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Between Kantianism and Empiricism: Otto Hölder's Philosophy of Geometry

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Résumé : La philosophie de la géométrie de Hölder, si l'on s'en tient à une lecture superficielle, est la part la plus problématique de son épistémologie. Il soutient que la géométrie est fondée sur l'expérience à la manière de Helmholtz, malgré les objections sérieuses de Poincaré. Néanmoins, je pense que la position de Hölder mérite d'être discutée pour deux motifs. Premièrement, ses implications méthodologiques furent importantes pour le développement de son épistémologie. Deuxièmement, Poincaré utilise l'opposition entre le kantisme et l'empirisme comme un argument pour justifier son conventionnalisme géométrique. Cependant, Hölder montre qu'une stratégie alternative n'est pas exclue : il sait tirer parti des objections kantiennes pour développer un empirisme cohérent. En même temps, surtout dans *Die mathematische Methode* [Hölder 1924], il adopte aussi bien les expressions que les conceptions de Kant. Dans mon article, je considère d'abord les arguments de Hölder pour la méthode déductive en géométrie dans *Anschauung und Denken in der Geometrie* [Hölder 1900], en relation avec sa façon d'aborder la théorie de la quantité [Hölder 1901]. Ensuite, j'examine son rapport avec Kant. À mon sens, les considérations méthodologiques de Hölder lui permettent de préfigurer une relativisation de l'*a priori.*

Abstract: Hölder's philosophy of geometry might appear to be the most problematic part of his epistemology. He maintains that geometry depends on experience also after Poincaré's fundamental criticism of Helmholtz. Nevertheless, I think that Hölder's view is worth discussing, for two reasons. Firstly, the related methodological considerations were crucial for the development of his epistemology. Secondly, Poincaré uses the opposition between Kantianism and empiricism to argue for his geometrical conventionalism. Nevertheless, Hölder shows that an alternative strategy is not excluded: he profits from Kantian objections in order to develop a consistent empiricism. At the same time, especially in *Die mathematische Methode* [Hölder 1924], he vindicates the Kantian view that mathematics is synthetic. In this paper, I will consider Hölder's

defence of the deductive method in geometry in *Anschauung und Denken in der Geometrie* [Hölder 1900] in connection with his approach to the theory of quantity [Hölder 1901]. Moreover, I will discuss his connection with Kant. My suggestion is that Hölder's methodological considerations enable him to foreshadow a relativized conception of the a priori.

Introduction

In 1899 Otto Hölder succeeded Sophus Lie in the chair of mathematics at the University of Leipzig. In the extended version of his inaugural lecture, *Anschauung und Denken in der Geometrie,* which was printed in 1900, Hölder first presents the proof that the so-called Archimedean axiom can be derived from Dedekind's continuity. This proof is the starting point of the axiomatic theory of quantity he develops in 1901 and of the theory of proportions he uses in 1908 to construct a numerical scale on a projective straight line. These developments are to be considered in connection with Hölder's philosophy of geometry. In fact, in *Anschauung und Denken,* Hölder takes part in a debate about the foundations of geometry which had begun in the second half of the 19th century, after non-Euclidean geometry was rediscovered.

Already during the 1820s, János Bolyai and Nikolaj Lobačevskij, independently of each other, developed a new geometry that is based upon the denial of Euclid's fifth postulate. Such development had been anticipated by mathematicians such as Gerolamo Saccheri, Johann Heinrich Lambert, and Adrien-Marie Legendre, who sought to prove Euclid's fifth postulate by denying it and finding a contradiction. Since these and many other attempts to prove Euclid's fifth postulate failed, the theory of parallel lines had lost credibility at the time Bolyai and Lobačevskij wrote. For this reason, their works remained largely unknown at that time and models of non-Euclidean geometry were first developed many years later, namely Eugenio Beltrami's model in 1868, Felix Klein's model in 1871, and Henri Poincaré's model in 1882.

One of the first to recognize the importance of non-Euclidean geometry was Carl Friedrich Gauss. However, his appreciation of the works of Bolyai and Lobačevskij is to be found only in his private correspondence, which was published posthumously in the second half of the 19th century. At the same time, Gauss called into question Kant's a priori geometry and developed the conviction that the geometry of space should be determined a posteriori. His opinion was that the necessity of Euclid's geometry cannot be proved. For this reason, in a letter to Olbers dated 28 April 1817, Gauss claimed that "for now geometry must stand, not with arithmetic which is pure a priori, but with mechanics" ([Gauß 1900, 177] Eng. trans. in [Gray 2006, 63]). In a letter to Bessel dated 9 April 1830, Gauss wrote:

> According to my most sincere conviction the theory of space has an entirely different place in knowledge from that occupied by

> pure mathematics. There is lacking throughout our knowledge of it the complete persuasion of necessity (also of absolute truth) which is common to the latter; we must add in humility that if number is exclusively the product of our mind, space has a reality outside our mind and we cannot completely prescribe its laws. [Gauß 1900, 201]; Eng. trans. in [Kline 1980, 87]

It is tempting to relate Gauss' opinion to his later claim that, since 1792, he had been developing the conviction that a non-Euclidean geometry would be consistent (see Gauss' letter to Schumacher dated 28 November 1846 in [Gauß 1900, 238]). However, Gauss' knowledge about non-Euclidean geometry before his reading of the works of Bolyai and Lobačevskij is hard to reconstruct, and the question as to whether his views about space and geometry presuppose the assumption of non-Euclidean geometry is controversial. The problem is that Gauss could hardly have possessed the concept of a non-Euclidean, three-dimensional space. Nevertheless, his empiricist insights were seminal in 19th-century philosophy of geometry and, at the time Hölder wrote, it was quite natural to associate Gauss' claims with a view which follows from Bernhard Riemann's classification of hypotheses underlying geometry [Riemann 1854] and which can be summarised as follows: since different geometries are logically possible, none of them can be deemed necessary or given a priori in our conception of space.

A related question is whether geometrical propositions can be empirically tested. Lobačevskij sought to detect whether the sum of the angles in a triangle is equal to or less than 180 degrees by means of measurements, and Sartorius von Waltershausen reports that Gauss also made such an attempt during his geodetic work [Sartorius von Waltershausen 1856, 81]. [1]

1. It might be questioned whether Gauss deliberately undertook an empirical test of Euclid's geometry. Ernst Breitenberger convincingly argues that the measurement Sartorius refers to does not suffice to draw such a conclusion. More probably, Gauss might have mentioned such measurement in his inner circle, because it incidentally confirms his conviction that Euclid's geometry is true within the limits of the best observational error of his time. The question of the empirical test is to be related to that of Gauss' views on space and geometry, because such a test would presuppose the assumption of non-Euclidean geometry as a possible alternative to Euclidean geometry. In fact, to support his reconstruction, Breitenberger maintains that Gauss could not have possessed the notion of a curved, three-dimensional space as developed only in 1854 by Bernhard Riemann [Breitenberger 1984, 285]. More recently, this point has been developed by Jeremy Gray, who points out that Gauss calls into question the necessity of Euclid's geometry, because he focuses on the problems concerning the definition of the plane and that of parallel lines. But he did not start with non-Euclidean three-dimensional space as Bolyai and Lobačevskij did [Gray 2006, 75]. On the other hand, the interpretation of Gauss' measurement as an empirical test of Euclid's geometry has been recently defended by Erhard Scholz. Scholz's claim is that, even though Gauss could not have known non-Euclidean geometry at that time, his study of the geometric properties of surfaces enabled him to make heuristic assumptions about physical space. Regarding Breitenberger's objection, Scholz

A more refined kind of empiricism was developed by the German physiologist and physicist Hermann von Helmholtz in a series of epistemological writings. The most important of them were reedited by Paul Hertz and Moritz Schlick in 1921 [Helmholtz 1921]. They are: *Über die Tatsachen, die der Geometrie zugrunde liegen* [Helmholtz 1868], *Über den Ursprung und die Bedeutung der geometrischen Axiome* [Helmholtz 1870], *Die Tatsachen in der Wahrnehmung* [Helmholtz 1878], *Zählen und Messen, erkenntnistheoretisch betrachtet* [Helmholtz 1887]. Helmholtz maintains that geometric knowledge rests upon general facts to be induced by experience, especially our experiences with solid bodies. Such experiences are required in order to determine the most basic kind of relationship between spatial magnitudes, namely, their congruence. If two such magnitudes are to be proved to be congruent, they must be brought to coincidence. The condition for observing this particular outcome is the well-known experience that solid bodies can be displaced without changes in shape and size. In mathematical terms, if spatial magnitudes are to be measured, their points must remain fixedly-linked during displacement. Helmholtz deems those bodies whose elements satisfy this requirement *rigid,* and maintains that the geometric properties we attribute to physical space depend on facts such as the free mobility of rigid bodies. Since physical bodies are only approximately rigid, and free mobility is presupposed as a precise, mathematical condition of measurement, it is clear that geometrical propositions cannot be directly tested. Moreover, in 1869, after his correspondence with Eugenio Beltrami, Helmholtz realizes that a non-Euclidean metric might satisfy the same conditions as Euclid's metric. He concludes that the geometry of our space is only approximately Euclidean or not notably different from Euclidean geometry. But he does not exclude that under different circumstances Bolyai-Lobačevskij's geometry might also be adopted.

In his 1870 paper Helmholtz discusses the consequences of his analysis of geometric knowledge for the Kantian philosophy of geometry as follows. He maintains that the principles of geometry cannot be a priori synthetic judgements in Kant's sense. Unlike a priori judgements, they are not necessarily valid, because their validity with regard to the empirical manifold is only hypothetical. Therefore Helmholtz's principles are supposed to be derived from experience. As an alternative, a strict Kantian might consider the concept of fixed geometric structure a transcendental one, in which case, however, the axioms of geometry would follow analytically from the same concept, because only structures satisfying those axioms could be acknowledged as fixed ones [Helmholtz 1921, 24, Eng. trans. 25].

Poincaré's well-known objection is that Helmholtz's definition of congruence between spatial magnitudes already presupposes geometry. Helmholtz believes that his definition follows from the fact that two such magnitudes can

writes: "*Es bedeutet dabei keine starke Zuspitzung, eine solche Genauigkeitsschranke als informelle Fassung der Abschätzung einer mit den Messergebnissen verträglichen oberen Schranke für den Betrag der Raumkrümmung zu interpretieren*" [Scholz 2004, 364ff.].

be brought into congruent coincidence. Apparently, the observation of this particular outcome presupposes some kind of displacements. Poincaré's question is: how are such displacements to be defined? If they are supposed to leave the magnitudes to be compared unvaried in shape and size, the concept of rigid figure is presupposed from the outset and geometrical empiricism is unable to escape circularity [Poincaré 1902, 60]. So Poincaré rejects Helmholtz's attempt to show how geometric notions can be derived from some facts: the foundations of geometry are to be better understood as general rules. On the other hand, he agrees with Helmholtz that geometrical axioms cannot be a priori synthetic judgements. Poincaré's view is that rules such as the free mobility of rigid bodies follow from the definition of such bodies and must be stipulated. Therefore he maintains that one geometry cannot be more true than another; it can only be more convenient [Poincaré 1902, 91].

Poincaré develops his argument in a series of papers which he collects in 1902 in his *La Science et l'Hypothèse* [Poincaré 1902]. Hölder does not seem to be acquainted with Poincaré's earlier papers at the time of *Anschauung und Denken.* Nevertheless, he takes into account a similar objection to Helmholtz, and yet develops a different conception of geometry. Poincaré's argument for the conventionality of geometry presupposes his rejection of both Kantianism and empiricism. I do not think that he rules out all solutions other than conventionalism. Hölder proceeds in a different way. He does not emphasize the opposition between Kantianism and empiricism, because he seeks to profit from Kantian objections in order to develop a consistent empiricism. His argument for empiricism depends on his analysis of mathematical method rather than on Helmholtz's physiological conception of spatial intuition. Hölder's analysis is strictly connected with his approach to the theory of measurement. Moreover, he develops his methodological considerations in his later work *Die mathematische Methode: Logisch erkenntnistheoretische Untersuchungen im Gebiete der Mathematik* [Hölder 1924]. There he makes explicit that he is also influenced by the Kantian view that mathematics is synthetic.

In the following, I will present Hölder's emendation of Helmholtz's empiricism. Moreover, I will analyze Hölder's methodological considerations and discuss his relationship with Kant. In the last section of my paper, I will suggest a comparison with the Marburg School of neo-Kantianism, especially Ernst Cassirer. Both Hölder and Cassirer develop a synthetic conception of geometry which is compatible with the use of deductive method, and which enables them to take into account non-Euclidean geometry.

1 Helmholtz's empiricism revised

Geometrical empiricism (broadly construed) was popular among German mathematicians in the second half of the 19th century. In particular, Hölder was influenced by Helmholtz and by Moritz Pasch. I will focus on Helmholtz, because his epistemological writings were widely discussed not only among mathematicians but also among philosophers and his assessment of Kant's philosophy of geometry was seminal for the debate Hölder refers to.

Hölder summarizes the different views on spatial intuition and geometry as follows. Kant's view is that a pure intuition endowed with subjective laws independently of experience makes experience first possible. By contrast, 19th-century thinkers, such as Julius Baumann and Wilhelm Wundt, maintain that spatial intuition can be induced by experience. Both standpoints look at intuition as a source of geometric knowledge. According to Hölder, Helmholtz avoids any commitment to the supposedly intuitive character of geometrical axioms and explains basic geometric concepts "in a more physical way" [Hölder 1900, 3, Eng. trans. 16]. At the same time, this explanation requires that our perceptions show some regularity, so that experiences form a general system in Kant's sense. Hölder writes:

> Of course, the advocates of this empirical viewpoint will not deny that any elaboration of experience comes out from assumptions, at least from the assumption of a certain conformity to laws of the investigated object, which we could not otherwise grasp conceptually (compare in particular [Helmholtz 1921, 148ff.]). Indeed, any single fact of experience, if expressed by means of concepts—and how could one want to express it otherwise—is the result of a mental elaboration of experience. But unlike Kant's conception, empiricism emphasizes the fact that according to the empirical viewpoint no law referring to external objects comes about independently from external experience. Kant says on the contrary that geometrical knowledge comes from intuition and that intuition is independent from experience. [Hölder 1900, 4, note 9, Eng. trans. 28]

Hölder emphasizes that the recognition of some facts presupposes both observation and reasoning. He therefore rejects the view that empirical laws can be generalized by induction. He writes:

> On the contrary, it seems to me that the intellectual activity that from the beginning goes hand in hand with experience, when we build concepts of experience, is a preliminary activity that should not be assigned neither to deduction nor to induction. [Hölder 1900, 6, note 16, Eng. trans. 31]

Hölder's account enables him to defend Helmholtz's geometrical empiricism from the charge of circularity. Hölder presents such objection as a Kantian one and summarizes it as follows. Since geometrical axioms are both exact and general rules, the role of single experiences in their development is only occasional. Their source, according to Kant, is a kind of intuition in which there is nothing that belongs to sensation. At the same time, Kant's pure intuition is a condition of experience. From this point of view, any attempt to infer geometrical propositions from experience would entail a vicious circle. For instance, Hölder mentions Helmholtz's definition of congruence. This definition had already been called into question by Poincaré. Hölder does not seem to be acquainted with Poincaré's remark. Nevertheless, a similar objection had been formulated also by Konrad Zindler in a paper entitled *Beiträge zur Theorie der mathematischen Erkenntnis,* from which Hölder quotes. Zindler maintains that Helmholtz's attempt to define the congruence of spatial magnitudes physically (i.e., by comparison of rigid bodies) already presupposes the geometric concept of congruence [Zindler 1889, 11].

Hölder's reply is that a consistent description of the empirical origin of geometric concepts is possible. Such description should be based on observations which do not presuppose geometry. For instance, approximately rigid bodies may be distinguished from non-rigid ones, because the former can be easily brought back to their initial position after displacement [Hölder 1900, 5ff., Eng. trans. 17], [Hölder 1924, 371].[2] Helmholtz's free mobility can be derived from the fact that parts of two such bodies can be brought to coincidence and that such experience can be repeated at any time. Hölder's point is that, since facts are basically singular, Helmholtz's facts underlying geometry are to be better understood as rules for inferring facts in any further cases [Hölder 1900, 4, note 12, Eng. trans. 29].[3]

The core of Hölder's geometrical empiricism is the claim that geometric concepts do not differ from empirical concepts such as those of mechanics and optics. All of these concepts are supposed to be related to the empirical manifold in a twofold way: they can be said to be derived from experience, and yet, once they have been defined, they make our understanding of experiences possible. Therefore general laws such as geometrical axioms have to be postulated. More precisely, our requirement is that such laws are valid with regard to the empirical manifold. On the one hand, geometrical axioms are exact statements and enable deductive inferences; on the other, their validity with regard to experience is but hypothetical. I think that Hölder recognizes a relativized a priori in the following consideration about the concept of space:

> It will no longer appear contradictory that, though we use this concept in some cases in order to interpret experience, we nevertheless consider it possible to check this concept—whose adequateness is hypothetical—for correspondence with experience in

2. In 1924, Hölder recalls that Poincaré developed a similar explanation in *La Science et l'Hypothèse* [Poincaré 1902, 79]: "Parmi les objects qui nous entourent, il y en a qui éprouvent fréquemment des déplacements susceptibles d'être [...] corrigés par un mouvement corrélatif de notre propre corps, ce sont le *corps solides*."

3. Even though Pasch's *Vorlesungen über neuere Geometrie* [Pasch 1882] are not mentioned in this regard by Hölder, he shows elsewhere that he is well acquainted with Pasch's writing and appreciates his attempt to provide an empirical explanation of primitive terms [Hölder 1900, 2, note 3, Eng. trans. 27]. Hölder might also have been influenced by Pasch's following consideration regarding geometrical principles: "Die Grundsätze sollen das von der Mathematik zu verarbeitende empirische Material vollständig umfassen, so dass man nach ihrer Aufstellung auf die Sinneswahrnehmungen nicht mehr zurückzugehen braucht" [Pasch 1882, 17].

> order to reshape the concept, if necessary, as we do with physical concepts. [Hölder 1900, 21, note 64; Eng. trans. 46]

In the following, Hölder points out that Kant also speaks of a priori with regard to principles of mechanics such as the conservation of matter and distinguishes between a pure part of natural science and an empirical one [Hölder 1900, 21, note 67, Eng. trans. 47]. It seems that space and geometry may be deemed a priori in the same sense. [4]

In *Die mathematische Methode* Hölder analyzes Archimedes' proof of the Law of the Lever to show that the underlying suppositions play the role of axioms. Hölder admits that it may not be possible to develop the whole of mechanics by deductive means [Hölder 1924, 50]. His argument is that geometry and mechanics are not different in principle, and this argument depends on his view that geometry and physics form a single system of concepts.

Hölder's view is made explicit in his discussion of Lobačevskij's experiment. Lobačevskij's conjecture is that the sum of the angles in a triangle might be proved to be less than 180 degrees, provided that the chosen triangle is sufficiently large. Therefore he needs astronomical measurements. At the time Hölder wrote, such procedure had already been criticized by Hermann Lotze and Poincaré. Hölder summarizes earlier criticisms as follows. In order to measure an angle directly, one should be placed at its vertex. Since the angles of an astronomical triangle cannot be measured simultaneously, it is clear that the result of measurement largely rests upon the adopted theories, including both optics and mechanics. Thus, if the sum of the angles is less than 180 degrees, we are not compelled to adopt Bolyai-Lobačevskij's geometry instead of Euclid's geometry. We may either suppose that light does not travel rigorously in a straight line or change the laws of motion, see [Poincaré 1902, 83 ff.]. Hölder's remark is that our entire system of concepts is involved: changes are not arbitrary, but for instance, in order to change the optical part of our system, we should find out that light travels according to the laws of the chosen geometry. Hölder emphasizes that not every geometry

4. Similarly, Hans Reichenbach distinguishes between axioms of coordination and axioms of connection. Axioms of connection are empirical and presuppose sufficiently well-defined concepts. Axioms of coordination are those non-empirical principles which make such definitions possible. The latter ones are deemed a priori relative to a theory, insofar as they can be distinguished from the former ones [Reichenbach 1920, 46 ff.]. See also [Friedman 1999, 61]: "Thus, for example, Gauss's proposed 'experiment' to determine the geometry of physical space *presupposes* the notion of 'straight line,' which notion is simply not well defined independently of the geometrical and optical principles supposedly being tested. The inadequacy of such an attempt thus makes it clear that at least some geometrical principles must be laid down antecedently as axioms of coordination before any empirical determination of space even makes sense. [. . .] Physical geometry is [. . .] revisable and can evolve and change with the progress of science. No geometry is necessary and true for all time. Nevertheless, at a given time and in the context of given scientific theory, the axioms of coordination—at that time and relative to the theory—are still quite distinct from the axioms of connection."

can be used to describe the same observations [Hölder 1924, 400]. So the view that geometry can be empirically tested is to be rejected. Nevertheless, the criticism of Lobačevskij's experiment does not provide a counterargument against Hölder's claim that geometry can be put to the test indirectly, namely, along with the remaining parts of our system of concepts.

To sum up, Hölder maintains that geometry does not differ from mechanics: they both need axioms, which are exact and general propositions, and yet are only hypothetically valid with regard to the empirical manifold. On the other hand, he draws a fundamental distinction between geometry and arithmetic, because arithmetic does not need axioms in the above mentioned sense. As we shall see, Hölder's motivation for his classification of the sciences is to be found in his analysis of mathematical method. But first of all, his considerations in this matter entail a rejection of intuition both as a source of truth and as an indispensable tool for proof.

2 Deductive method in geometry and Hölder's approach to the theory of proportions

The claim that arithmetic has no axioms goes back to Kant. Hölder quotes from the following passage of the *Kritik der reinen Vernunft*:

> On this successive synthesis of the productive imagination, in the generation of shapes, is grounded the mathematics of extension (geometry) with its axioms, which express the conditions of sensible intuition a priori, under which alone the schema of a pure concept of outer appearance can come about; e.g., between two points only one straight line is possible; two straight lines do not enclose a space, etc. These are the axioms that properly concern only magnitudes *(quanta)* as such.
> But concerning magnitude *(quantitas),* i.e., the answer to the question "How big something is?", although various of these propositions are synthetic and immediately certain *(indemonstrabilia)*, there are nevertheless no axioms in the proper sense. [Kant 1787, 204ff.; Eng. trans. 288]

Since geometry and arithmetic deal with different objects, namely, with spatial magnitudes and the magnitude of a quantity respectively, they also differ with regard to their methods. Kant's claim is that geometric objects are to be constructed, whereas the answer to the question: "How big something is?" is to be found by means of calculations. Geometrical axioms are finite in number and express those conditions of outer intuition a priori which are required for spatial magnitudes to be constructed. Kant maintains that arithmetic has no axioms in the proper sense, because it rests upon analytic judgements, not synthetic ones. According to him, arithmetic also entails a priori synthetic judgements. However, he calls these propositions *numerical formulas* and distinguishes them from axioms, because they are infinite in number and there is only one way to accomplish the corresponding calculation. By contrast, one and the same geometric construction can be realized in infinitely many ways in principle—for example by varying the length of the lines and the size of the angles in a given figure whilst leaving some given proportions unvaried.

Note that in this passage Kant makes it clear that the pure intuition of space as analyzed in the Transcendental Aesthetic does not entail geometrical axioms. These are now said to be grounded in the synthesis of the productive imagination, which is distinguished from the empirical, receptive one, because of its spontaneity. Therefore Kant sometimes (e.g., at B 155) identifies the productive imagination with the understanding. More precisely, the corresponding synthesis is required for the concepts of the understanding to be applied to the manifold of pure intuition. So Kant is not committed to the claim that our spatial intuition alone entails Euclid's axioms, even though he might have thought that these are the only possible ones.

Hölder, as most of his contemporaries, overlooks this aspect. Nevertheless, he is right to point out that Kant's characterisation of geometry apparently presupposes Euclid's method and the use of intuition as an indispensable tool for proof. By contrast, Hölder argues that intuition, although useful in praxis, can be replaced by a formal use of concepts. All proofs in geometry presuppose objects such as points, lines, etc. and their relationships to one another, and consist in an inference from the relations between the given objects to the relations between new objects. Intuitively, the proof may require that auxiliary objects be constructed and the inference is made by analogy: singular figures are supposed to exemplify all similar cases and such supposition depends on a case by case basis. By contrast, the consequences of a deductive inference follow necessarily from general premises to be postulated, namely, the axioms. Hölder's claim is that intuitive suppositions and concrete constructions in geometry are then replaced by geometrical axioms. He mentions, for example, the proof that the sum of the angles in a triangle is equal to 180 degrees. Given the sides of a triangle ABC and its angles α, β, and γ, the auxiliary object to be constructed is the parallel line to the base of the triangle from the opposite vertex. At the same time, two new angles γ' and α' are constructed, which obviously form two right angles, if summed with the angle β. So α should be identified with α', and γ with γ'. Hölder points out that the construction of the parallel line presupposes the existence of such a line through any given point, namely, Euclid's fifth postulate. The conclusion then follows from the assumption that alternate angles formed by a transversal of two parallel lines are equal, which is Euclid's proposition I.31.

The formulation of geometrical axioms enables rigorous proofs and makes the lack of precision of intuition inessential. Moreover Hölder maintains that geometric proofs can be developed deductively also in the case that the underlying assumptions contradict our intuitions. For example, Lobačevskij's assumptions entail that the sum of the angles in a triangle cannot be more than 180 degrees, and that such sum is less than 180 degrees in all triangles, if it is so in any given triangle.

Hölder gives further examples in *Die mathematische Methode* [Hölder 1924, 18ff.]. They show that mathematical reasoning consists of a "concatenation of relations" *(Verkettung der Relationen)* [Hölder 1924, 24]. In fact, already in *Anschauung und Denken,* [Hölder 1900] Hölder points out that the axiomatic method does not deal with objects, but rather with their relationships to one another. Some objects must be presupposed as given in intuition or experience, and yet they can be studied independently of their origin. By contrast, Hölder maintains that arithmetic objects exist independently of experience, because they depend on the indefinite repetition of an operation of our thought. Therefore arithmetic does not need axioms, but rather recurrent definitions. For example, the formula $ab = (a-1)b + b$ provides a definition of multiplication, provided that the concept of sum is already known. This definition is recurrent, because the concept at stake is the generalized rule that governs a series of prepositions like: one times b is b, two times b is one times b plus b, three times b is two times b plus b, etc. [Hölder 1924, 19, note 58]. Similarly, Grassmann's formula $a + (b + 1) = (a + b) + 1$ provides a definition of sum and means that the sum of a number a and the successor $b + 1$ of a number b is the successor of the number $a + b$ in the series of integer numbers [Hölder 1901, 1ff.].

A similar classification of the sciences is to be found in Hermann Grassmann's introduction to his *Ausdehnungslehre* [Graßmann 1844]. Grassmann distinguishes between real and formal sciences: the former ones reproduce something real and their truth depends on the correspondence between thought and its objects; the objects of the latter ones are produced by the acts of our thought and truth depends on the coherence of such acts with one another. These kinds of sciences differ at the very beginning: formal sciences begin with definitions whilst real sciences presuppose some principles. In this regard, Grassmann mentions the distinction between arithmetic and geometry [Graßmann 1844, XIX, and note].

Hölder agrees with Grassmann that the objects of geometry are given in intuition or experience, whereas the objects of arithmetic must be constructed. On the other hand, Hölder maintains that given objects can also be constructed. His example is the construction of the concept of measurable magnitude *(Maß)*. He therefore presupposes the axioms of order, those of equality, and an equivalent formulation for Dedekind's continuity: *If all quantities Q are divided into two classes such that each quantity belongs to one and only one class, and each member of the first class is less than each member of the second class, then there is a member ξ of Q such that each $\xi' < \xi$ belongs to the first class, and each $\xi'' > \xi$ belongs to the second class.* Dedekind calls *cuts* all partitions of rational numbers in two such classes. Each rational number corresponds to one and only one cut and irrational numbers can be defined by requiring that each cut corresponds to one and only one number [Dedekind 1872, 13].

Hölder shows how to use the Dedekind's continuity to deduce the divisibility of a given segment into equal parts and to prove a theorem which is equivalent to the Archimedean axiom: *Given two segments a and b, and $a < b$, there is an integer number n such that $na > b$* (see [Hölder 1900, 17, note 49, Eng. trans. 40]).

This proof is supposed to provide us with a construction of the concept of magnitude, because Hölder considers divisibility and being Archimedean as conditions of measurement. This way of proceeding enables him to improve Euclid's theory of proportions by making the requirement that the laws of addition hold for segments explicit. Once the Archimedean axiom has been deduced from the Dedekind's continuity and the theory of numbers is presupposed, Euclid's definition of proportions can be extended to non-commensurable segments and reformulated as follows: *The ratio of a to b equals the ratio of a' to b', if any multiple μa of a is equal to, less than or more than any multiple νb of b, when $\mu a'$ is equal to, less than or more than $\nu b'$.* The concept of measurable magnitude is not presupposed from the outset, because μ and ν are not considered as the magnitudes of a given quantity *(Maßzahlen)*, but rather as positive integers whatsoever [Hölder 1900, 17, Eng. trans. 24, and note 52].

Hölder's theory of proportions is developed in detail in the first part of his 1901 paper *Die Axiome der Quantität und die Lehre vom Mass.* He formulates a minimal set of axioms of quantity, including the axiom of continuity, by means of which he proves divisibility and the Archimedean property. He then presents the theory of irrational numbers and proceeds as follows. Suppose that a and b are quantities of the same kind and that μ/ν is called a *lower fraction* in relation to their ratio $a : b$, if $\nu a > \mu b$; and an *upper fraction,* if $\nu a \leq \mu b$. By the Archimedean property, there exist both a positive integer ν, such that $\nu a > 1b$, and a positive integer μ, such that $1a < \mu b$. Thus, in relation to the ratio $a : b$, there exist both lower fractions and upper ones, and in every case the lower fractions are less than the upper ones. It follows from the axiom of continuity that: *For every ratio of quantities $a : b$, i.e., for each pair of members of Q which are given in a determined order, there is one and only one cut, i.e., a determined number in the general sense of the word* [Hölder 1901, § 10].

This theorem justifies Newton's requirement that the ratio of quantities of the same kind be expressed by positive real numbers. Since order and operations with cuts can be developed arithmetically, the laws of addition apply to all objects that satisfy the axiom of continuity and the remaining axioms of quantity.

In the second part of the paper, Hölder presents a model of a non-Archimedean continuum. This is the most original part of the paper. So it is surprising that, in his introductory remarks, Hölder emphasizes above all the importance of the said theorem for the analysis of measurement. Nevertheless his remarks are interesting for our present topic, because they reflect his philosophical assumptions. Since Hölder develops a formal theory of quantity, nothing prevents us from considering his results independently of such assumptions. Nevertheless he vindicates a non-formalistic approach and maintains that the general concepts of arithmetic, such as the concept of number and that of sum, are developed by "purely logical" means and cannot be reduced to a symbolic calculus [Hölder 1901, 2, and note].

This claim is made clearer in *Die mathematische Methode.* Hölder maintains that those thoughts which may lead to the introduction of new symbols cannot be represented by another symbolic computation. Otherwise there would be an infinite regress. Hölder's view is that symbols represent concepts, and these can be constructed. He deems concepts whose development does not require assumptions other than the operations of our thought, such as the concept of coordination, series, number, and group, *purely synthetic,* and distinguishes them from the concepts of geometry and mechanics, which are deemed *hypothetico-synthetic* or simply *synthetic* and require special assumptions [Hölder 1924, 5ff., 295].

Hölder's approach to the theory of proportions shows the attempt to replace given magnitudes with arithmetical reasoning. Similarly, he maintains that projective metric must be complemented by the theory of irrational numbers [Hölder 1901, 19, note].

Already in 1847 Christian von Staudt had excluded metric concepts from the foundations of projective geometry by defining the harmonic relationship among four points by means of incidence relationships alone. This way of proceeding enables him to develop a projective metric independently of the axioms of congruence. In order to give projective coordinates, Staudt presupposes the fundamental theorem of projective geometry and the properties of involutions, and develops his calculus of jets, see [Nabonnand 2008, 202ff.]. As an alternative, a projective scale on a straight line can be constructed by determining the fourth harmonic to three given points. The resulting points can be coordinated with the series of rational numbers. As regards irrational points, their introduction requires that each point corresponds to one and only one number, which can be either rational or irrational. One such construction is sketched by Klein in the second part of his second paper *Über die sogenannte Nicht-Euklidische Geometrie* [Klein 1873, § 7], and developed by him in detail in his lectures on non-Euclidean geometry [Klein 1893, 315ff.]. Hölder himself deals with that subject-matter in an essay entitled *Die Zahlenskala auf der projektiven Geraden und die independente Geometrie dieser Geraden* [Hölder 1908]. In the introduction, he writes:

> The fundamental feature of inquiries such as those presented here lies in the development of proofs, which is to some extent purely logical: *the entire, intuitive content of the theory must be put in the postulates.* [Hölder 1908, 168]

A few years later, in 1911, Hölder also presents a calculus of segments which proceeds as the calculus Hilbert developed in the fifth chapter of his *Grundlagen der Geometrie* [Hilbert 1899]. In order to define operations with segments and give coordinates, Hilbert uses not so much the fundamental theorem of projective geometry, as, more simply, the theorem of Desargues. He presupposes all axioms of linear and plane geometry (connection, order, and the axiom of parallel lines), except the axioms of congruence and the Archimedean axiom [Hilbert 1899, 55ff.]. According to Hölder, this approach also enables the development of projective geometry in the manner of Staudt, that is, without metric foundations [Hölder 1911, 67, and note].

Hölder's approach sheds some light on his connection with Helmholtz. In the paper Hölder refers to, *Zählen und Messen, erkenntnistheoretisch betrachtet* [Helmholtz 1887], Helmholtz formulates additive principles for physical magnitudes. He does not presuppose Euclidean space, but rather the laws of arithmetic, on the one hand, and conditions such as the free mobility of rigid bodies, on the other. Despite the fact that Helmholtz considers the laws of arithmetic as axioms, Hölder agrees with him that arithmetic is to be connected with the theory of quantity ([Hölder 1901, 1, note]; on Hölder's connection with Helmholtz, see also [Michell 1993, 196]). Nevertheless he does not agree with Helmholtz's claim that numbers are but arbitrarily chosen symbols, on the one side, and that arithmetic is a method constructed upon psychological facts, on the other [Helmholtz 1921, 72]. Helmholtz believes that the series of natural numbers depends on our capacity to retain in our memory the sequence in which the acts of consciousness occurred in time. The time sequence is supposed to provide us with a natural basis for numerical symbolism, because it requires us to adopt a system of symbols allowing neither interruptions nor repetitions, as in the decimal system. Hölder's philosophy of arithmetic differs from Helmholtz's one, because it entails not so much a psychological approach to the theory of numbers, as a logical one. Since Hölder also rejects formalism, his conception of logic shows a philosophical aspect which has much in common with Kant's transcendental logic, see [Radu 2003, 343].

3 Hölder's connection with Kant: From *Anschauung und Denken* to *Mathematische Methode*

Since Hölder in *Anschauung und Denken* rejects both the assumption of geometric intuition and the apriority of Euclidean geometry, it might seem that he rejects Kant's standpoint altogether. Moreover, even if Hölder acknowledges that synthetic reasoning plays a role in the development of some concepts, he maintains that Kant's much discussed classification of judgements has become pointless [Hölder 1924, 8, 292, note]. Nevertheless, after introducing his distinction between hypothetico-synthetic and pure synthetic concepts in *Die mathematische Methode,* Hölder vindicates a Kantian conception of mathematics. Of course Hölder's view that geometrical axioms are only hypothetically valid for the empirical manifold contradicts Kant's claim that the axioms of geometry, including Euclid's fifth postulate, are necessary conditions of knowledge. On the other hand, Hölder rejects Helmholtz's more radical view that the laws of our thought depend on our habits and claims that mathematics is an a priori science in Plato's and Kant's sense. So Hölder admits a priori knowledge. What he calls into question is the border between a priori knowledge and experience as drawn by Kant. As Hölder puts it, the borderline must be drawn somewhere else [Hölder 1924, 7].

A priori knowledge in Hölder's sense is restricted to progression and continuity [Hölder 1924, 391]. Let us begin with Hölder's reflections on continuity. He maintains that this concept cannot be generated arithmetically. Dedekind's continuity entails that the totality of cuts corresponds to all the points of a straight line or linear continuity. By definition each cut requires a particular rule. Hölder's objection is that the totality of such rules might be contradictory [Hölder 1924, 193ff., 325, and note]. In order to prevent this problem, Hölder considers Dedekind's continuity as a definition in the theory of irrational numbers and as an axiom in the theory of quantity. Therefore he reintroduces Kant's pure intuition. Since one-dimensional continuum enables the construction of two-, three- or more dimensional continuum, it can be regarded as a form. And since it can neither be derived from purely synthetic concepts nor proved to be consistent, it must be given a priori [Hölder 1924, 351].

On the other hand, Hölder maintains that progression does not require further justification, because it is a product of thought. As he puts it, progression is the form of that process of thought which enables us to explain one concept by means of one other or to use the consequence of an inference as a premise for a further inference in those cases in which each inference presupposes the preceding one. For example, the principle of mathematical induction as formulated by Giuseppe Peano [Peano 1889] can be described as that inference from n to $n+1$ which generalizes a chain of inferences from 1 to 2, 2 to 3, etc., see also [Poincaré 1902, 20ff.]. According to Hölder, recurrent definitions occurring in arithmetic are grounded in progression, and the same holds true for those geometric constructions whose development requires us to indefinitely repeat some operations [Hölder 1924, 338].

The basic concepts of arithmetic (including the theory of irrational numbers) are supposed to exist owing to their purely logical development. This suggests that the consistency of geometry can be derived from that of arithmetic. By contrast, the consistency of a one-dimensional continuum cannot be taken for granted. Therefore the theory of quantity presupposes both progression as a form of thought and continuity as a form of intuition.

Hölder's view is problematic, for several reasons. Firstly, he does not take into account different conceptions of continuity such as Veronese's. As a consequence, the borderline he draws between a priori knowledge and experience might seem arbitrary. Secondly, it is questionable whether this way to reconsider Kant's philosophy of mathematics is appropriate. I think that Hölder prepares a subtler strategy when he points out that a priori knowledge in Kant's sense might turn out to be hypothetical. This suggests that the fundamental feature of a priori knowledge lies not so much in its intrinsic necessity, as in its constitutive function. Such function is predominant in Hölder's considerations about progression, but at least continuity is said to be both necessary and intuitive because it cannot be constructed. It is likely that Hölder might have had some hesitation about reintroducing Kant's pure intuition. In fact, he agrees with interpreters such as Cassirer, who points out that Kant uses the expression "form of intuition" in a very general sense: arithmetic is grounded not so much in our conception of time, which is only a kind of progression, as in the form of progression [Hölder 1924, 339, and note]; see also [Cassirer 1907, 34, note]. In this case it is clear that Kant's form of intuition might be described as a form of thought to be analyzed in mathematical logic. Kant might have sought for intuitive foundations of mathematics, rather than logical ones, because the syllogistic logic of his time did not suffice for such analysis. For that reason he used transcendental logic. We already mentioned that Hölder overlooks the role of Kant's productive imagination in the development of geometric knowledge and interprets Kant's form of outer intuition as it entailed Euclid's geometry. Hölder's distinction between time and progression enables him to avoid a similar misunderstanding at least with regard to the form of inner intuition.

For the above mentioned reasons, I think that Hölder's most helpful insights into the Kantian conception of a priori knowledge are to be found not so much in his general distinction between a priori and empirical knowledge, but rather in his analysis of mathematical method. In this regard, it is worth noting that there might be a connection between Hölder and the Marburg School of neo-Kantianism.

4 Hölder and the Marburg School of neo-Kantianism: Natorp and Cassirer

There is evidence that Hölder appreciates the philosophical logic of Paul Natorp, who studies the principles of the exact sciences in order to prove the possibility of knowledge. Not only is Natorp's major work on that subject-matter, *Die logischen Grundlagen der exakten Wissenschaften* [Natorp 1910], often quoted by Hölder in *Die mathematische Methode,* but it is mentioned by him in the preface as an example of methodological inquiry next to Bertrand Russell and Louis Couturat.[5]

Hölder might agree with Natorp that Kant's forms of intuition depend not so much on the assumption of pure intuitions, as on some processes of thought. For instance, Natorp analyzes the development of the series of natural numbers [Natorp 1910, 98ff.]. This section of Natorp's work is mentioned by Hölder in support of his own analysis of progression. Moreover he openly agrees with Natorp's rejection of formalism [Hölder 1924, 2ff.]. Natorp acknowledges that logic is the science of deduction, and yet his claim is that logic does not need to be developed deductively [Natorp 1910, 5]. Both Hölder and Natorp focus on methodological issues. On the other hand, they disagree on the solution to many such issues, especially in the philosophy of geometry. Natorp's attempt is to show that if spatial order is to be determined in one and the same way, the geometry of space must be Euclidean. He does not deny the formal-logical possibility of non-Euclidean geometry. Nevertheless he deems Euclid's axioms necessary conditions of possible experience [Natorp 1910, 312]. By contrast, it follows from Hölder's analysis of deductive method in geometry that geometers only infer those consequences which follow necessarily from some assumptions. Since the development of non-Euclidean geometry shows that such assumption cannot be necessary themselves, Hölder maintains that the constitutive function of the geometry underlying the interpretation of measurements is relative to scientific theories.

Hölder's views may have more in common with the philosophy of geometry developed by Cassirer in *Substanzbegriff und Funktionsbegriff: Untersuchungen über die Grundfragen der Erkenntniskritik* [Cassirer 1910]. In the second chapter, which is devoted to space and geometry, Cassirer maintains that the development of geometric concepts shows the characteristics of the deductive method. Individuals in axiomatic systems are replaced by their relationships to one another and cease to be objects of intuition. Moreover Cassirer deems initial assumptions hypothetical, because they might be true only with regard to their consequences [Cassirer 1907, 29], [Cassirer 1910, 123ff.].

Note that Cassirer's example for the realization of the deductive method in geometry is the same as Hölder's example, namely, the development of projective geometry in the manner of Staudt-Klein. Cassirer writes:

> [...] without any application of metrical concepts, a fundamental relation of position is established by a procedure which uses merely the drawing of straight lines. The logical ideal of a purely projective construction of geometry is thus reduced to a simpler requirement; it would be fulfilled by showing the possibility of deducing all the points of space in de-

5. It is astonishing that Hölder does not mention more recent studies. However, he declares that his writing was completed during the First World War. Since at that time he found difficulties to publish, *Die mathematische Methode* was published only in 1924 by Springer [Hölder 1924, IV].

> terminate order as members of a systematic totality, by means merely of this fundamental relation and its repeated application. [Cassirer 1910, 113, Eng. trans, 86]

Such possibility is realized by the construction of a projective scale on a straight line. Cassirer does not seem to be acquainted with Hölder's construction of 1908. Nonetheless he mentions the construction developed by Klein in his lectures on non-Euclidean geometry, [Klein 1893, 15]. Moreover Cassirer refers to projective metric as developed by Arthur Cayley [Cayley 1859] and henceforth applied to non-Euclidean geometry by Klein [Klein 1871]. Cassirer maintains that projective metric entails a deduction of the concept of space in its most general form, independently of the theory of parallel lines. In order to distinguish between Euclid's geometry and non-Euclidean geometries (*parabolic, elliptic,* and *hyperbolic* geometry, in Klein's terminology), special assumptions are required. So the choice among equivalent geometries cannot be determined by our conception of space. Since Euclid's geometry is included in a more general classification of hypotheses, Cassirer does not exclude the possibility that a different geometry might be used in physics [Cassirer 1910, 147].

There is no evidence that Hölder and Cassirer read each other's works at that time. And still in *Die mathematische Methode* Hölder mentions only Cassirer's earlier paper *Kant und die moderne Mathematik* [Cassirer 1907]. Nonetheless, since at least 1936-1937, Cassirer has been appreciating Hölder's description of mathematical reasoning and he has been adopting Hölder's expression "concatenation of relations" [Cassirer 1936-1937, 47], [Cassirer 1940, 82]. Cassirer's point is that mathematics provides knowledge, because its general concepts are developed progressively. By contrast, syllogistic inferences presuppose a hierarchy of concepts, some of which are given. Hölder's approach is mentioned by Cassirer as an example of purely logical approach, because it shows how the top-down direction of syllogistic reasoning can be reversed, so that given concepts can be constructed.

Concluding remarks

The comparison with Cassirer may shed some light on Hölder's claim that his approach to the theory of quantity is purely logical, and yet not formalistic: it is logical because of his constructive character, and it is not formalistic because of his bottom-up direction. As regards geometry, Cassirer's views have much in common with Hölder's views because Cassirer, unlike other neo-Kantians, acknowledges the hypothetical character of geometrical axioms. There is no evidence that Cassirer and Hölder influenced each other in the development of their views. Nonetheless, their agreement shows that Hölder's attempt to defend a conception of geometry other than conventionalism, and yet compatible with the use of axiomatic method, was not isolated. Both Hölder and Cassirer show that there might be an unexpected affinity between Kantianism and empiricism, provided that methodological issues are discussed independently of ontological assumptions. From both points of view, empirical measurements require more general laws, and Kantians and empiricists might also agree that the search for such laws must be freed from unjustified presuppositions such as habits or intellectual authorities. Of course such a comparison between Kantianism and empiricism is only possible after Kant's claim that the axioms of (Euclid's) geometry are necessary conditions of experience has been rejected. Nonetheless, Kant's a priori knowledge can be relativized, so that geometry can have a constitutive function relative to scientific theories. I think that a relativized conception of the a priori follows from Hölder's holistic view that geometry can provide rules for the interpretation of measurements only as a part of a complex system of concepts also including optics and mechanics. Hölder's related empiricist view is that such system can be empirically tested as a whole.

Despite the various aspects of Hölder's epistemology, I think that he cannot be charged with eclecticism. His classification of mathematical objects rests upon his analysis of mathematical method. It seems to me that this way of proceeding is basically Kantian. At the same time, especially as regards the problems concerning space, Hölder admits solutions other than those provided by Kant, because he takes into account scientific developments such as non-Euclidean geometry, projective geometry, and axiomatic method. In these regards, he finds points of agreement with various thinkers, such as Hermann Grassmann, Helmholtz, and Pasch, and reformulates their arguments from an original point of view.

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