,t}\end{aligned}$$

and $D^{\beta^\lambda}_{0,T} = E[M^\lambda]$, with

$$q^\lambda_t = \frac{\lambda M^{q^1}_t D^{\beta^1}_{0,t} q^1_t + (1-\lambda) M^{q^2}_t D^{\beta^2}_{0,t} q^2_t}{\lambda M^{q^1}_t D^{\beta^1}_{0,t} + (1-\lambda) M^{q^2}_t D^{\beta^2}_{0,t}}, \quad \beta^\lambda_t = \frac{\lambda M^{q^1}_t D^{\beta^1}_{0,t} \beta^1_t + (1-\lambda) M^{q^2}_t D^{\beta^2}_{0,t} \beta^2_t}{\lambda M^{q^1}_t D^{\beta^1}_{0,t} + (1-\lambda) M^{q^2}_t D^{\beta^2}_{0,t}}.$$

This follows from the facts that $M^{q^\lambda}_T E[M^\lambda] = M^\lambda = M^{q^\lambda}_T D^{\beta^\lambda}_{0,T}$ and $M^{q^\lambda} > 0$. Therefore, joint convexity of g^* and the definition of $(\beta^\lambda, q^\lambda)$ lead us to

$$\begin{aligned} 2\varepsilon + \lambda \alpha_{min}(M^1) + &(1-\lambda)\alpha_{min}(M^2) \\ &\geq E\left[\int_0^T \left(\lambda M^{q^1}_u D^{\beta^1}_{0,u} + (1-\lambda) M^{q^2}_u D^{\beta^2}_{0,u}\right) g^*_u(\beta^\lambda_u, q^\lambda_u)\, du\right] \\ &= E_{Q^{q^\lambda}}\left[\int_0^T D^{\beta^\lambda}_{0,u} g^*_u(\beta^\lambda_u, q^\lambda_u)\, du\right]. \end{aligned}$$

Therefore, taking first the infimum for $\beta \in \mathcal{D}_+$ such that $D^\beta_{0,T} = E[M^\lambda]$ on the right hand side, and then the limit on the left hand side as ε goes to 0 we have

$$\lambda \alpha_{min}(M^1) + (1-\lambda)\alpha_{min}(M^2) \geq \alpha_{min}(M^\lambda).$$

Closedness: Let $c \geq 0$ and (M^n) be a sequence in $\mathcal{SQ}$ converging to $M \in \mathcal{SQ}$ in L^1 and such that $\alpha_{min}(M^n) \leq c$ for every $n \in \mathbb{N}$. Let us show that $\alpha_{min}(M) \leq c$. For all $n \in \mathbb{N}$ let q^n be such that $M^{q^n}_T = M^n/E[M^n]$ and q be such that $M^q_T = M/E[M]$. Let $\varepsilon > 0$ be fixed. For every $n \in \mathbb{N}$, there exists $\beta^n \in \mathcal{D}_+$ such that $D^{\beta^n}_{0,T} = E[M^n]$ and

$$\varepsilon + \alpha_{min}(M^n) \geq E_{Q^{q^n}}\left[\int_0^T D^{\beta^n}_{0,u} g^*_u(\beta^n_u, q^n_u)\, du\right].$$

Since (M^n) converges to M in L^1, the sequence $(E[M^n])$ converges to $E[M]$, with $E[M^n] > 0$ and $E[M] > 0$. Therefore, $(M^{q^n}_t)$ converges to M^q_t in L^1 for all $t \in [0,T]$. We also introduce the martingales $M^n_t := E[M^n \mid \mathcal{F}_t]$ and $M_t := E[M \mid \mathcal{F}_t]$, $t \in [0,T]$. We choose a fast subsequence $(M^{n,m})$ such that $P(|M^{n,m}_T - M_T| \geq 1) < 2^{-n}/m$ and for all $m \in \mathbb{N}$, define the stopping time

$$\tau^m := \inf\left\{t \in [0,T] : |M^{n,m}_t - M_t| \geq m \text{ for some } n\right\}.$$

Then, (τ^m) is a localizing sequence of stopping times since

$$\begin{aligned} P(\tau^m = T) &\geq 1 - P(|M^{n,m}_T - M_T| \geq 1 \text{ for some } n) - P(|M_T| \geq m-1) \\ &\geq 1 - \frac{1}{m} - \frac{E[|M_T|]}{m-1} \longrightarrow 1. \end{aligned}$$

For every m, the sequence $(M^{n,m}_{\tau^m} - M_{\tau^m})$ is bounded, therefore $(M^{n,m}_{\tau^m})$ converges to (M_{τ^m}) in L^2. It follows by Burkholder-Davis-Gundy and Doob's inequalities that there exists a positive constant C such that

$$\begin{aligned} &E\left[\int_0^{\tau^m} \left|M^{n,m}_0 q^{n,m}_u M^{q^{n,m}}_u - M_0 q_u M^q_u\right|^2 du\right] \\ &\qquad = E\left[\langle M^{n,m} - M\rangle^2_{\tau^m}\right] \leq CE\left[\left(\sup_{t\in[0,\tau^m]} |M^{n,m}_t - M_t|\right)^2\right] \xrightarrow[n\to\infty]{} 0. \end{aligned}$$

Thus, up to a subsequence, $(q^{n,m} M^{q^{n,m}} 1_{[0,\tau^m]})$ converges $P \otimes dt$-a.s. to $qM^q 1_{[0,\tau^m]}$. But since the sequence of strictly positive martingales $\left(M^{q^{n,m}}\right)$ converges $P \otimes dt$-a.s. to $M^q > 0$, it follows that

$$\lim_{n\to\infty} q^{n,m} 1_{[0,\tau^m]} = q 1_{[0,\tau^m]} \quad P \otimes dt\text{-a.s.}$$

Since (τ^m) converges P-a.s. to T we obtain, by a diagonalization argument, another subsequence again denoted (q^n) which converges $P \otimes dt$-a.s. to q. As for the convergence of the sequence (β^n), since $(\exp(-\int_0^T \beta_u^n \, du)) = (E[M^n])$ converges to $E[M]$, it follows that the sequence $(\int_0^T \beta_u^n \, du)$ converges to $-\log(E[M])$, and $(E[\int_0^T \beta_u^n \, du])$ is uniformly bounded. Hence, we can apply a compactness argument, see for instance [35, Theorem 1.4] applied on the product space, to obtain a sequence $(\tilde{\beta}^n)$ in the asymptotic convex hull of (β^n) which converges $P \otimes dt$ to a positive predictable process β. In addition, $D_{0,T}^{\beta} = E[M]$ since the sequences $(\int_0^T \beta_u^n \, du)$ and $(\int_0^T \tilde{\beta}_u^n \, du)$ converge to the same limit. Now applying Fatou's lemma, convexity and lower-semicontinuity of g^* lead us to

$$\varepsilon + \liminf_{n\to\infty} \alpha_{min}(M^n) \geq E\left[\int_0^T \liminf_{n\to\infty} M_u^{q^n} D_{0,u}^{\beta^n} g_u^*(\beta_u^n, q_u^n)\, du\right]$$

$$\geq E\left[\int_0^T \liminf_{n\to\infty} M_u^{q^n} D_{0,u}^{\tilde{\beta}^n} g_u^*(\tilde{\beta}_u^n, q_u^n)\, du\right] = E\left[\int_0^T M_u^q D_{0,u}^{\beta} g_u^*(\beta_u, q_u)\, du\right] \geq \alpha_{\min}(M).$$

Once again the result is obtained by letting ε tend to 0.

□

We recover the robust representation of coherent (cash-subadditive) risk measures.

Corollary 2.3.13. *Under the assumptions of Theorem 2.3.12, if the generator g is positive homogeneous in the sense that*

$$g(\lambda y, \lambda z) = \lambda g(y, z) \quad \text{for all } \lambda > 0 \text{ and } (y, z) \in \mathbb{R} \times \mathbb{R}^d,$$

then $\mathcal{E}_0$ is also positive homogeneous and the dual representation of $\mathcal{E}_0$ reduces to

$$\mathcal{E}_0(X) = \sup_{(\beta,q)\in\mathcal{D}_+\times\mathcal{Q}} E_{Q^q}\left[D_{0,T}^{\beta} X\right], \quad X \in \mathcal{H}^1.$$

Proof. Let λ be strictly positive, and $(\mathcal{E}_0(\lambda X), Z)$ the minimal supersolution in $\mathcal{A}(\lambda X)$. By positive homogeneity of g, we have $(\mathcal{E}_0(\lambda X)/\lambda, Z/\lambda) \in \mathcal{A}(X)$, therefore $\mathcal{E}_0(\lambda X) \geq \lambda \mathcal{E}_0(X)$. Using the same reasoning on $\mathcal{A}(X)$ we have $\mathcal{E}_0(\lambda X) \leq \lambda \mathcal{E}_0(X)$, hence $\mathcal{E}_0$ is positive homogeneous.

The representation (2.3.13) follows from Theorem 2.3.12 since the convex conjugate of the positive homogeneous function g is the indicator of a closed convex set (i.e. it is either 0 or ∞.) □

Let us conclude this section with an example.

Example 2.3.14. *Let X be any random variable in $\mathcal{H}^1$. Consider the BSDE*

$$dY_t = -g(Y_t, Z_t), dt + Z_t \, dW_t, \quad Y_T = X \tag{2.35}$$

with generator g defined on $\mathbb{R} \times \mathbb{R}^d$ by

$$g(y,z) := \begin{cases} z^2/y & \text{if } y > 0,\ z \in \mathbb{R}^d \\ 0 & \text{if } y \leq 0,\ z = 0 \\ +\infty & \text{if } y \leq 0,\ z \in \mathbb{R}^d \setminus \{0\}\,. \end{cases}$$

The function g satisfies the conditions of Theorem 2.3.12. Therefore, the minimal supersolution $\mathcal{E}^g(X)$ of Equation (2.35) *admits the dual representation* (2.31)*. Moreover, defining*

$$\mathcal{K} := \left\{ (\beta, q) \in \mathcal{D}_+ \times \mathcal{Q} : \beta \geq \frac{1}{4} \|q\|^2 \right\},$$

one can check that g^* takes the value 0 on $\mathcal{K}$ and $+\infty$ on the complement of $\mathcal{K}$. Thus,

$$\mathcal{E}^g_0(X) = \sup_{(\beta,q)\in\mathcal{K}} E_{Q^q}\left[D^\beta_{0,T} X\right].$$

2.4 Cash-Subadditive Risk Measures and BSDE

The operator $\mathcal{E}_0$ studied in the previous section can be seen as a risk measure. In fact, when the generator does not depend on y, the functional ρ defined by $\rho(X) := \mathcal{E}_0(-X)$ is a convex risk measure in the sense of [60], and $u(X) := -\mathcal{E}_0(-X)$ defines a monetary utility function. If the generator g does depend on y and satisfies (DEC), then ρ is instead a cash-subadditive risk measure as defined in [54]. In particular, for all $m \geq 0$ holds $\rho(X - m) \leq \rho(X) + m$.

In this section we start with a cash-subadditive risk measure satisfying a given robust representation and show, in Theorem 2.4.5, that such a risk measure must be the minimal supersolution of a BSDE. Thus, we are given a dynamic cash-subadditive risk measure[4] of the form

$$\phi_t(X) := \operatorname*{ess\,sup}_{(\beta,q)\in\mathcal{D}_+\times\mathcal{Q}} E_{Q^q}\left[D^\beta_{t,T} X - \int_t^T D^\beta_{t,u} f(\beta_u, q_u)\, du \;\Big|\; \mathcal{F}_t \right], \quad t \in [0,T], \quad (2.36)$$

where X is a random variable in $\mathcal{H}^1$ and $f : \mathbb{R} \times \mathbb{R}^d \to (-\infty, \infty]$ a given proper function. A function f is said to be

(NORM) null at the origin if, $f(0,0) = 0$.

Remark 2.4.1. *Since $D^\beta_{s,t} D^\beta_{t,u} = D^\beta_{s,u}$, the penalty function α defined by Equation* (2.23) *satisfies the following cocycle property introduced in [11] for monetary convex risk measures:*

$$\alpha_{s,u}(\beta, q) = \alpha_{s,t}(\beta, q) + E_{Q^q}\left[D^\beta_{s,t} \alpha_{t,u}(\beta, q) \;\Big|\; \mathcal{F}_s \right] \quad \textit{for every} \quad (\beta, q) \in \mathcal{D}_+ \times \mathcal{Q}. \quad (2.37)$$

In the cash-additive case, the cocycle property takes the form

$$\alpha_{s,u}(q) = \alpha_{s,t}(q) + E_{Q^q}\left[\alpha_{t,u}(q) \,|\, \mathcal{F}_s\right].$$

Hence, the characterization of time-consistency in terms of the cocycle property given by [11, Theorem 3.3] shows that when g does not depend on y, $\mathcal{E}$ is time-consistent even if the normalization condition $g(0) = 0$ is not assumed, compare [47, Proposition 3.6].

In what follows we use the notation of the previous section. In particular, for any $q \in \mathcal{Q}$ we denote by M^q the martingale density process of the probability measure Q^q with respect to the reference measure P. We follow a method already put forth in [39] in the cash-additive case. The main idea is the following:

[4] Actually, this is only a risk measure up to a transformation as explained above.

Proposition 2.4.2. *For any $X \in \mathcal{H}^1$ and for each $(\beta, q) \in \mathcal{D}_+ \times \mathcal{Q}$ the process*

$$\varphi(X) := \left(D^{\beta}_{0,t}\phi_t(X) - \int_0^t D^{\beta}_{0,u} f(\beta_u, q_u)\, du \right)_{t \in [0,T]}$$

is a Q^q-supermartingale.

Proof. Let $0 \le s \le t \le T$. We start by showing that the set

$$\left\{ E_{Q^q}\left[D^{\beta}_{t,T} X - \int_t^T D^{\beta}_{t,u} f(\beta_u, q_u)\, du \,\Big|\, \mathcal{F}_t \right] : (\beta, q) \in \mathcal{D}_+ \times \mathcal{Q} \right\}$$

is directed upward. Let $(\beta^1, q^1), (\beta^2, q^2) \in \mathcal{D}_+ \times \mathcal{Q}$. Let us define the stopping time

$$\tau := \inf\left\{ s > t : L^1_t < L^2_t \right\},$$

with $L^i_t := E_{Q^{q^i}}[D^{\beta^i}_{t,T} X - \int_t^T D^{\beta^i}_{t,u} f(\beta^i_u, q^i_u)\, du \,|\, \mathcal{F}_t]$, $i = 1, 2$, and put $\hat{q} := q^1 1_{[0,\tau]} + q^2 1_{(\tau,T]}$ and $\hat{\beta} := \beta^1 1_{[0,\tau]} + \beta^2 1_{(\tau,T]}$. We have $(\hat{\beta}, \hat{q}) \in \mathcal{D}_+ \times \mathcal{Q}$ and, by definition, $\hat{L}_t \ge \max\{L^1_t, L^2_t\}$, with $\hat{L}_t := E_{Q^{\hat{q}}}[D^{\hat{\beta}}_{t,T} X - \int_t^T D^{\hat{\beta}}_{t,u} f(\hat{\beta}_u, \hat{q}_u)\, du \,|\, \mathcal{F}_t]$.

Therefore, by [60, Theorem A.32], there exists a sequence $(\beta^n, q^n) \subseteq \mathcal{D}_+ \times \mathcal{Q}$ such that

$$\phi_t(X) = \lim_{n\to\infty} E_{Q^{q^n}}\left[D^{\beta^n}_{t,T} X - \int_t^T D^{\beta^n}_{t,u} f(\beta^n_u, q^n_u)\, du \,\Big|\, \mathcal{F}_t \right].$$

In addition, this convergence is monotone. Therefore, $\phi_t(X)$ is integrable, and it is also Q^q-integrable for every $q \in \mathcal{Q}$ since $dQ^q/dP \in L^\infty$. Hence, for any $(\beta, q) \in \mathcal{D}_+ \times \mathcal{Q}$, it holds

$$\begin{aligned}
E_{Q^q}\left[\varphi_t(X) \,\big|\, \mathcal{F}_s\right] &= E_{Q^q}\left[D^{\beta}_{0,t} \lim_{n\to\infty} E_{Q^{q^n}}\left[D^{\beta^n}_{t,T} X - \int_t^T D^{\beta^n}_{t,u} f(\beta^n_u, q^n_u)\, du \,\Big|\, \mathcal{F}_t \right] \,\Big|\, \mathcal{F}_s \right] \\
&\quad - E_{Q^q}\left[\int_s^t D^{\beta}_{0,u} f(\beta_u, q_u)\, du \,\Big|\, \mathcal{F}_s \right] - \int_0^s D^{\beta}_{0,u} f(\beta_u, q_u)\, du \\
&= \lim_{n\to\infty} D^{\beta}_{0,s} E_{Q^q}\left[E_{Q^{q^n}}\left[D^{\beta}_{s,t} D^{\beta^n}_{t,T} X - \int_t^T D^{\beta}_{s,t} D^{\beta^n}_{t,u} f(\beta^n_u, q^n_u)\, du \right.\right. \\
&\quad \left.\left. - \int_s^t D^{\beta}_{s,u} f(\beta_u, q_u)\, du \,\Big|\, \mathcal{F}_t \right] \,\Big|\, \mathcal{F}_s \right] - \int_0^s D^{\beta}_{0,u} f(\beta_u, q_u)\, du,
\end{aligned}$$

where the second equation follows by dominated convergence theorem. We put $\bar{\beta}^n = \beta 1_{[0,t]} + \beta^n 1_{(t,T]}$ and $\bar{q}^n = q 1_{[0,t]} + q^n 1_{(t,T]}$. It follows that

$$\begin{aligned}
E_{Q^q}\left[\varphi_t(X) \,\big|\, \mathcal{F}_s\right] &= D^{\beta}_{0,s} \lim_{n\to\infty} E_{Q^{\bar{q}^n}}\left[D^{\bar{\beta}^n}_{s,T} X - \int_s^T D^{\bar{\beta}^n}_{s,u} f(\bar{\beta}^n_u, \bar{q}^n_u)\, du \,\Big|\, \mathcal{F}_s \right] \\
&\quad - \int_0^s D^{\beta}_{0,u} f(\beta_u, q_u)\, du \\
&\le D^{\beta}_{0,s}\phi_s(X) - \int_0^s D^{\beta}_{0,u} f(\beta_u, q_u)\, du = \varphi_s(X),
\end{aligned}$$

where the inequality follows by definition of $\phi(X)$ and the fact that $(\bar{\beta}^n, \bar{q}^n) \in \mathcal{D}_+ \times \mathcal{Q}$. $\square$

Next we give two consequences of the previous result.

Corollary 2.4.3. *Let $X \in \mathcal{H}^1$, suppose in addition that $\phi_0(X)$ admits a subgradient $(\beta, q) \in \mathcal{D}_+ \times \mathcal{Q}$, i.e. (β, q) is such that $\phi_0(X) = E_{Q^q}\left[D^\beta_{0,T}X - \int_0^T D^\beta_{0,u} f(\beta_u, q_u)\, du\right]$. Then for each $t \in [0, T]$ we have*

$$\phi_t(X) = E_{Q^q}\left[D^\beta_{t,T}X - \int_t^T D^\beta_{t,u} f(\beta_u, q_u)\, du \;\Big|\; \mathcal{F}_t\right], \tag{2.38}$$

that is, (β, q) is a subgradient of $\phi_t(X)$. Moreover, the process

$$\left(D^\beta_{0,t}\phi_t(X) - \int_0^t D^\beta_{0,u} f(\beta_u, q_u)\, du\right)_{t\in[0,T]}$$

is a Q^q-martingale.

Proof. Let $(\beta, q) \in \mathcal{D}_+ \times \mathcal{Q}$ be such that

$$\phi_0(X) = E_{Q^q}\left[D^\beta_{0,T}X - \int_0^T D^\beta_{0,u} f(\beta_u, q_u)\, du\right].$$

By the previous proposition and the choice of (β, q) we have for any $t \in [0, T]$

$$\begin{aligned} &E_{Q^q}\left[D^\beta_{0,t}\phi_t(X) - \int_0^t D^\beta_{0,u} f(\beta_u, q_u)\, du\right] \\ &\qquad \le \phi_0(X) = E_{Q^q}\left[D^\beta_{0,T}X - \int_0^T D^\beta_{0,u} f(\beta_u, q_u)\, du\right], \end{aligned}$$

from which ensues

$$\begin{aligned} E_{Q^q}\left[D^\beta_{0,t}\phi_t(X)\right] &\le E_{Q^q}\left[D^\beta_{0,T}X - \int_t^T D^\beta_{0,u} f(\beta_u, q_u)\, du\right] \\ &= E_{Q^q}\left[D^\beta_{0,t}\left(D^\beta_{t,T}X - \int_t^T D^\beta_{t,u} f(\beta_u, q_u)\, du\right)\right]. \end{aligned}$$

Since we have

$$\phi_t(X) \ge E_{Q^q}\left[D^\beta_{t,T}X - \int_t^T D^\beta_{t,u} f(\beta_u, q_u)\, du \;\Big|\; \mathcal{F}_t\right],$$

and $0 < D^\beta_{0,t} < \infty$ we conclude that

$$\phi_t(X) = E_{Q^q}\left[D^\beta_{t,T}X - \int_t^T D^\beta_{t,u} f(\beta_u, q_u)\, du \;\Big|\; \mathcal{F}_t\right] \quad Q^q\text{-a.s.}$$

From Equation (2.38) we have, for all $t \in [0,T]$,

$$D^\beta_{0,t}\phi_t(X) - \int_0^t D^\beta_{0,u} f(\beta_u, q_u)\, du = E_{Q^q}\left[D^\beta_{0,T} X - \int_0^T D^\beta_{0,u} f(\beta_u, q_u)\, du \,\Big|\, \mathcal{F}_t \right] \; Q^q\text{-a.s.}$$

□

Corollary 2.4.4. *Assume that the function f satisfies (*NORM*). Then, for every $X \in \mathcal{H}^1$ the process $(\phi_t(X))_{t\in[0,T]}$ is a P-supermartingale and admits a Doob-Meyer decomposition of the form $\phi(X) = \phi_0(X) + M - A$ where A is a càdlàg adapted and increasing process with $A_0 = 0$ and M a continuous local martingale.*

Proof. The P-supermartingale property of $\phi(X)$ follows from Proposition 2.4.2 and the fact that $f(0,0) = 0$. Let us show that $\phi(X)$ has a càdlàg modification which is still a P-supermartingale. Let $t \in [0,T]$, since $\phi(X)$ is a P-supermartingale, for all $s \in [t,T] \cap \mathbb{Q}$ we have $E[\phi_s(X) \,|\, \mathcal{F}_t] \le \phi_t(X)$. Hence, by Fatou's lemma and due to the fact that our filtration satisfies the usual conditions we obtain the inequality $\phi^+_t(X) \le \phi_t(X)$, where

$$\phi^+_t(X) := \lim_{s\downarrow t, s\in\mathbb{Q}} \phi_s(X).$$

On the other hand by continuity of martingales we have, for all $(\beta, q) \in \mathcal{D}_+ \times \mathcal{Q}$,

$$\phi^+_t(X) \ge E_{Q^q}\left[D^\beta_{t,T} X - \int_t^T D^\beta_{t,u} f(\beta_u, q_u)\, du \,\Big|\, \mathcal{F}_t \right] \; P\text{-a.s.},$$

so that taking the supremum with respect to β, q yields $\phi^+_t(X) \ge \phi_t(X)$ P-a.s., thus we have $\phi^+(X) = \phi(X)$ P-a.s. We conclude by [70, Proposition 1.3.14] that $\phi(X)$ has a càdlàg modification which is again a supermartingale. This path regularity of $\phi(X)$ ensures that it admits a Doob-Meyer decomposition. □

Now we want to link the dynamic risk measure defined by Equation (2.36) to a BSDE. In that regard, we assume that f is (CONV) and (LSC), and we denote by g the function defined on $\mathbb{R} \times \mathbb{R}^d$ by

$$g(y,z) := \sup_{\beta\ge 0; q\in\mathbb{R}^d} \{-\beta y + qz - f(\beta, q)\}.$$

The function g is (DEC) and if f is (NORM) then g is (POS).

Theorem 2.4.5. *Assume that the function f satisfies (*CONV*), (*LSC*) and (*NORM*). For all $X \in \mathcal{H}^1$, there exists a unique predictable d-dimensional process Z such that $(\phi(X), Z)$ is the minimal supersolution of the BSDE with generator g and terminal condition X.*

Proof. Supersolution property: Let $X \in \mathcal{H}^1$. We start by proving that there exists Z such that $(\phi(X), Z)$ is a supersolution of the BSDE with generator g and terminal condition X. By Corollary 2.4.4 there exist processes A and M such that $\phi_t(X) = \phi_0(X) + M_t - A_t$, and by martingale representation there exists a process $Z \in \mathcal{L}$ such that

$$\phi_t(X) = \phi_0(X) + \int_0^t Z_u\, dW_u - A_t. \tag{2.39}$$

By definition of $\phi(X)$ and Equation (2.39), $\int_0^t Z_u\, dW_u \ge E[X \,|\, \mathcal{F}_t] - \phi_0(X)$. Thus, $\int Z\, dW$ is a supermartingale as a local martingale bounded from below by a martingale.

Let $(\beta, q) \in \mathcal{D}_+ \times \mathcal{Q}$. Applying Itô's formula to $D^\beta_{0,t}\phi_t(X)$ leads us to

$$\begin{aligned} d\left(D^\beta_{0,t}\phi_t(X)\right) &= -\beta_t D^\beta_{0,t}\phi_t(X)\,dt + D^\beta_{0,t} d\phi_t(X) \\ &= -\beta_t D^\beta_{0,t}\phi_t(X)\,dt + D^\beta_{0,t}\left(-dA_t + Z_t\,dW_t\right) \\ &= -\beta_t D^\beta_{0,t}\phi_t(X)\,dt + D^\beta_{0,t}\left(-dA_t + Z_t q_t\,dt\right) + D^\beta_{0,t} Z_t\,dW^{Q^q}_t. \end{aligned}$$

Therefore,

$$\begin{aligned} &d\left(D^\beta_{0,t}\phi_t(X) - \int_0^t D^\beta_{0,u} f(\beta_u, q_u)\,du\right) \\ &\quad = D^\beta_{0,t}\left(-\beta_t\phi_t(X)\,dt - dA_t + Z_t q_t\,dt - f(\beta_t, q_t)\,dt\right) + D^\beta_{0,t} Z_t\,dW^{Q^q}_t. \end{aligned} \tag{2.40}$$

By the Q^q-supermartingale property proved in Proposition 2.4.2, we have

$$dA_t \geq \left(-\beta_t\phi_t(X) + q_t Z_t - f(\beta_t, q_t)\right)\,dt.$$

Since β and q were taken arbitrary, it holds

$$dA_t \geq g(\phi_t(X), Z_t)\,dt. \tag{2.41}$$

Hence Equation (2.39) gives, for all $0 \leq s \leq t \leq T$,

$$\phi_s(X) - \int_s^t g(\phi_u(X), Z_u)\,du + \int_s^t Z_u\,dW_u \geq \phi_t(X),$$

which shows that $(\phi(X), Z)$ is an admissible supersolution.

Minimality: Showing that the process $\phi(X)$ is minimal is done using exactly the same arguments as those used to prove Equation (2.33) in the second step of the proof of Theorem 2.3.12 and the first part of the proof of Theorem 2.3.6. Replacing 0 by t and the expectation by the conditional expectation in the proof of Equation (2.33) does not affect the reasoning. Recalling that since g is (CONV), (DEC) and (POS) the minimal supersolution is unique concludes the proof. □

Using the mean result of [38], if we restrict ourselves to the case of monetary risk measures, it is possible to extent the previous theorem as follows: given stopping times $0 \leq \sigma \leq \tau \leq T$, consider a functional $\rho_{\sigma,\tau} : L^\infty(\mathcal{F}_\tau) \to L^\infty(\mathcal{F}_\sigma)$ satisfying

a) monotonicity: if $\xi, \eta \in L^\infty(\mathcal{F}_\tau)$ and $\xi \leq \eta$, then $\rho_{\sigma,\tau}(\xi) \leq \rho_{\sigma,\tau}(\eta)$

b) convexity: $\rho_{\sigma,\tau}(\alpha\xi + (1-\alpha)\eta) \leq \alpha\rho_{\sigma,\tau}(\xi) + (1-\alpha)\rho_{\sigma,\tau}(\eta)$ for any $\alpha \in [0,1]$ and $\xi, \eta \in L^\infty(\mathcal{F}_\tau)$

c) cash-additivity: $\rho_{\sigma,\tau}(\xi + \eta) = \rho_{\sigma,\tau}(\xi) + \eta$ for any $\xi \in L^\infty(\mathcal{F}_\tau)$ and $\eta \in L^\infty(\mathcal{F}_\sigma)$

d) time-consistency $\rho_{\sigma,\nu}(\xi) = \rho_{\sigma,\tau}(\rho_{\tau,\nu}(\xi))$ for all $\xi \in L^\infty(\mathcal{F}_\nu)$

e) $\rho_{\sigma,\tau}(0) = 0$

f) $\rho_{\sigma,\tau}(\xi^n) \uparrow \rho_{\sigma,\tau}(\xi)$ for any $(\xi^n) \subseteq L^\infty(\mathcal{F}_\tau)$ with $\xi^n \uparrow \xi$

g) $\rho_{\sigma,\tau}(\xi 1_A + \eta 1_{A^c}) = u_{\sigma,\tau}(\xi)1_A + \rho_{\sigma,\tau}(\eta)1_{A^c}$ for all $\xi, \eta \in L^\infty(\mathcal{F}_\tau)$ and $A \in \mathcal{F}_\sigma$.

h) the convex conjugate $\rho^*_{t,T}$ of $\rho_{t,T}$ satisfies $\rho^*_{t,T}(P) = 0$, $t \in [0,T]$.

Then we have:

Corollary 2.4.6. *Assume the conditions a)-h) above hold for a dynamic convex risk measure ρ. Then, for all $X \in L^\infty$, there exists a function g satisfying (*CONV*), (*DEC*), (*LSC*) and (*POS*) and a unique predictable d-dimensional process Z such that the pair of processes $(\rho_{t,T}(X), Z_t)_{t\in[0,T]}$ is the minimal supersolution of the BSDE with terminal condition X and generator g.*

Proof. By [38, Theorem 3.2], there exists a proper function f satisfying (CONV); (LSC) as well as (POS) such that

$$\rho_{t,T}(X) = \operatorname*{ess\,sup}_{Q^q \in \mathcal{Q}} \left\{ E_{Q^q} \left[X - \int_t^T f_u(q_u)\, du \;\Big|\; \mathcal{F}_t \right] \right\}$$

and

$$\rho^*_{t,T}(Q^q) = E_{Q^q} \left[\int_t^T f_u(q_u)\, du \right].$$

Since $\rho^*_{t,T}(P) = 0$ and f satisfies (POS), we have $f_u(0) = 0$ $P \otimes dt$-almost everywhere. That is, f satisfies (NORM). The result then follows as an application of Theorem 2.4.5 to the case where f does not depend on β. ☐

The next result gives a condition on the dual problem under which the BSDE admits a solution. We refer the reader to [64] for a similar result in the case of constrained BSDEs.

Theorem 2.4.7. *Assume that the function f satisfies (*CONV*), (*LSC*) and (*NORM*). Let $X \in \mathcal{H}^1$, if $\phi_0(X)$ admits a subgradient $(\beta, q) \in \mathcal{D}_+ \times \mathcal{Q}$ then the minimal supersolution $(\phi(X), Z)$ is actually a solution.*

In addition, for $P \otimes dt$-almost all $(\omega, t) \in \Omega \times [0,T]$, $(\beta_t, q_t) \in \partial g(\omega, t, \phi_t(X), Z_t)$, subgradient of g with respect to $(\phi_t(X), Z_t)$.

Proof. Let $X \in \mathcal{H}^1$ and $(\beta, q) \in \mathcal{D}_+ \times \mathcal{Q}$ be a subgradient of $\phi_0(X)$. Then, by Corollary 2.4.3 and the decomposition appearing in Equation (2.40), we have

$$dA_t = (-\beta_t \phi_t(X) + q_t Z_t - f(\beta_t, q_t))\; dt.$$

Definition of g and Equation (2.41) give

$$g(\phi_t(X), Z_t)\, dt \geq (-\beta_t \phi_t(X) + q_t Z_t - f(\beta_t, q_t))\; dt = dA_t \geq g(\phi_t(X), Z_t)\, dt.$$

Then, $dA_t = g(\phi_t(X), Z_t)\, dt$, showing that $(\beta, q) \in \partial g(\phi(X), Z)$ $P \otimes dt$-a.s. Equation (2.39) yields

$$\phi_t(X) = X - \int_t^T g(\phi_u(X), Z_u)\, du + \int_t^T Z_u\, dW_u.$$

Hence $(\phi(X), Z)$ is a solution. ☐

We conclude by the following complete characterization of the minimal supersolution suggested by Corollary 2.3.7 and Theorem 2.4.5.

Theorem 2.4.8. *Assume that the function g satisfies (*CONV*), (*DEC*) and (*POS*), g^* satisfies (*NORM*). If $X \in L^\infty$, then the following are equivalent:*

(i) There exists a predictable d-dimensional process Z such that $(\mathcal{E}(X), Z)$ is the minimal supersolution of the BSDE with terminal condition X and driver g.

(ii) The functional $\mathcal{E}$ admits the representation

$$\mathcal{E}_t(X) = \operatorname*{ess\,sup}_{(\beta,q)\in\mathcal{D}_+\times\mathcal{Q}} E_{Q^q}\left[D^{\beta}_{t,T}X - \int\limits_t^T D^{\beta}_{t,u} g^*(\beta_u, q_u)\,du \,\middle|\, \mathcal{F}_t \right], \quad t\in[0,T].$$

Chapter 3

Portfolio Optimization under Nonlinear Utility

The theory of expected utility is of fundamental importance in finance and economy. Introduced by Bernoulli [8], the expected utility represents the level of satisfaction of a financial agent acting in a risky environment. In their seminal *Theory of Games and Economic Behavior*, von Neumann and Morgenstern [98] have provided an axiomatic foundation for decision making under risk based on *rational* principles; and by the work of Savage [95], under these axioms preferences can be modeled as expected utility. This description of expected utility has been seriously debated, mainly because it implies that the preference of a financial agent should be linear with respect to the probability distribution of the possible outcomes, contrary to the well known Allais paradox and Ellsberg paradox. On the other hand, expected utility does not capture uncertainty in the underlying probabilistic model. Many alternative approaches have been suggested to model decision beyond expected utility. A few examples include the concepts of capacity and weighted expected utility and, more recently, the recursive utility and the g-expectation. Following this trend, we consider in the present work the portfolio optimization of an agent whose utility is modeled by the maximal subsolution of a nonlinear backward stochastic differential equation (BSDE). Our principal aim is to give sufficient, and necessary conditions of existence of an optimal portfolio in this framework.

Amongst the numerous attempts that have been made in the literature to study portfolio optimization under nonlinear utility, the work of El Karoui et al. [56] on the optimization of stochastic differential utility is especially related to ours. This class of utility functions were introduced by Duffie and Epstein [50] and can be seen as solutions of nonlinear BSDEs. In a non-Markovian model, El Karoui et al. [56] prove existence of an optimal trading strategy and an optimal consumption policy and characterize the optimal wealth process and the utility as solutions of a forward-backward system. They assume that the generator of the BSDE satisfies a linear growth condition and is continuously differentiable in all variables, so that the utility itself is differentiable and satisfies a comparison principle. Their results are based on BSDE theory: Notably, the existence result follows from a penalization method which consists in approaching the problem by a sequence of penalized problems that can be solved, and then obtain the solution by compactness arguments.

The first contribution of this chapter is to give conditions that guarantee the existence of an optimal trading strategy for an agent whose utility is given as the maximal subsolution of a BSDE. We consider a non-Markovian incomplete market model where the agent also has a random terminal endowment, and the utility is modeled by a BSDE whose generator is convex, positive, lower semicontinuous and satisfies a normalization condition. The technique of the proof, inspired from Drapeau et al. [47], rests on localization arguments and compactness principles. We do not impose any artificial integrability with respect to the historical probability measure on the wealth process. Hence, the central idea here is to introduce an auxiliary function under which the image of the terminal conditions will be uniformly integrable in the set of subsolutions. To this end, we require the drift to satisfy a suitable integrability condition. This uniform integrability allows for the construction of a localizing sequence of stopping times that makes the value processes of the admissible subsolutions local submartingales. Thus, compactness results for sequences of martingales, see Delbaen and Schachermayer [35], and sequences of increasing finite variation processes can be used locally in time, and the candidate solutions obtained by almost sure convergence of the sequence of stopping times to the time horizon. The verification follows from Fatou's lemma and join convexity of the generator.

Analogous to the case of recursive utility studied by El Karoui et al. [56], there is an intrinsic link between the optimal wealth process and its utility: They can be seen as a maximal subsolution of a forward-backward system.

We also address the question of characterization of an optimal trading strategy. In the optimal stochastic control literature, such a characterization is usually a consequence of the stochastic maximum principle. One introduces a perturbation of the optimal diffusion and, by Itô's formula, obtains at the limit a variational equation which enables to characterize the optimal control, see for instance Peng [82] and Horst et al. [67]. This characterization follows from the fact that the expectation operator is linear, a property that our operator does not enjoy. The idea to get around this difficulty is to use the duality of BSDEs studied by Drapeau et al. [48], and transform the original control problem into a robust control problem with non-zero penalty term. Provided that the robust control problem admits a saddle point, the problem can be linearized and the maximum principle applies. The proof of the existence of a saddle point follows from the existence of an optimal trading strategy and a weak compactness argument introduced by Delbaen et al. [39] which is achieved under a growth condition on the generator of the BSDE.

The theory of BSDE duality fits quite well to our setting. It shows for instance that our maximization problem is nothing but the maximization of recursive utilities under model uncertainty. And because our generator depends on the value process, the uncertainty here also encompasses the uncertainty about the time value of money, see El Karoui and Ravanelli [54] and Drapeau et al. [48]. It also enables us to write and solve the dual problem and characterize its solution in terms of solutions of a BSDE, and shows that the dual optimizer is, in fact, the optimal probabilistic model.

Before presenting the structure of our work, let us give further references of related works. Using a convex duality approach, the expected utility maximization problem was studied by Kramkov and Schachermayer [74]. They give precise conditions on the utility function for a solution to exist. Cvitanić et al. [25] have extended their results to the non-zero random endowment case. A fully probabilistic method to study the problem has been investigated by Hu et al. [68]. For exponential utility, they characterize the value function and the optimal strategy of the problem with random endowment as the solution of a quadratic BSDE. Beyond the exponential utility case, Horst et al. [67] show that the problem can be solved via forward backward systems. Robust expected utilities have been considered by Bordigoni et al. [14] and Faidi et al. [57]. The latter authors consider a problem with non-zero penalty term and prove existence of and optimal model. Øksendal and Sulem [80] show that the robust control problem can be treated as a stochastic differential game, a consideration that is also implicitly used in this chapter of the thesis.

The next section of the chapter is dedicated to the setting of the probabilistic framework of our study and introduces the market model. Section 3.2 studies the primal problem: We prove existence of an optimal strategy and stability of the utility operator. The third section deals with the dual problem. Notably, we prove existence of a dual optimizer and characterize the dual and primal optimizers by means of BSDE solutions. In the last section, we draw the link between duality of BSDEs and the general theory of convex duality.

3.1 Setup and Market Model

Let $T \in (0, \infty)$ be a fixed time horizon, and let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ be a filtered probability space. The filtration $(\mathcal{F}_t)$ is generated by a d-dimensional Brownian motion W and satisfies the usual assumptions of completeness and right-continuity, with $\mathcal{F}_T = \mathcal{F}$. Statements concerning random variables or stochastic processes are understood in the P-almost sure or the $P \otimes dt$-almost sure sense, respectively. Indistinguishable processes are identified. When we make a statement without any precision regarding the probability measure, then we are referring to the probability measure P. Thus, by "M is a martingale" we mean "M is a P-martingale".

We write L^0 for the space of $\mathcal{F}$-measurable random variables endowed with the topology of convergence in probability with respect to the measure P. By $\mathcal{S} := \mathcal{S}(\mathbb{R})$ we denote the set of adapted processes with values in $\mathbb{R}$ which are càdlàg. For $p \in [1, \infty]$, the space $L^p(\Omega, \mathcal{F}, P)$ is denoted by L^p and for a different measure Q we write $L^p(Q)$ for $L^p(\Omega, \mathcal{F}, Q)$. The space L^p_+ is the space of positive random variables belonging to L^p. We further denote by $\mathcal{L}^p := \mathcal{L}^p(P)$ the set of predictable processes Z with values in $\mathbb{R}^{1\times d}$, endowed with the norm $\|Z\|_{\mathcal{L}^p} := E_P[(\int_0^T \|Z_s\|^2\, ds)^{p/2}]^{1/p}$. From [87], for every $Z \in \mathcal{L}^p$ the process $(\int_0^t Z_s\, dW_s)_{t\in[0,T]}$ is well defined and by means of Burkholder-Davis-Gundy's inequality, it is a continuous martingale. By $\mathcal{L}$ we denote the set of predictable processes valued in $\mathbb{R}^{1\times d}$ such that there exits a localizing sequence of stopping times (τ^n) with $Z1_{[0,\tau^n]} \in \mathcal{L}^1$, for all $n \in \mathbb{N}$. For $Z \in \mathcal{L}$, the stochastic process $(\int_0^t Z_u\, dW_u)_{t\in[0,T]}$ is a well defined continuous local martingale. Furthermore, for adequate integrands a and Z we write $\int a\, ds$ and $\int Z\, dW$ for $(\int_0^t a_s\, ds)_{t\in[0,T]}$ and $(\int_0^t Z_u\, dW_u)_{t\in[0,T]}$, respectively. The running maximum of a process X is denoted by $X^*_t = \sup_{s\in[0,t]} |X_s|$. Given a sequence (x_n) in some convex set, a sequence $(\tilde{x}_n)$ is said to be in the asymptotic convex hull of (x_n) if $\tilde{x}_n \in conv\{x^n, x^{n+1}, \dots\}$ for all n.

In the financial market, there are available for trading n stocks, $n \leq d$, with price dynamics

$$dS^i_t = S^i_t(\mu^i_t\, dt + \sigma^i_t\, dW_t), \quad i = 1, \dots, n,$$

such that μ^i and σ^i are predictable processes valued in $\mathbb{R}$ and $\mathbb{R}^d$, respectively. Let us denote by σ the $n \times d$ matrix with row vectors σ^i, the matrix[1] $\sigma\sigma'$ is assumed to be of full rank, so that the market price of risk θ takes the form $\theta_t = \sigma'_t(\sigma_t\sigma'_t)^{-1}\mu_t$, $t \in [0, T]$. For the rest of the chapter, we make the following standing assumption concerning θ:

- There exist constants $p > 1$ and $C_\theta > 0$ such that for all stopping times $0 \leq \tau \leq T$, one has

$$E\left[\left(\mathcal{E}\left(\int \theta\, dW\right)_\tau / \mathcal{E}\left(\int \theta\, dW\right)_T\right)^{\frac{1}{p-1}} \mid \mathcal{F}_\tau\right] \leq C_\theta, \tag{A}$$

where $\mathcal{E}(\int \theta\, dW)$ denotes the stochastic exponential of $\int \theta\, dW$. This is the so-called Muckenhoupt A_p condition. Under this assumption, by [72, Theorem 2.4], $\int \theta\, dW$ is a BMO martingale, and therefore $\frac{dQ}{dP} = \mathcal{E}(-\int \theta\, dW)_T$ defines a probability measure Q equivalent to P. This type of drift conditions are well-known, especially in the context of expected utility maximization, see for instance Delbaen et al. [37]. Let $x > 0$ be a fixed initial capital. A trading strategy is a predictable d-dimensional process π such that

[1] σ' is the transpose of σ.

$\pi\sigma \in \mathcal{L}(Q)$ and $X^\pi \geq 0$, where the wealth process X^π is given by

$$X_t^\pi = x + \int_0^t \pi_s \sigma_s (\theta_s \, ds + dW_s), \quad t \in [0, T]. \tag{3.1}$$

We denote by Π the set of trading strategies. For every $\pi \in \Pi$, X^π is a positive Q-local martingale and thus a Q-supermartingale. In particular, the market is free of arbitrage opportunities. The principal objective of this second chapter is to study the utility maximization from the terminal wealth of an agent who has a non-trivial endowment ξ and whose utility is modeled by a BSDE.

The generator we consider for the BSDEs is a jointly measurable function $g : \Omega \times [0, T] \times \mathbb{R}_+ \times \mathbb{R}^{1 \times d} \to \mathbb{R} \cup \{+\infty\}$, where $\Omega \times [0, T]$ is endowed with the predictable σ-algebra. Furthermore, similar to the previous chapter, a generator g is said to be

(CONV) convex, if $(y, z) \mapsto g(y, z)$ is convex,

(LSC) lower semicontinuous, if $(y, z) \mapsto g(y, z)$ is lower semicontinuous,

(NOR) normalized, if $g(y, 0) = 0$ for all $y \in \mathbb{R}_+$,

(POS) positive, if $g \geq 0$.

Given a random variable $H \in L^0$, a subsolution of the BSDE with generator g and terminal condition H is a pair (Y, Z) of processes satisfying

$$Y_s + \int_s^t g_u(Y_u, Z_u) \, du - \int_s^t Z_u \, dW_u \leq Y_t; \quad Y_T \leq H, \tag{3.2}$$

for all $0 \leq s \leq t \leq T$. Let $u : \mathbb{R}_+ \to \mathbb{R}$ be a continuous concave, strictly increasing function such that there exists $C > 0$, $|u(x)|^{p^2} \leq C(1 + |x|)$ for every $x > 0$, with p introduced in the condition (A) and such that $L \mapsto u^{-1}(E[u(L)])$ is concave on $\{L \in L_+^0 : E[u(L)] < +\infty\}$. Examples of such functions include $u(x) = x^r$ with $rp^2 < 1$, and $u(x) = -\exp(-rx)$ with $r > 0$, see [46, Section 3].

A value process $Y \in \mathcal{S}_+$ is said to be admissible if the process $u(Y)$ is a submartingale. We consider the operator

$$\mathcal{E}_0^g(H) := \sup \{Y_0 : (Y, Z) \in \mathcal{A}^u(H, g)\}$$

with

$$\mathcal{A}^u(H, g) := \{(Y, Z) \in \mathcal{S} \times \mathcal{L} : Y \text{ admissible and (3.2) holds}\},$$

the set of admissible subsolutions with respect to u. The reader will notice that the operator $\mathcal{E}_0^g(\cdot)$ depends on u. Similar to [47] the operator $\mathcal{E}_0^g(\cdot)$ is a nonlinear utility function. In particular, it is monotone, concave but not necessarily cash-additive. We study the investment problem

$$V(x) := \sup_{\pi \in \Pi} \mathcal{E}_0^g(\xi + X_T^\pi). \tag{3.3}$$

More precisely, we would like to give conditions of existence of a pair $(\bar{Y}, \bar{Z})$ along with a trading strategy $\bar{\pi} \in \Pi$ such that $(\bar{Y}, \bar{Z}) \in \mathcal{A}^u(\xi + X_T^{\bar{\pi}}, g)$ and for any other trading strategy $\pi \in \Pi$ one has

$$\bar{Y}_0 = V(x) = \mathcal{E}_0^g(\xi + X_T^{\bar{\pi}}) \geq \mathcal{E}_0^g(\xi + X_T^\pi).$$

Henceforth, the function V will be referred to as the value function of the optimization problem (3.3), and the triple $(\bar{X}, \bar{Y}, \bar{Z})$ with $\bar{X} = X^{\bar{\pi}}$, a maximal subsolution.

Example 3.1.1. *1. Certainty equivalent: Let X be an $\mathcal{F}_T$-measurable random variable such that $u(X)$ is integrable. The certainty equivalent $C_t(X)$ of X is defined as $C_t(X) := u^{-1}\left(E[u(X)\,|\,\mathcal{F}_t]\right)$, $t \in [0,T]$. Consider the utility maximization problem*

$$V(x) = \sup_{\pi\in\Pi} C_0(X_T^\pi + \xi). \tag{3.4}$$

The martingale representation theorem yields a process $N \in \mathcal{L}^1$ such that

$$E[u(X)\,|\,\mathcal{F}_t] = E[u(X)] + \int_0^t N_u\,dW_u, \quad \text{for all } t \in [0,T].$$

Applying Itô's formula to $Y_t = u^{-1}\left(E[u(X)\,|\,\mathcal{F}_t]\right)$, we have

$$dY_t = \frac{1}{u'(Y_t)} N_t\,dW_t - \frac{1}{2}\frac{u''(Y_t)}{(u'(Y_t))^3}\left|N_t\right|^2\,dt.$$

Hence, putting $Z_t = \frac{1}{u'(Y_t)}N_t$, the pair (Y,Z) solves the BSDE

$$Y_t = X + \frac{1}{2}\int_t^T \frac{u''(Y_u)}{u'(Y_u)}\left|Z_u\right|^2\,du - \int_t^T Z_u\,dW_u. \tag{3.5}$$

For $u(x) = x^r$, $r\in(0,1)$, the generator of the BSDE (3.5) *is given by $g(y,z) = \frac{1}{2}(1-r)\left|z\right|^2/y$ and satisfies the conditions (*Conv*), (*Lsc*), (*Nor*) and (*Pos*) on $(0,+\infty)\times\mathbb{R}^d$. By definition, we have $\mathcal{E}_0^g(X) \geq C_0(X)$. In addition, the admissibility condition implies $u(\mathcal{E}_0^g(X)) \leq E\left[u(\mathcal{E}_T^g(X))\right]$. Therefore, $\mathcal{E}_0^g(X) \leq C_0(X)$. Thus, the utility maximization problem* (3.4) *can be rewritten as $V(x) = \sup_{\pi\in\Pi}\mathcal{E}_0^g(X_T^\pi+\xi)$.*

*2. g-expectation: Let u be a utility function and g a function defined on $\mathbb{R}\times\mathbb{R}^d$ and satisfying (*Lsc*), (*Nor*) and (*Pos*) such that for every $\pi\in\Pi$ the BSDE with terminal condition $u(X_T^\pi+\xi)$ and generator g has a unique solution $(Y^\pi,Z^\pi)\in\mathcal{S}\times\mathcal{L}^2$. Denote by $\mathcal{E}_g[u(X_T^\pi+\xi)\,|\,\mathcal{F}_t] := Y_t^\pi$ the g-expectation of $u(X_T^\pi+\xi)$. The operator $\mathcal{E}_g[\cdot]$ is a nonlinear expectation which coincides with the classical expectation $E_P[\cdot]$ when $g=0$. Consider the utility maximization problem*

$$V(x) := \sup_{\pi\in\Pi} u^{-1}\left(\mathcal{E}_g[u(\xi+X_T^\pi)\,|\,\mathcal{F}_0]\right).$$

We further assume u to be twice continuously differentiable and that u' is bounded away from zero. For every $\pi\in\Pi$, we have

$$Y_t^\pi = u(X_T^\pi+\xi) + \int_t^T g(Y_u^\pi, Z_u^\pi)\,du - \int_t^T Z_u^\pi\,dW_u. \tag{3.6}$$

Applying Itô's formula to $\hat{Y}_t^\pi := u^{-1}(Y_t^\pi)$, we obtain

$$d\hat{Y}_t^\pi = -\left\{\frac{1}{u'(\hat{Y}_t^\pi)}g(u(\hat{Y}_t^\pi),\hat{Z}_t^\pi u'(\hat{Y}_t^\pi)) - \frac{1}{2}\frac{u''(\hat{Y}_t^\pi)}{u'(\hat{Y}_t^\pi)}\left|\hat{Z}_t^\pi\right|^2\right\}dt + \hat{Z}_t^\pi\,dW_t, \tag{3.7}$$

*with $\hat{Z}_t^\pi = Z_t^\pi/u'(Y_t^\pi)$ and $\hat{Y}_T^\pi = X_T^\pi+\xi$. For $u(x) = -\exp(-rx)$, $r>0$ and $g(y,z) = |z|$, the generator of the above BSDE takes the form $\hat{g}(y,z) = |z| + \frac{1}{2}(1-r)r^2\left|z\right|^2$ and it satisfies the properties (*Conv*), (*Lsc*), (*Nor*) and (*Pos*). Since g is positive, Y^π is a submartingale and we have $\mathcal{E}_0^{\hat{g}}(X_T^\pi+\xi) \geq \hat{Y}_0^\pi = u^{-1}(Y_0^\pi) = u^{-1}(\mathcal{E}_g(u(X_T^\pi +$*

$\xi) \mid \mathcal{F}_0)$). In addition, the admissibility condition implies $u(\mathcal{E}_0^{\hat{g}}(X_T^\pi+\xi)) \le E[u(\mathcal{E}_T^{\hat{g}}(X_T^\pi+\xi))] \le E[u(X_T^\pi+\xi)]$ by monotonicity of u. Since g is positive, taking expectation of both sides of (3.6) *yields $\mathcal{E}_g(u(X_T^\pi+\xi) \mid \mathcal{F}_0) \ge E[u(X_T^\pi+\xi)]$. Therefore, $\mathcal{E}_0^{\hat{g}}(X_T^\pi+\xi) \le u^{-1}(\mathcal{E}_g(u(X_T^\pi+\xi) \mid \mathcal{F}_0))$. Thus, the utility maximization problem* (3.4) *can be rewritten as $V(x) = \sup_{\pi\in\Pi} \mathcal{E}_0^{\hat{g}}(X_T^\pi+\xi)$.*

3.2 Maximal Subsolutions

3.2.1 Existence Results

In this section we give sufficient conditions of existence of an optimal trading strategy to Problem (3.3). In order to simplify the presentation, let us introduce the set

$$\mathcal{A}(x) := \{(X,Y,Z) : X \text{ satisfies (3.1) for some } \pi\in\Pi \text{ and } (Y,Z)\in\mathcal{A}^u(\xi+X_T,g)\}.$$

The function $V(x)$ can be written as

$$V(x) = \sup\{Y_0 : (X,Y,Z)\in\mathcal{A}(x)\}.$$

If g satisfies (NOR) and $\xi \ge 0$, the set $\mathcal{A}(x)$ is nonempty, and contains an element with positive value process. The triplet (X^0,Y^0,Z^0), with $Z^0=0$, $Y^0=X^0=x$ and with associated trading strategy $\pi=0$ is an element of $\mathcal{A}(x)$. Indeed, the pair (Y^0,Z^0) satisfies (3.2), and we have $Y_T^0 = x \le x+\xi = \xi+X_T^0$. Moreover, for all $(X,Y,Z)\in\mathcal{A}(x)$ the càdlàg process Y can jump only up, since by taking the limit as s tends to $t-$ in Equation (3.2) we have $Y_t \ge Y_{t-}$, for all $t\in[0,T]$. Before stating our existence result, let us prove the following lemmas.

Lemma 3.2.1. *Assume $\xi\in L^1_+(\Omega,\mathcal{F}_T,Q)$. Then there exists a constant $C\ge 0$ such that for all $(X,Y,Z)\in\mathcal{A}(x)$ with $Y\ge 0$, we have*

$$E[|u(\xi+X_T)|^p] \le C \quad \textit{and} \quad u(Y_t) \le E[u(\xi+X_T) \mid \mathcal{F}_t] \quad t\in[0,T].$$

Proof. Let (X,Y,Z) be in $\mathcal{A}(x)$, and q the Hölder conjugate of p. We first prove the L^p boundedness of $u(\xi+X_T)$. Using Hölder's inequality, we estimate as follows:

$$\begin{aligned} E[|u(\xi+X_T)|^p] &= E_Q\left[\frac{1}{\mathcal{E}(\int\theta\,dW)_T}|u(\xi+X_T)|^p\right] \\ &\le E_Q\left[\left(\frac{1}{\mathcal{E}(\int\theta\,dW)_T}\right)^q\right]^{\frac{1}{q}} E_Q[|u(\xi+X_T)|^{p^2}]^{\frac{1}{p}}. \end{aligned}$$

Since there exists a positive constant C such that

$$|u(\xi+X_T)|^{p^2} \le C(1+\xi+X_T),$$

we have

$$E[|u(\xi+X_T)|^p] \le C^{1/p} E\left[\mathcal{E}\left(\int\theta\,dW\right)_T\left(\frac{1}{\mathcal{E}(\int\theta\,dW)_T}\right)^q\right]^{\frac{1}{q}} E_Q[1+\xi+X_T]^{\frac{1}{p}}.$$

Thus, since $q-1=\frac{1}{p-1}$, it follows from the Muckenhoupt A_p condition and the Q-supermartingale property of X, that

$$E\left[|u(\xi+X_T)|^p\right] \le C^{1/p} C_\theta^{1/q}(1+E_Q[\xi]+x)^{\frac{1}{p}},$$

hence the first estimate.

For the second estimate, first notice that $u(\xi + X_T)$ is integrable, and since u is increasing and (Y, Z) satisfies Equation (3.2), we have $u(Y_T) \le u(\xi + X_T)$. Since the value process Y is admissible, we have $u(Y_t) \le E[u(Y_T) \,|\, \mathcal{F}_t] \le E[u(\xi + X_T) \,|\, \mathcal{F}_t]$ for all $t \in [0, T]$. □

The previous lemma gives two *a priori* estimates for subsolutions of Equation (3.2). In particular, it shows that the family of random variables $u(\xi + X_T)$, when (X, Y, Z) runs through $\mathcal{A}(x)$, is uniformly integrable.

Remark 3.2.2. *a) Due to the admissibility condition and the previous lemma, it holds $V(x) \in \mathbb{R}$ for every $x > 0$. In fact, for any $(X, Y, Z) \in \mathcal{A}(x)$, since $(x, x, 0) \in \mathcal{A}$, we can assume $Y_0 \ge x$. By admissibility,*

$$u(Y_0) \le E[u(Y_T)] \le E[u(\xi + X_T)].$$

Lemma 3.2.1 and Jensen's inequality give

$$u(Y_0)^p \le E[|u(\xi + X_T)|^p] \le C.$$

b) If a subsolution $(X, Y, Z) \in \mathcal{A}(x)$ is such that $\log(Y)$ is a submartingale, then since $Y_0 \ge x$, we have $E[\log(Y_t)] \ge \log(x)$ for all $t \in [0, T]$. Hence, $Y_t = 0$ with probability zero. Therefore, the function $u = \log$ can be used to defined admissibility of subsolutions.

The next lemma describes the set of subsolutions.

Lemma 3.2.3. *If g satisfies (*CONV*), then the set $\mathcal{A}(x)$ is convex.*

Proof. Let (X^1, Y^1, Z^1) and (X^2, Y^2, Z^2) be two elements of $\mathcal{A}(x)$; and $\lambda_1, \lambda_2 \in (0, 1)$ such that $\lambda_1 + \lambda_2 = 1$. Then, by joint convexity of g, $(\lambda_1 Y^1 + \lambda_2 Y^2, \lambda_1 Z^1 + \lambda_2 Z^2)$ satisfies Equation (3.2) and the terminal condition $\lambda_1 Y_T^1 + \lambda_2 Y_T^2 \le \lambda_1 X_T^1 + \lambda_2 X_T^2 + \xi$ is also satisfied. In addition, since $u^{-1}(E[u(\cdot)])$ is concave, for all $0 \le s \le t \le T$, we have

$$\begin{aligned} u^{-1}(E[u(\lambda_1 Y_t^1 + \lambda_2 Y_t^2) \,|\, \mathcal{F}_s]) &\ge \lambda_1 u^{-1}(E[u(Y_t^1) \,|\, \mathcal{F}_s]) + \lambda_2 u^{-1}(E[u(Y_t^2) \,|\, \mathcal{F}_s]) \\ &\ge \lambda_1 u^{-1}(u(Y_s^1)) + \lambda_2 u^{-1}(u(Y_s^2)) \\ &= \lambda_1 Y_s^1 + \lambda_2 Y_s^2, \end{aligned}$$

where the second inequality comes from the facts that Y^1 and Y^2 are admissible and u^{-1} increasing. Hence because u is increasing, we have

$$E[u(\lambda_1 Y_t^1 + \lambda_2 Y_t^2) \,|\, \mathcal{F}_s] \ge u(\lambda_1 Y_s^1 + \lambda_2 Y_s^2),$$

which implies that $\lambda_1 Y^1 + \lambda_2 Y^2$ is admissible. Put $X^1 = X^{\pi^1}$ and $X^2 = X^{\pi^2}$. The process $\lambda_1 X^1 + \lambda_2 X^2$ is a wealth process, since

$$\lambda_1 X_t^1 + \lambda_2 X_t^2 = x + \int_0^t (\lambda_1 \pi_u^1 + \lambda_2 \pi_u^2) \sigma_u \, dW_u^Q.$$

□

The following existence theorem is the first main result of this chapter.

Theorem 3.2.4. *Assume that the generator g satisfies (*CONV*), (*LSC*), (*NOR*) and (*POS*); and that the random endowment ξ belongs to L_+^∞. Then there exists a trading strategy $\bar{\pi} \in \Pi$ with associated wealth process $\bar{X}$ and a pair $(\bar{Y}, \bar{Z}) \in \mathcal{A}^u(\xi + \bar{X}_T, g)$ such that $\bar{Y}_0 = V(x)$.*

Proof. Let $((X^n, Y^n, Z^n))$ be a sequence in $\mathcal{A}(x)$ such that $Y_0^n \uparrow V(x)$. The proof goes in several steps. We start by making some transformations on the maximizing sequence $((X^n, Y^n, Z^n))$.

Step 1 Preliminary transformations. The sequence $((X^n, Y^n, Z^n))$ can be considered to be such that for all $n \in \mathbb{N}$, $Y_0^n \geq x$ and $Y^n \geq X^n$. In fact, since the set $\mathcal{A}(x)$ contains the triple $(x, x, 0)$, by definition of $V(x)$ it holds $V(x) \geq x$. Hence, we can assume without loss of generality that $Y_0^n \geq x$, for all n. For each $n \in \mathbb{N}$, define the stopping time δ^n by

$$\delta^n := \inf\{t \geq 0 : Y_t^n \leq X_t^n\} \wedge T,$$

and put

$$\hat{Y}^n := Y^n 1_{[0,\delta^n)} + Y_{\delta^n}^n 1_{[\delta^n,T]}; \quad \hat{Z}^n := Z^n 1_{[0,\delta^n]}$$

and

$$\hat{X}^n := X^n 1_{[0,\delta^n]} + X_{\delta^n}^n 1_{[\delta^n,T]}.$$

The triple $(\hat{X}^n, \hat{Y}^n, \hat{Z}^n)$ belongs to $\mathcal{A}(x)$. In fact, for all $s, t \in [0, T]$ with $0 \leq s \leq t \leq T$, on the set $\{s \leq \delta^n \leq t\}$ we have

$$\begin{aligned}
&\hat{Y}_s^n + \int_s^t g_u(\hat{Y}_u^n, \hat{Z}_u^n)\, du - \int_s^t \hat{Z}_u^n\, dW_u \\
&\quad = Y_s^n + \int_s^{\delta^n} g_u(Y_u^n, Z_u^n)\, du - \int_s^{\delta^n} Z_u^n\, dW_u + \int_{\delta^n}^t g_u(Y_{\delta^n}^n, 0)\, du \\
&\quad \leq Y_{\delta^n}^n = \hat{Y}_{\delta^n}^n.
\end{aligned}$$

On the sets $\{s \geq \delta^n\}$ and $\{t \leq \delta^n\}$ the proof is the same. Now for the forward process, let $t \in [0, T]$. On the set $\{\delta^n \leq t\}$, putting $\hat{\pi}^n := \pi^n 1_{[0,\delta^n]}$, we have

$$\hat{X}_t^n = X_{\delta^n}^n = x + \int_0^{\delta^n} X_u^n \pi_u^n \sigma_u\, dW_u^Q + \int_{\delta^n}^t 0\, dW_u^Q = x + \int_0^t \hat{X}_u^n \hat{\pi}_u^n \sigma_u\, dW_u^Q.$$

On $\{t \leq \delta^n\}$ there is nothing to prove. In order to show that the terminal condition is satisfied, notice that on the set $\{\delta^n < T\}$ it holds $Y_{\delta^n}^n = X_{\delta^n}^n$. This is because $Y_0^n \geq x$, X^n is continuous and Y^n only jumps upward. Thus,

$$\hat{Y}_T^n = Y_{\delta^n}^n = X_{\delta^n}^n \leq X_{\delta^n}^n + \xi = \hat{X}_T^n + \xi$$

and on the set $\{\delta^n = T\}$ it holds

$$\hat{Y}_T^n = Y_T^n \leq \xi + X_T^n = \xi + \hat{X}_T^n.$$

In addition, for all $n \in \mathbb{N}$, $\hat{\pi}^n$ is a trading strategy and $u(\hat{Y}^n)$ is a P-submartingale. In fact, for all $0 \leq s \leq t \leq T$, due to the admissibility of Y^n, we have

$$E[u(\hat{Y}_t^n) - u(\hat{Y}_s^n) \,|\, \mathcal{F}_s] = E[u(Y_{(s \vee \delta^n) \wedge t}^n) - u(Y_s^n) \,|\, \mathcal{F}_s] \geq 0.$$

Hence $\hat{Y}^n$ is admissible. Therefore, we have

$$((\hat{X}^n, \hat{Y}^n, \hat{Z}^n)) \subseteq \mathcal{A}(x)$$

with $\hat{Y}_0^n \uparrow V(x)$ and for all $t \in [0, T]$, $\hat{X}_t^n \leq \hat{Y}_t^n$. In the sequel of the proof we shall simply write (X^n, Y^n, Z^n) for $(\hat{X}^n, \hat{Y}^n, \hat{Z}^n)$, for every $n \in \mathbb{N}$.

Step 2 An estimate for the value process. Now we provide a bound on the value process that will be a key ingredient for the localization in the subsequent step. Since (X^n_T) is a sequence of positive random variables, by [34, Lemma A1.1] there exists a sequence denoted $(\tilde{X}^n_T)$ in the asymptotic convex hull of (X^n_T) and an $\mathcal{F}_T$-measurable random variable X such that

$$\lim_{n\to\infty} \tilde{X}^n_T = X \quad Q\text{-a.s.}$$

Let $(\tilde{X}^n)$ be the sequence in the asymptotic convex hull associated to $(\tilde{X}^n_T)$. For each $n \in \mathbb{N}$ the process $\tilde{X}^n$ is positive and inherits the Q-supermartingale property of X^n, that is, $E_Q[\tilde{X}^n_T] \leq x$. Hence, it follows from Fatou's lemma that

$$x \geq \liminf_{n\to\infty} E_Q[\tilde{X}^n_T] \geq E_Q[\liminf_{n\to\infty} \tilde{X}^n_T] = E_Q[X].$$

By continuity of the function u and Q-almost sure convergence of $(\tilde{X}^n_T)$ it follows that $(u(\xi + \tilde{X}^n_T))$ converges to $u(\xi + X)$ Q-a.s., and therefore P-a.s. by equivalence of measures. Moreover, due to Lemmas 3.2.3 and 3.2.1, the family $(u(\xi + \tilde{X}^n_T))_n$ is uniformly integrable. Therefore, we can conclude using the dominated convergence theorem that

$$\lim_{n\to\infty} u(\xi + \tilde{X}^n_T) = u(\xi + X) \quad \text{in } L^1. \tag{3.8}$$

For all $n \in \mathbb{N}$ and $t \in [0, T]$ define

$$M^n_t := E[u(\xi + \tilde{X}^n_T) \,|\, \mathcal{F}_t] \quad \text{and} \quad M_t := E[u(\xi + X) \,|\, \mathcal{F}_t].$$

We denote by $((\tilde{X}^n, \tilde{Y}^n, \tilde{Z}^n))$ the sequence in the asymptotic convex hull of $((X^n, Y^n, Z^n))$ associated to $(\tilde{X}^n_T)$. By Lemma 3.2.3, $((\tilde{X}^n, \tilde{Y}^n, \tilde{Z}^n)) \subseteq \mathcal{A}(x)$, and Lemma 3.2.1 leads to

$$u(\tilde{Y}^n_t) \leq M^n_t \leq (M^n_\cdot)^*_T \quad \text{for all } t \in [0, T],$$

which implies, since u^{-1} is increasing, that $\tilde{Y}^n_t \leq u^{-1}((M^n_\cdot)^*_T)$. Thus, $(\tilde{Y}^n_\cdot)^*_T \leq u^{-1}((M^n_\cdot)^*_T)$; recall that $\tilde{Y}^n_t \geq \tilde{X}^n_t \geq 0$. Using again the fact that u^{-1} is increasing and the inequalities

$$(M^n_\cdot)^*_T \leq (M^n_\cdot - M_\cdot + M_\cdot)^*_T \leq (M^n_\cdot - M_\cdot)^*_T + M^*_T$$

we finally have

$$(\tilde{Y}^n_\cdot)^*_T \leq u^{-1}((M^n_\cdot - M_\cdot)^*_T + M^*_T).$$

Step 3 Local bound for the control process. Here we obtain an estimate that will enable us to use a compactness argument for the space $\mathcal{L}^1$. That estimate stems from the fact that Y^n can be shown to be a local submartingale. We start by introducing a localization of the value processes. Since the sequence (M^n_T) converges in L^1, for a given $k \in \mathbb{N}$ we may, and do, choose a subsequence $(M^{n,k})_n$ such that

$$E[|M^{n,k}_T - M_T|] \leq \frac{2^{-n}}{k} \quad n \in \mathbb{N}. \tag{3.9}$$

Let $((\tilde{X}^{n,k}, \tilde{Y}^{n,k}, \tilde{Z}^{n,k}))_n$ be the subsequence of $((\tilde{X}^n, \tilde{Y}^n, \tilde{Z}^n))_n$ associated to $(M^{n,k}_T)_n$. Now, introduce the sequence of stopping times

$$\tau^k = \inf\Big\{t \geq 0 : (\tilde{Y}^{n,k}_\cdot)^*_t \geq k, \text{ for some } n \in \mathbb{N}\Big\} \wedge T.$$

Let us show that (τ^k) is in fact a localizing sequence.

$$
\begin{aligned}
P[\tau^k = T] &= P\left[(\tilde{Y}^{n,k}_{\cdot})^*_T < k, \text{ for all } n \in \mathbb{N}\right] \\
&\geq P\left[u^{-1}((M^{n,k}_{\cdot} - M_{\cdot})^*_T + M^*_T) < k, \text{ for all } n \in \mathbb{N}\right] \\
&= 1 - P\left[(M^{n,k}_{\cdot} - M_{\cdot})^*_T + M^*_T \geq u(k), \text{ for some } n \in \mathbb{N}\right] \\
&\geq 1 - P\left[\left\{(M^{n,k}_{\cdot} - M_{\cdot})^*_T \geq 1, \text{ for some } n \in \mathbb{N}\right\} \cup \{(M_{\cdot})^*_T > u(k)-1\}\right] \\
&= 1 - P\left[(M^{n,k}_{\cdot} - M_{\cdot})^*_T \geq 1, \text{ for some } n \in \mathbb{N}\right] - P\left[(M_{\cdot})^*_T > u(k)-1\right] \\
&\geq 1 - \sum_n P\left[(M^{n,k}_{\cdot} - M_{\cdot})^*_T \geq 1\right] - P\left[(M_{\cdot})^*_T > u(k)-1)\right] \\
&\geq 1 - \sum_n E\left[|M^{n,k}_T - M_T|\right] - \frac{E\left[(M_{\cdot})^*_T\right]}{u(k)-1} \qquad (3.10) \\
&\geq 1 - \frac{1}{k} - \frac{E[(M_{\cdot})^*_T]}{u(k)-1} \longrightarrow 1 \\
&\qquad\qquad\qquad k \longrightarrow \infty,
\end{aligned}
$$

where we used Markov's inequality to obtain (3.10). Therefore (τ^k) is a localizing sequence.

Let $n, k \in \mathbb{N}$, for all $t \in [0,T]$, $\tilde{Y}^{n,k}_{t\wedge\tau^k}$ is integrable. It follows from Jensen's inequality, since u^{-1} is convex, that for all $0 \leq s \leq t \leq T$,

$$
\begin{aligned}
E\left[\tilde{Y}^{n,k}_{t\wedge\tau^k} \,|\, \mathcal{F}_s\right] &= E\left[u^{-1} \circ u(\tilde{Y}^{n,k}_{t\wedge\tau^k}) \,|\, \mathcal{F}_s\right] \\
&\geq u^{-1}\left(E\left[u(\tilde{Y}^{n,k}_{t\wedge\tau^k}) \,|\, \mathcal{F}_s\right]\right). \qquad (3.11)
\end{aligned}
$$

On the set $\{\tau^k \leq s\}$ it holds $E[u(\tilde{Y}^{n,k}_{\tau^k}) \,|\, \mathcal{F}_s] = u(\tilde{Y}^{n,k}_{\tau^k})$, and recalling that $u(\tilde{Y}^{n,k})$ is a submartingale, on the set $\{\tau_k > s\}$ it holds $E[u(\tilde{Y}^{n,k}_{t\wedge\tau^k}) \,|\, \mathcal{F}_s] \geq u(\tilde{Y}^{n,k}_{s\wedge\tau^k})$ by the optional sampling theorem. As u^{-1} is increasing, (3.11) leads to

$$
E\left[\tilde{Y}^{n,k}_{t\wedge\tau^k} \,|\, \mathcal{F}_s\right] \geq u^{-1}\left(u\left(\tilde{Y}^{n,k}_{s\wedge\tau^k}\right)\right) = \tilde{Y}^{n,k}_{s\wedge\tau^k}.
$$

Hence for all $n \in \mathbb{N}$, $\tilde{Y}^{n,k,\tau^k} := \tilde{Y}^{n,k}_{\cdot\wedge\tau^k}$ is a submartingale and $E[\tilde{Y}^{n,k}_{\tau^k} \,|\, \mathcal{F}_{\cdot}]$ is a martingale. By Doob-Meyer decomposition, see [87, Theorem 3.3.13], the càdlàg submartingale $\tilde{Y}^{n,k,\tau^k}$ admits the unique decomposition

$$
\tilde{Y}^{n,k}_{t\wedge\tau^k} = \tilde{Y}^{n,k}_0 + \tilde{A}^{n,k}_{t\wedge\tau^k} + \tilde{N}^{n,k}_{t\wedge\tau^k}, \quad t \in [0,T], \qquad (3.12)
$$

where $\tilde{A}^{n,k,\tau^k}$ is an increasing predictable process starting at 0 and $\tilde{N}^{n,k,\tau^k}$ is a local martingale. Moreover, by Equation (3.2) and Lemma 3.2.3 there exists an increasing càdlàg process $\tilde{K}^{n,k}$ with $\tilde{K}^{n,k}_0 = 0$ such that

$$
\tilde{Y}^{n,k}_t = \tilde{Y}^{n,k}_0 + \int_0^t g_u(\tilde{Y}^{n,k}_u, \tilde{Z}^{n,k}_u)\, du + \tilde{K}^{n,k}_t - \int_0^t \tilde{Z}^{n,k}_u \, dW_u;
$$

where $\int g(\tilde{Y}^{n,k}, \tilde{Z}^{n,k})\, du + \tilde{K}^{n,k}$ is increasing, since g fulfills (POS), and is predictable. In addition $\int \tilde{Z}^{n,k}\, dW$ is a local martingale. By uniqueness of Doob-Meyer decomposition the processes $-\int \tilde{Z}^{n,k} 1_{[0,\tau^k]}\, dW$ and $\tilde{N}^{n,k,\tau^k}$ as well as $\int g(\tilde{Y}^{n,k}, \tilde{Z}^{n,k}) 1_{[0,\tau^k]}\, du + \tilde{K}^{n,k,\tau^k}$ and $\tilde{A}^{n,k,\tau^k}$ are indistinguishable. Then, from Equation (3.12) and $\tilde{Y}^{n,k}_t \geq 0$ we have for all $t \in [0,T]$

$$\int_0^{t\wedge\tau^k} \tilde{Z}^{n,k}_u \, dW_u = \tilde{Y}^{n,k}_0 - \tilde{Y}^{n,k}_{t\wedge\tau^k} + \tilde{A}^{n,k}_{t\wedge\tau^k} \leq V(x) + \tilde{A}^{n,k}_{\tau^k}, \tag{3.13}$$

where the last inequality comes from the fact that $(\tilde{Y}^{n,k}_0)_n$ increases to $V(x)$. On the other hand, since $(\tilde{Y}^{n,k}, \tilde{Z}^{n,k})$ satisfies (3.2) and g satisfies (POS),

$$\begin{aligned} \int_0^{t\wedge\tau^k} \tilde{Z}^{n,k}_u \, dW_u &\geq \tilde{Y}^{n,k}_0 - \tilde{Y}^{n,k}_{t\wedge\tau^k} + \int_0^{t\wedge\tau^k} g(\tilde{Y}^{n,k}_u, \tilde{Z}^{n,k}_u)\, du \\ &\geq -\tilde{Y}^{n,k}_{t\wedge\tau^k} \\ &\geq -E[\tilde{Y}^{n,k}_{\tau^k} \,|\, \mathcal{F}_{t\wedge\tau^k}], \end{aligned} \tag{3.14}$$

where the last inequality comes from the fact that $\tilde{Y}^{n,k,\tau^k}$ is a submartingale. Therefore, $\int \tilde{Z}^{n,k} 1_{[0,\tau^k]}\, dW$ is a supermartingale, as a local martingale bounded from below by the martingale $-E[\tilde{Y}^{n,k}_{\tau^k} \,|\, \mathcal{F}_{\cdot\wedge\tau^k}]$. Hence, the inequalities (3.13) and (3.14) above lead to

$$\left| \int_0^{t\wedge\tau^k} \tilde{Z}^{n,k}_u \, dW_u \right| \leq V(x) + \left|\tilde{Y}^{n,k}_{t\wedge\tau^k}\right| + \tilde{A}^{n,k}_{\tau^k},$$

which implies

$$\left(\int_0^{\cdot} \tilde{Z}^{n,k}_u \, dW_u \right)^*_{T\wedge\tau^k} \leq V(x) + k + \tilde{A}^{n,k}_{\tau^k}.$$

The random variable $\tilde{A}^{n,k}_{\tau^k}$ is bounded in L^1, since we have $\tilde{A}^{n,k}_{\tau^k} = \tilde{Y}^{n,k}_0 - \tilde{Y}^{n,k}_{\tau^k} + \int_0^{\tau^k} \tilde{Z}^{n,k}_u \, dW_u$ with $(\tilde{Y}^{n,k}_0)_n$ increasing; $\int \tilde{Z}^{n,k} 1_{[0,\tau^k]}\, dW$ a P-supermartingale and $\tilde{Y}^{n,k}_{\tau^k}$ bounded. Hence, by Burkholder-Davis-Gundy's inequality, $(\tilde{Z}^{n,k} 1_{[0,\tau^k]})_n$ is bounded in $\mathcal{L}^1$.

Step 4 Construction of the candidates $\bar{Z}$ and $\bar{Y}$. Now we are ready to construct the candidates maximizers for the control and the value processes. These constructions are based on compactness principles for the spaces $\mathcal{L}^1$ and L^1. Since $(\tilde{Z}^{n,k} 1_{[0,\tau^k]})_n$ is $\mathcal{L}^1$ bounded, there exists, by means of [35, Theorem A], a sequence again denoted $(\tilde{Z}^{n,k} 1_{[0,\tau^k]})_n$ in the asymptotic convex hull of $(\tilde{Z}^{n,k} 1_{[0,\tau^k]})_n$ which converges in $\mathcal{L}^1$ along a localizing sequence $(\sigma^{n,k})_n$, and therefore $P \otimes dt$-a.s., to a process $\bar{Z}^k$. We obtain $\bar{Z}$ by implementing a diagonalization procedure such as in step 7 of the proof of [47, Theorem 4.1]: For another $k' > k$, we can find a subsequence $(\tilde{Z}^{n,k'})_n$ such that $(\tilde{Z}^{n,k'} 1_{[0,\tau^{k'}]} 1_{[0,\sigma^{n,k'}]})_n$ converges to a process $\bar{Z}^{k'}$ in $\mathcal{L}^1$ and $P \otimes dt$-a.s. By the same method, we can define the process $\bar{Z}$ by

$$\bar{Z}_0 = 0; \quad \bar{Z} = \sum_{k=1}^{\infty} \bar{Z}^k 1_{(\tau^{k-1},\tau^k]},$$

and put $\tilde{Z}^n = \tilde{Z}^{n,n}$ and $\sigma^{n,n} = \sigma^n$. Hence $(\tilde{Z}^n 1_{[0,\tau^n]} 1_{[0,\sigma^n]})$ converges to $\bar{Z}$ in $\mathcal{L}^1$ and $P \otimes dt$-a.s., but we also have $(\tilde{Z}^n 1_{[0,\tau^k]} 1_{[0,\sigma^k]})_n$ converges to $\bar{Z}^k$ for all k. Thus, by

Burkholder-Davis-Gundy's inequality,

$$\int_0^{t\wedge\tau^k\wedge\sigma^k} \tilde{Z}_s^n\,dW_s \longrightarrow \int_0^{t\wedge\tau^k\wedge\sigma^k} \bar{Z}_s\,dW_s, \quad \text{for all } t,\ P\text{-a.s. and for each } k.$$

Taking the limit as $k \to \infty$ we have, for all t,

$$\int_0^t \tilde{Z}_u^n\,dW_u \longrightarrow \int_0^t \bar{Z}_u\,dW_u, \quad P\text{-a.s.} \tag{3.15}$$

Let $(\tilde{Y}^n)$ be a sequence in the asymptotic convex hull of (Y^n) corresponding to $(\tilde{Z}^n)$. For all $t \in [0,T]$ and $k \in \mathbb{N}$, we have $\tilde{Y}^n_{t\wedge\tau^k} = \tilde{Y}^n_0 + \tilde{A}^n_{t\wedge\tau^k} - \int_0^{t\wedge\tau^k} \tilde{Z}^n_u\,dW_u$. The sequence $(\tilde{A}^n_{T\wedge\tau^k})$ is bounded in L^1 as a consequence of the L^1-boundedness of $(A^n_{T\wedge\tau^k})_n$. Therefore, by Helly's theorem, we can find a subsequence in the asymptotic convex hull of $(\tilde{A}^{n,\tau^k})_n$ still denoted $(\tilde{A}^{n,\tau^k})_n$ such that, for k fixed, $(\tilde{A}^n_{t\wedge\tau^k})_n$ converges to $\tilde{A}_{t\wedge\tau^k}$ for all $t \in [0,T]$, P-a.s. and such that $\tilde{A}^{\tau^k}$ is an increasing positive integrable process with $\tilde{A}_0 = 0$. In particular, $(\tilde{A}^n_T)$ converges to $\tilde{A}_T$ P-a.s. Letting k go to infinity, $(\tilde{A}_{t\wedge\tau^k})_k$ converges to $\tilde{A}_t$, for all $t \in [0,T)$, P-a.s. Therefore we put

$$\tilde{Y}_t := \lim_{k\to\infty}\lim_{n\to\infty} \tilde{Y}^n_{t\wedge\tau^k} = V(x) + \tilde{A}_t - \int_0^t \hat{Z}_u\,dW_u; \quad t \in [0,T). \tag{3.16}$$

and for all $t \in [0,T)$, define

$$\bar{Y}_t := \lim_{s\downarrow t\, s\in\mathbb{Q}} \tilde{Y}_s = V(x) + \lim_{s\downarrow t\, s\in\mathbb{Q}} \tilde{A}_s - \int_0^t \hat{Z}_u\,dW_u$$

and $\bar{Y}_T := \tilde{Y}_T$. We claim that

$$\bar{Y} = \tilde{Y} \quad P\otimes dt\text{-a.s.} \tag{3.17}$$

This is because the jumps of $\tilde{Y}$ and $\bar{Y}$ coincide with the jumps of $\tilde{A}$, and being increasing, the latter process has countably many jumps.

Step 5 Construction of the candidate $\bar{X}$. Recall that since g satisfies (CONV), by Lemma 3.2.3 for all $n \in \mathbb{N}$ the triple $(\tilde{X}^n, \tilde{Y}^n, \tilde{Z}^n)$, element of the asymptotic convex hull of $((X^n, Y^n, Z^n))_n$ is in $\mathcal{A}(x)$; and from *Step 1* we have $0 \leq \tilde{X}^n_t \leq \tilde{Y}^n_t$. Moreover, for each $n \in \mathbb{N}$ the process $\tilde{X}^n$ admits the representation

$$\tilde{X}^n_t = x + \int_0^t \tilde{\nu}^n_u\,dW^Q_u, \quad t \in [0,T]$$

for some predictable process $\tilde{\nu}^n \in \mathcal{L}^1(Q)$. Hence for all $t \in [0,T]$, for all $n \in \mathbb{N}$, we have

$$\left|\int_0^t \tilde{\nu}^n_u\,dW^Q_u\right| = \left|\tilde{X}^n_t - x\right| \leq |\tilde{Y}^n_t| + x,$$

which implies, taking $(\tilde{\nu}^{n,k})_n$ to be the subsequence corresponding to $(M^{n,k})_n$, recall (3.9),

$$\left(\int_0^{\cdot} \tilde{\nu}_u^{n,k}\, dW_u^Q\right)^*_{T\wedge\tau^k} \leq (\tilde{Y}_\cdot^{n,k})^*_{T\wedge\tau^k} + x \leq k + x. \tag{3.18}$$

Therefore, by Burkholder-Davis-Gundy's inequality $(\tilde{\nu}^{n,k} 1_{[0,\tau^k]})_n$ is bounded in $\mathcal{L}^1(Q)$. With this local $\mathcal{L}^1(Q)$ bound at hand, we can use similar arguments as in *Step 4* to obtain a process $\bar{\nu}$ such that

$$\int_0^{t\wedge\tau^k} \tilde{\nu}_u^n\, dW_u^Q \longrightarrow \int_0^{t\wedge\tau^k} \bar{\nu}_u\, dW_u^Q \quad \text{for all } t,\ Q\text{-a.s. and for each k} \tag{3.19}$$

and

$$\int_0^t \tilde{\nu}_u^n\, dW_u^Q \longrightarrow \int_0^t \bar{\nu}_u\, dW_u^Q \quad \text{for all } t \in [0,T]\,,\ Q\text{-a.s.}$$

Put

$$\bar{X}_t = x + \int_0^t \bar{\nu}_u\, dW_u^Q. \tag{3.20}$$

Step 6 Verification. It follows from the definition of $\bar{Y}$ that $\bar{Y}_0 \geq V(x)$; let us verify that $(\bar{X}, \bar{Y}, \bar{Z})$ actually belongs to $\mathcal{A}(x)$. We start by showing that $\bar{X}$ is a wealth process. From $\tilde{X}^n \geq 0$ for all $n \in \mathbb{R}$, follows $\bar{X} \geq 0$. Since $\sigma\sigma'$ is of full rank, we can find a predictable process $\bar{\pi}$ such that $\bar{\pi}\sigma = \bar{\nu}$. Hence, from (3.18) and (3.19), $\bar{\pi}\sigma 1_{[0,\tau^k]} \in \mathcal{L}^1(Q)$ for all $k \in \mathbb{N}$ and therefore $\bar{\pi}\sigma \in \mathcal{L}(Q)$ and $d\bar{X}_t = \bar{\pi}_u\sigma_u(\theta_u du + dW_u)$. Next let us show that $(\bar{Y}, \bar{Z}) \in \mathcal{A}^u(\xi + \bar{X}_T, g)$. To that end, we use an argument from [47] . By (3.17), there exists a set $B \subseteq \Omega \times [0,T]$ with $P \otimes dt(B^c) = 0$ such that $\bar{Y}_t(\omega) = \tilde{Y}_t(\omega)$ for all $(\omega, t) \in B$. Then, there exists a set $D \subseteq \{\omega : (\omega, t) \in B,\ \text{for some } t\}$ with $P(D) = 1$ such that for all $\omega \in D$ the set $I(\omega) := \{t \in [0,T] : (\omega, t) \in B\}$ is a set of Lebesgue measure T and $\bar{Y}_t(\omega) = \tilde{Y}_t(\omega)$ for all $t \in I(\omega)$. Denote by λ_i^n, $n \leq i \leq \Lambda^n$, the convex weights of the convex combination $\tilde{Z}^n$. Let $s, t \in I$, $s \leq t$, where I; s and t depend on $\omega \in D$. Using subsequently Fatou's lemma and (CONV) we are led to

$$\begin{aligned}
&\bar{Y}_s + \int_s^t g_u(\bar{Y}_u, \bar{Z}_u)\, du - \int_s^t \bar{Z}_u\, dW_u \\
&\leq \lim_{k\to\infty} \liminf_{n\to\infty} \left(\tilde{Y}^n_{s\wedge\tau^k} + \int_{s\wedge\tau^k}^{t\wedge\tau^k} g_u(\tilde{Y}_u^n, \tilde{Z}_u^n 1_{[0,\sigma^n]}(u))\, du - \int_{s\wedge\tau^k}^{t\wedge\tau^k} \tilde{Z}_u^n\, dW_u \right) \\
&\leq \lim_{k\to\infty} \liminf_{n\to\infty} \sum_{i=n}^{\Lambda^n} \lambda_i^n \left(Y^i_{s\wedge\tau^k} + \int_{s\wedge\tau^k}^{t\wedge\tau^k} g_u(Y_u^i, Z_u^i)\, du - \int_{s\wedge\tau^k}^{t\wedge\tau^k} Z_u^i\, dW_u \right) \\
&\leq \lim_{k\to\infty} \liminf_{n\to\infty} \sum_{i=n}^{\Lambda^n} \lambda_i^n Y^i_{t\wedge\tau^k} = \lim_{k\to\infty} \liminf_{n\to\infty} \tilde{Y}^n_{t\wedge\tau^k} \\
&= \lim_{k\to\infty} \tilde{Y}_{t\wedge\tau^k} = \tilde{Y}_t = \bar{Y}_t.
\end{aligned} \tag{3.21}$$

If s or t are not in I, then there exist two sequences (s_n) and (t_n) in I such that $s_n \downarrow s$, $t_n \downarrow t$ and $s_n \leq t_n$. Equation (3.21) holds for each s_n, t_n. Namely,

$$\bar{Y}_{s_n} + \int_{s_n}^{t_n} g(\bar{Y}_u, \bar{Z}_u)\, du - \int_{s_n}^{t_n} \bar{Z}_u\, dW_u \leq \bar{Y}_{t_n}$$

holds for all $n \in \mathbb{N}$. Since $\bar{Y}$ is right continuous and the integrals are continuous, taking the limit as n tends to infinity yields the desired result for s and t. Therefore, the pair $(\bar{Y}, \bar{Z})$ satisfies the inequality (3.2) with terminal condition $H = \xi + \bar{X}_T$ since for all $n \in \mathbb{N}$, $\tilde{Y}_T^n \leq \xi + \tilde{X}_T^n$; and $(\tilde{Y}_T^n)$ and $(\tilde{X}_T^n)$ converges P-a.s. to $\bar{Y}_T$ and $\bar{X}_T$, respectively. Now let us show that $\bar{Y}$ is admissible and is a càdlàg process. Due to Lemmas 3.2.1 and 3.2.3 and positivity of u we have for all $n \in \mathbb{N}$ and $t \in [0,T]$

$$\begin{aligned} u\left(\tilde{Y}_t^n\right)^p &\leq E\left[u\left(\xi + \tilde{X}_T^n\right) \mid \mathcal{F}_t\right]^p \\ &\leq E\left[u(\xi + \tilde{X}_T^n)^p \mid \mathcal{F}_t\right], \end{aligned}$$

where we used Jensen's inequality. Taking expectation on both sides leads to $E[u(\tilde{Y}_t^n)^p] \leq E[u(\xi + \tilde{X}_T^n)^p] \leq C$. Hence, the family $(u(\tilde{Y}_t^n))_n$ is uniformly integrable, for all $t \in [0,T]$. Since for all n the process $\tilde{Y}^n$ is admissible, we have $u(\tilde{Y}_s^n) \leq E[u(\tilde{Y}_t^n) \mid \mathcal{F}_s]$, $0 \leq s \leq t \leq T$. Taking the limit as n goes to infinity, we obtain by means of continuity of u and dominated convergence theorem $u(\tilde{Y}_s) \leq E[u(\tilde{Y}_t) \mid \mathcal{F}_s]$, i.e. $u(\tilde{Y})$ is a submartingale. The continuity property of the function u and definition of $\hat{Y}$ imply

$$u(\bar{Y}_t) = \lim_{s \uparrow t, s \in \mathbb{Q}} u(\tilde{Y}_s),$$

therefore by [70, Proposition 1.3.14], $u(\bar{Y})$ is a càdlàg submartingale, and $\bar{Y}$ is thus càdlàg as well. Hence $(\bar{X}, \bar{Y}, \bar{Z}) \in \mathcal{A}(x)$ and consequently $V(x) = \bar{Y}_0$, which ends the proof.

□

Remark 3.2.5. *a) Unlike in [47] and [65] where minimal supersolutions of BSDEs are studied, we cannot guarantee that the stochastic integral of the process $\bar{Z}$ is a supermartingale even for a bounded terminal condition ξ. This is due to the fact that the random variable $\bar{X}_T$ may not be integrable.*

b) In the above result, the assumption $\xi \in L^1_+(\Omega, \mathcal{F}_T, Q)$ can be replaced by $\xi \in L^2_+(\Omega, \mathcal{F}_T, P)$. This would cost a stronger integrability condition on the process θ. Indeed, if the martingale $\mathcal{E}(-\int \theta\, dW)$ satisfies the reverse Hölder inequality R_2 that is, there is a positive constance C such that for all stopping times $\tau \leq T$ it holds

$$E\left[\mathcal{E}\left(-\int \theta_u\, dW_u\right)_T^2 \mid \mathcal{F}_\tau\right]^{\frac{1}{2}} \leq C\mathcal{E}\left(-\int \theta_u\, dW_u\right)_\tau,$$

then by [41, Proposition 3] we have $E_Q[\xi] = E[\mathcal{E}(-\int_0^\cdot \theta_u\, dW_u)_T \xi] \leq CE[\xi^2]$ and therefore the first estimate of Lemma 3.2.1 remains valid.

c) The random endowment ξ can be allowed to take values in $\mathbb{R}$ if we assume the existence of $(X, Y, Z) \in \mathcal{A}(x)$ such that $Y_0 \geq x$.

We finish this section with a direct consequence of Theorem 3.2.4 and its proof. Namely, existence of a maximal subsolution of a decoupled controlled FBSDE:

Corollary 3.2.6. *Assume that the generator g satisfies* (CONV)*,* (LSC)*,* (NOR) *and* (POS)*; and $\xi \in L^\infty_+$. Then the system*

$$\begin{cases} Y_s & \leq Y_t - \int_s^t g(Y_u, Z_u)\, du + \int_s^t Z_u\, dW_u, \quad Y_T \leq \xi + X_T^\pi \\ X_t^\pi & = x + \int_0^t \pi_u \sigma_u \left(\theta_u\, du + dW_u\right), \quad \pi \in \Pi \end{cases} \tag{3.22}$$

admits a maximal subsolution. That is, there exists a control $\pi^* \in \Pi$ and a triple (X^{π^*}, Y^*, Z^*) satisfying (3.22) *with $u(Y^*)$ being a submartingale such that for any control $\pi \in \Pi$ and any processes (X^π, Y, Z) satisfying* (3.22) *with $u(Y)$ a submartingale, we have $Y_0^* \geq Y_0$.*

Proof. This follows from Theorem 3.2.4. □

3.2.2 Stability Results

In this section we assess the stability of maximal subsolutions with respect to the terminal condition and the generator. We will show that maximal subsolutions have a monotone stability with respect to both data. These stability results, already proved in [47] for minimal supersolution, will enable us to obtain a robust representation of the operator $\mathcal{E}_0^g$.

Proposition 3.2.7. *Assume that the generator g satisfies* (CONV)*,* (LSC)*,* (NOR) *and* (POS)*. Let $(\xi^n) \subseteq L^\infty_+$. If (ξ^n) decreases pointwise to a random variable ξ, then $\mathcal{E}_0^g(\xi) = \lim_{n\to\infty} \mathcal{E}_0^g(\xi^n)$.*

Proof. First notice that the operator $\mathcal{E}_0^g(\cdot)$ is increasing. Indeed, if $\xi' \leq \xi$ then $\mathcal{A}^r(\xi', g) \subseteq \mathcal{A}^r(\xi, g)$, which implies $\mathcal{E}_0^g(\xi') \leq \mathcal{E}_0^g(\xi)$. Since the sequence $(\xi^n) \subseteq L^\infty_+$ is decreasing, the limit ξ belongs to L^∞_+. By monotonicity, $(\mathcal{E}_0^g(\xi^n))$ is a decreasing sequence, bounded from below by $\mathcal{E}_0^g(\xi)$. Thus, we can define $Y_0 := \lim_{n\to\infty} \mathcal{E}_0^g(\xi^n) \geq \mathcal{E}_0^g(\xi)$. By monotonicity and the condition (NOR), $\mathcal{E}_0^g(\xi) \geq \mathcal{E}_0^g(0) > -\infty$. Theorem 3.2.4 yields a maximal subsolution $(\bar{Y}^n, \bar{Z}^n) \in \mathcal{A}^r(\xi^n, g)$ with $\bar{Y}_0^n = \mathcal{E}_0^g(\xi^n)$ for all $n \in \mathbb{N}$. We can use the method introduced in the proof of Theorem 3.2.4 to obtain a pair $(\bar{Y}, \bar{Z}) \in \mathcal{A}^r(\xi, g)$ with

$$Y_0 = \lim_{n\to\infty} \mathcal{E}_0^g(\xi^n) = \bar{Y}_0 = \mathcal{E}_0^g(\xi).$$

The sequence $(\bar{Y}_0^n)$ is not increasing as in the proof of Theorem 3.2.4 but decreasing. Nevertheless we can obtain an estimate such as that of (3.13) using $\bar{Y}_0^n \leq Y_0^1$. Finally, $\mathcal{E}_0^g(\xi)$ is optimal. In fact, let $(Y, Z) \in \mathcal{A}^r(\xi, g)$ be any subsolution. Since $\xi \leq \xi^n$ for all $n \in \mathbb{N}$, we have $(Y, Z) \in \mathcal{A}^r(\xi^n, g)$. Thus, $Y_0 \leq \mathcal{E}_0^g(\xi^n)$ for all n. Taking the limit as n tends to infinity, we conclude $Y_0 \leq \mathcal{E}_0^g(\xi)$. □

Proposition 3.2.8. *Let $\xi \in L^\infty_+$ be a terminal condition, and (g^n) be a sequence of generators decreasing pointwise to g. Assume that each function satisfies* (CONV)*,* (LSC)*,* (NOR) *and* (POS)*. Then $\mathcal{E}_0^g(\xi) = \lim_{n\to\infty} \mathcal{E}_0^{g^n}(\xi)$.*

Proof. Since (g^n) is increasing, $(\mathcal{E}_0^{g^n}(\xi))$ is decreasing and bounded from below by $\mathcal{E}_0^g(\xi)$. Define $Y_0 := \lim_{n\to\infty} \mathcal{E}_0^{g^n}(\xi) \geq \mathcal{E}_0^g(\xi)$. Y_0 is finite since $\mathcal{E}_0^g(\xi) \leq Y_0 \leq \mathcal{E}_0^{g^1}(\xi)$. For all n, there exists $(\bar{Y}^n, \bar{Z}^n) \in \mathcal{A}^r(\xi, g^n)$ such that $\mathcal{E}_0^{g^n}(\xi) = \bar{Y}_0^n$. Then by the method introduced in the proof of Theorem 3.2.4 we can obtain a candidate $(\bar{Y}, \bar{Z})$, maximal subsolution of the system with parameters g and ξ. The verification that $(\bar{Y}, \bar{Z})$ is indeed an element of $\mathcal{A}^r(\xi, g)$ relies on Fatou's lemma and monotone convergence theorem, since $g^n \uparrow g$. See the proof of [47, Theorem 4.14] for similar arguments. The subsolution $(\bar{Y}, \bar{Z})$ is maximal, since $\bar{Y}_0 = Y_0$. □

3.3 Representation and Characterization

In the previous section we obtained existence of optimal trading strategies of our control problem. This was a rather abstract result, and only gave us little information on how one could compute such an optimizer or how it depends on the other parameters. The point of this section is to find a characterization of the optimal controls of Problem (3.3).

3.3.1 Robust Representation

We consider the set

$$\mathcal{D} := \left\{ \beta : \beta \text{ predictable and } \int_0^T |\beta_u| \, du < \infty \right\}.$$

For any $\beta \in \mathcal{D}$ and $q \in \mathcal{L}$, we define, for $0 \le s \le t \le T$

$$\frac{dQ^q}{dP} = \exp\left(\int_0^T q_u \, dW_u - \frac{1}{2} \int_0^T \|q_u\|^2 \, du \right) \quad \text{and} \quad D^\beta_{s,t} := e^{-\int_s^t \beta_u \, du}, \quad t \in [0,T].$$

We also define the set

$$\mathcal{Q} := \left\{ q \in \mathcal{L} : \frac{dQ^q}{dP} \in L^1_+ \right\}.$$

For any admissible trading strategy $\pi \in \Pi$, the associated wealth process is given by $dX^\pi_t = \pi_t \sigma_t (\theta_t \, dt + dW_t)$, with $X^\pi_0 = x$ and $X^\pi \ge 0$. Let $0 \le s \le t \le T$, and consider the functional

$$\mathcal{E}^g_{s,t}(H) := \operatorname{ess\,sup} \{ Y_s : (Y,Z) \in \mathcal{A}^u(H,g) \}, \quad H \in L^0(\mathcal{F}_t).$$

Recall that $\mathcal{A}^u(H,g)$ is the set of subsolutions $(Y,Z) \in \mathcal{S}_+ \times \mathcal{L}$ of the BSDE with terminal condition H and generator g such that $u(Y)$ is a submartingale. In particular, $\mathcal{E}^g_0(H) = \mathcal{E}^g_{0,T}(H)$ for all $H \in L^0(\mathcal{F}_T)$. Let $\tau \le \gamma$ be two stopping times valued in $[0,T]$. For any $\pi \in \Pi$, define

$$\Theta_{\tau,\gamma}(\pi) := \left\{ \pi' \in \Pi : \pi' 1_{[\tau,\gamma]} = \pi 1_{[\tau,\gamma]} \right\}$$

and

$$Y_\tau(X^\pi_\tau) := \operatorname*{ess\,sup}_{\pi' \in \Theta_{0,\tau}(\pi)} \mathcal{E}^g_{\tau,T}\left(X^{\pi'}_T + \xi \right), \tag{3.23}$$

where $\xi \in L^\infty_+$ is the random endowment. We define the convex conjugate g^* of the generator g by

$$g^*(\beta, q) := \sup_{y \in \mathbb{R}_+, z \in \mathbb{R}^d} \{ \beta y + qz - g(y,z) \}, \quad \beta \in \mathbb{R}, q \in \mathbb{R}^d.$$

Consider the condition

(ADM) $g(y,z) \ge -1/2 \|z\|^2 u''(y)/u'(y)$ on $\mathbb{R}_+ \times \mathbb{R}^d$.

The following theorem gives a robust representation of $\mathcal{E}^g_{0,\tau}$.

Theorem 3.3.1. *Assume that the generator g satisfies* (CONV)*,* (LSC)*,* (NOR)*,* (POS) *and* (ADM)*. Then, for every $\pi \in \Pi$ and any stopping time $0 \le \tau \le T$, the following robust representation holds:*

$$\mathcal{E}^g_{0,\tau}(Y_\tau(X^\pi_\tau)) = \inf_{(\beta,q) \in \mathcal{D} \times \mathcal{Q}} E_{Q^q}\left[D^\beta_{0,\tau} Y_\tau(X^\pi_\tau) + \int_0^\tau D^\beta_{0,u} g^*_u(\beta_u, q_u) \, du \right], \quad \pi \in \Pi. \tag{3.24}$$

For the proof of the theorem we need the following lemma.

Lemma 3.3.2. *Assume $H \in L^\infty$. Let f be a function satisfying (*ADM*) and such that the BSDE with terminal condition H and generator f has a solution $(Y,Z) \in \mathcal{S} \times \mathcal{L}^1$ satisfying $Y \geq c$ for some $c > 0$. Then, $u(Y)$ is a submartingale.*

Proof. By Itô's formula it holds

$$u(Y_t) = u(Y_0) + \int_0^t \left(u'(Y_u) f(Y_u, Z_u) + \frac{1}{2} u''(Y_u) Z_u^2 \right) du - \int_0^t u'(Y_u) Z_u \, dW_u, \tag{3.25}$$

for all $t \in [0,T]$. Therefore since $Y > 0$, due to (ADM) we have $u'(Y_u)f(Y_u,Z_u) + \frac{1}{2}u''(Y_u)Z_u^2 \geq 0$ so that the second term of the right hand side in (3.25) defines an increasing process. Thus, as $H \in L^\infty$ and $Y \geq c$, $u(Y)$ is a submartingale. In other words, (Y,Z) is an admissible subsolution of the BSDE with terminal condition H and generator f. □

proof of Theorem 3.3.1. Let $\tau \leq T$ be a stopping time. For every $\pi \in \Pi$ and $(\beta, q) \in \mathcal{D} \times \mathcal{Q}$, if $\mathcal{A}^u(Y_\tau(X_\tau^\pi), g) \neq \emptyset$, let $(Y,Z) \in \mathcal{A}^u(Y_\tau(X_\tau^\pi), g)$. There exists a càdlàg increasing process K with $K_0 = 0$ such that on $\{t \leq \tau\}$,

$$Y_t = Y_s + \int_s^t g_u(Y_u, Z_u) \, du - \int_s^t Z_u \, dW_u + K_t - K_s, \quad 0 \leq s \leq t.$$

Define the localizing sequence of stopping times (σ_n) by

$$\sigma_n := \inf \left\{ t \geq 0 : \left| \int_0^t D_{0,u}^\beta Z_u \, dW_u \right| \geq n \right\} \wedge T.$$

Applying Itô's formula to $D_{0,t}^\beta Y_t$ and Girsanov's theorem such as in [54], we have

$$Y_0 \leq E_{Q^q} \left[D_{0,t\wedge\sigma_n}^\beta Y_{t\wedge\sigma_n} + \int_0^{t\wedge\sigma_n} D_{0,u}^\beta g_u^*(\beta_u, q_u) \, du \right], \quad \text{for all } n \in \mathbb{N} \text{ and } t \in [0,T].$$

Since g satisfies (NOR) the function g^* is positive. Using the fact that (σ_n) is a localizing sequence, there is n large enough such that $\tau \leq \sigma_n$; and since $Y_\tau \leq Y_\tau(X_\tau^\pi)$ and D^β is positive, we have

$$Y_0 \leq E_{Q^q} \left[D_{0,\tau}^\beta Y_\tau(X_\tau^\pi) + \int_0^\tau D_{0,u}^\beta g_u^*(\beta_u, q_u) \, du \right].$$

Therefore,

$$\mathcal{E}_{0,\tau}^g(Y_\tau(X_\tau^\pi)) \leq \inf_{(\beta,q)\in\mathcal{D}\times\mathcal{Q}} E_{Q^q} \left[D_{0,\tau}^\beta Y_\tau(X_\tau^\pi) + \int_0^\tau D_{0,u}^\beta g_u^*(\beta_u, q_u) \, du \right]. \tag{3.26}$$

If $\mathcal{A}^u(Y_\tau(X_\tau^\pi), g) = \emptyset$, (3.26) is obvious.

On the other hand, for each $k \in \mathbb{N}$ and $\pi \in \Pi$ we define $H^k(\pi) := Y_\tau(X^\pi_\tau) \wedge k$, which is a bounded $\mathcal{F}_\tau$-random variable. Defining for every $n \in \mathbb{N}$ the function g^n on $\mathbb{R}_+ \times \mathbb{R}^d$ by

$$g^n(y,z) := \sup_{|\beta|\leq n; \|q\|\leq n} \{\beta y + qz - g^*(\beta, q)\} \vee -\frac{1}{2} u''(y) \|z\|^2 / u'(y),$$

the sequence (g^n) converges pointwise to g as a consequence of the Fenchel-Moreau theorem. In addition, for each $n \in \mathbb{N}$ the function g^n satisfies the quadratic growth condition

$$g^n(y,z) \leq C_n(1 + |y| + \|z\|^2), \quad y \in \mathbb{R},\ z \in \mathbb{R}^d,\ C_n \geq 0.$$

Fixing $n \in \mathbb{N}$, for every $k \in \mathbb{N}$ there exists $(Y^{n,k}, Z^{n,k}) \in \mathcal{S} \times \mathcal{L}^1$ solution of the BSDE with terminal condition $H^k(\pi)$ and driver g^n, see for instance [28]. It follows from [54] that there exist predictable processes (β^n, q^n) satisfying $|\beta^n| \leq C_n$ and $\int q^n \, dW \in BMO$ such that on $\{t \leq \tau\}$

$$Y^{n,k}_t = E_{Q^{q^n}} \left[D^{\beta^n}_{t,\tau} H^k(\pi) + \int_t^\tau D^{\beta^n}_{0,u} g^{n,*}_u(\beta^n_u, q^n_u) \, du \mid \mathcal{F}_t \right], \quad P\text{-a.s.}, \tag{3.27}$$

where $g^{n,*}$ is the convex conjugate of g^n. In particular, since g satisfies (NOR), we have $\beta y - g^*(\beta, q) \leq 0$ for all β, q so that g^n also satisfies (NOR). Thus, it holds $g^{n,*} \geq 0$, and from $(x, x, 0) \in \mathcal{A}(x)$ it follows $H^k(\pi) \geq x$, which yields $Y^{n,k}_t \geq E_{Q^{q^n}}[D^{\beta^n}_{t,\tau} x] > 0$. Since $g^n(y,z) \geq -\frac{1}{2} u''(y) \|z\|^2 / u'(y)$, it follows from Lemma 3.3.2 that $u(Y^{n,k})$ is a submartingale. That is, $(Y^{n,k}, Z^{n,k})$ is an admissible subsolution of the BSDE with generator g^n and terminal condition $H^k(\pi)$. Therefore, $\mathcal{E}^{g^n}_0(H^k(\pi)) \geq Y^{n,k}_0$. Taking the limit as k goes to infinity, it follows from the monotone stability of Proposition 3.2.7 and the monotone convergence theorem that

$$\mathcal{E}^{g^n}_{0,\tau}(Y_\tau(X^\pi_\tau)) \geq E_{Q^{q^n}} \left[D^{\beta^n}_{0,\tau} Y_\tau(X^\pi_\tau) + \int_0^\tau D^{\beta^n}_{0,u} g^{n,*}_u(\beta^n_u, q^n_u) \, du \right] \quad \text{for all } n \in \mathbb{N}.$$

Since $(\beta^n, q^n) \in \mathcal{D} \times \mathcal{Q}$ for each n, we have

$$\mathcal{E}^{g^n}_{0,\tau}(Y_\tau(X^\pi_\tau)) \geq \inf_{(\beta,q)\in\mathcal{D}\times\mathcal{Q}} E_{Q^q} \left[D^\beta_{0,\tau} Y_\tau(X^\pi_\tau) + \int_0^\tau D^\beta_{0,u} g^{n,*}_u(\beta_u, q_u) \, du \right].$$

Using $g^* \leq g^{n,*}$ for all $n \in \mathbb{N}$ and then taking the limit as n goes to infinity, the monotone stability of Proposition 3.2.8 yields the second inequality, which concludes the proof. □

Proposition 3.3.3. *Under the assumptions of Theorem 3.3.1, for any $[0,T]$-valued stopping time τ, it holds*

$$V(x) = \sup_{\pi\in\Pi} \inf_{(\beta,q)\in\mathcal{D}\times\mathcal{Q}} E_{Q^q} \left[D^\beta_{0,\tau} Y_\tau(X^\pi_\tau) + \int_0^\tau D^\beta_{0,u} g^*_u(\beta_u, q_u) \, du \right]. \tag{3.28}$$

Proof. We have

$$V(x) = \sup_{\pi\in\Pi} \mathcal{E}^g_{0,\tau}(Y_\tau(X^\pi_\tau)). \tag{3.29}$$

In fact,

$$\begin{aligned}\sup_{\pi\in\Pi} \mathcal{E}^g_{0,\tau}(Y_\tau(X^\pi_\tau)) &= \sup_{\pi\in\Pi} \mathcal{E}^g_{0,\tau} \left(\operatorname*{ess\,sup}_{\pi'\in\Theta_{0,\tau}(\pi)} \mathcal{E}^g_{\tau,T}\left(X^{\pi'}_T + \xi\right) \right) \\ &= \sup_{\pi\in\Pi} \sup_{\pi'\in\Theta_{0,\tau}(\pi)} \mathcal{E}^g_{0,T}\left(X^{\pi'}_T + \xi\right) = V(x),\end{aligned}$$

where we used monotonicity and flow property of the operators $\mathcal{E}^g_{s,t}(\cdot)$, $0 \le s \le t \le T$, see [47, Proposition 3.6]. By Theorem 3.3.1 the proof is done. $\square$

3.3.2 Existence of a Saddle Point

Considering the dual representation of $\mathcal{E}^g_{0,\tau}$ derived in Theorem 3.3.1, a pair $(\beta, q) \in \mathcal{D} \times \mathcal{Q}$ is said to be a subgradient of $\mathcal{E}^g_{0,\tau}$ at $Y_\tau(X^\pi_\tau)$ if

$$\mathcal{E}^g_{0,\tau}(Y_\tau(X^\pi_\tau)) = E_{Q^q}\left[D^\beta_{0,\tau} Y_\tau(X^\pi_\tau) + \int_0^\tau D^\beta_{0,u} g^*_u(\beta_u, q_u)\,du\right].$$

In the case where the generator only depends on z, equivalence between existence of a subgradient of a monetary utility function and quadratic growth of the driver g was proved by Delbaen et al. [39]. The following result uses their compactness argument. We will also need the conditions

(QG) quadratic growth: $g : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ and $\forall \eta > 0$ there exists $C > 0$: $g(y, z) \le C(1 + |y| + \|z\|^2)$ for all $y \in \mathbb{R}$: $|y| \ge \eta$ and $z \in \mathbb{R}^d$.

Theorem 3.3.4. *Assume that g satisfies (ADM), (CONV), (LSC), (QG), (NOR) and (POS). Then, $\mathcal{E}^g_0$ admits a local subgradient: For any $[0, T]$-valued stopping time τ and any $\pi \in \Pi$, $\mathcal{E}^g_{0,\tau}$ admits a subgradient $(q^\tau, \beta^\tau) \in \mathcal{D} \times \mathcal{Q}$ at $Y_\tau(X^\pi_\tau)$.*

Proof. Let $\pi \in \Pi$ be fixed for the rest of the proof. Let $\eta > 0$ in (QG). Due to Theorem 3.3.1, we have

$$\mathcal{E}^g_{0,\tau}(Y_\tau(X^\pi_\tau)) = \inf_{\frac{dQ^q}{dP} D^\beta_{0,\tau} \in \mathcal{K}} \left\{ E_{Q^q}\left[D^\beta_{0,\tau} Y_\tau(X^\pi_\tau) + \int_0^\tau D^\beta_{0,u} g^*_u(\beta_u, q_u)\,du\right]\right\},$$

where

$$\mathcal{K} := \left\{\frac{dQ^q}{dP} D^\beta_{0,\tau} : (\beta, q) \in \mathcal{D} \times \mathcal{Q}\right\} \subseteq L^1.$$

For every $k \ge 0$ the set

$$\Gamma_\tau := \left\{\frac{dQ^q}{dP} D^\beta_{0,\tau} \in \mathcal{K} : E_{Q^q}\left[\int_0^\tau D^\beta_{0,u} g^*_u(\beta_u, q_u)\,du\right] \le k\right\} \tag{3.30}$$

is convex, see [48]. Let us show that it is $\sigma(L^1, L^\infty)$-compact. Let $(\beta, q) \in \mathbb{R} \times \mathbb{R}^d$ be given. By definition, we have

$$\begin{aligned} g^*(\beta, q) &= \sup_{y \in \mathbb{R}, z \in \mathbb{R}^d} \{\beta y + qz - g(y, z)\} \\ &\ge \sup_{|y| \ge \eta, z \in \mathbb{R}^d} \{\beta y + qz - g(y, z)\} \\ &\ge \sup_{|y| \ge \eta, z \in \mathbb{R}^d} \left\{\beta y + qz - C(1 + y + \|z\|^2)\right\} \\ &\ge \sup_{|y| \ge \eta} \{\beta y - Cy\} + b\|q\|^2 - C, \end{aligned}$$

with $b = \frac{1}{4C}$. If $|\beta| > C$, then let $n \in \mathbb{N}$ be big enough such that $y := n\beta$ satisfies $|y| \ge \eta$. Then,

$$\sup_{|y| \ge \eta} \{-\beta y - C|y|\} \ge n|\beta|(|\beta| - C),$$

so that $g^*(\beta, q) = \infty$. Therefore, we can restrict ourselves to $(\beta, q) \in \mathcal{D} \times \mathcal{Q}$ with $|\beta| \leq C$. Hence, we can find a positive constant a such that

$$g^*(\beta, q) \geq a\beta + b\,\|q\|^2 - C. \tag{3.31}$$

Since β is bounded, $D^{\beta}_{0,u} = e^{-\int_0^u \beta_r\,dr}$ is bounded as well. Thus multiplying both sides of (3.31) by $D^{\beta}_{0,u}$ and integrating with respect to $Q^q \otimes dt$ lead to

$$E_{Q^q}\left[\int_0^{\tau} D^{\beta}_{0,u} g^*(\beta_u, q_u)\,du\right] \geq A_1 + A_2 E_{Q^q}\left[\int_0^{\tau} \|q_u\|^2\,du\right],$$

where A_1 and A_2 are positive constants which do not depend on β and q. Arguing similar to the proof of [39, Theorem 2.2], we can find a positive constant c such that

$$\left\{\frac{dQ^q}{dP} D^{\beta}_{0,\tau} \in \mathcal{K} : E_{Q^q}\left[\int_0^{\tau} D^{\beta}_{0,u} g^*_u(\beta_u, q_u)\,du\right] \leq k\right\}$$
$$\subseteq \left\{\frac{dQ^q}{dP} D^{\beta}_{0,\tau} \in \mathcal{K} : E\left[\frac{dQ^q}{dP} \log \frac{dQ^q}{dP}\right] \leq c\right\}$$

and therefore, we can conclude using the de la Vallée Poussin theorem that the left hand side in the above inclusion is L^1- uniformly integrable. We take a maximizing sequence $(\frac{dQ^{q^n}}{dP} D^{\beta^n}_{0,\tau})_n$ for the functional $\mathcal{E}^g_{0,\tau}(Y_\tau(X^\pi_\tau))$. Since $Y_\tau(X^\pi_\tau)$ is positive, it follows that the sequence $(E_{Q^{q^n}}[\int_0^\tau D^{\beta^n}_{0,u} g^*_u(\beta^n_u, q^n_u)\,du])_n$ admits a subsequence which is bounded from above. Therefore, the previous step shows that the sequence $(\frac{dQ^{q^n}}{dP} D^{\beta^n}_{0,\tau})_n$ is uniformly integrable. In addition, applying a compactness argument of Komlos type, we can find a sequence denoted $(\tilde{M}^n_T)$ in the asymptotic convex hull of $(\frac{dQ^{q^n}}{dP} D^{\beta^n}_{0,\tau})_n$ which converges P-a.s. to the limit $M_T \in L^0_+$. The sequence $(\tilde{M}^n_T)$ is as well uniformly integrable and therefore converges to M_T in L^1. By the arguments used in the proof of [48, Theorem 3.10], it is possible to show that for all $n \in \mathbb{N}$ there exist $\tilde{q}^n$ and $\tilde{\beta}^n$ such that $\tilde{M}^n_T = \frac{dQ^{\tilde{q}^n}}{dP} D^{\tilde{\beta}^n}_{0,\tau}$ and, up to other convex combinations, the sequences $(\tilde{q}^n)$ and $(\tilde{\beta}^n)$ converge $P \otimes dt$-a.s. to some q^τ and β^τ, respectively and $M_T = \frac{dQ^{q^\tau}}{dP} D^{\beta^\tau}_{0,\tau}$. Since $|\tilde{\beta}^n| \leq C$ for all n, it holds $|\beta^\tau| \leq C$. By Fatou's lemma and convexity, we have

$$\mathcal{E}^g_{0,\tau}(Y_\tau(X^\pi_\tau)) = \liminf_{n\to\infty} E_{Q^{q^n}}\left[D^{\beta^n}_{0,\tau} Y_\tau(X^\pi_\tau) + \int_0^\tau D^{\beta^n}_{0,u} g^*(\beta^n_u, q^n_u)\,du\right]$$
$$\geq E\left[\liminf_{n\to\infty} \frac{dQ^{\tilde{q}^n}}{dP}\left(D^{\tilde{\beta}^n}_{0,\tau} Y_\tau(X^\pi_\tau) + \int_0^\tau D^{\tilde{\beta}^n}_{0,u} g^*(\tilde{\beta}^n_u, \tilde{q}^n_u)\,du\right)\right].$$

Lower-semicontinuity of g^* yields

$$\mathcal{E}^g_{0,\tau}(Y_\tau(X^\pi_\tau)) \geq E_{Q^{q^\tau}}\left[D^{\beta^\tau}_{0,\tau} Y_\tau(X^\pi_\tau) + \int_0^\tau D^{\beta^\tau}_{0,u} g^*(\beta^\tau_u, q^\tau_u)\,du\right].$$

Since $|\beta^\tau| \leq C$ and $M_T \in L^1$, we have $\beta^\tau \in \mathcal{D}$ and $q^\tau \in \mathcal{Q}$. □

Corollary 3.3.5. *Under the assumptions of Theorem 3.3.4, for any optimal strategy* $\pi^* \in \Pi$ *and any* $[0, T]$*-valued stopping time* τ *one has*

$$V(x) = \mathcal{E}^g_{0,\tau}\left(Y_\tau(X^{\pi^*}_\tau)\right). \tag{3.32}$$

In addition, Problem (3.3) *admits a local saddle point in the sense that, there exists* $(\beta^\tau, q^\tau) \in \mathcal{D} \times \mathcal{Q}$ *satisfying*

$$V(x) = E_{Q^{q^\tau}}\left[D^{\beta^\tau}_{0,\tau} Y_\tau(X^{\pi^*}_\tau) + \int_0^\tau D^{\beta^\tau}_{0,u} g^*_u(\beta^\tau_u, q^\tau_u)\, du\right]$$
$$= \inf_{(\beta,q)\in\mathcal{D}\times\mathcal{Q}} \sup_{\pi\in\Pi} E_{Q^q}\left[D^{\beta}_{0,\tau} Y_\tau(X^{\pi}_\tau) + \int_0^\tau D^{\beta}_{0,u} g^*_u(\beta_u, q_u)\, du\right].$$

Proof. By definition of $Y_\tau(X^{\pi^*}_\tau)$, monotonicity and the flow property of $\mathcal{E}^g_{s,t}$; $0 \le s \le t \le T$, we have

$$\mathcal{E}^g_{0,\tau}(Y_\tau(X^{\pi^*}_\tau)) = \mathcal{E}^g_{0,\tau}\left(\operatorname*{ess\,sup}_{\pi\in\Theta_{0,\tau}(\pi^*)} \mathcal{E}^g_{\tau,T}\,(X^\pi_T + \xi)\right)$$
$$= \sup_{\pi\in\Theta_{0,\tau}(\pi^*)} \mathcal{E}^g_{0,T}\,(X^\pi_T + \xi) \ge V(x),$$

since $\pi^* \in \Theta_{0,\tau}(\pi^*)$. Thus, Equation (3.32) is a consequence of Equation (3.29).

It follows from Theorem 3.3.4 and Equation (3.32) that there exists $(\beta^\tau, q^\tau) \in \mathcal{D} \times \mathcal{Q}$ such that

$$V(x) = E_{Q^{q^\tau}}\left[D^{\beta^\tau}_{0,T} Y_\tau(X^{\pi^*}_\tau) + \int_0^\tau D^{\beta^\tau}_{0,u} g^*_u(\beta^\tau_u, q^\tau_u)\, du\right] \tag{3.33}$$

and for every $\pi \in \Pi$ exists $(\beta(\pi), q(\pi)) \in \mathcal{D} \times \mathcal{Q}$ such that

$$\mathcal{E}^g_{0,\tau}\,(Y_\tau(X^\pi_\tau)) = E_{Q^{q(\pi)}}\left[D^{\beta(\pi)}_{0,\tau} Y_\tau(X^\pi_\tau) + \int_0^\tau D^{\beta(\pi)}_{0,u} g^*_u(\beta_u(\pi), q_u(\pi))\, du\right].$$

Thus, taking the supremum with respect to π on both sides yields

$$\mathcal{E}^g_{0,\tau}\left(Y_\tau(X^{\pi^*}_\tau)\right) = \sup_{\pi\in\Pi} E_{Q^{q(\pi)}}\left[D^{\beta(\pi)}_{0,\tau} Y_\tau(X^\pi_\tau) + \int_0^\tau D^{\beta(\pi)}_{0,u} g^*_u(\beta_u(\pi), q_u(\pi))\, du\right]$$
$$\ge \inf_{(\beta,q)\in\mathcal{D}\times\mathcal{Q}} \sup_{\pi\in\Pi} E_{Q^q}\left[D^{\beta}_{0,\tau} Y_\tau(X^{\pi}_\tau) + \int_0^\tau D^{\beta}_{0,u} g^*_u(\beta_u, q_u)\, du\right].$$

Since we always have $\inf\sup \ge \sup\inf$, it follows that

$$\sup_{\pi\in\Pi} \inf_{(\beta,q)\in\mathcal{D}\times\mathcal{Q}} E_{Q^q}\left[D^{\beta}_{0,\tau} Y_\tau(X^{\pi}_\tau) + \int_0^\tau D^{\beta}_{0,u} g^*_u(\beta_u, q_u)\, du\right]$$
$$= E_{Q^{q^\tau}}\left[D^{\beta^\tau}_{0,\tau} Y_\tau(X^{\pi^*}_\tau) + \int_0^\tau D^{\beta^\tau}_{0,u} g^*_u(\beta^\tau_u, q^\tau_u)\, du\right]$$
$$= \inf_{(\beta,q)\in\mathcal{D}\times\mathcal{Q}} \sup_{\pi\in\Pi} E_{Q^q}\left[D^{\beta}_{0,\tau} Y_\tau(X^{\pi^*}_\tau) + \int_0^\tau D^{\beta}_{0,u} g^*_u(\beta_u, q_u)\, du\right].$$

The proof is complete. □

Remark 3.3.6. *If g defined on the space $\mathbb{R} \times \mathbb{R}^d$ satisfies* (ADM)*,* (CONV)*,* (NOR)*,* (POS) *and* (QG)*, then one can take $\tau = T$ in Equation* (3.23)*, that is $Y_\tau(X^\pi_\tau) = X^\pi_T + \xi$, and work on the whole time interval $[0, T]$ in the proof of Theorem 3.3.4 and the subsequent corollary. The main reason for working with stopping times is to allow for generators that satisfy the conditions* (CONV)*,* (NOR) *and* (POS) *only on a subset $I \times \mathbb{R}^d$, where $I \subseteq \mathbb{R}_+$ is an open interval as in the following example.*

Example 3.3.7 (Certainty equivalent)**.** *Let us come back to the certainty equivalent example of Section 3.1. For $u(x) = \log(x)$, Equation* (3.5) *becomes*

$$Y_t = X - \frac{1}{2}\int_t^T \frac{|Z_u|^2}{Y_u}\,du + \int_t^T Z_u\,dW_u, \quad t \in [0, T].$$

The generator $g(y, z) = \frac{1}{2}|z|^2/y$ satisfies (LSC)*,* (CONV)*,* (NOR) *and* (POS) *on $(0, \infty) \times \mathbb{R}^d$ and it can be extended on $\mathbb{R}_+ \times \mathbb{R}^d$ to a generator satisfying the same conditions by putting*

$$g(y, z) = \begin{cases} \frac{1}{2}\frac{|z|^2}{y} & \text{if } y > 0 \\ 0 & \text{if } z = 0 \\ +\infty & \text{if } y = 0, z \neq 0. \end{cases}$$

Hence, Theorem 3.2.4 ensures the existence of an optimal trading strategy $\pi^* \in \Pi$. However, if we consider the function on $\mathbb{R}_+ \times \mathbb{R}^d$, we can not guarantee, with our method, that the set Γ_τ defined in (3.30) *is weakly compact and therefore that the problem admits a saddle point. A way around is to introduce a stopping time $0 < \tau \leq T$ and work locally on $[0, \tau]$ as follows: Let $\pi^* \in \Pi$ be an optimal strategy and put $Y^\pi_t := u^{-1}(E[u(X^\pi_T + \xi) \mid \mathcal{F}_t])$. Since $x > 0$, there exists $m \in \mathbb{N}$ such that $x \geq \frac{1}{m}$. Define the stopping time τ by*

$$\tau := \inf\left\{t \geq 0 : X^{\pi^*}_t \leq \frac{1}{m}\right\} \wedge T.$$

We can restrict the study to subsolutions $(Y, Z) \in \mathcal{A}^u(X^\pi_T + \xi)$ satisfying $Y \geq X^\pi$, for all $t \in [0, T]$. Hence we have $Y^{\pi^*}_{\tau\wedge t} \geq X^{\pi^*}_{\tau\wedge t} \geq \frac{1}{m}$. Applying martingale representation theorem and Itô's formula such as in Example 3.1.1, we can find a process $Z^{\pi^*} \in \mathcal{L}^1$ such that

$$Y^{\pi^*}_t = Y^{\pi^*}_\tau - \int_t^\tau g_u(Y^{\pi^*}_u, Z^{\pi^*}_u)\,du + \int_t^\tau Z^{\pi^*}_u\,dW_u \quad \text{on } \{t \leq \tau\}.$$

Since the set $\{Y_\tau : (X, Y, Z) \in \mathcal{A}(x)\}$ is upward directed, using the arguments of Theorem 3.2.4 we can find a strategy $\bar{\pi} \in \Pi$ such that

$$Y_\tau(X^{\pi^*}_\tau) := \operatorname*{ess\,sup}_{\pi' \in \Theta_{0,\tau}(\pi^*)} \mathcal{E}_{\tau,T}(X^{\pi'}_T + \xi) = Y^{\bar{\pi}}_\tau = \mathcal{E}^g_{\tau,T}(X^{\bar{\pi}}_T + \xi)$$

with $\bar{\pi} \in \Theta_{0,\tau}(\pi^*)$, i.e. $\bar{\pi}1_{[0,\tau]} = \pi^* 1_{[0,\tau]}$ and $\bar{\pi} \in \Pi$. Moreover, since $\Theta_{0,\tau}(\pi^*) = \Theta_{0,\tau}(\bar{\pi})$, we have $Y_\tau(X^{\pi^*}_\tau) = Y_\tau(X^{\bar{\pi}}_\tau)$. By $Y^{\bar{\pi}}_{t\wedge\tau} \geq X^{\bar{\pi}}_{t\wedge\tau} = X^{\pi^*}_{t\wedge\tau} \geq \frac{1}{m} > 0$, we also have

$$\begin{aligned} Y^{\bar{\pi}}_0 &= Y^{\bar{\pi}}_\tau - \int_0^\tau g_u(Y^{\bar{\pi}}_u, Z^{\bar{\pi}}_u)\,du + \int_0^\tau Z^{\bar{\pi}}_u\,dW_u \\ &= Y_\tau(X^{\bar{\pi}}_\tau) - \int_0^\tau g_u(Y^{\bar{\pi}}_u, Z^{\bar{\pi}}_u)\,du + \int_0^\tau Z^{\bar{\pi}}_u\,dW_u. \end{aligned}$$

For almost every (ω, t) such that $t \leq \tau(\omega)$ the function g is differentiable at $(Y_t^{\bar\pi}(\omega), Z_t^{\bar\pi}(\omega))$ and it admits a unique subgradient $(\bar\beta_t(\omega), \bar q_t(\omega))$ given by

$$\bar q_t = \frac{Z_t^{\bar\pi}}{Y_t^{\bar\pi}} \quad \textit{and} \quad \bar\beta_t = -\frac{|Z_t^{\bar\pi}|^2}{2(Y_t^{\bar\pi})^2} \quad \textit{on } \{t \leq \tau\}.$$

Since $Y_{t\wedge\tau}^{\bar\pi} \geq 1/m$ and $Z^{\bar\pi} \in \mathcal{L}^1$, it follows that $(\bar\beta, \bar q) \in \mathcal{D}\times\mathcal{Q}$ and we have $g_t(Y_t^{\bar\pi}, Z_t^{\bar\pi}) = \bar\beta_t Y_t^{\bar\pi} + \bar q_t Z_t^{\bar\pi} - g_t^*(\bar\beta_t, \bar q_t)$. Thus, using the arguments leading to Equation 3.27, one has

$$Y_0^{\bar\pi} = E_{Q^{\bar q}}\left[D_{0,\tau}^{\bar\beta} Y_\tau(X_\tau^{\bar\pi}) + \int_0^\tau g_u^*(\bar\beta_u, \bar q_u)\, du\right]. \tag{3.34}$$

But since for every $(\beta, q) \in \mathcal{D}\times\mathcal{Q}$ it holds

$$Y_0^{\bar\pi} \leq E_{Q^{q}}\left[D_{0,\tau}^{\beta} Y_\tau(X_\tau^{\bar\pi}) + \int_0^\tau g_u^*(\beta_u, q_u)\, du\right],$$

it follows,

$$\begin{aligned} Y_0^{\bar\pi} &= \inf_{(\beta,q)\in\mathcal{D}\times\mathcal{Q}} E_{Q^{q}}\left[D_{0,\tau}^{\beta} Y_\tau(X_\tau^{\bar\pi}) + \int_0^\tau g_u^*(\beta_u, q_u)\, du\right] \\ &= \mathcal{E}_{0,\tau}^g(Y_\tau(X_\tau^{\bar\pi})) \end{aligned} \tag{3.35}$$

where the second equality above follows from the representation theorem 3.3.1. By the identity $Y_\tau(X_\tau^{\pi^*}) = Y_\tau(X_\tau^{\bar\pi})$, one has $\mathcal{E}_{0,\tau}^g(Y_\tau(X_\tau^{\bar\pi})) = \mathcal{E}_{0,\tau}^g(Y_\tau(X_\tau^{\pi^*}))$, so that it follows from (3.34) *and* (3.35) *that $\mathcal{E}_{0,\tau}^g(Y_\tau(X_\tau^{\pi^*}))$ admits the subgradient $(\bar\beta, \bar q)$. Therefore, the utility maximization problem $V(x) = \sup_{\pi\in\Pi} C_0(X_T^\pi + \xi)$ can be written as a robust control problem admitting a local saddle point in the sense of Corollary 3.3.5. In fact,*

$$\begin{aligned} &\inf_{(\beta,q)\in\mathcal{D}\times\mathcal{Q}} \sup_{\pi\in\Pi} E_{Q^{q}}\left[D_{0,\tau}^{\beta} Y_\tau(X_\tau^{\pi}) + \int_0^\tau g_u^*(\beta_u, q_u)\, du\right] \\ &\leq \sup_{\pi\in\Pi} E_{Q^{\bar q}}\left[D_{0,\tau}^{\bar\beta} Y_\tau(X_\tau^{\pi}) + \int_0^\tau g_u^*(\bar\beta_u, \bar q_u)\, du\right] \\ &\leq \mathcal{E}_{0,\tau}^g(Y_\tau(X_\tau^{\pi^*})) = E_{Q^{\bar q}}\left[D_{0,\tau}^{\bar\beta} Y_\tau(X_\tau^{\pi^*}) + \int_0^\tau g_u^*(\bar\beta_u, \bar q_u)\, du\right] \\ &\leq \inf_{(\beta,q)\in\mathcal{D}\times\mathcal{Q}} E_{Q^{q}}\left[D_{0,\tau}^{\beta} Y_\tau(X_\tau^{\pi^*}) + \int_0^\tau g_u^*(\beta_u, q_u)\, du\right] \\ &\leq \sup_{\pi\in\Pi} \inf_{(\beta,q)\in\mathcal{D}\times\mathcal{Q}} E_{Q^{q}}\left[D_{0,\tau}^{\beta} Y_\tau(X_\tau^{\pi}) + \int_0^\tau g_u^*(\beta_u, q_u)\, du\right] \\ &\leq \inf_{(\beta,q)\in\mathcal{D}\times\mathcal{Q}} \sup_{\pi\in\Pi} E_{Q^{q}}\left[D_{0,\tau}^{\beta} Y_\tau(X_\tau^{\pi}) + \int_0^\tau g_u^*(\beta_u, q_u)\, du\right]. \end{aligned}$$

To justify the second inequality above, notice that with the arguments leading to (3.26)*, we have*

$$\mathcal{E}_{\tau,T}^g(X_T^\pi + \xi) \leq E_{Q^{\bar q}}\left[D_{\tau,T}^{\bar\beta}(X_T^\pi + \xi) + \int_\tau^T g_u^*(\bar\beta_u, \bar q_u)\, du \,|\, \mathcal{F}_\tau\right].$$

Therefore,

$$
\begin{aligned}
&\sup_{\pi\in\Pi} E_{Q^{\bar q}}\left[D^{\bar\beta}_{0,\tau} Y_\tau(X^\pi_\tau) + \int_0^\tau g^*_u(\bar\beta_u,\bar q_u)\,du\right]\\
&= \sup_{\pi\in\Pi} E_{Q^{\bar q}}\left[D^{\bar\beta}_{0,\tau} \operatorname*{ess\,sup}_{\pi'\in\Theta_{0,\tau}(\pi)} \mathcal{E}^g_{\tau,T}(X^{\pi'}_T+\xi) + \int_0^\tau g^*_u(\bar\beta_u,\bar q_u)\,du\right]\\
&\le \sup_{\pi\in\Pi} \sup_{\pi'\in\Theta_{0,\tau}(\pi)} E_{Q^{\bar q}}\left[D^{\bar\beta}_{0,\tau} \mathcal{E}^g_{\tau,T}(X^{\pi'}_T+\xi) + \int_0^\tau g^*_u(\bar\beta_u,\bar q_u)\,du\right]\\
&= \sup_{\pi\in\Pi} E_{Q^{\bar q}}\left[D^{\bar\beta}_{0,\tau} \mathcal{E}^g_{\tau,T}(X^{\pi}_T+\xi) + \int_0^\tau g^*_u(\bar\beta_u,\bar q_u)\,du\right]\\
&\le E_{Q^{\bar q}}\left[D^{\bar\beta}_{0,\tau} E_{Q^{\bar q}}\left[D^{\bar\beta}_{\tau,T}(X^{\pi}_T+\xi) + \int_\tau^T g^*_u(\bar\beta_u,\bar q_u)\,du \,\Big|\, \mathcal{F}_\tau\right] + \int_0^\tau g^*_u(\bar\beta_u,\bar q_u)\,du\right]\\
&\le E_{Q^{\bar q}}\left[D^{\bar\beta}_{0,T}(X^{\pi}_T+\xi) + \int_0^T g^*_u(\bar\beta_u,\bar q_u)\,du\right]\\
&\le V(x) = E_{Q^{\bar q}}\left[D^{\bar\beta}_{0,\tau} Y_\tau(X^{\pi^*}_\tau) + \int_0^\tau g^*_u(\bar\beta_u,\bar q_u)\,du\right].
\end{aligned}
$$

3.3.3 Characterization

We conclude this section by providing a characterization of an optimal trading strategy and a corresponding optimal model in the framework of the stochastic maximum principle. It dates back to the work of Bismut in the 1970s. The maximum principle has been widely used in the context of expected utility maximization to characterize optimal strategies, see for instance Horst et al. [67]. Applying the perturbation techniques yielding the stochastic maximum principle as developed by Peng [82] to the control problem (3.3) as it is does not give much information on the optimal solution because of the nonlinearity of the operator $\mathcal{E}^g_0$. This is where the dual representation for BSDEs becomes useful, in helping to linearize the problem by transforming it into a robust control problem under a linear operator. In the following we denote by $\partial g^*/\partial a$ and $\partial g^*/\partial b$, when they exist, the derivative of the function $g^* : \mathbb{R}\times\mathbb{R}^d \to \mathbb{R}$ with respect to the first and the second variable, respectively.

Since for every $\pi\in\Pi$ the process X^π is a positive Q-martingale, we can write X^π as

$$X^\pi_t = x + \int_0^t \sigma_u\tilde\pi_u X^\pi_u\,dW^Q_u \tag{3.36}$$

for some predictable process $\tilde\pi$ satisfying $\{\int_0^T |\sigma_u\tilde\pi_u|^2\,du < \infty\} = \{X^\pi_T > 0\}$, see [72, Chapter 1]. The next theorem gives a characterization of the optimal model (q^*,β^*) and of the process $\tilde\pi^*$ associated to the optimal strategy π^*.

Theorem 3.3.8. *Assume that the driver g is strictly convex, satisfies (*ADM*), (*LSC*), (*NOR*), (*POS*), and (*QG*). Further assume that $\xi\in L^\infty_+$. Then, for every saddle point $(\pi^*,(\beta^*,q^*))$*

there exists a pair (p,k) depending on $\tilde{\pi}^*$, β^* and q^* such that $p_t\theta_t + p_t q_t^* + k_t = 0$ $P\otimes dt$ - a.s. and which solves the BSDE

$$dp_t = -(\theta_t p_t + p_t q_t^* + k_t)\tilde{\pi}_t^* \sigma_t \, dt + k_t \, dW_t^{Q^{q^*}}, \quad p_T = D_{0,T}^{\beta^*} \quad Q^{q^*} \text{a.s.}$$

Furthermore, g^* is differentiable at (β^*, q^*) and satisfy

$$-\frac{\partial g_t^*}{\partial a}(\beta_t^*, q_t^*) + Y_t = 0 \quad \text{and} \quad -\frac{\partial g_t^*}{\partial b}(\beta_t^*, q_t^*) + Z_t = 0; \quad P\otimes dt\text{-a.s.}, \tag{3.37}$$

where (Y,Z) solves the BSDE

$$dY_t = g(Y_t, Z_t)\, dt - Z_t \, dW_t, \quad Y_T = X_T^{\pi^*} + \xi. \tag{3.38}$$

Proof. By assumptions and Remark 3.3.6 the control problem admits as saddle point $(\pi^*, (\beta^*, q^*))$, that is,

$$\begin{aligned} V(x) =& E_{Q^{q^*}}\left[D_{0,T}^{\beta^*}(X_T^{\pi^*} + \xi) + \int_0^T D_{0,u}^{\beta^*} g_u^*(\beta_u^*, q_u^*)\, du \right] \qquad (3.39)\\ =& \inf_{(\beta,q)\in\mathcal{D}\times\mathcal{Q}} \sup_{\pi\in\Pi} E_{Q^q}\left[D_{0,T}^{\beta}(X_T^{\pi} + \xi) + \int_0^T D_{0,u}^{\beta} g_u^*(\beta_u, q_u)\, du \right]. \end{aligned}$$

It follows from (3.39) that $X_T^{\pi^*}$ is Q^{q^*}-integrable. Put

$$Y_t := \operatorname*{ess\,inf}_{(\beta,q)\in\mathcal{D}\times\mathcal{Q}} E_{Q^q}\left[D_{t,T}^{\beta}(X_T^{\pi^*} + \xi) + \int_t^T D_{t,u}^{\beta} g_u^*(\beta_u, q_u)\, du \,\Big|\, \mathcal{F}_t \right], \quad t\in[0,T].$$

By [48, Corollary 4.3], for all $t \in [0,T]$, we have

$$Y_t := E_{Q^{q^*}}\left[D_{t,T}^{\beta^*}(X_T^{\pi^*} + \xi) + \int_t^T D_{t,u}^{\beta^*} g_u^*(\beta_u^*, q_u^*)\, du \,\Big|\, \mathcal{F}_t \right]$$

so that applying martingale representation theorem and Itô's formula, we can find a predictable process Z such that (Y,Z) solves the linear BSDE

$$dY_t = (\beta_t^* Y_t + q_t^* Z_t - g_t^*(\beta_t^*, q_t^*))\, dt - Z_t\, dW_t, \quad Y_T = X_T^{\pi^*} + \xi.$$

Moreover, by [48, Theorem 4.6], for almost every (ω, t), the subgradients $\partial g(\omega, t, Y_t, Z_t)$ with respect to (Y_t, Z_t) contain (β_t^*, q_t^*). Hence, (Y,Z) also solves the BSDE (3.38).

Characterization of $\tilde{\pi}^*$: For any $\pi\in\Pi$ define

$$Y_t^{\pi} := E_{Q^{q^*}}\left[D_{t,T}^{\beta^*}(X_T^{\pi} + \xi) + \int_t^T D_{t,u}^{\beta^*} g_u^*(\beta_u^*, q_u^*)\, du \,\Big|\, \mathcal{F}_t \right], \quad t\in[0,T].$$

It follows from the saddle point property that

$$V(x) = \sup_{\pi\in\Pi} Y_0^{\pi} = Y_0^{\pi^*}.$$

Let $\pi\in\Pi$ be a bounded strategy such that for every $\varepsilon\in(0,1)$, $\pi^* + \varepsilon\pi \in \Pi$ and let $\tilde{\pi}$ be the process associated to π, see (3.36). Then, by optimality of π^*,

$$0 = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left(Y_0^{\pi^* + \varepsilon\pi} - Y_0^{\pi^*} \right) = E_{Q^{q^*}}\left[D_{0,T}^{\beta^*} \eta_T \right],$$

where $\eta_t := \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left(X_t^{\pi^*+\varepsilon\pi} - X_t^{\pi^*}\right)$ solves the SDE

$$\begin{aligned} d\eta_t &= \theta_t \left(\tilde{\pi}_t^* \sigma_t \eta_t + X_t^{\pi^*}\sigma_t \tilde{\pi}_t\right) dt + \left(\tilde{\pi}_t^* \sigma_t \eta_t + X_t^{\pi^*}\sigma_t \tilde{\pi}_t\right) dW_t \\ &= \alpha_t(\theta_t + q_t^*)\, dt + \alpha_t\, dW_t^{Q^{q^*}}, \quad \eta_0 = 0 \quad Q^{q^*}\text{-a.s.} \end{aligned}$$

with $\alpha_t = (\tilde{\pi}_t^* \sigma_t \eta_t + X_t^{\pi^*}\sigma_t\tilde{\pi}_t)$. In fact, this follows by applying the dominated convergence theorem [87, Theorem IV.32], since

$$\begin{aligned} X_t^{\pi^*+\varepsilon\pi} &= x\mathcal{E}\left(\tilde{\pi}^* + \varepsilon\tilde{\pi} dW^Q\right)_t \\ &\leq x\exp\left(\int_0^t \tilde{\pi}_u^*\, dW_u^Q + \left|\int_0^t \tilde{\pi}_u\, dW_u^Q\right| + \frac{1}{2}\int_0^t (\tilde{\pi}_u^* + \tilde{\pi}_u)^2\, du\right), \end{aligned}$$

where $dQ/dP = \mathcal{E}(-\int \theta\, dW)_T$. Let (p,k) be the solution of the linear BSDE with bounded terminal condition

$$dp_t = -(\theta_t p_t + q_t^* p_t + k_t)\tilde{\pi}_t^* \sigma_t\, dt + k_t\, dW_t^{Q^{q^*}}, \quad p_T = D_{0,T}^{\beta^*} \quad Q^{q^*}\text{-a.s.}$$

which is known as the adjoint equation. Observe that since $\beta^* \in \mathcal{D}$, $D_{0,T}^{\beta^*}$ is bounded. Applying Itô's formula to $\eta_t p_t$ yields

$$\eta_t p_t = \int_0^t X_u^{\pi^*}\tilde{\pi}_u\sigma_u(p_u\theta_u + p_u q_u^* + k_u)\, du + \int_0^t \left\{\eta_u k_u + p_u(\tilde{\pi}_u^*\sigma_u\eta_u + X_u^{\pi^*}\sigma_u\tilde{\pi}_u)\right\} dW_u^{Q^{q^*}}. \tag{3.40}$$

Since we cannot ensure that the second term of the left hand side of Equation (3.40) is a true Q^{q^*}-martingale, we introduce the following localization:

$$\tau^n := \inf\left\{t \geq 0 : \left|\int_0^t \left\{\eta_u k_u + p_u(\tilde{\pi}_u^*\sigma_u\eta_u + X_u^{\pi^*}\sigma_u\tilde{\pi}_u)\right\} dW_u^{Q^{q^*}}\right| > n\right\} \wedge T.$$

Hence, taking expectation with respect to Q^{q^*} on both sides of (3.40), we have

$$E_{Q^{q^*}}\left[p_{\tau^n}\eta_{\tau^n}\right] = E_{Q^{q^*}}\left[\int_0^{\tau^n} X_u^{\pi^*}\tilde{\pi}_u\sigma_u(p_u\theta_u + p_u q_u^* + k_u)\, du\right]. \tag{3.41}$$

By definition of $\mathcal{D}$, the family $(D_{0,\tau^n}^{\beta^*})_n$ is dominated by the bounded random variable $e^{\int_0^T (\beta_u^*)^-\, du}$. Moreover, for any $\delta > 0$ there exists $\varepsilon > 0$ such that

$$\eta_{\tau^n} \leq \frac{1}{\varepsilon}(X_{\tau^n}^{\pi^*+\varepsilon\pi} - X_{\tau^n}^{\pi^*}) + \delta \leq \frac{1}{\varepsilon}X_{\tau^n}^{\pi^*+\varepsilon\pi} + \delta.$$

Because we can restrict ourselves to subsolutions $(Y,Z) \in \mathcal{A}^u(X_T^\pi + \xi)$ satisfying $Y \geq X^\pi$, we can further estimate η_{τ^n} by

$$\eta_{\tau^n} \leq \frac{1}{\varepsilon}Y_{\tau^n}^{\pi^*+\varepsilon\pi} + \delta \leq \frac{1}{\varepsilon}E_{Q^{q^*}}\left[D_{0,\tau^n}^{\beta^*}(X_T^{\pi^*+\varepsilon\pi} + \xi) + \int_{\tau^n}^T g_u^*(\beta_u^*, q_u^*)\, du \,|\, \mathcal{F}_{\tau^n}\right] + \delta$$

where the second inequality follows from the same arguments which led to Equation (3.26) in the proof of Theorem 3.3.1. Hence,

$$\eta_{\tau^n} \leq \frac{1}{\varepsilon}E_{Q^{q^*}}\left[e^{\int_0^T (\beta_u^*)^-\, du}(X_T^{\pi^*+\varepsilon\pi} + \xi) + \int_0^T g_u^*(\beta_u^*, q_u^*)\, du \,|\, \mathcal{F}_{\tau^n}\right] + \delta.$$

Since the right hand side above is Q^{q^*}-uniformly integrable, taking the limit in (3.41) and using dominated convergence theorem and Fatou's lemma give

$$E_{Q^{q^*}}\left[\int_0^T X_u^{\pi^*}\tilde{\pi}_u\sigma_u(p_u\theta_u + p_u q_u^* + k_u)\,du\right] \le E_{Q^{q^*}}\left[D_{0,T}^{\beta^*}\eta_T\right] = 0,$$

recall that both p and η are Q^{q^*}-a.s. continuous processes. Arguing as above with $-\pi$ instead of π, we have

$$E_{Q^{q^*}}\left[\int_0^T X_u^{\pi^*}\tilde{\pi}_u\sigma_u(p_u\theta_u + p_u q_u^* + k_u)\,du\right] = 0.$$

Thus, since π was taken arbitrary, this leads to

$$p_t\theta_t + p_t q_t^* + k_t = 0 \quad P\otimes dt\text{-a.s},$$

since $Q^{q^*} \sim P$.

Characterization of β^* and q^*:

The function g satisfies (LSC) and $(\beta^*, q^*) \in \partial g(Y, Z)$ imply that $(Y, Z) \in \partial g^*(\beta^*, q^*)$, and since g is strictly convex, it holds $\partial g^*(\beta^*, q^*) = \{(Y, Z)\}$ so that by [91, Theorem 25.1], g^* is differentiable at (β^*, q^*). Hence, β^* and q^* are the points verifying

$$-\frac{\partial g^*}{\partial a}(\beta_t^*, q_t^*) + Y_t = 0 \quad \text{and} \quad -\frac{\partial g^*}{\partial b}(\beta_t^*, q_t^*) + Z_t = 0 \quad P\otimes dt\text{-a.s.}$$

□

3.4 Link to Conjugate Duality

In this final section we show the inherent link between duality of BSDEs and the theory of conjugate duality in optimization as presented, for instance, in Ekeland and Témam [52]. We will exploit the general method of conjugate duality in convex optimization to study the problem at hands. In Proposition 3.4.2 below we write the dual problem to (3.3). The main result of this section, Theorem 3.4.3, shows that even without the condition (QG) which enabled us to have weak compactness, the robust control problem still satisfies a minimax property. Consider the probability measure $Q = Q^\theta$ introduced in Section 3.1. Recall that $\mathcal{H}^1(Q)$ is the set of Q-martingales X such that $E_Q[\sup_{t\in[0,T]}|X|_t] < \infty$. We introduce the sets

$$\mathcal{C} := \{X_T^\pi : \pi \in \Pi\}\cap\mathcal{H}^1(Q),\ \mathcal{M} := \{M \in \text{BMO}_{++}(Q) : E_Q[MX_T^\pi] \le x \text{ for all } \pi \in \Pi\}$$

and $\bar{\mathcal{Q}} := \{q \in \mathcal{L} : \frac{dQ^q}{dP} \in \mathcal{M}\}$. Let us define the perturbation function F on $\mathcal{C}\times\mathcal{C}$ with values in $\mathbb{R}$ by

$$F(X_T^\pi, H) := \mathcal{E}_0^g\left(X_T^\pi + \xi + H\right).$$

For all $H \in \mathcal{C}$ we put

$$u(H) := \sup_{\pi\in\Pi} F(X_T^\pi, H).$$

The space BMO(Q) can be identified with the dual of the space $\mathcal{H}^1(Q)$. We extent the function F to the Banach space $\mathcal{H}^1(Q)\times\mathcal{H}^1(Q)$ by setting $F(X_T^\pi, H) = -\infty$ whenever H or X_T^π does not belong to $\mathcal{C}$. It holds $u(0) = V(x)$, the value function of the primal control problem. Since $\mathcal{E}_0^g$ is concave increasing, the function u is as well concave increasing, and from $u(0) = V(x) < \infty$ follows that $u(H) < \infty$ for all $H \in \mathcal{C}$. Define the concave conjugate F^* of F on BMO$(Q) \times$ BMO(Q) with values in $\bar{\mathbb{R}}$ by

$$F^*(M', M) := \inf_{H, X_T^\pi \in \mathcal{H}^1(Q)} \left\{ E_Q\left[M' X_T^\pi\right] + E_Q\left[MH\right] - F(X_T^\pi, H) \right\}.$$

The function F^* is concave and upper semicontinuous. For each $M' \in$ BMO(Q), put

$$v(M') := \inf_{M \in \text{BMO}(Q)} \{-F^*(M', M)\}. \tag{3.42}$$

For $M' = 0$ Equation (3.42) is the dual problem, and the relation $u(0) \leq v(0)$ follows as an immediate consequence of the definition of F^*. Since the functional $\mathcal{E}_0^g$ is increasing and $\mathcal{E}_0^g(0) > -\infty$ we have $u(0) \geq \mathcal{E}_0^g(0) > -\infty$. Hence $v(0) > -\infty$.

Lemma 3.4.1. *Assume that the driver g defined on $\mathbb{R}_+ \times \mathbb{R}^d$ satisfies (*CONV*), (*LSC*), (*NOR*) and (*POS*). Then, the function F is $\sigma(\mathcal{H}^1(Q) \times \mathcal{H}^1(Q), BMO(Q) \times BMO(Q))$-upper semicontinuous.*

Proof. Let us first show that $\mathcal{C}$ is closed in $\mathcal{H}^1(Q)$. For any sequence $(X_T^n) \subseteq \mathcal{C}$ converging to X_T in $\mathcal{H}^1(Q)$, the process $X_t := E_Q[X_T \mid \mathcal{F}_t]$ defines a positive Q-martingale starting at x. By martingale representation theorem, there exists $\nu \in \mathcal{L}^1(Q)$ such that $X_t = x + \int_0^t \nu_u \, dW_u^Q$, but since $\sigma\sigma'$ is of full rank, we can find a predictable process π such that $\pi\sigma = \nu$. Therefore, $dX_t = \pi_t \sigma_t (\theta_t dt + dW_t)$. That is, $X \in \mathcal{C}$.

Now it suffices to show that the function F is $\sigma(\mathcal{H}^1(Q) \times \mathcal{H}^1(Q), \text{BMO}(Q) \times \text{BMO}(Q))$-upper semicontinuous on $\mathcal{C} \times \mathcal{C}$ because the extension to $\mathcal{H}^1(Q) \times \mathcal{H}^1(Q)$ would also be weakly upper semicontinuous. Hence, we need to show that for every $c \geq 0$ the concave level set $\{(\alpha, \gamma) \in \mathcal{C} \times \mathcal{C} : F(\alpha, \gamma) \geq c\}$ is closed in $\mathcal{C} \times \mathcal{C}$. Let $c \geq 0$ be fixed and let us show that $\{\zeta \in \mathcal{H}^1(Q) : \mathcal{E}_0^g(\zeta) \geq c\}$ is $\mathcal{H}^1(Q)$-closed. Let (ζ^n) be a sequence converging in $\mathcal{H}^1(Q)$ to ζ and such that $\mathcal{E}_0^g(\zeta^n) \geq c$ for every $n \in \mathbb{N}$. Put $\eta^n := \sup_{m \geq n} \zeta^m$, $n \in \mathbb{N}$. The sequence (η^n) decreases to ζ and by Proposition 3.2.7, $(\mathcal{E}_0^g(\eta^n))$ converges to $\mathcal{E}_0^g(\zeta)$ and is decreasing. Hence, since $\mathcal{E}_0^g(\zeta) = \lim_{n \to \infty} \mathcal{E}_0^g(\eta^n) = \inf_n \mathcal{E}_0^g(\eta^n)$, it holds

$$\begin{aligned} \mathcal{E}_0^g(\zeta) &= \inf_{n \in \mathbb{N}} \mathcal{E}_0^g\left(\sup_{m \geq n} \zeta^m\right) \\ &\geq \inf_{n \in \mathbb{N}} \sup_{m \geq n} \mathcal{E}_0^g(\zeta^m) = \limsup_{n \to \infty} \mathcal{E}_0^g(\zeta^n). \end{aligned}$$

Now for every sequence $(\alpha^n, \gamma^n) \subseteq \mathcal{C} \times \mathcal{C}$ converging to $(\alpha, \gamma) \in \mathcal{C} \times \mathcal{C}$ in $\mathcal{H}^1(Q) \times \mathcal{H}^1(Q)$ such that $F(\alpha^n, \gamma^n) \geq c$ for every $n \in \mathbb{N}$ one has

$$\begin{aligned} c &\leq \limsup_{n \to \infty} F(\alpha^n, \gamma^n) = \limsup_{n \to \infty} \mathcal{E}_0^g(\alpha^n + \gamma^n + \xi) \\ &\leq \mathcal{E}_0^g(\alpha + \gamma + \xi) = F(\alpha, \gamma). \end{aligned}$$

This concludes the proof. □

For any $M \in \mathcal{M}$, define by $\mathcal{E}_0^*$ the convex conjugate of $\mathcal{E}_0^g$ relative to the dual pair $(\mathcal{H}^1(Q), \text{BMO}(Q))$. It follows from [48, Remark 3.8] that for each $M \in \mathcal{M}$, there exists $(\beta, q) \in \mathcal{D} \times \mathcal{Q}$, with q unique such that $M = D_{0,T}^\beta dQ^q / dP$, and $D_{0,T}^\beta = E[M]$. We put

$$\mathcal{E}_0^*(\beta, q) := \inf_{\{M \in \mathcal{M} : E[M] = D_{0,T}^\beta\}} \mathcal{E}_0^*(M).$$

Proposition 3.4.2. *Assume that the driver g defined on $\mathbb{R}_+ \times \mathbb{R}^d$ satisfies (*CONV*), (*LSC*), (*NOR*) and (*POS*). Further assume that $\xi \in L_+^\infty$. Then the dual problem to* (3.3) *is given by*

$$v(0) = \inf_{(\beta, q) \in \mathcal{D} \times \bar{\mathcal{Q}}} \left\{ \mathcal{E}_0^*(\beta, q) - E_Q\left[\frac{dQ^q}{dP} D_{0,T}^\beta \xi\right] \right\} - x \tag{3.43}$$

and the primal problem

$$u(0) = \sup_{X_T^\pi \in \mathcal{C}} \inf_{(\beta,q)\in\mathcal{D}\times\bar{\mathcal{Q}}} E_{Q^q}\left[D_{0,T}^{\beta}\frac{dQ}{dP}(X_T^\pi + \xi) + \int_0^T D_{0,u}^{\beta} g_u^*(\beta_u, q_u)\,du\right].$$

Proof. For every $M \in L^\infty$ one has

$$\begin{aligned} F^*(0,M) &= \inf_{H\in\mathcal{H}^1(Q), X_T^\pi\in\mathcal{C}} \{E_Q[MH] - F(X_T^\pi, H)\} \\ &= \inf_{H'\in\mathcal{H}^1(Q), X_T^\pi\in\mathcal{C}} \{E_Q[M(H' - X_T^\pi - \xi) - F(X_T^\pi, H' - X_T^\pi - \xi)]\}. \end{aligned}$$

In fact, $\{H' - X_T^\pi - \xi : X_T^\pi \in \mathcal{C},\ H' \in \mathcal{H}^1(Q)\} \subseteq \mathcal{H}^1(Q)$ and, reciprocally, for any $H \in \mathcal{H}^1(Q)$ we can write $H = H' - x - \xi = H' - X_T^0 - \xi$ for some $H' \in \mathcal{H}^1(Q)$. Hence,

$$F^*(0,M) = \inf_{H'\in\mathcal{H}^1(Q)} \{E_Q[MH'] - \mathcal{E}_0^g(H')\} - \sup_{X_T^\pi\in\mathcal{C}} E_Q[M(X_T^\pi + \xi)].$$

It is clear that if there exists $X_T^\pi \in \mathcal{C}$ such that $E_Q[MX_T^\pi] > x$, then $F^*(0,M) = -\infty$. Thus, the supremum in Equation (3.42) can by restricted to $\mathcal{M}$, and $F^*(0,M)$ takes the form

$$F^*(0,M) = \mathcal{E}_0^*(M) - E_Q[M\xi] - x.$$

Therefore, the dual problem (3.42) to the control problem (3.3) is given by

$$\begin{aligned} v(0) &= \inf_{M\in\mathcal{M}} \{\mathcal{E}_0^*(M) - E_Q[M\xi]\} - x \\ &= \inf_{(\beta,q)\in\mathcal{D}\times\bar{\mathcal{Q}}} \inf_{\{M: E[M]=D_{0,T}^\beta\}} \left\{\mathcal{E}_0^*(M) - E_Q\left[\frac{dQ^q}{dP} D_{0,T}^\beta \xi\right]\right\} - x \\ &= \inf_{(\beta,q)\in\mathcal{D}\times\bar{\mathcal{Q}}} \left\{\mathcal{E}_0^*(\beta,q) - E_Q\left[\frac{dQ^q}{dP} D_{0,T}^\beta \xi\right]\right\} - x. \end{aligned} \tag{3.44}$$

Now, let us introduce the following Lagrangian L, which is such that $-L$ is the H-conjugate of the function F, i.e.

$$L(X_T^\pi, M) = \sup_{H\in\mathcal{C}} \{F(X_T^\pi, H) - E_Q[MH]\}.$$

It is well known in convex duality theory, see for instance [52], that the following hold:

$$F^*(M', M) = \inf_{X_T^\pi\in\mathcal{C}} \{E_Q[M'X_T^\pi] - L(X_T^\pi, M)\}$$

and, since F is $\sigma(\mathcal{H}^1(Q)\times\mathcal{H}^1(Q), \mathrm{BMO}(Q)\times\mathrm{BMO}(Q))$-upper semicontinuous, the Fenchel-Moreau theorem and definition of L yield

$$F(X_T^\pi, H) = \inf_{M\in\mathcal{M}} \{E_Q[MH] + L(X_T^\pi, M)\}. \tag{3.45}$$

In particular,

$$v(0) = \inf_{M\in\mathcal{M}} \sup_{X_T^\pi\in\mathcal{C}} \{L(X_T^\pi, M)\} \quad \text{and} \quad u(0) = \sup_{X_T^\pi\in\mathcal{C}} \inf_{M\in\mathcal{M}} \{L(X_T^\pi, M)\}.$$

Let $\pi \in \Pi$ and $M \in \mathcal{M}$. By definition of the Laplacian, we have

$$\begin{aligned} L(X_T^\pi, M) &= \sup_{H\in\mathcal{H}^1(Q)} \{F(X_T^\pi, H) - E_Q[MH]\} \\ &= \sup_{H'\in\mathcal{H}^1(Q)} \{F(X_T^\pi, H' - X_T^\pi - \xi) - E_Q[M(H' - X_T^\pi - \xi)]\} \\ &= \sup_{H'\in\mathcal{H}^1(Q)} \{\mathcal{E}_0^g(H') - E_Q[MH']\} + E_Q[M(X_T^\pi + \xi)] \\ &= E_Q[M(X_T^\pi + \xi)] - \mathcal{E}_0^*(M). \end{aligned}$$

But by the proof of [48, Theorem 3.10], the function

$$\alpha_{\text{min}} : M \mapsto \inf_{\{\beta\in\mathcal{D}:E[M]=D^{\beta}_{0,T}\}} E_{Q^q}\left[\int_0^T D^{\beta}_{0,u} g^*_u(\beta_u, q_u)\, du\right]$$

is convex and $\sigma(\mathcal{H}^1(Q), \text{BMO(Q)})$-lower semicontinuous; that is, it is the minimal penalty function. Hence, $-\mathcal{E}^*_0(M) = \alpha_{\text{min}}(M)$ and therefore,

$$L(X^{\pi}_T, M) = \inf_{\{\beta\in\mathcal{D}:E[M]=D^{\beta}_{0,T}\}} E_{Q^q}\left[D^{\beta}_{0,T}\frac{dQ}{dP}(X^{\pi}_T + \xi) + \int_0^T D^{\beta}_{0,u} g^*_u(\beta_u, q_u)\, du\right].$$

In particular, this implies

$$u(0) = \sup_{X^{\pi}_T\in\mathcal{C}} \inf_{(\beta,q)\in\mathcal{D}\times\bar{\mathcal{Q}}} E_{Q^q}\left[D^{\beta}_{0,T}\frac{dQ}{dP}(X^{\pi}_T + \xi) + \int_0^T D^{\beta}_{0,u} g^*_u(\beta_u, q_u)\, du\right].$$

□

Next, we show that the control problem (3.3) satisfies the minimax property even if we do not assume any growth condition on the generator g. Notice that it does not ensure existence of a saddle point. We refer to [4] for some similar results in robust utility maximization.

Theorem 3.4.3. *Assume that the driver g satisfies (*CONV*), (*LSC*), (*NOR*) and (*POS*). Then, the value functions of the primal problem and dual problem coincide. More precisely, it holds*

$$\inf_{M\in\mathcal{M}} \sup_{X^{\pi}_T\in\mathcal{C}} \{L(X^{\pi}_T, (\beta, q))\} = \sup_{X^{\pi}_T\in\mathcal{C}} \inf_{M\in\mathcal{M}} \{L(X^{\pi}_T, (\beta, q))\}.$$

Proof. The main argument of the proof is the Fenchel-Rockafellar theorem applied on the Banach space $\mathcal{H}^1(Q)$. By definition $\mathcal{M} = \mathcal{C}^*$, the polar cone of $\mathcal{C}$ with respect to the dual pair $(\mathcal{H}^1(Q), \text{BMO}(Q))$. Moreover, since $\mathcal{C}$ is a cone, $\mathcal{M}$ is the polar of $\mathcal{C}$, i.e. $\mathcal{M} = \mathcal{C}^{\circ}$. Consider the convex-indicator function

$$\delta_{\mathcal{C}}(H) = \begin{cases} 0 & \text{if} \quad H \in \mathcal{C} \\ \infty & \text{if} \quad H \in \mathcal{H}^1(Q) \setminus \mathcal{C}. \end{cases}$$

We can rewrite u as

$$u(0) = \sup_{H\in\mathcal{H}^1(Q)} \{F(H, 0) - \delta_{\mathcal{C}}(H)\}.$$

Since $\mathcal{C}$ is $\sigma(\mathcal{H}^1(Q), \text{BMO}(Q))$-closed (see proof of Lemma 3.4.1), the function $F(\cdot, 0) - \delta_{\mathcal{C}}(\cdot)$ is concave and $\sigma(\mathcal{H}^1(Q), \text{BMO}(Q))$-upper semicontinuous. Hence, by [90, Corollary 1] we have

$$u(0) = \inf_{M\in\text{BMO}(Q)} \{\delta^*_{\mathcal{C}}(\beta, q) - F^*(0, M)\}.$$

The function $\delta_{\mathcal{C}}$ obeys the conjugacy relation $\delta^*_{\mathcal{C}} = \delta_{\mathcal{C}^{\circ}} = \delta_{\mathcal{M}}$, see [93, Section 11.E]. Thus,

$$\begin{aligned} u(0) &= \inf_{M\in\text{BMO}(Q)} \{\delta_{\mathcal{M}}(M) - F^*(0, M)\} \\ &= \inf_{M\in\mathcal{M}} \{-F^*(0, M)\} = v(0). \end{aligned}$$

This concludes the proof. □

Part II

Non-dominated Case

Chapter 4

Representation of Increasing Convex Functionals with Countably Additive Measures

4.1 Introduction

Let $\phi : X \to \mathbb{R} \cup \{+\infty\}$ be an increasing convex functional on a linear space of functions $f : \Omega \to \mathbb{R}$. More precisely, ϕ is convex and satisfies $\phi(f) \geq \phi(g)$ for $f \geq g$, where the second inequality is understood pointwise. By $I(\phi)$ we denote the algebraic interior of the effective domain $\operatorname{dom} \phi := \{f \in X : \phi(f) < +\infty\}$; that is, $I(\phi)$ consists of all $f \in \operatorname{dom} \phi$ with the property that for every $g \in X$, there is an $\varepsilon > 0$ such that $f + \lambda g \in \operatorname{dom} \phi$ for each $0 \leq \lambda \leq \varepsilon$.

If X is a linear space of bounded measurable functions on a measurable space $(\Omega, \mathcal{F})$ containing all indicator functions 1_A, $A \in \mathcal{F}$, it follows from standard convex duality arguments (see Section 4.2) that

$$\phi(f) = \max_{\mu \in ba^+(\mathcal{F})} \left(\langle f, \mu \rangle - \phi^*_X(\mu) \right) \quad \text{for all } f \in I(\phi), \tag{4.1}$$

where $ba^+(\mathcal{F})$ is the set of all finitely additive measure μ on $\mathcal{F}$ satisfying $\mu(\Omega) < \infty$, $\langle f, \mu \rangle$ denotes the integral $\int f d\mu$, and ϕ^*_X is the convex conjugate of ϕ, given by

$$\phi^*_X(\mu) := \sup_{f \in X} \left(\langle f, \mu \rangle - \phi(f) \right). \tag{4.2}$$

In applications, a representation like (4.1) is often more useful if it is in terms of countably instead of finitely additive measures. In this chapter we provide such representations under different assumptions. The following proposition is a non-linear extension of the Daniell–Stone theorem (see e.g. Theorem 4.5.2 in [49]) and provides context to our main results, Theorems 4.1.3 and 4.1.7 below. All proofs are given in Sections 4.2 and 4.3.

For a non-empty set Ω, we call a linear subspace X of $\mathbb{R}^\Omega$ a Stone vector lattice if for all $f, g \in X$, the point-wise minima $f \wedge g$ and $f \wedge 1$ also belong to X. By $\sigma(X)$ we denote the smallest σ-algebra on Ω making all functions $f \in X$ measurable with respect to the Borel σ-algebra on $\mathbb{R}$ and by $ca^+(X)$ all (countably additive) measures on $\sigma(X)$ satisfying $\langle f, \mu \rangle < \infty$ for every $f \in X^+ := \{g \in X : g \geq 0\}$. $\phi^*_X : ca^+(X) \to \mathbb{R} \cup \{+\infty\}$ is defined as in (4.2). If a sequence (f^n) in X converges pointwise from above to $f \in X$, we write $f^n \downarrow f$. Analogously, $f^n \uparrow f$ means pointwise convergence from below.

Proposition 4.1.1. *Let X be a Stone vector lattice over a non-empty set Ω and $\phi : X \to \mathbb{R} \cup \{+\infty\}$ an increasing convex function. Then the implications* (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi) *hold among the conditions*

(i) *There exists an $f \in I(\phi)$ such that $\phi(f^n) \downarrow \phi(f)$ for every sequence (f^n) in X satisfying $f^n \downarrow f$*

(ii) *$\phi(f^n) \downarrow \phi(f)$ for each $f \in I(\phi)$ and every sequence (f^n) in X satisfying $f^n \downarrow f$*

(iii) *For each $f \in I(\phi)$ and every sequence (f^n) in X^+ satisfying $f^n \downarrow 0$ there exists an $\varepsilon > 0$ such that $\phi(f + \varepsilon f^n) \downarrow \phi(f)$*

(iv) *$\phi(f) = \max_{\mu \in ca^+(X)}(\langle f, \mu\rangle - \phi^*_X(\mu))$ for all $f \in I(\phi)$*

(v) *$\phi(f) = \sup_{\mu \in ca^+(X)}(\langle f, \mu\rangle - \phi^*_X(\mu))$ for all $f \in I(\phi)$*

(vi) *$\phi(f^n) \uparrow \phi(f)$ for each $f \in I(\phi)$ and every sequence (f^n) in X satisfying $f^n \uparrow f$.*

We are interested in representations of the form (iv) and (v). If ϕ is real-valued and linear, (i) is Daniell's condition [27] and equivalent to each of (ii), (iii) and (vi). However, for an increasing convex ϕ, (i)–(iii) do not necessarily follow from (vi). Also, in general (iii) is weaker than (ii), and there exist examples which do not satisfy any of the conditions (i)–(vi). These points are illustrated in the following

Example 4.1.2. Consider the Stone vector lattice l^∞ of all bounded functions $f : \mathbb{N} \to \mathbb{R}$, where we use the convention $\mathbb{N} = \{1, 2, ...\}$. Denote by $ca_1^+(\mathbb{N})$ the set of all probability measures on $\mathbb{N}$ and by $ba_1^+(\mathbb{N})$ the set of all finitely additive probability measures on $\mathbb{N}$, that is, all finitely additive measures μ on $\mathbb{N}$ satisfying $\mu(\mathbb{N}) = 1$.

1. $s(f) := \sup_m f(m)$ defines an increasing convex functional $s : l^\infty \to \mathbb{R}$ which clearly fulfills (vi). It can easily be checked that the convex conjugate of s is $s^*_{l^\infty}(\mu) = 0$ if μ belongs to $ba_1^+(\mathbb{N})$ and $s^*_{l^\infty}(\mu) = \infty$ for all $\mu \in ba^+(\mathbb{N}) \setminus ba_1^+(\mathbb{N})$. One obviously has

$$s(f) = \sup_{\mu \in ca_1^+(\mathbb{N})} \langle f, \mu \rangle , \tag{4.3}$$

and it follows from (4.1) that

$$s(f) = \max_{\mu \in ba_1^+(\mathbb{N})} \langle f, \mu \rangle . \tag{4.4}$$

(4.3) is of the form (v). Moreover, for all $f \in l^\infty$ attaining their supremum, the supremum in (4.3) is attained by a Dirac measure. But if $f \in l^\infty$ does not attain its supremum, then $s(f)$ cannot be written in the form (iv). So s satisfies (v)–(vi) but not (i)–(iv).

2.

$$p(f) = \sup_{\mu \in ca^+(\mathbb{N})} \langle f, \mu \rangle \tag{4.5}$$

defines an increasing convex functional $p : l^\infty \to \mathbb{R} \cup \{+\infty\}$ mapping f to 0 or $+\infty$ depending on whether $s(f) \leq 0$ or $s(f) > 0$. So f belongs to $I(p)$ if and only if $s(f) < 0$, in which case the supremum in (4.5) is attained. It is easy to see that p fulfills (iii) but not (ii). So by Proposition 4.1.1, it satisfies (iii)–(vi) but violates (i)–(ii).

3. Now pick an increasing $f \in l^\infty$ that does not attain its supremum, and choose a $\mu \in ba_1^+(\mathbb{N})$ which maximizes (4.4). Then one has for all n,

$$s(f) = \left\langle f 1_{[1,n]}, \mu \right\rangle + \left\langle f 1_{(n,\infty)}, \mu \right\rangle \leq f(n)\mu[1,n] + s(f)(1 - \mu[1,n]).$$

It follows that $\mu[1,n] = 0$ for all n. So the positive linear functional $l : l^\infty \to \mathbb{R}$, given by $l(f) := \langle f, \mu \rangle$, satisfies $l(1_{[1,n]}) = 0 < l(1) = 1$, showing that it violates condition (vi), and therefore also (i)–(v).

In the following we introduce four conditions, called (A), (B), (C) and (D), for an increasing convex functional $\phi : X \to \mathbb{R} \cup \{+\infty\}$ on a Stone vector lattice X of functions $f : \Omega \to \mathbb{R}$ on a topological space Ω. If X consists of continuous functions, (A) and (B) both imply a max-representation like (iv). From each of (C) and (D) we derive a sup-representation similar to (v) in the case where X is the set of all bounded measurable functions on a Hausdorff space Ω equipped with its Borel σ-algebra.

(A) For all $f \in I(\phi)$ and every sequence (f^n) in X^+ satisfying $f^n \downarrow 0$, there exists an $\varepsilon > 0$ such that for each $\delta > 0$, there are $m \in \mathbb{N}$, $g \in \mathbb{R}^\Omega_+$ and an increasing convex function $\hat{\phi} : Y \to \mathbb{R}$ on a convex subset $Y \subseteq \mathbb{R}^\Omega_+$ containing $\{0, f^m g, (\varepsilon - g)^+, \varepsilon f^n : n \geq m\}$ so that

(i) $\{g < \varepsilon\}$ is relatively compact

(ii) $\hat{\phi}(f^m g) \leq \delta$ and

(iii) $\hat{\phi}(\varepsilon f^n) \geq \phi(f + \varepsilon f^n) - \phi(f)$ for all $n \geq m$.

(B) For all $f \in I(\phi)$ and every sequence (f^n) in X^+ satisfying $f^n \downarrow 0$, there exist functions $g, g^1, g^2, \dots$ in $\mathbb{R}^\Omega_+$ and numbers $m, m^1, m^2, \dots$ in $\mathbb{N}$ together with an increasing convex function $\hat{\phi} : Y \to \mathbb{R}$ on a convex subset $Y \subseteq \mathbb{R}^\Omega_+$ containing $\{0, f^n/m, g, g^n : n \geq m\}$ so that

(i) $\{f^m > g/n\}$ is relatively compact and contained in $\{m^n g^n \geq 1\}$ for all $n \geq m$

(ii) $\hat{\phi}(0) = 0$

(iii) $\hat{\phi}(f^n/m) \geq \phi(f + f^n/m) - \phi(f)$ for all $n \geq m$.

Theorem 4.1.3. *Let X be a Stone vector lattice of continuous functions $f : \Omega \to \mathbb{R}$ on a topological space Ω and $\phi : X \to \mathbb{R} \cup \{+\infty\}$ an increasing convex function satisfying at least one of the conditions* (A) *and* (B). *Then*

$$\phi(f) = \max_{\mu \in ca^+(X)} \left(\langle f, \mu \rangle - \phi^*_X(\mu) \right) \quad \textit{for all } f \in I(\phi).$$

As a special case of Theorem 4.1.3, one obtains the following variant of the Daniell–Stone theorem:

Corollary 4.1.4. *If X is a Stone vector lattice of continuous functions $f : \Omega \to \mathbb{R}$ on a topological space Ω, then every positive linear functional $\phi : X \to \mathbb{R}$ satisfying condition* (A) *or* (B) *is of the form $\phi(f) = \langle f, \mu \rangle$, $f \in X$, for a measure $\mu \in ca^+(X)$.*

In various situations, a measure on a σ-algebra $\mathcal{F}$ of subsets of a topological space Ω can be shown to possess regularity properties. Let us call a finite measure μ on $\mathcal{F}$ closed regular if

$$\mu(A) = \sup \{\mu(B) : B \in \mathcal{F}, B \text{ is closed and } B \subseteq A\} \quad \text{for all } A \in \mathcal{F}$$

and regular if

$$\mu(A) = \sup \{\mu(B) : B \in \mathcal{F}, B \text{ is closed, compact and } B \subseteq A\} \quad \text{for all } A \in \mathcal{F}.$$

If X is a Stone vector lattice of real-valued functions containing the constant functions, then every measure $\mu \in ca^+(X)$ is finite. Moreover, standard arguments (see Section 4.2 for details) yield the following:

Proposition 4.1.5. *Let X be a family of continuous functions $f : \Omega \to \mathbb{R}$ on a topological space Ω. Then every finite measure μ on $\sigma(X)$ is closed regular. Furthermore, if μ is a finite measure on $\sigma(X)$ and there exists a sequence (K_n) of compact sets in $\sigma(X)$ such that $\mu(K_n) \to \mu(\Omega)$, then μ is regular.*

Examples 4.1.6.
1. (Tightness conditions)
Let $\phi : C_b \to \mathbb{R} \cup \{+\infty\}$ be an increasing convex functional on the set C_b of all bounded continuous functions $f : \Omega \to \mathbb{R}$ on a topological space Ω. Let V be a linear space containing all functions of the form $f1_K$ and $f1_{K^c}$ for $f \in C_b$ and K a compact subset of Ω. If ϕ has an extension to an increasing convex function $\psi : V \to \mathbb{R} \cup \{+\infty\}$ with the property that for every $f \in I(\phi)$, there exists a $\delta > 0$ and a sequence (K_n) of compact sets such that

$$\psi(f + \delta 1_{K_n^c}) \downarrow \psi(f), \tag{4.6}$$

then for every $f \in I(\phi)$ and each sequence $(f^n) \in C_b^+$ satisfying $f^n \downarrow 0$, there exists an $\varepsilon > 0$ such that $\phi(f + \varepsilon) < +\infty$ and

$$\psi(f + \varepsilon f^1 1_{K_n^c}) - \psi(f) \le \psi(f + \varepsilon ||f^1||_\infty 1_{K_n^c}) - \psi(f) \to 0 \quad \text{as } n \to \infty.$$

It follows that condition (A) holds with $\hat{\phi}(h) = \phi(f + h) - \phi(f)$, and one obtains form Theorem 4.1.3 that

$$\phi(f) = \max_{\mu \in ca^+(C_b)} \langle f, \mu \rangle - \phi^*_{C_b}(\mu)) \quad \text{for all } f \in I(\phi). \tag{4.7}$$

If Ω is Hausdorff, all compact sets $K \subseteq \Omega$ are closed and therefore, belong to the Borel σ-algebra $\mathcal{F}$. Assume $\phi : B_b \to \mathbb{R}$ is an increasing convex functional defined on the space B_b of all bounded measurable functions $f : \Omega \to \mathbb{R}$ with the property that for every constant $M \ge 1$, there exists a sequence (K_n) of compact sets such that

$$\phi(M 1_{K_n^c}) \downarrow \phi(0). \tag{4.8}$$

Then one deduces as in the proof of (i) ⇒ (ii) of Proposition 4.1.1 that ϕ satisfies condition (4.6). If in addition, ϕ satisfies the translation property: $\phi(f + m) = \phi(f) + m$ for all $f \in B_b$ and $m \in \mathbb{R}$, then (4.8) holds if and only if for every $M \ge 1$, there exists a sequence of compacts (K_n) such that

$$\phi(-M 1_{K_n}) \downarrow \phi(-M).$$

This is slightly weaker than the tightness condition used in Proposition 4.28 of [60] to derive a max-representation for convex risk measures. Note that if ϕ has the translation property, then $\phi^*_{C_b}(\mu) = +\infty$ for $\mu \in ca^+(C_b) \setminus ca_1^+(C_b)$. So ϕ has a representation in terms of probability measures:

$$\phi(f) = \max_{\mu \in ca_1^+(C_b)} (\langle f, \mu \rangle - \phi^*(\mu)). \tag{4.9}$$

In the special case where Ω is a metric space, C_b generates the Borel σ-algebra $\mathcal{F}$, and for every compact set K_n, there exists a sequence $(h^{m,n})$ of $[0,1]$-valued functions in C_b such that $h^{m,n} \uparrow 1_{K_n^c}$. Therefore, if $\phi : C_b \to \mathbb{R} \cup \{+\infty\}$ is an increasing convex functional with an increasing convex extension ψ satisfying (4.6), then for any $f \in I(\phi)$ and $\mu \in ca^+(\mathcal{F})$ maximizing (4.7), there exists a $\delta > 0$ and a sequence (K_n) of compact sets such that

$$\delta \left\langle 1_{K_n^c}, \mu \right\rangle = \lim_m \delta \left\langle h^{m,n}, \mu \right\rangle \le \lim_m \phi(f + \delta h^{m,n}) - \phi(f) \le \psi(f + \delta 1_{K_n^c}) - \psi(f)) \downarrow 0.$$

So it follows from Proposition 4.1.5 that μ is regular, and as a result, the representations (4.7) and (4.9) can be written as maxima over regular finite measures or regular probability measures on $\mathcal{F}$, respectively.

2. (Adapted spaces and cones)
Let $\psi : V \to \mathbb{R} \cup \{+\infty\}$ be an increasing convex functional, where V is an adapted space [23] or an adapted cone [76] of continuous functions $f : \Omega \to \mathbb{R}$ on a topological space Ω. That is, V is either a linear space satisfying

(i) $V = V^+ - V^+$ (where $V^+ = \{f \in V : f \geq 0\}$)

(ii) For every $\omega \in \Omega$ there exists an $f \in V^+$ such that $f(\omega) > 0$

(iii) For every $f \in V^+$, there exists a $g \in V^+$ such that for each $\varepsilon > 0$ the set $\{f > \varepsilon g\}$ is relatively compact,

or V is a convex cone with the properties

(i) $V = V^+ \cup \{0\}$

(ii) For every $\omega \in \Omega$ there exists an $f \in V$ such that $f(\omega) > 0$

(iii) For every $f \in V$, there exists a $g \in V$ such that for each $\varepsilon > 0$ the set $\{f > \varepsilon g\}$ is relatively compact.

In both cases,

$$X := \{f : \Omega \to \mathbb{R} \text{ continuous } : |f| \leq g \text{ for some } g \in V\}$$

is a Stone vector lattice containing V, and

$$\phi(f) := \inf \{\psi(g) : f \leq g, \, g \in V\}$$

defines an increasing convex extension $\phi : X \to \mathbb{R} \cup \{+\infty\}$ of ψ. Moreover, for $f \in I(\phi)$ and a sequence (f^n) in X^+ satisfying $f^n \downarrow 0$, there is an $\varepsilon > 0$ such that $\phi(f + \varepsilon f^1) < +\infty$. It follows from (iii) that there exists a $g \in V^+$ such that $\phi(f + g) < +\infty$ and the set $\left\{f^1 > g/n\right\}$ is relatively compact for all $n \in \mathbb{N}$. By compactness, one obtains from (ii) that there exist functions $g^n \in V^+$ and numbers m^n, $n \in \mathbb{N}$ such that $\phi(f + g^n) < +\infty$ and $m^n g^n \geq 1$ on $\left\{f^1 > g/n\right\}$. This shows that condition (B) holds with $\hat{\phi}(h) = \phi(f + h) - \phi(f)$. So by Theorem 4.1.3,

$$\phi(f) = \max_{\mu \in ca^+(X)} (\langle f, \mu \rangle - \phi_X^*(\mu)), \quad \text{for all } f \in I(\phi). \tag{4.10}$$

Moreover, it follows from the definition of ϕ that $I(\psi) \subseteq I(\phi)$ and $\psi_V^*(\mu) = \phi_X^*(\mu)$ for $\mu \in ca^+(X)$. Therefore

$$\psi(f) = \max_{\mu \in ca^+(X)} (\langle f, \mu \rangle - \psi_V^*(\mu)), \quad \text{for all } f \in I(\psi). \tag{4.11}$$

(4.10) and (4.11) are non-linear versions of the linear representation results, Proposition 2 in [23] and Proposition 11 of [76]. But in contrast to [23, 76], here X does not have to be locally compact. As a special case of (4.11), one also recovers e.g. the max-representation of sublinear distributions given in Lemma 3.4 of [85].

The next result gives a sup-representation for increasing convex functionals ϕ on the space B_b of all bounded measurable functions $f : \Omega \to \mathbb{R}$ on a Hausdorff space Ω with Borel σ-algebra $\mathcal{F}$. The following two conditions are variants of (vi) in Proposition 4.1.1. We call a sequence (K_n) of subsets of Ω or a sequence (f^n) of real-valued functions on Ω increasing if $K_n \subseteq K_{n+1}$ or $f^n \leq f^{n+1}$ for all n, respectively.

(C) ϕ is real-valued and there exists an increasing sequence (K_n) of compact subsets of Ω such that $\phi(f^n) \uparrow \phi(f)$ for every increasing sequence (f^n) in B_b and $f \in B_b$ such that $|f - f^n| 1_{K_m} = 0$ for all $n \geq m$.

(D) There exists an increasing sequence (K_n) of compact subsets of Ω such that $\phi(f^n) \uparrow \phi(f)$ for every increasing sequence (f^n) in B_b and $f \in B_b$ such that $|f - f^n| 1_{K_m} \leq 1/m$ for all $n \geq m$.

By C_b we denote the set of all bounded continuous functions $f : \Omega \to \mathbb{R}$ and by U_b all bounded upper semicontinuous functions $f : \Omega \to \mathbb{R}$. We define the lower regularization of ϕ by

$$\phi_r(f) := \sup \{\phi(g) : g \in U_b,\, g \leq f\},$$

and say ϕ is lower regular if $\phi = \phi_r$. $ca_r^+(\mathcal{F})$ is the collection of all regular finite measures on $\mathcal{F}$. For

$$\phi^*_{C_b}(\mu) := \sup_{f \in C_b} (\langle f, \mu \rangle - \phi(f)) \quad \text{and} \quad \phi^*_{U_b}(\mu) := \sup_{f \in U_b} (\langle f, \mu \rangle - \phi(f)),$$

one obviously has $\phi^*_{C_b}(\mu) \leq \phi^*_{U_b}(\mu)$, $\mu \in ca_r^+(\mathcal{F})$.

Theorem 4.1.7. *Let Ω be a Hausdorff space with Borel σ-algebra $\mathcal{F}$ and $\phi : B_b \to \mathbb{R} \cup \{+\infty\}$ an increasing convex function. If ϕ satisfies* (C) *or* (D)*, then*

$$\phi(f) = \sup_{\mu \in ca_r^+(\mathcal{F})} \left(\langle f, \mu \rangle - \phi^*_{C_b}(\mu)\right) \quad \textit{for all } f \in C_b, \tag{4.12}$$

$$\phi(f) \leq \sup_{\mu \in ca_r^+(\mathcal{F})} \left(\langle f, \mu \rangle - \phi^*_{C_b}(\mu)\right) \quad \textit{for all } f \in U_b, \tag{4.13}$$

$$\phi_r(f) \leq \sup_{\mu \in ca_r^+(\mathcal{F})} \left(\langle f, \mu \rangle - \phi^*_{C_b}(\mu)\right) \textit{for all } f \in B_b, \tag{4.14}$$

and both inequalities become equalities if $\phi^*_{C_b}(\mu) = \phi^*_{U_b}(\mu)$ for all $\mu \in ca_r^+(\mathcal{F})$.

In particular, if ϕ is lower regular and $\phi^*_{C_b}(\mu) = \phi^*_{U_b}(\mu)$ for all $\mu \in ca_r^+(\mathcal{F})$, then

$$\phi(f) = \sup_{\mu \in ca_r^+(\mathcal{F})} \left(\langle f, \mu \rangle - \phi^*_{C_b}(\mu)\right) \quad \textit{for all } f \in B_b. \tag{4.15}$$

For positive linear functionals, Theorem 4.1.7 yields the following:

Corollary 4.1.8. *Let Ω be a Hausdorff space with Borel σ-algebra $\mathcal{F}$ and $\phi : B_b \to \mathbb{R}$ a positive linear functional satisfying* (C)*. Then there exists a $\mu \in ca_r^+(\mathcal{F})$ such that*

$$\phi(f) = \langle f, \mu \rangle \quad \textit{for all } f \in C_b. \tag{4.16}$$

If Ω is a metric space with Borel σ-algebra $\mathcal{F}$, one also has

$$\phi(f) \leq \langle f, \mu \rangle \textit{ for all } f \in U_b \quad \textit{and} \quad \phi_r(f) \leq \langle f, \mu \rangle \textit{ for all } f \in B_b, \tag{4.17}$$

and the inequalities are equalities if $\phi^*_{C_b}(\nu) = \phi^*_{U_b}(\nu)$ for all $\nu \in ca_r^+(\mathcal{F})$.

In particular, if Ω is a metric space with Borel σ-algebra $\mathcal{F}$, ϕ is lower regular, and $\phi^*_{C_b}(\nu) = \phi^*_{U_b}(\nu)$ for all $\nu \in ca_r^+(\mathcal{F})$, then

$$\phi(f) = \langle f, \mu \rangle \quad \textit{for all } f \in B_b. \tag{4.18}$$

Remarks 4.1.9.

1. To have a representation of the form (4.15) or (4.18), it is necessary that ϕ be lower regular. Indeed, for every $f \in B_b$ and $\delta > 0$, there exists a measurable partition (A_m) of Ω and numbers $a_1 < \cdots < a_M$ such that the step function $g = \sum_{m=1}^M a_m 1_{A_m}$ satisfies

$g \le f \le g + \delta$. Furthermore, for each $\mu \in ca_r^+(\mathcal{F})$, one can choose closed sets $F_m \subseteq A_m$ such that $\langle g, \mu \rangle \le \langle h, \mu \rangle + \delta$ for the upper semicontinuous function $h = a_1 1_{\Omega \setminus \bigcup_{m=2}^M F_m} + \sum_{m=2}^M a_m 1_{F_m} \le g$. It follows that $\langle f, \mu \rangle \le \langle h, \mu \rangle + \delta(\langle 1, \mu \rangle + 1)$. So any linear functional of the form (4.18) is lower regular, and as a supremum of lower regular functionals, (4.15) is again lower regular.

2. If Ω is a Hausdorff space with Borel σ-algebra $\mathcal{F}$, it follows from 1. that for all $\mu \in ca_r^+(\mathcal{F})$ and $f \in B_b$, there exists a sequence (f^n) in U_b such that $f^n \le f$ and $\langle f^n, \mu \rangle \uparrow \langle f, \mu \rangle$. As a result, one obtains for every increasing functional $\phi : B_b \to \mathbb{R} \cup \{+\infty\}$ and $\mu \in ca_r^+(\mathcal{F})$,

$$\phi^*_{U_b}(\mu) = \sup_{f \in U_b} \left(\langle f, \mu \rangle - \phi(f) \right) = \phi^*_{B_b}(\mu) = \sup_{f \in B_b} \left(\langle f, \mu \rangle - \phi(f) \right).$$

Similarly, if $\mu \in ca_r^+(\mathcal{F})$ has the property that for all $f \in U_b$, there exists a sequence (f^n) in C_b such that $f^n \le f$ and $\langle f^n, \mu \rangle \uparrow \langle f, \mu \rangle$, then $\phi^*_{C_b}(\mu) = \phi^*_{U_b}(\mu)$ for every increasing functional $\phi : B_b \to \mathbb{R} \cup \{+\infty\}$. This provides a sufficient condition for the inequalities in (4.13), (4.14) and (4.17) to be equalities.

The remainder of the chapter is organized as follows: In Section 4.2 we prove representation (4.1), Proposition 4.1.1, Theorem 4.1.3, Corollary 4.1.4 and Proposition 4.1.5. In Section 4.3 we give the proofs of Theorem 4.1.7 and Corollary 4.1.8.

4.2 Derivation of Max-Representations

Proof of the representation (4.1)
It is immediate from the definition of ϕ^*_X that

$$\phi(f) \ge \sup_{\mu \in ba^+(\mathcal{F})} \left(\langle f, \mu \rangle - \phi^*_X(\mu) \right) \quad \text{for every } f \in X. \tag{4.1}$$

On the other hand, for $f \in I(\phi)$, the directional derivative

$$\phi'(f; g) := \lim_{\varepsilon \downarrow 0} \frac{\phi(f + \varepsilon g) - \phi(f)}{\varepsilon}$$

is a real-valued increasing sublinear function of $g \in X$. So it follows from the Hahn–Banach extension theorem that there exists a positive linear functional $\psi : X \to \mathbb{R}$ satisfying

$$\psi(g) \le \phi'(f; g) \le \phi(f + g) - \phi(f)$$

for all $g \in X$. Since $\psi(\lambda 1) = \lambda \psi(1)$, $\lambda \in \mathbb{R}$, one obtains by monotonicity that ψ is continuous with respect to the sup-norm on X. Therefore, it can be represented as $\psi(g) = \langle g, \nu \rangle$ for the finitely additive measure $\nu \in ba^+(\mathcal{F})$ given by $\nu(A) := \psi(1_A)$, $A \in \mathcal{F}$. It follows that $\phi(f) + \phi^*_X(\nu) = \langle f, \nu \rangle$, which together with (4.1), implies $\phi(f) = \max_{\mu \in ba^+(\mathcal{F})} \left(\langle f, \mu \rangle - \phi^*_X(\mu) \right)$. □

Proof of Proposition 4.1.1
To prove (i) ⇒ (ii), let $f, g \in I(\phi)$ such that f fulfills (i). Then for all $\lambda \in (0,1)$ and every sequence (f^n) in X^+ satisfying $f^n \downarrow 0$, one has

$$\begin{aligned} \phi(g + f^n) &\le \lambda \phi\left(f + \frac{1}{\lambda} f^n \right) + (1-\lambda) \phi\left(\frac{g - \lambda f}{1 - \lambda} \right) \\ &= \lambda \phi\left(f + \frac{1}{\lambda} f^n \right) + (1-\lambda) \phi\left(g + \frac{\lambda}{1-\lambda}(g - f) \right). \end{aligned}$$

Since f satisfies (i),

$$\phi\left(f + \frac{1}{\lambda} f^n\right) \downarrow \phi(f) \quad \text{for fixed } \lambda \in (0,1) \text{ and } n \to \infty.$$

Moreover, there exists a $\delta > 0$ such that $x \mapsto \phi(g + x(g - f))$ is a real-valued convex function on the interval $(-\delta, \delta)$. As a consequence, it is continuous at 0, and one obtains

$$\lambda\phi(f) + (1-\lambda)\phi\left(g + \frac{\lambda}{1-\lambda}(g - f)\right) \to \phi(g) \quad \text{for } \lambda \downarrow 0.$$

This shows that $\phi(g + f^n) \downarrow \phi(g)$.

(ii) $\Rightarrow$ (iii) is obvious. To prove (iii) $\Rightarrow$ (iv), note first that it follows from the definition of ϕ^*_X that

$$\phi(f) \geq \sup_{\mu \in ca^+(X)} \left(\langle f, \mu\rangle - \phi^*_X(\mu)\right) \quad \text{for all } f \in X.$$

Moreover, for $f \in I(\phi)$, one deduces as in the proof of the representation (4.1) that there exists a positive linear functional $\psi : X \to \mathbb{R}$ satisfying

$$\psi(g) \leq \phi'(f; g) \leq \phi(f + g) - \phi(f), \quad g \in X.$$

If (iii) holds, then for every sequence (f^n) in X^+ satisfying $f^n \downarrow 0$, there exists an $\varepsilon > 0$ such that

$$\varepsilon\psi(f^n) \leq \phi(f + \varepsilon f^n) - \phi(f) \downarrow 0.$$

So one obtains from the Daniell–Stone theorem a $\nu \in ca^+(X)$ such that $\psi(g) = \langle g, \nu\rangle$ for all $g \in X$. It follows that $\phi(f) + \phi^*_X(\nu) = \langle f, \nu\rangle$, which implies

$$\phi(f) = \max_{ca^+(X)} \left(\langle f, \mu\rangle - \phi^*_X(\mu)\right).$$

(iv) $\Rightarrow$ (v) is clear, and (v) $\Rightarrow$ (vi) follows since by the monotone convergence theorem, the mapping $f \mapsto \langle f, \mu\rangle - \phi^*_X(\mu)$ satisfies (v) for every $\mu \in ca^+(X)$. □

Proof of Theorem 4.1.3

Choose a function $f \in I(\phi)$ and a sequence (f^n) in X^+ satisfying $f^n \downarrow 0$. If we can show that there exists an $\varepsilon > 0$ such that $\phi(f + \varepsilon f^n) \downarrow \phi(f)$, the theorem follows from Proposition 4.1.1. Let us first assume ϕ satisfies (A). Then there exists a $\lambda > 0$ such that for every $\delta > 0$, there are $m \in \mathbb{N}$, $g \in \mathbb{R}^\Omega_+$ and an increasing convex function $\hat{\phi} : Y \to \mathbb{R}$ on a convex subset $Y \subseteq \mathbb{R}^\Omega_+$ containing $\{0, f^m g, (\lambda - g)^+, \lambda f^n : n \geq m\}$ such that $\{g < \lambda\}$ is relatively compact, $\hat{\phi}(f^m g) \leq \delta$, and $\hat{\phi}(\lambda f^n) \geq \phi(f + \lambda f^n) - \phi(f)$ for all $n \geq m$. Since $x \mapsto \hat{\phi}(x(\lambda - g)^+)$ is a real-valued increasing convex function on the interval $[0, 1]$, it must be continuous at 0. In particular, there exists an $x \in (0, 1]$ such that

$$\hat{\phi}(x(\lambda - g)^+) \leq \hat{\phi}(0) + \delta \leq \hat{\phi}(f^m g) + \delta \leq 2\delta.$$

For $n \geq m$, one has $\lambda f^n \leq f^m g + f^n(\lambda - g)^+$, and by Dini's lemma, f^n converges to 0 uniformly on the closure of $\{g < \lambda\}$. So there exists an $n \geq m$ such that

$$f^n(\lambda - g)^+ \leq x(\lambda - g)^+,$$

and one obtains

$$\begin{aligned}\phi\left(f + \frac{\lambda}{2} f^n\right) - \phi(f) &\leq \hat{\phi}\left(\frac{\lambda}{2} f^n\right) \\ &\leq \hat{\phi}\left(\frac{f^m g + x(\lambda - g)^+}{2}\right) \leq \frac{\hat{\phi}(f^m g) + \hat{\phi}(x(\lambda - g)^+)}{2} \leq 2\delta. \quad (4.2)\end{aligned}$$

Since $\delta > 0$ was arbitrary, this shows that $\phi(f + \lambda f^n/2) \downarrow \phi(f)$.

If instead of (A), f satisfies condition (B), there exist functions $g, g^1, g^2, \dots$ in $\mathbb{R}^\Omega_+$ and numbers $m, m^1, m^2, \dots$ in $\mathbb{N}$ together with an increasing convex function $\hat{\phi} : Y \to \mathbb{R}$ on a convex subset $Y \subseteq \mathbb{R}^\Omega_+$ containing $\{0, f^n/m, g, g^n : n \geq m\}$ such that $\{f^m > g/n\}$ is relatively compact and contained in $\{m^n g^n \geq 1\}$ for all $n \geq m$, $\hat{\phi}(0) = 0$, and $\hat{\phi}(f^n/m) \leq \phi(f + f^n/m) - \phi(f)$ for all $n \geq m$. Since $x \mapsto \hat{\phi}(xg)$ is a real-valued increasing convex function on the interval $[0,1]$, it is continuous at 0. In particular, for given $\delta > 0$, there exists an integer $k \geq 2m$ such that $\hat{\phi}(2g/km) \leq \delta$. Similarly, there exists an integer $l \geq 2m^k$ such that $\hat{\phi}(2m^k g^k/lm) \leq \delta$. By Dini's Lemma, f^n converges uniformly to 0 on the closure of the set $\{f^m > g/k\}$. So there exists an $n \geq m$ such that $f^n \leq 1/l$ on $\{f^m > g/k\}$. Since $\{f^m > g/k\}$ is contained in $\{m^k g^k \geq 1\}$ and $f^n \leq f^m \leq g/k$ on $\{f^m \leq g/k\}$, one has $(f^n - g/k)^+ \leq m^k g^k/l$. Therefore,

$$f^n \leq g/k + (f^n - g/k)^+ \leq g/k + m^k g^k/l,$$

and

$$\phi\left(f + \frac{f^n}{m}\right) - \phi(f) \leq \hat{\phi}\left(\frac{f^n}{m}\right) \leq \hat{\phi}\left(\frac{g}{km} + \frac{m^k g^k}{lm}\right) \leq \frac{\hat{\phi}(2g/km) + \hat{\phi}(2m^k g^k/lm)}{2} \leq \delta. \quad (4.3)$$

This shows that $\phi(f + f^n/m) \downarrow \phi(f)$, and the proof is complete. □

Proof of Corollary 4.1.4

It follows from Theorem 4.1.3 that there exists a $\mu \in ca^+(X)$ such that $\phi^*_X(\mu) < +\infty$. If ϕ is linear, this implies that $\langle f, \mu \rangle = \phi(f)$ for all $f \in X$. □

Proof of Proposition 4.1.5

Fix a finite measure μ on $\sigma(X)$ and call a set $A \in \sigma(X)$ closed regular if

$$\mu(A) = \sup\left\{\mu(B) : B \in \sigma(X), B \text{ is closed and } B \subseteq A\right\}.$$

The collection of sets

$$\mathcal{G} := \{A \in \sigma(X) : \ A \text{ and } \Omega \setminus A \text{ are closed regular}\}$$

forms a sub-σ-algebra of $\sigma(X)$. For a closed set $F \subseteq \mathbb{R}$ and $f \in X$, $f^{-1}(F)$ is a closed subset of Ω. Moreover, $\mathbb{R} \setminus F$ can be written as a countable union $\bigcup_n F_n$ of closed sets $F_n \subseteq \mathbb{R}$. Therefore, $\Omega \setminus f^{-1}(F)$ equals $\bigcup_n f^{-1}(F_n)$, which can be approximated with the closed sets $\bigcup_{n=1}^N f^{-1}(F_n)$. This shows that $f^{-1}(F)$ belongs to $\mathcal{G}$. Since the sets $f^{-1}(F)$ generate $\sigma(X)$, one obtains $\mathcal{G} = \sigma(X)$, which means that μ is closed regular.

If there exists a sequence (K_n) of compact sets in $\sigma(X)$ such that $\mu(K_n) \to \mu(\Omega)$, then $\mu(A \cap K_n) \to \mu(A)$ for every $A \in \mathcal{F}$. Moreover, for every n there exists a closed set $B_n \subseteq A \cap K_n$ in $\sigma(X)$ such that $\mu(B_n) \geq \mu(A \cap K_n) - 1/n$. Since every closed subset of a compact set is compact, this shows that μ is regular. □

4.3 Derivation of Sup-Representations

For a sequence of compact Hausdorff spaces (H_n), consider the sequence spaces

$$U := \left\{u \in \prod_n C(H_n) : ||u|| < \infty\right\} \quad \text{and} \quad V := \left\{\nu \in \prod_n ca_r(H_n) : ||\nu|| < \infty\right\},$$

where $C(H_n)$ denotes the set of all real-valued continuous functions on H_n, $ca_r(H_n) = ca_r^+(H_n) - ca_r^+(H_n)$, where $ca_r^+(H_n)$ are all finite regular measures on the Borel σ-algebra of H_n, and the norms are defined as follows:

$$||u|| := \sup_n ||u_n||_\infty \text{ for the sup-norm } ||.||_\infty \text{ and}$$

$$||\nu|| := \sum_n ||\nu_n||_{\mathrm{tv}} < \infty \text{ for the total variation norm } ||.||_{\mathrm{tv}}.$$

By the Riesz representation theorem (see e.g. Theorem IV.6.3 in [51]), $ca_r(H_n)$ is the topological dual of $C(H_n)$. Therefore, (U, V) is a dual pair under the bilinear form $\langle u, \nu \rangle := \sum_n \langle u_n, \nu_n \rangle$. By V^+ we denote the set of all $\nu \in V$ belonging to $\prod_n ca_r^+(H_n)$. For a function $\psi : U \to \mathbb{R} \cup \{+\infty\}$, we consider the following two conditions:

(C') ψ is real-valued and $\lim_n \psi(u^n) = \psi(u)$ for every increasing sequence (u^n) in U and $u \in U$ such that $u^n_m = u_m$ for all $n \geq m$.

(D') $\lim_n \psi(u^n) = \psi(u)$ for every increasing sequence (u^n) in U and $u \in U$ such that $\lim_n ||u^n_m - u_m||_\infty = 0$ for every m.

Note that U contains l^∞ as a subspace, and on l^∞ the following holds:

Lemma 4.3.1. *Every increasing convex function $\psi : l^\infty \to \mathbb{R} \cup \{+\infty\}$ satisfying* (C') *or* (D') *is $\sigma(l^\infty, l^1)$-lower semicontinuous.*

Proof. To prove the lemma one has to show that all lower level sets of ψ are $\sigma(l^\infty, l^1)$-closed. By the Krein–Šmulian theorem (see e.g. Theorem V.5.7 in [51]), it is enough to show that the sets

$$D_{a,b} = \{x \in l^\infty : \psi(x) \leq a,\ ||x||_\infty \leq b\}, \quad a, b \in \mathbb{R},$$

are $\sigma(l^\infty, l^1)$-closed, which they are if and only if they are $\sigma(l^\infty, l^1(\eta))$-closed, where $l^1(\eta)$ is the l^1-space with respect to the probability measure η on $\mathbb{N}$ given by $\eta(n) = 2^{-n}$, and the pairing on $(l^\infty, l^1(\eta))$ is $\langle x, y \rangle = \sum_n x_n y_n 2^{-n}$. The embedding of l^∞ in $l^1(\eta)$ is continuous with respect to $\sigma(l^\infty, l^1(\eta))$ and $\sigma(l^1(\eta), l^\infty)$. So it is sufficient to show that the sets $D_{a,b}$ are $\sigma(l^1(\eta), l^\infty)$-closed. But by convexity, this follows if it can be shown that they are norm-closed in $l^1(\eta)$. To do that, consider a sequence (x^n) in $D_{a,b}$ converging to x in the $l^1(\eta)$-norm. Then $||x||_\infty \leq b$, and $y^n_m := \inf_{j \geq n} x^j_m$ defines a sequence (y^n) in $D_{a,b}$ which increases component-wise to x.

Under (C') ψ is real-valued, and since every real-valued convex function on $\mathbb{R}^m$ is continuous, one has

$$\psi(x_1, \dots, x_m, y^1_{m+1}, y^1_{m+2}, \dots) = \lim_n \psi(y^n_1, \dots, y^n_m, y^1_{m+1}, y^1_{m+2}, \dots) \leq \lim_n \psi(y^n) \leq a$$

for all $m \geq 1$. Therefore, it follows from (C') that x belongs to $D_{a,b}$. If ψ satisfies (D'), one obtains $\psi(x) = \lim_n \psi(y^n) \leq a$. So x belongs to $D_{a,b}$. □

We also need the following

Lemma 4.3.2. *For every $y \in l^1$, the set $A_y = \{\nu \in V : ||\nu_n||_{\mathrm{tv}} \leq |y_n|\}$ is $\sigma(V, U)$-compact.*

Proof.

$$\tilde{U} := \left\{ u \in \prod_n C(H_n) : \sum_n ||u_n||_\infty < \infty \right\}$$

is a Banach space with topological dual

$$\tilde{V} := \left\{ \nu \in \prod_n ca_r(H_n) : \sup_n ||\nu_n||_{\mathrm{tv}} < \infty \right\}.$$

Therefore, one obtains from the Banach–Alaoglu theorem that the norm ball $\{\nu \in \tilde{V} : \sup_n ||\nu_n||_{\mathrm{tv}} \leq 1\}$ is $\sigma(\tilde{V}, \tilde{U})$-compact. But for $y \in l^1$, the mapping $(\nu_n) \mapsto (\nu_n y_n)$ continuously embeds $\tilde{V}$ in V with respect to $\sigma(\tilde{V}, \tilde{U})$ and $\sigma(V, U)$. It follows that A_y is $\sigma(V, U)$-compact. □

Now we are ready to prove a representation result for increasing convex functionals on U.

Proposition 4.3.3. *Every increasing convex function* $\psi : U \to \mathbb{R} \cup \{+\infty\}$ *satisfying* (C') *or* (D') *has a representation of the form*

$$\psi(u) = \sup_{\nu \in V^+} (\langle u, \nu \rangle - \psi^*(\nu)) \quad \textit{for} \quad \psi^*(\nu) := \sup_{u \in U} (\langle u, \nu \rangle - \psi(u)).$$

Proof. In the case $\psi \equiv +\infty$, the proposition is clear. So let us assume that $\psi(u) < +\infty$ for at least one $u \in U$. Then it is enough to show that

$$\psi(u) = \sup_{\nu \in V} (\langle u, \nu \rangle - \psi^*(\nu)), \quad u \in U, \tag{4.1}$$

since it follows from the monotonicity of ψ that $\psi^*(\nu) = +\infty$ for all $\nu \in V \setminus V^+$. But (4.1) is a consequence of the Fenchel–Moreau theorem (see e.g. Theorem 3.2.2 in [99]) if we can show that ψ is $\sigma(U, V)$-lower semicontinuous, or equivalently, all lower level sets $D_a = \{u \in U : \psi(u) \leq a\}$ are $\sigma(U, V)$-closed. Moreover, since every D_a is convex, it follows from the Hahn–Banach separation theorem together with the Mackey–Arens theorem (see e.g. Theorem IV.3.2 in [96]) that it is $\sigma(U, V)$-closed if we can show that it is closed in the Mackey topology $\tau(U, V)$. So let (u^α) be a net in D_a such that $u^\alpha \to \hat{u} \in U$ in $\tau(U, V)$. We know from Lemma 4.3.2 that for every $y \in l^1$, the set

$$A_y := \{\nu \in V : ||\nu_n||_{\mathrm{tv}} \leq |y_n|\}$$

is $\sigma(V, U)$-compact. Therefore, one has

$$\sum_n ||u_n^\alpha - \hat{u}_n||_\infty |y_n| \leq \sup_{\nu \in A_y} |\langle u^\alpha - \hat{u}, \nu \rangle| \to 0. \tag{4.2}$$

If ψ satisfies (C'), we define the projections $\pi_n : U \to l^\infty$ as follows: for $m > n$,

$$\pi_n(u)_m := \underline{u}_m := \min_{z \in H_m} u_m(z),$$

and for $m = 1, \ldots, n$,

$$\begin{aligned}
\pi_n(u)_1 &:= \min\{x \in \mathbb{R} : x \geq \underline{u}_1,\ \psi(x, u_2, \ldots, u_n, \underline{u}_{n+1}, \ldots) \\
&\qquad = \psi(u_1, u_2, \ldots, u_n, \underline{u}_{n+1}, \ldots)\} \\
\pi_n(u)_2 &:= \min\{x \in \mathbb{R} : x \geq \underline{u}_2,\ \psi(\pi_n(u)_1, x, u_3, \ldots, u_n, \underline{u}_{n+1}, \ldots) \\
&\qquad = \psi(u_1, \ldots, u_n, \underline{u}_{n+1}, \ldots)\} \\
&\ \ldots \\
\pi_n(u)_n &:= \min\{x \in \mathbb{R} : x \geq \underline{u}_n,\ \psi(\pi_n(u)_1, \ldots, \pi_n(u)_{n-1}, x, \underline{u}_{n+1}, \ldots) \\
&\qquad = \psi(u_1, \ldots, u_n, \underline{u}_{n+1}, \ldots)\}.
\end{aligned}$$

Since $x \mapsto \psi(x, u_2, \dots, u_n, \underline{u}_{n+1}, \dots)$ is a convex function from $\mathbb{R}$ to $\mathbb{R}$, it is continuous. Therefore, the minimum in the definition of $\pi_n(u)_1$ is attained, and

$$\psi(\pi_n(u)_1, u_2, \dots, u_n, \underline{u}_{n+1}, \dots) = \psi(u_1, u_2, \dots, u_n, \underline{u}_{n+1}, \dots).$$

Analogously, the other minima are attained, and

$$\psi \circ \pi_n(u) = \psi(u_1, \dots, u_n, \underline{u}_{n+1}, \dots) \quad \text{for all } u \in U.$$

Since ψ is increasing, $\pi_n(u^\alpha)$ is in D_a for all α, and by (4.2), one has for each $y \in l^1$,

$$|\langle \pi_n(u^\alpha) - \pi_n(\hat{u}), y \rangle| \le \sum_m ||u_m^\alpha - \hat{u}_m||_\infty |y_m| \to 0,$$

showing that $\pi_n(u^\alpha) \to \pi_n(\hat{u})$ in $\sigma(l^\infty, l^1)$. From Lemma 4.3.1 we know that ψ restricted to l^∞ is $\sigma(l^\infty, l^1)$-lower semicontinuous. Therefore, $\pi_n(\hat{u})$ is in D_a for all n, and one obtains from (C') that

$$\psi \circ \pi_n(\hat{u}) = \psi(\hat{u}_1, \dots, \hat{u}_n, \underline{\hat{u}}_{n+1}, \dots) \uparrow \psi(\hat{u}) \quad \text{for } n \to \infty.$$

This shows that $\hat{u}$ belongs to D_a, which completes the proof in the case where ψ satisfies (C').

If ψ fulfills (D'), we fix $n \ge 1$ and note that due to (4.2), there exists an α_0 such that

$$u_m^\alpha \ge \hat{u}_m - \frac{1}{n} \quad \text{for all } \alpha \ge \alpha_0 \text{ and } m = 1, \dots, n.$$

It follows that

$$\left(\hat{u}_1 - \frac{1}{n}, \dots, \hat{u}_n - \frac{1}{n}, \underline{u}_{n+1}^\alpha - \frac{1}{n}, \dots\right) \text{ is in } D_a \text{ for all } \alpha \ge \alpha_0.$$

As above, one deduces from (4.2) that

$$\left(\underline{u}_{n+1}^\alpha - \frac{1}{n}, \underline{u}_{n+2}^\alpha - \frac{1}{n}, \dots\right) \to \left(\underline{\hat{u}}_{n+1} - \frac{1}{n}, \underline{\hat{u}}_{n+2} - \frac{1}{n}, \dots\right) \text{ in } \sigma(l^\infty, l^1).$$

So, since

$$x \mapsto \psi\left(\hat{u}_1 - \frac{1}{n}, \dots, \hat{u}_n - \frac{1}{n}, x_1, x_2, \dots\right)$$

defines an increasing convex mapping on l^∞ with property (D'), one obtains from Lemma 4.3.1 that

$$\left(\hat{u}_1 - \frac{1}{n}, \dots, \hat{u}_n - \frac{1}{n}, \underline{\hat{u}}_{n+1} - \frac{1}{n}, \underline{\hat{u}}_{n+2} - \frac{1}{n}, \dots\right) \text{ belongs to } D_a \text{ for all } n \ge 1.$$

Now it follows from (D') that $\hat{u}$ is in D_a, and the proof is complete. □

Proof of Theorem 4.1.7

We first prove (4.12). It is immediate from the definition of $\phi^*_{C_b}$ that

$$\phi(f) \ge \sup_{\mu \in ca_r^+} \left(\langle f, \mu \rangle - \phi^*_{C_b}(\mu)\right)$$

for all $f \in C_b$. We show the other inequality in the following three steps:

Step 1: For $H_n = K_n$, define the function $\psi : U = \prod_n C(H_n) \to \mathbb{R} \cup \{+\infty\}$ by

$$\psi(u) := \phi\left(\sum_n u_n 1_{K_n \setminus K_{n-1}}\right), \quad \text{where } K_0 := \emptyset.$$

Then ψ is increasing and convex. Moreover, it fulfills (C') or (D') if ϕ satisfies (C) or (D), respectively. So it follows from Proposition 4.3.3 that ψ can be represented as

$$\psi(u) = \sup_{\nu \in V^+} \left(\langle u, \nu \rangle - \psi^*(\nu) \right).$$

Step 2: For every $\nu \in V^+$, $\mu_\nu(A) = \sum_n \nu_n(A \cap K_n)$ defines an element of $ca_r^+(\mathcal{F})$. Indeed, μ_ν is a finite measure since $||\nu|| = \sum_n ||\nu_n||_{\text{tv}} < \infty$. Moreover, for given $A \in \mathcal{F}$ and $\varepsilon > 0$, there exist compact sets $F_n \subseteq A \cap K_n$ such that $\nu_n(F_n) \geq \nu_n(A \cap K_n) - 2^{-n-1}\varepsilon$. So for $m \in \mathbb{N}$ large enough, $F = \bigcup_{n=1}^m F_n$ is compact, $F \subseteq A$ and $\mu_\nu(F) \geq \mu_\nu(A) - \varepsilon$.

Step 3: Since ϕ satisfies (C) or (D), one has for each $f \in C_b$,

$$\phi(f) = \phi(f 1_{\bigcup_n K_n}) = \psi(f\,|_{K_1}, f\,|_{K_2}, ...).$$

Therefore,

$$\phi(f) = \sup_{\nu \in V^+} \left(\sum_n \langle f\,|_{K_n}, \nu_n \rangle - \psi^*(\nu) \right) = \sup_{\nu \in V^+} \left(\langle f, \mu_\nu \rangle - \psi^*(\nu) \right),$$

and it is enough to show that $\phi^*_{C_b}(\mu_\nu) \leq \psi^*(\nu)$ for all $\nu \in V^+$ to complete the proof of (4.12). But this readily follows from

$$\begin{aligned} \phi^*_{C_b}(\mu_\nu) &= \sup_{f \in C_b} \left(\langle f, \mu_\nu \rangle - \phi(f) \right) = \sup_{f \in C_b} \left(\sum_n \langle f 1_{K_n}, \nu_n \rangle - \psi(f\,|_{K_1}, f\,|_{K_2}, ...) \right) \\ &\leq \sup_{u \in U} \left(\langle u, \nu \rangle - \psi(u) \right) = \psi^*(\nu). \end{aligned}$$

To show (4.13) we fix an $f \in U_b$ and a constant $\varepsilon > 0$. For every $\delta > 0$, there exists a measurable partition (A_m) of Ω and real numbers $a_1 < \cdots < a_M$ such that the step function $g = \sum_{m=1}^M a_m 1_{A_m}$ satisfies $g \leq f \leq g + \delta$, and by passing to the upper semicontinuous hull, one can assume g to be upper semicontinuous. If ϕ satisfies (C), then $x \mapsto \phi(f + x)$ defines a convex function from $\mathbb{R}$ to $\mathbb{R}$. So it has to be continuous, and since ϕ is increasing, one can ensure that $\phi(g) \geq \phi(f) - \varepsilon$ by choosing $\delta > 0$ small enough. If ϕ satisfies (D) and $\phi(f) < +\infty$, one obtains directly that $\phi(g) \geq \phi(f) - \varepsilon$ for $\delta > 0$ small enough. On the other hand, if ϕ satisfies (D) and $\phi(f) = +\infty$, then $\phi(g) \geq \varepsilon$ for $\delta > 0$ small enough. Now denote

$$\begin{aligned} U^M &:= \left\{ u \in \prod_n C(K_n)^M : \sup_{n,m} ||u_{nm}||_\infty < \infty \right\}, \\ V^M &:= \left\{ \nu \in \prod_n ca_r(K_n)^M : \sum_{n,m} ||\nu_{nm}||_{\text{tv}} < \infty \right\}, \end{aligned}$$

and define $\psi : U^M \to \mathbb{R} \cup \{+\infty\}$ by

$$\psi(u) := \phi\left(\sum_n \sum_{m=1}^M u_{nm} 1_{B_{nm}} \right), \quad \text{where } K_0 := \emptyset \text{ and } B_{nm} := (K_n \setminus K_{n-1}) \cap A_m.$$

Then ψ is increasing, convex and satisfies (C') or (D'). Therefore, it follows from Proposition 4.3.3 that

$$\psi(u) = \sup_{\nu \in (V^M)^+} \left(\langle u, \nu \rangle - \psi^*(\nu) \right), \text{ where } \quad \psi^*(\nu) = \sup_{u \in U^M} \left(\langle u, \nu \rangle - \psi(u) \right).$$

If $\phi(h) = +\infty$ for all $h \in C_b$, then $\phi^*_{C_b} \equiv -\infty$, and (4.13) is obvious. So let us assume there exists an $h \in C_b$ such that $\phi(h) < +\infty$. Then it follows that $\nu_{nm}(K_n \setminus \bar{B}_{nm}) = 0$ for all $\nu \in (V^M)^+$ satisfying $\psi^*(\nu) < +\infty$. Indeed, assume $\nu_{nm}(K_n \setminus \bar{B}_{nm}) > 0$. Then, since ν_{nm} is regular, there exists a closed set $F \subseteq K_n \setminus \bar{B}_{nm}$ with $\nu_{nm}(F) > 0$. By Theorem 2.48 in Aliprantis and Border (2006), K_n is normal. So it follows from Urysohn's lemma that there exists a continuous function $\varphi : K_n \to [0,1]$ which is 1 on F and 0 on $\bar{B}_{nm}$. This gives

$$\begin{aligned}\psi^*(\nu) &\geq \sup_{x\in\mathbb{R}_+}\left(\sum_i\sum_{j=1}^M \langle h1_{K_i}, \nu_{ij}\rangle + \langle x\varphi, \nu_{nm}\rangle - \phi(h + x\varphi 1_{B_{nm}})\right)\\ &= \sup_{x\in\mathbb{R}_+}\left(\sum_i\sum_{j=1}^M \langle h1_{K_i}, \nu_{ij}\rangle + \langle x\varphi, \nu_{nm}\rangle - \phi(h)\right) = +\infty.\end{aligned}$$

Now define $u \in U^M$ by $u_{nm} = a_m$. It follows from (C) or (D) that

$$\phi(g) = \phi(g1_{\bigcup_n K_n}) = \psi(u).$$

Therefore, since g is upper semicontinuous,

$$\begin{aligned}\phi(g) &= \sup_{\nu\in(V^N)^+}\left(\sum_n\sum_{m=1}^M \langle u_{nm}, \nu_{nm}\rangle - \psi^*(\nu)\right)\\ &\leq \sup_{\nu\in(V^N)^+}\left(\sum_n\sum_{m=1}^M \left\langle g1_{\bar{B}_{nm}}, \nu_{nm}\right\rangle - \psi^*(\nu)\right) = \sup_{\nu\in(V^N)^+}\left(\langle g, \mu_\nu\rangle - \psi^*(\nu)\right),\end{aligned}$$

where μ_ν is given by $\mu_\nu(A) := \sum_n \sum_{m=1}^M \nu_{nm}(A \cap K_n)$. It follows as above that μ_ν belongs to $ca_r^+(\mathcal{F})$, and for all $\nu \in (V^M)^+$, one has

$$\begin{aligned}\phi^*_{C_b}(\mu_\nu) &= \sup_{l\in C_b}\left(\langle l, \mu_\nu\rangle - \phi(l)\right)\\ &= \sup_{l\in C_b}\left(\sum_n\sum_{m=1}^M \langle l1_{K_n}, \nu_{nm}\rangle - \psi(l1_{K_1}, \ldots, l1_{K_1}, l1_{K_2}, \ldots)\right)\\ &\leq \sup_{u\in U^M}\left(\langle u, \nu\rangle - \psi(u)\right) = \psi^*(\nu).\end{aligned}$$

So in the case $\phi(f) < +\infty$, one obtains

$$\phi(f) - \varepsilon \leq \phi(g) \leq \sup_{\mu\in ca_r^+}\left(\langle g, \mu\rangle - \phi^*_{C_b}(\mu)\right) \leq \sup_{\mu\in ca_r^+}\left(\langle f, \mu\rangle - \phi^*_{C_b}(\mu)\right),$$

and if $\phi(f) = +\infty$,

$$\varepsilon \leq \phi(g) \leq \sup_{\mu\in ca_r^+}\left(\langle g, \mu\rangle - \phi^*_{C_b}(\mu)\right) \leq \sup_{\mu\in ca_r^+}\left(\langle f, \mu\rangle - \phi^*_{C_b}(\mu)\right).$$

Since $\varepsilon > 0$ was arbitrary, this yields (4.13). On the other hand, it follows from the definition of $\phi^*_{U_b}$ that $\phi(f) \geq \sup_{\mu\in ca_r^+}(\langle f, \mu\rangle - \phi^*_{U_b}(\mu))$. So if $\phi^*_{C_b}(\mu) = \phi^*_{U_b}(\mu)$ for all $\mu \in ca_r^+$, the inequality in (4.13) becomes an equality.

Finally, by Remark 4.1.9.1, $\hat{\phi}(f) := \sup_{\mu\in ca_r^+}(\langle f, \mu\rangle - \phi^*_{C_b}(\mu))$ is lower regular on B_b. So one obtains from the second part of the proof that for all $f \in B_b$,

$$\phi_r(f) = \sup\{\phi(g) : g \in U_b, g \leq f\} \leq \sup\{\hat{\phi}(g) : g \in U_b, g \leq f\} = \hat{\phi}(f),$$

with equality if $\phi^*_{C_b}(\mu) = \phi^*_{U_b}(\mu)$ for all $\mu \in ca_r^+$. This completes the proof. □

Proof of Corollary 4.1.8
By Theorem 4.1.7, one has

$$\phi(f) = \sup_{\mu \in ca_r^+(\mathcal{F})} (\langle f, \mu \rangle - \phi^*_{C_b}(\mu)) \quad \text{for all } f \in C_b.$$

So $\phi^*_{C_b}(\mu) < +\infty$ for at least one $\mu \in ca_r^+(\mathcal{F})$. Since ϕ is linear, this implies that $\phi(f) = \langle f, \mu \rangle$ for all $f \in C_b$, and $\phi^*_{C_b}(\mu) = 0$. Moreover, if Ω is a metric space, μ is completely determined by the values $\langle f, \mu \rangle$, $f \in C_b$ (see e.g., [10]). So one obtains from (4.13) and (4.14) that $\phi(f) \le \langle f, \mu \rangle$ for all $f \in U_b$ and $\phi_r(f) \le \langle f, \mu \rangle$ for all $f \in B_b$, with equality if $\phi^*_{C_b}(\nu) = \phi^*_{U_b}(\nu)$ for all $\nu \in ca_r^+(\mathcal{F})$. □

4.4 Representation in the Unbounded Case

In this section, we derive a variant of the representation Theorem 4.1.7 for a space of unbounded random variables. We assume that there exists an increasing sequence (K_n) of compact subsets of Ω such that $\Omega = \bigcup_n K_n$.

For a weight function $(\rho_n)_{n \in \mathbb{N}} \subseteq [1, \infty)$ with respective weight function

$$\rho = \sum_{n \ge 1} \rho_n 1_{\Delta K_n}$$

from Ω to $\mathbb{R}$, we denote by B_ρ, C_ρ and U_ρ the spaces of functions $f : \Omega \to \mathbb{R}$ which are measurable, continuous and upper semicontinuous, respectively, and such that

$$|f| \le c\rho \quad \text{for a constant } c > 0.$$

Let $l^{1,+}_\rho$ be the space of positive sequences (y_j) which satisfy $\sum_{j \ge 1} y_j \rho_j < \infty$. Denote by ca^ρ_r the space of all regular signed σ-additive measures $\mathbb{Q}$ on $\mathcal{F}$ such that the sequence $(\mathbb{Q}(\Delta K_n))$ is in $l^{1,+}_\rho$ and $\mathbb{Q}(A) = \mathbb{Q}(A \cap K)$ for all $A \in \mathcal{F}$. The set of probability measures in ca^ρ is denoted by $\mathcal{M}^\rho$. For a functional $\phi : B_\rho \to \mathbb{R} \cup \{+\infty\}$ the conditions (C) and (D) are defined analogous to the bounded case with the set B_b replaced by the set B_ρ.

Theorem 4.4.1. *Assume that Ω is a Hausdorff space with Borel σ -algebra $\mathcal{F}$ and $\phi : B_\rho \to \mathbb{R} \cup \{+\infty\}$an increasing convex function such that $\phi(f^n) \uparrow \phi(f)$ for every increasing sequence (f^n) in B_ρ with pointwise limit $f \in B_\rho$ such that $f^n \ge f - 1/n$ on K_n for all $n \in \mathbb{N}$. Then,*

$$\phi(f) = \sup_{\mu \in ca_r^{\rho,+}} \{\langle f, \mu \rangle - \phi^*_{C_\rho}(\mu)\}, \quad f \in C_\rho, \tag{4.3}$$

$$\phi(f) \le \sup_{\mu \in ca_r^{\rho,+}} \{\langle f, \mu \rangle - \phi^*_{C_\rho}(\mu)\}, \quad f \in U_\rho, \tag{4.4}$$

where $ca_r^{\rho,+}$ denotes the positive elements of ca^ρ_r. Moreover, if $\phi^*_{C_\rho} = \phi^*_{U_\rho}$, (4.4) *becomes an equality.*

Proof. Define $h : \Omega \to \mathbb{R}$ by $h(\omega) = \rho_n$ if $\omega \in \Delta K_n$. Then, $B_\rho = hB_b$ and the function $\phi_b : B_b \to (\infty, \infty]$ defined by $\phi_b(f) = \phi(hf)$ inherits some properties of ϕ, that is, ϕ_b is convex and increasing on B_b. Further for every increasing sequence (k_n) in $\mathbb{N}$ such that $\rho_n/k_n \le 1/n$ for all $n \in \mathbb{N}$, one has $\phi_b(f^n) \uparrow \phi_b(f)$ for every increasing sequence (f^n) in B_ρ with pointwise limit $f \in B_\rho$ such that $f^n \ge f - 1/k_n$ on K_n for all $n \in \mathbb{N}$. Thus, Theorem 4.1.7 implies

$$\phi_b(f) \le \sup_{\mu \in ca_r^+} \{\langle f, \mu \rangle - \phi^*_{b,C_b}(\mu)\}, \quad f \in U_b,$$

where ca_r^+ is the set of regular finite measures on $\mathcal{B}(\Omega)$. Hence, for each $f \in U_b$ we get

$$\begin{aligned}\phi(f) = \phi_b(f/h) &\le \sup_{\mu \in ca_r^+} \{\langle f/h, \mu\rangle - \phi^*_{b,C_b}(\mu)\}\\ &= \sup_{\mu \in ca_r^+} \Big\{ \sum_{n\ge1} \langle (f/\rho_n) 1_{\Delta K_n}, \mu\rangle - \phi^*_{b,C_b}(\mu)\Big\}\\ &= \sup_{\mu \in ca_r^+} \{\langle f, \nu_\mu\rangle - \phi^*_{b,C_b}(\mu)\},\end{aligned}$$

for the regular measure $\nu_\mu(\cdot) := \sum_{n\ge1} \mu(\cdot \cap \Delta K_n)/\rho_n$ on $\mathcal{B}(\Omega)$ for which $(\nu_\mu(\Delta K_n)) \in l^1_\rho$. Moreover, one has

$$\begin{aligned}\phi^*_{b,C_b}(\mu) &= \sup_{f\in C_b} \{\langle f, \mu\rangle - \phi_b(f)\} = \sup_{f\in C_\rho} \{\langle f/h, \mu\rangle - \phi(f)\}\\ &= \sup_{f\in C_\rho} \Big\{ \sum_{n\ge1} \langle (f/\rho_n) 1_{\Delta K_n}, \mu\rangle - \phi(f)\Big\} = \sup_{\xi\in C_\rho} \{\langle f, \nu_\mu\rangle - \phi(f)\} = \phi^*_{C_\rho}(\nu_\mu).\end{aligned}$$

This shows (4.4). In case that $\phi^*_{C_\rho} = \phi^*_{U_\rho}$, the reverse inequality in (4.4) follows by definition of the convex conjugate. The representation (4.3) follows analogously. □

4.5 Probabilistic Version of the Main Result

Let $\mathcal{P}$ be a family of regular probability measures on the Borel σ-algebra $\mathcal{F}$ of a normal Hausdorff space Ω. By $L^0(\mathcal{P})$ we denote the space of all measurable functions from Ω to $\mathbb{R}$ where two of them are identified if they coincide $\mathcal{P}$ -q.s. Denote by $L^\infty(\mathcal{P})$ the space elements of $L^0(\mathcal{P})$ such that

$$\|f\|_{\infty,\mathcal{P}} := \inf\{m \in \mathbb{R} : \mathcal{P}(|f| > m) = 0\} < \infty.$$

An element in $L^0(\mathcal{P})$ belongs to $U_b(\mathcal{P})$ or $C_b(\mathcal{P})$ if it has a $\mathcal{P}$-modification which belongs to U_b or C_b, respectively. Equalities and inequalities in $L^\infty(\mathcal{P})$ are understood in the $\mathcal{P}$-q.s. sense. Each $\phi : B_b \to \mathbb{R}$ for which $\phi(f) = \phi(g)$ whenever $f = g$ $\mathcal{P}$-q.s. corresponds to a function $\phi : L^\infty(\mathcal{P}) \to \mathbb{R}$. We need the following version of condition (C):

(C") ϕ is real-valued and there exists an increasing sequence (K_n) of compact subsets of Ω such that $\phi(f^n) \uparrow \phi(f)$ for every increasing sequence (f^n) in $L^\infty(\mathcal{P})$ and $f \in L^\infty(\mathcal{P})$ such that $\|(f - f^n)1_{K_m}\|_{\infty,\mathcal{P}} = 0$ for all $n \ge m$.

Let $S := \mathrm{supp}(\mathcal{P})$ be the support of $\mathcal{P}$. Here, $\mathrm{supp}(\mathcal{P})$ is the unique closed set S for which $\mathcal{P}(S^c) = 0$ and $\mathcal{P}(S \cap O) > 0$ whenever O is open and $S \cap O \ne \emptyset$. It turns out that $S = \cup_{\mathbb{P}\in\mathcal{P}} \mathrm{supp}(\mathbb{P})$, where each support $\mathrm{supp}(\mathbb{P})$ exists, since the elements of $\mathcal{P}$ are regular, see [2, Theorem 12.14]. Denote by $ca_r^+(S)$ the collection of all regular finite measures μ on $\mathcal{F}$ such that $\mathrm{supp}(\mu) \subseteq S$ and by $ca_r^+(\mathcal{P}, S)$ the elements μ of $ca_r^+(S)$ which are absolutely continuous with respect to $\mathcal{P}$, i.e., for each $A \in \mathcal{F}$ with $\mu(A) > 0$ one has $\nu(A) > 0$ for some $\nu \in \mathcal{P}$. Then $\langle f, \mu\rangle$ is well-defined for each $f \in L^\infty(\mathcal{P})$ and all $\mu \in ca_r^+(\mathcal{P}, S)$. On the other hand, for every $f \in C_b(\mathcal{P})$ and $\mu \in ca^+(\mathcal{P}, S)$, we define the integral of f with respect to μ as

$$\langle f, \mu\rangle := \langle \tilde{f}, \mu\rangle,$$

with $\tilde{f} \in C_b$ and $\tilde{f} = f$ $\mathcal{P}$ -q.s. The following lemma shows that the above integral is well defined on $C_b(\mathcal{P})$.

Lemma 4.5.1. *For any $f \in C_b(\mathcal{P})$ and $\mu \in ca^+(S)$ the integral $\langle f, \mu\rangle$ is uniquely defined.*

Proof. Let $f \in C_b(\mathcal{P})$ and $\tilde{f}_1$, $\tilde{f}_2$ two $\mathcal{P}$ -modifications of f which are continuous on Ω. Then, $\tilde{f}_1 =_p \tilde{f}_2$ on S; where $=_p$ denotes the pointwise equality. Otherwise, if $\emptyset \neq \{\omega \in S : \tilde{f}_1(\omega) < \tilde{f}_2(\omega)\} = \{\omega \in \Omega : \tilde{f}_1(\omega) < \tilde{f}_2(\omega)\} \cap S$, then since $\{\omega \in \Omega : \tilde{f}_1(\omega) < \tilde{f}_2(\omega)\}$ is open, it has strict positive measure under some $\mu \in \mathcal{P}$. Analogously, we have $\{\omega \in S : \tilde{f}_2(\omega) < \tilde{f}_1(\omega)\} = \emptyset$. Therefore, $\langle f, \mu\rangle = \langle \tilde{f}_1, \mu\rangle = \langle \tilde{f}_1 1_S, \mu\rangle + \langle \tilde{f}_1 1_{S^c}, \mu\rangle = \langle \tilde{f}_1 1_S, \mu\rangle = \langle \tilde{f}_2 1_S, \mu\rangle = \langle \tilde{f}_2, \mu\rangle$. □

We also define the lower regularization ϕ_r of a function ϕ by

$$\phi_r(f) := \sup\{\phi(g) : g \in U_b(\mathcal{P}) \text{ and } g \leq f \ \mathcal{P}\text{-q.s.}\}, \quad f \in L^\infty(\mathcal{P}).$$

The function ϕ is said to be lower regular if $\phi_r = \phi$ on $L^\infty(\mathcal{P})$.

Theorem 4.5.2. *Let $\mathcal{P}$ be a family of probability measures on the Borel σ-algebra $\mathcal{F}$ of a normal Hausdorff space Ω. If $\phi : L^\infty(\mathcal{P}) \to \mathbb{R}$ is an increasing convex function which satisfies* (C”) *then*

$$\phi(f) = \sup_{\mu \in ca_r^+(S)} \big(\langle f, \mu\rangle - \phi^*_{C_b}(\mu)\big) \quad \text{for all } f \in C_b(\mathcal{P}). \tag{4.5}$$

If in addition $\phi^*_{C_b}(\mu) = \phi^*_{U_b}(\mu)$ for all $\mu \in ca_r^+$, then

$$\phi(f) = \sup_{\mu \in ca_r^+(\mathcal{P},S)} \big(\langle f, \mu\rangle - \phi^*_{C_b}(\mu)\big) \quad \text{for all } f \in U_b(\mathcal{P}). \tag{4.6}$$

Moreover, if $\phi^*_{C_b}(\mu) = \phi^*_{U_b}(\mu)$ for all $\mu \in ca_r^+$ and ϕ is lower regular the representation (4.6) *holds on $L^\infty(\mathcal{P})$.*

Proof. For $\mu \in ca_r^+ \setminus ca_r^+(S)$, there exists a compact set A such that $\mu(A) > 0$ whereas $A \cap S = \emptyset$. Since S is closed and Ω normal, it follows from Urysohn's lemma that there exists a positive function $f \in C_b$ taking value 0 on S and 1 on A. Notice that since S is the support of $\mathcal{P}$, it holds $f = 0$ $\mathcal{P}$ -q.s. Hence $\phi^*_{C_b}(\mu) \geq \langle nf, \mu\rangle - \phi(nf) \geq n\mu(A) - \phi(0)$ for all $n \in \mathbb{N}$ which shows that $\phi^*_{C_b}(\mu) = +\infty$. Thus, the representation (4.6) follows from (4.12). In fact, since $\phi : B_b \to \mathbb{R}$ satisfies (C), by Theorem 4.1.7,

$$\phi(f) = \sup_{\mu \in ca_r^+} \big(\langle f, \mu\rangle - \phi^*_{C_b}(\mu)\big) = \sup_{\mu \in ca_r^+(S)} \big(\langle f, \mu\rangle - \phi^*_{C_b}(\mu)\big) \quad \text{for all } f \in C_b.$$

In addition, $\phi(f)$ and the right hand side of the previous equation do not change if f is replaced by a $\mathcal{P}$-modification the representation (4.5) holds.

Now, let us assume that $\phi^*_{C_b}(\mu) = \phi^*_{U_b}(\mu)$ for all $\mu \in ca_r^+$. For $\mu \in ca_r^+$, if μ is not absolutely continuous with respect to $\mathcal{P}$, then by regularity of μ there exists a compact set A such that $\mu(A) > 0$ whereas $\nu(A) = 0$ for all $\nu \in \mathcal{P}$. Hence $\phi^*_{U_b}(\mu) \geq \langle n1_A, \mu\rangle - \phi(n1_A) = n\mu(A) - \phi(0)$ for all $n \in \mathbb{N}$ which shows that $\phi^*_{C_b}(\mu) = \phi^*_{U_b}(\mu) = +\infty$. Thus, again by Theorem 4.1.7 it holds,

$$\phi(f) = \sup_{\mu \in ca_r^+} \big(\langle f, \mu\rangle - \phi^*_{C_b}(\mu)\big) = \sup_{\mu \in ca_r^+(\mathcal{P},S)} \big(\langle f, \mu\rangle - \phi^*_{C_b}(\mu)\big) \quad \text{for all } f \in U_b.$$

Since $\phi(f)$ and the right hand side of the previous equation do not change if f is replaced by a $\mathcal{P}$-modification the representation (4.6) holds.

The function $\bar{\phi}(f) := \sup_{\mu \in ca_r^+(\mathcal{P},S)}(\langle f, \mu\rangle - \phi^*_{C_b}(\mu))$ is lower regular on $L^\infty(\mathcal{P})$. In fact, by Remark 4.1.9, $\hat{\phi}(f) := \sup_{\mu \in ca_r^+}(\langle f, \mu\rangle - \phi^*_{C_b}(\mu))$ is lower regular on B_b and $\bar{\phi}$ coincide with $\hat{\phi}$ on B_b. Let $f \in B_b$, we have

$$\begin{aligned}\bar{\phi}_r(f) &= \sup\{\bar{\phi}(g) : \ g \in U_b(\mathcal{P}) \text{ and } g \leq f \ \mathcal{P}\text{-q.s.}\}\\ &= \sup\{\bar{\phi}(g) : \ g \in U_b \text{ and } g \leq f\}\\ &= \sup\{\hat{\phi}(g) : \ g \in U_b \text{ and } g \leq f\} = \hat{\phi}(f) = \bar{\phi}(f).\end{aligned}$$

Therefore, $\bar{\phi}_r(f) = \bar{\phi}(f)$ for all $f \in B_b$. So from the first part of the proof, for all $f \in L^\infty(\mathcal{P})$ it holds $\phi_r(f) = \bar{\phi}(f)$, which concludes the proof. □

Remark 4.5.3. *If the functional ϕ is translation invariant in the sense that $\phi(f + c) = \phi(f) + c$ for all $c \in \mathbb{R}$, then the representations results of Theorems 4.1.7; 4.5.2 and 4.4.1 can be written in with respect to probability measures.*

Remark 4.5.4. *In case that $\mathcal{P}$ is tight there exists an increasing sequence of compact sets (K_n) for which $\nu(K_n^c) \leq 1/n$ for all $\nu \in \mathcal{P}$, i.e., $\bigcup_n K_n = \Omega$ $\mathcal{P}$-q.s. In that context a related result to Theorem 4.5.2 has been shown by Bion-Nadal and Kervarec [12] . They provide a representation result for certain monotone convex functions on the $L^1(\mathcal{P})$-closure of C_b.*

Chapter 5

Fundamental Theorem of Asset Pricing under Ambiguity

5.1 Introduction

The Fundamental Theorem of Asset Pricing (FTAP) characterizes efficient financial markets which do not allow for arbitrage opportunities by means of the existence of pricing/martingale measures. If the financial market is modeled via a probability measure, then all null sets are neglected and the same has to hold for the respective pricing measures. In this article we provide a FTAP under model-uncertainty, when an arbitrary set of possible market models is taken into account. We are particularly interested in the case where these probabilistic models are not dominated by a measure; that is, the neglected sets are not the null sets of a probability measure.

In the dominated case, the FTAP dates back to Harrison and Kreps [62] in the context of a finite sample space. In finite discrete time on an arbitrary sample space, Dalang, Morton and Willinger [26] link the exclusion of arbitrage with the existence of equivalent martingale measures. In continuous time the situation is more involved and it has been shown in a series of works by Delbaen and Schachermayer [36] that a d-dimensional price process admits no free lunch with vanishing risk (NFLVR) if and only if the process is a sigma-martingale under an equivalent probability measure. Roughly speaking, a market admits a FLVR if the outcome of an arbitrage opportunity can be approximated by a sequence of outcomes of trading strategies in the topology of uniform convergence. Kardaras [71] introduced the concept of no arbitrage of the first kind (NA1), which is a market efficiency condition weaker than the NFLVR. He characterized NA1 by the existence of an equivalent local martingale deflator.

Most of the results in the dominated case are eventually built on hyperplane separation arguments in L^p-spaces, which lead to pricing measures with Radon-Nikodym derivatives in L^q. This method does not transfer to the non-dominated case directly and new techniques are required. By means of measurable selection techniques, the FTAP for finitely many assets in finite discrete time has been shown by Bouchard and Nutz [15] and was extended to markets with transaction costs by Bayraktar and Zhang [6]. A continuous-time version under NA1 is provided by Biagini, Bouchard, Kardaras and Nutz [9]. In a model-free setup and based on topological arguments Riedel [89] proposed an FTAP in a one period market model, which has been extended to multi-periods by Burzoni, Frittelli and Maggis [18]. The specific situation in which one observes the prices of infinitely many options was investigated by Acciaio, Beiglböck, Penkner, and Schachermayer [1] under the weak concept of no model-independent arbitrage, for a market comprising an option with super-linearly growing payoff and modeled on $\mathbb{R}^T$, $T \in \mathbb{N}$. Cox and Obłój [24] examined the continuous time case, they obtained bounds for prices for exotic options and characterized option markets satisfying the condition of no weak free lunch with vanishing risk by the existence of pricing measures consistent with the prices of the observed options. We refer to Davis, Obłój and Raval [31] and to Davis and Hobson [29] for results along the same lines.

The findings in this last chapter of the thesis strongly rely on results by Cheridito, Kupper and Tangpi [21] on the robust representation of increasing convex functionals with countably additive measures. The market is specified by a set $\mathcal{P}$ of probabilistic models on a topological state space Ω, which up to $\mathcal{P}$-null sets is the countable union of compacts (K_n). This assumption is satisfied, if Ω is σ-compact, or $\mathcal{P}$ is a tight family of probability measures, or $\Omega = C([0,T],\mathbb{R}^d)$ and each probability measure in $\mathcal{P}$ has support on the Hölder continuous paths. Trading outcomes attainable with zero initial wealth are modeled by an arbitrary cone G of continuous functions. The set of pricing measures $\mathcal{M}(G)$ consists of those probability measures under which each trading outcome in G has a negative expectation. Our main result given in Theorem 5.3.2 states that the set of pricing measures is nonempty and equivalent to $\mathcal{P}$ if and only if the market does not allow for a free lunch with disappearing risk (FLDR). A FLDR is a sequence of attainable trading outcomes which approximates the outcome of an arbitrage opportunity uniformly on each compact K_n and such that losses are controlled. Although the notion of NFLDR is slightly stronger than NFLVR, it implies equivalent pricing measures for arbitrary cones of attainable outcomes that are continuous. This allows to consider simple trading strategies or markets with uncountable many options. Furthermore, the trading outcomes are not necessarily bounded, and markets including options whose payoffs are of linear growth are covered by Theorem 5.3.2.

The key idea in the proof of Theorem 5.3.2 is to study the functional $\sup_{\mathbb{Q}\in\mathcal{M}(G)} E_{\mathbb{Q}}[\cdot]$, that is strongly connected to the superhedging functional ϕ. By the Fenchel-Moreau theorem, the two functionals coincide if ϕ is lower semicontinuous in a reasonable topology. Since for an arbitrary G, the superhedging functional generally fails to satisfy the required lower semicontinuity property, we consider a regularized version ψ of ϕ, which coincides with $\sup_{\mathbb{Q}\in\mathcal{M}(G)} E_{\mathbb{Q}}[\cdot]$. Moreover ψ does not assume the value $-\infty$ or equivalently $\mathcal{M}(G) \neq \emptyset$ if and only if G satisfies NFLDR.

In Section 5.4 we discuss several applications of Theorem 5.3.2 to financial markets in discrete and continuous time. The set of (local) martingale measures for finitely many dynamically traded assets and an arbitrary set of statically traded options is nonempty if there does not exist a FLDR with simple trading strategies. Furthermore, the notion of NFLDR allows for a characterization of equivalent martingale measures under which the canonical process on the product space is an $\mathcal{H}^1$ -martingale.

The rest of the chapter is organized as follows: After introducing the setting and notation we discuss our main abstract FTAP in Section 5.3. Several applications of the main theorem in discrete and continuous time are finally given in Section 5.4.

5.2 Setting and Notation

Let $\mathcal{P}$ be a set of reference probability measures defined on the Borel sigma-algebra $\mathcal{F}$ of a normal Hausdorff space Ω. Throughout, we assume there exists an increasing sequence of compact subsets $K_n \subseteq \Omega$ such that

$$\Omega = \bigcup_{n\geq 1} K_n \quad \mathcal{P}\text{-q.s.} \tag{5.1}$$

and that the elements of $\mathcal{P}$ are regular probability measures. Define the set function $\mathcal{P}(A) := \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{P}(A)$ for all $A \in \mathcal{F}$. Recall that a property is said to hold $\mathcal{P}$-quasi surely if it holds P-almost surely for all $P \in \mathcal{P}$. Define $K := \bigcup_{n\geq 1} K_n$ and denote

$\Delta K_n := K_n \setminus K_{n-1}$, where by convention $K_0 := \emptyset$. For instance, the condition (5.1) is satisfied if the set $\mathcal{P}$ is tight, which in particular holds when $\mathcal{P}$ consists of one single element on a Polish space. Further examples are discussed in Section 5.4.

Let $L^0(\mathcal{P})$ be the space of all measurable functions from Ω to $\mathbb{R}$ where two of them are identified if they coincide $\mathcal{P}$-q.s.. Equalities and inequalities between random variables will be understood in the $\mathcal{P}$-q.s. sense. Let $L^\infty(\mathcal{P})$ be the subspace of those $f \in L^0(\mathcal{P})$ for which

$$\|f\|_{\infty,\mathcal{P}} := \inf\{m \in \mathbb{R} : \mathcal{P}(|f| > m) = 0\} < \infty.$$

For weights $(\rho_n)_{n\in\mathbb{N}} \subseteq [1,\infty)$ with respective weight function $\rho = \sum_{n\geq 1} \rho_n 1_{\Delta K_n}$ from K to $\mathbb{R}$, we denote by B_ρ, C_ρ and U_ρ the spaces of functions $f : K \to \mathbb{R}$ which are measurable, continuous and upper semicontinuous, respectively, and such that

$$|f| \leq c\rho$$

for a constant $c > 0$. An element in $L^0(\mathcal{P})$ belongs to $B_\rho(\mathcal{P})$, $U_\rho(\mathcal{P})$ or $C_\rho(\mathcal{P})$ if it has a $\mathcal{P}$-modification whose restriction to K belongs to B_ρ, U_ρ or C_ρ, respectively. In case that ρ is bounded, i.e. $\sup_{j\geq 1} \rho_j < \infty$, then B_ρ is equal to the space of all bounded measurable functions B_b. Analogously, U_b and C_b denote the spaces of bounded upper semicontinuous and continuous functions, respectively.

Let $l^{1,+}_\rho$ be the space of positive sequences (y_j) which satisfy $\sum_{j\geq 1} y_j\rho_j < \infty$. Denote by ca^ρ the space of all regular signed σ-additive measures $\mathbb{Q}$ on $\mathcal{F}$ such that the sequence $(\mathbb{Q}(\Delta K_n))$ is in $l^{1,+}_\rho$ and $\mathbb{Q}(A) = \mathbb{Q}(A \cap K)$ for all $A \in \mathcal{F}$. The set of probability measures in ca^ρ is denoted by $\mathcal{M}^\rho$. Let $S = \text{supp}(\mathcal{P})$ be the support of $\mathcal{P}$ on K. Here, supp($\mathcal{P}$) is the unique closed set S for which $\mathcal{P}(S^c) = 0$ and $\mathcal{P}(S \cap O) > 0$ whenever O is open and $S \cap O \neq \emptyset$. It turns out that $S = \cup_{\mathbb{P}\in\mathcal{P}}\text{supp}(\mathbb{P})$, where each support supp($\mathbb{P}$) exists, since the elements of $\mathcal{P}$ are regular, see [2, Theorem 12.14]. Finally, let $\mathcal{M}^\rho_\mathcal{P}$ be the set of those elements $\mathbb{Q} \in \mathcal{M}^\rho$ such that $\text{supp}(\mathbb{Q}) \subseteq S$. Similar to Chapter 4 Section 4.4, for any $f \in C_\rho(\mathcal{P})$ and $\mathbb{Q} \in \mathcal{M}^\rho_\mathcal{P}$, we define the expectation of f with respect to $\mathbb{Q}$ as

$$E_\mathbb{Q}[f] := E_\mathbb{Q}[\tilde{f}],$$

with $\tilde{f}|_K \in C_\rho$ and $\tilde{f} = f$ $\mathcal{P}$ -q.s. This expectation is well define. We repeat the argument of Section 4.4 on the relative topology of K for the sake of completeness.

Lemma 5.2.1. *For any $f \in C_\rho(\mathcal{P})$ and $\mathbb{Q} \in \mathcal{M}^\rho_\mathcal{P}$ the expectation $E_\mathbb{Q}[f]$ is uniquely defined.*

Proof. Let $f \in C_\rho(\mathcal{P})$ and $\tilde{f}_1$, $\tilde{f}_2$ two $\mathcal{P}$ -modifications of f whose restrictions to K belong to C_ρ. Then, $\tilde{f}_1 =_p \tilde{f}_2$ on S; where $=_p$ denotes the pointwise equality. Otherwise, if $\emptyset \neq \{\omega \in S : \tilde{f}_1(\omega) < \tilde{f}_2(\omega)\} = \{\omega \in K : \tilde{f}_1(\omega) < \tilde{f}_2(\omega)\} \cap S$, then since $\{\omega \in K : \tilde{f}_1(\omega) < \tilde{f}_2(\omega)\}$ is open, it has strict positive measure under some $\mathbb{P} \in \mathcal{P}$. Analogously, we have $\{\omega \in S : \tilde{f}_2(\omega) < \tilde{f}_1(\omega)\} = \emptyset$. Therefore, $E_\mathbb{Q}[f] = E_\mathbb{Q}[\tilde{f}_1] = E_\mathbb{Q}[\tilde{f}_1 1_S] + E_\mathbb{Q}[\tilde{f}_1 1_{S^c}] = E_\mathbb{Q}[\tilde{f}_1 1_S] = E_\mathbb{Q}[\tilde{f}_2 1_S] = E_\mathbb{Q}[\tilde{f}_2]$. ☐

5.3 Main Result

Throughout this section we fix a weight function $\rho : \mathbb{N} \to [1,\infty)$. Traded outcomes attainable with zero initial wealth are modeled by a convex cone

$$G \subseteq C_\rho(\mathcal{P})$$

such that $0 \in G$. By $\mathcal{M}^\rho_\mathcal{P}(G)$ we denote the set of those $\mathbb{Q} \in \mathcal{M}^\rho_\mathcal{P}$ which satisfy

$$\mathbb{E}_\mathbb{Q}[g] \leq 0 \quad \text{for all } g \in G.$$

The set $\mathcal{M}^\rho_\mathcal{P}(G)$ can be viewed as the set of pricing measures for the financial market G. In order to state our fundamental theorem of asset pricing we need the following no arbitrage concept.

Definition 5.3.1. *We say that G admits a free lunch with disappearing risk (FLDR) with respect to ρ if there exists $f \in B_\rho(\mathcal{P})_+ \setminus \{0\}$ such that for every $y \in l^{1,+}_\rho$ there is a sequence (g^n) in G which satisfies*

$$\lim_{n\to\infty} \sum_{j\geq 1} y_j ||(g^n - f)^- 1_{\Delta K_j}||_{\infty,\mathcal{P}} = 0.$$

Let us consider the superhedging functional $\phi : B_\rho(\mathcal{P}) \to [-\infty, \infty]$, which is given by

$$\phi(f) := \inf\{x \in \mathbb{R} : x + g \geq f \text{ for some } g \in G\}.$$

In case that ϕ is $\sigma(C_\rho, ca^\rho)$-lower semicontinuous, it follows from the Fenchel-Moreau theorem that $\phi(f) = \sup_{\mathbb{Q}\in\mathcal{M}^\rho_\mathcal{P}} E_\mathbb{Q}[f]$ for all $f \in C_\rho$, which further shows that the set of pricing measures is non-empty. However, since we cannot guarantee that the function ϕ is lower-semicontinuous in general, we consider a regularized version of ϕ. For $f \in B_\rho(\mathcal{P})$ let $\psi(f)$ be defined as the infimum over all $x \in \mathbb{R}$ such that for every $y \in l^{1,+}_\rho$ there exists a sequence (g^n) in G which satisfies

$$\lim_{n\to\infty} \sum_{j\geq 1} y_j ||(x + g^n - f)^- 1_{\Delta K_j}||_{\infty,\mathcal{P}} = 0.$$

Note that ψ is well-defined since $||f||_{\infty,\mathcal{P}} = ||g||_{\infty,\mathcal{P}}$ for any two measurable functions $f, g : \Omega \to \mathbb{R}$ which coincide $\mathcal{P}$-q.s.. In particular, $\psi : B_b \to [-\infty, \infty]$ is uniquely determined.

The main result of this chapter is the following abstract version of the FTAP.

Theorem 5.3.2. *The following statements are equivalent:*

(i) *G admits no FLDR with respect to ρ;*

(ii) *Every $\mathbb{P} \in \mathcal{P}$ is dominated by a $\mathbb{Q} \in \mathcal{M}^\rho_\mathcal{P}(G)$.*

Moreover, under (i) or (ii), the function ψ is the lower-semicontinuous hull of ϕ, that is

$$\psi(f) = \sup_{\mathbb{Q}\in\mathcal{M}^\rho_\mathcal{P}(G)} E_\mathbb{Q}[f] \qquad \text{for all } f \in U_\rho. \tag{5.2}$$

Proof. (i) $\Rightarrow$ (ii): First notice that the function $\psi : B_\rho(\mathcal{P}) \to [-\infty, \infty]$ satisfies the following properties:

a) $\psi(f) \geq \psi(g)$ whenever $f \geq g$

b) $\psi(f + m) = \psi(f) + m$ for all $m \in \mathbb{R}$

c) $\psi(\lambda f) = \lambda\psi(f)$ for all $\lambda \in \mathbb{R}_+ \setminus \{0\}$

d) ψ is convex

e) $\psi(f) > -\infty$ for every $f \in B_\rho(\mathcal{P})$

f) $\psi(f^n) \uparrow \psi(f)$ for every increasing sequence (f^n) in $B_\rho(\mathcal{P})$ and $f \in B_\rho(\mathcal{P})$ such that $\lim_{n\to\infty} ||(f - f^n)1_{\Delta K_j}||_{\infty,\mathcal{P}} = 0$ for all $j \in \mathbb{N}$.

Indeed, it is clear that ψ maps $B_\rho(\mathcal{P})$ to $[-\infty,\infty]$ and satisfies a)–d). In order to show e), it suffices to show that $\psi(-\rho) > -\infty$ because it that case, for any $f \in B_\rho(\mathcal{P})$ there would exists a constant $c > 0$ such that by a) and c) we would have $\psi(f) \geq c\psi(-\rho) > -\infty$. Assume $\psi(-\rho) = -\infty$, let $\delta > 0$ be a constant and $(y_j) \in l_\rho^{1,+}$. Since $\rho \geq 1$, it holds $\sum_{j\geq 1} y_j \left\|(\rho+\delta)1_{\Delta K_j}\right\|_{\infty,\mathcal{P}} < \infty$. Fix $\varepsilon > 0$ and $N \in \mathbb{N}$ such that

$$\sum_{j\geq N} y_j \left\|(\rho+\delta)1_{\Delta K_j}\right\|_{\infty,\mathcal{P}} < \frac{\varepsilon}{2}.$$

Since ρ is bounded on K_N, we can find $x \in \mathbb{R}_+$ such that $x \geq \rho 1_{K_N} + \delta$. That is, $(x-\rho-\delta)^- 1_{\Delta K_j} = 0$ for every $j \leq N$. By $\psi(-\rho) < -x$, there exists $(g^n) \subseteq G$ and $n \in \mathbb{N}$ such that

$$\sum_{j\geq 1} y_j \left\|(-x+g^n+\rho)^- 1_{\Delta K_j}\right\|_{\infty,\mathcal{P}} \leq \frac{\varepsilon}{2}.$$

Thus,

$$\begin{aligned}
&\sum_{j\geq 1} y_j \left\|(g^n-\delta)^- 1_{\Delta K_j}\right\|_{\infty,\mathcal{P}} \\
&\quad\leq \sum_{j\geq 1} y_j \left\|(g^n+\rho-x)^- 1_{\Delta K_j}\right\|_{\infty,\mathcal{P}} + \sum_{j\geq 1} y_j \left\|(x-\rho-\delta)^- 1_{\Delta K_j}\right\|_{\infty,\mathcal{P}} \\
&\quad\leq \frac{\varepsilon}{2} + \sum_{j\geq N} y_j \left\|(x-\rho-\delta)^- 1_{\Delta K_j}\right\|_{\infty,\mathcal{P}} \\
&\quad\leq \frac{\varepsilon}{2} + \sum_{j\geq N} y_j \left\|(\rho+\delta) 1_{\Delta K_j}\right\|_{\infty,\mathcal{P}} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Hence, (g^n) is a FLDR. To show f), fix a number $x > \lim_n \psi(f^n) \leq \psi(f)$. For given $y \in l_\rho^{1,+}$, there exists (g^n) in G such that

$$\lim_{n\to\infty} \sum_{j\geq 1} y_j ||(x+g^n-f^n)^- 1_{\Delta K_j}||_{\infty,\mathcal{P}} = 0.$$

Moreover, it follows from Lebesgue's dominated convergence theorem that

$$\lim_{n\to\infty} \sum_{j\geq 1} y_j ||(f-f^n) 1_{\Delta K_j}||_{\infty,\mathcal{P}} = 0,$$

and therefore

$$\begin{aligned}
&\sum_{j\geq 1} y_j \; ||(x+g^n-f)^- 1_{\Delta K_j}||_{\infty,\mathcal{P}} \\
\leq\;& \sum_{j\geq 1} y_j ||(x+g^n-f^n)^- 1_{\Delta K_j}||_{\infty,\mathcal{P}} + \sum_{j\geq 1} y_j ||(f-f^n) 1_{\Delta K_j}||_{\infty,\mathcal{P}} \to 0.
\end{aligned}$$

This shows that $\lim_n \psi(f^n) = \psi(f)$. So it follows from Theorem 4.4.1 and Remark 4.5.3 that

$$\psi(f) \leq \sup_{\mathbb{Q}\in\mathcal{M}^\rho} \left\{\mathbb{E}_\mathbb{Q}[f] - \psi^*_{C_\rho}(\mathbb{Q})\right\} \quad \text{for all } f \in U_\rho, \tag{5.3}$$

where the convex conjugate is given by $\psi^*_{C_\rho}(\mathbb{Q}) := \sup_{f\in C_\rho} \{\mathbb{E}_\mathbb{Q}[f] - \psi(f)\}$. Since ψ is positive homogeneous, one has $\psi^*_{C_\rho}(\mathbb{Q}) = 0$ if $\mathbb{E}_\mathbb{Q}[f] \leq \psi(f)$ for all $f \in C_\rho$ and $\psi^*_{C_\rho}(\mathbb{Q}) = \infty$ else.

We next show

$$\psi(f) = \sup_{\mathbb{Q}\in\mathcal{M}^\rho_\mathcal{P}(G)} \mathbb{E}_\mathbb{Q}[f] \quad \text{for all } f \in U_\rho. \tag{5.4}$$

To that end, let us first prove that

$$\psi(f) \leq \sup_{\mathbb{Q}\in\mathcal{M}^\rho_\mathcal{P}} \left\{\mathbb{E}_\mathbb{Q}[f] - \psi^*_{C_\rho}(\mathbb{Q})\right\} \quad \text{for all } f \in U_\rho. \tag{5.5}$$

Indeed, fix $\mathbb{Q} \in \mathcal{M}^\rho \setminus \mathcal{M}^\rho_\mathcal{P}$. Since $\mathbb{Q}$ is regular there exists a compact set A such that $\mathbb{Q}(A) > 0$ whereas $A \cap S = \emptyset$. Since S is closed and Ω normal, by Urysohn's lemma there exists a positive function $f \in C_b \subseteq C_\rho$ taking value 0 on S and 1 on A. Hence, $f = 0$ $\mathcal{P}$-q.s. so that

$$\psi^*_{C_\rho}(\mathbb{Q}) \geq E_\mathbb{Q}[nf] - \psi(nf) \geq nE_\mathbb{Q}[1_A] - \psi(0) \quad \text{for all } n \in \mathbb{N},$$

which implies $\psi^*_{C_\rho}(\mathbb{Q}) = +\infty$. In combination with (5.3) we derive (5.5).

Fix $\mathbb{Q} \in \mathcal{M}^\rho_\mathcal{P} \setminus \mathcal{M}^\rho_\mathcal{P}(G)$, so that $\mathbb{E}_\mathbb{Q}[g] > 0$ for some $g \in G$. since mg is superreplicable at zero cost for all $m \in \mathbb{R}_+$, one has $\phi(mg) \leq 0$, and so for any $\mathcal{P}$ -modification $\tilde{g}$ of g such that $\tilde{g}|_K \in C_\rho$ we have

$$\phi^*_{C_\rho}(\mathbb{Q}) \geq \sup_{m\in\mathbb{R}_+} \{\mathbb{E}_\mathbb{Q}[m\tilde{g}] - \phi(m\tilde{g})\} = \infty.$$

This shows that $\phi^*_{C_\rho} = \infty$ outside of $\mathcal{M}^\rho_\mathcal{P}(G)$. Since ϕ dominates ψ on C_ρ it follows that $\phi^*_{C_\rho} \leq \psi^*_{C_\rho}$, which implies $\psi^*_{C_\rho} = \psi^*_{U_\rho} = \infty$ outside of $\mathcal{M}^\rho_\mathcal{P}(G)$.

Furthermore, $\psi^*_{C_\rho} = \psi^*_{U_\rho} = 0$ on $\mathcal{M}^\rho_\mathcal{P}(G)$. To that end fix $\mathbb{Q} \in \mathcal{M}^\rho_\mathcal{P}(G)$ and let $f \in U_\rho$. If $\psi(f) < \infty$, let $x \in \mathbb{R}$ be such that for any $y \in l^{1,+}_\rho$ there is $(g^n) \subseteq G$ for which $\sum_{i\geq 1} y_i\|(x + g^n - f)^- 1_{\Delta K_i}\|_{\infty,\mathcal{P}} \to 0$. For $y_i := \mathbb{Q}(\Delta K_i)$, since $E_\mathbb{Q}[g^n] \leq 0$ we get

$$E_\mathbb{Q}[f]-x \leq E_\mathbb{Q}\Big[\sum_{i\geq 1}(x+g^n-f)^- 1_{\Delta K_i}\Big] \leq \sum_{i\geq 1}\mathbb{Q}(\Delta K_i)\|(x+g^n-f)^- 1_{\Delta K_i}\|_{\infty,\mathcal{P}} \to 0,$$

so that $E_\mathbb{Q}[f] \leq x$. Therefore, $E_\mathbb{Q}[f] \leq \psi(f)$ which in turn yields $0 \leq \psi^*_{C_\rho}(\mathbb{Q}) \leq \psi^*_{U_\rho}(\mathbb{Q}) \leq 0$.

Combining (5.5) with the previous arguments gives

$$\psi(f) \leq \sup_{\mathbb{Q}\in\mathcal{M}^\rho_\mathcal{P}(G)} \left\{\mathbb{E}_\mathbb{Q}[f] - \psi^*_{U_\rho}(\mathbb{Q})\right\} \leq \psi(f) \quad \text{for all } f \in U_\rho,$$

by definition of the convex conjugate $\psi^*_{U_\rho}$, which implies (5.4). In particular, $\mathcal{M}^\rho_\mathcal{P}(G)$ is not empty.

It remains to show that every $\mathbb{P} \in \mathcal{P}$ is dominated by a $\mathbb{Q} \in \mathcal{M}^\rho_\mathcal{P}(G)$. To do that, note that every $\mathbb{Q} \in \mathcal{M}^\rho_\mathcal{P}(G)$ has a unique Lebesgue decomposition

$$\mathbb{Q}[A] = \mathbb{Q}[A \cap N] + \int_A Z^\mathbb{Q}_\mathbb{P} d\mathbb{P}, \text{ where } \mathbb{P}[N] = 0 \text{ and } Z^\mathbb{Q}_\mathbb{P} \geq 0 \text{ } \mathbb{P}\text{-a.s..}$$

Define

$$\alpha := \sup\left\{\mathbb{P}\left(Z^\mathbb{Q}_\mathbb{P} > 0\right) : \mathbb{Q} \in \mathcal{M}^\rho_\mathcal{P}(G)\right\}.$$

There exists a sequence $(\mathbb{Q}^n)$ in $\mathcal{M}^\rho_\mathcal{P}(G)$ such that $\mathbb{P}(Z^{\mathbb{Q}^n}_\mathbb{P} > 0)$ converges to α. Choose a countable convex combination (λ^n) in $[0,1]$, i.e. $\sum_{n\geq 1}\lambda_n = 1$, such that $\mathbb{Q} = \sum_{n\geq 1}\lambda^n\mathbb{Q}^n$ is an element of $\mathcal{M}^\rho_\mathcal{P}$. But then $\mathbb{P}(Z^\mathbb{Q}_\mathbb{P} > 0) = \alpha$ and the previous supremum is attained. We claim that $\alpha = 1$. Otherwise, there would exist a closed set A with $\mathbb{P}(A) > 0$ and $\mathbb{Q}(A) = 0$ for all $\mathbb{Q} \in \mathcal{M}^\rho_\mathcal{P}(G)$. But then $\psi(1_A) = 0$ by (5.4), so that for every $y \in l^{1,+}_\rho$ and $n \in \mathbb{N}$, there exists a sequence (g^n) in G such that

$$\sum_{j\geq 1} y_j\big\|\,(1/n + g^n - 1_A)^-\, 1_{\Delta K_j}\big\|_{\infty,\mathcal{P}} \leq \frac{1}{n}.$$

But then

$$\lim_{n\to\infty}\sum_{j\geq 1} y_j \big\| (g^n - 1_A)^- 1_{\Delta K_j}\big\|_{\infty,\mathcal{P}} = 0,$$

which contradicts (i). Hence, there is a $\mathbb{Q} \in \mathcal{M}^\rho_{\mathcal{P}}(G)$ that dominates $\mathbb{P}$.

(ii) $\Rightarrow$ (i): Let $f \in B_\rho(\mathcal{P})_+$ such that for every $y \in l^{1,+}_\rho$ there exists a sequence (g^n) in G which satisfies

$$\lim_{n\to\infty}\sum_{j\geq 1} y_j ||(g^n - f)^- 1_{\Delta K_j}||_{\infty,\mathcal{P}} = 0.$$

Then for every $\mathbb{Q} \in \mathcal{M}^\rho_{\mathcal{P}}(G)$ there exists a sequence (g^n) in G such that

$$\lim_{n\to\infty}\sum_{j\geq 1} \mathbb{Q}(\Delta K_j)\ ||(g^n - f)^- 1_{\Delta K_j}||_{\infty,\mathcal{P}} = 0.$$

Since $\mathbb{E}_{\mathbb{Q}}[g^n] \leq 0$ for all $n \in \mathbb{N}$ one obtains

$$\mathbb{E}_{\mathbb{Q}}[f] \leq \lim_{n\to\infty} \mathbb{E}_{\mathbb{Q}}\left[\sum_{j\geq 1}(f - g^n)1_{\Delta K_j}\right] \leq \lim_{n\to\infty}\sum_{j\geq 1} \mathbb{Q}(\Delta K_j)\ ||(g^n - f)^- 1_{\Delta K_j}||_{\infty,\mathcal{P}} = 0,$$

which shows that G does not admit FLDR. □

5.4 Robust FTAP in Discrete and Continuous Time

In this section we study applications of Theorem 5.3.2 to financial markets in discrete and continuous time.

5.4.1 The State Space $\mathbb{R}^{dT}$

Let $X^i_t(\omega) = \omega^i(t)$, where $i \in I := \{1, \dots d\}$ and $t \in \mathbb{T} := \{0, \dots, T\}$, be the price process of d (discounted) assets on the state space $\Omega = \mathbb{R}^{dT}$, which generates the natural filtration $\mathcal{F}_t := \sigma(X_s : 0 \leq s \leq t)$, $t \in \mathbb{T}$. Then $\mathbb{R}^{dT}$ is the union over the compacts

$$K_n = \Big\{\omega \in \mathbb{R}^{dT} : \max_{i\in I,\, t\in\mathbb{T}} |\omega^i(t)| \leq n\Big\}.$$

In particular $\Omega = \bigcup_{n\in\mathbb{N}} K_n$ $\mathcal{P}$-q.s. for any reference set $\mathcal{P}$ of regular probability measures on $\mathcal{F}_T$. We consider the weights $\rho_n := n$ and associate the function $\rho : \mathbb{R}^{dT} \to \mathbb{N}$ defined as $\rho := \sum_{n\geq 1} \rho_n 1_{\Delta K_n}$. In particular, $X^i_t \in C_\rho$ for all $i \in I$ and $t \in \mathbb{T}$.

Denote by $H \subseteq C_\rho$ a convex cone of options attainable with zero initial wealth priced at 0. By investing dynamically in the assets and statically in the options the set of outcomes attainable with zero initial wealth becomes

$$G = \Big\{\sum_{t=1}^{T} \vartheta_t(X_t - X_{t-1}) + \sum_{l=1}^{L} h_l : \vartheta_t \in C_b(\mathcal{F}_{t-1}), h_l \in H, L \in \mathbb{N}\Big\}$$

when only “continuous” trading strategies are taken into account. Considering all measurable trading strategies leads to

$$\hat{G} = \Big\{\sum_{t=1}^{T} \vartheta_t(X_t - X_{t-1}) + \sum_{l=1}^{L} h_l : \vartheta_t \in B_b(\mathcal{F}_{t-1}), h_l \in H, L \in \mathbb{N}\Big\}.$$

The set of martingale measures $\mathcal{M}_{\mathcal{P}}(X, H)$ consists of those regular probability measures $\mathbb{Q}$ on $\mathcal{F}_T$ under which X is a $\mathbb{Q}$-martingale, $E_{\mathbb{Q}}[|h|] < \infty$ and $E_{\mathbb{Q}}[h] \leq 0$ for all $h \in H$.

Theorem 5.4.1. *The following statements are equivalent:*

(i) *G admits no FLDR with respect to ρ;*

(ii) *Every $\mathbb{P} \in \mathcal{P}$ is dominated by a $\mathbb{Q} \in \mathcal{M}^{\rho}_{\mathcal{P}}(G)$;*

(iii) *Every $\mathbb{P} \in \mathcal{P}$ is dominated by a $\mathbb{Q} \in \mathcal{M}_{\mathcal{P}}(X, H)$;*

(iv) *$\hat{G}$ admits no FLDR with respect to ρ.*

Proof. (i) $\Rightarrow$ (ii) follows from Theorem 5.3.2 since $G \subseteq C_\rho$.

(ii) $\Rightarrow$ (iii): Fix $\mathbb{P} \in \mathcal{P}$ which is dominated by a $\mathbb{Q} \in \mathcal{M}^{\rho}_{\mathcal{P}}(G)$. Each $h \in H \subseteq C_\rho$ is bounded by $|h| \leq c\rho$ for some $c > 0$, so that $E_{\mathbb{Q}}[|h|] < \infty$. Further, for every $A \in \mathcal{F}_{t-1}$ for $t \in \{1, \ldots, T\}$, there exists a sequence (ϑ^n) in $C_b(\mathcal{F}_{t-1})$ such that $\vartheta^n \to 1_A$ $\mathbb{Q}$-a.s., which by dominated convergence implies

$$E_{\mathbb{Q}}[1_A(X_t - X_{t-1})] = \lim_{n\to\infty} E_{\mathbb{Q}}[\vartheta^n(X_t - X_{t-1})] = 0.$$

This shows $\mathbb{Q} \in \mathcal{M}_{\mathcal{P}}(X, H)$.

(iii) $\Rightarrow$ (iv): Fix $\mathbb{Q} \in \mathcal{M}_{\mathcal{P}}(X, H)$. Then

$$\rho \leq \max_{i\in I, t\in\mathbb{T}} |X^i_t| + 1$$

since for $\omega \in \Delta K_n$ one has $\rho(\omega) \leq n$ and $\omega \notin K_{n-1}$ so that $|X^i_t(\omega)| \geq n - 1$ for some $i \in I$ and $t \in \mathbb{T}$, which implies $\max_{i\in I,\, t\in\mathbb{T}} |X^i_t(\omega)| \geq n - 1$. Thus $\sum_{n\geq 1} \rho_n \mathbb{Q}(\Delta K_n) = E_{\mathbb{Q}}[\rho] < \infty$, and therefore $\mathbb{Q} \in \mathcal{M}^{\rho}_{\mathcal{P}}$. We finally argue along the same lines as (ii) $\Rightarrow$ (i) in the proof of Theorem 5.3.2 because $E_{\mathbb{Q}}[\vartheta_t(X_t - X_{t-1})] = 0$ for each $\mathbb{Q} \in \mathcal{M}_{\mathcal{P}}(X, H)$ and all $\vartheta \in B_b(\mathcal{F}_{t-1})$ with $t \in \{1, \ldots, T\}$.

Finally, (iv) $\Rightarrow$ (i) is obvious. □

5.4.2 The Product Space

Let $X^i_t(\omega) = \omega^i(t)$, where $t \in [0, T]$ and $i \in I := \{1, \ldots, d\}$, be the canonical process on $(\mathbb{R}^d)^{[0,T]}$ endowed with the product topology. Let

$$K_n := \Big\{\omega \in \Omega : \sup_{t\in[0,T],\, i\in I} |\omega^i(t)| \leq n\Big\}, \quad n \in \mathbb{N},$$

which are compact by the Tychonoff theorem. In this subsection we consider the state space

$$\Omega := \bigcup_{n\geq 1} K_n,$$

on which the paths of the canonical process are bounded, i.e. $\sup_{t\in[0,T]} |X_t(\omega)| < \infty$ for all $\omega \in \Omega$.

Lemma 5.4.2. *Ω is a normal Hausdorff space.*

Proof. The space Ω endowed with the relative topology of $(\mathbb{R}^d)^{[0,T]}$ is Hausdorff.

By [2, Theorem 2.49], it sufficies to show that Ω is Lindelöf and regular, in the sense that for every closed subset F and any point x that does not belong to F, there exist neighborhoods of F and x which are disjoint.

a) Let us show that Ω is Lindelöf. Let $(C_j)_{j\in J}$ be an open cover of Ω. Then, since $\Omega = \bigcup_{n\geq 1} K_n$ it follows from compactness that $K_n \subseteq \bigcup_{j\in J_n} C_j$ where $J_n \subseteq J$ is finite for all $n \in \mathbb{N}$. But then $\Omega = \bigcup_{n\geq 1} \bigcup_{j\in J_n} C_j$ and $\bigcup_{n\geq 1} J_n$ is countable.

b) Let us show that Ω is regular. Fix $x \in \Omega$ and $F \subseteq \Omega$ closed such that $x \notin F$. Since F^c is open in the product topology there exists a family $(U_t)_{t\in[0,T]}$ of open subsets of $\mathbb{R}^d$ such that $U_t = \mathbb{R}^d$ except for finitely many $t \in \{t_1, \ldots, t_m\} \subseteq [0,T]$ and $x \in \Pi_{t\in[0,T]} U_t \subseteq F^c$. Since $\mathbb{R}^d$ is regular, there exist open sets $V_t \subseteq \mathbb{R}^d$ such that $x_t \in V_t \subseteq \mathrm{cl}(V_t) \subseteq U_t$ for all $t \in [0,T]$ and $V_t = \mathbb{R}^d$ except for $t \in \{t_1, \ldots, t_m\}$. Here, $\mathrm{cl}(V_t)$ denotes the closure of V_t. Hence

$$x \in V = \Pi_{t\in[0,T]} V_t \subseteq \Pi_{t\in[0,T]} \mathrm{cl}(V_t) \subseteq \Pi_{t\in[0,T]} U_t \subseteq F^c,$$

so that $W = \Pi_{t\in[0,T]} W_t \supseteq (\Pi_{t\in[0,T]} \mathrm{cl}(V_t))^c \supseteq F$ where $W_t = (\mathrm{cl}(V_t))^c$ if $t \in \{t_1, \ldots, t_m\}$ and $W_t = \mathbb{R}^d$ else. By construction, $W \cap V = \emptyset$. □

Let $(\mathcal{F}_t)_{t\in[0,T]}$ be the natural filtration $\mathcal{F}_t = \sigma(X_s^i; i \in I,\, t \in [0,T],\, s \le t)$ and fix a reference set $\mathcal{P}$ of regular probability measures on $\mathcal{F}_T$. Similar to the previous subsection we consider the weights $\rho_n := n$ and associate the function $\rho := \sum_{n\ge1} \rho_n 1_{\Delta K_n}$, so that $X_t^i \in C_\rho$ for all $i \in I$ and $t \in [0,T]$. Denote by $H \subseteq C_\rho$ a convex cone of options attainable with zero initial wealth priced at 0 and define

$$G = \Big\{ \sum_{n=1}^{N} \vartheta_{s_n}(X_{t_n} - X_{s_n}) + \sum_{l=1}^{L} h_l : \\ 0 \le s_n \le t_n \le T,\, \vartheta_s \in C_b(\mathcal{F}_s), h_l \in H, \text{ and } L, N \in \mathbb{N} \Big\}$$

and

$$\hat{G} = \Big\{ \sum_{n=1}^{N} \vartheta_{s_n}(X_{t_n} - X_{s_n}) + \sum_{l=1}^{L} h_l : \\ 0 \le s_n \le t_n \le T,\, \vartheta_s \in B_b(\mathcal{F}_s), h_l \in H \text{ and } L, N \in \mathbb{N} \Big\}.$$

Denote by $\mathcal{M}_{\mathcal{P}}(X,H)$ the set of those regular probability measures $\mathbb{Q}$ on $\mathcal{F}_T$ under which X is a $\mathbb{Q}$-martingale with $\sup_{t\in[0,T]} |X_t| \in L^1(\mathbb{Q})$, $E_{\mathbb{Q}}[|h|] < \infty$ and $E_{\mathbb{Q}}[h] \le 0$ for all $h \in H$.

Theorem 5.4.3. *The following statements are equivalent:*

(i) G admits no FLDR with respect to ρ;

(ii) Every $\mathbb{P} \in \mathcal{P}$ is dominated by a $\mathbb{Q} \in \mathcal{M}_{\mathcal{P}}^{\rho}(G)$;

(iii) Every $\mathbb{P} \in \mathcal{P}$ is dominated by a $\mathbb{Q} \in \mathcal{M}_{\mathcal{P}}(X,H)$;

(iv) $\hat{G}$ admits no FLDR with respect to ρ.

Proof. For $\mathbb{Q} \in \mathcal{M}_{\mathcal{P}}^{\rho}(G)$ one has

$$E_{\mathbb{Q}}\Big[\sup_{t\in[0,T]} |X_t| \Big] \le \sum_{n\ge1} E_{\mathbb{Q}}\Big[\sup_{t\in[0,T],\, i\in I} \big|X_t^i\big| 1_{\Delta K_n} \Big] \le \sum_{n\ge1} n\mathbb{Q}(\Delta K_n) < \infty.$$

Conversely, if $\mathbb{Q} \in \mathcal{M}_{\mathcal{P}}(X,H)$ then

$$\rho \le \sup_{t\in[0,T],\, i\in I} |X_t^i| + 1 \le d \sup_{t\in[0,T]} |X_t| + 1,$$

which shows that $\sum_{n\ge1} n\mathbb{Q}(\Delta K_n) = E_{\mathbb{Q}}[\rho] < \infty$.

The rest is analogous to the proof of Theorem 5.4.1. □

As a direct consequence of the previous result we deduce the following characterization of $\mathcal{H}^1$-martingales on the product space Ω.

Corollary 5.4.4. *Given a regular probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$ the following statements are equivalent:*

(i) There exists a regular probability measure $\mathbb{Q}$ on $\mathcal{F}_T$ such that $\mathbb{Q} \sim \mathbb{P}$ and the canonical process $(X_t)_{t\in[0,T]}$ is a $\mathbb{Q}$-martingale with $\sup_{t\in[0,T]} |X_t| \in L^1(\mathbb{Q})$;

(ii) $\left\{ \sum_{n=1}^N \vartheta_{s_n}(X_{t_n} - X_{s_n}) : 0 \le s_n \le t_n \le T,\, \vartheta_s \in C_b(\mathcal{F}_s), N \in \mathbb{N} \right\}$ admits no FLDR with respect to ρ.

Proof. The result follows from Theorem 5.4.3 for $\mathcal{P} = \{\mathbb{P}\}$ and $H = \emptyset$. ☐

5.4.3 The State Space $C([0,T]; \mathbb{R}^d)$

Let $X_t(\omega) = \omega_t$ be the canonical process on $\Omega = C([0,T]; \mathbb{R}^d)$ endowed with the supremum norm $\|\omega\|_\infty := \sup_{t\in[0,T]} |\omega_t|$. Denote by $(\mathcal{F}_t)_{t\in[0,T]}$ the right-continuous version of the canonical filtration $\sigma(X_s; s \le t)$, $t \in [0,T]$. We consider a set $\mathcal{P}$ of regular reference probability measures on $\mathcal{F}_T$ such that

$$C\left([0,T]; \mathbb{R}^d\right) = \bigcup_{n\ge1} K_n \quad \mathcal{P}\text{-q.s.}$$

for an increasing sequence of compacts (K_n). For instance

a) $\mathcal{P}$ is tight, or

b) every $\mathbb{P} \in \mathcal{P}$ has support on the Hölder continuous paths so that $C([0,T]; \mathbb{R}^d)$ is $\mathcal{P}$-q.s. the union of the compact sets

$$K_n = \left\{ \omega \in C([0,T]; \mathbb{R}^d) : \|w\|_\infty \le n,\, \sup_{s\neq t} \frac{|\omega(s) - \omega(t)|}{|s-t|^{1/n}} \le n \right\}.$$

We consider the weight function $\rho_n = 1$ for all $n \ge 1$, so that C_ρ equals the space C_b of all bounded continuous functions on $C([0,T], \mathbb{R}^d)$, and we write $C_b(\mathcal{P})$ for $C_\rho(\mathcal{P})$.

The financial market consists of the assets $X = (X^1, \ldots, X^d)$ and a convex cone $H \subseteq C_b(\mathcal{P})$ of statically traded contingent claims. The convex cone of outcomes of simple strategies G is then given by

$$\sum_{n=1}^N \vartheta_{s_n}(X_{t_n}^{m_n} - X_{s_n}^{m_n}) + \sum_{l=1}^L h_l$$

where $0 \le s_n \le t_n \le T$, $m_n \in \mathbb{N}$, $\vartheta_{s_n} \in C_b(\mathcal{F}_{s_n})$, $h_l \in H$, $L, N \in \mathbb{N}$, and X^m is the stopped process $X_t^{\tau_m} := X_{t\wedge\tau_m}$ for the stopping time $\tau^m := \inf\{t > 0 : |X_t| > m\} \wedge T$ for all $m \in \mathbb{N}$. The set G is a subset of $C_b(\mathcal{P})$.

Let $\mathcal{M}_{\mathcal{P}}^{\text{loc}}(X, H)$ denote the set of probability measures $\mathbb{Q}$ such that supp$(\mathbb{Q}) \subseteq S$, the canonical process X is a local $\mathbb{Q}$-martingale and which satisfy $E_{\mathbb{Q}}[|h|] < \infty$ and $\mathbb{E}_{\mathbb{Q}}[h] \le 0$ for all $h \in H$.

Theorem 5.4.5. *The following statements are equivalent:*

(i) *G admits no FLDR with respect to ρ;*

(ii) *Every $\mathbb{P} \in \mathcal{P}$ is dominated by a $\mathbb{Q} \in \mathcal{M}_{\mathcal{P}}^{\text{loc}}(X, H)$.*

Proof. Since $G \subseteq C_b(\mathcal{P})$ it follows from Theorem 5.3.2 that G admits no FLDR with respect to ρ if and only if each $\mathbb{P} \in \mathcal{P}$ is dominated by a $\mathbb{Q} \in \mathcal{M}^\rho_\mathcal{P}(G)$. Hence it is enough to show that $\mathcal{M}^\rho_\mathcal{P}(G) = \mathcal{M}^{\text{loc}}_\mathcal{P}(X,H)$. On the one hand, for $\mathbb{Q} \in \mathcal{M}^\rho_\mathcal{P}(G)$ it follows $\mathbb{E}_\mathbb{Q}[h] \leq 0$ for all $h \in H$ and $\mathbb{E}_\mathbb{Q}[\vartheta(X^m_t - X^m_s)] = 0$ for all $\vartheta \in C_b(\mathcal{F}_s)$, so that $\mathbb{E}_\mathbb{Q}[1_A(X^m_t - X^m_s)] = 0$ for all $A \in \mathcal{F}_s$ by approximating 1_A with a sequence (ϑ^n) of continuous functions in $C_b(\mathcal{F}_s)$. This shows that X^m is a $\mathbb{Q}$-martingale for all $m \in \mathbb{N}$, that is, $\mathbb{Q} \in \mathcal{M}^{\text{loc}}_\mathcal{P}(X,H)$. Conversely, if $\mathbb{Q} \in \mathcal{M}^{\text{loc}}_\mathcal{P}(X,H)$ the process X^m is a $\mathbb{Q}$-martingale which implies

$$\mathbb{E}_\mathbb{Q}[\vartheta(X^m_t - X^m_s)] = \mathbb{E}_\mathbb{Q}[\vartheta\, \mathbb{E}_\mathbb{Q}[X^m_t - X^m_s | \mathcal{F}_s]] = 0 \quad \text{for all } \vartheta \in C_b(\mathcal{F}_s).$$

Since in addition $\mathbb{E}_\mathbb{Q}[h] \leq 0$ for all $h \in H$, we conclude that $\mathbb{Q} \in \mathcal{M}^\rho_\mathcal{P}(G)$. □

5.4.4 The State Space $D([0,T];\mathbb{R}^d)$

Let $X_t(\omega) = \omega_t$ be the canonical process on $D([0,T];\mathbb{R}^d)$ endowed with the Skorohod topology. Denote by $(\mathcal{F}_t)_{t\in[0,T]}$ the right-continuous version of the canonical filtration $\sigma(X_s; s \leq t)$, $t \in [0,T]$. We consider a set $\mathcal{P}$ of regular reference probability measures on $\mathcal{F}_T$ such that

$$D([0,T];\mathbb{R}^d) = \bigcup_{n\geq 1} K_n \quad \mathcal{P}\text{-q.s.}$$

for an increasing sequence of compacts (K_n). For instance $\mathcal{P}$ is tight, or each $\mathbb{P} \in \mathcal{P}$ has support on some K_n. A characterization of compact sets in the Skorohod topology is given in [10, Theorem 14.3]. Further, the process X is assumed to be $\mathcal{P}$-locally bounded, in the sense that there exists a sequence of stopping times (τ_m) such that $X^m := X^{\tau_m}$ is bounded, and $\mathcal{P}(\bigcup_{m\geq 1}\{\tau_m < T\}) = 0$. We consider the weight function $\rho_n = 1$ for all $n \geq 1$.

The financial market consists of the risky assets $X = (X^1,\dots,X^d)$ and a convex cone of statically traded contingent claims H satisfying $H \subseteq C_b(\mathcal{P})$. The space G consists of all outcomes of a simple trading strategies given by

$$\sum_{n=1}^N \vartheta_{s_n}(X^{m_n}_{t_n} - X^{m_n}_{s_n}) + \sum_{l=1}^L h_l$$

where $0 \leq s_n \leq t_n \leq T$, $m_n \in \mathbb{N}$, $\vartheta_{s_n} \in C_b(\mathcal{F}_{s_n})$, $h_l \in H$, and $L, N \in \mathbb{N}$. Notice that G is not a subspace of $C_b(\mathcal{P})$. By $\mathcal{M}^{\text{loc}}_\mathcal{P}(X,H)$ we denote the set of probability measures $\mathbb{Q}$ such that supp$(\mathbb{Q}) \subseteq S$, the canonical process X is a local $\mathbb{Q}$-martingale and which satisfy $E_\mathbb{Q}[|h|] < \infty$ and $\mathbb{E}_\mathbb{Q}[h] \leq 0$ for all $h \in H$.

Theorem 5.4.6. *The following statements are equivalent:*

(i) *G admits no FLDR with respect to ρ;*

(ii) *Every $\mathbb{P} \in \mathcal{P}$ is dominated by a $\mathbb{Q} \in \mathcal{M}^{\text{loc}}_\mathcal{P}(X,H)$.*

Proof. (i) ⇒ (ii): Define

$$\tilde{G} := \{\tilde{g} \in C_b(\mathcal{P}) : \tilde{g} \leq g,\, g \in G\}$$

which is a convex cone in $C_b(\mathcal{P})$. Since G admits no FLDR with respect to ρ, the same holds true for $\tilde{G}$. By Theorem 5.3.2 each $\mathbb{P} \in \mathcal{P}$ is dominated by a $\mathbb{Q} \in \mathcal{M}^\rho_\mathcal{P}(\tilde{G})$. Hence, it is enough to show that $\mathcal{M}^\rho_\mathcal{P}(\tilde{G}) \subseteq \mathcal{M}^{\text{loc}}_\mathcal{P}(X,H)$. Fix $\mathbb{Q} \in \mathcal{M}^\rho_\mathcal{P}(\tilde{G})$. Since $H \subseteq \tilde{G}$ one has $E_\mathbb{Q}[h] \leq 0$ for all $h \in H$. By [10, Chapter 3], X^m_t is continuous at ω in the Skorohod topology if and only if ω is continuous at t and, moreover, there exists a dense subset Π of $[0,T]$ such that $\mathbb{Q}(X^m_t \neq X^m_{t-}) = 0$ for all $t \in \Pi$. Fix $s,t \in \Pi$ with $s \leq t$ and define by

$\bar{X}_t^m$ the lower semicontinuous hull of X_t^m and $\hat{X}_s^m$ the upper semicontinuous hull of X_s^m. Then, $\mathbb{Q}(X_t^m \neq \bar{X}_t^m) = \mathbb{Q}(X_s^m \neq \hat{X}_s^m) = 0$, so that $\vartheta(\bar{X}_t^m - \hat{X}_s^m) = \vartheta(X_t^m - X_s^m)$ $\mathbb{Q}$-a.s. for all $\vartheta \in C_b(\mathcal{F}_s)$. In addition, since $D([0,T],\mathbb{R}^d)$ is a Polish space, there is a sequence $(g^n) \subseteq C_b$ which increases pointwise to the lower semicontinuous function $\vartheta(\bar{X}_t^m - \hat{X}_s^m)$. Then $g^n \leq \vartheta(\bar{X}_t^m - \hat{X}_s^m) \leq \vartheta(X_t^m - X_s^m)$ so that $(g^n) \subseteq \tilde{G}$. Hence, by monotone convergence

$$\begin{aligned} E_{\mathbb{Q}}[\vartheta(X_t^m - X_s^m)] &= E_{\mathbb{Q}}[\vartheta(\bar{X}_t^m - \hat{X}_s^m)] \\ &= \lim_{n\to\infty} E_{\mathbb{Q}}[g^n] \leq 0. \end{aligned}$$

Since X^m is càdlàg and Π dense in $[0,T]$, it follows that $E_{\mathbb{Q}}[\vartheta(X_t^m - X_s^m)] = 0$ for all $0 \leq s \leq t \leq T$, and $\vartheta \in C_b(\mathcal{F}_s)$. Therefore, $E_{\mathbb{Q}}[1_A(X_t^m - X_s^m)] = 0$ for all $A \in \mathcal{F}_s$ by approximating 1_A by a sequence (ϑ^n) of continuous functions. This shows that X^m is a $\mathbb{Q}$-martingale for all $m \in \mathbb{N}$, that is, $\mathbb{Q} \in \mathcal{M}_{\mathcal{P}}^{\mathrm{loc}}(X,H)$.

(ii) $\Rightarrow$ (i): Similar to the proof of Theorem 5.4.5 one has $\mathcal{M}_{\mathcal{P}}^{\mathrm{loc}}(X,H) \subseteq \mathcal{M}_{\mathcal{P}}^{\rho}(G)$. The statement (i) then follows from Theorem 5.3.2. □

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